FINAL REPORT

NASA Grant NGR 44-005-039

Submitted to the

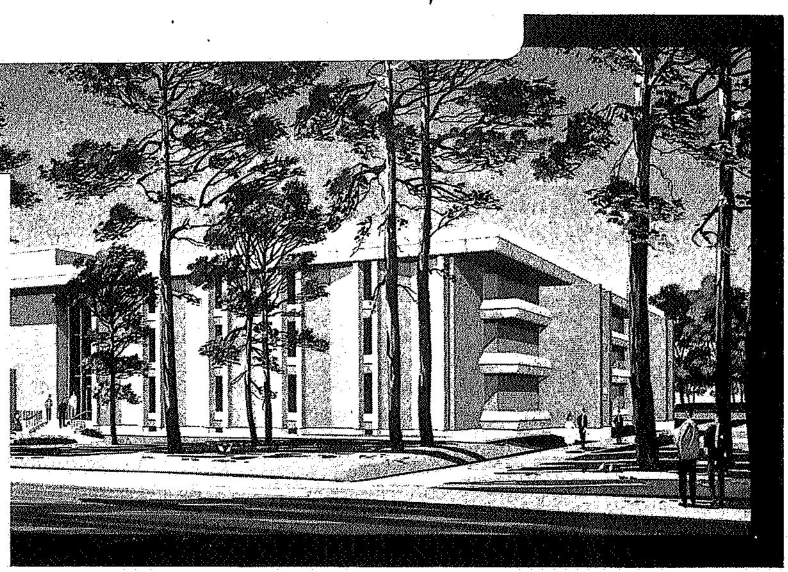
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

by

THE UNIVERSITY OF HOUSTON

CULLEN COLLEGE OF ENGINEERING

UNIVERSITY OF HOUSTON



HOUSTON, TEXAS

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June 1, 1968

CHAPTER I

Background of Grant

Grant NGR-44-005-039 was given by the National Aeronautics and Space Administrations to the University of Houston on May 17, 1966. The principal investigator was R. D. Shelton. The grant was for the amount of $51,356 and was to run for one year. A renewal was received on February 26, 1967 to support the project for a second year with the amount of $52,458.

The title of the grant was "Advancement of the General Theory of Multiplexing with Applications to Space Communications". Originally, the grant consisted of three major tasks: 1) development of new methods of signal multiplexing, 2) performance comparison of new multiplexing systems with conventional systems, and 3) application of the best of these new techniques to the design of a modulation and multiplexing system for the Apollo Applications Program. In the second year the grant activities consisted primarily of applications of techniques developed in the first year to space communication system design, and further analysis of problem areas uncovered during the first year.

Most of the grant results have been previously reported in detail. Table 1-1 lists the seven technical papers presented by grant personnel based on their work. Copies have been sent to NASA. In addition, three lengthy semi-annual progress reports have presented detailed results not covered in technical papers. The six Master's theses and three doctoral dissertations listed in Table 1-2 also cover work done on the grant. Close coordination was maintained with personnel of the Information Systems Division of the Manned Spacecraft Center by means of monthly letter progress reports and two oral presentations. Finally, this report contains an extensive amount of material not previously included in the reports to NASA.

In addition to the technical accomplishments summarized above, there are many other benefits for a grant of this type. Table 1-2 lists twelve degrees granted to personnel supported at one time by the grant. Two other persons listed, B. H. Batson and N. K. Tomaras, never received salaries from the grant but were supervised by grant personnel, and their work was directed toward the goals of the grant. It should be noted that the three doctorates awarded to grant personnel are the first Ph.D. degrees awarded in electrical engineering by the University of Houston. Thus, this grant was instrumental in development of the doctoral program at the University. In Table 1-3 are listed eight other students who were assisted toward degrees by grant support. Several faculty members made contributions to the grant: Doctors R. D. Shelton, H. S. Hayre, E. L. Michaels, J. D. Bargainer, and N. M. Shehadeh. Their time was provided free by the University, except during the summer terms. One of the greatest potential benefits of this program is the familiarity these men gained with NASA systems and their problems, as these faculty members now supervise the thesis research of a number of NASA engineers. Finally, one fulltime secretary assisted greatly with the clerical work - Mrs. S. Mitchell during the first year and Mrs. A. Roach during the second.

TABLE 1-1.

CONFERENCE PAPERS PRESENTED BY GRANT PERSONNEL

April 1967	"Orthogonal Multiplexing Systems Based on Easily-Generated Waveform"	SWIEEECO Dallas	T.Williams R.D.Shelton
June 1967	"Analog and Digital Computer Simulations of Multiplex Systems Performance"	IEEE International Communications Conference, Minneapolis	S.Riter T.Williams R.D.Shelton
June 1967	"Communications Systems for Manned Interplanetary Explorations"	1967 Symposium of the American Astronautical Society, Huntsville, Ala.	S.Riter R.D.Shelton
October 1967	"T.V. Systems for Manned Interplanetary Missions"	EASTCON Washington, D.C.	E.L.Michaels
March 1968	"Some Binary Cyclic Codes"	Princeton Information Theory Conference	N.M.Shehadeh
March 1968	"A Jump-Search Procedure for Sequential Decoding"	Princeton Information Theory Conference	N.M.Shehadeh C.Q.Ho
September 1968	"Noise Performance of Practical Phase Reference Detectors for PSK Signals"	EASTCON Washington, D.C.	S.Riter

TABLE 1-2.

DEGREES GRANTED TO GRANT PERSONNEL WITH THESIS SUBJECT

DATE	NAME	DEGREE	THESIS SUBJECT
September 1967	L.Puigjaner	M.S.	Analytic Signals in Multiplexing
June 1967	W.L.Hon	B.S.	
June 1967	S.Sloan	B.S.	
June 1967	C.Osborne	B.S.	
June 1967	W.Trainor	B.S.	
June 1967	J.Froeschner	B.S.	
June 1967	S.Riter	M.S.	Optimum Multiplexing for a Space Communication System
June 1967	R.D.Shelton	Ph.D.	A Study of Optimum Multiplexing Systems
August 1967	B.H.Batson	M.S.	Signal Design for a Communication System
August 1967	T.Williams	Ph.D.	Realization of Optimum Multiplexing Systems
June 1968	M.A.Smither	M.S.	Data Compression
June 1968	S.Riter	Ph.D.	Modulation Optimization for Space Communication
June 1968	N.K.Tomaras	M.S.	Review of Television Systems
June 1968	W.L.Hon	M.S.	Carrier Synchronization

TABLE 1-3.

ADDITIONAL STUDENTS SUPPORTED BY GRANT

NAME	MONTHS ON GRANT	WORKING TOWARD DEGREE
S.Z.H.Taqvi	16	Ph.D.
I.D.Tripathi	12	Ph.D.
C.Q.Ho	9	Ph.D.
P.Weinreb	4	Ph.D.
S.Wade (technical writer)	12	M.A.
S.Vaharami	4	B.S.
M.L.Butler (draftsman)	9	B.S.
J.C.Hennessy (draftsman)	9	B.S.

CHAPTER II

ORGANIZATION OF FINAL REPORT

This report can be divided into two parts. The first part, Chapters III through IX represents work conducted by Dr. T. Williams and is concerned with the realization of optimum orthogonal multiplexing systems. The second part, Chapters IX through XI, represents work conducted by Dr. S. Riter and represents an application of the theory of orthogonal multiplexing to the solution of a number of unresolved problems common to systems such as the Apollo unified S-band communication system.

Part I.

In Part I. several sets of orthonormal functions suitable for use in orthogonal multiplexing systems are derived. The functions are selected on the basis of ease of implementation since orthogonal multiplexing systems perform equally well for all signal waveshapes in channels with white Gaussian noise.

The functions are analyzed and mathematical relationships derived for message distortion due to time, frequency, and amplitude truncation. The orthomux system was also simulated on an IBM 7090 digital computer using DSL-90 simulation language to evaluate the effects of realistic channel models. Examples of the computer programs are enclosed.

It was found that the real exponential set is the most easily realized signal set from the standpoint of equipment simplicity. A discussion of the performance of this set given along with the performance of a set derived using powers of t.

Part II.

In Part II. Dr. Riter developes general design criteria for realizing a multiplexing system patterned after the USB system, and answers a number of previously unresolved questions concerning the choice and performance of the data detector and the effect of probabilistic coding of the data signals. From this work a few general conclusions can be drawn.

For the system mentioned the best multiplexing process is FDM. The best modulation technique is PM however, under certain circumstances AM will perform satisfactorily and certainly deserves serious consideration in the design of any system. Whether PM or AM is chosen coherent detection will be used to recover the information signal and consequently the designer must insure that sufficient energy is placed in the carrier reference signal. It is shown in Chapter IX that if a second order phase lock loop is used to recover the reference then a signal to noise ratio of 10 to 13 decibels in the noise bandwidth of the phase lock loop is in a sense optimum.

In Chapter X the effects of random phase and timing errors on the detection of the PSK signals are studied. It is shown that of the two the phase errors are considerably more significant. It is also shown that at low signal to noise ratio the optimum estimator of the phase of a PSK signal can be realized with a Costas loop, and that at high signal to noise ratios although the receiver structure is different the performance of the two receivers is for all practical purposes identical. In addition, it is shown that at low signal to noise ratios the Costas loop gives slightly improved performance over the squaring loop (the more conventional circuit) but that at higher signal to noise ratios the performance is again nearly identical. It is concluded that although the squaring loop is not the best detector on a statistical basis that due to it's ease of implementation, current use in many practical systems, and demonstrated satisfactory performance one is hardly justified in going to more sophisticated techniques if a few simple design rules are followed.

In Chapter XI relationships for determining the improvement in performance for the PSK channel with random coding are developed and discussed. In general it is shown that startling improvements in performance are obtainable if the designer is willing to pay the price of high equipment complexity and lower information rate.

CHAPTER III

ORTHOGONAL MULTIPLEXING

Introduction

Multiplexing is the combining of several messages in such a way that they can be transmitted over a channel on a single carrier and then be individually recovered at the receiver. The only multiplexing techniques which have been used to any great extent in the past are frequency division multiplexing and time division multiplexing. Since the recovery of the individual signals at the receiver is based on the orthogonality of the signals, and since there exist many types of orthogonal functions, then many other multiplexing schemes may be devised. Ballard (1962) analyzed a system based on the Legendre polynomials. The general orthogonal multiplexing technique was referred to as "orthomux." A search of the literature shows that very little has been done in the orthomux field to determine the relative advantages of the various possible systems with regard to implementation and performance.

The operation of an orthogonal multiplexing system is based on the integral

$$\int_0^T o_i(t)\, o_j(t)\,dt = \begin{cases} K, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases} \qquad [3\text{-}1]$$

where $o_i(t)$ is the i^{th} channel transmitted signal and $o_j(t)$ is the j^{th} channel signal generated by the receiver, and usually the signals are orthonormal; that is, each signal is amplitude normalized so that K is one. This distributes the power equally among the signals in order to create an equal signal-to-noise ratio condition. The system operation is explained by Figures (3-1) and (3-2). The orthomux system is a pulsed system in that each block in the transmitter and receiver operates over a period of time T and then resets and starts another cycle. Identical orthogonal function generators are used in the transmitter and receiver to produce the set of orthogonal functions required for system operation. The transmitter and receiver operate in frequency

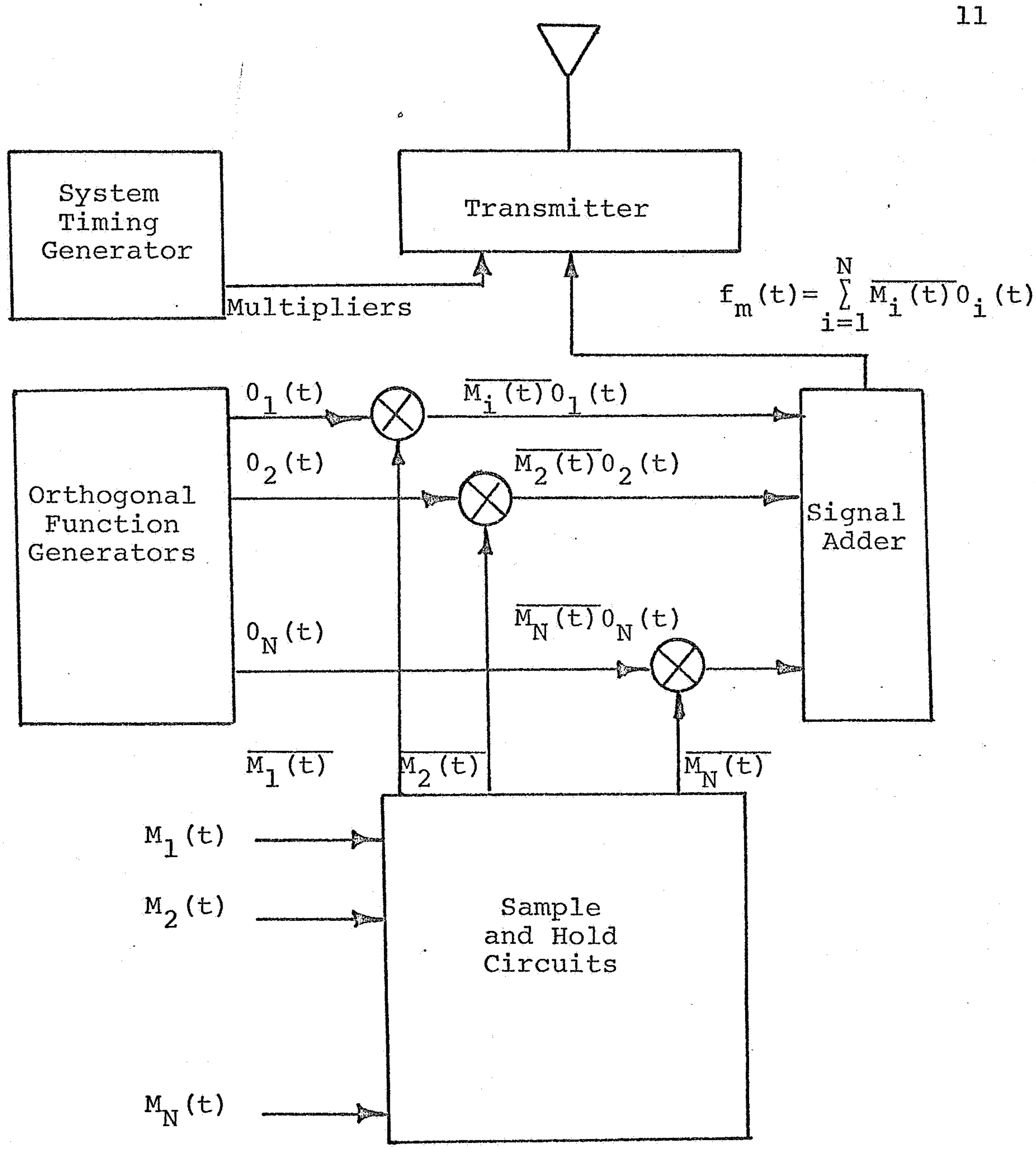


Figure 3-1. General orthogonal multiplex system Transmitter

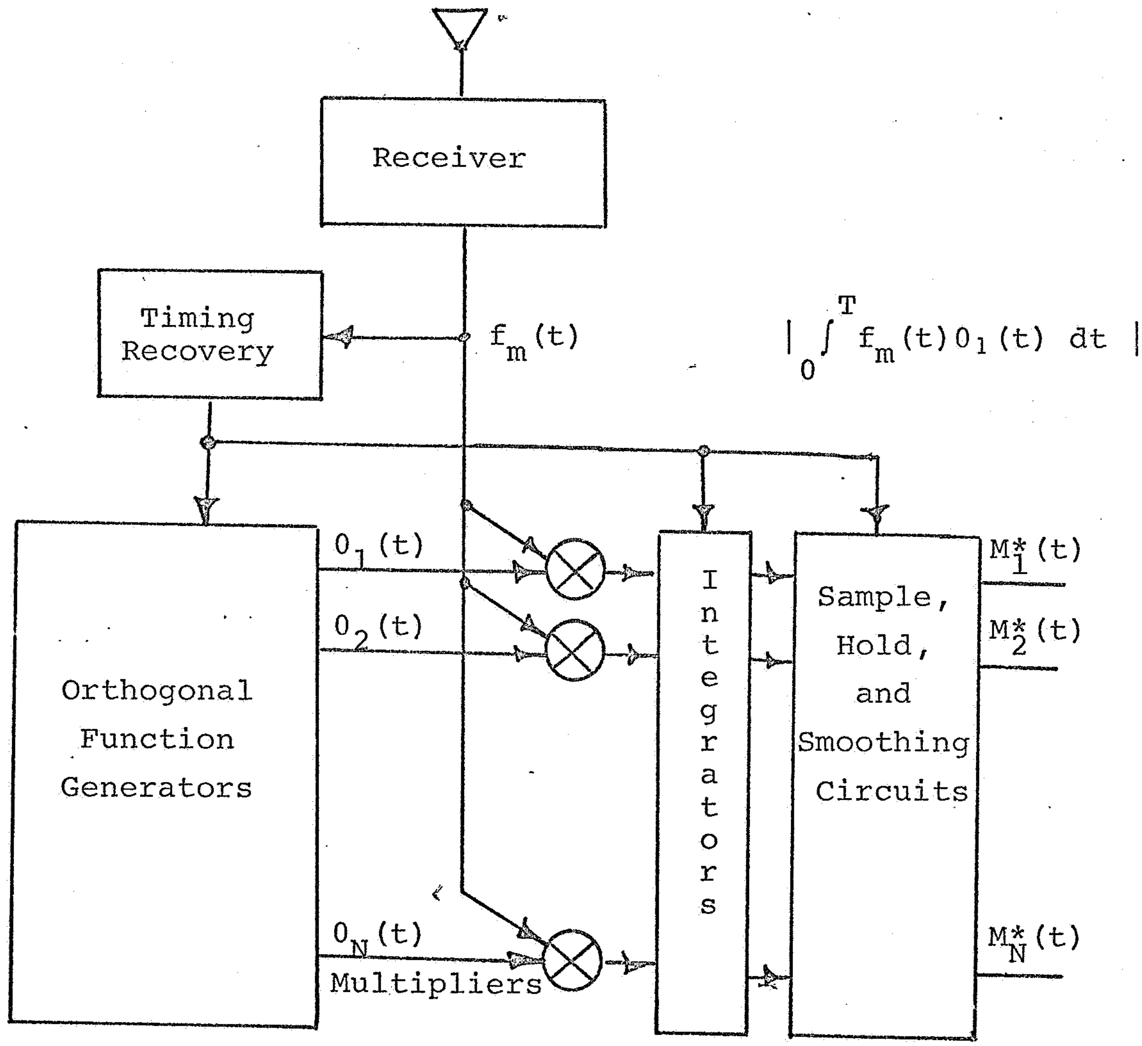


Figure 3-2. General orthogonal multiplex system receiver

synchronism, with receiver operation delayed in time enough to accomodate signal propagation time. A single message would travel through channel one, for instance, of the transmitter system in this manner:

(1) The message input of channel one of the transmitter $M_1(t)$ is sampled and the sampled value is held for one cycle of operation as the constant value $\overline{M_1(t)}$;

(2) The message sample is multiplied with the orthogonal signal $0_1(t)$ from channel one to form $\overline{M_1(t)}\ 0_1(t)$; this product is then added to the signals of the other channels to form $f_m(t)$. This composite signal is sent to the link transmitter for final processing before transmission over the path to the receiver. This might entail shifting the entire signal spectrum to a high frequency for transmission over a radio link, for example. The entire process described above is repeated continuously for each channel of the system. A timing signal generated by the transmitter controls the sequence of operations in the transmitter and the receiver. At the receiver, the received message is processed to recover the composite message signal $f_m(t)$, system timing information, and any other information necessary to recover the individual channel messages. The synchronized receiver generates a set of orthogonal functions identical to those in the receiver but delayed in time by propagation time, as shown in Figure (3-2). The message in channel one is recovered in the following way:

   (a) The first orthogonal function is multiplied with the composite input message to form $0_1(t)f_m(t)$;

   (b) This product is integrated to produce the expression

$$\int_0^T O_1(t)\,[f_m(t) + \eta]\,dt, \text{ and } \eta = \text{Noise} \qquad [3\text{-}2]$$

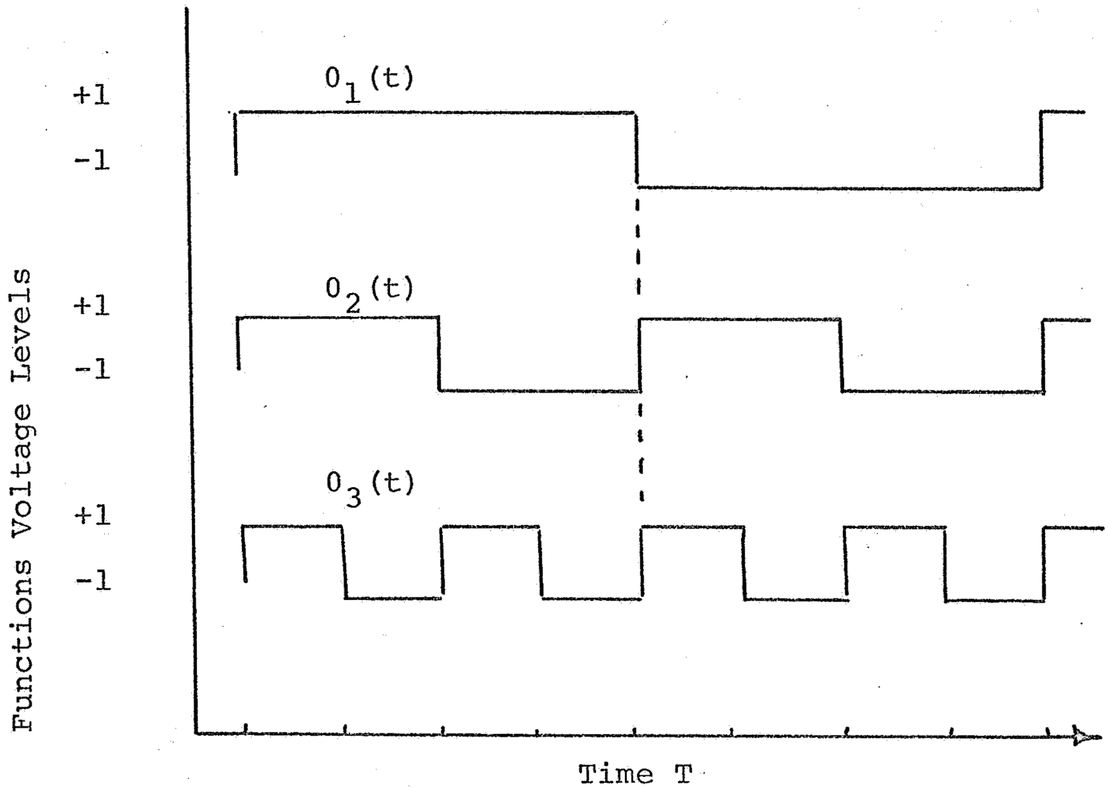


Figure 3-3. A set of binary orthogonal waveforms

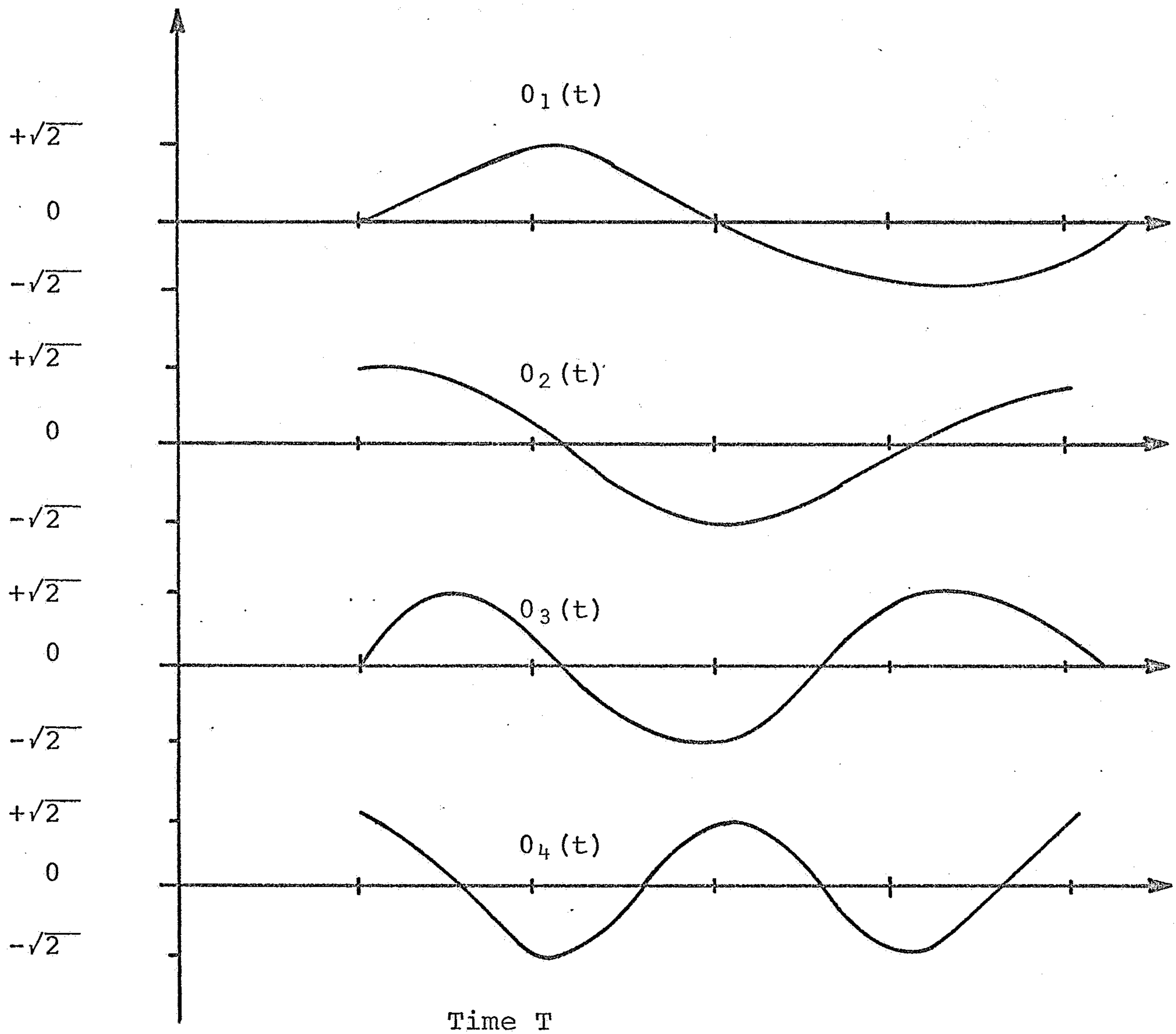


Figure 3-4. Sinusoidal Orthogonal Waveforms Suitable for Use in a Frequency Division Multiplexing System

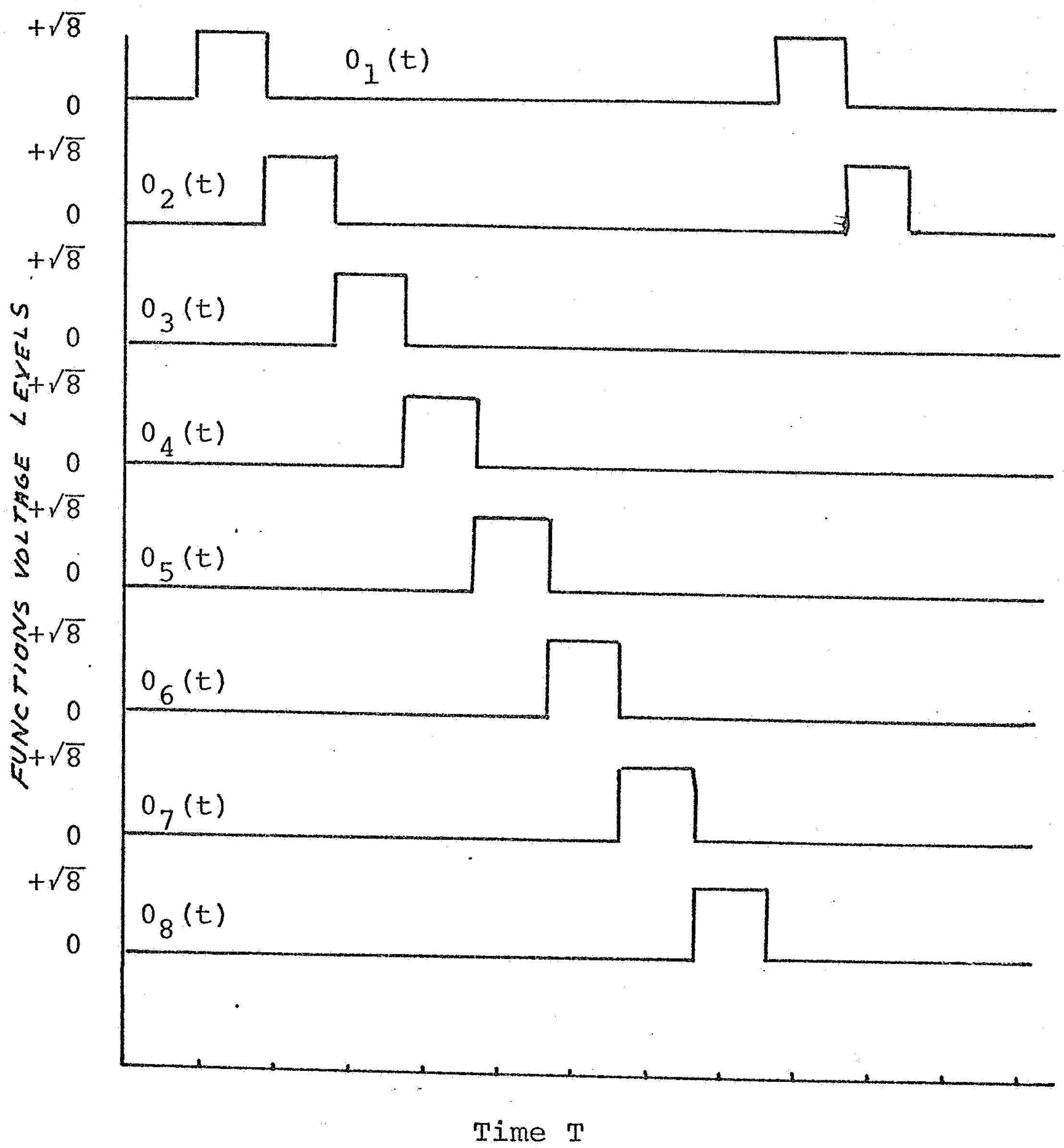


Figure 3-5. Orthogonal waveforms for time division multiplexing

which upon expansion of $f_m(t)$ becomes

$$\int_0^T O_1(t)\,[\overline{M_1(t)}O_1(t) + \overline{M_2(t)}O_2(t) + \ldots +$$

$$\overline{M_N(t)}O_N(t)]\,dt + \int_0^T [O_1(t)\,n]\,dt \qquad [3\text{-}3]$$

(c) Finally, the receiver samples the integrator output at time t=T, and the sampled output is smoothed as one of a stream of levels of period T, by a smoothing network whose output is $M_1^*(t)$.

The difference between $M_1^*(t)$ and $M_1(t)$ is the channel error. The smoothing filter is not needed in the case of digital messages. Figures (3-3), (3-4), and (3-5) show functions suitable for use in an orthomux system.

Davenport and Root show that the optimum process for the detection of a signal in the presence of additive white Gaussian noise is a correlation process, where the shape of the signal and time or arrival of the signal at the receiver are known, and the information is amplitude modulated. Specifically, the correlation process maximizes the signal-to-noise ratio or in the case of a digital signal, it minimizes the probability of error. This is true for the simple baseband channel with additive Gaussian noise. However, if the channel differs from this simple channel with unrestricted bandwidth, then the optimum multiplexing system depends on the assumptions made about the channel itself. For the simple baseband channel with additive white Gaussian noise all the orthomux systems perform equally well (Davenport, 1958). Thus for the simple baseband channel the optimum orthomux system would be the system which is relatively simple to construct. This seems to be the system based on the real exponential set, since the exponentials are very easy to generate. For a channel which band-limits the frequency spectrum of the transmitted signals, consideration must be given to the problem of the resultant distortion of the received message. In the case of a peak power-limited channel, the optimum multiplexing system uses binary (digital) waveforms. These waveforms have an optimum peak-to-average power ratio of one. The types of binary combination methods leading to the simplest implementation are also discussed.

Each of the types of signals which are optimum in terms of the considerations discussed above are investigated as to their implementation in the following chapters. In addition several other sets are considered because of one or more interesting properties. First a brief review of previous applicable work in the area of orthogonal multiplexing is given.

Review of Previous Investigations

Interest in orthogonal multiplexing has come about because of the following advantages:

(1) Orthogonality of the signals gives a theoretical minimum of zero crosstalk between channels.

(2) A correlation detection process assures maximum rejection of noise and interference.

(3) Orthogonal multiplexing is optimum in several practical ways which depend on the channel characteristics. The different orthogonal sets allow the designer to select a set on the basis of ease of implementation or optimum in the sense of peak-to-average-power ratio, for instance.

Ballard (1962) pointed out that Legendre functions may be generated using cascaded solid state circuits. The first three Legendre functions are:

$$P_0(x) = 1 \qquad [3\text{-}4]$$

$$P_1(x) = X \qquad [3\text{-}5]$$

$$P_2(x) = \tfrac{1}{2}[3x^2-1] \qquad [3\text{-}6]$$

and so forth, where

$$x = \frac{2}{T}[t - \frac{T}{2}] \text{ , for } 0 \le t \le T \qquad [3\text{-}7]$$

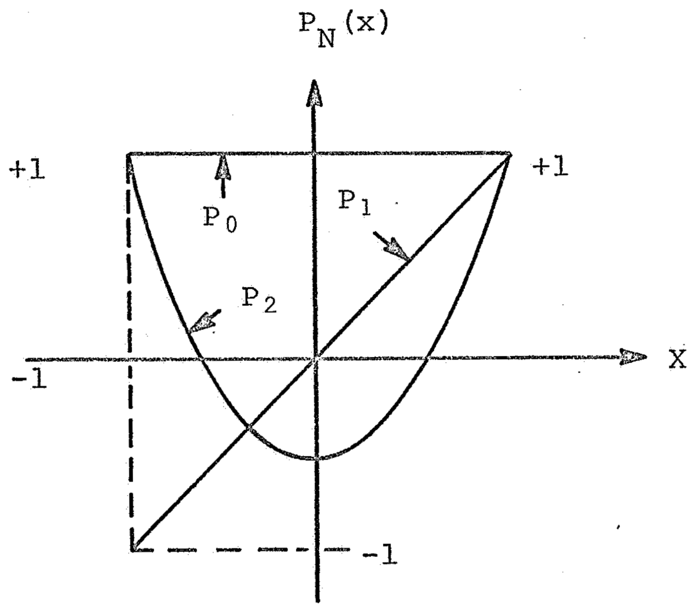


Figure 3-6. The First Three Legendre Polynomials

Higher ordered polynomials are obtained using the Rodrigue's formula

$$P_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N} [x^2 - 1]^N \text{, for } n=0,1,2,\ldots \quad [3\text{-}8]$$

Note that all the functions have zero mean and a peak value of one except for $P_0(x)$. The first three functions are shown in Figure (3-6). Even order functions have even symmetry and odd order functions have odd symmetry, and their orthogonality interval extends from -1 to +1.

Equations for the frequency spectrum of the Legendre polynomials are tabulated in Ballard's paper and the effect of filtering or frequency truncation on crosstalk is mentioned in addition to the details of implementation.

Karp and Higuchi (1963) analyzed a system based on the modified Hermite polynomials which are said to have superior time-bandwidth compression properties. Their interval of orthogonality is from $t=-\infty$ to $t=+\infty$, but the functions can be truncated in time at the expense of introducing crosstalk into the system. This system is complex and therefore its implementation is not simple. The generating function for the modified Hermite polynomials is (Karp, 1963)

$$H_N(t) = \frac{(-1)^N a^{(N-\frac{1}{2})}}{\pi^{\frac{1}{4}}\sqrt{N!}\; 2^{N/2}} e^{t^2/2a^2} [\frac{d^N}{dt^N}(e^{-t^2/a^2})] \quad [3\text{-}9]$$

Titsworth (1963) has described a Boolean-Function-Multiplexed system with the advantage that the average and peak powers are the same since it is a digital system.

CHAPTER IV

EXPONENTIAL ORTHOGONAL FUNCTIONS

In this chapter exponential orthogonal functions are analyzed with a view to their application in pulsed multiplex systems. The interest in the exponential functions with real exponents is based on their being relatively easy to be generated. Methods of implementation and performance improvement techniques for this system are given later.

A sequence of functions $g_i(t)$, is given below:

$$
\begin{aligned}
g_1(t) &= e^{-pt} \\
g_2(t) &= e^{-2pt} \\
&\vdots \qquad\qquad \text{for } 0 \leq t \leq \infty \qquad\qquad [4\text{-}1] \\
g_N(t) &= e^{-Npt}
\end{aligned}
$$

with p as a real positive constant, may be used to obtain a set of orthogonal functions $0_i(t)$ over an interval given below:

$$O_1(t) = \sqrt{2p}\; e^{-pt} \qquad [4\text{-}2]$$

$$O_2(t) = 4\sqrt{p}\; e^{-pt} - 6\sqrt{p}\; e^{-2pt} \qquad [4\text{-}3]$$

$$O_3(t) = \sqrt{6p}(3e^{-pt} - 12e^{-2pt} + 10e^{-3pt}) \qquad [4\text{-}4]$$

or

$$O_N(t) = \sum_{k=1}^{N} C_{Nk}\; e^{-kpt} \quad \text{for } 0 \leq t \leq \infty \qquad [4\text{-}5]$$

The method by which this set of functions shown in Figure (4-1) is obtained is described in Appendix B.

Time Truncation Crosstalk

Since the exponentials decay rapidly, it is possible to truncate the signal at a finite time T and preserve most of their desirable characteristics. This truncation causes crosstalk which can be calculated using the basic orthogonality integral

$$\phi_{NM}(T) = \int_0^T O_N(t)\ O_M(t)\ dt \qquad [4\text{-}6]$$

The crosstalk is calculated by substituting the defining equations for the orthogonal signals in Equation (4-6)

$$\phi_{NM}(T) = \int_0^T \Big(\sum_{k=1}^{N} C_{Nk}\ e^{-kpt}\Big)\Big(\sum_{\lambda=1}^{M} C_{M\lambda}\ e^{-\lambda pt}\Big)dt \qquad [4\text{-}7]$$

$$\phi_{NM}(T) = \sum_{k=1}^{N}\sum_{\lambda=1}^{M} C_{Nk}\ C_{M\lambda} \int_0^T e^{-(k+\lambda)pt}\ dt \qquad [4\text{-}8]$$

$$\phi_{NM}(T) = \sum_{k=1}^{N}\sum_{\lambda=1}^{M} \frac{C_{Nk}\ C_{M\lambda}}{(k+\lambda)p} - \sum_{k=1}^{N}\sum_{l=1}^{M} \frac{C_{Nk}\ C_{M\lambda}}{(k+\lambda)p}\ e^{-[k+\lambda]pt}$$

[4-9]

or $$\phi_{NM}(T) = \delta_{NM} - E_{NM} \qquad [4\text{-}10]$$

and $$\delta_{NM} = \sum_{k=1}^{N}\sum_{\lambda=1}^{M} \frac{C_{Nk}\ C_{M\lambda}}{(k+\lambda)p} \qquad [4\text{-}11]$$

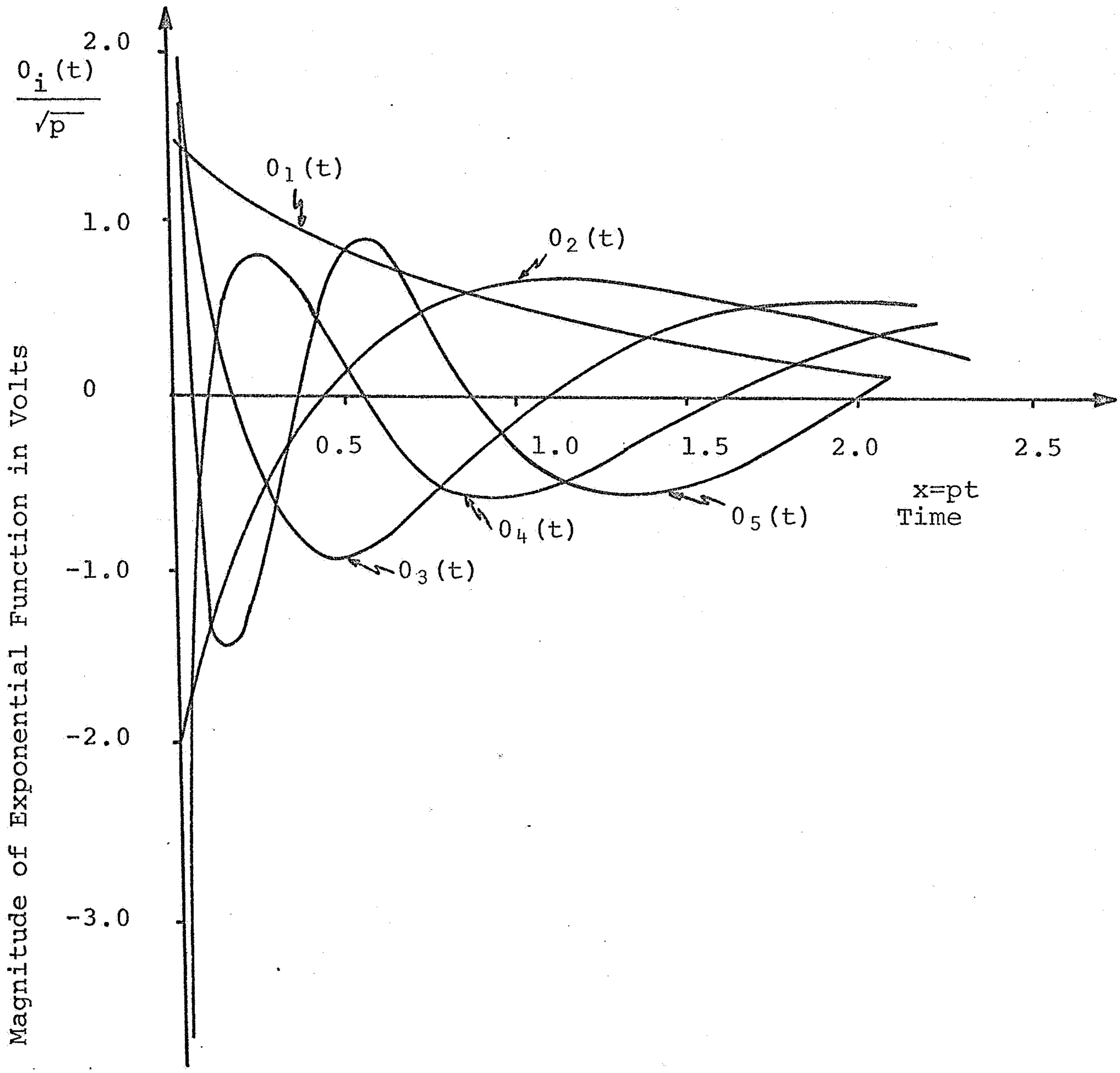


Figure 4-1. Normalized Plot of Real Exponential set vs pT

where

$$\delta_{NM} = \begin{cases} 1, & N = M \\ 0, & N \neq M \end{cases} \qquad [4\text{-}12]$$

$E_{NM}(T)$ = crosstalk term in M^{th} receiver channel due to the N^{th} channel signal.

$\phi_{NM}(T)$ = total output of the M^{th} channel due to the N^{th} channel signal.

Figures (4-2) and (4-3) show $E_{NM}(T)$ (crosstalk) variation with pT.

Spectrum

The complex line spectrum of a periodic function with basic minimum frequency ω_1 is given by

$$C_\beta = \frac{1}{T}\int_0^T f(t)e^{-j\beta\omega_1 t}\,dt \qquad [4\text{-}13]$$

where $\beta = 0,1,2,\ \ldots$

T = Period of $f(t)$

The power spectrum of the N^{th} exponential function is found as follows:

$$C_\beta = \frac{1}{T}\int_0^T \sum_{k=1}^{N} C_{Nk}\, e^{-kpt}\, e^{-j\beta\omega_1 t}\,dt \qquad [4\text{-}14]$$

$$C_\beta = \frac{1}{T}\sum_{k=1}^{N} C_{Nk}\left[\frac{1 - e^{-T(kp+j\beta\omega_1)}}{kp + j\beta\omega_1}\right] \qquad [4\text{-}15]$$

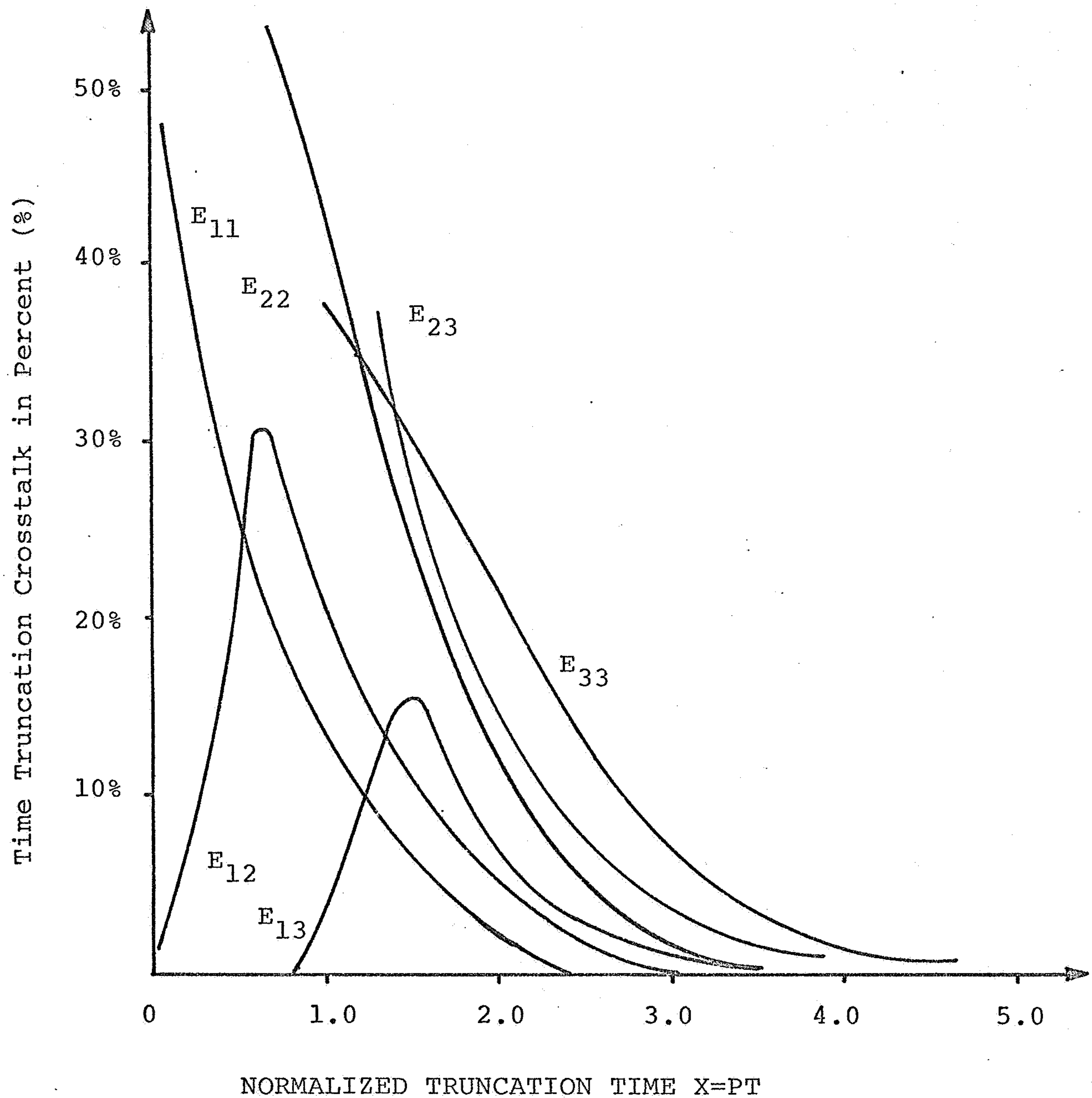


Figure 4-2. Crosstalk caused by time truncation of exponentially generated orthogonal functions

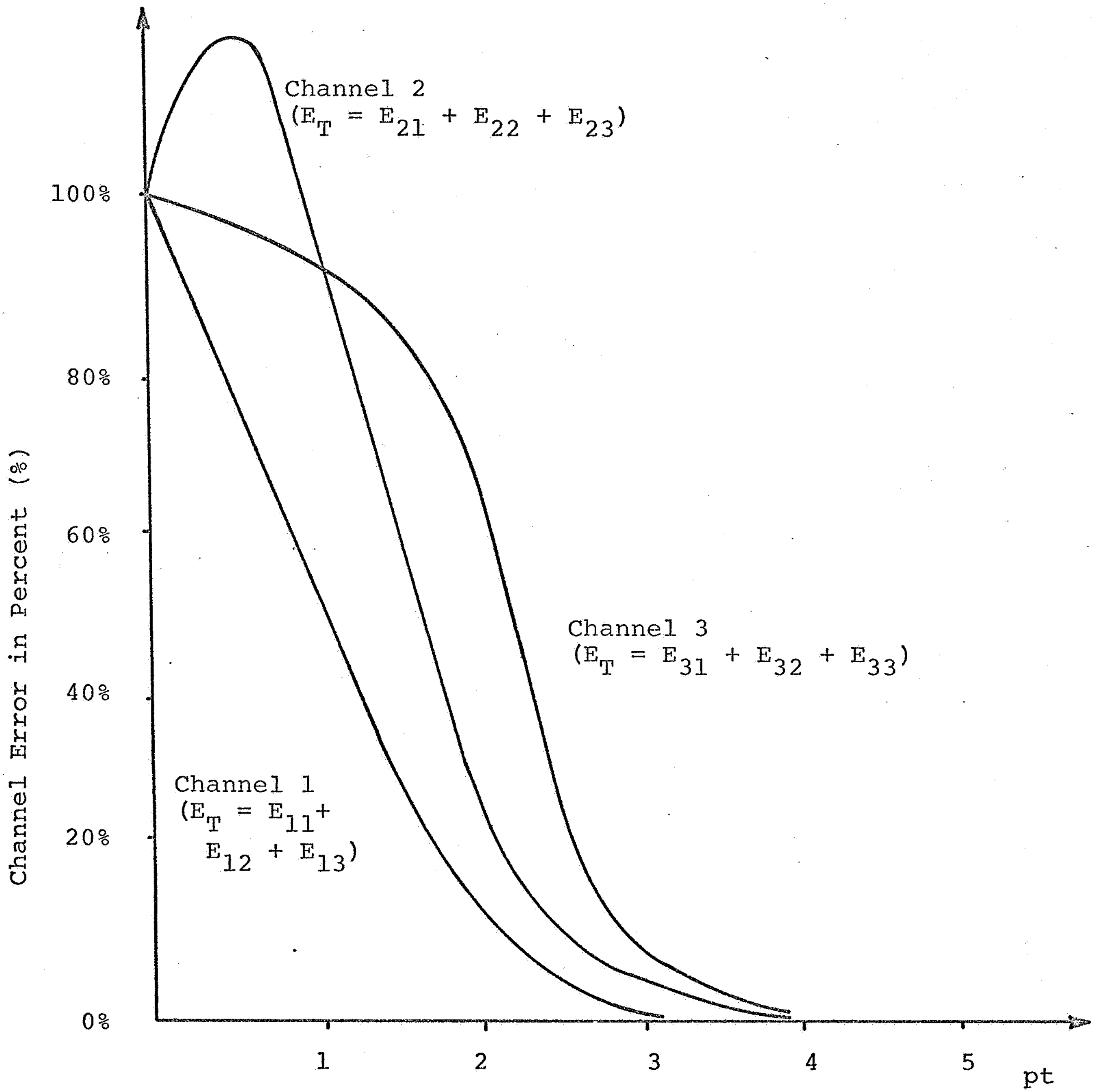


Figure 4-3. Total time truncation crosstalk in each channel for a three channel system using the real exponential set

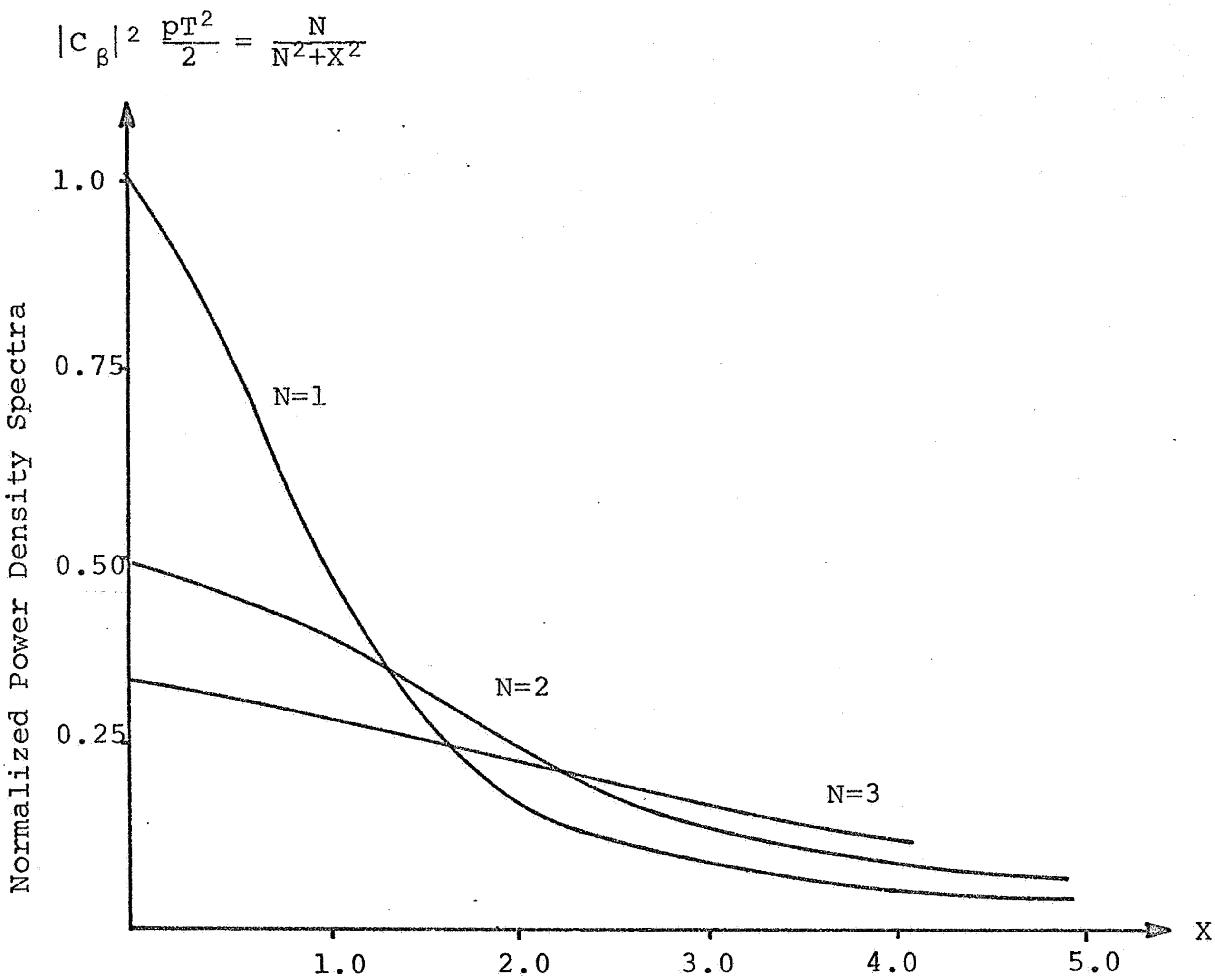


Figure 4-4. Envelope of the normalized power density spectra versus $X = \frac{\beta\omega_1}{p}$ for the real exponential set.

As T becomes large enough to reduce the time truncation crosstalk to tolerable limits (less than 5 per cent for example), the above equation can be approximated by:

$$C_\beta \doteq \frac{1}{T}\sum_{k=1}^{N}\frac{C_{Nk}}{(kp+j\beta\omega_1)} = \frac{1}{pT}\sum_{k=1}^{N}\frac{C_{Nk}}{(k+jX)} \qquad [4\text{-}16]$$

A substitution of the numerical values for the coefficients in Equation (4-16) results in an expression for the power spectrum as given in Equation (4-17).

$$|C_\beta|^2 = \frac{2}{pT^2}\left(\frac{N}{X^2+N^2}\right) \qquad [4\text{-}17]$$

where N is the index number of the orthogonal function. This equation is plotted in Figure (4-4). The highest order exponential of a particular function dominates the bandwidth expression because in the time domain the highest order exponent is most compressed, or has very short decay time.

Frequency Truncation Crosstalk

Crosstalk is introduced when the function under study is passed through a filter which causes amplitude or phase distortion. The effects of frequency truncation are studied by passing the exponential function through a one pole RC filter. The impulse response of the network is given as

$$h(t) = \frac{1}{RC}\,e^{-t|RC}, \quad \text{for } t \geq 0 \qquad [4\text{-}18]$$

The real exponential function inputs $e_i(t)$ are given by

$$e_i(t) = O_N(t) = \sum_{k=1}^{N} C_{Nk}\,e^{-kpt}, \quad 0 \leq t \leq T \qquad [4\text{-}19]$$

for which the output would be

$$e_{ON}(\tau) = \int_0^{\tau} \sum_{k=1}^{N} C_{Nk}\, e^{-kpt}\left[\frac{1}{RC} e^{-\frac{1}{RC}(\tau-t)}\right] dt \qquad [4\text{-}20]$$

$$e_{ON}(\tau) = \frac{e^{-\tau/RC}}{RC} \sum_{k=1}^{N} C_{Nk}\left[\frac{1 - e^{-(kp-1/RC)\tau}}{kp - 1/RC}\right] \qquad [4\text{-}21]$$

Figure (4-5) shows the effect of a one pole filter on the input signal given by

$$fm(t) = O_1(t) + O_2(t) + O_3(t) \qquad [4\text{-}22]$$

where the $O_i(t)$ are the first three of the exponential signals. The graph shows the input signal and the output for two values of bandwidth. The crosstalk resulting from bandlimiting is obtained by substituting Equation (4-20) into Equation (4-6) as

$$\phi_{NM}(T) = \int_0^{T} \sum_{k=1}^{N} C_{Nk}\left[\frac{e^{-\tau/RC} - e^{-kp\tau}}{RCkp - 1}\right]\left(\sum_{\lambda=1}^{M} C_{M\lambda} e^{-\lambda p\tau}\right) d\tau \qquad [4\text{-}23]$$

$$\phi_{NM}(T) = \sum_{k=1}^{N} \sum_{\lambda=1}^{M} \frac{C_{Nk}\ C_{M\lambda}}{(RCkp-1)(\lambda p + 1/RC)}\left(1 - e^{-T(\lambda p+1/RC)}\right) - \sum_{k=1}^{N} \sum_{\lambda=1}^{M} \frac{C_{Nk}\ C_{M\lambda}}{(RCkp-1)(\lambda p + 1/RC)}\left[1 - e^{-T(kp+\lambda p)}\right] \qquad [4\text{-}24]$$

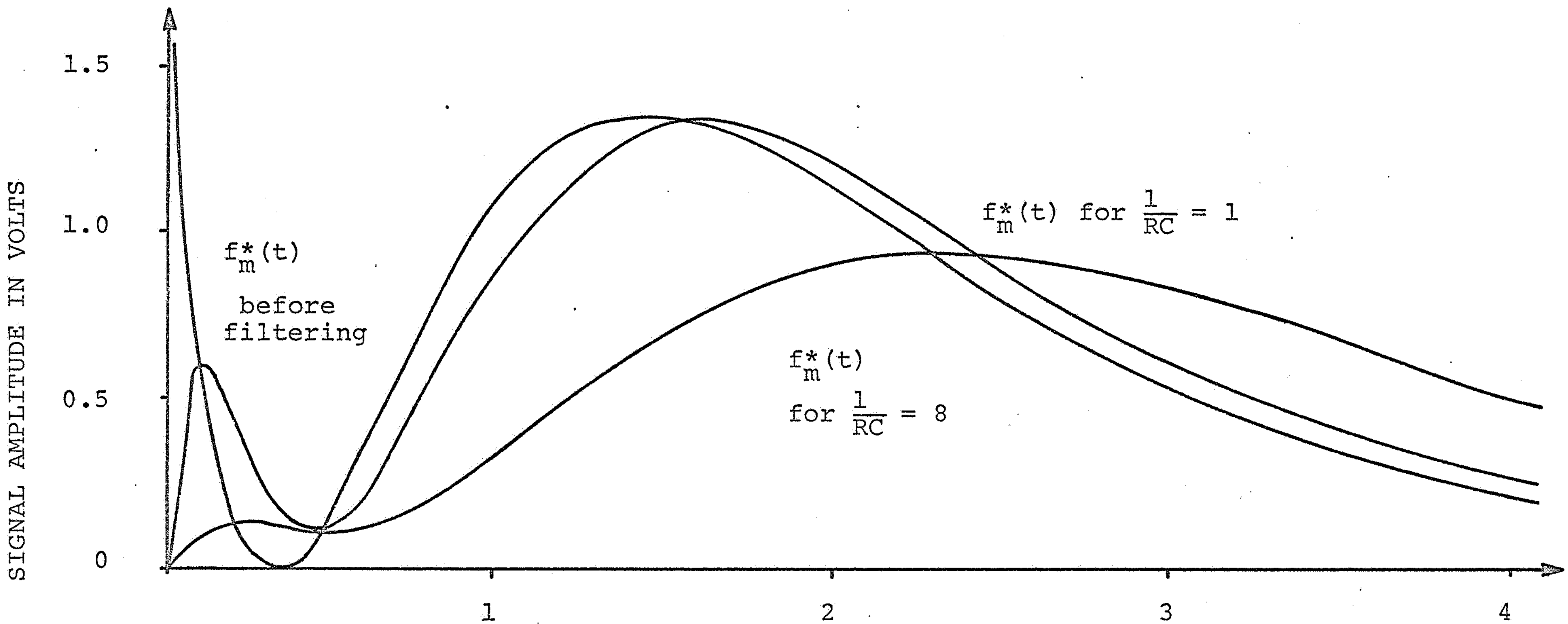


Figure 4-5. The effect of one pole filter on the composite transmitted waveform for several values of RC

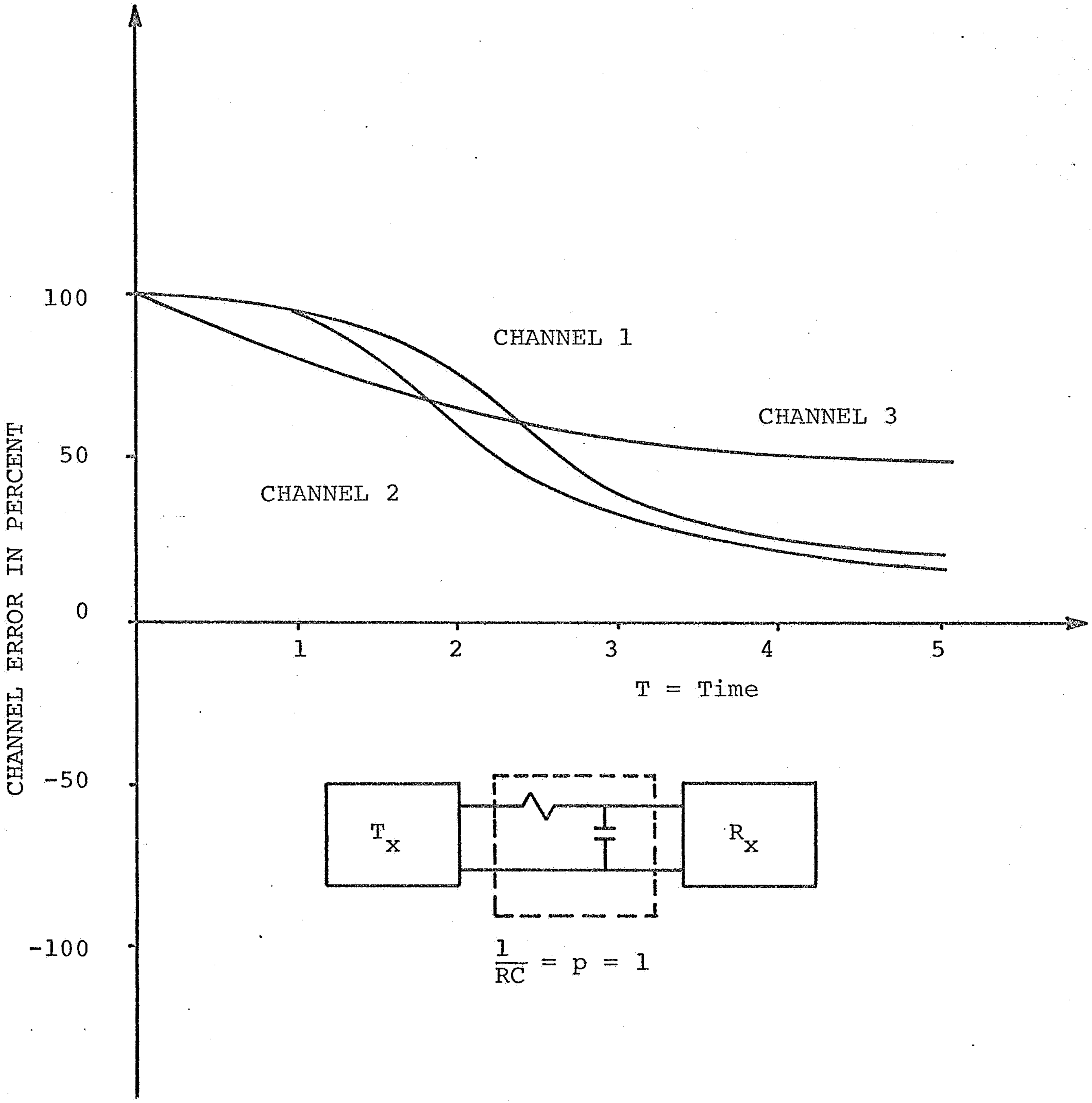


Figure 4-6. System error for one pole filter channel model 1/RC equal to one

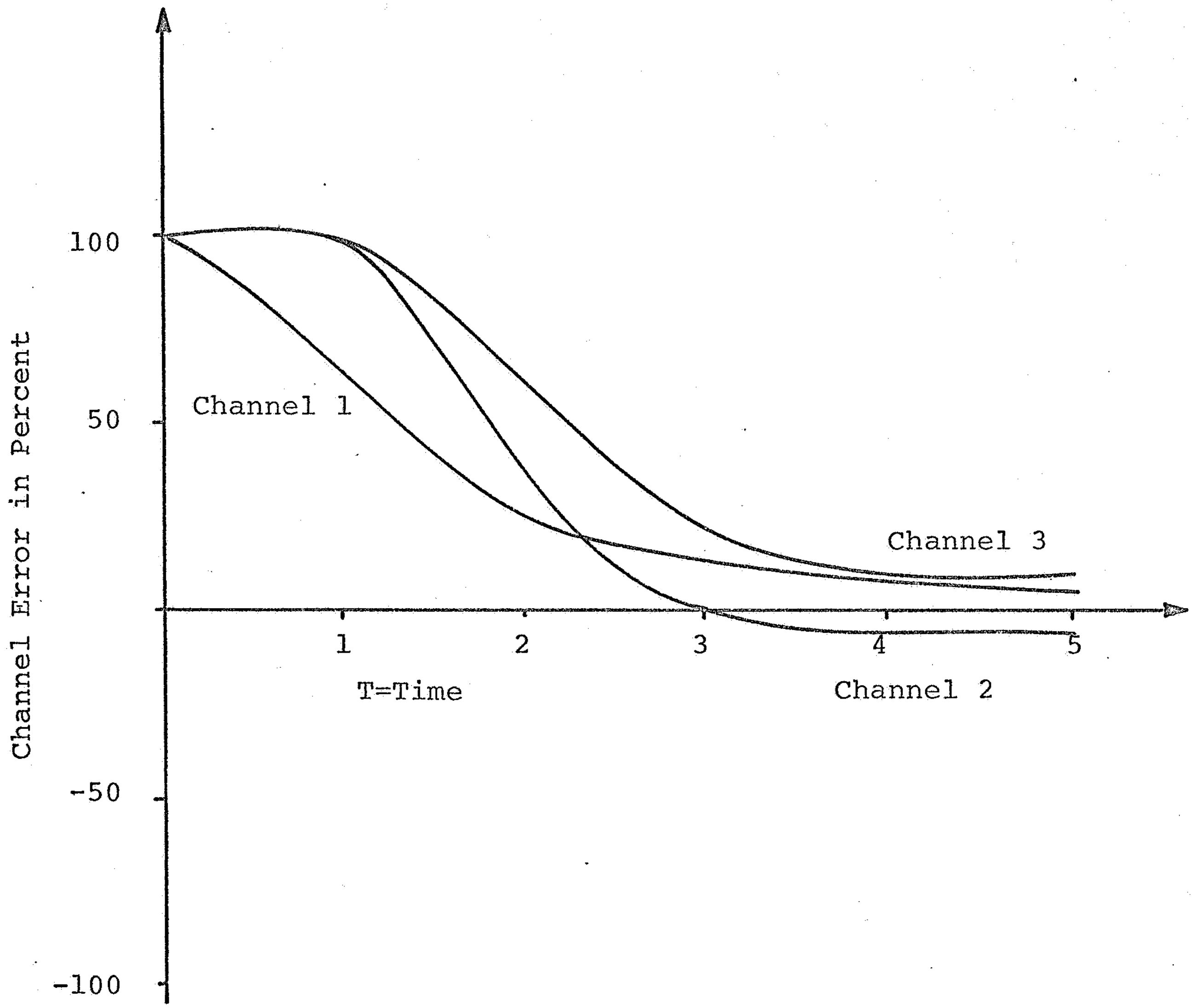


Figure 4-7. System error for a one pole channel model with 1/RC equal to eight

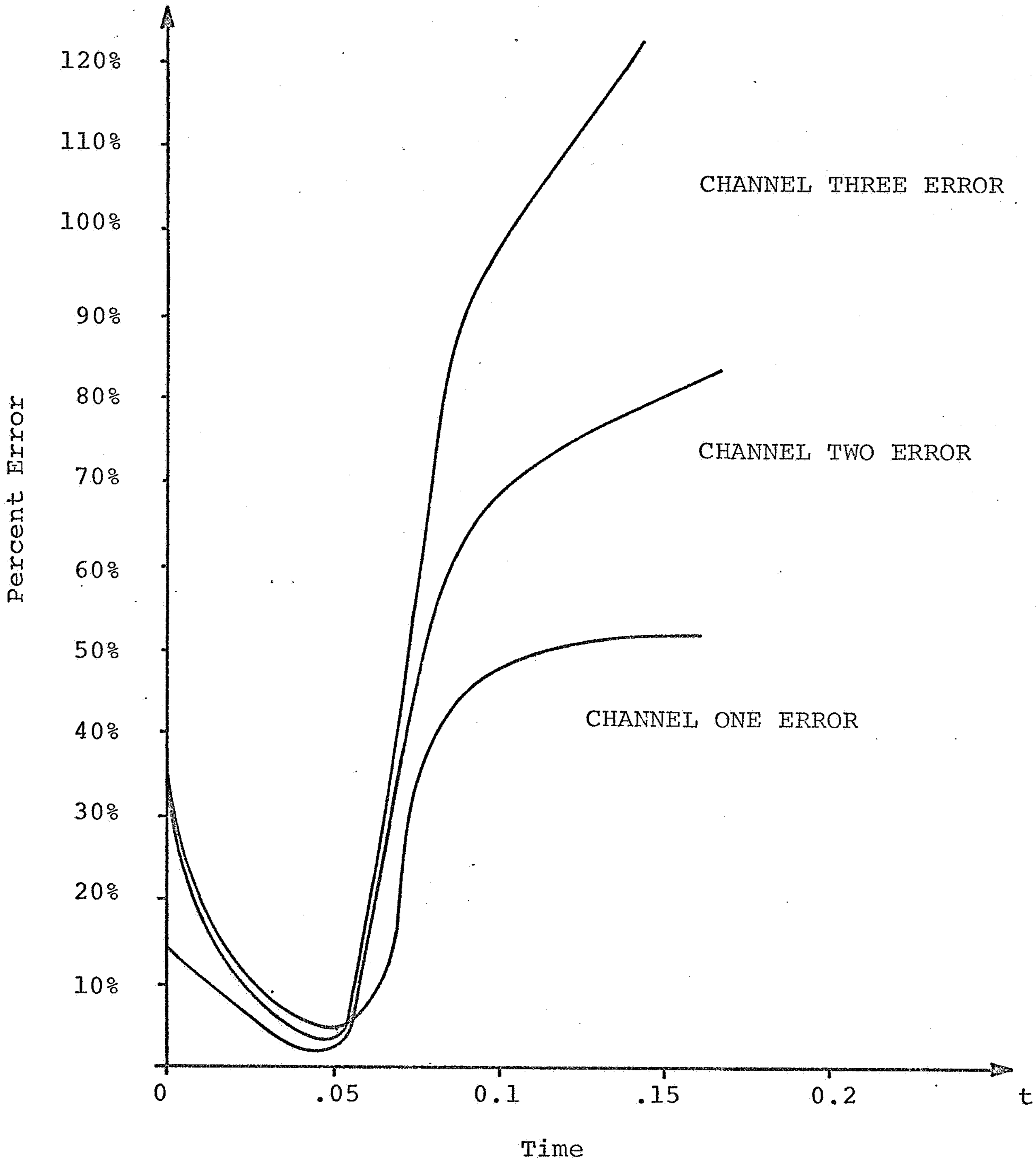


Figure 4-8. Error due to the delay caused by the channel

The error introduced in a message is shown in Figures (4-6) and (4-7) for two values of bandwidth.

One way to reduce the error due to the delay of the signal caused by the filter is to delay the operation of the receiver by an equal amount. The effect of this is shown in Figure (4-8). For the correct amount of delay in the receiver operation the error is greatly reduced. At the point of minimum error the receiver is said to be synchronized with the incoming signal.

A Lower Bound on pT

The crosstalk effects of time truncation crosstalk on the real exponential set may be studied as given in Equation (4-9).

$$E_{NM}(T) = \sum_{k=1}^{N} \sum_{\lambda=1}^{M} \frac{C_{Nk}\ C_{M\lambda}\ e^{-(k+\lambda)pT}}{(k+\lambda)p} \qquad [4\text{-}25]$$

For large values of pT, only the first term of Equation (4-25) is significant as shown in Figure (4-2) and may be rewritten as

$$E_{NM}(T) = \varepsilon_{NM}(T) > \frac{C_{N1}\ C_{M1}\ e^{-2pT}}{2p} \qquad [4\text{-}26]$$

or

$$e^{2pT} > \frac{C_{N1}\ C_{M1}}{2p\varepsilon_{NM}(T)} \qquad [4\text{-}27]$$

or

$$T > \frac{1}{2p}[\mathrm{Ln.}(\frac{C_{N1}\ C_{M1}}{p}) - \mathrm{Ln.}(2\varepsilon_{NM}(T)] \qquad [4\text{-}28]$$

The coefficients of the initial terms in Equations (4-2) through (4-4) are given by the sequence $C_{11} = \sqrt{2p}$, $C_{21} = 4\sqrt{p}$, $C_{31} = 3\sqrt{6p}$, and the general expression for the N^{th} channel is therefore given by

$$C_{N1} = N\sqrt{2Np} \tag{4-29}$$

and for the M^{th} channel the expression is

$$C_{M1} = M\sqrt{2Mp} \tag{4-30}$$

The worst case of crosstalk occurs for M=N channels so that Equation (4-28) becomes

$$T > \frac{1}{2p}\{\mathrm{Ln.}[2N^3] - \mathrm{Ln.}[2\epsilon_{NN}(T)]\} \tag{4-31}$$

$$T > \frac{1}{2p}\{\mathrm{Ln.}[2] + 3\mathrm{Ln.}[N] - \mathrm{Ln.}[2] - \mathrm{Ln.}[\epsilon_{NN}(T)]\} \tag{4-32}$$

$$T > \frac{1}{2p}\{3\mathrm{Ln.}[N] - \mathrm{Ln.}[\epsilon_{NN}(T)]\} \tag{4-33}$$

or

$$pT > \frac{1}{2}[3\mathrm{Ln.}(N) - \mathrm{Ln.}[\epsilon_{NN}(T)]] \tag{4-34}$$

This equation is plotted in Figure (4-9) for several values of $\epsilon_{NN}(T)$, and the lower bound on pT necessary for a required amount of crosstalk due to time truncation can be readily selected from this figure.

The bandwidth of a system based on such an exponential set is now compared with that of common multiplex systems. The Nyquist sampling theorem states

$$T \leq \frac{1}{2B_m} \tag{4-35}$$

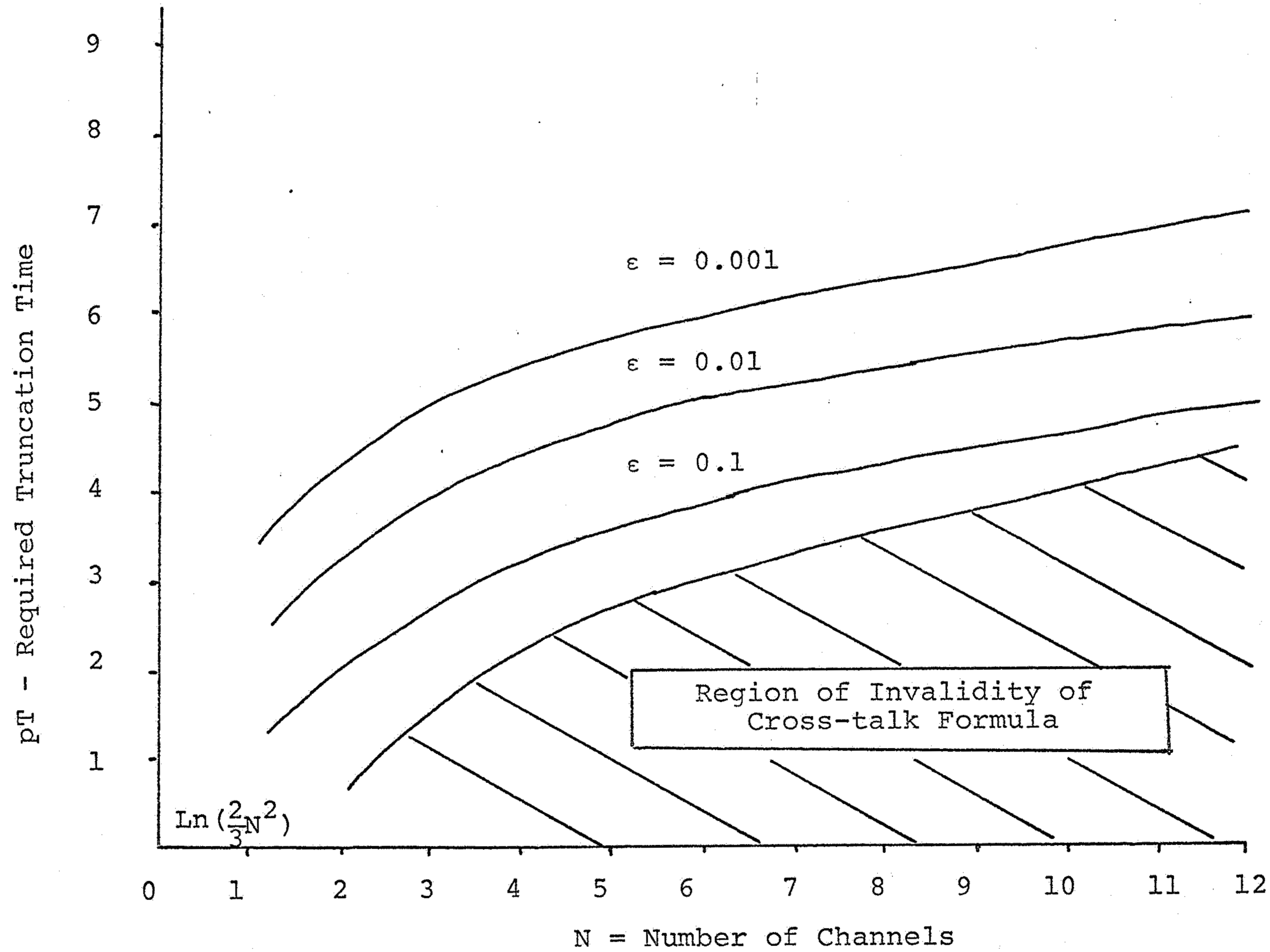


Figure 4-9. Truncation time required for specified crosstalk level $pT > \frac{1}{2} \left| 3Ln, N-Ln, \varepsilon \right|$ where $\varepsilon = Max.E_{NM}$.

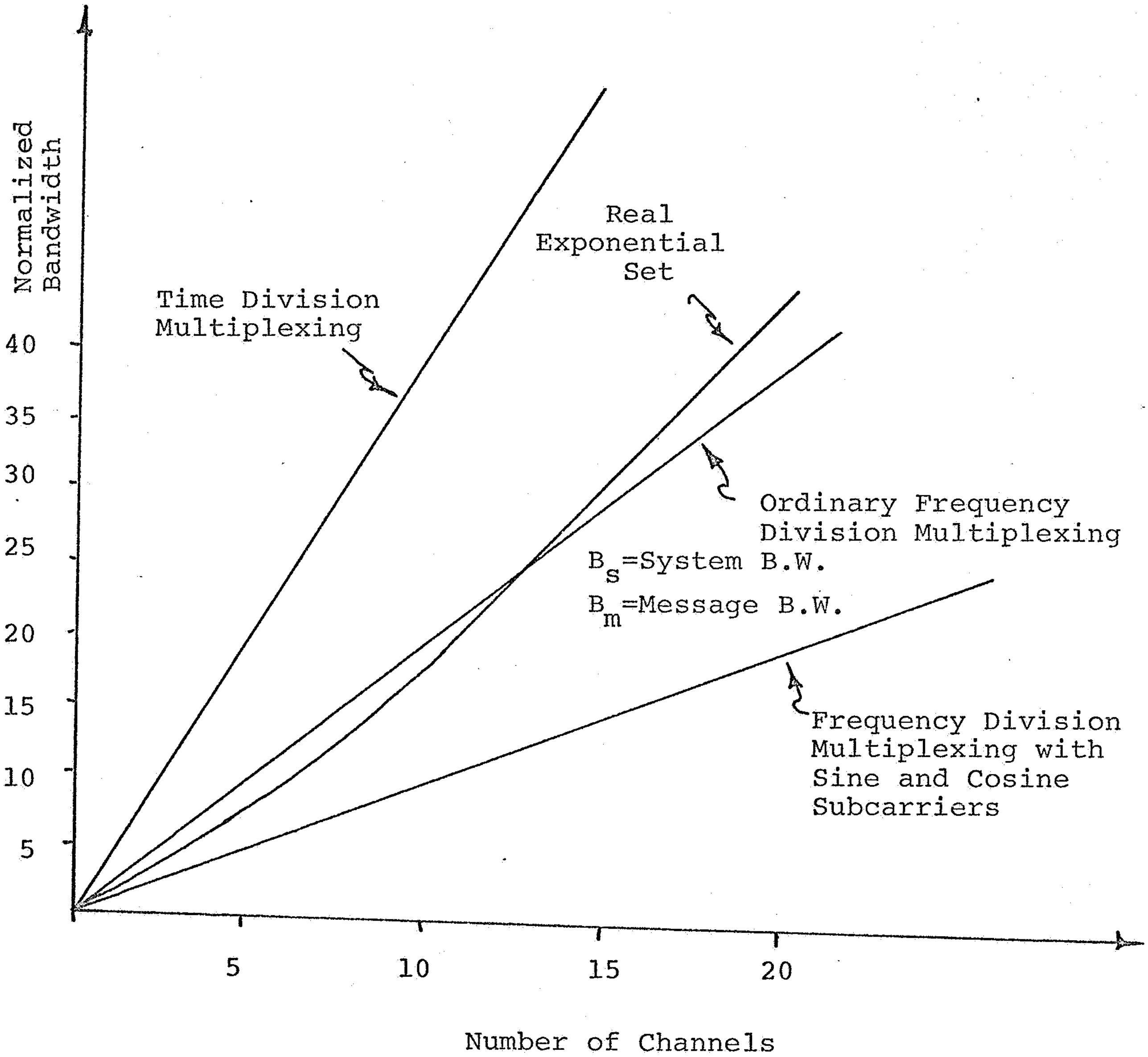


Figure 4-10. Normalized bandwidth versus number of channels

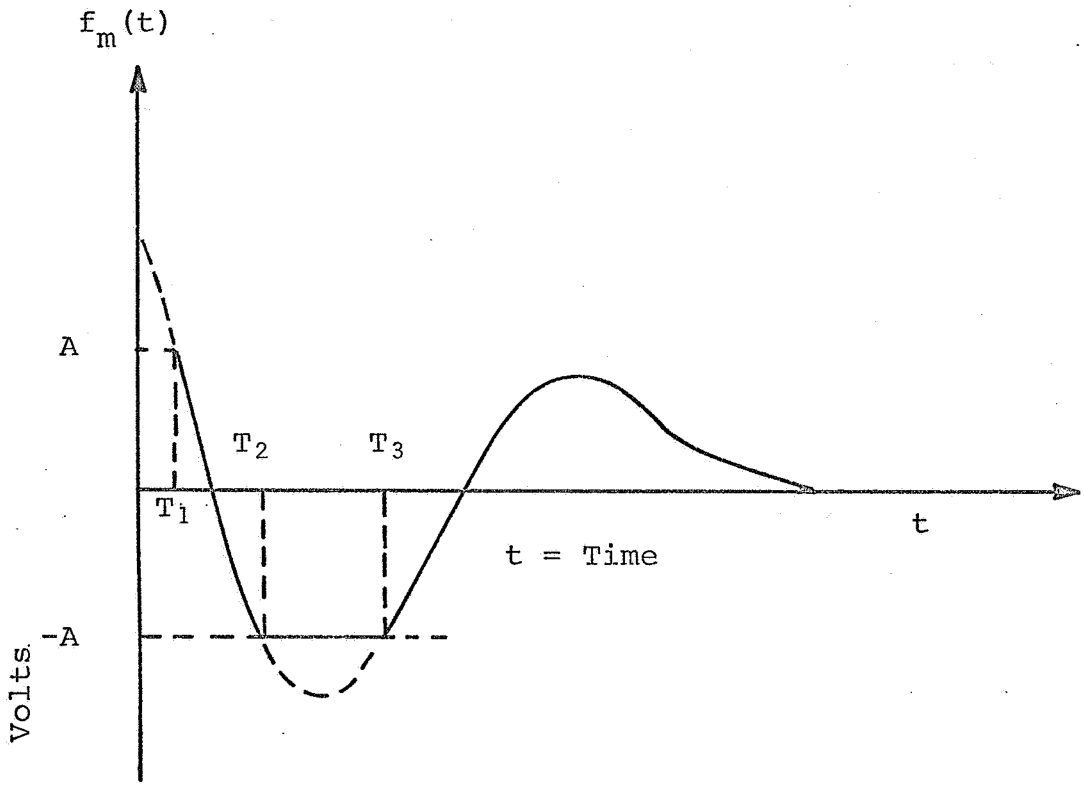


Figure 4-11. The function is shown limited at a peak amplitude of $f_m(t)=|A|$ for $0 \leq t \leq T_1$ and $T_2 \leq t \leq T_3$.

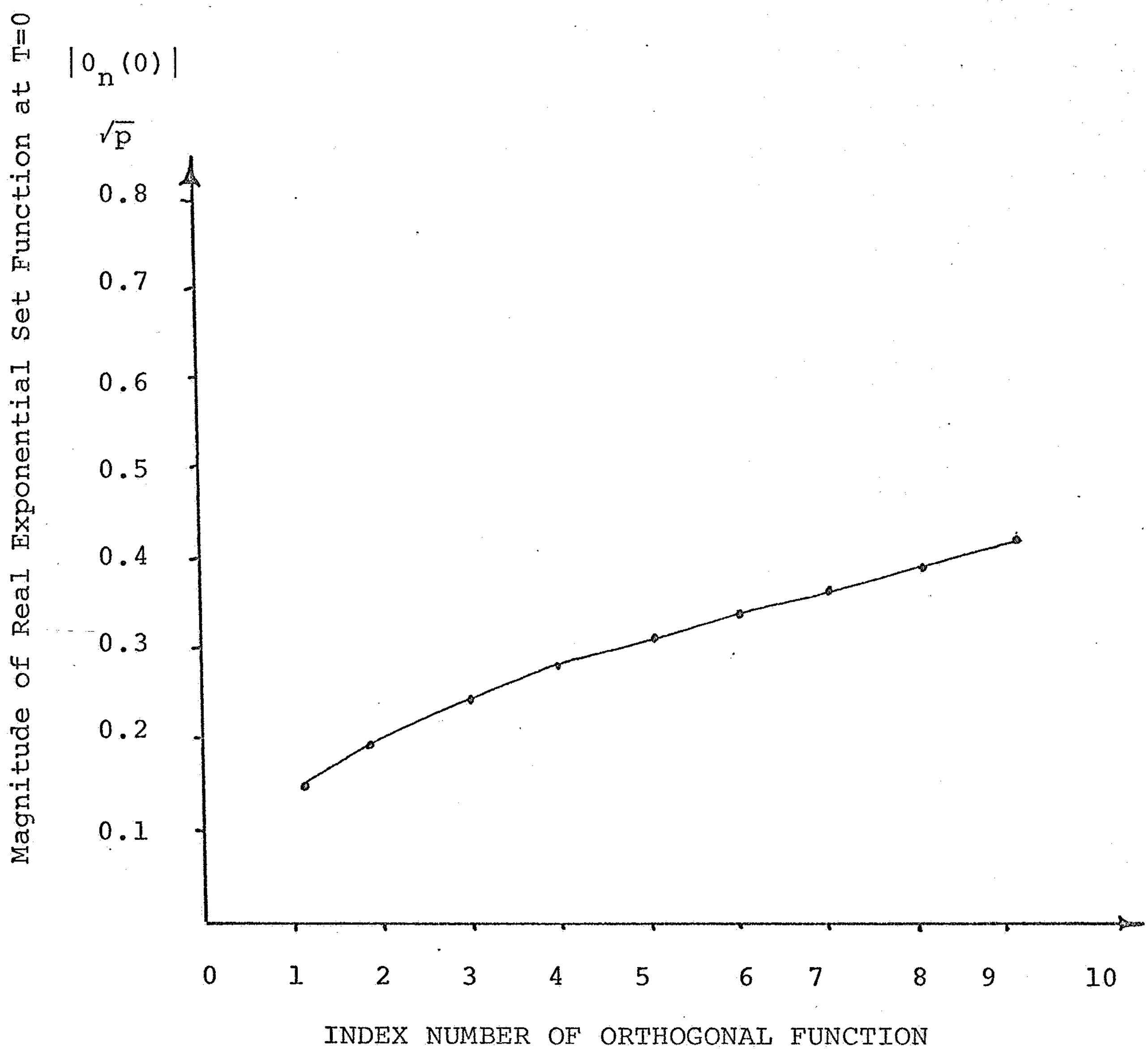


Figure 4-12. The magnitude of real exponential function at T=0 versus the index number

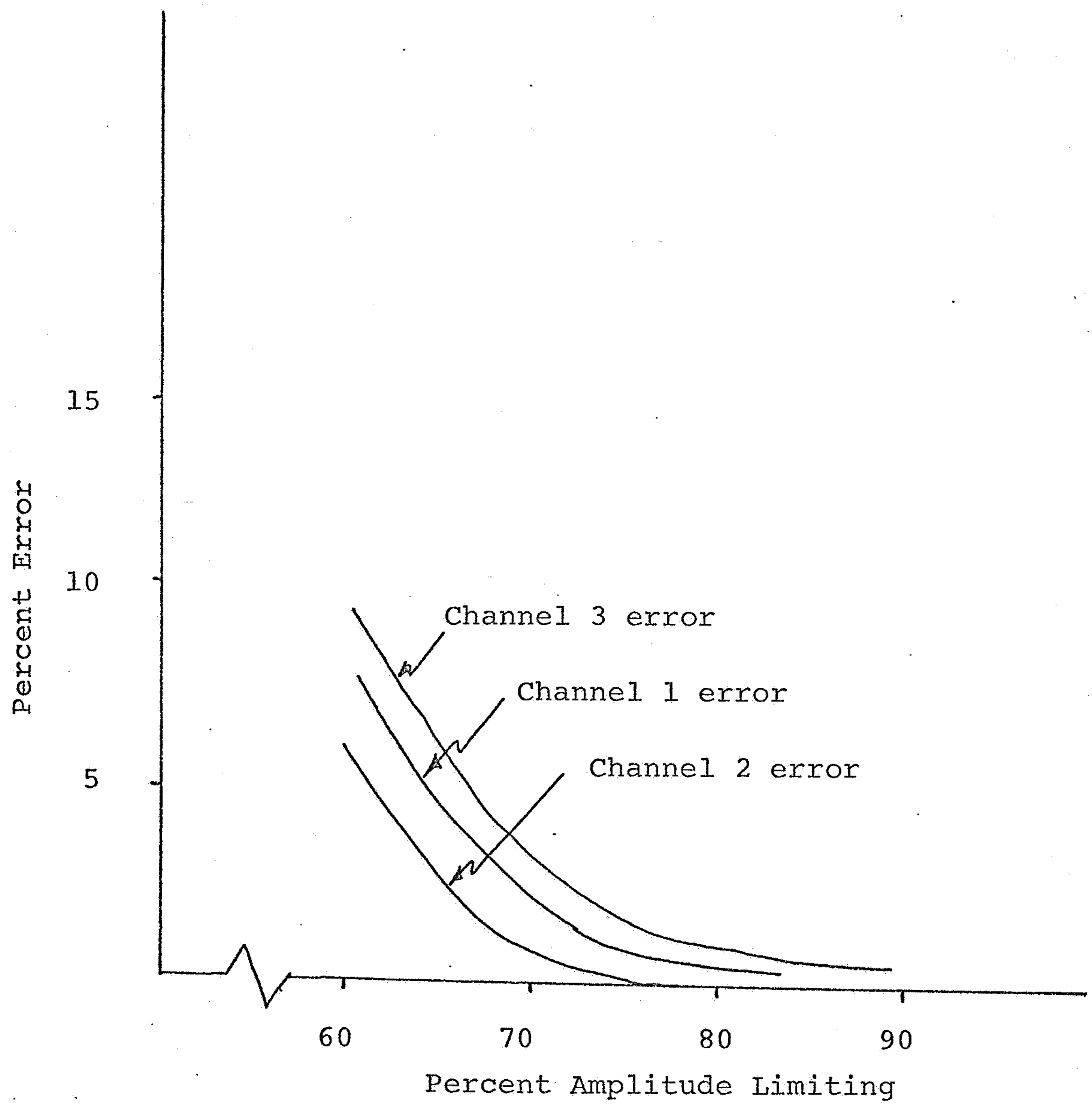


Figure 4-13. The effect of amplitude limiting on channel response. The amplitude is limited to a percentage of the peak value in both the positive and negative swings

where B_m is the message bandwidth and T is the sampling period. Equation (4-17) yields an expression for system bandwidth given by

$$B_s = \frac{Np}{2\pi} \qquad [4\text{-}36]$$

or

$$p = \frac{2\pi B_s}{N}$$

A minimum sampling rate given by Equation (4-35) is used for purposes of comparing the various systems. A substitution of Equations (4-35) and (4-37) in Equation (4-34) yields

$$\left(\frac{2\pi B_s}{N}\right)\left(\frac{1}{2B_m}\right) > \frac{1}{2}[3Ln.(N) - Ln.(\epsilon_{NN}(T))] \qquad [4\text{-}38]$$

or

$$B_s = \frac{NB_m}{2\pi}[3Ln.(N) - Ln.(\epsilon_{NN}(T))] \qquad [4\text{-}39]$$

Equivalent expressions for other systems are given below

$B_s = 2NB_m$, for ordinary frequency division multiplexing. [4-40]

$B_s = 4NB_m$, for time division multiplexing. [4-41]

The results in the form of bandwidth versus number of channels are shown in Figure (4-10).

Peak Limiting Distortion

Figure (4-11) illustrates peak amplitude limiting of the composite waveform given by

$$f_m(t) = \sum_{N=1}^{R} \overline{m_N(t)}O_N(t) \qquad [4\text{-}42]$$

$\overline{m_N(t)}$ = sample value of $m_N(t)$, the N^{th} channel message

$O_N(t)$ = N^{th} orthogonal function

R = number of channels

Clipping results in a system when the peak amplitude capability of the system is exceeded, and it may occur during one or more intervals of each period of operation of the repetitive system. The peak signal amplitude of the real exponential set occurs at the t=0 and increases as the number of functions increase. The peak amplitude of the individual members of the real exponential set at t=0 is shown by Figure (4-12). Amplitude limiting of $f_m(t)$ causes distortion, and the equation for the receiver output in the case of amplitude limiting to the value of

$$f_m(t) = \pm A$$

where A = constant in the interval $0 \leq t \leq T$, is

$$m_J^*(T) = \int_0^{T_1} AO_J(t)dt + \int_{T_1}^{T} f_m(t)O_J(t)dt \qquad [4\text{-}43]$$

where T_1 = time at which limiting ceases

T = pulse period

$m_J^*(T)$ = output of J^{th} channel at receiver.

An expansion of Equation (4-43) yields

$$m_J^*(T) = A \int_0^{T_1} \sum_{k=1}^{J} G_{Jk}e^{-kpt}\,dt +$$

$$\sum_{N=1}^{R} \sum_{\lambda=1}^{N} \sum_{k=1}^{J} m_N(t)C_{N1}C_{Jk} \int_{T_1}^{T} e^{-(\lambda+k)pt}dt \qquad [4\text{-}44]$$

and on evaluation of this integral one obtains

$$m_J^*(T) = A\sum_{k=1}^{J}\left[\frac{1 - e^{-kpT_1}}{kp}\right] +$$

$$\sum_{N=1}^{R}\sum_{\lambda=1}^{N}\sum_{k=1}^{J}\overline{m_N(t)}\,C_{N1}C_{Jk}\left[\frac{e^{-(\lambda+k)pT_1} - e^{-(\lambda+k)pT}}{(\lambda + k)p}\right] \quad [4-45]$$

Figure (4-13) shows the effect of peak amplitude limiting on channel response in the form of percent error. For a three-channel system, the error is less than 5 per cent if the amplitude is allowed to reach 70 per cent of its peak value.

CHAPTER V

THE ORTHONORMAL POLYNOMIAL SET

The circuitry necessary to generate polynomial sets is relatively simple because its elements are linearly independent functions like

$$\{t^0, t^1, t^2, \ldots, t^n, \ldots\}. \qquad [5\text{-}1]$$

This set can be generated using analog integrators and amplifiers. A general polynomial would be of the form

$$\phi_N(t) = \sum_{k=0}^{N} \alpha_{Nk} t^k \qquad [5\text{-}2]$$

An example of such a set are the Legendre polynomials which form the basis for an Orthomux system described by Ballard (1962). The first few Legendre polynomials are

$$O_0(x) = 1 \qquad [5\text{-}3]$$

$$O_1(x) = x \qquad [5\text{-}4]$$

$$O_2(x) = \frac{1}{2}[3x^2 - 1] \qquad [5\text{-}5]$$

with the orthogonality interval of $-1 \leq x \leq 1$. If the following substitution is made for x in the above equations;

$$x = (t - \frac{T}{2}) \qquad [5\text{-}6]$$

the orthogonality interval becomes

$$0 \leq t \leq T$$

and the polynomials take on the form

$$O_0(t) = 1 \qquad [5\text{-}7]$$

$$O_1(t) = 2t/T - 1 \qquad [5\text{-}8]$$

$$O_2(t) = 6t^2/T^2 - 6t/T + 1 \qquad [5\text{-}9]$$

The system should have equal power in each channel, so it is necessary to normalize these functions, and it is done by evaluating the integrals

$$K_N^2 \int_0^T O_N^2(t)\,dt = 1 \qquad [5\text{-}10]$$

to obtain K_N as

$$K_0 = \sqrt{1/T}$$

$$K_1 = \sqrt{3/T}$$

$$\vdots$$

$$K_N = \sqrt{(2N+1)/T} \qquad [5\text{-}11]$$

Therefore these functions 0_1, 0_2, ..., 0_N,... do not have the same value at t=T. The orthonormal Legendre polynomials become

$$O_0(t) = \sqrt{1/T} \qquad [5\text{-}12]$$

$$O_1(t) = \sqrt{3/T}\left(\frac{2t}{T} - 1\right) \qquad [5\text{-}13]$$

$$O_2(t) = \sqrt{5/T}\ (6t^2/T^2 - 6t/T + 1) \qquad [5\text{-}14]$$

and this set would have equal power distribution in each channel. It is necessary to produce stable positive and negative voltages to set the initial values of the ordinary Legendre polynomials. A different voltage must be derived from the reference voltages for each of the polynomials in the modified set because the initial values are all different.

<u>Zero Initial Value Set</u>

In this section another polynomial set has been constructed using the Gram-Schmidt procedure, where each function has an initial value of zero as shown in Appendix B. The first three polynomials are

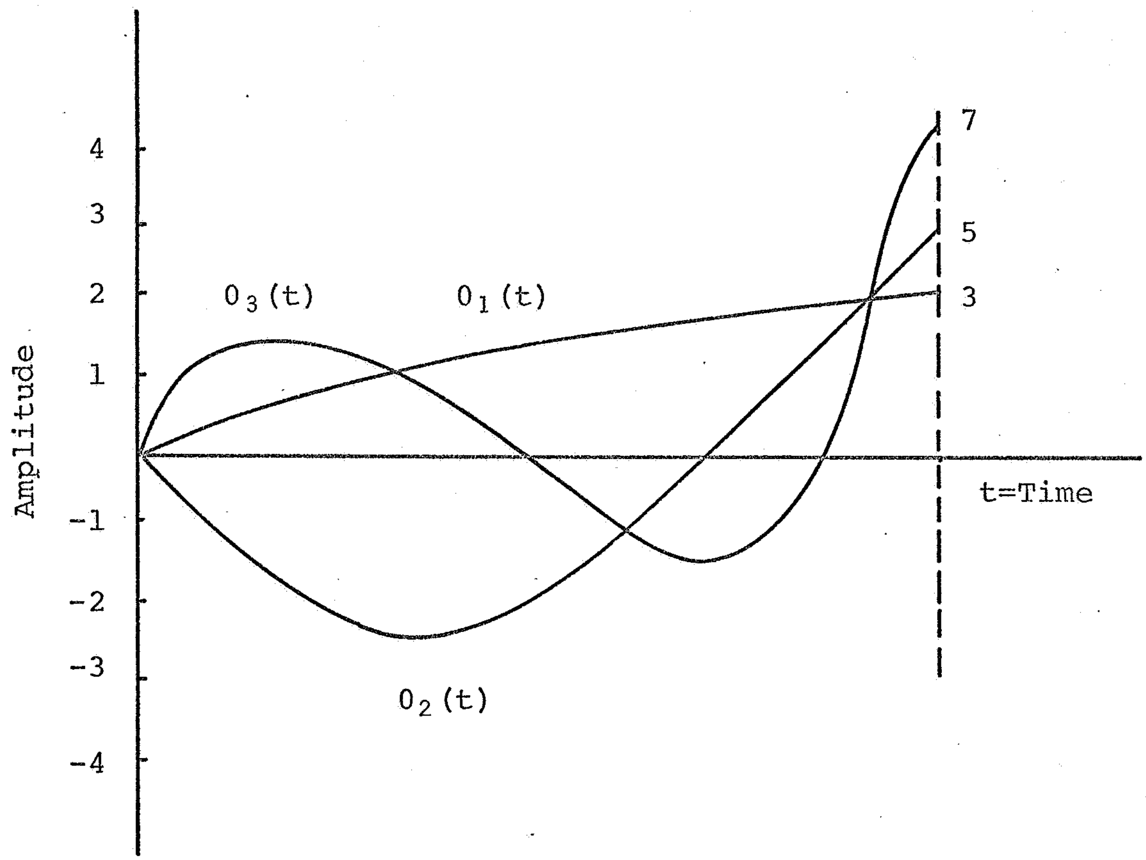


Figure 5-1. The First Three Polynomials of the Zero Initial Value Set

$$O_1(t) = \sqrt{3}t \quad [5\text{-}15]$$

$$O_2(t) = \sqrt{5}(4t^2 - 3t) \quad [5\text{-}16]$$

$$O_3(t) = \sqrt{7}(15t^3 - 20t^2 + 6t) \quad [5\text{-}17]$$

where T is normalized to unity. The base set used here is $(t^1, t^2, \ldots, t^n, \ldots)$ and Figure (5-1) shows the first three functions of the zero initial value set.

System Bandwidth Requirements

An expression for the power density spectrum is for the polynomials functions discussed above. In general,

$$O_N(t) = \sum_{k=1}^{N} \alpha_{Nk} t^k \quad [5\text{-}18]$$

and the Fourier coefficients α_{Nk} are given below

$$a_\beta = 2\int_0^1 t^k \cos(\beta\omega_1 t)\,dt \quad , \quad \beta = 0,1,2,\ldots \quad [5\text{-}19]$$

$$b_\beta = 2\int_0^1 t^k \sin(\beta\omega_1 t)\,dt \quad , \quad \beta = 1,2,\ldots \quad [5\text{-}20]$$

$$c_\beta^2 = a_\beta^2 + b_\beta^2 \quad [5\text{-}21]$$

Integration by parts yields the following:

$$a_\beta = 2\{\frac{1}{\beta\omega_1} t^k \sin(\beta\omega_1 t)\Big|_0^1 - \frac{k}{\beta\omega_1}\int_0^1 t^{k-1} \sin(\beta\omega_1 t)\,dt\} \quad [5\text{-}22]$$

$$b_\beta = 2\{-\frac{1}{\beta\omega_1} t^k \cos(\beta\omega_1 t)\Big|_0^1 + \frac{k}{\beta\omega_1}\int_0^1 t^{k-1} \cos(\beta\omega_1 t)\,dt\} \quad [5\text{-}23]$$

which can be easily calculated by a computer program. The first two calculations give the following results for $O_1(t)$

$$|c_\beta|^2 = 3/(\beta\pi)^2 \quad [5\text{-}24]$$

and for $O_2(t)$,

$$|c_\beta|^2 = [5/(\beta\pi)^2][1 + 16/(\beta\pi)^2] \quad [5\text{-}25]$$

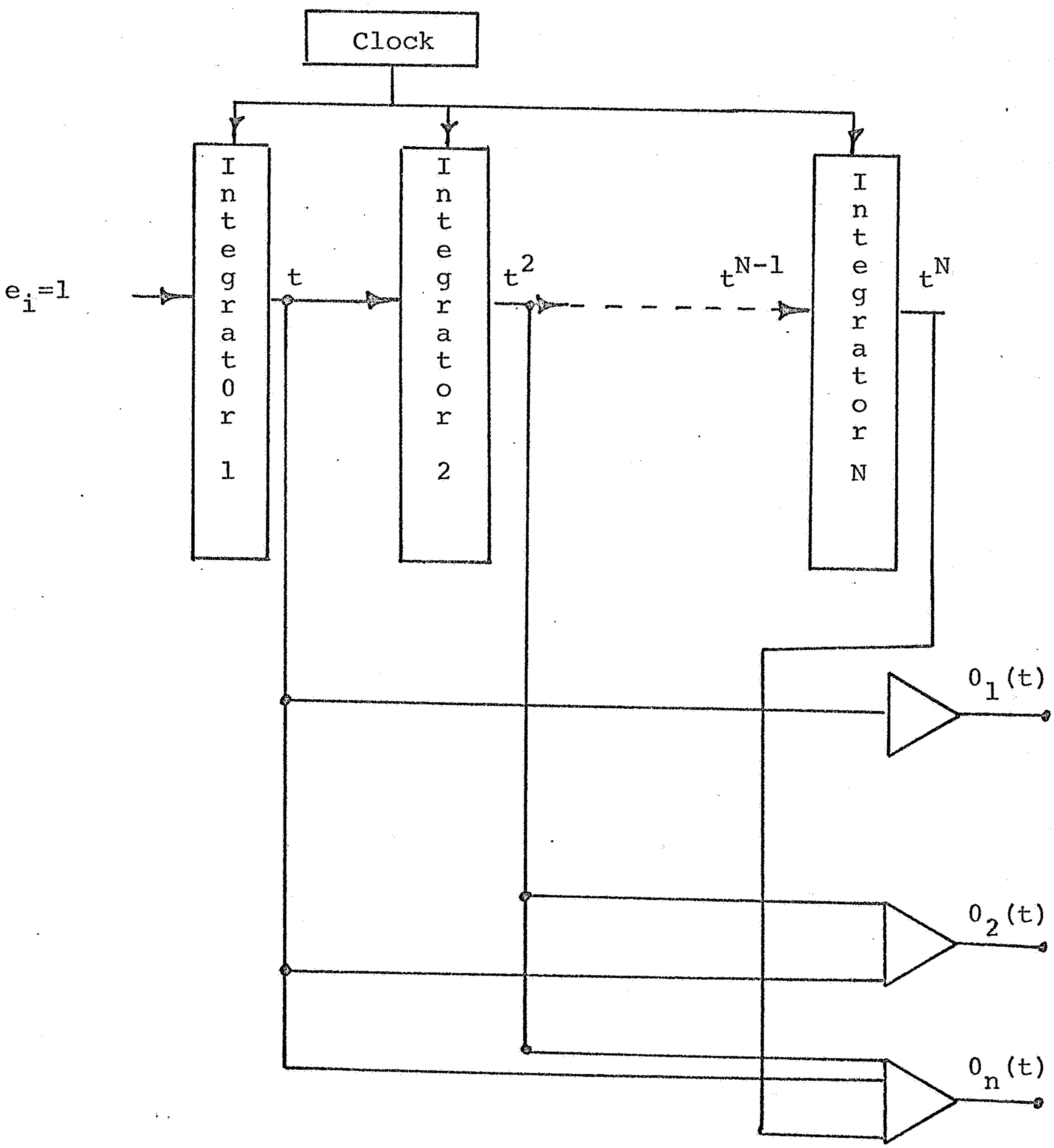


Figure 5-2. An implementation of the polynomial set

The dominant term in the above equations for large values of β is given by

$$|c_\beta|^2 \approx \frac{2N + 1}{(\beta\pi)^2} \qquad [5\text{-}26]$$

where N is the index number of the function and $\beta = 0,1,2,$ ... . An evaluation of the crosstalk in a system due to frequency truncation is easily done by a computer program for each filter. The bandwidth of the simulated filter should be varied from a minimum value given by $f_0 = 1/T$, which is the system repetition rate, to an upper limit of approximately ten times the minimum value or until the distortion reaches tolerable values. There is no crosstalk due to time truncation for a system based on the zero initial value set of polynomials since the interval of orthogonality will be the same as the system period. Thus crosstalk would occur due only to amplitude and frequency truncation and interference.

Peak Voltage

The zero initial value set has unit energy in each channel for the orthogonality interval, and the peak value for $O_n(t)$ occurs at t equal to unity and is given by

$$O_N(t=1) = \sqrt{2N + 1} \qquad [5\text{-}27]$$

This makes it evident that the problem of peak voltages poses a very serious limitation, and it becomes more serious when the composite signal is formed by summing the various individual polynomials.

Implementation

The orthomux system based on polynomials can be built using analog integrating circuits. Figure (5-2) shows a block diagram of such a system based on such polynomials.

CHAPTER VI

AN ORTHOGONAL DIGITAL SYSTEM

The preceding chapters dealt with analog signal sets suitable for use in an orthogonal multiplexing system, and now a possible orthomux system using orthogonal digital waveforms is discussed. Formation of sets of orthogonal digital functions suitable for such a system will be emphasized.

A digital orthomux system is designed in the same manner as an analog system, in that the code words used are orthogonal to each other. An example of an orthonormal set suitable for use as a basis for an orthomux system is:

$$R = |R_0, R_1, \ldots| \qquad [6\text{-}1]$$

$$R_N = \mathrm{Sgn}\ |\mathrm{Sin}(2^N \pi t)| \qquad [6\text{-}2]$$

This is the family of Radamacher functions, and their region of orthogonality is $0 \leq t \leq 1$. The Sgn function is defined as

$$\begin{aligned} \mathrm{Sgn}(a) &= 1 & (a > 0) \\ \mathrm{Sgn}(a) &= -1 & (a < 0) \\ \mathrm{Sgn}(a) &= 0 & (a = 0) \end{aligned} \qquad [6\text{-}3]$$

The first three Radamacher functions are shown in Figure 6-1. A simple method of forming an orthogonal binary set is by employing the Hadamard matrix. A Hadamard matrix is a square matrix whose elements are restricted to take on values of plus or minus one, and has the property that the row vectors (as well as the column vectors) are mutually orthogonal. Formation of Hadamard matrices of the order 2^k where k is an integer equal to or greater than zero is possible using the relationship

$$H_{2^{k+1}} = \begin{vmatrix} H_{2^{k+1}} & H_{2^k} \\ H_{2^k} & \overline{H_{2^k}} \end{vmatrix} \qquad [6\text{-}4]$$

For example, H_1 can be immediately written as $H_1=[1]$ (or $H_1=[0]$. If $H_1=[1]$ is selected, the matrix H_2 takes the form

$$H_2 = \begin{vmatrix} H_1 & H_1 \\ H_1 & \overline{H_1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \qquad [6\text{-}5]$$

and the matrix H_4 is given by

$$H_4 = \left|\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right| = \begin{vmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{vmatrix} \qquad [6\text{-}6]$$

and so on. A useful definition of correlation for binary waveforms is

$$P = A - B \qquad [6\text{-}7]$$

where
A = number of like terms, and
B = number of unlike terms.

Application of this definition to Equation [6-6] shows that either the row vectors or column vectors are a suitable set of functions for an orthogonal multiplexing system. Another set of orthogonal waveforms is given by

$$\begin{aligned} F_1 &= [\,1, \quad 1, \quad 1, \quad -1] \\ F_2 &= [\,1, \quad 1, \quad -1, \quad 1] \\ F_3 &= [\,1, \quad -1, \quad 1, \quad 1] \\ F_4 &= [-1, \quad 1, \quad 1, \quad 1] \end{aligned} \qquad [6\text{-}8]$$

If the value zero is substituted for the positive ones in Equation [6-8], the resulting orthogonal set is recognized as that used in time division multiplexing. In all these orthogonal digital codes, n information bits are transmitted in 2^n symbols. Time division multiplexing can have the optimum peak-to-average power ratio of unity. However, it requires more accurate synchronization of the receiver in order to prevent excess message distortion, since the energy of each message is concentrated in a narrow time span. Systems using Equation |6-6| as a basis have the advantage of having the impulse noise distributed evenly among the channels and thus the synchronization problem becomes less critical.

Implementation of the digital system is accomplished by using cascaded bistable multivibrator circuits which give the outputs of Equation |6-2| as shown in Figure (6-1). The additional waveforms which must be formed to complete the orthogonal set are formed by combinational logic blocks. An eight-channel code is given by

$$
\begin{aligned}
DC &= [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1] \\
A &= [1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0] \\
B &= [1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0] \\
C &= [1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0] \\
D &= [1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1] \\
E &= [1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1] \\
F &= [1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1] \\
G &= [1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0]
\end{aligned}
\qquad [6\text{-}9]
$$

where DC is a level, and A, B, and C are formed by cascading an astable multivibrator and two bistable multivibrators. D, E, F, and G are derived from A, B, and C, using and-or logic circuits which satisfy the Boolean functions given by

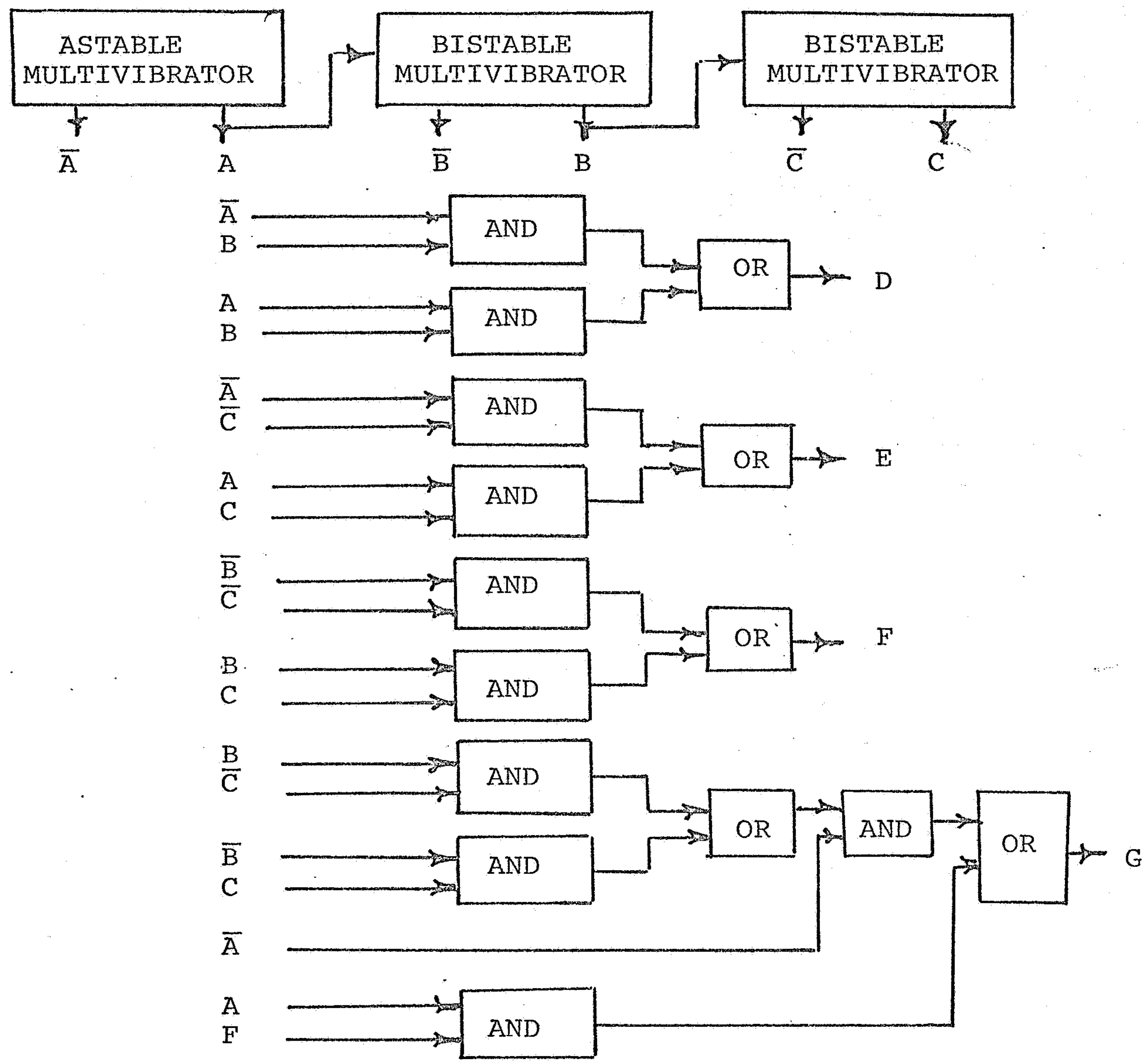


Figure 6-1. Logic circuitry for realization of an orthogonal digital set

$$D = \bar{A}\,\bar{B} + AB \qquad [6\text{-}10]$$

$$E = \bar{A}\,\bar{C} + AC \qquad [6\text{-}11]$$

$$F = \bar{B}\,\bar{C} + BC \qquad [6\text{-}12]$$

$$G = \bar{A}\,(B\bar{C} + \bar{B}C) + A\,F \qquad [6\text{-}13]$$

A message can be impressed on an orthogonal function by transmitting either the function or its inverse, which is formed by passing the function through a simple inverting amplifier or by ordinary analog methods. A digital system does not require analog multipliers in either the transmitter or receiver, since multiplication by plus or minus one can be accomplished by a selective inverting amplifier controlled by the appropriate orthogonal waveform. In the case of a digital system it is possible to devise a method of combining the orthogonal functions by means of a Boolean logic transformation. Titsworth (1963) has proposed a system employing a majority logic in the transmitter to combine the channel waveforms and correlation dectection in the receiver. The advantage of such a system is that the peak-to-average power ratio is unity as is the case with time division multiplexing, and hard limiting at the receiver is possible. This is not an orthomux system and, according to Titsworth, performance is about two decibels poorer than the systems previously discussed.

CHAPTER VII

TECHNIQUES TO IMPROVE SYSTEM PERFORMANCE

The performance of orthomux communication systems may be limited because of the following factors:

(1) the finite time truncation of a signal set with an infinite orthogonality interval,
(2) large peak-to-average signal voltage ratio, and
(3) excessive bandwidth requirement of the signal set.

These factors are often responsible for distortion of the received messages, since crosstalk is the inevitable result of time, amplitude, or frequency truncation of the transmitted signal. If the resultant distortion becomes intolerable, it is necessary to take steps to improve the system performance, and this chapter is a discussion of methods to achieve this objective.

Finite Time Truncation

It is sometimes desirable to use a signal set which is orthogonal over an infinite interval. The reason for this is that the system parameters can be selected so that the crosstalk is tolerable at the minimum truncation time, and therefore will be less for longer truncation times. Thus, a basic system may be operated at any repetition rate below some maximum value. In some cases it may be desirable to completely eliminate the crosstalk due to time truncation, and this can often be accomplished by applying the Gram-Schmidt procedure to the linearly independent set of functions which are used to construct an orthonormal set of functions under consideration. For instance, consider the real exponential set given by Equation |4-5| which has an orthogonality interval of $0 \leq t \leq \infty$. The linearly independent set is given by e^{-pt}, $e^{-2pt}, \ldots, e^{-npt}$. The first two functions of the real exponential set orthonormal over the interval $0 \leq t \leq T$ were calculated using the Gram-Schmidt procedure outlined in Appendix B, and are given by:

$$\phi_1 = (2p/B)^{\frac{1}{2}} e^{-pt} \qquad [7\text{-}1]$$

$$\phi_2 = \frac{(2C/3B)e^{-pt} - e^{-pt}}{[(D/4p) - (2C^2/9pB)]^{\frac{1}{2}}} \qquad [7\text{-}2]$$

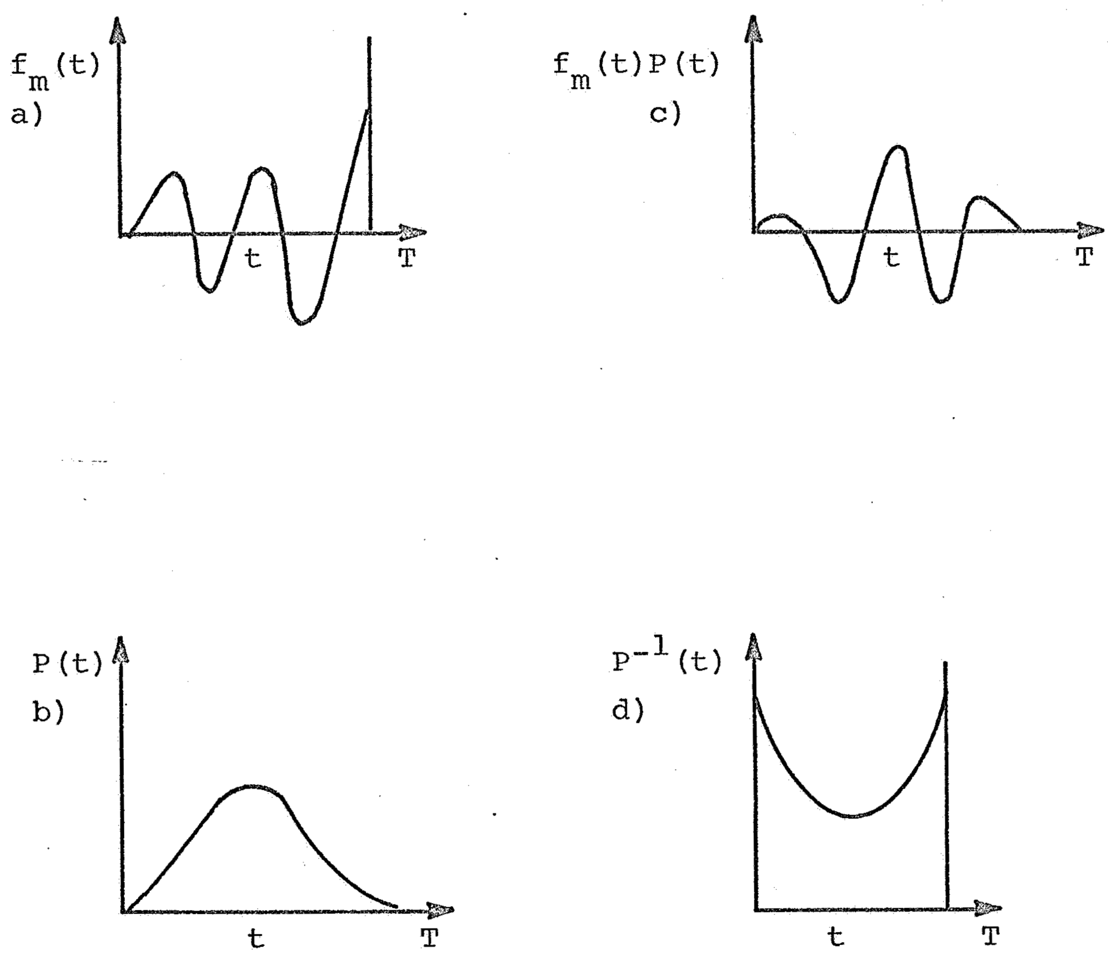


Figure 7-1. Waveforms formed in the transmitter and receiver

where

$$B = 1 - e^{-2pt}$$

$$C = 1 - e^{-3pt}$$

$$D = 1 - e^{-4pt}$$

T = Orthogonality Interval, and

p = System Parameters

These functions may be expressed in the following form:

$$\phi_1 = C\, e^{-pt} \qquad [7\text{-}3]$$

$$\phi_2 = C\, e^{-pt} + C_2 e^{-2pt} \qquad [7\text{-}4]$$

where

$$C_1 = (2p/B)^{1/2}$$

$$C_2 = \frac{2C/3B}{[(D/4p) - (2C^2/9pB)]}$$

and

It is evident that reducing the orthogonality interval from an infinite to a finite value does not increase the complexity of the circuitry required to generate this set of functions. The crosstalk due to time truncation is completely removed and the system performance improved by a corresponding amount, however. This general result for ϕ_i may be verified by allowing the interval T to increase without bound until in the limit B = C = D = 1, and the orthogonal functions become

$$\phi_1 = \sqrt{2p}e^{-pt} \qquad [7\text{-}5]$$

$$\phi_2 = 4\sqrt{p}e^{-pt} - 6\sqrt{p}e^{-2pt} \qquad [7\text{-}6]$$

Peak-To-Average Ratio

A practical communication system has a finite allowable peak-to-average power ratio and if operation is attempted outside this limit, the transmitted signal is subject to amplitude limiting resulting in message distortion. A certain amount of amplitude limiting is tolerable in order to assure the maximum utilization of the transmitter capability. A knowledge of the amplitude distribution of the message signals is very useful in this connection and can sometimes be obtained. The simulation program given in Appendix C is used to determine the exact message distortion of each channel as a function of amplitude limiting. Typical values of peak signal levels are given below for several orthonormal sets with N representing the number of channels and T the magnitude of the orthogonality interval:

(1) for time division multiplexing V (peak), the composite signal peak is given by

$$V(\text{peak}) = \sqrt{N/T} \qquad |7\text{-}7|$$

(2) for frequency division multiplexing,

$$V(\text{peak}) \leq N\sqrt{N/T} \qquad |7\text{-}8|$$

(3) for the real exponential set which is orthonormal over the interval $0 \leq t \leq \infty$, the peak value ϕ_k of the k^{th} signal of the set, is given by

$$\phi_k(\text{peak}) = \sqrt{2pk} \qquad |7\text{-}9|$$

and the composite signal peak is given by

$$V(\text{peak}) = \sum_{k=1}^{N} \phi_k = \sqrt{2p} \sum_{k=1}^{N} \sqrt{k} \qquad |7\text{-}10|$$

(4) for the polynomial set given by Equations |7-14|, |7-15|, and |7-16| the composite peak value is

$$V(\text{peak}) = \sum_{k=1}^{N} \frac{\sqrt{2k+1}}{\sqrt{T^{2k+1}}} \qquad |7\text{-}11|$$

There are several methods which may be capable of lowering the peak signal value of an orthomux system. Each particular orthomux system needs to be examined, since all the techniques are not suitable for all systems. In the case of frequency division multiplexing, it is clearly desirable to use sine functions rather than cosine functions as the orthogonal set, for the peaks of the sine functions occur at different points in the orthogonality interval and thus the peak value of the sum of the sine function subcarriers is less than the cosine functions. The peak value for the cosine set is

$$V(\text{peak}) = \sum_{k=1}^{N} A \cos(k\omega t) = NA \qquad [7\text{-}12]$$

where for the sine set it is

$$V(\text{peak}) = \sum_{k=1}^{N} A \text{ sine } (k\omega t) < NA \qquad [7\text{-}13]$$

where

N = total number of channels, and
A = peak value of the individual channels.

The peak value for the normalized sine function set is shown in Table 7-1.

The system based on both the sine and cosine functions utilizes bandwidth two times as effectively as the sine-only system and also has a lower peak signal value for an equal number of channels. The peak signal value for the sine and cosine subcarrier system for eight channels was calculated to be

$$V(\text{peak}) = 7.59$$

which is less than the value listed for the eight-channel system in Table 7-1. The peak of the sine and cosine system contains less energy than that of the sine-only system, and this is responsible for further reducing the distortion due to amplitude limiting. Table 7-2 shows the effect of amplitude limiting in terms of message distortion for the sine-and-cosine system. It is therefore preferable from bandwidth and

Table 7-1. The peak signal value of the orthogonal system based on sine functions only.

Number of Channels N	V(Peak)	θ(Radians)
1	1.41	1.57
2	2.48	.93
3	3.5	.67
4	4.51	.52
5	5.53	.43
6	6.55	.36
7	7.57	.31
8	8.57	.27
9	9.6	.25
10	10.6	.23

peak-to-average power considerations to use both the sine and cosine functions for frequency division multiplexing systems.

The peak signal value of orthogonal systems based on functions such as the real exponential set and the polynomial set can be reduced by shifting the level of the signal so that the constant or D.C. value is removed. This makes the set orthogonal to any constant as well. An example of such an orthonormal polynomial set is given by

$$\phi_1 = 1/\sqrt{T} \qquad [7\text{-}14]$$

$$\phi_2 = \sqrt{3/T^3}\,[2t - T] \qquad [7\text{-}15]$$

$$\phi_3 = \sqrt{5/T^5}\,[6t^2 - 6Tt + T^2] \qquad [7\text{-}16]$$

where $0 \leq t \leq T$ is the orthogonality interval, ϕ_1 may be a constant, and the other ϕ_is will still be orthogonal. It is of course possible to shift this set by a constant amount in the transmitter so that the positive and negative peaks are equal, but the equipment necessary to remove the messages in the receiver would be very complex. Another way to improve the condition is to multiply the composite signal $f_m(t)$

$$f_m(t) = m_1(t)O_1(t) + m_2(t)O_2(t) + \ldots \qquad [7\text{-}17]$$

where

$m_i(t)$ = message in the i^{th} channel, and

$O_i(t)$ = i^{th} channel orthogonal function

by another function P(t), which has a small or zero value at the point in time where the composite signal peaks. This is particularly effective for the real exponential set where the peak value of each function occurs at t = 0. It is

TABLE 7-2. Message distortion as a function of amplitude limiting for an eight-channel system. The odd numbered channels are sine channels, the even numbered channels are cosine channels.

Channel Number	Percent Message Distortion for limiting at .8 V(Peak)	Percent Message Distortion for limiting at .6 V(Peak)
1	2.7	8.5
2	10.0	28.0
3	5.2	15.6
4	8.8	24.9
5	7.1	20.3
6	7.0	19.7
7	8.3	22.2
8	4.7	13.1

necessary to generate the reciprocal of P(t) in the receiver in order to recover the messages. The transmitted signal is thus

$$Tx = f_m(t)P(t) \qquad [7\text{-}18]$$

and the functions generated in the receiver are given by

$$(Rx)_j = P^{-1}(t)O_j(t) \qquad [7\text{-}19]$$

where

$$P^{-1}(t) = 1/P(t)$$

$$j = j^{th} \text{ channel}$$

The j^{th} channel detection operation takes the form given by

$$\int_0^T (f_m(t)P(t))(P^{-1}(t)O_j(t))dt \qquad [7\text{-}20]$$

which becomes

$$\int_0^T f_m(t)O_j(t)dt = m_j(t) \qquad [7\text{-}21]$$

and thus the message is recovered correctly. Figure (7-1) depicts such an operation. The improvement is dependent on the function selected for P(t).

System Bandwidth Considerations

The bandwidth requirements of a system often determine the suitability of a given set. A comparison of the bandwidth requirements of orthogonal multiplexing systems based on practical functions, such as sines, cosines, and real exponentials shows that the system based on both sines and cosines (sometimes called super frequency division multiplexing) is considerably better in this respect (see Figure (4-10)). It is possible to compress the spectrum of a signal by smoothing it in the time domain. One way this can be accomplished is by inverting the time domain waveforms of odd functions such as the odd ordered Legendre polynomials.

CHAPTER VIII

SYSTEM IMPLEMENTATION

The orthogonal multiplexing system contains several functions whose complexity of realization in the transmitter and receiver is considered next. Specific functional operations considered here are function generation, multiplication, filtering, and signal detection.

Function Generation

Suitable functions for use in orthogonal multiplexing are the exponentials, sines, polynomials of time, and digital signals, or their products. The real exponential linearly independent set uses these functions

$$\{e^{-pt}, e^{-2pt}, \ldots , e^{-kpt}\} \tag{8-1}$$

These functions are the response of an RC network to an impulse function input (Figure (8-1)). These may also be produced by a variation of this circuit which is shown in Figure (8-2). The capacitor is charged rapidly by a short duration pulse from a low impedance source. The diode conducts when the pulse is positive and as the pulse voltage discharges through the resistor to produce

$$e_0 = Ee^{-t/RC}, \qquad t \geq 0 \tag{8-2}$$

The values of RC are selected to be equal to 1/np for n=1,2, ...and thus all the basic functions are generated for the real exponential set. In combining the basic exponentials to form a function such as

$$O_2 = C_{21}e^{-pt} + C_{22}e^{-2pt} \tag{8-3}$$

it is necessary to use operational amplifiers with the gains adjusted to give the coefficients the correct value. It is simpler to build these amplifiers for "A.C. coupled" or "capacitively coupled" rather than direct coupled service. It is possible to use capacitively coupled amplifiers provided the orthogonal set has no d.c. component in any member function, which implies that these functions must be orthogonal to any constant term. A set of orthonormal functions has been formed which are based on the real exponential functions given by

$$e^{-0pt},\ e^{-pt},\ e^{-2pt},\ \ldots,\ e^{-kpt} \tag{8-4}$$

These orthonormal functions also have a finite interval of orthogonality, and the first three are given by

$$\phi_1 = 1/\sqrt{T} \qquad 8\text{-}5$$

$$\phi_2 = \frac{e^{-pt} - (A/pT)}{[(B/2p) - (A^2/p^2T]} \tag{8-6}$$

$$\phi_3 = \frac{e^{-2pt} - K_1 e^{-pt} + K_2}{\{g_3, g_3\}^{\frac{1}{2}}} \tag{8-7}$$

where

$$A = 1 - e^{-pt}$$

$$B = 1 - e^{-2pt}$$

$$K_1 = \frac{2pTC - 3AB}{3BpT - 6A^2}$$

$$K_2 = \frac{A}{3pT}\left(\frac{2pTC - 3AB}{BpT - 2A^2}\right) - \frac{B}{2pT}$$

$$\{g_3, g_3\}^{\frac{1}{2}} = \left[\frac{-24K_1K_2A + 6(K_1^2 + K_2)B - 8K_1C + 3D + 12K_2^2Tp}{12p}\right]^{\frac{1}{2}}$$

This set has a lower peak-to-average ratio than the real exponential set discussed in Chapter IV by the amount the elimination of the constant or d.c. term removes.

Polynomials in powers of t are easily formed, as shown in Figure (8-3). Functions of the form

$$y = Ke^{-a^2t^2} \tag{8-8}$$

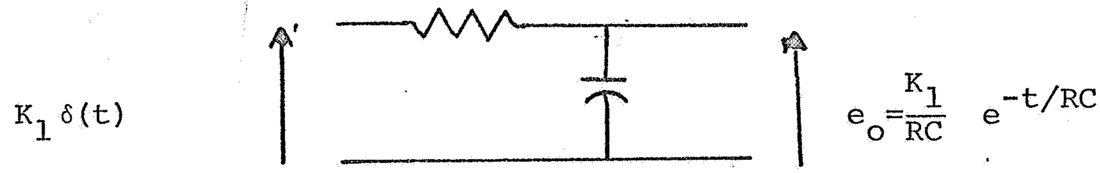


Figure 8-1. Exponential functions generating RC network

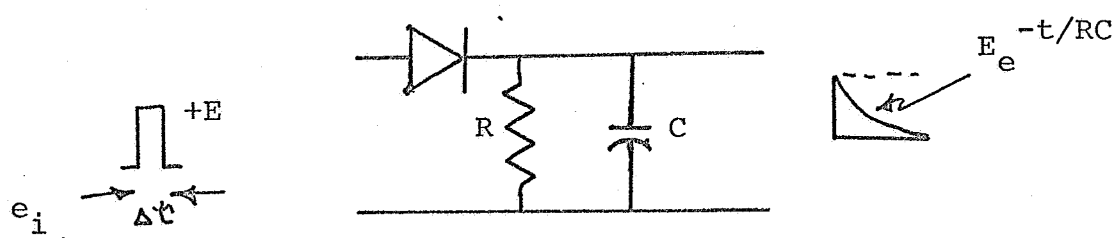


Figure 8-2. Exponential functions generating RC network with diode isolation

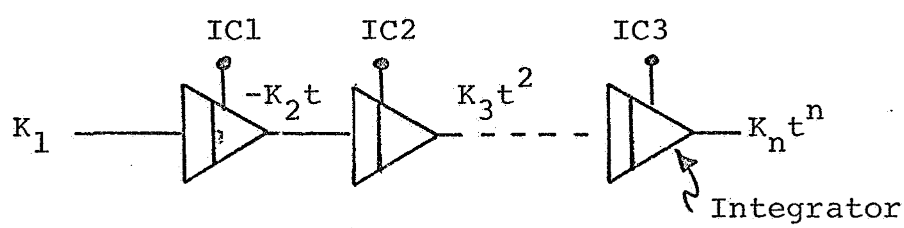


Figure 8-3. Polynomial generating system

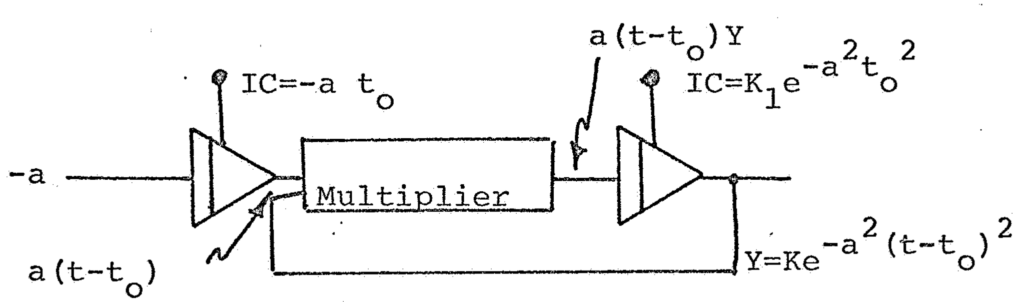


Figure 8-4. Function generation circuit

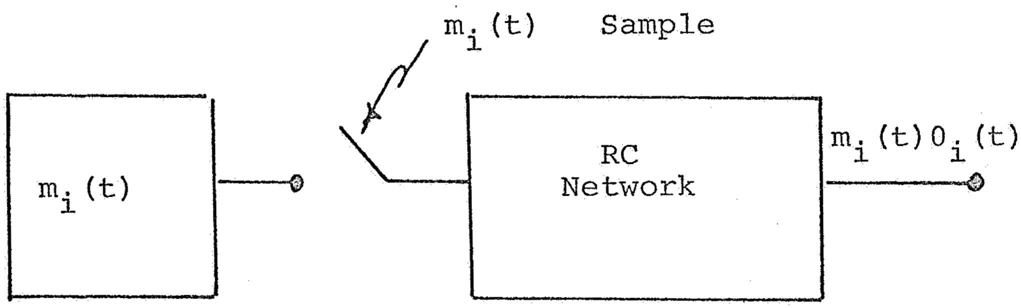


Figure 8-5. Function generation without multiplication

are generated by the circuit of Figure (8-4) with electronic multipliers in order to achieve a high speed operation in a communication system. Accurate electronic multipliers require non-linear components the cost of which may be prohibitive.

An alternate solution is to approximate the function with a series expansion. If it is desired to use the normal pulse from $0 \leq t \leq T$ to lower the peak power requirements of the zero initial value set, then the $e^{-a^2t^2}$ signal can be formed as a sum of even powers of t from already available functions. The set of waveforms commonly used in analog multiplexing systems is the sine-cosine set. This has resulted in the development of sinusoidal signal sources which are stable and are capable of operating over a wide range of frequencies. One particular form of stable sinusoidal sources is used widely in the frequency synthesizer. An example of a system for generating orthogonal digital signals is given in Chapter VI.

Multiplication

Circuits used to satisfactorily perform four quadrant multiplication are expensive, and therefore the use of these devices should be avoided wherever possible. Multiplication is also necessary if the correlation detection process is used in the receiver.

The need for multiplication in the transmitter of the orthomux system based on the real exponential set can be avoided by using the sampled value of the message waveform as the input. This is shown diagrammatically in Figure (8-5) and the waveform for

$$\overline{m_1(t)}O_1(t) = \overline{\sin(\omega t)}e^{-t/RC} \qquad [8\text{-}9]$$

where the bar denotes the sampled value. This technique is shown in the photograph of Figure (8-6).

Filtering

Distortionless transmission and reception of a waveform is desirable in a communication system and is achieved by the use of synchronous correlation detection. Amplitude distortion or non-linear phase shift causes crosstalk or interference in the receiving channels. It is often necessary to limit the signal energy level outside a bandpass in order to avoid interference with other systems. The design of a filter to minimize distortion is the next requirement. For distortionless transmission, a bandpass filter must have (Panter, 1965)

$$H(j\omega) = K, \quad \omega_1 \leq \omega \leq \omega_2 \qquad [8\text{-}10]$$

$$H(j\omega) = 0 \quad \text{otherwise} \qquad [8\text{-}11]$$

$$\theta(\omega) = \omega t_0 \pm n\pi \qquad [8\text{-}12]$$

The output of this ideal filter is a replica of the input waveform but delayed in time. The operation of the receiver of the orthogonal multiplex system can be delayed to compensate for this and thus the undistorted messages are recovered. Such an ideal filter cannot be realized but modern filter theory does allow the realization of filters which give maximally flat amplitude or phase response. The Butterworth filter has a maximally flat amplitude response characteristic, and the transfer function is given by

$$T(S) = \left[\frac{T_0}{S^N + b_{n-1}S^{N-1} + \ldots + b_1 S + b_0}\right] \qquad [8\text{-}13]$$

The solutions to the denominator polynomial in Equation |8-13| maximally flat amplitude case lead to the Butterworth polynomials. Equation |8-13| also represents the general form of the Bessel, or maximally flat delay filter. The denominator in this case yields the Bessel polynomial. These filters are tabulated in Weinberg (1962). A set of functions that have characteristics between the Butterworth and Bessel filters are the transitional Butterworth-Thompson filters (Peless and Murakami, 1957). The crosstalk resulting from filtering can be formulated as in Equation |4-23| and is easily calculated, using a computer program such as the one in Appendix C. "Polynomial Functions and Channel Filter".

Signal Detection

The correlation process of signal detection is used in the orthomux receiver. Actually only one value is calculated, at τ equal to zero, thus implying that the functions are occurring in synchronism. This is the ideal case when the receiver is perfectly synchronized. The response of a correlation detector in the case of the real exponential set

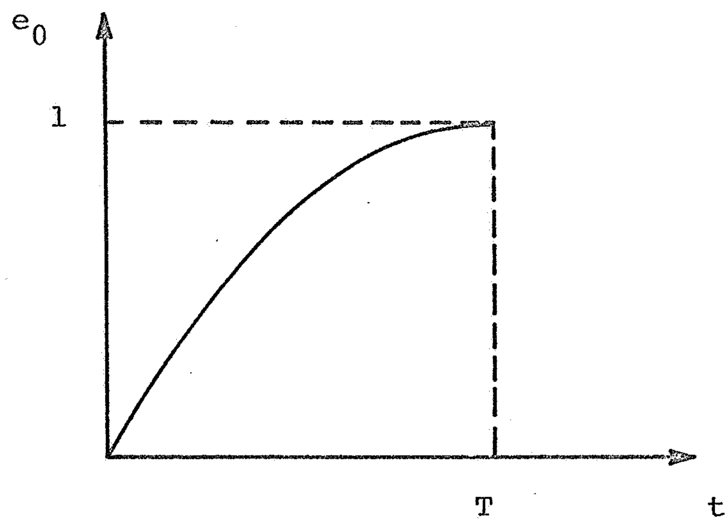


Figure 8-7. Correlation detection output

is discussed here. A message input of unity in the first channel yields an output shown in Figure (8-7), where crosstalk terms are ignored:

$$\int_0^T (\sqrt{2p}e^{-pt})(\sqrt{2p}e^{-pt})dt = (1-e^{-2pT}) \qquad [8\text{-}14]$$

The detector output approaches the correct value as an asymptote. The correlation detector requires a multiplier and integrator with inputs as shown. At the end of the integration period the output of the integrator is sampled and the integrator is reset. The sampled outputs are held and passed through a smoothing filter to recover the analog input message.

CHAPTER IX

MULTIPLEXING AND MODULATION DESIGN CRITERIA

A. Introduction

In this chapter, factors affecting the choice of "good" modulation and multiplexing schemes for a communication link where the transmitter is required to transmit N binary data signals, one binary PN waveform and a carrier reference signal, as is shown in Figure 9-1 are discussed. The approach used is the so-called "orthumux" approach discussed previously. The characteristics of these signals are as follows:

1. Binary data signals

The binary data signals consist of a sequence of ones and minus ones which have an equally likely probability of occurrence. It is known (Wozencraft, 1965) that the most efficient signal set $\{s_1(t), s_2(t)\}$ for transmission of this type of information over a peak power limited channel is a binary antipodal set of signals with equal energy E and period T. That is, a set for which

$$\rho = \frac{1}{E}\int_0^T s_1(t)s_2(t)dt = -1 \tag{9-1}$$

where ρ is called the correlation coefficient. One such signal set is:

$$s_1(t) = \sqrt{2E}\ \sin\omega t \tag{9-2}$$

$$s_2(t) = -\sqrt{2E}\ \sin\omega t$$

where ω is called the carrier radian frequency and is chosen such that

$$\omega = \frac{2\pi n}{T} \tag{9-3}$$

where n is an integer. This set is used in many practical telemetry systems and is called phase shift keying (PSK). It is assumed in the work to follow that the data signals available to the transmitter are in this form.

2. Pseudo-noise (PN) waveform

The PN waveform is one of a family of PN binary waveforms. The generation, use, detection, and characteristics

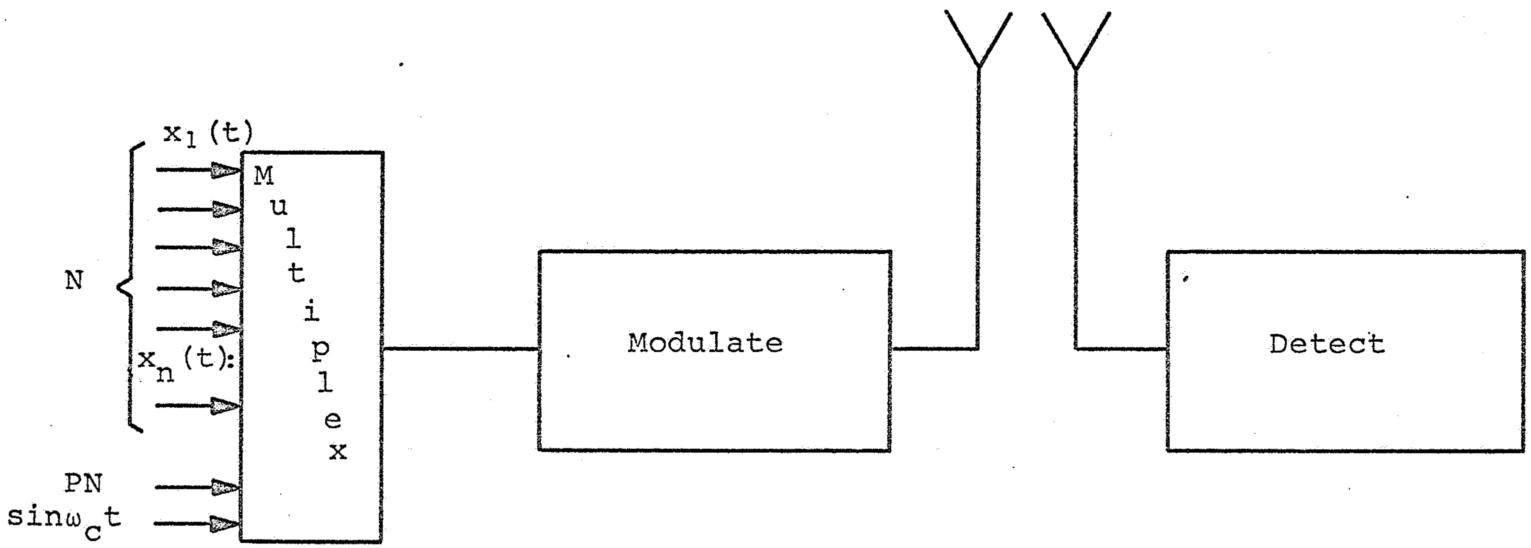


Figure 9-1. The Communication System

of these waveforms is described in great detail by Golomb, et al. (1964). Characteristics of these waveforms important to the work which follows are their power spectrum and autocorrelation function. If the period of the PN waveform is p, the time slot occupied by each digit is t_0 seconds long, and the waveform is of unit amplitude then it can be shown by Fourier series analysis (Golomb, 1965) that the power density spectrum of the waveform is of the form

$$F_{PN}(\omega) = \frac{p+1}{p^2} \cdot \left\{ \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \right\}^2 \cdot \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \delta\left(\omega - \frac{2\pi n}{pt_0}\right) + \frac{\delta(\omega)}{p^2} \qquad (9\text{-}4)$$

where $\delta(\omega)$ is a unit impulse function. A number of characteristics of the spectrum should be noted. First it is a line spectrum with components at zero frequency (D C) and multiples of the fundamental. The power at DC is $1/p^2$ and is small in comparison with the other components except for large p. Second, the envelope of the spectrum is determined by the digit period, t_0, while the density of the spectral lines is a function of the period of the sequence, p. Increasing p while leaving t_0 constant, makes the spectrum more dense but does not change the envelope of the spectrum or the bandwidth necessary to transmit the waveform.

The most important characteristics of PN binary waveforms are their correlation properties. Let the autocorrelation function of a periodic waveform be defined as

$$R(x) = \int_0^T s(t)s(t+x)dt \qquad (9\text{-}5)$$

where s(t) is a periodic waveform of period T. Then it can be shown that R(x) for the PN binary sequence is of the form shown in Figure 9-2.

Indeed, PN sequences are chosen so as to yield the indicated autocorrelations.

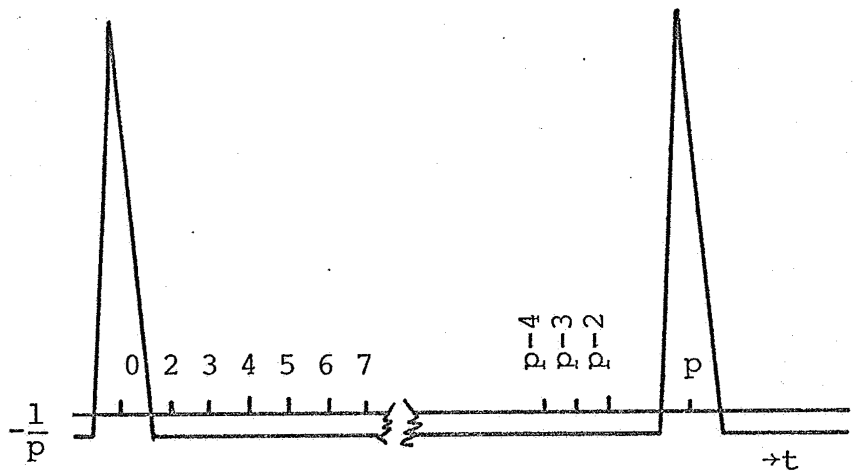


Figure 9-2. Autocorrelation Function of a PN Binary Waveform

The characteristics mentioned above are for so-called prime PN waveforms. Often, it is necessary to transmit combinations of primed PN waveforms. It turns out (Golomb, et al., 1965) that the resulting correlation functions have multiple peaks but that they can be treated as the superposition of single spike autocorrelation functions, and that the power density spectrum can be approximated by the power density spectrum of a prime waveform. Since a combination of waveforms can be approximated by a prime waveform, the waveforms discussed in this work will always be assumed to be prime.

3. Carrier reference signal

In many communication receivers, it is necessary that a reference signal be available to perform certain auxiliary functions. Since the reference signal provides information about the phase and frequency of the received signal, it is an additional information channel and will be treated as such in the work to follow. It turns out that the accuracy with which this reference is reproduced by the receiver will be an important factor in the selection of modulation and multiplexing techniques. Unless stated otherwise, the reference signal will be represented by a sine wave with radian frequency ω_c and zero initial phase angle.

B. Multiplexing N Binary Data Signals

The special case of multiplexing only N binary PSK signals yields a simple result which will greatly simplify later work, and hence, will be treated here. Two practical means of multiplexing the N signals exist: frequency division multiplexing (FDM) and time division multiplexing (TDM). In this section, it is shown that superior performance is always obtained using TDM when transmitting over a peak power limited channel.

1. For FDM, the orthogonal waveforms may be chosen from the set

$$0_n(t) = \sin((1+n)\omega t+\theta_n) \tag{9-6}$$

where n is a positive integer and θ_n is the phase angle of the nth waveform. The composite waveform may then be written as

$$f_{FDM}(t) = \sum_{\text{chosen } n} a_n m_n(t)\sin((1+n)\omega t+\theta_n) \tag{9-7}$$

As indicated by Equation (9-7), this is simply the sum of N PSK signals of maximum amplitude a_n.

For the special case of equal energy signals in all the channels Equation (9-7) may be written as

$$f_{FDM}(t) = \frac{A}{N} \sum_{\text{chosen } n} m_n(t)\sin((1+n)\omega t+\theta_n) \tag{9-8}$$

If the channel is peak power limited to a maximum value of 2E watts then

$$\sqrt{2E} \geq \sum_{n=0}^{N-1} a_n \tag{9-9}$$

or

$$a_n \leq 1/N \sqrt{2E} \tag{9-10}$$

The average power per signal is

$$E\{(1/N \sqrt{2E} \sin(1+n)\omega t)^2\} = E/N^2 \tag{9-11}$$

where $E\{x\}$ denotes the expected value of x. Since the N binary data signals are uncorrelated, the total average power in the FDM waveform is the sum of the average power in each of the individual waveforms or

$$P_{FDM,avg} = N(E/N^2) = E/N \tag{9-12}$$

One of the quantities used to rate multiplexing systems is the peak to average power ratio, Γ. In general, it is desired to obtain as small a value of Γ as possible. For the FDM signal under consideration, assuming that the phase angles of the N sinusoids are such that the peak values of all the sinusoids occur at the same instant,

$$\Gamma_{FDM} = (2E)/(E/N) = 2N \tag{9-13}$$

It is shown in Chapter X that for the case of additive white Gaussian noise of spectral density N_0, the optimum detector for a PSK signal is a correlation detector which makes hard decision at the end of each bit period. The probability of an error on a particular bit is

$$P_e^b = \tfrac{1}{2}(1-\mathrm{erf}\ \sqrt{2ET/N_0}) \tag{9-14}$$

where ET is the average energy per bit and erf is called the error function, and is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2|2)dy \tag{9-15}$$

It follows that the probability of error for any bit of the composite FDM waveform is

$$P^b_{e,FDM} = \tfrac{1}{2}(1-\mathrm{erf}\ \frac{1}{N} \sqrt{2ET/N_0}) \tag{9-16}$$

2. For TDM the orthogonal waveforms are chosen from the set

$$0_n(t) = u\left(t+\frac{nT}{N}\right) - u\left(t+\frac{(n+1)T}{N}\right) \qquad 0 \leq n \leq N-1 \tag{9-17}$$

where u(t) is the unit step function. The composite waveform may be written as

$$f_{TDM}(t) = \sum_{n=0}^{N-1} 0_n(t) m_n(t) a_n \sin\omega t \tag{9-18}$$

Once again let the channel be peak power limited to a maximum value of 2E watts then

$$a_n = \sqrt{2E} \tag{9-19}$$

Regardless of the relative length of the time slot assigned to each waveform the total average power is

$$\begin{aligned} P_{TDM,avg} &= E\{f^2{}_{TDM}\} \\ &= E \end{aligned} \tag{9-20}$$

and the peak to average power ratio is

$$\Gamma_{TDM} = 2 \tag{9-21}$$

The case of equal power in all the signals for the FDM waveform is analogous to allocating equal time slots to each of the TDM channels.

To calculate the probability of error on any bit, substitute T/N for T in Equation (9-14) to obtain

$$P^b_{e,TDM} = \tfrac{1}{2}(1-\mathrm{erf}\ \sqrt{2ET/N_0N}) \tag{9-22}$$

3. It is seen from Equations (9-13) and (9-21) that an improved (smaller) value of Γ is obtained in the TDM case. If both systems operate at the same information rate, 1/T, then since the error function is a monotonically increasing function of argument

$$P^b_{e,TDM} \leq P^b_{e,FDM} \tag{9-23}$$

for the equal energy case. The above result can be easily extended to the case of unequal division of energy between the channels.

Since by assumption there is no bandwidth constraint on the N binary data signals and since there is no cross-talk or interchannel interference if synchronization is perfect it is seen from the above that the best performance is obtained by using TDM, that is, by forming one composite PSK signal. For this reason, in the work to follow, the N binary data signals previously defined are replaced by a single PSK signal as is shown in Figure 9-3.

C. Multiplexing a PN Sequence and a PSK Data Signal

In this section, the problem of multiplexing the PN sequence, r(t), and a PSK data signal is discussed. It is not necessary to include the carrier reference signal since as will be shown in the next section the carrier reference signal may be obtained by the choice of an appropriate modulation technique.

Using the orthomux approach the problem reduces to solving the following integral equations for $0_1(t)$ and $0_2(t)$

$$\int_0^{pt_0} m(t)\sin(\omega_c t+\theta)\ 0_1(t)r(t)0_2(t)dt = 0 \qquad (9\text{-}24)$$

$$\int_0^{T} m(t)\sin(\omega_c t+\phi)\ 0_1(t)r(t)0_2(t)dt = 0 \qquad (9\text{-}25)$$

Unfortunately, finding solutions to Equations (9-24) and (9-25) is very difficult. A much more tractable approach is to think of practical multiplexing techniques and then see if the proper orthogonal functions can be found.

One very useful solution to the above system of equations makes use of the natural orthogonality of each waveform. From Equation (9-4) it is seen that r(t) may be approximated by

$$r(t) \simeq \sum_{n=1}^{\infty} \frac{1}{p} \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} (\sin n\omega t+\theta_n) \qquad (9\text{-}26)$$

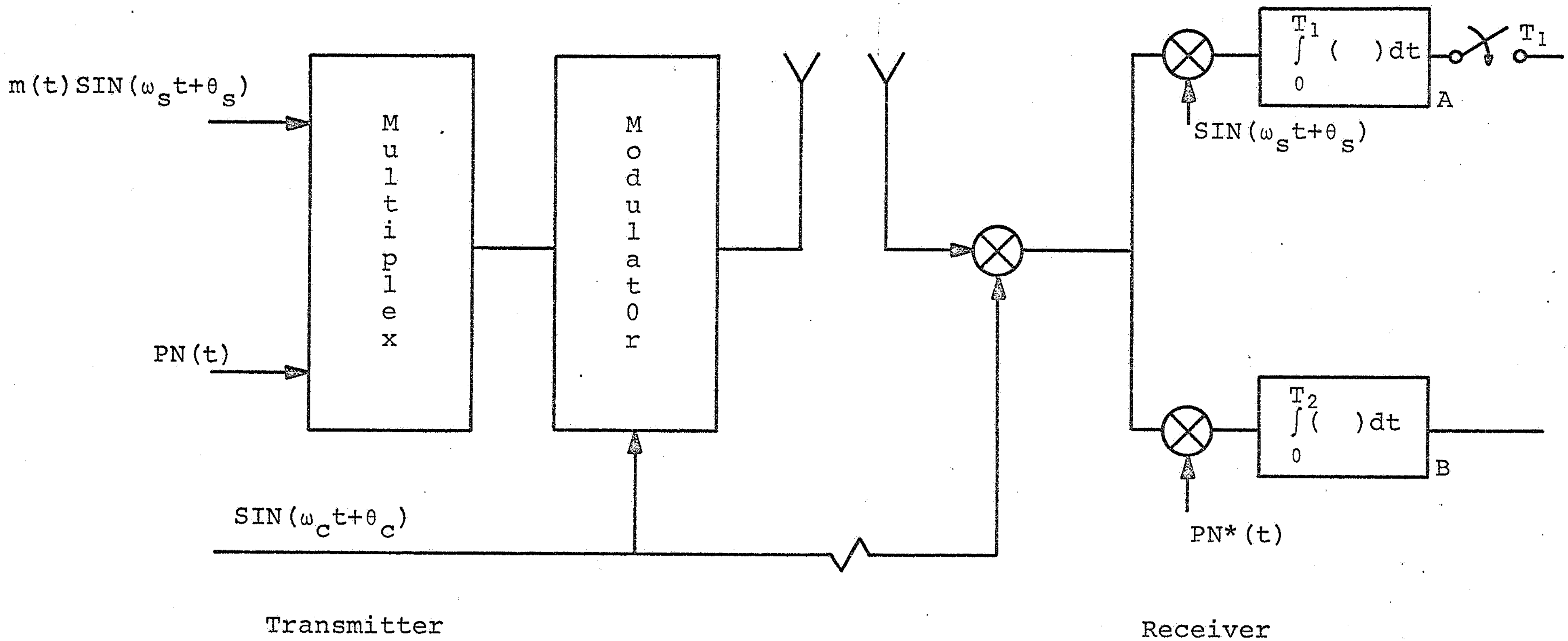


Figure 9-3. A Simplified Realization of the Desired Communication System

where

$$\omega = \frac{2\pi}{pt_0}$$

Substituting Equation (9-26) into Equation(9-24) yields

$$\int_0^{pt_0} m(t)\sin(\omega_c t+\phi)0_1(t) \sum_{n=1}^{\infty} \frac{1}{p}\left\{\frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}}\right\}\sin(n\omega t+\theta_n)0_2(t)dt = 0 \tag{9-27}$$

Let

$$0_1(t) = 1 \quad \text{for all } t \tag{9-28}$$

$$0_2(t) = 1 \quad \text{for all } t \tag{9-29}$$

$$\omega_c = m\omega \tag{9-30}$$

where m is an integer. Then reversing the order of summation and integration in Equation (9-27) and observing that m(t) is constant over the integration period yields

$$m(t) \sum_{n=1}^{\infty} \frac{1}{p} \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \int_0^{pt_0} \sin(n\omega t+\theta_n)\sin(m\omega t+\phi)dt = 0 \tag{9-31}$$

Since

$$\int_0^{\pi} \sin(mx) \cdot \sin(nx)dx = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \end{cases} \tag{9-32}$$

Equation (9-31) reduces to

$$\frac{m(t)}{p} \left\{ \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \right\} \frac{pt_0}{2} \cos(\theta_n+\phi) = 0 \tag{9-33}$$

It is seen from Equation (9-33) that this choice of $0_1(t)$ and $0_2(t)$ gives no interchannel interference if the radian frequency of the data signal is chosen such that $\omega_c=(2m\pi)/p$ and conversely any other choice of ω_c will yield a finite interference term. It is readily seen that the same conditions also yields a solution to Equation (9-25).

The orthogonal waveforms in a strict sense are not really $0_1(t)$ and $0_2(t)$ but are $\sin(\omega_c t+\phi)$ and $r(t)$. The composite waveform $f_m(t)$ is then

$$f_m(t) = r(t)+m(t)\sin(\omega_c t+\phi) \quad . \tag{9-34}$$

Using Equation (9-26) it is easily seen that $m(t)$ can be recovered by multiplying $f_m(t)$ by $\sin(\omega_c t+\phi)$ and integrating over a period T seconds long. The information in $r(t)$ is conveyed by its phase. The receiver derives this information by calculating the correlation function of the transmitted $r(t)$ and a replica of $r(t)$ stored at the receiver. It is seen from Equation (9-24) that this is obtained with no interference if the cross-correlation between $f_m(t)$ and $r(t)$ is calculated.

The above technique is easily realized and in fact many practical systems utilize techniques of this sort. It is generally considered good practice to place the data signal at a null in the PN spectrum. In view of this, the preceding arguments might be viewed as a practical example of the fact (Rowe, 1965) that orthogonality in the time domain implies orthogonality in the frequency domain and consequently is a justification in the time domain of what designers have often done by intuition in the frequency domain.

Often for practical reasons, the optimum choice of ω_c is not possible. If ϕ and θ_n were known, then Equation (9-26) could be used to measure the interchannel interference. Unfortunately, $r(t)$ and $m(t)$ are usually uncorrelated and consequently, only an upper bound on the interference may be calculated. The bounds are

$$I_m \leq \left\{ \frac{t_0}{2p} \; \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \right\}^2 \tag{9-35}$$

$$I_r \leq \left\{ \frac{T}{2p} \; \frac{\sin\frac{\omega t_0}{2}}{\frac{\omega t_0}{2}} \right\}^2 \tag{9-36}$$

where I_m is the fraction of the energy in the data channel due to the code and I_r is the fraction of the energy in the code channel due to the data. Minimizing these bounds will, from a practical point of view, give the best performance.

Sin($n\omega t$) and sin($m\omega t$) are other functions that might be chosen for $0_1(t)$ and $0_2(t)$. It is not hard to see that this merely represents a high pass representation of the same process and that performance would be identical. Orthogonal sets other than sine and cosine functions might be used to perform the multiplexing process. While not discounting the existence of some easily realized and easily implemented sets for this system, none are presently known to exist. For this reason, in the work to follow, the composite waveform will be assumed to be of the form given in Equation (9-34).

D. Factors Effecting the Choice of Modulation Technique

In the previous section it was shown that a system which forms the composite waveform described by Equation (9-34) is in a sense optimum. Such a technique is a form of a general class of multiplexing systems called frequency division multiplexing systems (FDM). In this section, the factors affecting the choice of a carrier modulation technique for the FDM waveform is examined.

In most practical applications, the bit period, t_0, of the PN sequence is very short, consequently the bandwidth of the composite FDM waveform is very large. For this reason, a modulation process which is bandwidth expanding, for example, wide band frequency modulation FM, is not suitable. This limits the discussion of modulation techniques to bandwidth conserving processes for example, narrow band angle modulation or amplitude modulation. Single sideband suppressed carrier (SSSC) amplitude modulation might also be considered, however, since generation of a coherent reference is a necessary criterion for detection of SSSC it is not practical to use it as a carrier modulation technique for a multiplexed signal. If it were used the relations developed for AM could be modified to apply to the SSSC case. The mathematical representations of these processes for the amplitude, frequency and phase modulated cases follow:

$$s_{AM}(t) = \sqrt{2E}\ (a_0+a_1r(t)+a_2m(t)\sin(\omega_s t+\theta_s))\sin(\omega_c t+\theta_c) \tag{9-37}$$

$$s_{FM}(t) = \sqrt{2E}\ \sin(\omega_c t+\theta_c+\int_0^t\{\beta_1 r(t')+\beta_2 m(t')\sin(\omega_s t'+\theta_s)\}dt') \tag{9-38}$$

$$s_{PM}(t) = \sqrt{2E}\ \sin(\omega_c t+\theta_c+\beta_1 r(t)+\beta_2 m(t)\sin(\omega_s t+\theta_s)) \tag{9-39}$$

where the set a_0, a_1, and a_2 and the set β_1, and β_2 are constants called the modulation indices, which determine the fraction of the total energy E allotted to each message and are different in each case, and θ_c is the carrier phase shift. In the AM case, the a_0 term provides the carrier reference signal. In both angle modulation cases this reference is provided by a suitable choice of β_1, β_2 as will be shown later.

1. Peak to average power ratio.

The peak to average power ratio, Γ, is a measure of the transmitter efficiency. A small peak to average power ratio means that the average power is close to the peak power and that the transmitter is being utilized efficiently. In the angle modulated case

$$P_{AVG} = \lim_{T\to\infty} \frac{1}{T} \int_0^T s^2(t)dt = E \tag{9-40}$$

The peak power is 2E. Hence, Γ for the angle modulated case is 2. For the AM case

$$P_{AVG} = \lim_{T\to\infty} \frac{1}{T} \int_0^T s^2(t)dt \tag{9-41}$$

$$= \lim_{T\to\infty} \frac{2E}{T} \int_0^T \{\{a_0^2+a_1^2r^2(t)+a_2m^2(t)\sin^2(\omega_s t+\theta_s)$$

$$+ 2a_0a_1r(t)+2a_0a_2m(t)\sin(\omega_s t+\theta_s)+2a_1a_2m(t)r(t)\sin(\omega_s t+\theta_s)\}$$

$$\cdot \sin^2(\omega_c t+\theta_c)\}dt \quad .$$

The fourth and fifth terms of Equation (9-41) represent the time averages of non ergodic processes (i.e. the product of ergodic random processes with sinusoids). In this case it is customary (Rowe, 1965) to define the average power as the average power of the individual processes. This implies that one must first take the expected value of these terms

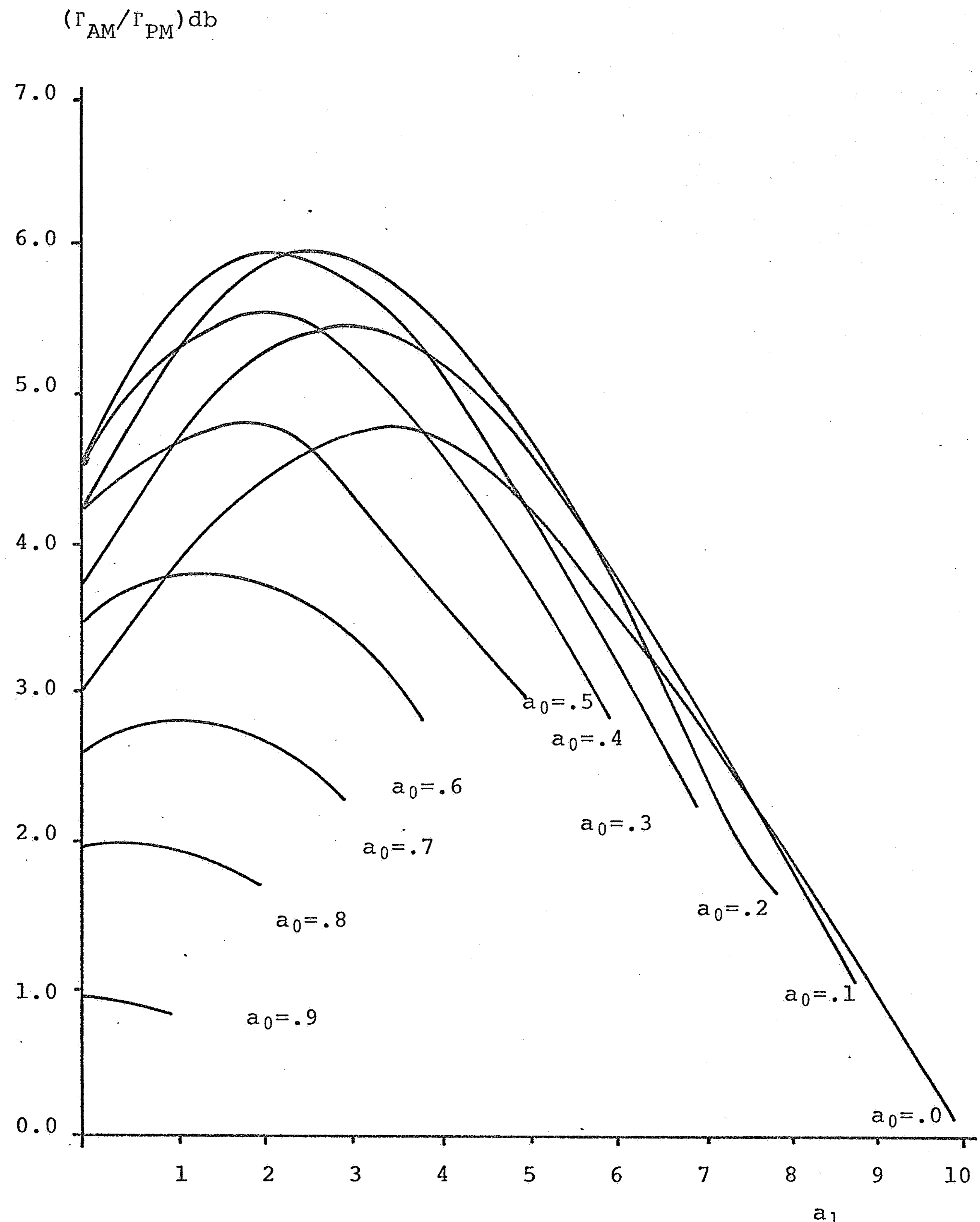


Figure 9-4. The Ratio of $\Gamma_{AM}:\Gamma_{PM}$ versus a_1

and then average over all time. The result is that there is no contribution to the average power due to them. The sixth term of Equation (9-41) is zero by the orthogonality condition of Equation (9-25). Using the above results and observing that $m^2(t) = r^2(t) = 1$, Equation (9-41) can be evaluated to give

$$P_{AVG} = E(a_0^2+a_1^2+a_2^2|2) \quad . \tag{9-42}$$

The peak power is the same as for the angle modulated case giving a peak to average power ratio of

$$\Gamma_{AM} = \frac{4}{2a_0^2+2a_1^2+a_2^2} \quad . \tag{9-43}$$

The ratio of Γ_{AM} to Γ_{PM} in db is plotted versus a_1 in Figure 9-4 with a_0 as a parameter. It is seen that a better peak to average power ratio is always obtained using angle modulation. It should be observed however that for systems that require large fractions of energy to be devoted to carrier synchronization this improvement of angle modulation over amplitude modulation is not as great as might be expected, and is certainly not convincing enough to lead one to choose one over the other without investigating other criteria.

2. Recoverable Energy and Interchannel Interference.

Perhaps the best criterion for evaluation of the various modulation techniques is to compare the interchannel interference and the energy available in each message after demultiplexing for the same division of energy between the messages.

In both angle and amplitude modulation fm*(t) can be recovered by coherent detection as is indicated in Figure 9-3. For the time being, it is assumed in both cases that a perfect phase reference is available at the receiver.

a. Amplitude Modulation

In the AM case, the carrier reference signal is

$$s_{c,AM}(t) = \sqrt{2E}\, a_0+a_1/p \sin(\omega_c t+\theta_c) \tag{9-44}$$

The average power in the carrier reference signal is

$$E_{c,AM} = (\overline{s_{c,AM}(t)})^2 = E(a_0^2+\frac{2a_1a_0}{p} + \frac{a_1^2}{p^2}) \tag{9-45}$$

For relatively long PN sequences $p>>1$, and $E_{c,AM} \simeq Ea_0^2$. The output of the coherent product detector is

$$fm^*(t) = s_{AM}(t)\sin(\omega_c t+\theta_c) \tag{9-46}$$

which neglecting double frequency terms can be written as

$$fm^*(t) = \tfrac{1}{2}(a_0+a_1 r(t)+a_2 m(t)\sin(\omega_s t+\theta_s)) \tag{9-47}$$

Assuming perfect synchronization the signal at point A of Figure 9-3 at t=T, the bit length of the message, is

$$f_A(T_m) = \sqrt{2E}\ \tfrac{1}{2}\{ \int_0^T a_0\sin(\omega_s t+\theta_s)dt \tag{9-48}$$

$$+ a_1 \int_0^T r(t)\sin(\omega_s t+\theta_s)dt+a_2 \int_0^T m(t)\sin^2(\omega_s t+\theta_s)dt\}.$$

Since $r(t)$ and $\sin(\omega_s t+\theta_s)$ are orthogonal over the interval 0 to T Equation (9-48) can be written as

$$f_A(T_m) = \sqrt{E/2}\ a_2 m(t) \int_0^T \sin^2(\omega_s t+\theta_s)dt \quad . \tag{9-49}$$

As can be seen from Equation (9-49) there is no interchannel interference and the usable available signal power is

$$E_{m,AM} = \overline{(\sqrt{E/2}\ a_2 m(t)\sin(\omega_s t+\theta_s))^2} = (E/4)a_2^2 \quad . \tag{9-50}$$

Likewise, the signal at point B may be written

$$f_B(p) = \sqrt{2E}\ \tfrac{1}{2}\{ \int_0^p a_0 r(t)dt+a_1 \int_0^p r^2(t)dt \tag{9-51}$$

$$+ a_2 \int_0^p m(t)r(t)\sin(\omega_s t+\theta_s)dt\} \quad .$$

Since $m(t)\sin(\omega_s t+\theta_s)$ and $r(t)$ are orthogonal over the period 0 to p and since from Equation (9-4) the DC value of $r(t)$ is proportional to $1/p$, Equation (9-51) can be written as

$$f_B(p) = \sqrt{E/2}\ \{a_0/p+a_1 \int_0^p r^2(t)dt\} \ . \qquad (9\text{-}52)$$

As can be seen from Equation (9-52), there is an inter-channel interference term of amplitude $\sqrt{(a_0^2E)|(2p^2)}$. The usable available signal power is

$$E_{r,AM} = \overline{(\sqrt{E/2}\ a_1 r(t))^2} = (E/2)a_1^2 \qquad . \qquad (9\text{-}53)$$

The signal to interference power ratio at the output of the correlator is

$$S/I = \frac{a_1^2p^2}{a_0^2p^2} = (a_1/a_0)^2p^4 \qquad . \qquad (9\text{-}54)$$

Fortunately, in most applications $p>>1$ and consequently, $S/I \simeq \infty$.

b. Angle Modulation.

For the angle modulation case, only the case of phase-modulation is treated here. Both FM and PM are similar, and PM is more easily analyzed and compared with AM.

As in the AM case, the reference is recovered by using a narrow band phase locked loop which is locked to the discrete frequency component of the signal $s_{PM}(t)$. To obtain an expression for the discrete frequency component, first expand Equation (9-39) using standard trigonometric identities to give

$$\begin{aligned} s_{PM}(t) = \sqrt{2E}\ \{&\sin(\omega_c t+\theta_c)\cos(\beta_1 r(t))\cos(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \\ &- \sin(\omega_c t+\theta_c)\sin(\beta_1 r(t))\sin(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \\ &+ \cos(\omega_c t+\theta_c)\sin(\beta_1 r(t))\cos(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \\ &+ \cos(\omega_c t+\theta_c)\cos(\beta_1 r(t))\sin(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \ . \end{aligned} \qquad (9\text{-}55)$$

Observing that

$$\cos(\beta_1 r(t)) = \cos(\beta_1) \tag{9-56}$$

$$\cos(\beta_2 m(t))\sin(\omega_s t+\theta_s) = \cos(\beta_2)\sin(\omega_s t+\theta_s) \tag{9-57}$$

$$\sin(\beta_1 r(t)) = r(t)\sin(\beta_1) \tag{9-58}$$

$$\sin(\beta_2 m(t))\sin(\omega_s t+\theta_s) = m(t)\sin(\beta_2)\sin(\omega_s t+\theta_s) \tag{9-59}$$

it is seen that the second and fourth terms of Equation (9-55) are random processes and hence, contain no discrete energy at ω_c. Examining the first and third terms, using the above relations and the well-known Fourier Bessel expansion (Gradshteyn and Rhyzhick, 1965)

$$\cos(\beta\sin\omega t) = J_0(\beta)+2\sum_1^{\infty} J_{2\lambda}(\beta)\cos 2\lambda\omega t \tag{9-60}$$

the first term of Equation (9-55) may be written as

$$\sin(\omega_c t+\theta_c)\cos(\beta_1 r(t))\cos(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \tag{9-61}$$

$$= \sin(\omega_c t+\theta_c)\cos(\beta_1)\{J_0(\beta_2)+2\sum_1^{\infty} J_{2\lambda}(\beta_2)$$

$$\cos 2\lambda\ \omega_s t+\theta_s \}$$

and the third term as

$$\cos(\omega_c t+\theta_c)\sin(\beta_1 r(t))\cos(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \tag{9-62}$$

$$= \cos(\omega_c t+\theta_c)r(t)\sin\beta_1\{(J_0(\beta_2))\}$$

$$+ 2\sum_1^{\infty} J_{2\lambda}(\beta_2)\cos(2\lambda(\omega_s t+\theta_s))\} \quad .$$

Remembering that $r(t)$ has a DC component of $1/p$, the expression for the discrete energy at ω_c is

$$s_{c,PM}(t) = \sqrt{2E}\, J_0(\beta_2)\{1/p \sin\beta_1 \cos(\omega_c t+\theta_c) + \cos\beta_1 \sin(\omega_c t+\theta_c)\} \quad . \tag{9-63}$$

The phase lock loop can be designed to track either the sine or cosine term. In general $p>>1$, hence

$$1/p \sin\beta_1 < \cos\beta_1 \quad . \tag{9-64}$$

Consequently, the sine term should be tracked. It follows that the average usable power in the carrier reference signal is

$$E_{c,PM} = \overline{(\sqrt{2E}\, J_0(\beta_2)\cos\beta_1 \sin(\omega_c t+\theta_c))^2} = EJ_0^2(\beta_2)\cos^2\beta_1 \quad . \tag{9-65}$$

To coherently detect PM the reference signal should be

$$0(t) = \cos(\omega_c t+\theta_c) \quad . \tag{9-66}$$

The output of the coherent detector neglecting double frequency terms is

$$fm^*(t) = \sqrt{E/2}\, \sin(\beta_2 m(t)\sin(\omega_s t+\theta_s)+\beta_1 r(t)) \quad . \tag{9-67}$$

If $\beta_1+\beta_2$ is small, then

$$fm^*(t) \simeq \sqrt{E/2}\, (\beta_1 r(t)+\beta_2 m(t)\sin(\omega_s t+\theta_s)) \tag{9-68}$$

and the relations are the same as those derived for the AM case. If the small angle approximation is not made $fm^*(t)$ can be written as

$$fm^*(t) = \sqrt{E/2}\, \sin(\beta_1 r(t))\cos(\beta_2 m(t)\sin(\omega_s t+\theta_s)) + \sqrt{E/2}\, \cos(\beta_1 r(t))\sin(\beta_2 m(t)\sin(\omega_s t+\theta_s)) \quad . \tag{9-69}$$

Since $m(t) = \pm 1$ and $r(t) = \pm 1$ Equation (9-69) may be written as

$$fm^*(t) = \sqrt{E/2}\, r(t)\sin\beta_1\cos(\beta_2\sin(\omega_s t+\theta_s)) + \sqrt{E/2}\, m(t)\cos\beta_1\sin(\beta_2\sin(\omega_s t+\theta_s)) \quad . \tag{9-70}$$

The signal at point A is at t=T

$$f_A(T) = \int_0^T \sqrt{E/2}\, r(t)\sin\beta_1\cos(\beta_2\sin(\omega_s t+\theta_s))\sin(\omega_s t+\theta_s)dt + \sqrt{E/2}\, m(t)\cos\beta_1\sin(\beta_2\sin(\omega_s t+\theta_s))\sin(\omega_s t+\theta_s)dt \quad . \tag{9-71}$$

Using Equation (9-60) and the following Fourier Bessel expansion (Gradshteyn and Rhyzhick, 1965)

$$\sin(\beta\sin\omega t) = 2\sum_{\nu=1}^{\infty} J_{2\nu-1}(\beta)\sin(2\nu-1)\omega t \tag{9-72}$$

Equation (9-71) can be written as

$$\begin{aligned} f_A(T) = \int_0^T \sqrt{E/2}\, r(t)\sin\beta_1\{&J_0(\beta_2)+2J_2(\beta_2)\cos 2(\omega_s t+\theta_s) \\ &+ 4J_4(\beta_2)\cos 4(\omega_s t+\theta_s)+\ldots\}\sin(\omega_s t+\theta_s)dt \\ &+ \int_0^T \sqrt{E/2}\, m(t)\cos\beta_1\{2J_1(\beta_2)\sin(\omega_s t+\theta_s) \\ &+ 2J_3(\beta_2)\sin 2(\omega_s t+\theta_s) \\ &+ 2J_5(\beta_2)\sin 5(\omega_s t+\theta_s)+\ldots\} \\ &\cdot \sin(\omega_s t+\theta_s)dt \quad . \end{aligned} \tag{9-73}$$

Since $r(t)$ and $\sin(\omega_s t+\theta_s)$ are orthogonal over the interval 0 to T the first integral is identically equal to zero. Since $\sin(\omega_s t+\theta_s)$ is orthogonal to $\sin(n(\omega_s t+\theta_s))$ for $n>1$ over the interval of integration all terms but the first term of the second integral are zero. Equation (9-73) can then be written as

$$f_A(T) = \sqrt{E/2}\, m(t)\cos(\beta_1)J_1(\beta_2) \int_0^T \sin^2(\omega_s t+\theta_s)dt \qquad (9\text{-}74)$$

As can be seen from Equation (9-74), there is no interchannel interference and the usable available signal power is

$$E_{m,PM}(t) = \overline{(\sqrt{E/2}\, m(t)\cos\beta_1 J_1(\beta_2)\sin(\omega_s t+\theta_s)^2} \qquad (9\text{-}75)$$

$$= \tfrac{1}{2}\, E\cos^2\beta_1 J_1^2(\beta_2) \quad .$$

Likewise, the signal at point A may be written

$$f_B(t) = \sqrt{E/2}\ \{ \int_0^p r^2(t)\sin\beta_1\cos(\beta_2\sin\omega_s t+\theta_s)dt \qquad (9\text{-}76)$$

$$+ \int_0^p r(t)m(t)\cos\beta_1\sin(\beta_2\sin(\omega_s t+\theta_s))dt\}$$

which can be expanded using the Fourier Bessel expansions to give

$$\begin{aligned} f_B(p) = \sqrt{E/2}(&\int_0^p \sin\beta_1\{J_0(\beta_2)+2J_2(\beta_2)\cos 2(\omega_s t+\theta_s) \qquad (9\text{-}77) \\ &+ 4J_4(\beta_2)\cos 4(\omega_s t+\theta_s)+\ldots\} \\ &+ \int_0^p m(t)r(t)\cos\beta_1\{2J_1(\beta_2)\sin(\omega_s t+\theta_s) \\ &+ 2J_3(\beta_2)\sin 3(\omega_s t+\theta_s) \\ &+ 2J_5(\beta_2)\sin 5(\omega_s t+\theta_s) \\ &+\ldots\}dt) \quad . \end{aligned}$$

By orthogonality arguments similar to those used above Equation (9-77) can be written as

$$f_B(p) = \sqrt{E/2}\ \sin\beta_1 J_0(\beta_2)p \quad . \tag{9-78}$$

As can be seen from Equation (9-78), there is no inter-channel interference and the usable available signal power is

$$E_{r,PM} = \tfrac{1}{2}2E\sin^2\beta_1 J_0^2(\beta_2) \quad . \tag{9-79}$$

The amount of available energy in each channel for the AM and PM case is summarized in Table 9-1.

TABLE 9-1

RECOVERABLE ENERGY

	AM	PM
Reference	Ea_0^2	$2EJ_0^2(\beta_2)\cos^2(\beta_1)$
PN Sequence	$Ea_1^2/2$	$\frac{1}{2}EJ_0^2(\beta_2)\sin^2(\beta_1)$
Data	$Ea_2^2/4$	$\frac{1}{2}EJ_1^2(\beta_2)\cos^2(\beta_1)$

where $a_0+a_1+a_2 = 1$

$$\beta_1+\beta_2\leq 2.4 \quad .$$

In general, there is no unique solution for the values of β and one type of modulation does not appear to be clearly superior for all divisions of energy. An interesting optimization problem is to devise a technique to optimize over the choice of modulation techniques and over the choice of modulation indices.

It is interesting to note that relationships of the same form as those above have been previously obtained using an approximate steady state analysis. The above work shows that the product detector only reduces the energy in the message channels but does not generate any interfering signals and does not distort the waveform, if the channels utilize correlation detectors. This of course assumes that the product detectors are ideal.

3. The Effect of a Noisy Carrier Reference Signal on The Performance of a Coherent Detector.

It was previously assumed that a perfect carrier reference was available to perform the coherent demodulation indicated in Figure 9-3. In practical systems, the receiver must somehow make an estimate of the carrier frequency and phase, and this estimate must of necessity contain some error. Let the estimate be

$$0(t) = \cos(\omega_c t+\theta_c+\theta_\varepsilon) \quad . \tag{9-80}$$

That is, let the error be θ_ε radians. Then for the PM case the signal at point A in Figure 9-2 is

$$f_A(t) = s_{PM}(t)\cos(\omega_c t+\theta_c+\theta_\varepsilon) \tag{9-81}$$

which neglecting double frequency terms can be written as

$$f_A(t) = \sqrt{E/2}\,\sin(\theta_\varepsilon+\beta_1 r(t)+\beta_2 m(t)\sin(\omega_s t+\theta_s)) \quad . \tag{9-82}$$

Expanding Equation (9-82) in a manner similar to that used for Equation (9-83) yields

$$\begin{aligned} f_A(t) = \sqrt{E/2}\,\sin\theta_\varepsilon\{&\cos\beta_1\cos(\beta_2\sin(\omega_s t+\theta_s)) \\ &+ m(t)r(t)\sin\beta_1\sin\beta_2\sin(\omega_s t+\theta_s)\} \\ &+ \sqrt{E/2}\,\cos\theta_\varepsilon\{r(t)\sin\beta_1\cos(\beta_2\sin(\omega_s t+\theta_s)) \\ &+ m(t)\cos\beta_1\sin(\beta_2\sin(\omega_s t+\theta_s))\} \quad . \end{aligned} \tag{9-83}$$

The first and second terms of Equation (9-83) are orthogonal to $r(t)$ and $\sin(\omega_s t+\theta_s)$ over the integration periods of both correlators hence, the effective signals to each correlator using the results of Equation (9-74) and (9-78) are

$$f_B(t) = \sqrt{E/2}\ \cos\theta_\varepsilon r(t)\sin\beta_1 J_0(\beta_2) \tag{9-84}$$

$$f_A(t) = \sqrt{E/2}\ \cos\theta_\varepsilon m(t)\sin(\omega_s t+\theta_s)\cos\beta_1 J_1(\beta_2). \tag{9-85}$$

From the above relationships it can be seen that the effect of the error in the phase estimate is to reduce the amplitude of the received signal by $\cos\theta_\varepsilon$. Unfortunately, θ_ε is a random variable whose distribution function is dependent on the technique used to obtain the carrier phase reference. The mean and variance of $\cos\theta_\varepsilon$ is calculated for the most important carrier tracking scheme in Appendix D and the results are plotted in Figure 9-5. In addition, the probability that $\cos\theta_\varepsilon$ in no more than δ less than its expected value is plotted in Figure 9-6. The effect of the random phase error on the performance of the data detector will be discussed in Chapter X, however, the curves of Figures 9-5 and 9-6 are useful in determining the allocation of available energy between the carrier synchronization signal and the other channels in a practical system. For example, practical situations might easily arise where reducing the energy in a message channel and using this energy in the synchronization channel will actually increase the signal to noise ratio in the message channel because of decreased phase error in the reference signal. The designer can anticipate this effect by using Figure 9-5 to overbound the loss in channel power due to phase jitter and Figure 9-6 to estimate what percentage of the time this bound is valid.

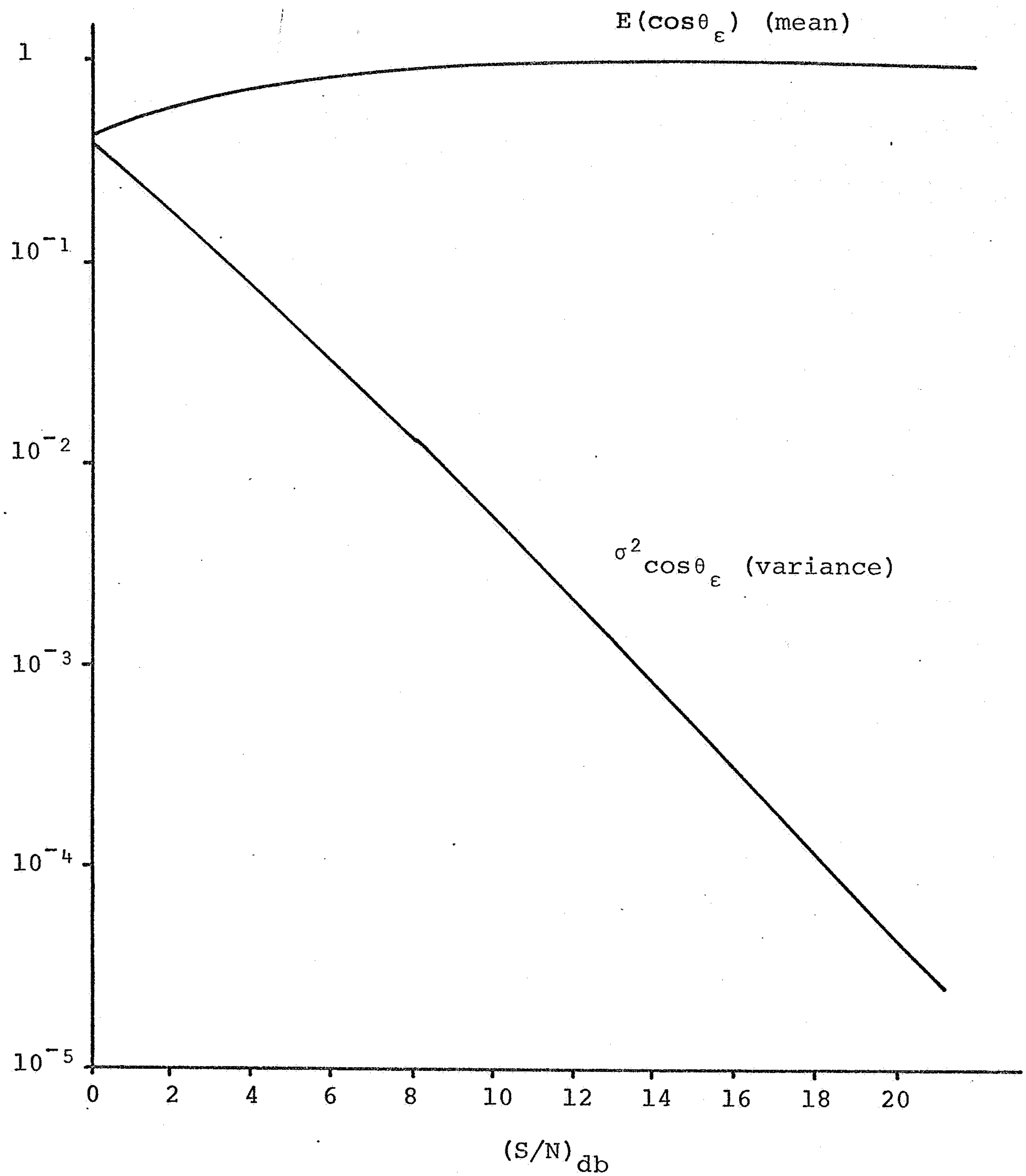


Figure 9-5. Mean and Variance of the Expected Value of the cosine of the Phase Error of the Output of a Second Order Phase Lock Loop

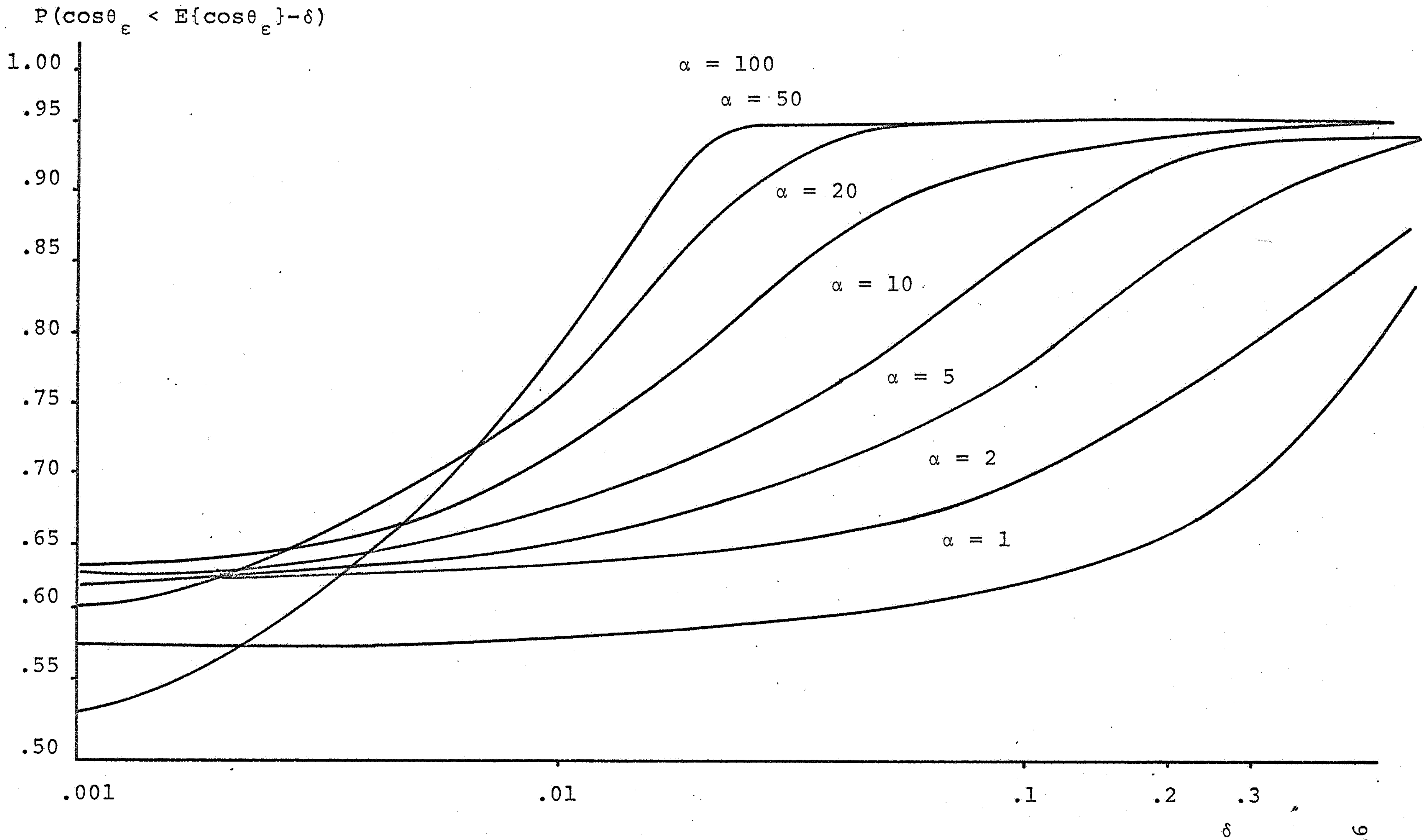


Figure 9-6. The Probability That $\cos\theta_\varepsilon$ is no Less Than Delta Smaller Than its Expected Value

CHAPTER X

THE EFFECT OF PHASE AND TIMING ERRORS ON THE PERFORMANCE OF PSK DATA DETECTORS

A. Introduction

In this chapter, the problem of the reception of PSK data signals is discussed for the case of a noisy phase reference at the coherent demodulator and the data detector and a noisy timing signal. The performances of two practical phase angle detectors are analyzed. The optimum phase angle detector is formulated and compared with the two practical detectors. Finally, the effects of random timing errors are discussed. The results are then applied to the evaluation of the performance of a practical telemetry demodulator.

B. Background

It is well-known that the optimum choice of signals for binary signalling in the presence of white Gaussian noise is a binary antipodal signal set. A realization of this set is to choose

$$s_1(t) = +\sqrt{2E}\,\sin(\omega_s t+\theta_s) \tag{10-1}$$

$$s_2(t) = -\sqrt{2E}\,\sin(\omega_s t+\theta_s) \tag{10-2}$$

It is also known that if a perfect phase reference and if perfect timing information is available, the integrate and dump detector or correlation detector is the optimum receiver. A complete realization of this scheme is shown in Figure 10-1.If, for example, s_1 is transmitted, then the receiver observes the following waveform

$$\Omega(t) = \sqrt{2E}\,\sin(\omega_s t+\theta_s)+n(t) \tag{10-3}$$

where $n(t)$ represents additive white Gaussian noise with spectral density $N_0/2$ watts per Hertz. The detector multiplies the received signal by a reference signal $\sin(\omega_s t+\theta_s)$, integrates the product over it's bit period, T seconds, and forms a so-called decision function

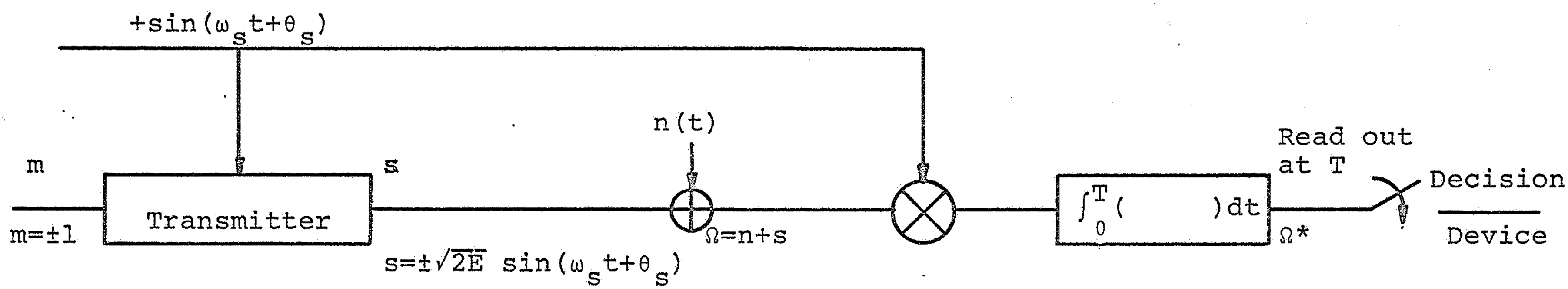


Figure 10-1. A Coherent PSK System

$$\Omega^*(T) = \int_0^T \sqrt{2E}\, \sin^2(\omega_s t+\theta_s)dt + \int_0^T n(t)\sin(\omega_s t+\theta_s)dt \tag{10-4}$$

If $\Omega^*(T)$ is positive, the decision device declares that $s_1(t)$ was transmitted otherwise it declares that $s_2(t)$ was transmitted. Since $\Omega^*(T)$ is a Gaussian random variable to calculate the probability of an incorrect decision one need only calculate the mean and variance of $\Omega^*(T)$. The mean of $\Omega^*(T)$ is

$$\begin{aligned} E\{\Omega^*(T)\} &= \sqrt{2E} \int_0^T E\{\sin^2(\omega_s t+\theta_s)\}dt \\ &+ \int_0^T E\{n(t)\}\sin(\omega_s t+\theta_s)dt \\ &= \sqrt{E/2}\; T \end{aligned} \tag{10-5}$$

where $E\{x\}$ denotes the expected value of x. Likewise, the variance can be shown to be

$$\sigma^2_{\Omega^*(T)} = \frac{N_0 T}{4} \tag{10-6}$$

where N_0 is the spectral density of the input noise. It follows that the probability of an incorrect decision is the probability that $\Omega^*(T)$ is negative or

$$P_e = \int_{-\infty}^{0} \frac{\exp-(2(x-\sqrt{E/2}\;T)^2/N_0T)}{\sqrt{\pi N_0 T/2}}\, dx \tag{10-7}$$

For the case of equally likely message probabilities, that is

$$P(s_1) = P(s_2) = \tfrac{1}{2} \tag{10-8}$$

it can be easily shown that the average probability of error on any bit is

$$P_e^b = P_e \tag{10-9}$$

Equation (10-7) cannot be integrated in closed form but is tabulated in terms of the so-called error function erf x where

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\frac{1}{2}y^2} dy \tag{10-10}$$

Utilizing Equations (10-7), (10-9), and (10-11) and making the appropriate changes in variables the probability of bit error for the case of equally likely message probabilities can be written as

$$P_e^b = \tfrac{1}{2}(1-\text{erf } \sqrt{2ET/N_0}) \tag{10-11}$$

C. <u>The Effect of Random Phase Errors</u>

In reality the perfect phase reference implied in the derivation of Equation (10-11) is never available. To account for phase reference errors let the reference signal differ in phase from the received signal by θ_ε radians, that is

$$0(t) = \sin(\omega_s t+\theta_s+\theta_\varepsilon) \tag{10-12}$$

If s_1 was transmitted, then it is easily seen using trigonometric identities that

$$\Omega^*(T) = \sqrt{E/2} \int_0^T \cos\theta_\varepsilon dt + \int_0^T n(t)\sin(\omega_s t+\theta_\varepsilon)dt \tag{10-13}$$

Comparing Equations (10-13) and (10-4), it is seen that the effect of the phase error is to reduce the energy in the received signal, E, by $\cos^2\theta_\varepsilon$. If there were a fixed phase error, that is, if $\cos\theta_\varepsilon$ were a constant, then the resulting probability of error using the arguments of the previous section could be written as

$$P_e^b = \tfrac{1}{2}(1-\text{erf}\sqrt{2ET\cos^2\theta_\varepsilon/N_0}) \tag{10-14}$$

In practical systems θ_ε is not a constant but a random variable whose value is continuously estimated by some detection scheme. Since most detection schemes ultimately utilize a phase lock loop to estimate θ_ε the distribution function of θ_ε can be written in terms of the distribution function of the output of a phase lock loop.

The important measure of system performance is as in the ideal case the average probability of bit error. Evaluation of $E\{P_e^b\}$ can of course be accomplished by calculating the probability density function of $\Omega^*(T)$ and then calculating the probability of an incorrect decision. Such an approach presents formidable problems since the probability distribution function of the integral of a non-Gaussian process must be evaluated, and this as Papoulis (1965) points out is "hopelessly complicated".

One approach to the problem is to assume that $\cos(\theta_\varepsilon(t))$ is random over the integration period and that samples of $\cos(\theta_\varepsilon(t))$ are uncorrelated. This assumption tends to overestimate the randomness of $\Omega^*(T)$ and should provide a tight upper bound on $E\{P_e^b\}$. This assumption appears to be born out by qualitative observation of the process (Hon, 1968) and leads to a closed form solution for $E\{P_e^b\}$.

The decision function can in this case be thought of as

$$\Omega^*(T) = \lim_{\substack{m\to\infty \\ \Delta t\to 0 \\ m\Delta t\to T}} \sum_{r=0}^{m} (\sqrt{E/2}\,\cos\theta_\varepsilon(t+r\Delta t)\Delta t + n(t+r\Delta t)\sin(\omega_s t+r\Delta t)+\theta_\varepsilon)\Delta t) \quad . \tag{10-15}$$

Since the samples are to be assumed independent it follows from the central-limit theorem (Papoulis, 1965) that the distribution of $\Omega^*(T)$ approaches a normal distribution if the distribution of $\Omega(t)$ is reasonably well behaved. Hence, we can calculate $E\{P_e^b\}$ in terms of the error function providing we can evaluate the mean and variance of $\Omega^*(T)$. It can be shown (Lindsey, 1965) that the statistics of the output of the integrator do in fact approach a Gaussian distribution for $\alpha>1$ and that this approximation is extremely tight for $\alpha>3$. Since conventional phase lock loops exhibit a threshold effect for $2\leq\alpha\leq3$ (Gardner and Kent, 1967), the above approximations may be used for systems operating above threshold providing linear phase lock loop theory is used to predict where threshold occurs. The above approximations will not hold for small α ($\alpha<2$) however, for this case practical loops completely lose coherence and detection is not possible.

The expected value of $\Omega^*(T)$ is

$$E\{\Omega^*(T)\} = \sqrt{E/2}\int_0^T E\{\cos\theta_\varepsilon(t)\}dt + \int_0^T E\{n(t)\sin(\omega_s t+\theta_\varepsilon(t))\}dt$$

Remembering that

$$E\{n(t)\sin(\omega_s t+\theta_\varepsilon(t))\} = 0 \tag{10-17}$$

and substituting Equation D-6 of Appendix D into Equation (10-16) gives

$$E\{\Omega^*(T)\} = \sqrt{E/2}\,\frac{I_1(\alpha)}{I_0(\alpha)}\,T \tag{10-18}$$

where $I_n(\alpha)$ is the modified Bessel function of order n. Likewise

$$\begin{aligned} E\{\Omega^{*2}(T)\} = {} & E/2\ E\{\int_0^T \cos(\theta_\varepsilon(t_1)dt_1 \int_0^T \cos(\theta_\varepsilon(t_2))dt_2\} \\ & + \sqrt{2E}\ E\{\int_0^T \cos(\theta_\varepsilon(t_1))dt_1 \int_0^T n(t_2)\sin(\omega_s t_2+\theta_\varepsilon(t_2))dt_2\} \\ & + E\{(\int_0^T n(t_1)\sin(\omega_s t+\theta_\varepsilon(t_1))dt_1)^2\} \end{aligned} \tag{10-19}$$

which can be rewritten as

$$E\{\Omega^{*2}(T)\} = E/2 \int_0^T \int_0^T E\{\cos(\theta_\varepsilon(t_1))\cos(\theta_\varepsilon(t_2))\}dt_1 dt_2 + (N_0/4)T \quad . \tag{10-20}$$

Since samples of $\cos\theta_\varepsilon(t)$ are assumed to be independent we can write

$$\begin{aligned} E\{\cos(\theta_\varepsilon(t_1))\cos(\theta_\varepsilon(t_2))\} &= R(t_1,t_2) \\ &= E\{\cos^2\theta_\varepsilon\}T\delta(t_1-t_2) \end{aligned} \tag{10-21}$$

where $R(t_1,t_2)$ is the autocorrelation function of $\cos(\theta_\varepsilon(t))$ and $\delta(x)$ is the Dirac delta function. Substituting Equation (10-21) into Equation (10-20), and performing the integrations yields

$$\begin{aligned} E\{\Omega^{*2}(T)\} &= E/2 \int_0^T TE\{\cos^2\theta_\varepsilon\}dt + N_0T/4 \\ &= ET^2/2\ E\{\cos^2\theta_\varepsilon\} + N_0T/4 \quad . \end{aligned} \tag{10-22}$$

Using Equation D-7, Equation (10-22) may be rewritten as

$$E\{\Omega^2(t)\} = ET^2/2\ \{\tfrac{1}{2}+\tfrac{1}{2}\ \frac{I_2(\alpha)}{I_0(\alpha)}\} + N_0T/4 \quad . \tag{10-23}$$

It follows immediately that

$$\sigma^2{}_{\Omega^*(T)} = ET^2/2\ (\tfrac{1}{2}+\tfrac{1}{2}\ \frac{I_2(\alpha)}{I_0(\alpha)} - (\frac{I_1(\alpha)}{I_0(\alpha)})^2) + N_0T/4 \quad . \tag{10-24}$$

and that

$$P_e^b = \tfrac{1}{2}(1-\mathrm{erf}\{\frac{I_1(\alpha)}{I_0(\alpha)} \sqrt{\frac{2ET}{N_0+2ET(\tfrac{1}{2}+\tfrac{1}{2}\frac{I_2(\alpha)}{I_1(\alpha)} - (\frac{I_1(\alpha)}{I_0(\alpha)})^2)}}\}) \tag{10-25}$$

or (10-26)

$$P_e^b = \tfrac{1}{2}\left(1-\mathrm{erf}\left\{\frac{I_1(\alpha)}{I_0(\alpha)} \sqrt{\left\{\left((2S/N)^{-1}+\tfrac{1}{2}+\tfrac{1}{2}\frac{I_2(\alpha)}{I_0(\alpha)} - \left(\frac{I_1(\alpha)}{I_0(\alpha)}\right)\right)\right\}^{-1}}\right\}\right)$$

where

$$S/N = ET/N_0$$

and is the signal to noise ratio in a bandwidth 2/T. Equation (10-26) is plotted in Figure 10-2 versus ET/N_0 with α as a parameter.

For the case of infinite signal to noise ratio there is an irreducible probability of bit error. To see this consider

(10-27)

$$\lim_{S/N\to\infty} P_e^b = \tfrac{1}{2}\left(1-\mathrm{erf}\left(\left(\frac{I_1(\alpha)}{I_0(\alpha)}\right) \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\frac{I_2(\alpha)}{I_0(\alpha)} - \left(\frac{I_1(\alpha)}{I_0(\alpha)}\right)^2\right)^{-1}}\right)\right) \quad .$$

This indicates that the designer quickly reaches a point where increasing the signal to noise ratio of the PSK signal will not increase performance unless the quality of the reference signal is improved. This characteristic is shown in the curves of Figure 10-2. The irreducible error is plotted versus α for low values of α in Figure 10-3.

The effect of random phase errors in the reference signal used to coherently demodulate the radio frequency carrier can also be accounted for using Equations (10-26) and (10-27). The quantity α is, in this case, the signal to noise ratio in the bandwidth of the detector used to detect the carrier reference signal. The designer has control over this quantity since he can choose how much energy is to be allotted to this reference signal. Figure 10-2 indicates that if α is chosen high enough say greater than ten, then there is really no significant degradation from the ideal when a noisy reference is used. It is also seen from the curves that no particular advantage is gained in choosing α larger than twenty. A good design rule might be that the modulation indices should always be chosen such that

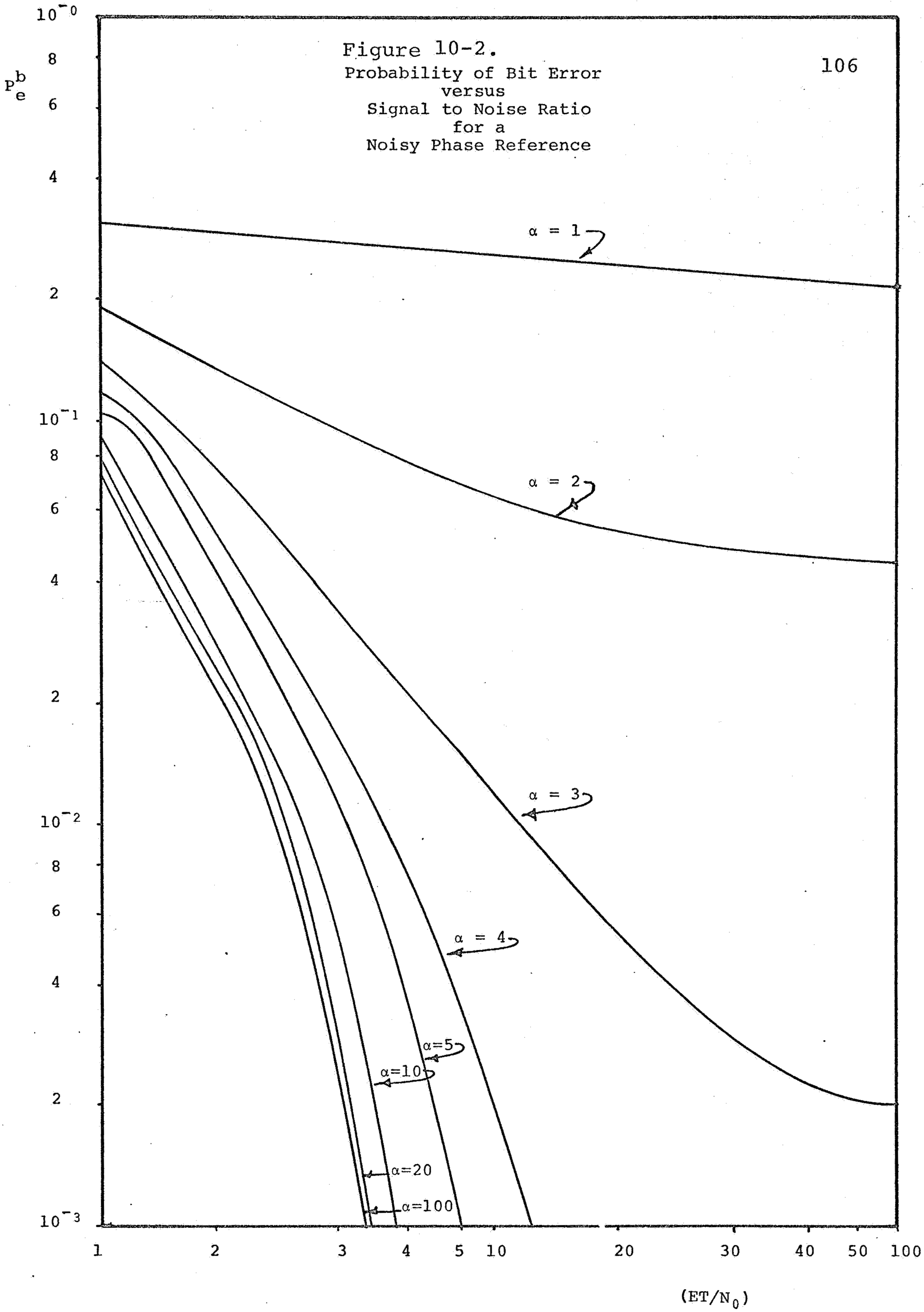


Figure 10-2.
Probability of Bit Error
versus
Signal to Noise Ratio
for a
Noisy Phase Reference
P_e^b
10^{-0}
8
6
4
2
10^{-1}
8
6
4
2
10^{-2}
8
6
4
2
10^{-3}
α = 1
α = 2
α = 3
α = 4
α=5
α=10
α=20
α=100
1
2
3
4
5
10
20
30
40
50
100
(ET/N_0)

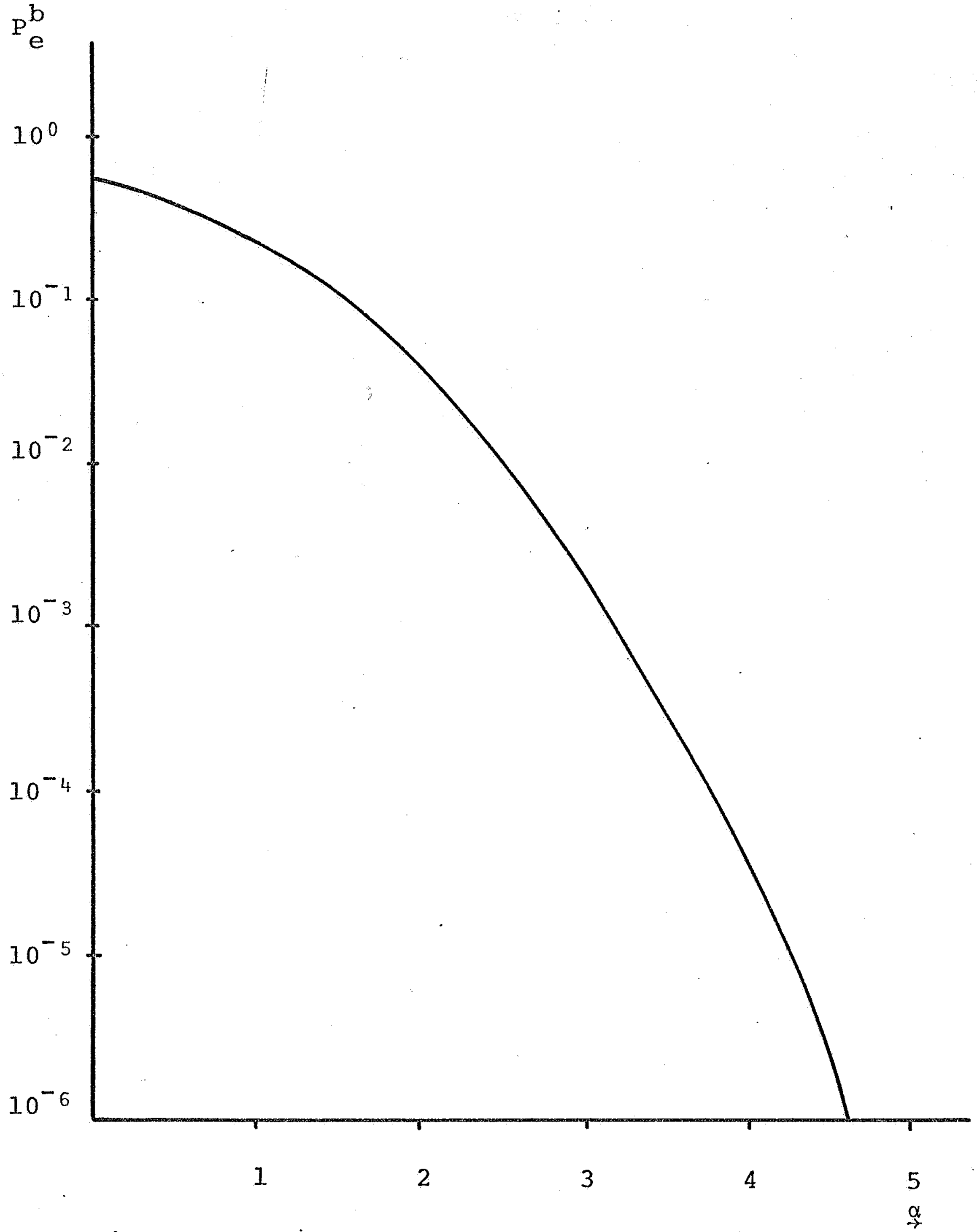


Figure 10-3. Irreducible Error versus Signal to Noise Ratio in Carrier Tracking Loop

$$10 \leq \alpha \leq 20 \quad . \tag{10-28}$$

D. The Squaring Loop

In the previous section, no mention was made of the method of obtaining the synchronization signal. One obvious technique is to transmit a discrete subcarrier reference signal. Such a technique is often described as auxiliary channel synchronization. The receiver then uses a phase lock loop to track the discrete frequency component and to generate a reference signal. Such a system is easily analyzed using the techniques of the previous section. It has however, two limitations. First, for the power limited channel the reference signal is provided at a cost of decreased energy in the information signal and secondly, the synchronization signal is often disturbed by the channel in a manner different from that of the information signal.

In view of the above, a receiver which estimates the reference signal from only the information is desirable (Golomb, et.al., 1963). In order to perform this estimation the signal must be transmitted through some nonlinear device. To understand why the nonlinearity is necessary, it should be remembered that the information signal is the product of a sine wave and a random binary bit stream. The signal hence, has no finite energy at discrete frequencies and consequently, there is no component for the phase lock loop to lock onto (Gardner and Kent, 1967). Passing the signal through a nonlinearity will generate energy at discrete frequencies which can be tracked by the phase lock loop.

One practical tracking device which uses a square law nonlinearity is the so-called "squaring loop" depicted in Figure 10-4. The circuit utilizes a square law device to generate the nonlinearity. The discussion to follow could be suitably modified to apply to a simple rectifier or any even power law device.

Let the input to the squaring loop be

$$\Omega(t) = \pm\sqrt{2E}\,\sin(\omega_s t+\theta_s)+n(t) \tag{10-29}$$

where $n(t)$ represent white Gaussian noise with spectral density $(N_0/2)$ watts per Hertz. Let the filter be sufficiently wideband that the information signal is not appreciably altered. The filter can be thought of as representing the minimum of the IF bandwidth of the circuitry preceding the square

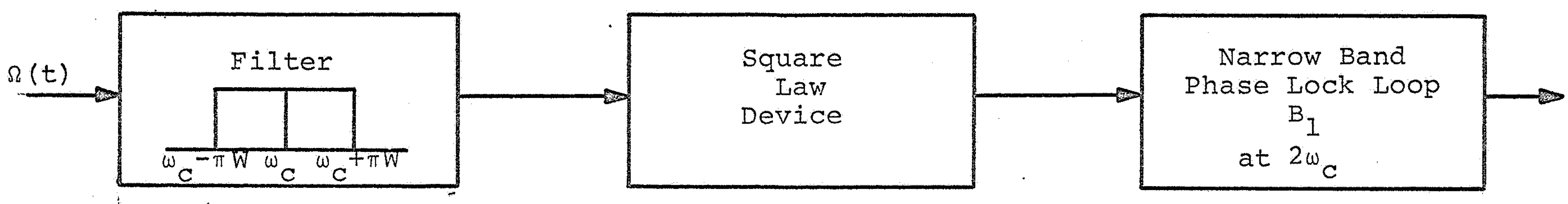


Figure 10-4.

A Squaring Loop

law device and the bandwidth of the device itself. The input to the square law device is then

$$\Omega'(t) = \pm\sqrt{2E}\,\sin(\omega_s t+\theta_s)+n'(t) \tag{10-30}$$

where $n'(t)$ is the filtered version of $n(t)$. The output of the square law device is

$$\Omega'^2(t) = E+E\,\cos 2(\omega_s t+\theta_s)\pm 2\sqrt{2E}\,n'(t)\sin(\omega_s t+\theta_s)+n'^2(t). \tag{10-31}$$

As can be seen from Equation (10-31) $\Omega'^2(t)$ contains a discrete spectral component at $2\omega_s$ with energy $(E^2|2)$ watts. This component can be tracked by the phase lock loop to obtain a pure sine wave at $2\omega_s$. This sine wave can then be divided in frequency by two to obtain a coherent reference at ω_s, with phase error $\frac{1}{2}\theta_\varepsilon$ where θ_ε is the phase error at the output of the phase lock loop. Using the techniques of Section C of this chapter and Appendix D the distribution, moments, and effect of θ_ε on the probability of bit error can be evaluated if the effective signal to noise ratio in the bandwidth (B_1) of the phase lock loop can be determined.

In order to evaluate the effective signal to noise ratio in the bandwidth of the phase lock loop consider the autocorrelation function of $\Omega'^2(t)$, $R_{\Omega'^2}(x)$. Using results for the output of a square law device when the input is signal plus noise (Davenport and Root, 1958) it can be shown that

$$R_{\Omega'^2}(x) = R_{s^2}(x)+R_{n'^2}(x)+4R_s(x)R_{n'}(x)+2\sigma^2_{n'}\sigma^2_s \tag{10-32}$$

where $R_{s^2}(x)$ is the autocorrelation function of the squared information signal, $R_{n'^2}(x)$ is the autocorrelation function of the squared noise, $R_s(x)$ is the autocorrelation function of the information signal, $R_{n'}(x)$ is the autocorrelation function of the noise, σ^2_s is the variance of the information signal, and $\sigma^2_{n'}$ is the variance of the noise. It is tedious but straightforward to show that Equation (10-33) can be written as

$$\begin{aligned} R_{\Omega'^2}(x) = {} & E^2+E^2/2\,\cos 2\omega_s x+N_0{}^2/4 \\ & + 2(R_{n'}(x))\ (R_{n'}(x)) \\ & + 4(R_{n'}(x))\ (E(1-\tfrac{|x|}{T})\cos(\omega_s x)) \\ & + N_0{}^2/4 + 2\sigma^2_{n'}\sigma^2_s \end{aligned} \tag{10-33}$$

for $|x| \leq T$. If $\Omega'^2(t)$ is assumed to be wide sense stationary then the Fourier transform of $R_{\Omega'^2}(x)$ is the power density spectrum of $\Omega'^2(t)$. It is then clear that the first, third, and last terms of Equation (10-33) lead to energy at zero frequency and therefore, energy outside the effective bandwidth of the phase lock loop. The second term is the signal term and upon being transformed gives two delta functions of weight $E^2/4$ at $\pm 2\omega_c$. It follows that the total signal power is $E^2/2$. The fourth and fifth terms of Equation (10-33) contribute to the noise and can be evaluated by making use of the convolution law. That is

$$F\{R_{n'}(x)R_{n'}(x)\} = F\{R_{n'}(x)\}*F\{R_{n'}(x)\} \qquad (10\text{-}34)$$

and

$$F\{R_{n'}(x) \ (E(1-\tfrac{|x|}{T})\cos\omega_s x\} \qquad (10\text{-}35)$$

$$= F\{R_{n'}(x)\}*F\{E(1-\tfrac{|x|}{T})\cos\omega_s x\}$$

where F(x) denotes the Fourier tranform of x, and * denotes frequency domain convolution as defined in standard textbooks on operational analysis. Since the unfiltered input noise was assumed to have spectral density $N_0/2$ watts per Hertz it follows that the filtered noise has power density spectrum

$$F\{R_{n'}(x)\} = \begin{cases} N_0/2, \omega_s/2\pi - W/2 \leq f \leq \omega_s/2\pi + W/2 \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \qquad (10\text{-}36)$$

The indicated correlation leads to a spectral density of the noise at the input to the phase lock loop which is not white. If, however, $W \gg B_1$ as is the case in most practical systems, the contribution of the noise can be overbounded in the vicinity of $2\omega_c$ by a uniform density of height $2\cdot 2(N_0{}^2/4)$. Since W was assumed to be large enough to pass the information without degradation, the contribution of the correlation of signal with noise can be overbounded in the vicinity of $2\omega_c$ by a uniform density of height $4\cdot(N_0/2)\cdot E$.

As a result of the above, the effective signal to noise ratio in the bandwidth of the loop, α, can be written as

$$\alpha = \frac{E^2/2}{2B_1\{2N_0E+N_0{}^2W\}} \quad . \tag{10-37}$$

Defining a = WT where T is the bit length of the information signal and K = B_1/W Equation (10-37) can be written as

$$\alpha = 1/2ak \quad \frac{(\frac{ET}{N_0})^2}{2a+4\,(\frac{ET}{N_0})} \tag{10-38}$$

where ET/N_0 is the signal to noise ratio of the information signal in a bandwidth 2/T, and is the parameter against which the performance of the data detector is evaluated.

Typically a $\geq$ 2 in order for the data signal to be passed with negligible loss. The noise bandwidth of practical phase lock loops is usually in a range of 10 to 1000 Hz.

Equation (10-38) is a powerful design tool since as is shown by Figures 10-2 and 10-3 increasing α decreases the expected value of the probability of bit error and the irreducible error. For this reason based on Equation (10-38) one might be tempted to choose K and a as small as possible. If a is chosen too small the information signal is reduced in energy by the filter and the preceding analysis is no longer valid and the performance is degraded. If B_1 is too small, then the loop might take too long to lock onto the spectral component at $2\omega_c$ (Gardner and Kent, 1967). The tradeoffs between these quantities require a thorough study of the behavior of phase lock loops and consequently are beyond the scope of this work. However, Equation (10-38) should be included in the usual equations used for designing tracking loops. Equation (10-38) can be substituted into Equation (10-26) to determine the performance of a detector using a squaring loop to derive its reference.

E. The Costas Loop

Another configuration which can be used to derive a coherent phase reference is the "Costas loop", (Costas, 1956), a modification of which is shown in Figure 10-5. Although originally envisioned as a demodulator for double sideband suppressed carrier amplitude modulation it was never used in

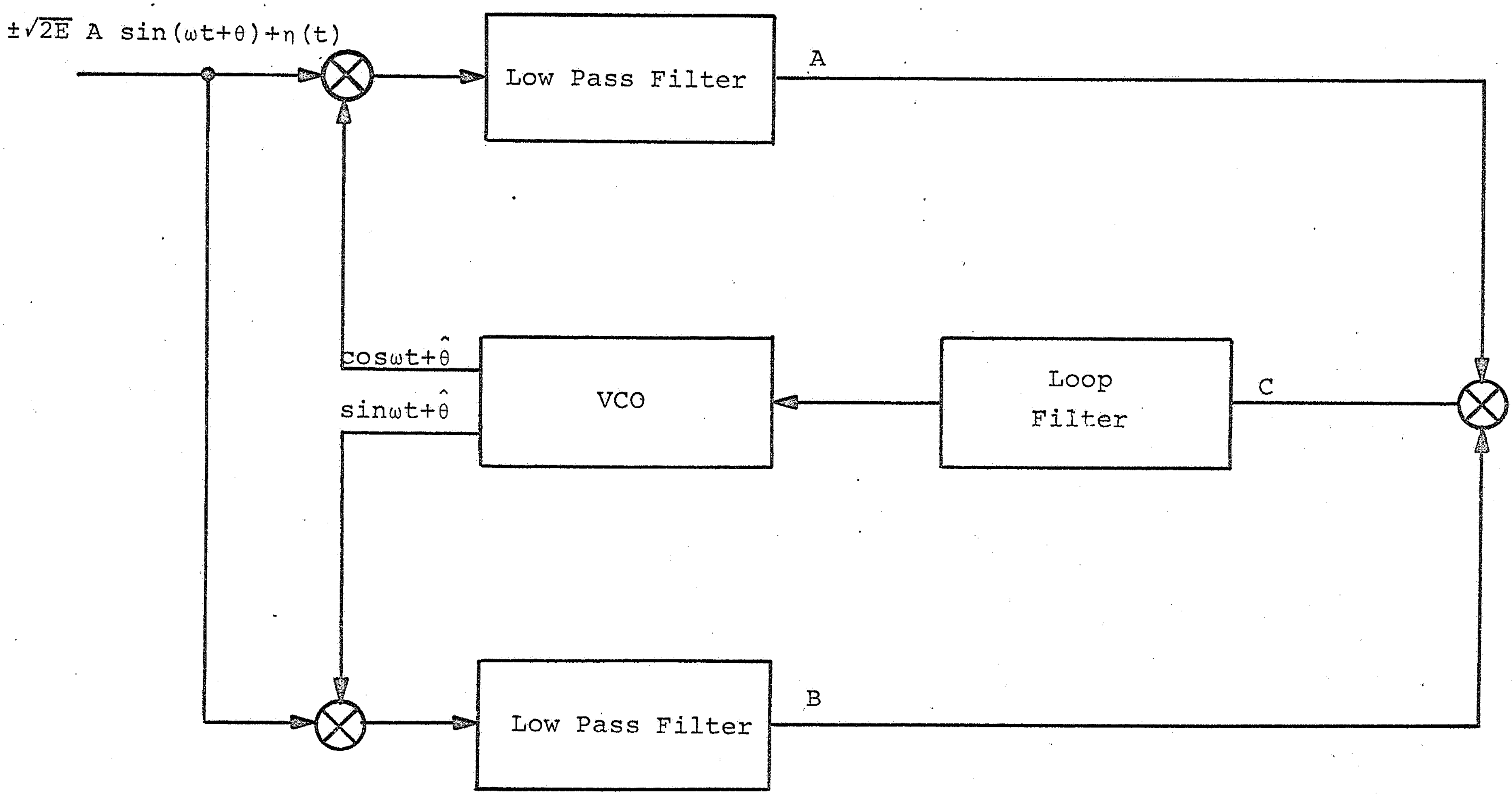


Figure 10-5. The Costas Loop

this role. The loop generates the reference signal by substituting a multiplier for the square law device and then uses feedback circuitry to drive to zero the phase error at the output of the multiplier. (Point C, Figure 10-5).

Let the input to the loop be as before

$$\Omega(t) = \pm\sqrt{2E}\ \sin(\omega_s t)+n(t) \quad . \tag{10-39}$$

Assuming that there is a phase error of θ_ε radians and that the bandwidth W of the low pass filters is such that the double frequency terms of the multiplier are rejected and the lower sidebands are passed without significant attenuation the signals at points A and B of Figure 10-5 can be written

$$A(t) = \Omega(t)\sin(\omega t+\theta_\varepsilon) = \pm\sqrt{2E}\ (\tfrac{1}{2})\cos\theta_\varepsilon+n'_A(t) \tag{10-40}$$

$$B(t) = \Omega(t)\cos(\omega t+\theta_\varepsilon) = \pm\sqrt{2E}\ (-\tfrac{1}{2})\sin\theta_\varepsilon+n'_B(t) \tag{10-41}$$

where $n'_A(t)$ and $n'_B(t)$ are the filtered versions of the translated noise.

In the noise free case, the signal at point C is

$$\begin{aligned} C(t) &= A(t)B(t) = -E/2\ \sin 2\theta_\varepsilon \\ &\simeq -E\theta_\varepsilon \qquad \text{for small } \theta_\varepsilon \end{aligned} \tag{10-42}$$

Equation (10-42) is a function of θ_ε regardless of the sign of the modulation and consequently, can be used as the control signal for the center branch. In view of the above, the Costas loop can be thought of as a modified phase lock loop. If the signal to noise ratio at the effective input to the loop, point C, is calculated then phase lock loop theory and the work of Appendix D can be used to determine the statistics of θ_ε. Consequently, the expected value of the probability of bit error can be obtained when the loop is used to derive a coherent reference.

Since the first and second terms of Equation (10-40) are assumed to be orthogonal, the power density spectrum at point A can be written as the spectrum of the signal term plus the spectrum of the filtered noise, that is

$$\Phi_A(f) = E/2 \cos^2\theta_\varepsilon \Phi_m(f)+\Phi_n(f) \tag{10-43}$$

where

$$\Phi_n(f) = \begin{cases} \frac{N_0}{4} & |f| < W \\ 0 & \text{otherwise} \end{cases} \tag{10-44}$$

and where $\Phi_m(f)$ is the spectral density of the binary message that modulates $\sin\omega_s t$. Likewise, the spectral density at point B is

$$\Phi_B(f) = E/2 \sin^2\theta_\varepsilon \Phi_m(f)+\Phi_n(f) \quad . \tag{10-45}$$

The spectral density at point C is the convolution of $\Phi_A(f)$ and $\Phi_B(f)$, that is

$$\Phi_C(f) = \Phi_A(f)*\Phi_B(f) \quad . \tag{10-46}$$

Since convolution is a linear operation, the distributive law can be used to write Equation (10-46) as

$$\begin{aligned} \Phi_C(f) = {} & E^2/4 \cos^2\theta_\varepsilon \sin^2\theta_\varepsilon \Phi_m(f)*\Phi_m(f) \\ & + E/2 \cos^2\theta_\varepsilon \Phi_m(f)*\Phi_n(f) \\ & + E/2 \sin^2\theta_\varepsilon \Phi_n(f)*\Phi_m(f) \\ & + \Phi_n(f)*\Phi_n(f) \quad . \end{aligned} \tag{10-47}$$

Substituting Equation (10-44) into Equation (10-47), remembering that the first term gives a discrete component at DC, and performing the indicated operations gives

$$\Phi_C(f) \simeq E^2/4\{\sin^2\theta_\varepsilon \cos^2\theta_\varepsilon\}\delta(f) + (EN_0/8)\cos^2\theta_\varepsilon + (EN_0/8)\sin^2\theta_\varepsilon + (N_0{}^2/16)2W, \text{ for } f < B_1 < W$$
$$= E^2/16\{1-\cos^2\theta_\varepsilon\}\delta(f) + EN_0/8 + N_0{}^2/8\ W, \text{ for } f < B_L < W \quad . \tag{10-48}$$

It follows that the signal to noise ratio at the input to the loop is approximately

$$\alpha = \frac{E^2}{2B_1\{2EN_0 + 2N_0{}^2W\}} \tag{10-49}$$

or using the definitions of the previous section

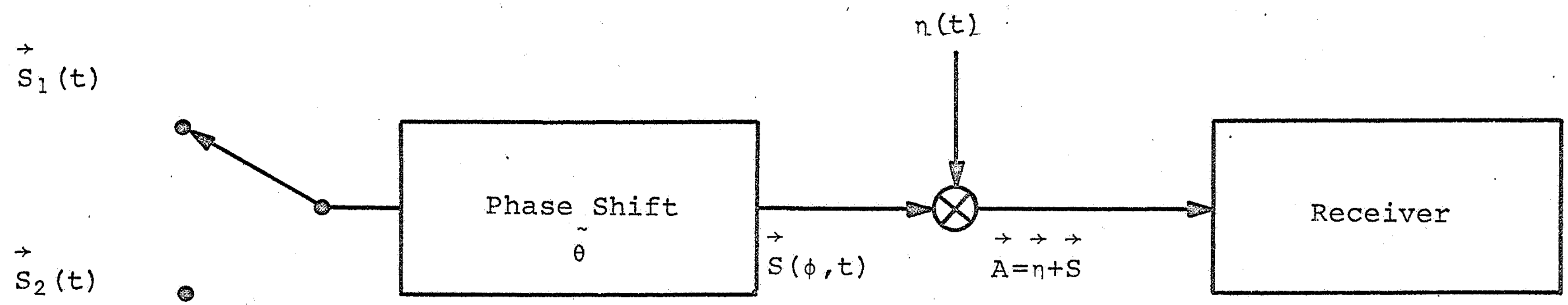
$$\alpha = 1/2ak \qquad \frac{(\frac{ET}{N_0})^2}{2\frac{ET}{N_0} + 2a} \quad . \tag{10-50}$$

Comparing Equations (10-50) and (10-48), shows that the Costas loop will give slightly improved performance over the squaring loop for identical filter bandwidths. This improvement is not so great as to be the only criterion in choosing one over the other. In fact, there are so many variables involved here that a more important criterion might be ease of implementing the various filters. Indeed little if anything is written about the Costas loop other than some veiled references by authors to its' apparent utility (Viterbi, 1966), consequently, a more detailed investigation of its' shortcomings and potentialities is certainly warranted.

F. The Optimum Phase Detector

In previous sections attention was directed toward the problem of analyzing the performance of practical techniques for determining the phase angle of a PSK signal. This discussion ignored the problem of whether or not these techniques were optimum in some statistical sense. In this section, the problem is rephrased more rigorously, the optimum detector is derived and its' performance compared with more practical systems.

Consider the communication systems of Figure 10-6. At time t_0 the transmitter chooses either signal $s_1(t)$ or $s_2(t)$



$S_1(t) = \sqrt{2E}\ \mathrm{Sin}\,\omega t$ for $0 \leq t \leq T$

$S_2(t) = -\sqrt{2E}\ \mathrm{Sin}\,\omega t$ for $0 \leq t \leq T$

Figure 10-6. A PSK Channel With Random Phase Error and inserts it into the channel. The channel shifts the signal by an unknown amount $\tilde{\theta}$ and adds white Gaussian noise. Since it is most important that the detector be able to lock onto the received phase it is assumed that the unknown phase, θ, is a random variable with a uniform distribution between 0 and 2π radians. That is, the receiver is assumed to have no prior knowledge of the phase angle. The receiver estimates $\tilde{\theta}$ by operating on the received signal $\Omega(t)$ in such a manner as to satisfy the criterion of optimality. In this case, the criterion chosen is the so-called maximum a posteriori probability criterion (Hancock, 1966). This amounts to choosing the value of θ which maximizes the probability density function of $\tilde{\theta}$ given that $\vec{R}$ was received, that is, to maximize $p(\tilde{\theta}|R)$. Another criterion that might be used is to minimize the expected value of the mean square error, that is, to minimize $E\{(\tilde{\theta}-\hat{\theta})^2\}$ where $\hat{\theta}$ is the estimated value of the phase angle. Other functions of the error could also be minimized. It can be shown (Viterbi, 1966) that if the a posteriori probability density function is unimodal then the maximum a posteriori probability estimate of $\tilde{\theta}$ and the minimum mean square error estimate of $\tilde{\theta}$ is realized by the same receiver structure.

According to Bayes rule, we can write

$$p(\tilde{\theta}|\vec{R}) = \frac{p(\vec{R}|\tilde{\theta})p(\tilde{\theta})}{p(\vec{R})} \quad . \tag{10-51}$$

Since the logarithm is a monotonically increasing function of its' positive argument maximizing Equation (10-51) is equivalent to maximizing

$$\ln(p(\theta|\vec{R})) = \ln(p(R|\tilde{\theta}))+\ln(p(\tilde{\theta}))-p(\vec{R}) \quad . \tag{10-52}$$

The second term of Equation (10-52) is a constant since $\tilde{\theta}$ was assumed to be uniformly distributed and the third term is not a function of $\tilde{\theta}$ consequently, maximizing Equation (10-51) with respect to $\tilde{\theta}$ is equivalent to maximizing the first term of Equation (10-52) with respect to $\tilde{\theta}$. In Appendix E it is shown that at time t_0+T

$$p(\vec{R}|\tilde{\theta}) = C\cosh(\sqrt{E/N_0{}^2}\int_{t_0}^{t_0+T}\sin(\omega t+\hat{\theta})R(t)dt) \tag{10-53}$$

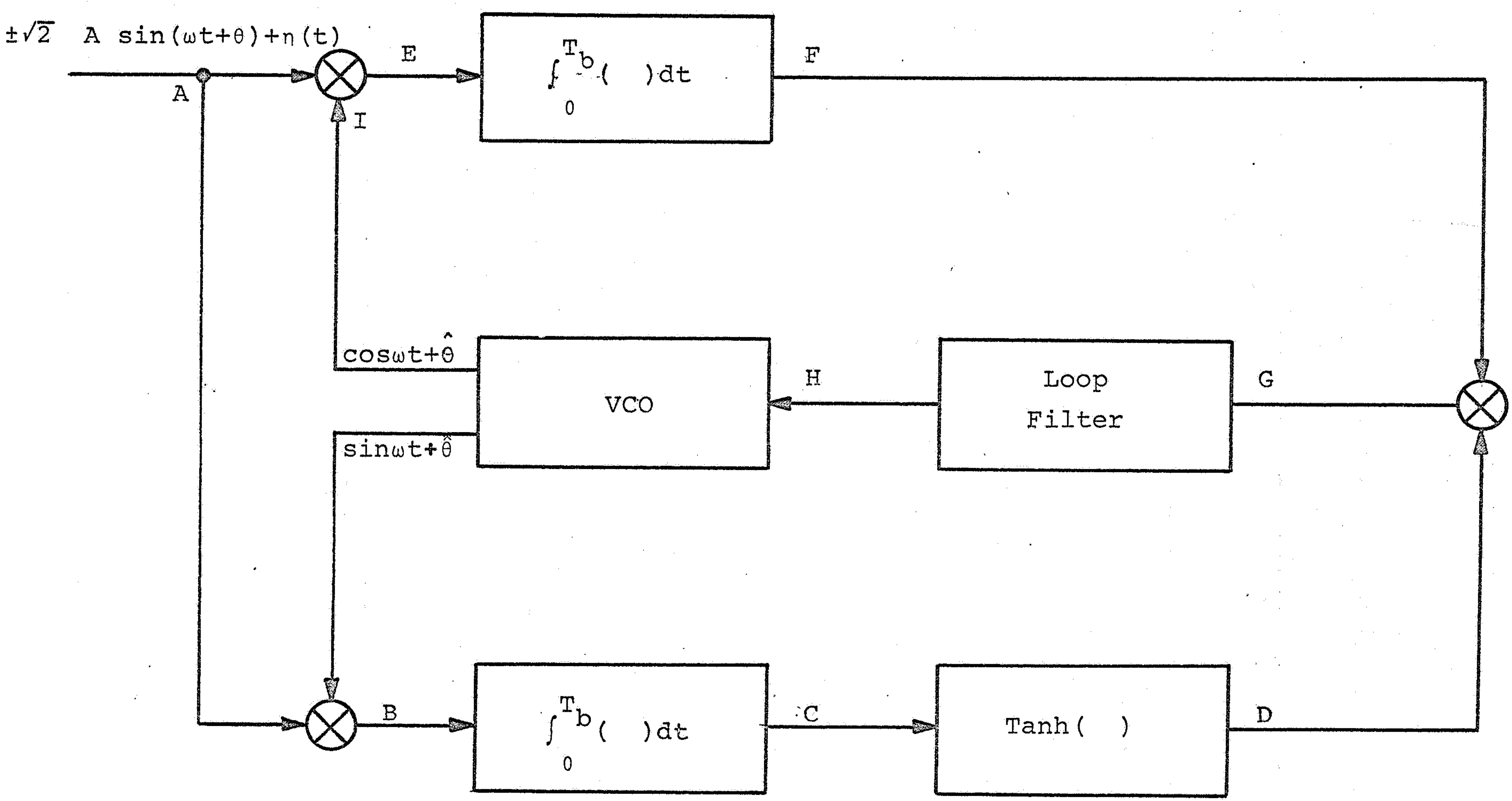


Figure 10-7. An Optimum Subcarrier Phase Detector

where C is not a function of $\tilde{\theta}$. Equation (10-53) is indeed a unimodal function of $\tilde{\theta}$ centered around $\hat{\theta}=\tilde{\theta}$ as can easily be verified. It follows that a necessary and sufficient condition for the maximum a posteriori probability estimate at $t = t_0+T$ of $\tilde{\theta}$ is $\hat{\theta}$ such that

$$d/d\theta\left(\ln\left(\cosh\sqrt{2E/N_0{}^2}\int_{t_0}^{t_0+T}\sin(\omega t+\hat{\theta})R(t)dt\right)\right)\Bigg|_{\hat{\theta}=\tilde{\theta}} = 0 \tag{10-54}$$

which reduces to

$$\left\{\mathrm{Tanh}\sqrt{2E/N_0{}^2}\int_{t_0}^{t_0+T}\sin(\omega t+\theta)R(t)dt\right\}\cdot\left\{\int_{t_0}^{t_0+T}\cos(\omega t+\theta)R(t)dt\right\}\Bigg|_{\hat{\theta}=\tilde{\theta}} = 0 \tag{10-55}$$

Since Equation (10-54) is unimodal Equation (10-55) is also the minimum mean square error estimate of $\tilde{\theta}$.

Equation (10-54) suggests the configuration of Figure 10-7 where once again a loop filter and voltage controlled oscillator are used to drive the error signal to zero. All of the elements are easily realized except for the box labeled "Tanh". For large values of its' argument the hyperbolic tangent is equal to the sign of its' argument while for small values of its' argument the hyperbolic tangent is approximately equal to its' argument. It follows that at large signal to noise ratio the hyperbolic tangent may be realized by a hard limiter while at small signal to noise ratios the hyperbolic tangent may be realized by a direct connection from input to output.

For the high signal to noise ratio case, the lower path (A,B,C,D) from the input to the output of the box labeled "Tanh", that is, a hard limiter, corresponds to the optimum detector of the polarity of a PSK signal. The upper path from the input and back through the loop filter and voltage controlled oscillator to the input (A,E,F,G,H,I,A) has been shown to be the optimum detector for the phase of a sine wave given the amplitude. The detector then in the high signal to noise ratio case estimates in an optimum fashion the phase of the received signal as if it knew its' polarity and then multiplies this result which has an ambiguity in polarity by the optimum estimate of the polarity given that the phase is known. This technique is intuitively appealing. Indeed if one knew nothing of statistics but did know of the mentioned optimum estimators of phase and polarity he might be tempted to try such a technique. It is also worth noting that this circuit needs no additional equipment to detect the data since the output of the hard limiter at time t_0+T is the optimum estimate of the data.

To calculate the performance of the optimum detector in the high signal to noise ratio case once again consider the upper path as a second order phase lock loop. For such a loop the probability distribution function of the phase error is given in Appendix E provided the estimate of the polarity of the PSK signal is correct. If the estimate of the polarity of the PSK signal is not correct then it is easily seen that the probability distribution function of the phase error is

$$p(\theta|E) = \begin{cases} p(\theta+\pi) & \theta \geq 0 \\ p(\theta-\pi) & \theta < 0 \end{cases} \qquad (10\text{-}56)$$

since now the phase lock loop will be in error by π radians. It follows that the unconditional distribution function of the phase error is

$$p'_{OPT}(\theta) = (1-P_E)p(\theta)+P_E p(\theta|E) \qquad (10\text{-}57)$$

where P_E is the probability of an incorrect estimate of the polarity of the PSK signal. Obviously the probability of an incorrect estimate must be very close to the probability of bit error as previously calculated and hence must be very small for even moderate signal to noise from which it follows that $P'_{OPT}(\theta) \simeq p(\theta)$. As in the previous cases, the quantity α may be used to rate the quality of the phase reference. It is then easily seen that

$$\alpha_{optimum} = 1/2ak(ET/N_0) \quad \text{for } ET/N_0 >> 1 \quad . \qquad (10\text{-}58)$$

From Equations (10-38) and (10-50) for the high signal to noise ratio case, it is seen that for the "Costas loop"

$$\alpha = 1/4ak\ ET/N_0 = \tfrac{1}{2}\ \alpha_{optimum} \tag{10-59}$$

and that for the squaring loop

$$\alpha = 1/8ak\ ET/N_0 = \tfrac{1}{4}\ \alpha_{optimum} \tag{10-60}$$

This indicates a 3 dB improvement of the optimum detector over the "Costas loop", and a 6 dB improvement over the squaring loop at high signal to noise ratios.

If the integrators of Figure 10-7 are thought of as low pass filters, the "Costas loop" may be used to approximate the optimum phase detector. It follows that α for this case is given by Equation (10-50).

From a practical point of view, the data detector using a "Costas loop" to derive a reference signal will give near optimum performance over all signal to noise ratios. At low signal to noise ratios its' performance is exactly that of the optimum detector; while at high signal to noise ratios although its' reference signal has a lower value of α the probability of bit error will be approximately the same.

G. <u>The Effect of Timing Errors</u>

In previous sections, the performance of PSK data detectors was investigated subject to the condition that a perfect timing signal was available. That is, that the detector knew exactly the length and starting time of each bit. In practical systems, the detector will probably know the length of each bit but it must somehow or other estimate its' starting time. Consequently, the integration period of the data detector never exactly coincides with the period in time occupied by the information bit. It is this type of error and its' effect on system performance which will be considered here.

To understand the effect of this type of error on system performance consider Figure 10-8 where it is assumed that there is an error of x seconds in the estimation of the starting point of a random binary waveform. The waveform with the resulting data detector output is shown being preceded and followed by waveforms of the same and opposite polarities. If the next bit is of the same polarity, the output of the data detector is

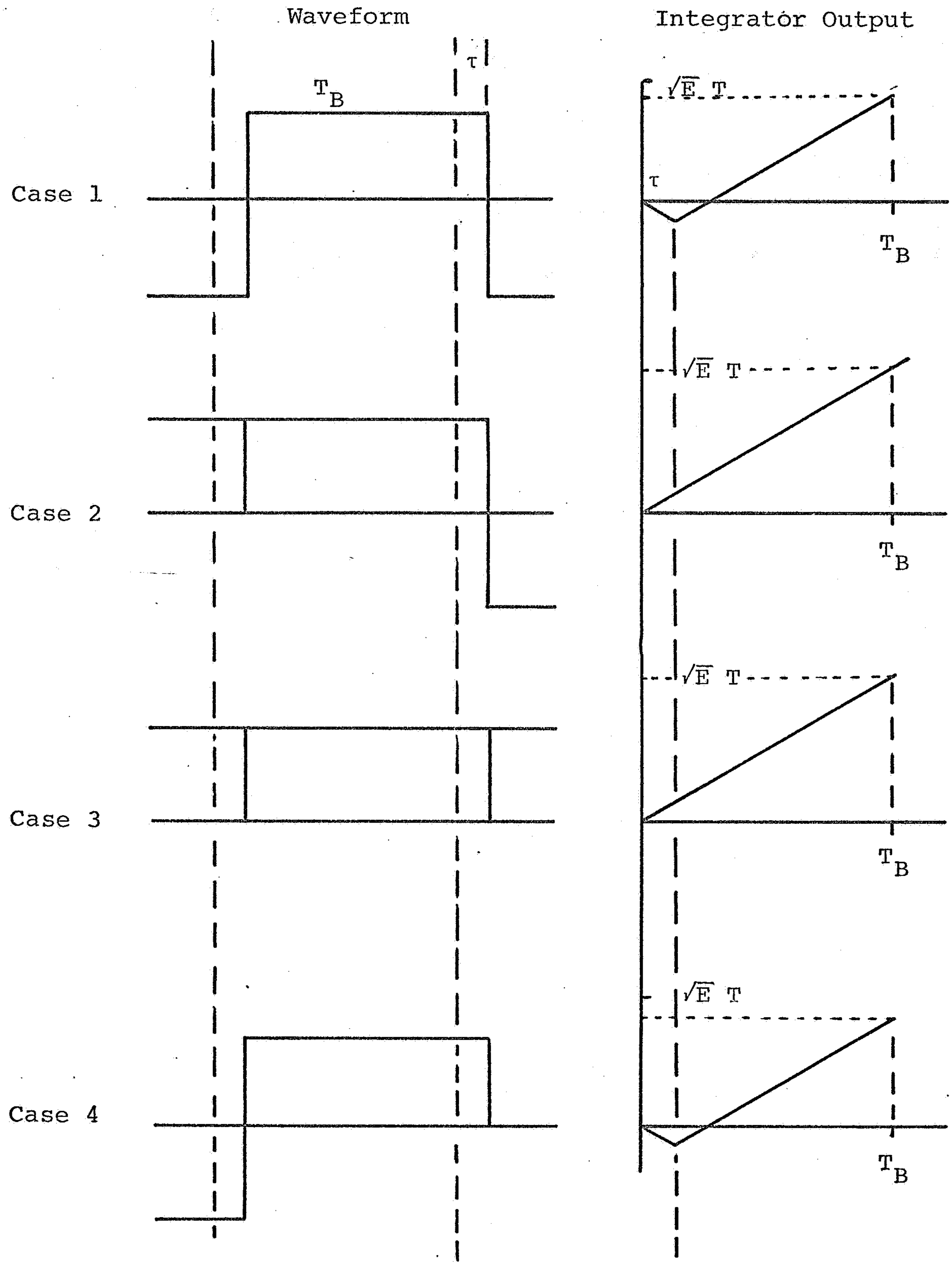


Figure 10-8. The Effect of a Timing Error τ Seconds

unchanged. From Figure 10-8 it can be shown that given a timing error of x seconds the average probability of bit error is

$$P(E|x) = \tfrac{1}{4}(2-\mathrm{erf}\sqrt{2ST/N_0}-\mathrm{erf}\sqrt{2ST(1-|x|/T)/N_0}) \quad . \qquad (10\text{-}61)$$

If the probability distribution function of x were known then the average probability of bit error could be evaluated by averaging Equation (10-61) over all values of x. That is, by calculating

$$P_e^b = \int_{\text{All } x} P(E/x)p(x)dx \qquad (10\text{-}62)$$

where p(x) is the probability distribution function of x. The determination of the distribution function of x depends on the technique used to estimate x and there appears to be no universally accepted technique for doing this. Consequently, a general solution of Equation (10-61) is not possible. Indeed even for very simple distribution functions Equation (10-61) cannot be integrated without using numerical techniques. It is profitable however, to look at the case of x uniformly distributed over some fraction of the bit period of the data. The evaluation of x for this case involves approximations which can be used with other density functions to obtain a feel for the effect of the timing errors on performance.

If x is uniformly distributed over some fraction of the bit period, a, then the probability of bit error is

$$P_e^b = 1/8aT \int_{-aT}^{+aT} (2-\mathrm{erf}\sqrt{2ST/N_0}-\mathrm{erf}\sqrt{2ST(1-|x|/T)/N_0})dx \qquad (10\text{-}63)$$

Equation (10-63) cannot be evaluated in closed form, however, it can be tightly bounded by making use of the well-known uniform bound on the error function

$$(1-\mathrm{erf}\alpha) \leq e^{-\alpha^2/2} \quad . \qquad (10\text{-}64)$$

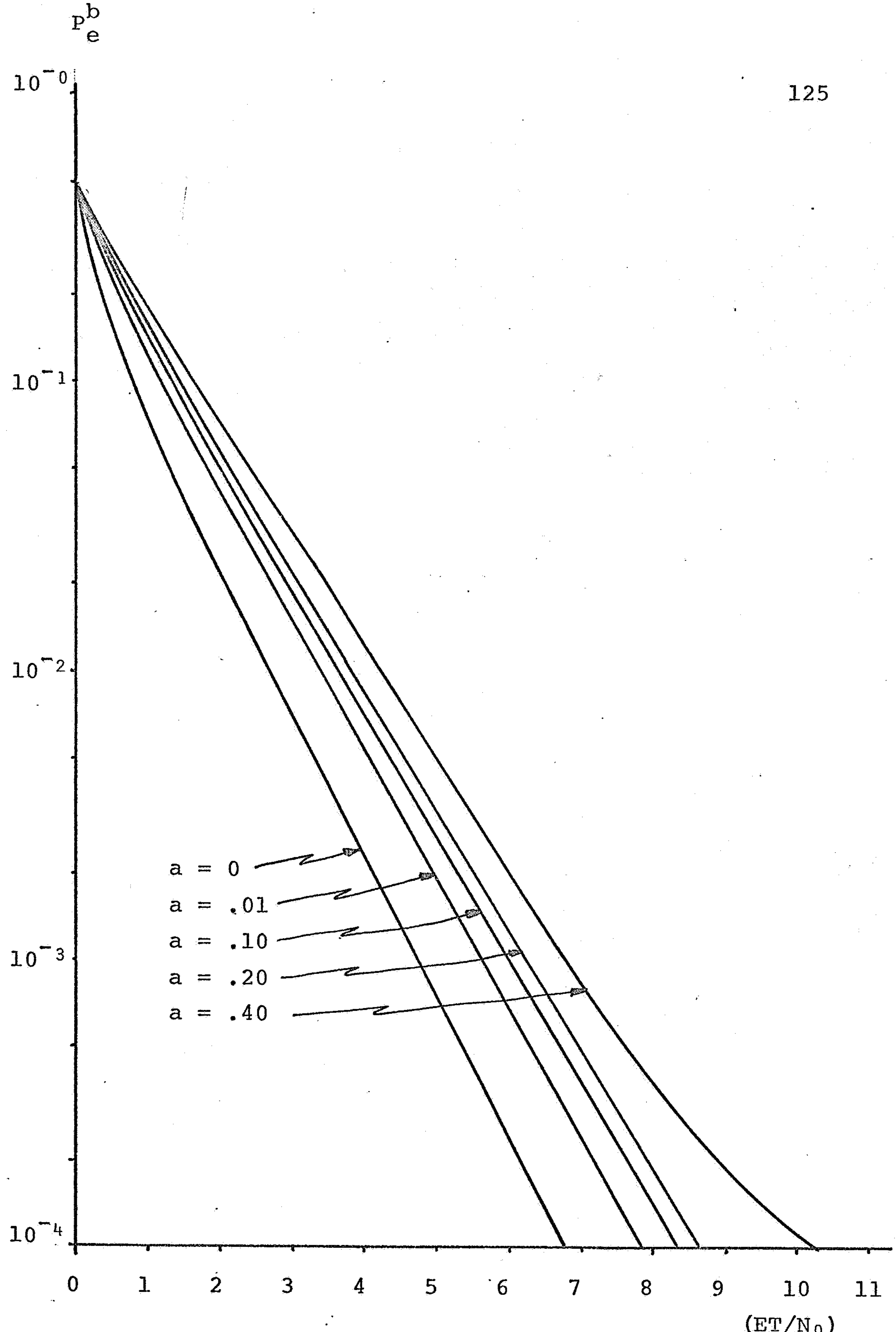


Figure 10-9. The Probability of Bit Error versus Signal to Noise Ratio for a Uniformly Distributed Timing Error.

Substituting Equation (10-64) into Equation (10-63) and performing the integration yields

$$P_e^b \le \tfrac{1}{4}\{1-\mathrm{erf}\sqrt{2ST/N_0}\}+\frac{\exp-(ST/N_0)}{4a(ST/N_0)}\ (1-\exp\{a(ST/N_0)\}) \tag{10-65}$$

Equation (10-65) is plotted versus ST/N_0 the signal to noise ratio with a as a parameter in Figure 10-9, along with the probability of bit error for the ideal case. It should be remembered that these curves represent an upper bound on P_e^b, and consequently, represent the worst case condition for a particular timing error. It can be seen that there is no thresholding effect as is caused in the case of phase reference errors. In addition, it is seen that the relative effect of timing errors in comparison with phase reference errors is small. This conclusion is borne out by observation of practical systems (Hon, 1968).

H. Performance of Practical Detectors

In this section, the results of the previous sections are used to evaluate the performance of practical data demodulators using a squaring and a Costas loop. The bandwidths chosen are typical of the bandwidths encountered in MSFN signal data demodulators and the bit rates correspond to the two telemetry bit rates used in the Apollo unified S band communication system.

The predetection bandwidth, W, and the phase lock loop bandwidths B_1 are shown for the two data rates in Table 10-1. Three bandwidths are shown for the phase lock loop. It is customary to use the larger bandwidth to lock onto the reference and then switch to one of the smaller bandwidths.

TABLE 10-1

Bandwidths of a Practical Receiver

	Hz
$W(51.2(10)^3\text{KBPS})$	$150(10)^3$
$W(1.6(10)^3\text{KBPS})$	$6(10)^3$
$B_{1}1$	25
$B_{1}2$	100
$B_{1}3$	350

Using Equations (10-38) and (10-50) α is plotted versus signal to noise ratio for the two information rates, the three bandwidths, and the two detectors in Figures 10-10 and 10-11. The curves dramatically bring out the importance of a small ratio between the bandwidth of the phase lock loop and the bandwidth of the signal. As can be seen this is much more important than whether or not one uses a squaring loop or a Costas loop. Using Equation (10-26) the probability of bit error for the various combinations can be calculated. The results are shown in Figures 10-12 and 10-13. For the high bit rate case the difference between the two detection schemes cannot be seen on the graphs, in fact, it is seen that the bandwidths are really not critical. The performance for the smallest bandwidth is for all practical purposes the same as the ideal performance predicted by Equation (10-11).

For the low bit rate case it is seen that the Costas loop significantly outperforms the squaring loop at low signal to noise ratios and that the choice of bandwidth is very important. This might be expected since at low signal to noise ratios, as was shown, the Costas loop is a good approximation of the optimum detector.

It might appear that higher performance is achieved when using the higher bit rate; this is not correct. It should be remembered that for equal amplitude signals switching from the higher to lower bit rate gives a signal to noise improvement of $\sqrt{51.2/1.6} = 5.65$. For example, when the high speed bit stream reaches its lowest acceptable quality $P_e^b = 10^{-3}$ the signal to noise ratio is 4.8, switching to the low speed bit stream will give a signal to noise ratio of 27 which will certainly achieve the desired error rate.

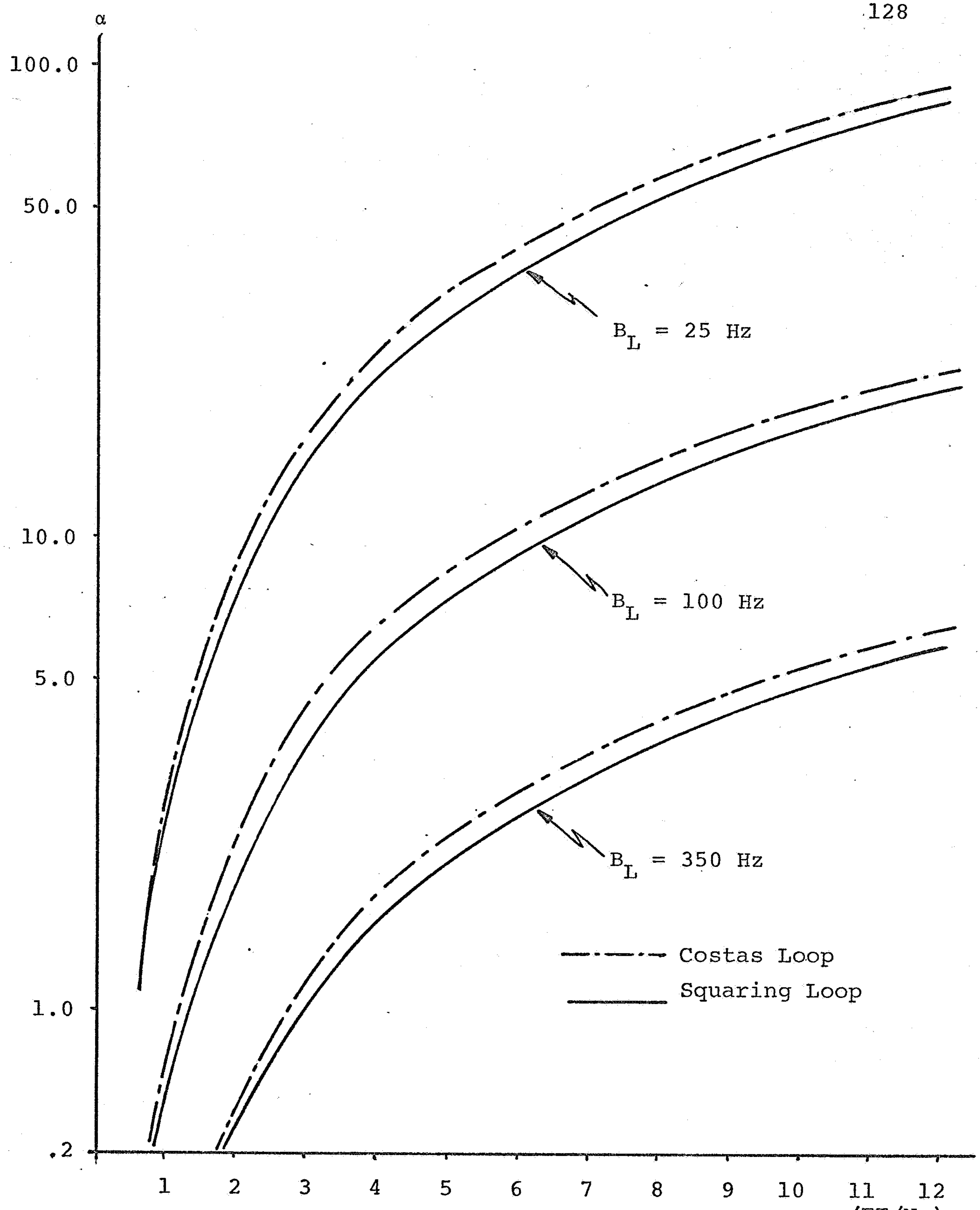


Figure 10-10. Alpha versus Signal to Noise Ratio
$a\_ = 3.75$
$T^{-1} = 1.6(10)^3$

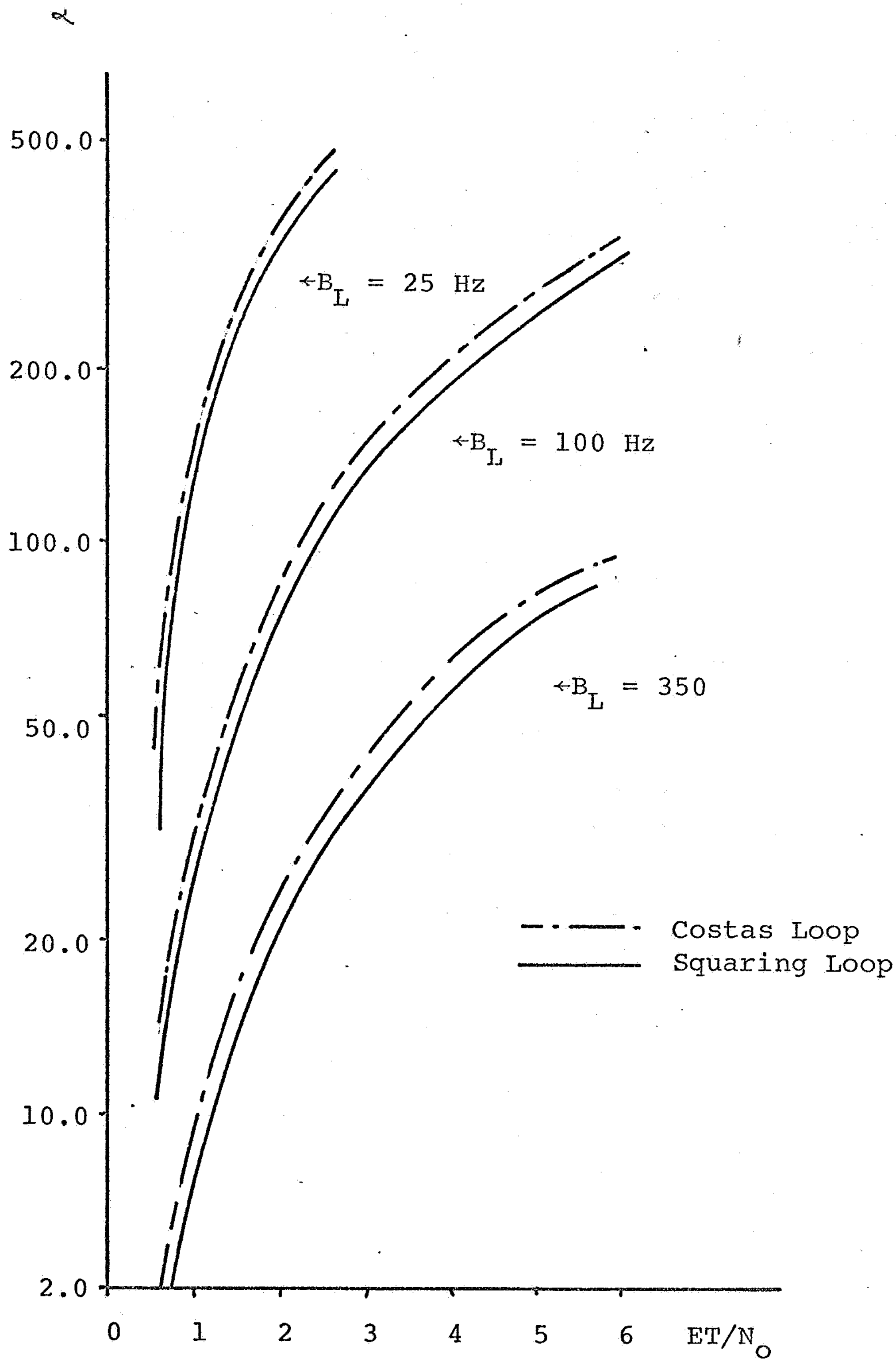


Figure 10-11. Alpha versus Signal to Noise Ratio
$a = 2.929$
$T^{-1} = 51.2(10)^3$

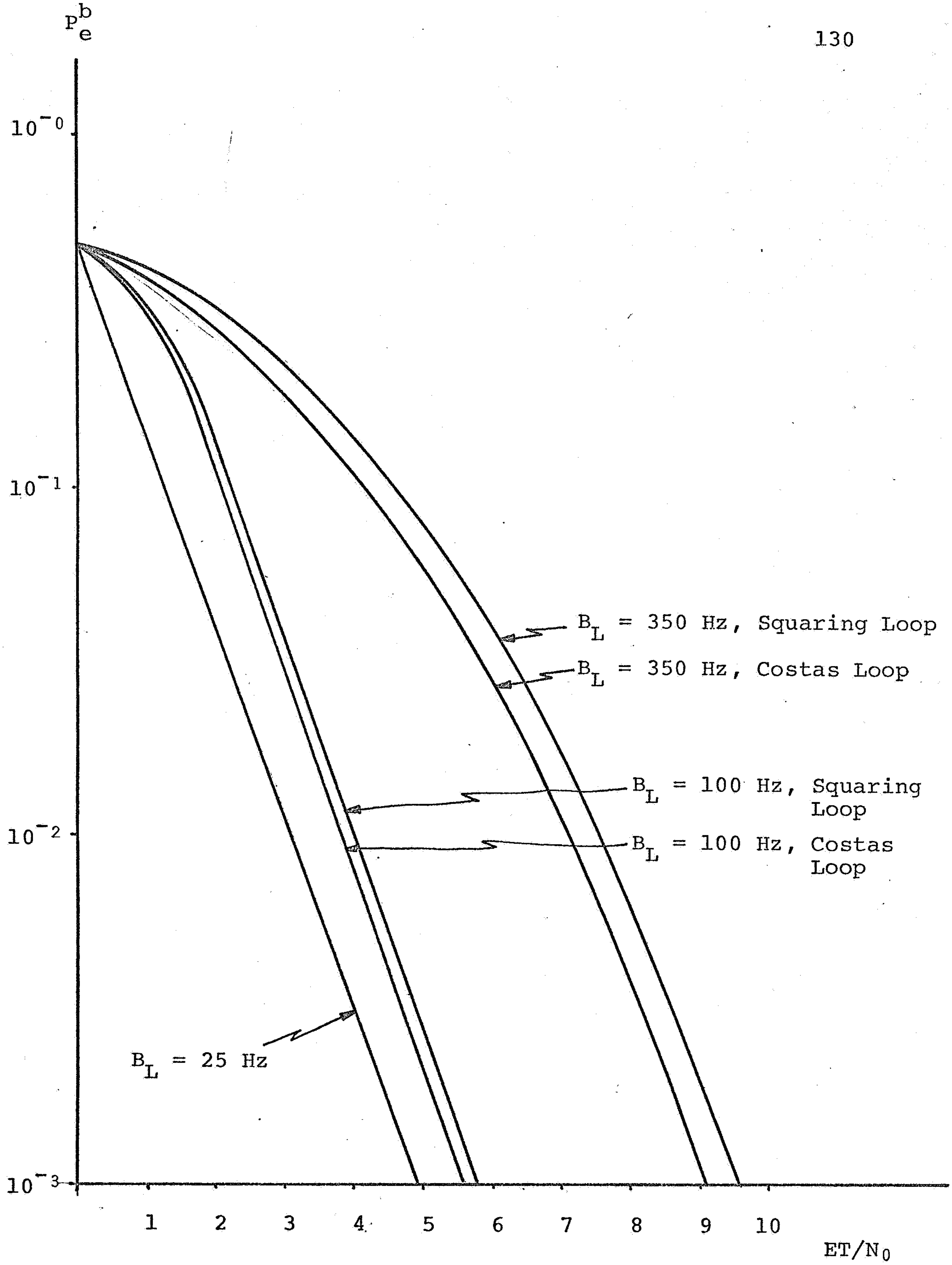


Figure 10-12. Probability of Bit Error versus Signal to Noise Ratio $a = 3.75$

$T^{-1} = 1.6(10)^3$

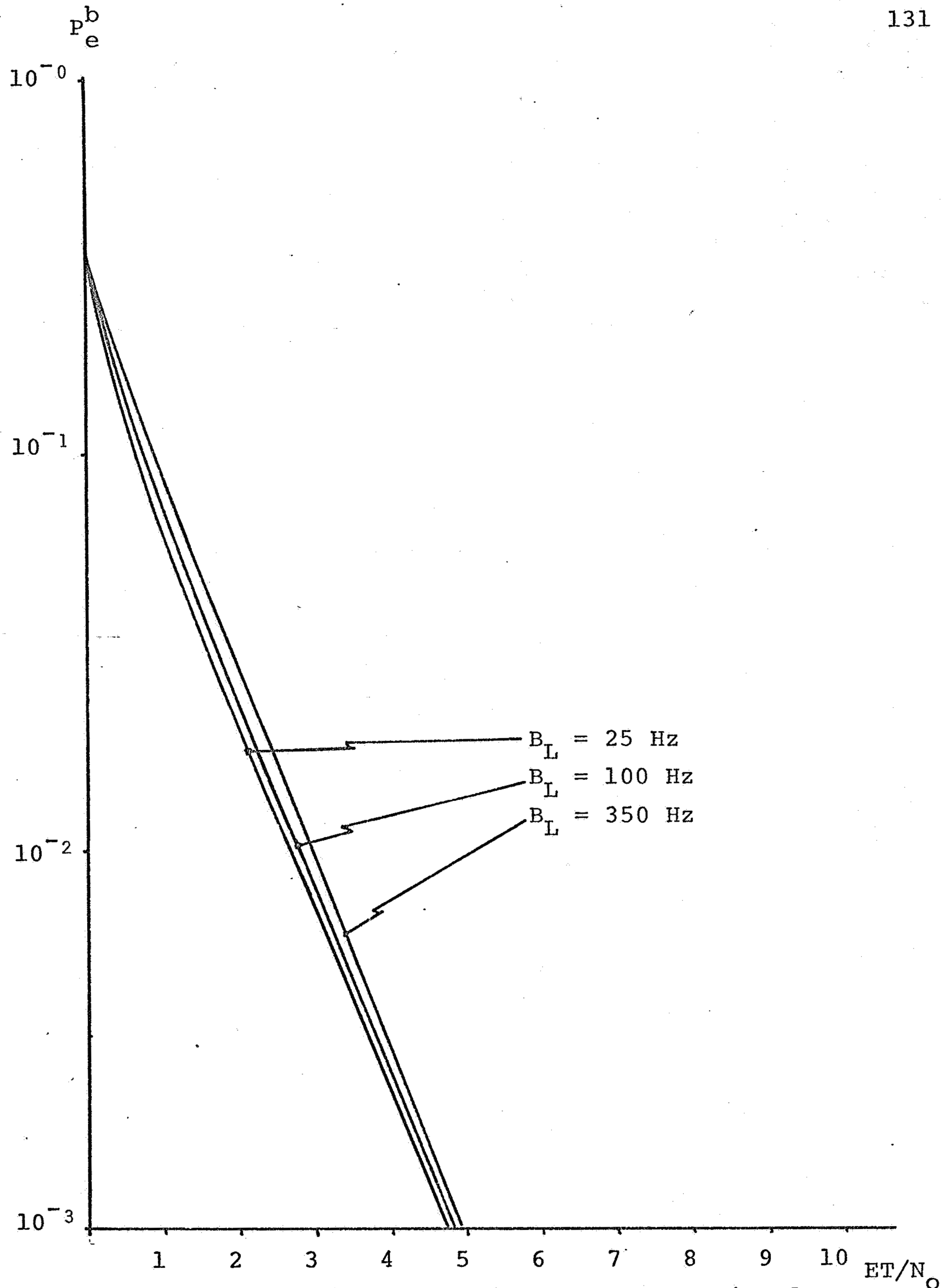


Figure 10-13. Probability of Error versus Signal to Noise Ratio a = 2.929
$T^{-1}=51.2(10)^{-3}$

CHAPTER XI

THE EFFECT OF CODING ON THE PERFORMANCE OF A PSK DATA CHANNEL

A. Introduction

Since the fundamental coding theorem of information theory was first proven by Shannon in 1948, communication engineers have been concerned with trying to implement systems which will achieve the predicted arbitrarily small probability of error at rates less than the channel capacity. In this chapter, practical relationships between the rate R, and the previously discussed signal design parameters for the multiplexed PSK signal set will be investigated.

B. Background

The system studied here is the complete system from input to output of Figure 11-1. In this system the encoder observes a block of L input bits and then produces a block of N symbols as an output. The rate used will not be the information rate in symbols per second, or input bits per second but the ratio of the length of a block of input symbols to a block of output symbols. That is, the rate, R, is L/N. For the purposes here the most useful form of the coding theorem is due to Fano (1961) and states that for a discrete memoryless channel at rates below the channel capacity, the average probability of error, P_e, for codes of length N is bounded above and below by

$$e^{-N\{E(R_L)+0(N)\}} \leq P_e \leq 2E^{-NE(R)} \qquad (11\text{-}1)$$

where $E(R_L)$ and $E(R)$ are positive functions of the transistion probabilities and the rate R, and $0(N)$ is a function of N going quickly to zero for large N. In principle, Equation (11-1) says that any desired error probability can be achieved provided N is chosen large enough.

The primary utility of Equation (11-1) is due to the fact that the bound predicted is independent of the choice of code. This independence is a result of the use of so-called random coding arguments (Gallager, 1965) in the derivation of Equation (11-1). For example, the coder shown in Figure 10-1

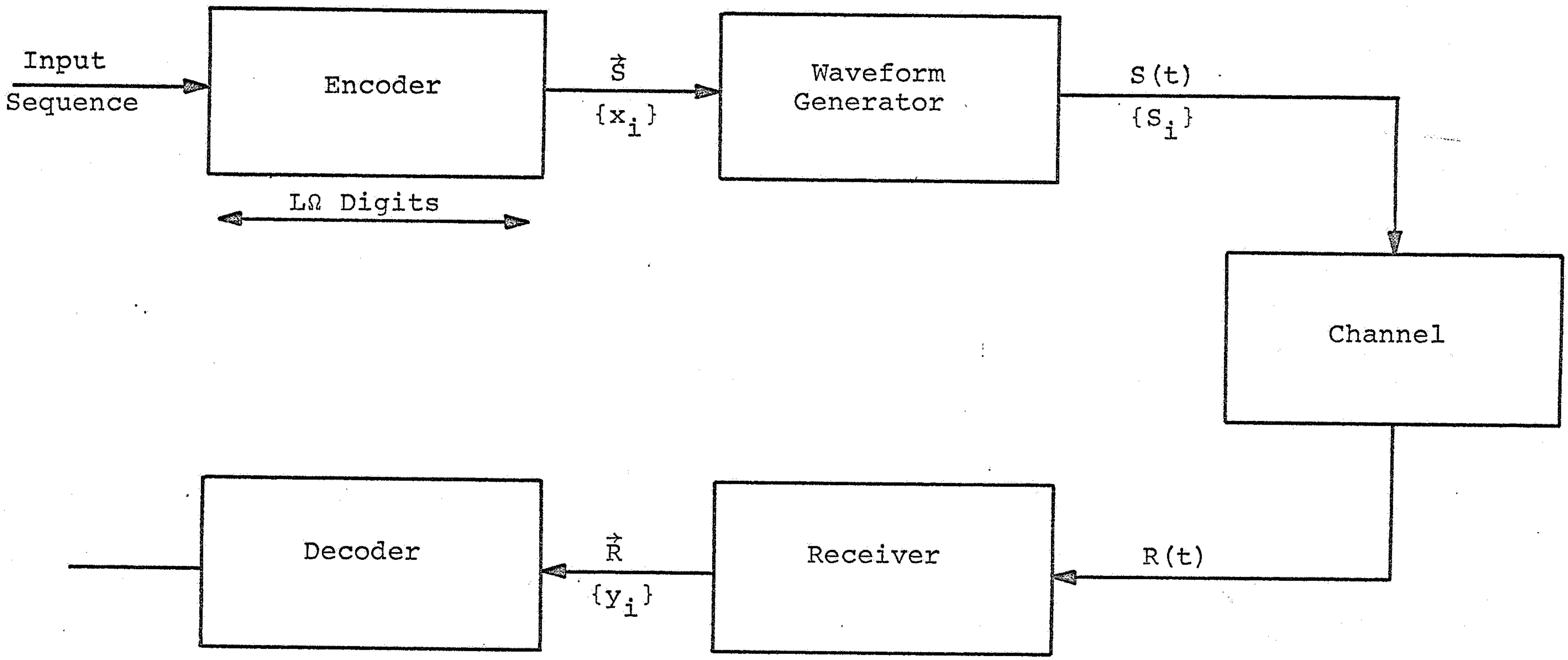


Figure 11-1. A Coded Communication System

can be thought of as a device which in the binary case maps each of 2^L input sequences into one of 2^N output sequences. A person attempting to construct codes is interested in the optimum way to perform this mapping based upon some constraints. Obviously some choices are very good and some are very bad, however, the bound of Equation (11-1) is independent of the choice and merely represents the average attainable error if the choice of mappings were made at random. It seems reasonable then to assume that with a little intelligence and luck the coding theorist can choose a mapping as good as the average or random choice consequently, the signal designer can use the information obtainable from Equation (11-1) without worrying about the subsequent choice of a code.

Unfortunately for all but a few simple pathological cases the evaluation of E(R) is all but hopeless. (Gallager, 1966). It can be shown, however, that (Wozencraft and Kennedy, 1966)

$$E(R) \geq R_0 - R \tag{11-2}$$

where R_0 is alternately known as the computational cut-off rate or the zero-rate intercept of E(R). R_0 is a function of the choice of modulation techniques and is more easily evaluated than E(R) or $E(R_L)$. It follows from Equation (11-2) that

$$P_e \leq 2e^{-N\{R_0-R\}} \tag{11-3}$$

Equation (11-3) says that for any rate less than the computational cutoff rate arbitrarily small error probabilities may be obtained if N is chosen large enough.

In a very elegant and complex derivation, Gallager (1966) has shown that

$$R_0 = \max_{\{p_i\}}\{-\log \sum_j \{ \sum p_i \sqrt{p_{i_j}} \}^2\} \tag{11-4}$$

where p_i is the probability of sending the ith symbol and p_{i_j} is the probability of receiving the ith symbol given that the jth symbol is sent. The maximization is over the input probabilities p_i and hence R_0 is independent of the input probabilities.

It might appear that R_0 is a dummy variable created by elegant manipulation of obscure relationships with little, except perhaps some contrived, physical significance. It turns out that not only is R_0 related to the choice of the modulation and detection technique but that it also appears to be related to the actual number of computations needed to decode the message block, the computer storage necessary to store the message blocks during decoding, and the necessary decoding times (Lebow and McHugh, 1967). Thus, evaluation of this quantity provides more than just information about the word error probability, since it also tells the designer something about the computational equipment necessary to attain a particular error probability. In addition, it has been shown analytically and verified with simulations (Lebow and McHugh, 1967) that for the case of convolutional encoding and sequential decoding R_0 also represents the rate at which the number of computations necessary to decode a block of data becomes infinite.

C. <u>R_0 For the PSK Channel</u>

In this section R_0 is calculated for the PSK channel of Chapter X. For this case i=1,2 and j=1,2 in Equation (11-4). Expanding Equation (11-2) yields

$$R_0 = \max_{\{p_i\}}\{-\log\{((p_1\sqrt{p_{11}})+(p_2\sqrt{p_{12}}))^2 + ((p_1\sqrt{p_{21}})+(p_2\sqrt{p_{22}}))^2\}\} \quad . \tag{11-5}$$

The maximum value of Equation (11-5) occurs for p_i such that

$$\frac{dR_0}{dp_1} = 0 \quad . \tag{11-6}$$

It can be shown by performing the indicated differentiation that p_1 must be a solution of

$$\frac{dR_0}{dp_1} = \{p_1(\sqrt{p_{11}}-\sqrt{p_{12}})+\sqrt{p_{12}}\}\{\sqrt{p_{11}}-\sqrt{p_{12}}\} \qquad (11\text{-}7)$$
$$+ \{p_1(\sqrt{p_{21}}-\sqrt{p_{22}}+\sqrt{p_{22}}\}\{\sqrt{p_{11}}-\sqrt{p_{12}}\} = 0 \quad .$$

The solution to Equation (11-7) is $p_1=p_2=\frac{1}{2}$. Assuming the channel to be symmetric, it can be seen that the transition probabilities correspond to the average probability of bit error, P_e^b, in the following manner.

$$p_{21} = P_e^b \qquad (11\text{-}8)$$

$$p_{22} = 1-P_e^b \qquad (11\text{-}9)$$

$$R_0 = \log_2 \frac{1}{\frac{1}{4}(p_{11}+p_{12}+p_{21}+p_{22})+\frac{1}{2}\sqrt{p_{11}p_{12}}+\frac{1}{2}\sqrt{p_{21}p_{22}}} \qquad (11\text{-}10)$$
$$= \log_2 \frac{2}{1+2\sqrt{P_e^b(1-P_e^b)}}$$

R_0 is plotted versus P_e^b in Figure 11-2. For the case of a PSK signal observed with a perfect phase reference

$$1-P_e^b \simeq 1 \qquad (11\text{-}11)$$

and

$$R_0 \simeq \log_2 \frac{2}{1+2\sqrt{\frac{1}{2}(1-\mathrm{erf}(2ST/N_0))}} \quad . \qquad (11\text{-}12)$$

Since the effect of an imperfect phase reference is to reduce P_e^b the probability of bit error, R_0 for the case of an imperfect phase reference may, using the results of Chapter X, be written as

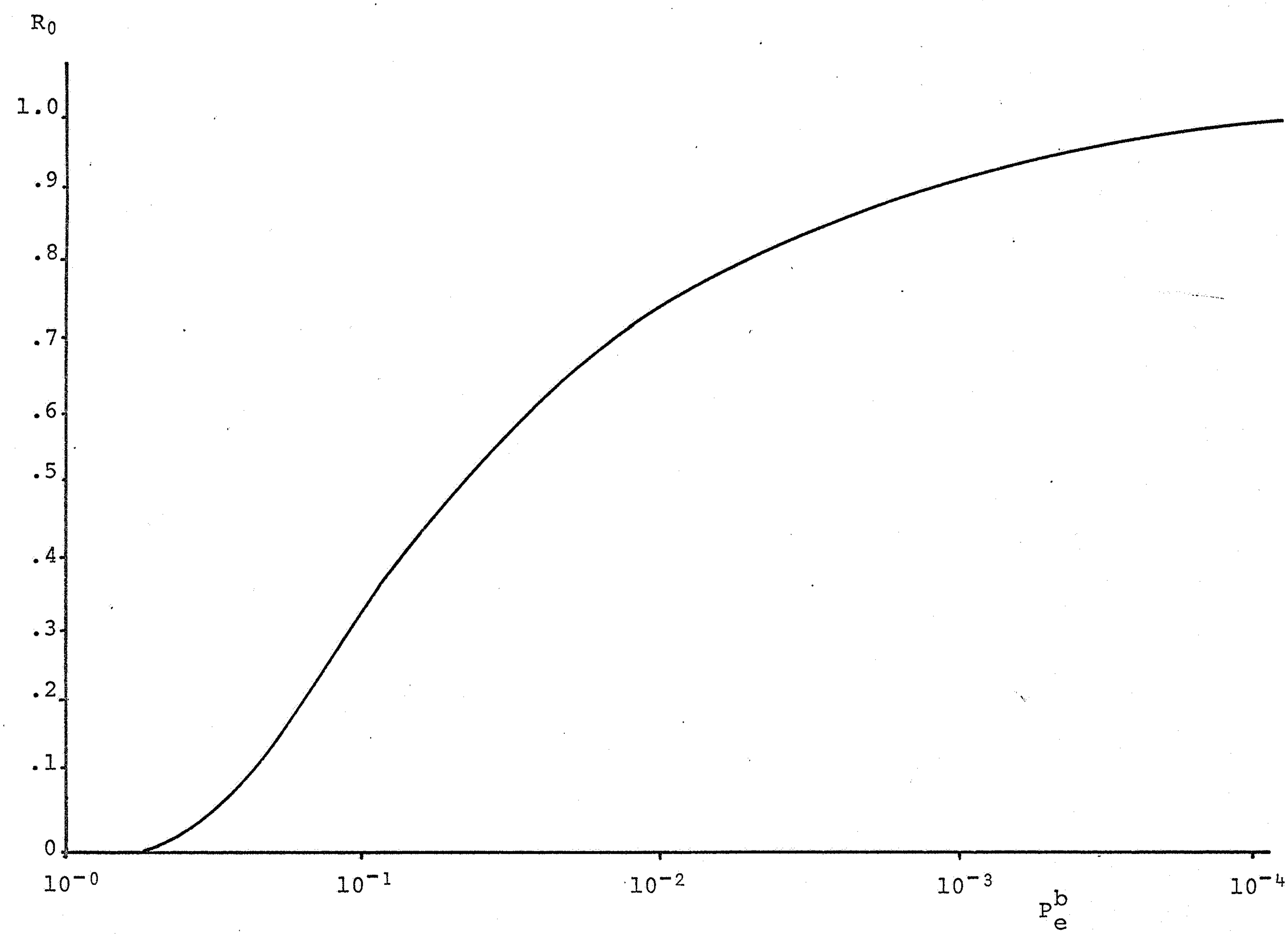


Figure 11-2. R_0 versus The Probability of Bit Error for a Binary Symmetric Channel

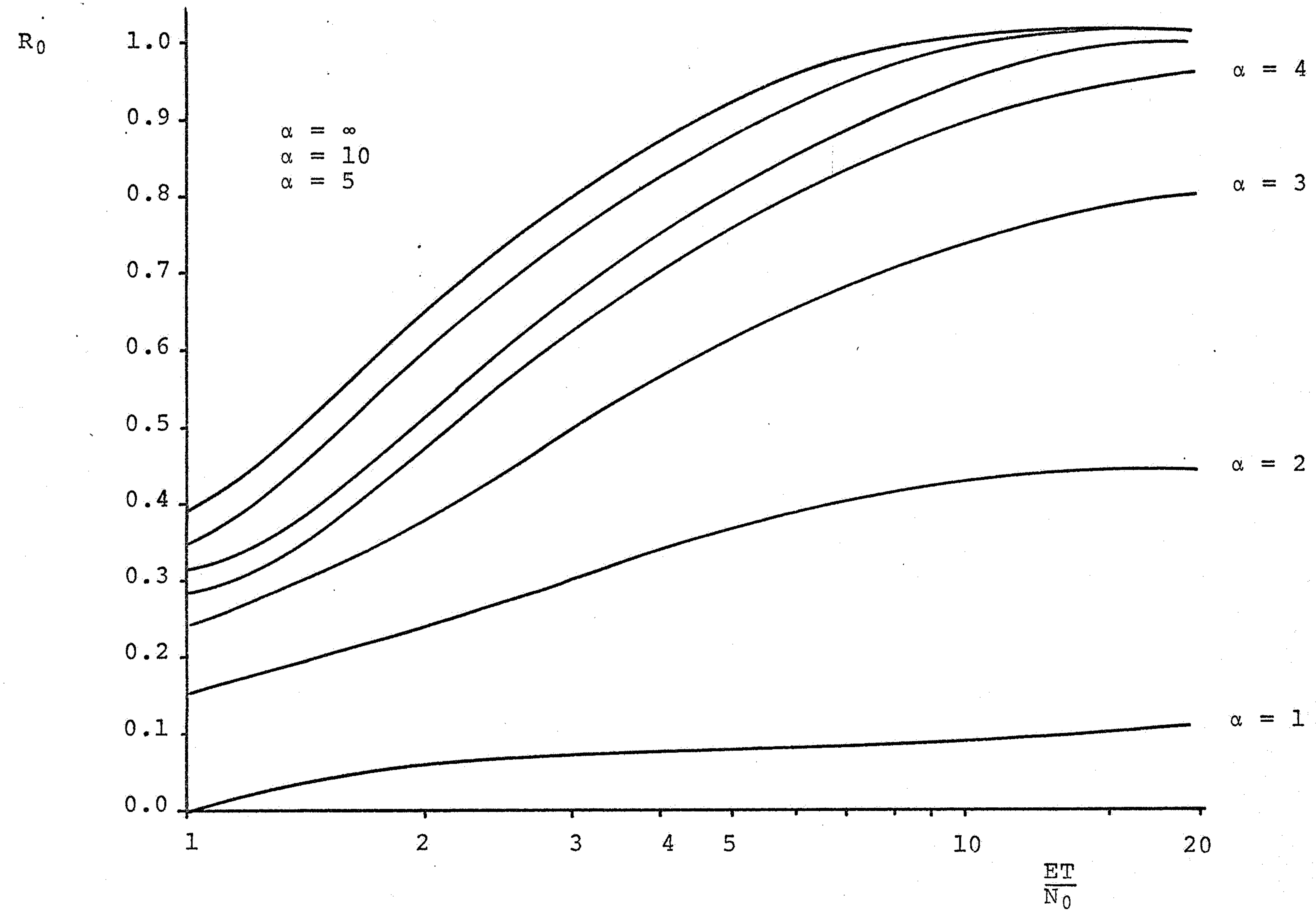


Figure 11-3. R_0 versus Signal to Noise Ratio for a Noisy Phase Reference

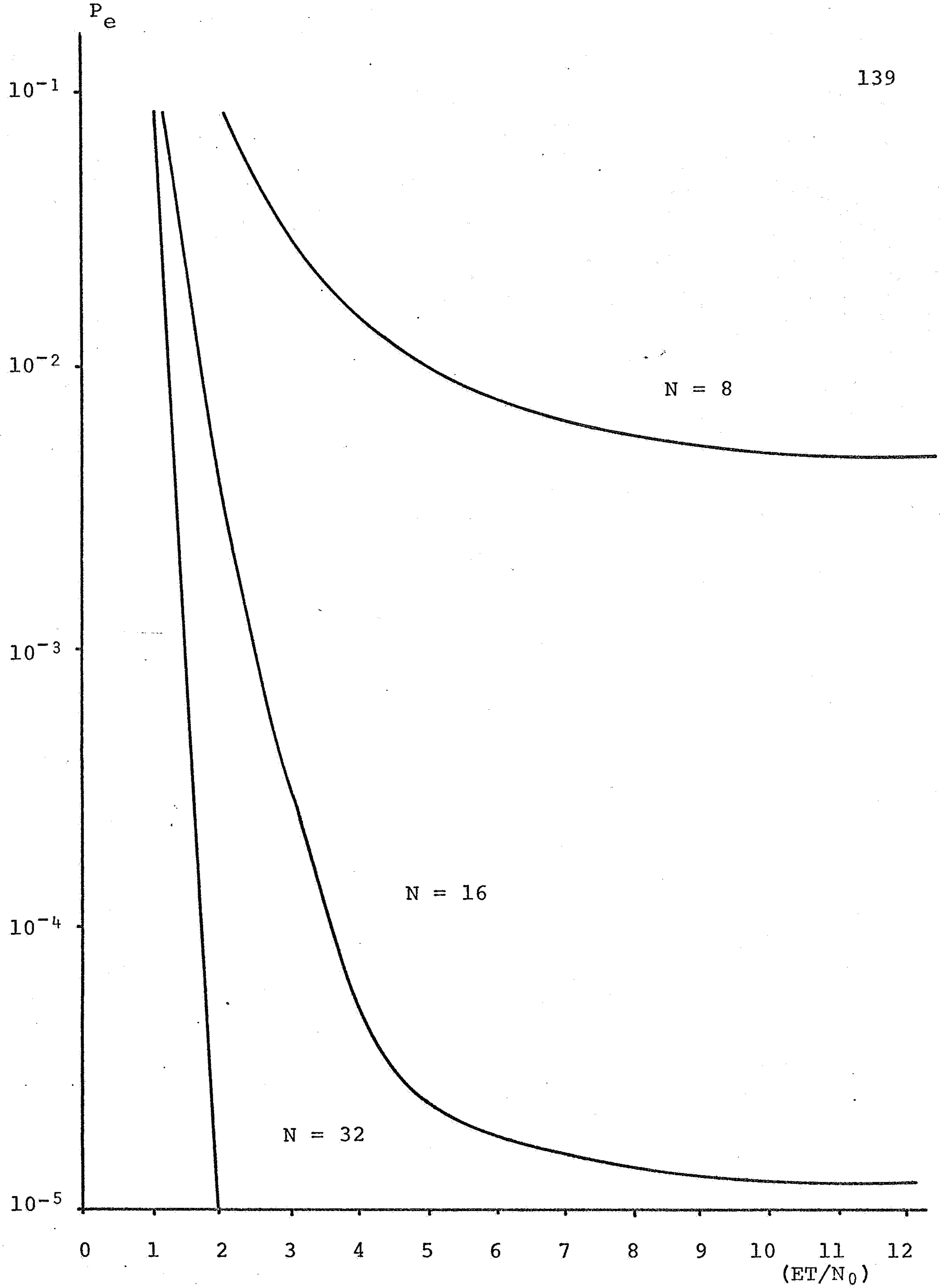


Figure 11-4. The Probability of Error versus Signal to Noise Ratio For Rate .25 Codes of Length N

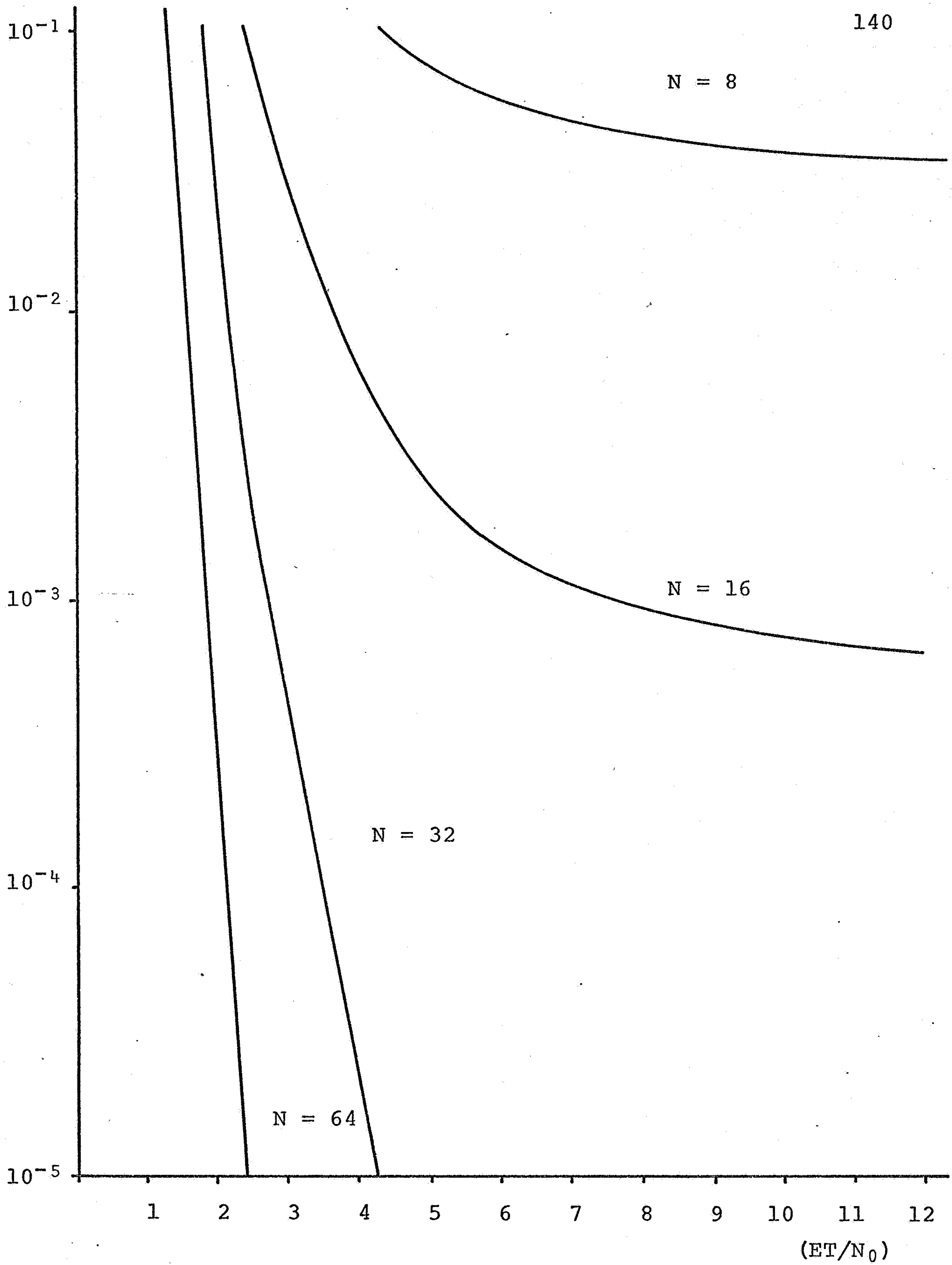


Figure 11-5. The Probability of Error versus Signal to Noise Ratio For Rate 0.5 Codes of Length N

$$R_0 = \log_2 2-\log_2 (1+2\sqrt{\tfrac{1}{2}(1-\text{erf}((\tfrac{I_1(\alpha)}{I_0(\alpha)})((2S/N)^{-1}+\tfrac{1}{2}+\tfrac{1}{2}\tfrac{I_2(\alpha)}{I_0(\alpha)}-(\tfrac{I_1(\alpha)}{I_0(\alpha)})^2)^{-\frac{1}{2}}))}) \qquad (11\text{-}13)$$

where α is calculated according to the techniques of Chapter X. R_0 is plotted versus S/N with α as a parameter along with R_0 for an ideal phase reference, $\alpha=\infty$, in Figure 11-3.

The above equations can be used to calculate a bound on the performance of a PSK channel versus the signal to noise ratio. Examples of such relationships are shown in Figures 11-4 and 11-5 for rate ¼ and ½ codes of various lengths. These figures show dramatically the improvement that can be obtained by increasing the code length.

D. A Comparison of Coded and Uncoded Communication

The work of the preceding section leaves unanswered the question of whether or not anything is gained by coding blocks of data. The only equitable way to compare a coded and uncoded system is on the basis of equal information transfer and such comparisons are discussed in this section.

For the case of PSK data transmission with a perfect phase reference the probability of error on any bit has been shown to be

$$P_e^b = \tfrac{1}{2}(1-\text{erf}\sqrt{2ET/N_0}) \quad . \qquad (11\text{-}14)$$

If it is assumed that the probability of error on any bit is independent of the probability of error on any other bit then it is easily seen that the probability of correctly transmitting a block of N bits is

$$P_C = (1-P_e^b)^N \qquad (11\text{-}15)$$

and consequently, the probability of an error in a block N bits long is

$$P_e = 1-(1-P_e^b)^N \qquad (11\text{-}16)$$

$$P_e = 1-(\tfrac{1}{2}-\tfrac{1}{2}\text{erf}\sqrt{2ET/N_0})^N .$$

The equivalent of a coded system of length N and rate R is an uncoded block of NR bits. To compare equitably on the basis of information transfer the two blocks a block of N coded bits of length R must be compared with a block of NR uncoded bits where each bit is now no longer T seconds long but is T/R seconds long. This will result in an equal information transfer rate. It follows that the probability of error on a block is

$$P_e = 1-(\tfrac{1}{2}-\tfrac{1}{2}\mathrm{erf}\sqrt{2ET/N_0R})^{NR} . \tag{11-17}$$

The probability of error for coded and uncoded blocks with equal information transfer is shown in Figures 11-6, 11-7, 11-8, and 11-9 for various block lengths with rate ½ codes. These figures bring out the considerable improvement that is obtained by going to longer and longer block lengths. While with the uncoded blocks performance decreases with increasing block length startling improvements are obtained for long block lengths with the coded systems.. For the shorter block lengths the figures do not show the big improvement obtained for long blocks. It should be remembered that the curves for the coded case represent an upper bound on the probability of error and not the probability of error itself. Thus while one may conclude from Figure 11-6 for example that for a rate ½ code of length 128 one obtains superior performance with coding for all signal to noise ratios one cannot conclude from Figure 11-9 that an uncoded system gives superior performance for signal to noise ratios greater than 7. To determine performance in these cases the actual code used must be specified and the resulting probability calculated.

It might be argued that it is unrealistic to require perfect transmission of complete blocks of data as the criterion of performance. On the contrary, just the opposite is true. It is customary in most telemetry systems to group together large blocks of data for transmission and to only use the block of data if it is believed that the transmission of the entire block was perfect. In this case, coding large blocks is in the general scheme of things and requires little extra equipment to perform the coding process. Of course, decoding the data is quite another matter, and decoding large blocks

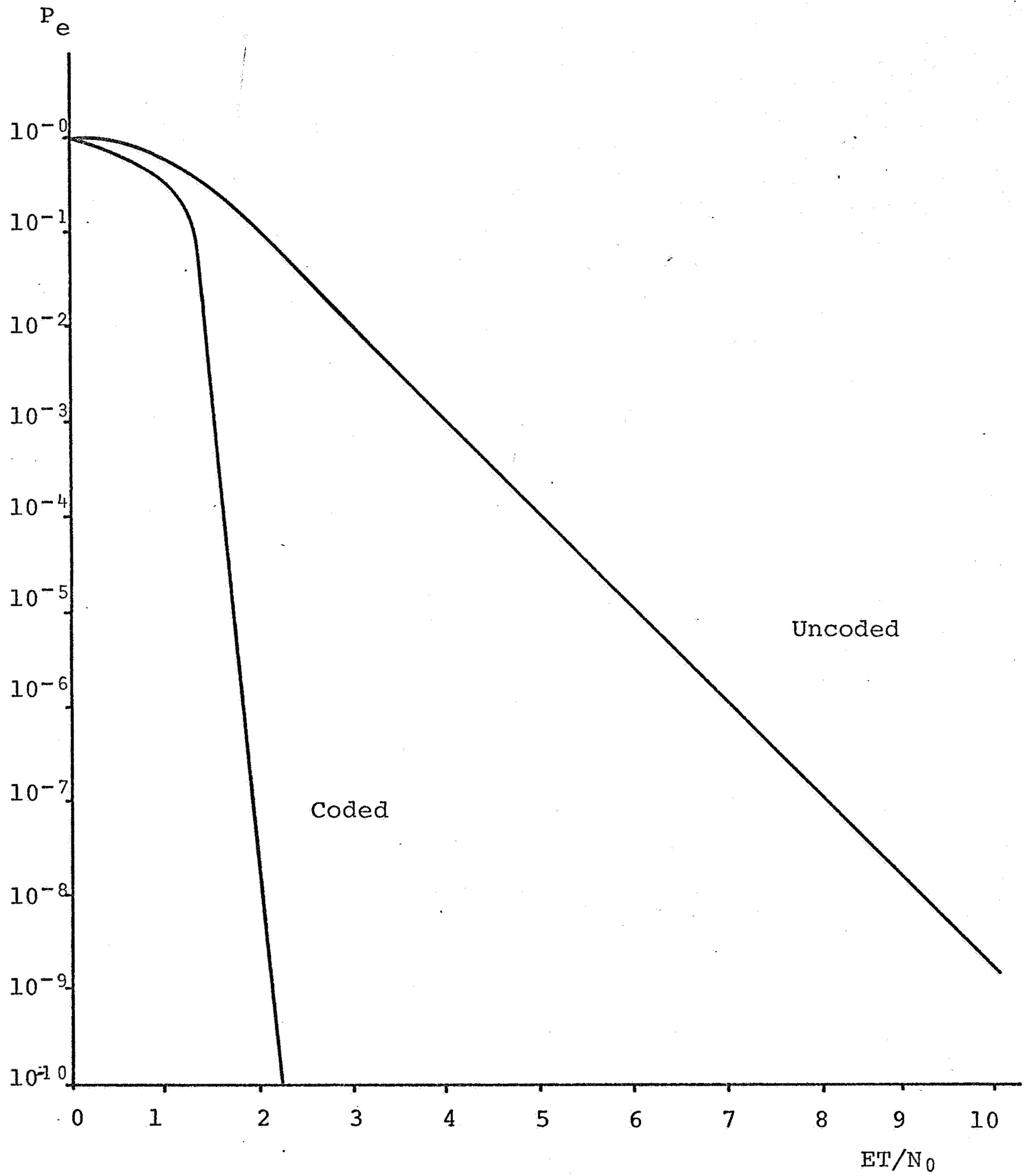


Figure 11-6. The Probability of Error versus Signal to Noise Ratio For a Rate .5, Length 128 Code and Equivalent Uncoded Block

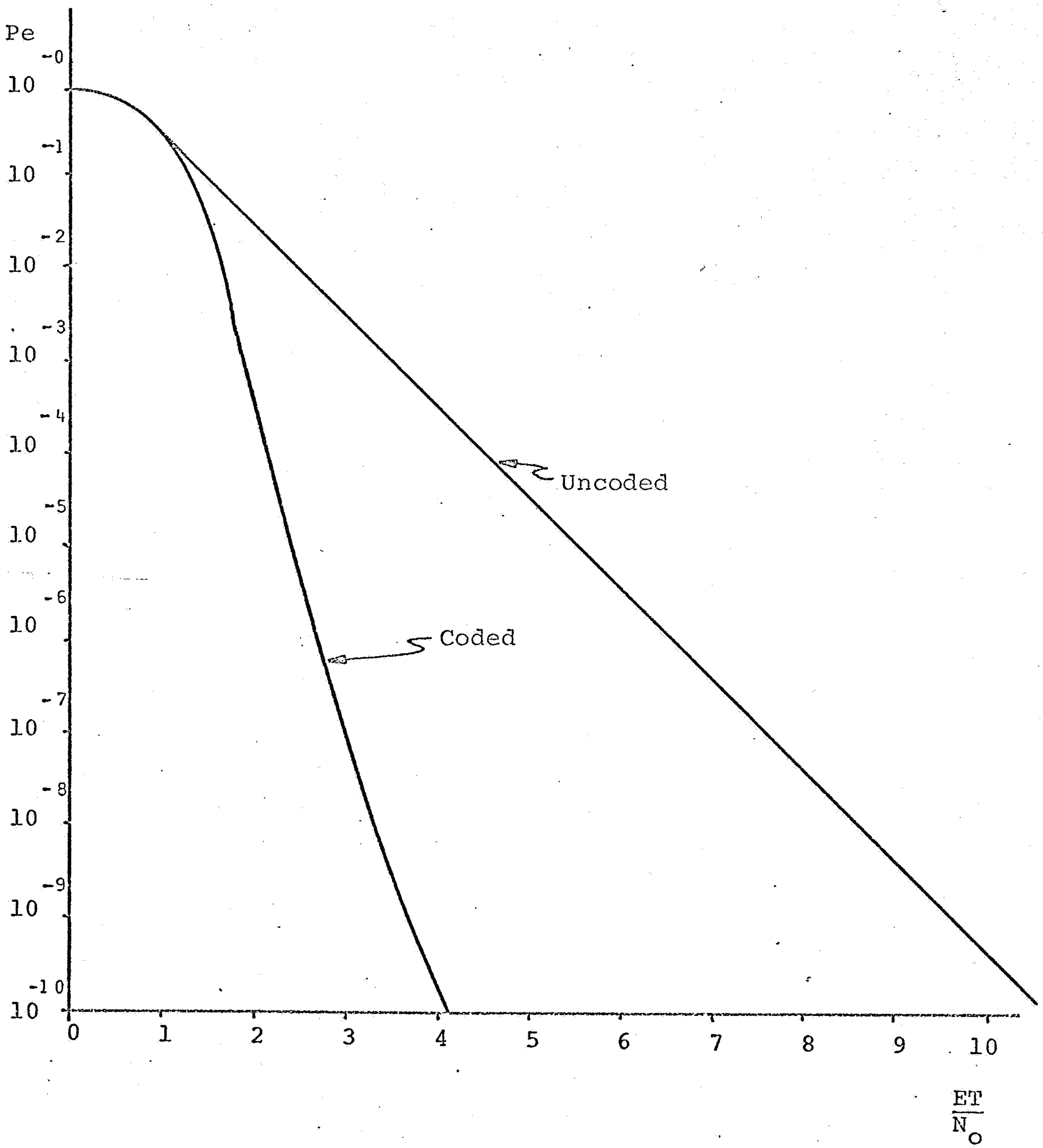


Figure 11-7. The Probability of Error versus Signal to Noise Ratio For a Rate .5, Length 64 Code and Equivalent Uncoded Block

of data is a quite complicated process which is beyond the scope of this work.

E. A Practical Communication System

In this section previously derived results of this chapter are applied to a practical communication system. The system is a phase shift keyed communication system where the transmitter can transmit at one of two information rates $1.6(10)^3$ bits per second or $51.2(10)^3$ bits per second. The data is transmitted in 128 bit blocks and it is always desired to transmit at as high a rate as possible. This means that the transmitter will always operate at the higher rate unless the performance falls below some predetermined level at which time it will switch to the lower rate. Switching is arbitrarily chosen to occur when the probability of bit error becomes greater than 10^{-3}. Switching to the lower speed increases the effective signal to noise ratio by $\sqrt{51.2/1.6} = 5.65$ and decreases the information transfer rate by a factor of 32. It is desired to see if by using probabilistic coding a rate of information transfer between these two values can be achieved.

If the high speed telemetry link is coded when the probability of bit error reaches a value of 10^{-3} the performance can be calculated using Equations (11-13) and (11-3). The results of such a calculation for a rate ½ and ¼ code are shown in Table 11-1 where the probability of block error is shown for various signal to noise along with the corresponding probability of bit error for the uncoded rates. The information transfer rates for the rate ½ and ¼ codes are $25.6(10)^3$ and $12.8(10)^3$ respectively. The probability of block error for the uncoded case can be calculated from Equation (11-16); it is of course much higher than the indicated probability of bit error. It is seen that a significant improvement in information transfer rate and in data quality is obtained by going to the coded system. The designer of course pays for this in the expensive equipment necessary in order to decode the signal. The decision on whether or not this cost is justified cannot really be discussed here. In space communications systems where for example it might cost a million dollars to obtain one decibel improvement in signal to noise ratio and where the quality of the data might jeopardize the success of a mission costing billions of dollars it would probably be profitable to code the data. The relationships of this chapter provide some of the tools the designer needs to make this cost decision.

TABLE 11-1

IMPROVEMENT FROM CODING

ET/N_0	P_e^b at 1.6KBPS	P_e^b at 51.2KNPS	P_e $r=\frac{1}{4}, N=128$	P_e $r=\frac{1}{2}, N=128$
1	$2.4(10)^{-4}$	$6.4(10)^{-2}$	$1.4(10)^{-9}$	*
2	$4.7(10)^{-7}$	$2.0(10)^{-2}$	$4.0(10)^{-22}$	$3(10)^{-8}$
3	$9.6(10)^{-10}$	$6.7(10)^{-3}$	$4.2(10)^{-29}$	$1.2(10)^{-15}$
4	$4.8(10)^{-12}$	$2.2(10)^{-3}$	$1.8(10)^{-33}$	$1.4(10)^{-19}$
5	$9.7(10)^{-15}$	$7.5(10)^{-3}$	$2.4(10)^{-37}$	$7.0(10)^{-24}$

*$R_0 < R$ for $ET/N_0 \leq 1.1$

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APPENDIX A

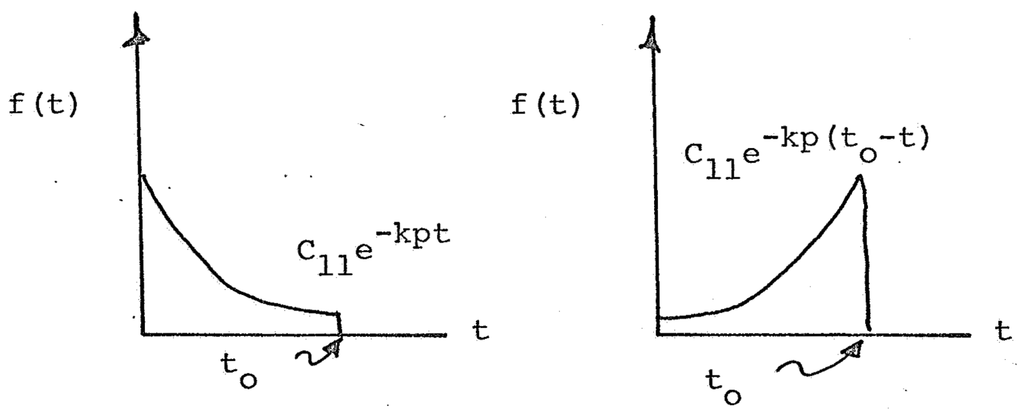
PERFORMANCE IN THE PRESENCE OF NOISE

In many communications applications the noise is Gaussian and has a power spectrum that is almost flat up to frequencies much higher than the significant signal frequencies. This type noise is called white Gaussian and is defined as the stationary, zero-mean Gaussian process with a power spectrum of

$$G(f) = \frac{n_0}{2} \text{ watts/Hz.}, \ -\infty \leq f \leq \infty \qquad \text{(A-1)}$$

The filter which maximizes the ratio of the peak value of the signal amplitude to the rms noise is called a matched filter, since the impulse response of the filter is matched to the signal pulse shape. This is demonstrated in connection with the freedom of selection of signal waveshape. Finally, the results of a digital computer simulation of an orthogonal multiplexing system operating with additive noise, which has a Gaussian amplitude distribution is presented at the end of the appendix. Assume an input signal $f(t)$ to a linear filter with a transfer function $H(\omega)$. The output of the filter is given by

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}F(\omega)H(\omega)d\omega \qquad \text{(A-2)}$$



(a) (b)

Figure A-1. Exponential waveforms referred to in the text

If the filter input is white Gaussian noise, the output is

$$N = \frac{n_0}{2} \int_{-\infty}^{\infty} |H(\omega)|^2 df = \frac{n_0}{2} \int_{-\infty}^{\infty} h^2(\tau) d\tau \tag{A-3}$$

The ratio of the signal squared to the mean noise power squared at time t_0 is

$$\frac{g^2(t_0)}{N} = \frac{S}{N} = \frac{\left[\int_{-\infty}^{\infty} f(\tau) h(t_0-\tau) d\tau\right]^2}{\frac{n_0}{2} \int_{-\infty}^{\infty} h^2(\tau) d\tau} \tag{A-4}$$

For the case of the orthogonal multiplexing system, the signal f(t) is known and the energy E can be considered a known constant

$$E = \int_{-\infty}^{\infty} f^2(t) dt = \int_{-\infty}^{\infty} |F(\omega)|^2 df \tag{A-5}$$

If S/N is divided by E, it gives the following expression which must be maximized.

$$\frac{\left[\int_{\infty}^{\infty} f(\tau) h(t_0-\tau) d\tau\right]^2}{\int_{-\infty}^{\infty} f^2(t) dt \int_{-\infty}^{\infty} h^2(\tau) d\tau} \tag{A-6}$$

Schwarz's inequality states that

$$\left[\int_{-\infty}^{\infty} f(\tau) h(t_0-\tau) d\tau\right]^2 \leq \int_{-\infty}^{\infty} f^2(t) dt \int_{-\infty}^{\infty} h^2(\tau) d\tau \tag{A-7}$$

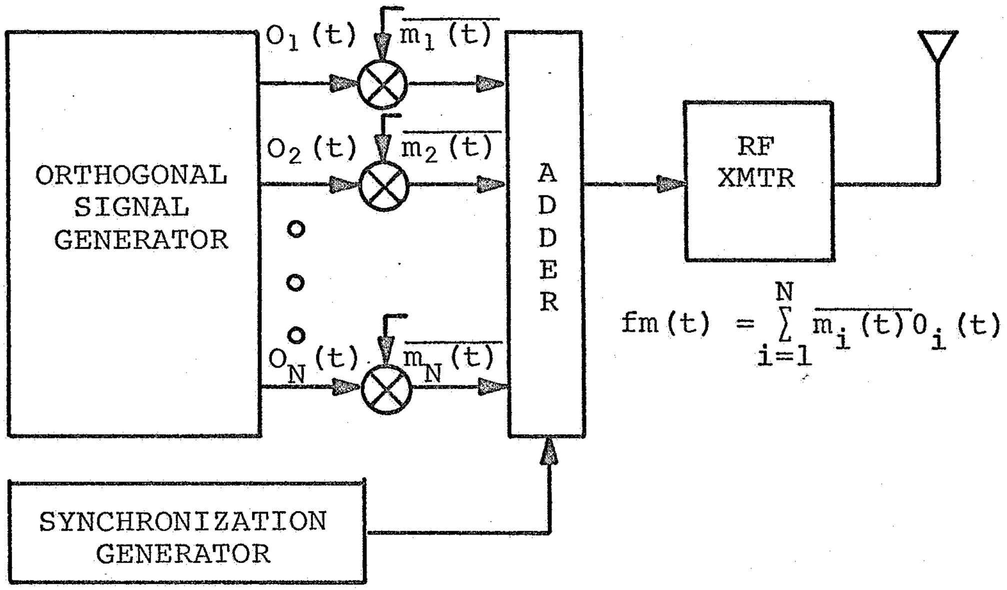


Figure A-2. Transmitter block diagram

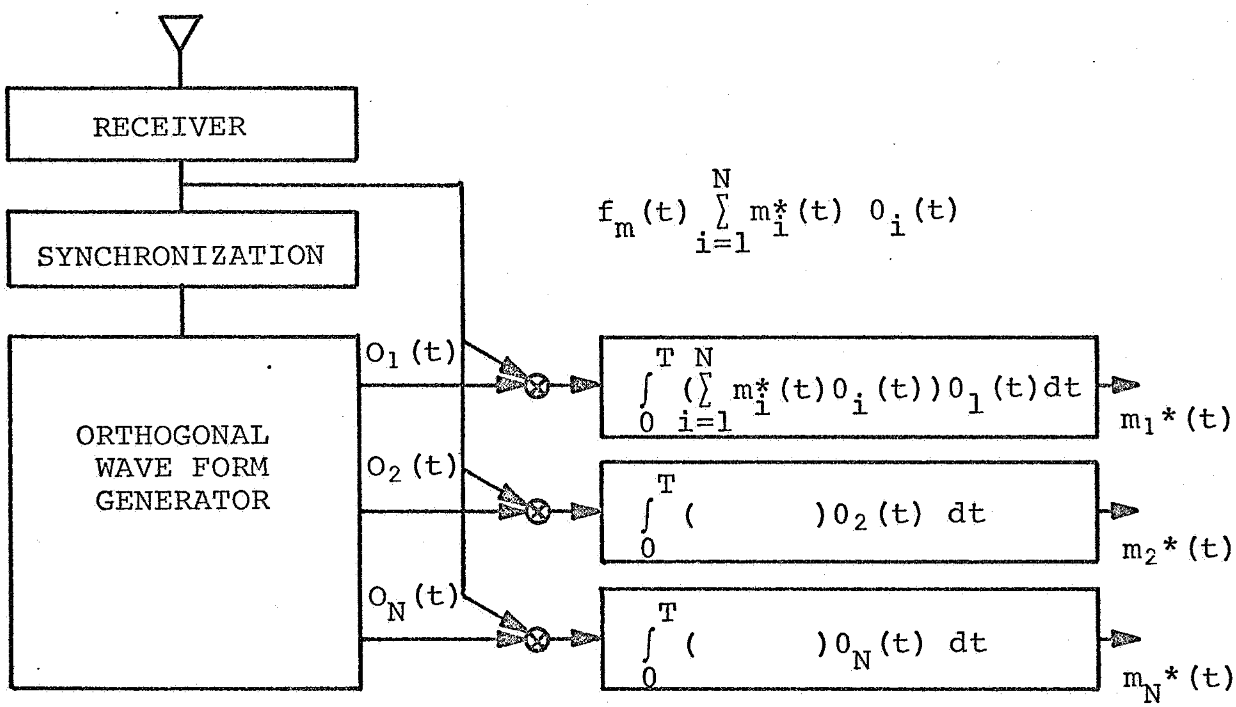


Figure A-3. Receiver block diagram

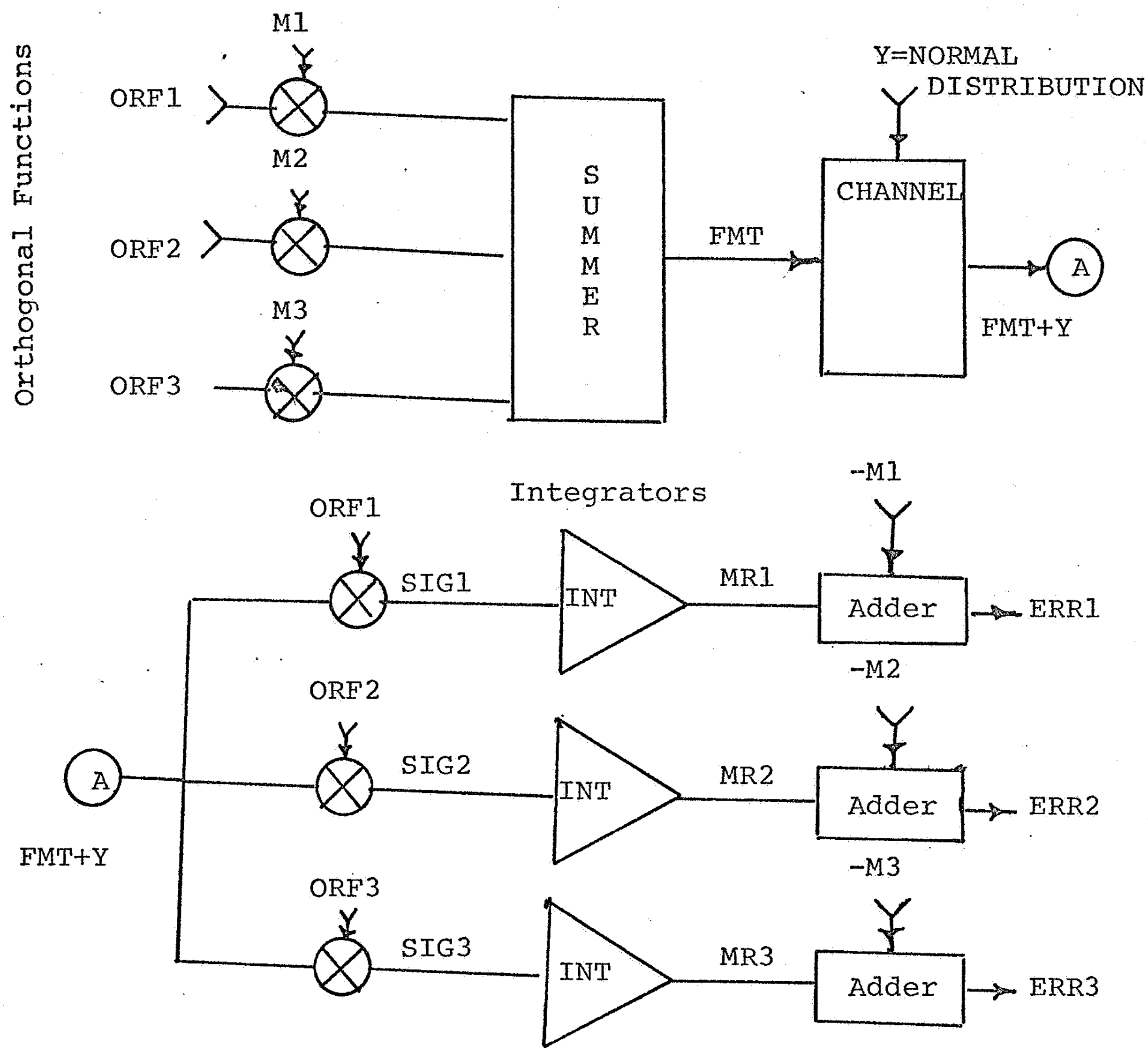


Figure A-4. Computer Program Notation

Using the equality the expression for S/N may be maximized to yield

$$f(\tau) = h(t_0-\tau) \quad \text{or} \quad h(t) = f(t_0-t) \tag{A-8}$$

and in the frequency domain $H(\omega) = F^*(\omega)e^{-j\omega t_o}$ where $F^*(j\omega)$ is the complex conjugate of $F(j\omega)$. This is the impulse response of the matched filter, or the filter which has an impulse response which is a replica of the signal but inverted, as shown in Figure A-1.

The signal-to-noise ratio of the system using the matched filter is found by substituting Equation(A-8) into Equation (A-4).

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} f^2(\tau)\,d\tau}{n_0/2} = \frac{2E}{n_0} \tag{A-9}$$

Thus the maximum value of the peak signal-to-noise ratio is dependent only on the signal energy and the noise spectral density and is independent of the pulse shape. This allows great freedom in the selection of signal waveshapes for orthogonal multiplexing systems.

A digital computer program written in the Digital Simulation Language is included to show how well the orthomux system performs in the presence of noise with Gaussian amplitude distribution. The results for several ratios of signal-to-noise are included. The results for each run are identified by the identical heading for P1. The computer program follows the orthomux system layout, as shown in

Figures A-2, A-3, and A-4. Figures A-2 and A-3 are the conventional block diagrams of an orthomux system with the notation used throughout the thesis, while Figure A-4 gives the notation used in the computer program. The notation is almost identical and the flow of the computer routine can be easily identified. The digital simulation language is non-procedural and consists essentially of a group of subroutines which can be closely related to analog computer blocks. In this case, the computer printout shows the error does not increase significantly as the noise power is increased from S/N=10 to S/N=1/5. The value of Y, the normally distributed amplitude function, is also printed to show the magnitude and random variation of this quantity.

```
$IEDIT          SYSLB3,SRCH
$IBLDR MAIN
$IBLCK CENTRL
$IEDIT
TITLE DSL/90 ADDITIVE GAUSSIAN NOISE SIMULATION 3/12/67
      ORF1=C11*EXP(-TIME)
      ORF2=C21*EXP(-TIME)+C22*EXP(-2.*TIME)
      ORF3=C31*EXP(-TIME)+C32*EXP(-2.*TIME)+C33*EXP(-3.*TIME)
      FMT=ORF1*M1+ORF2*M2+ORF3*M3+Y
      Y=NORMAL(P1,P2,P3)
      SIG1=FMT*ORF1
      MR1=INTGRL(0.,SIG1)
      SIG2=FMT*ORF2
      MR2=INTGRL(0.,SIG2)
      SIG3=FMT*ORF3
      MR3=INTGRL(0.,SIG3)
      ERR1=MR1-M1
      ERR2=MR2-M2
      ERR3=MR3-M3
CONTRL FINTIM=5.,DELT=.01
PARAM M1=1.,M2=1.,M3=1.,P1=1,P2=0.,P3=.245
CONST C11=1.4142,C21=4.,C22=-6.,C31=7.3485,C32=-29.394,C33=24.495
RANGE ORF1,ORF2,ORF3,FMT,Y
INTEG MILNE
PRINT .1,Y,FMT,MR1,MR2,MR3,ERR1,ERR2,ERR3
END
TITLE SAME SYSTEM P1=3
PARAM P1=3
END
TITLE SAME SYSTEM P1=5
PARAM P1=5
END
TITLE SAME SYSTEM P1=7
PARAM P1=7
END
TITLE SAME SYSTEM P1=9,P3=.347,S/N=5
PARAM P1=9,P3=.347
END
TITLE SAME SYSTEM P1=11,P3=.548,S/N=2
PARAM P1=11,P3=.548
END
TITLE SAME SYSTEM P1=13,P3=.778,S/N=1
PARAM P1=13,P3=.778
END
TITLE SAME SYSTEMP P1=15,P3=1.095,S/N=1/2
PARAM P1=15,P3=1.095
END
TITLE SAME SYSTEM,P1=17,P3=1.73,S/N=1/5
PARAM P1=17,P3=1.73
END
TITLE SAME SYSTEM,P1=19,P3=2.45,S/N=1/10
PARAM P1=19,P3=2.45
END
STOP
```

SAME SYSTEM P1=5, S/N = 10

TIME	Y	FMT	MR1	MR2
0.	2.8823E-01	2.1519E 00	0.	0.
1.000E-01	3.3551E-01	1.0519E 00	1.6695E-01	-2.0775E-01
2.000E-01	2.4399E-01	4.1103E-01	2.1466E-01	-2.4933E-01
3.000E-01	1.7847E-01	1.6759E-01	2.2261E-01	-2.5358E-01
4.000E-01	2.9505E-01	3.2434E-01	2.2159E-01	-2.5340E-01
5.000E-01	-2.0401E-01	-1.8195E-02	2.3135E-01	-2.5219E-01
6.000E-01	-3.1899E-01	7.3861E-02	2.5517E-01	-2.4297E-01
7.000E-01	-2.2702E-01	3.8227E-01	2.9034E-01	-2.2108E-01
8.000E-01	-2.9881E-01	5.1205E-01	3.3890E-01	-1.8102E-01
9.000E-01	-3.0625E-01	6.7829E-01	3.9255E-01	-1.2647E-01
1.000E 00	-2.9362E-01	8.3099E-01	4.5076E-01	-5.7354E-02
1.100E 00	1.5061E-01	1.3806E 00	5.0900E-01	2.0859E-02
1.200E 00	3.6726E-01	1.6697E 00	5.6623E-01	1.0582E-01
1.300E 00	-1.9706E-01	1.1482E 00	6.1928E-01	1.9129E-01
1.400E 00	2.3942E-01	1.6017E 00	6.6850E-01	2.7637E-01
1.500E 00	-4.9559E-01	8.6211E-01	7.1315E-01	3.5819E-01
1.600E 00	6.6434E-02	1.4020E 00	7.5309E-01	4.3513E-01
1.700E 00	-3.2987E-01	9.6977E-01	7.8943E-01	5.0829E-01
1.800E 00	3.3375E-01	1.5869E 00	8.2097E-01	5.7421E-01
1.900E 00	-5.5341E-01	6.4566E-01	8.4823E-01	6.3312E-01
2.000E 00	-2.9794E-01	8.4176E-01	8.7191E-01	6.8576E-01
2.100E 00	-3.3803E-03	1.0737E 00	8.9229E-01	7.3226E-01
2.200E 00	2.3385E-03	1.0153E 00	9.0989E-01	7.7334E-01
2.300E 00	-3.6710E-01	5.8139E-01	9.2503E-01	8.0937E-01
2.400E 00	-1.2883E-01	7.5598E-01	9.3790E-01	8.4057E-01
2.500E 00	1.9275E-01	1.0154E 00	9.4841E-01	8.6645E-01
2.600E 00	-3.0659E-01	4.5613E-01	9.5750E-01	8.8915E-01
2.700E 00	2.5072E-01	9.5602E-01	9.6486E-01	9.0775E-01
2.800E 00	3.4032E-01	9.9105E-01	9.7131E-01	9.2423E-01
2.900E 00	-3.1443E-01	2.8474E-01	9.7625E-01	9.3699E-01
3.000E 00	3.8525E-02	5.8923E-01	9.8026E-01	9.4745E-01
3.100E 00	8.8271E-02	5.9363E-01	9.8400E-01	9.5727E-01
3.200E 00	-1.5705E-01	3.0604E-01	9.8673E-01	9.6459E-01
3.300E 00	-3.6930E-01	5.4506E-02	9.8895E-01	9.7042E-01
3.400E 00	-2.7669E-01	1.1074E-01	9.9080E-01	9.7537E-01
3.500E 00	-1.5205E-01	2.0175E-01	9.9278E-01	9.8071E-01
3.600E 00	3.6479E-02	3.5928E-01	9.9418E-01	9.8451E-01
3.700E 00	-1.9415E-01	1.9013E-01	9.9512E-01	9.8706E-01
3.800E 00	3.2140E-01	5.8947E-01	9.9591E-01	9.8921E-01
3.900E 00	-1.7308E-01	7.0965E-02	9.9641E-01	9.9057E-01
4.000E 00	1.9833E-02	2.4187E-01	9.9692E-01	9.9199E-01
4.100E 00	-2.5003E-01	-4.8125E-02	9.9752E-01	9.9363E-01
4.200E 00	5.7196E-01	7.5546E-01	9.9790E-01	9.9468E-01
4.300E 00	-1.0834E-01	5.8373E-02	9.9821E-01	9.9554E-01
4.400E 00	-1.6342E-02	1.3506E-01	9.9843E-01	9.9615E-01
4.500E 00	-2.0395E-02	1.1656E-01	9.9878E-01	9.9712E-01
4.600E 00	3.2605E-01	4.5072E-01	9.9882E-01	9.9723E-01
4.700E 00	-1.5744E-01	-4.4266E-02	9.9895E-01	9.9761E-01
4.800E 00	4.5295E-02	1.4795E-01	9.9909E-01	9.9799E-01
4.900E 00	-1.3178E-01	-3.8689E-02	9.9923E-01	9.9838E-01
5.000E 00	1.1365E-01	1.9894E-01	9.9933E-01	9.9866E-01

VARIABLE	MINIMUM	MAXIMUM
ORF1	9.5288E-03	1.4142E 00
ORF2	-2.0000E 00	6.6667E-01

P1=5, S/N =10

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
2.0324E-01	-8.3305E-01	-1.2077E 00	-7.9676E-01
2.1526E-01	-7.8534E-01	-1.2493E 00	-7.8474E-01
2.1218E-01	-7.7739E-01	-1.2536E 00	-7.8782E-01
2.1304E-01	-7.7841E-01	-1.2534E 00	-7.8696E-01
2.0319E-01	-7.6865E-01	-1.2522E 00	-7.9681E-01
1.7879E-01	-7.4483E-01	-1.2430E 00	-8.2121E-01
1.4627E-01	-7.0966E-01	-1.2211E 00	-8.5373E-01
1.0970E-01	-6.6110E-01	-1.1810E 00	-8.9030E-01
8.1882E-02	-6.0745E-01	-1.1265E 00	-9.1812E-01
6.7335E-02	-5.4924E-01	-1.0574E 00	-9.3266E-01
6.9885E-02	-4.9100E-01	-9.7914E-01	-9.3012E-01
9.0025E-02	-4.3377E-01	-8.9418E-01	-9.0997E-01
1.2502E-01	-3.8072E-01	-8.0871E-01	-8.7498E-01
1.7289E-01	-3.3150E-01	-7.2363E-01	-8.2711E-01
2.2968E-01	-2.8685E-01	-6.4181E-01	-7.7032E-01
2.9204E-01	-2.4691E-01	-5.6487E-01	-7.0796E-01
3.5896E-01	-2.1057E-01	-4.9171E-01	-6.4104E-01
4.2530E-01	-1.7903E-01	-4.2579E-01	-5.7470E-01
4.8949E-01	-1.5177E-01	-3.6688E-01	-5.1051E-01
5.5071E-01	-1.2809E-01	-3.1424E-01	-4.4929E-01
6.0788E-01	-1.0771E-01	-2.6774E-01	-3.9212E-01
6.6080E-01	-9.0107E-02	-2.2666E-01	-3.3920E-01
7.0916E-01	-7.4973E-02	-1.9063E-01	-2.9084E-01
7.5253E-01	-6.2099E-02	-1.5943E-01	-2.4747E-01
7.8964E-01	-5.1586E-02	-1.3355E-01	-2.1036E-01
8.2307E-01	-4.2495E-02	-1.1085E-01	-1.7693E-01
8.5111E-01	-3.5138E-02	-9.2251E-02	-1.4889E-01
8.7648E-01	-2.8691E-02	-7.5768E-02	-1.2352E-01
8.9650E-01	-2.3752E-02	-6.3009E-02	-1.0350E-01
9.1317E-01	-1.9737E-02	-5.2545E-02	-8.6827E-02
9.2905E-01	-1.6002E-02	-4.2731E-02	-7.0954E-02
9.4091E-01	-1.3269E-02	-3.5496E-02	-5.9092E-02
9.5072E-01	-1.1047E-02	-2.9576E-02	-4.9278E-02
9.5901E-01	-9.1997E-03	-2.4626E-02	-4.0992E-02
9.6803E-01	-7.2195E-03	-1.9291E-02	-3.1973E-02
9.7450E-01	-5.8160E-03	-1.5492E-02	-2.5497E-02
9.7889E-01	-4.8769E-03	-1.2938E-02	-2.1113E-02
9.8260E-01	-4.0905E-03	-1.0792E-02	-1.7404E-02
9.8497E-01	-3.5918E-03	-9.4261E-03	-1.5028E-02
9.8744E-01	-3.0782E-03	-8.0145E-03	-1.2559E-02
9.9033E-01	-2.4814E-03	-6.3705E-03	-9.6719E-03
9.9219E-01	-2.1001E-03	-5.3164E-03	-7.8106E-03
9.9371E-01	-1.7894E-03	-4.4557E-03	-6.2850E-03
9.9480E-01	-1.5699E-03	-3.8464E-03	-5.2013E-03
9.9652E-01	-1.2231E-03	-2.8826E-03	-3.4834E-03
9.9672E-01	-1.1815E-03	-2.7663E-03	-3.2751E-03
9.9741E-01	-1.0450E-03	-2.3855E-03	-2.5921E-03
9.9808E-01	-9.1077E-04	-2.0103E-03	-1.9175E-03
9.9878E-01	-7.7209E-04	-1.6223E-03	-1.2188E-03
9.9929E-01	-6.7160E-04	-1.3408E-03	-7.1069E-04

SAME SYSTEM P1=9,P3=.347,S/M=5

TIME	Y	FMT	MR1	MR2
0.	4.0790E-01	2.2716E 00	0.	0.
1.000E-01	3.3153E-02	7.4951E-01	1.6592E-01	-2.0631E-01
2.000E-01	-1.4577E-02	1.5247E-01	2.1370E-01	-2.4780E-01
3.000E-01	9.6192E-02	8.5318E-02	2.2160E-01	-2.5206E-01
4.000E-01	3.6119E-01	3.9049E-01	2.2375E-01	-2.5252E-01
5.000E-01	5.4911E-01	7.3493E-01	2.2863E-01	-2.5177E-01
6.000E-01	2.2614E-02	4.1546E-01	2.5336E-01	-2.4220E-01
7.000E-01	1.3534E-01	7.4463E-01	2.8912E-01	-2.2009E-01
8.000E-01	4.1573E-01	1.2266E 00	3.3883E-01	-1.7893E-01
9.000E-01	8.3177E-02	1.0677E 00	3.9641E-01	-1.2042E-01
1.000E 00	-4.5575E-01	6.6886E-01	4.5390E-01	-5.2149E-02
1.100E 00	2.1836E-01	1.4484E 00	5.1292E-01	2.7132E-02
1.200E 00	4.7981E-01	1.7823E 00	5.7049E-01	1.1258E-01
1.300E 00	2.4564E-01	1.5909E 00	6.2478E-01	2.0003E-01
1.400E 00	4.4618E-01	1.8084E 00	6.7509E-01	2.8683E-01
1.500E 00	1.7114E-01	1.5288E 00	7.2060E-01	3.7035E-01
1.600E 00	6.1677E-01	1.9524E 00	7.6164E-01	4.4941E-01
1.700E 00	5.1382E-01	1.8135E 00	7.9795E-01	5.2248E-01
1.800E 00	-4.2894E-01	8.2426E-01	8.3050E-01	5.9049E-01
1.900E 00	2.5912E-02	1.2250E 00	8.5847E-01	6.5094E-01
2.000E 00	-6.2149E-01	5.1821E-01	8.8188E-01	7.0298E-01
2.100E 00	-7.6138E-01	3.1572E-01	9.0194E-01	7.4877E-01
2.200E 00	-1.3662E-01	8.7630E-01	9.1919E-01	7.8904E-01
2.300E 00	-1.0329E-02	9.3916E-01	9.3385E-01	8.2393E-01
2.400E 00	1.4586E-01	1.0307E 00	9.4609E-01	8.5338E-01
2.500E 00	-2.9571E-02	7.9312E-01	9.5699E-01	8.8041E-01
2.600E 00	-5.0131E-01	2.6140E-01	9.6560E-01	9.0191E-01
2.700E 00	-4.9492E-01	2.1038E-01	9.7264E-01	9.1972E-01
2.800E 00	-3.9259E-01	2.5814E-01	9.7926E-01	9.3664E-01
2.900E 00	-1.1882E-01	4.8934E-01	9.8438E-01	9.4988E-01
3.000E 00	-4.0034E-01	1.5037E-01	9.8865E-01	9.6100E-01
3.100E 00	8.9478E-02	5.9484E-01	9.9227E-01	9.7050E-01
3.200E 00	-8.0015E-02	3.8307E-01	9.9533E-01	9.7861E-01
3.300E 00	2.8344E-01	7.0725E-01	9.9795E-01	9.8560E-01
3.400E 00	-2.1893E-01	1.6850E-01	9.9997E-01	9.9101E-01
3.500E 00	-3.0591E-01	4.7890E-02	1.0016E 00	9.9545E-01
3.600E 00	2.3259E-01	5.5539E-01	1.0030E 00	9.9917E-01
3.700E 00	-1.3933E-01	1.6494E-01	1.0044E 00	1.0030E 00
3.800E 00	-9.0142E-02	1.7793E-01	1.0054E 00	1.0058E 00
3.900E 00	-5.8120E-01	-3.3716E-01	1.0064E 00	1.0085E 00
4.000E 00	4.4470E-01	6.6673E-01	1.0072E 00	1.0107E 00
4.100E 00	2.5172E-01	4.5362E-01	1.0080E 00	1.0127E 00
4.200E 00	-4.4496E-02	1.3901E-01	1.0085E 00	1.0141E 00
4.300E 00	1.0924E-01	2.7596E-01	1.0090E 00	1.0156E 00
4.400E 00	5.7449E-01	7.2581E-01	1.0093E 00	1.0165E 00
4.500E 00	5.1550E-02	1.8900E-01	1.0096E 00	1.0174E 00
4.600E 00	2.8727E-01	4.1200E-01	1.0097E 00	1.0175E 00
4.700E 00	4.7449E-01	5.8757E-01	1.0098E 00	1.0189E 00
4.800E 00	-2.3268E-01	-1.3003E-01	1.0101E 00	1.0187E 00
4.900E 00	-1.2259E-01	-2.9509E-02	1.0102E 00	1.0189E 00
5.000E 00	3.4965E-01	4.3404E-01	1.0103E 00	1.0192E 00

VARIABLE	MINIMUM	MAXIMUM
ORF1	9.5288E-03	1.4142E 00
ORF2	-2.0000E 00	6.6667E-01

P1 = 9, S/N = 5

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
2.0156E-01	-8.3408E-01	-1.2063E 00	-7.9844E-01
2.1330E-01	-7.8630E-01	-1.2478E 00	-7.8670E-01
2.1029E-01	-7.7840E-01	-1.2521E 00	-7.8971E-01
2.0856E-01	-7.7625E-01	-1.2525E 00	-7.9144E-01
2.0362E-01	-7.7137E-01	-1.2518E 00	-7.9638E-01
1.7829E-01	-7.4664E-01	-1.2422E 00	-8.2171E-01
1.4512E-01	-7.1088E-01	-1.2201E 00	-8.5488E-01
1.0784E-01	-6.6117E-01	-1.1789E 00	-8.9216E-01
7.7927E-02	-6.0359E-01	-1.1204E 00	-9.2207E-01
6.3602E-02	-5.4610E-01	-1.0521E 00	-9.3640E-01
6.6208E-02	-4.8708E-01	-9.7287E-01	-9.3379E-01
8.6431E-02	-4.2951E-01	-8.8742E-01	-9.1357E-01
1.2218E-01	-3.7522E-01	-7.9997E-01	-8.7782E-01
1.7099E-01	-3.2500E-01	-7.1317E-01	-8.2901E-01
2.2888E-01	-2.7940E-01	-6.2965E-01	-7.7112E-01
2.9296E-01	-2.3836E-01	-5.5059E-01	-7.0704E-01
3.5975E-01	-2.0205E-01	-4.7752E-01	-6.4025E-01
4.2814E-01	-1.6950E-01	-4.0951E-01	-5.7186E-01
4.9401E-01	-1.4153E-01	-3.4906E-01	-5.0599E-01
5.5453E-01	-1.1812E-01	-2.9702E-01	-4.4547E-01
6.1084E-01	-9.8058E-02	-2.5123E-01	-3.8916E-01
6.6275E-01	-8.0807E-02	-2.1096E-01	-3.3725E-01
7.0955E-01	-6.6147E-02	-1.7607E-01	-2.9045E-01
7.5051E-01	-5.3997E-02	-1.4662E-01	-2.4949E-01
7.8925E-01	-4.3013E-02	-1.1959E-01	-2.1075E-01
8.2091E-01	-3.4403E-02	-9.8088E-02	-1.7909E-01
8.4776E-01	-2.7359E-02	-8.0278E-02	-1.5224E-01
8.7380E-01	-2.0744E-02	-6.3363E-02	-1.2620E-01
8.9458E-01	-1.5616E-02	-5.0116E-02	-1.0542E-01
9.1229E-01	-1.1351E-02	-3.9001E-02	-8.7710E-02
9.2766E-01	-7.7346E-03	-2.9499E-02	-7.2340E-02
9.4094E-01	-4.6720E-03	-2.1393E-02	-5.9058E-02
9.5253E-01	-2.0477E-03	-1.4402E-02	-4.7473E-02
9.6160E-01	-2.7537E-05	-8.9866E-03	-3.8401E-02
9.6910E-01	1.6204E-03	-4.5465E-03	-3.0899E-02
9.7543E-01	2.9918E-03	-8.3230E-04	-2.4567E-02
9.8196E-01	4.3895E-03	2.9685E-03	-1.8041E-02
9.8679E-01	5.4124E-03	5.7609E-03	-1.3214E-02
9.9160E-01	6.4234E-03	8.5299E-03	-8.4003E-03
9.9540E-01	7.2144E-03	1.0704E-02	-4.5982E-03
9.9898E-01	7.9534E-03	1.2741E-02	-1.0206E-03
1.0014E 00	8.4582E-03	1.4136E-02	1.4428E-03
1.0040E 00	8.9786E-03	1.5577E-02	3.9958E-03
1.0056E 00	9.2944E-03	1.6454E-02	5.5565E-03
1.0072E 00	9.6193E-03	1.7358E-02	7.1686E-03
1.0073E 00	9.6541E-03	1.7455E-02	7.3422E-03
1.0082E 00	9.8322E-03	1.7952E-02	8.2333E-03
1.0097E 00	1.0117E-02	1.8747E-02	9.6607E-03
1.0100E 00	1.0185E-02	1.8938E-02	1.0005E-02
1.0104E 00	1.0261E-02	1.9152E-02	1.0392E-02

SAME SYSTEM P1=11,P3=.548,S/N=2

TIME	Y	FMT	MR1	MR2
0.	6.4391E-01	2.5076E 00	0.	0.
1.000E-01	3.1674E-01	1.0331E 00	1.7173E-01	-2.1341E-01
2.000E-01	7.2904E-02	2.3995E-01	2.2278E-01	-2.5733E-01
3.000E-01	1.0978E 00	1.0869E 00	2.2760E-01	-2.6001E-01
4.000E-01	-1.6483E 00	-1.6190E 00	2.2537E-01	-2.5943E-01
5.000E-01	-1.3363E-01	5.2189E-02	2.3268E-01	-2.5807E-01
6.000E-01	5.0365E-01	8.9649E-01	2.5106E-01	-2.5084E-01
7.000E-01	4.6960E-01	1.0789E 00	2.8854E-01	-2.2748E-01
8.000E-01	-5.1276E-01	2.9810E-01	3.3722E-01	-1.8727E-01
9.000E-01	3.0634E-01	1.2909E 00	3.9052E-01	-1.3309E-01
1.000E 00	-1.9603E-01	9.2858E-01	4.4807E-01	-6.4841E-02
1.100E 00	6.7746E-01	1.9075E 00	5.0453E-01	1.0904E-02
1.200E 00	7.4659E-01	2.0491E 00	5.6097E-01	9.4693E-02
1.300E 00	-3.0548E-01	1.0397E 00	6.1623E-01	1.8373E-01
1.400E 00	7.5789E-01	2.1201E 00	6.6559E-01	2.6905E-01
1.500E 00	1.7529E-02	1.3752E 00	7.0854E-01	3.4772E-01
1.600E 00	-4.9909E-01	8.3650E-01	7.4795E-01	4.2369E-01
1.700E 00	7.4431E-01	2.0440E 00	7.8303E-01	4.9430E-01
1.800E 00	9.9243E-02	1.3484E 00	8.1564E-01	5.6243E-01
1.900E 00	-1.9050E-01	1.0086E 00	8.4375E-01	6.2314E-01
2.000E 00	-2.8155E-01	8.5814E-01	8.6679E-01	6.7438E-01
2.100E 00	4.7473E-01	1.5518E 00	8.8693E-01	7.2033E-01
2.200E 00	-2.9292E-01	7.2001E-01	9.0400E-01	7.6017E-01
2.300E 00	7.2304E-01	1.6715E 00	9.1871E-01	7.9520E-01
2.400E 00	-1.2746E-02	8.7206E-01	9.3096E-01	8.2489E-01
2.500E 00	1.8518E-01	1.0079E 00	9.4154E-01	8.5093E-01
2.600E 00	-9.3003E-02	6.6971E-01	9.5051E-01	8.7333E-01
2.700E 00	8.8702E-01	1.5923E 00	9.5776E-01	8.9154E-01
2.800E 00	-1.3289E 00	-6.7820E-01	9.6434E-01	9.0847E-01
2.900E 00	-6.0594E-01	-6.7687E-03	9.6938E-01	9.2149E-01
3.000E 00	-6.6423E-01	-1.1352E-01	9.7328E-01	9.3168E-01
3.100E 00	-9.6823E-01	-4.6287E-01	9.7735E-01	9.4236E-01
3.200E 00	-2.6624E-01	1.9665E-01	9.8025E-01	9.5002E-01
3.300E 00	4.0322E-01	8.2703E-01	9.8244E-01	9.5587E-01
3.400E 00	-1.5981E-01	2.2761E-01	9.8447E-01	9.6131E-01
3.500E 00	7.2426E-01	1.0781E 00	9.8622E-01	9.6602E-01
3.600E 00	-6.6477E-01	-3.4197E-01	9.8754E-01	9.6960E-01
3.700E 00	-6.4023E-02	2.3025E-01	9.8872E-01	9.7281E-01
3.800E 00	-4.5606E-01	-1.8798E-01	9.8983E-01	9.7585E-01
3.900E 00	2.8000E-01	5.2404E-01	9.9075E-01	9.7837E-01
4.000E 00	-8.4936E-02	1.3710E-01	9.9149E-01	9.8040E-01
4.100E 00	2.4325E-01	4.4515E-01	9.9223E-01	9.8245E-01
4.200E 00	1.9607E-01	3.7958E-01	9.9295E-01	9.8443E-01
4.300E 00	-9.4564E-01	-7.7893E-01	9.9326E-01	9.8530E-01
4.400E 00	3.5999E-01	5.1139E-01	9.9382E-01	9.8683E-01
4.500E 00	3.3567E-01	4.7312E-01	9.9405E-01	9.8749E-01
4.600E 00	7.1975E-02	1.9581E-01	9.9412E-01	9.8767E-01
4.700E 00	-5.0653E-01	-3.9346E-01	9.9431E-01	9.8822E-01
4.800E 00	9.8279E-03	1.1245E-01	9.9439E-01	9.8843E-01
4.900E 00	5.1518E-01	6.0827E-01	9.9456E-01	9.8891E-01
5.000E 00	-1.6688E-01	-6.2482E-02	9.9474E-01	9.8941E-01

VARIABLE	MINIMUM	MAXIMUM
ØRF1	9.5288E-03	1.4142E 00
ØRF2	-2.0000E 00	6.6567E-01

Pl = 11, S/V = 2

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
2.0823E-01	-8.2827E-01	-1.2134E 00	-7.9177E-01
2.1988E-01	-7.7722E-01	-1.2573E 00	-7.8012E-01
2.1817E-01	-7.7240E-01	-1.2600E 00	-7.8183E-01
2.1985E-01	-7.7463E-01	-1.2594E 00	-7.8015E-01
2.1232E-01	-7.6732E-01	-1.2581E 00	-7.8768E-01
1.9353E-01	-7.4894E-01	-1.2508E 00	-8.0647E-01
1.5890E-01	-7.1146E-01	-1.2275E 00	-8.4110E-01
1.2226E-01	-6.6278E-01	-1.1873E 00	-8.7774E-01
9.4605E-02	-6.0948E-01	-1.1331E 00	-9.0540E-01
8.0081E-02	-5.5193E-01	-1.0648E 00	-9.1992E-01
8.2409E-02	-4.9547E-01	-9.8910E-01	-9.1759E-01
1.0224E-01	-4.3903E-01	-9.0531E-01	-8.9776E-01
1.3870E-01	-3.8377E-01	-8.1627E-01	-8.6130E-01
1.8669E-01	-3.3441E-01	-7.3095E-01	-8.1331E-01
2.4121E-01	-2.9146E-01	-6.5228E-01	-7.5879E-01
3.0290E-01	-2.5205E-01	-5.7631E-01	-6.9710E-01
3.6745E-01	-2.1697E-01	-5.0570E-01	-6.3255E-01
4.3596E-01	-1.8436E-01	-4.3757E-01	-5.6404E-01
5.0204E-01	-1.5625E-01	-3.7686E-01	-4.9796E-01
5.6161E-01	-1.3321E-01	-3.2562E-01	-4.3839E-01
6.1809E-01	-1.1307E-01	-2.7967E-01	-3.8191E-01
6.6946E-01	-9.5999E-02	-2.3983E-01	-3.3054E-01
7.1648E-01	-8.1286E-02	-2.0480E-01	-2.8352E-01
7.5777E-01	-6.9037E-02	-1.7511E-01	-2.4223E-01
7.9510E-01	-5.8460E-02	-1.4907E-01	-2.0490E-01
8.2807E-01	-4.9489E-02	-1.2667E-01	-1.7193E-01
8.5570E-01	-4.2236E-02	-1.0834E-01	-1.4430E-01
8.8158E-01	-3.5662E-02	-9.1528E-02	-1.1842E-01
9.0200E-01	-3.0621E-02	-7.8505E-02	-9.7996E-02
9.1823E-01	-2.6715E-02	-6.8325E-02	-8.1772E-02
9.3551E-01	-2.2648E-02	-5.7637E-02	-6.4488E-02
9.4807E-01	-1.9755E-02	-4.9976E-02	-5.1933E-02
9.5776E-01	-1.7560E-02	-4.4129E-02	-4.2239E-02
9.6688E-01	-1.5530E-02	-3.8685E-02	-3.3117E-02
9.7483E-01	-1.3785E-02	-3.3981E-02	-2.5166E-02
9.8093E-01	-1.2463E-02	-3.0403E-02	-1.9068E-02
9.8645E-01	-1.1281E-02	-2.7189E-02	-1.3549E-02
9.9171E-01	-1.0167E-02	-2.4146E-02	-8.2900E-03
9.9609E-01	-9.2469E-03	-2.1626E-02	-3.9052E-03
9.9964E-01	-8.5099E-03	-1.9600E-02	-3.6313E-04
1.0032E 00	-7.7654E-03	-1.7548E-02	3.2420E-03
1.0067E 00	-7.0509E-03	-1.5574E-02	6.7252E-03
1.0083E 00	-6.7356E-03	-1.4699E-02	8.2769E-03
1.0110E 00	-6.1849E-03	-1.3171E-02	1.0994E-02
1.0122E 00	-5.9458E-03	-1.2506E-02	1.2180E-02
1.0125E 00	-5.8817E-03	-1.2326E-02	1.2500E-02
1.0135E 00	-5.6854E-03	-1.1778E-02	1.3484E-02
1.0139E 00	-5.6113E-03	-1.1570E-02	1.3857E-02
1.0147E 00	-5.4398E-03	-1.1090E-02	1.4722E-02
1.0156E 00	-5.2625E-03	-1.0594E-02	1.5618E-02

SAME SYSTEM P1=13,P3=.778,S/N=1

TIME	Y	FMT	MR1	MR2
0.	9.1374E-01	2.7774E 00	0.	0.
1.000E-01	-4.9892E-01	2.1743E-01	1.6429E-01	-2.0435E-01
2.000E-01	-1.2128E-02	1.5492E-01	2.1497E-01	-2.4805E-01
3.000E-01	8.5239E-01	8.4152E-01	2.1589E-01	-2.4886E-01
4.000E-01	1.0389E 00	1.0682E 00	2.2299E-01	-2.4953E-01
5.000E-01	7.7780E-01	9.6362E-01	2.3012E-01	-2.4848E-01
6.000E-01	-1.2036E-01	2.7249E-01	2.5629E-01	-2.3840E-01
7.000E-01	-1.8319E-01	4.2610E-01	2.9163E-01	-2.1654E-01
8.000E-01	2.7377E-01	1.0846E 00	3.3778E-01	-1.7823E-01
9.000E-01	7.7682E-01	1.7614E 00	3.9495E-01	-1.2029E-01
1.000E 00	5.2716E-02	1.1773E 00	4.5100E-01	-5.3456E-02
1.100E 00	3.3378E-01	1.5638E 00	5.0871E-01	2.4140E-02
1.200E 00	-3.9008E-01	9.1239E-01	5.6626E-01	1.0955E-01
1.300E 00	-1.5772E 00	-2.3196E-01	6.2004E-01	1.9629E-01
1.400E 00	-5.7037E-01	7.9185E-01	6.7064E-01	2.8363E-01
1.500E 00	4.2111E-01	1.7788E 00	7.1517E-01	3.6517E-01
1.600E 00	-6.4484E-02	1.2711E 00	7.5572E-01	4.4333E-01
1.700E 00	-4.9536E-01	8.0430E-01	7.9145E-01	5.1521E-01
1.800E 00	7.9029E-01	2.0434E 00	8.2046E-01	5.7589E-01
1.900E 00	7.3622E-01	1.9353E 00	8.4907E-01	6.3770E-01
2.000E 00	-1.0929E 00	4.6770E-02	8.7301E-01	6.9091E-01
2.100E 00	-3.4600E-01	7.3110E-01	8.9332E-01	7.3725E-01
2.200E 00	2.4562E-01	1.2585E 00	9.0980E-01	7.7573E-01
2.300E 00	-6.5488E-02	8.8300E-01	9.2429E-01	8.1024E-01
2.400E 00	-1.0353E 00	-1.5050E-01	9.3700E-01	8.4104E-01
2.500E 00	1.3671E 00	2.1898E 00	9.4739E-01	8.6662E-01
2.600E 00	-1.2241E-01	6.4030E-01	9.5599E-01	8.8808E-01
2.700E 00	1.1580E-01	8.2110E-01	9.6369E-01	9.0757E-01
2.800E 00	8.5475E-01	1.5955E 00	9.7108E-01	9.2646E-01
2.900E 00	1.0189E 00	1.6171E 00	9.7640E-01	9.4022E-01
3.000E 00	-2.8777E-01	2.6294E-01	9.8043E-01	9.5073E-01
3.100E 00	-2.8914E-01	2.1622E-01	9.8316E-01	9.5789E-01
3.200E 00	-1.7833E-01	2.8475E-01	9.8607E-01	9.6558E-01
3.300E 00	4.7075E-01	8.9456E-01	9.8804E-01	9.7086E-01
3.400E 00	1.0695E-01	4.9437E-01	9.8959E-01	9.7501E-01
3.500E 00	1.8777E 00	2.2315E 00	9.9088E-01	9.7849E-01
3.600E 00	-7.0606E-01	-3.8326E-01	9.9294E-01	9.8405E-01
3.700E 00	1.5553E-01	4.4981E-01	9.9401E-01	9.8697E-01
3.800E 00	7.8780E-01	1.0559E 00	9.9465E-01	9.8873E-01
3.900E 00	-2.4530E-01	-1.2563E-03	9.9555E-01	9.9120E-01
4.000E 00	5.8674E-01	8.0877E-01	9.9589E-01	9.9212E-01
4.100E 00	-8.1558E-01	-6.1368E-01	9.9643E-01	9.9362E-01
4.200E 00	8.1300E-01	9.9651E-01	9.9718E-01	9.9568E-01
4.300E 00	1.0786E 00	1.2454E 00	9.9773E-01	9.9721E-01
4.400E 00	3.3165E-01	4.8305E-01	9.9803E-01	9.9806E-01
4.500E 00	-4.3339E-01	-2.9594E-01	9.9804E-01	9.9807E-01
4.600E 00	1.0432E-01	2.2906E-01	9.9818E-01	9.9845E-01
4.700E 00	-7.8214E-01	-6.6897E-01	9.9826E-01	9.9870E-01
4.800E 00	1.6312E-01	2.6577E-01	9.9841E-01	9.9910E-01
4.900E 00	-8.8430E-01	-7.9991E-01	9.9866E-01	9.9981E-01
5.000E 00	1.8974E-01	2.7413E-01	9.9873E-01	1.0000E 00

VARIABLE	MINIMUM	MAXIMUM
ORF1	9.5288E-03	1.4142E 00

PI=13, S/N=1

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
1.9970E-01	-8.3571E-01	-1.2043E 00	-8.0030E-01
2.1148E-01	-7.8503E-01	-1.2480E 00	-7.8852E-01
2.1158E-01	-7.8411E-01	-1.2489E 00	-7.8842E-01
2.0519E-01	-7.7701E-01	-1.2495E 00	-7.9481E-01
1.9795E-01	-7.6988E-01	-1.2485E 00	-8.0205E-01
1.7124E-01	-7.4380E-01	-1.2384E 00	-8.2876E-01
1.3838E-01	-7.0837E-01	-1.2165E 00	-8.6162E-01
1.0386E-01	-6.6222E-01	-1.1782E 00	-8.9614E-01
7.4140E-02	-6.0515E-01	-1.1203E 00	-9.2586E-01
6.0402E-02	-5.4900E-01	-1.0535E 00	-9.3960E-01
6.3111E-02	-4.9120E-01	-9.7586E-01	-9.3689E-01
8.3271E-02	-4.3374E-01	-8.9045E-01	-9.1673E-01
1.1877E-01	-3.7996E-01	-8.0380E-01	-8.8123E-01
1.6786E-01	-3.2936E-01	-7.1637E-01	-8.3214E-01
2.2434E-01	-2.8483E-01	-6.3483E-01	-7.7566E-01
2.8771E-01	-2.4428E-01	-5.5667E-01	-7.1229E-01
3.5339E-01	-2.0855E-01	-4.8479E-01	-6.4661E-01
4.1454E-01	-1.7954E-01	-4.2411E-01	-5.8546E-01
4.8185E-01	-1.5093E-01	-3.6230E-01	-5.1815E-01
5.4369E-01	-1.2699E-01	-3.0909E-01	-4.5631E-01
6.0067E-01	-1.0668E-01	-2.6275E-01	-3.9933E-01
6.5030E-01	-9.0203E-02	-2.2427E-01	-3.4970E-01
6.9661E-01	-7.5707E-02	-1.8976E-01	-3.0339E-01
7.3940E-01	-6.2996E-02	-1.5896E-01	-2.6060E-01
7.7607E-01	-5.2608E-02	-1.3338E-01	-2.2393E-01
8.0768E-01	-4.4013E-02	-1.1192E-01	-1.9232E-01
8.3707E-01	-3.6306E-02	-9.2429E-02	-1.6293E-01
8.6615E-01	-2.8921E-02	-7.3541E-02	-1.3385E-01
8.8775E-01	-2.3597E-02	-5.9780E-02	-1.1225E-01
9.0450E-01	-1.9567E-02	-4.9274E-02	-9.5505E-02
9.1607E-01	-1.6845E-02	-4.2118E-02	-8.3927E-02
9.2869E-01	-1.3935E-02	-3.4416E-02	-7.1315E-02
9.3743E-01	-1.1956E-02	-2.9141E-02	-6.2571E-02
9.4439E-01	-1.0408E-02	-2.4990E-02	-5.5613E-02
9.5026E-01	-9.1193E-03	-2.1515E-02	-4.9740E-02
9.5976E-01	-7.0628E-03	-1.5945E-02	-4.0244E-02
9.6476E-01	-5.9909E-03	-1.3031E-02	-3.5244E-02
9.6780E-01	-5.3457E-03	-1.1268E-02	-3.2196E-02
9.7209E-01	-4.4457E-03	-8.8016E-03	-2.7906E-02
9.7372E-01	-4.1095E-03	-7.8755E-03	-2.6284E-02
9.7635E-01	-3.5664E-03	-6.3771E-03	-2.3650E-02
9.7998E-01	-2.8226E-03	-4.3215E-03	-2.0023E-02
9.8269E-01	-2.2703E-03	-2.7912E-03	-1.7312E-02
9.8419E-01	-1.9659E-03	-1.9450E-03	-1.5806E-02
9.8421E-01	-1.9628E-03	-1.9344E-03	-1.5785E-02
9.8491E-01	-1.8234E-03	-1.5455E-03	-1.5090E-02
9.8535E-01	-1.7363E-03	-1.3017E-03	-1.4653E-02
9.8607E-01	-1.5928E-03	-8.9990E-04	-1.3929E-02
9.8735E-01	-1.3392E-03	-1.9004E-04	-1.2650E-02
9.8770E-01	-1.2694E-03	5.7220E-06	-1.2297E-02

SAME SYSTEM P1=15,P3=1.095,S/N=1/2

TIME	Y	FMT	MR1	MR2
0.	1.2854E 00	3.1491E 00	0.	0.
1.000E-01	-6.3669E-01	7.9668E-02	1.6408E-01	-2.0370E-01
2.000E-01	5.6079E-01	7.2784E-01	2.0898E-01	-2.4289E-01
3.000E-01	-2.7088E-01	-2.8175E-01	2.2005E-01	-2.4832E-01
4.000E-01	-1.0273E 00	-9.9805E-01	2.1455E-01	-2.4732E-01
5.000E-01	7.8073E-01	9.6655E-01	2.2983E-01	-2.4540E-01
6.000E-01	-2.2278E-01	1.7007E-01	2.4685E-01	-2.3911E-01
7.000E-01	-7.0295E-01	-9.3663E-02	2.8061E-01	-2.1790E-01
8.000E-01	8.6686E-01	1.6777E 00	3.2577E-01	-1.8070E-01
9.000E-01	-3.7580E-01	6.0874E-01	3.7625E-01	-1.2922E-01
1.000E 00	2.7256E 00	3.8502E 00	4.3402E-01	-6.0772E-02
1.100E 00	-1.6565E 00	-4.2648E-01	4.9762E-01	2.4581E-02
1.200E 00	-7.1738E-01	5.8510E-01	5.5598E-01	1.1122E-01
1.300E 00	-1.1489E 00	1.9629E-01	6.0823E-01	1.9547E-01
1.400E 00	2.8661E-01	1.6489E 00	6.5632E-01	2.7862E-01
1.500E 00	8.3526E-01	2.1930E 00	7.0147E-01	3.6130E-01
1.600E 00	-1.3021E 00	3.3466E-02	7.4047E-01	4.3644E-01
1.700E 00	-1.0216E 00	2.7806E-01	7.7701E-01	5.0996E-01
1.800E 00	-1.6721E-01	1.0860E 00	8.0970E-01	5.7832E-01
1.900E 00	-6.6167E-02	1.1329E 00	8.3589E-01	6.3491E-01
2.000E 00	3.2698E-01	1.4667E 00	8.6049E-01	6.8962E-01
2.100E 00	1.2029E 00	2.2800E 00	8.8221E-01	7.3918E-01
2.200E 00	4.2293E-01	1.4359E 00	9.0052E-01	7.8190E-01
2.300E 00	4.7558E-01	1.4241E 00	9.1515E-01	8.1673E-01
2.400E 00	-1.2956E 00	-4.1079E-01	9.2789E-01	8.4761E-01
2.500E 00	4.3893E-01	1.2616E 00	9.3898E-01	8.7492E-01
2.600E 00	-1.9192E 00	-1.1565E 00	9.4867E-01	8.9914E-01
2.700E 00	-7.3677E-01	-3.1467E-02	9.5553E-01	9.1648E-01
2.800E 00	-3.6645E-02	6.1408E-01	9.6169E-01	9.3223E-01
2.900E 00	-5.9415E-01	5.0166E-03	9.6643E-01	9.4450E-01
3.000E 00	1.2598E 00	1.8105E 00	9.7107E-01	9.5660E-01
3.100E 00	-1.6199E 00	-1.1145E 00	9.7517E-01	9.6738E-01
3.200E 00	-4.2469E-01	3.8396E-02	9.7882E-01	9.7703E-01
3.300E 00	-2.0741E 00	-1.6503E 00	9.8105E-01	9.8299E-01
3.400E 00	-9.2060E-01	-5.3317E-01	9.8318E-01	9.8870E-01
3.500E 00	-5.4092E-01	-1.8712E-01	9.8444E-01	9.9208E-01
3.600E 00	3.4550E-01	6.6830E-01	9.8635E-01	9.9726E-01
3.700E 00	5.5136E-02	3.4941E-01	9.8713E-01	9.9938E-01
3.800E 00	2.1478E 00	2.4159E 00	9.8816E-01	1.0022E 00
3.900E 00	-1.7496E 00	-1.5056E 00	9.8898E-01	1.0044E 00
4.000E 00	-9.1035E-01	-6.8831E-01	9.8911E-01	1.0048E 00
4.100E 00	1.1514E-01	3.1705E-01	9.8966E-01	1.0063E 00
4.200E 00	-3.7928E-02	1.4558E-01	9.8967E-01	1.0064E 00
4.300E 00	6.8394E-01	8.5066E-01	9.9033E-01	1.0082E 00
4.400E 00	-2.1828E 00	-2.0314E 00	9.9053E-01	1.0087E 00
4.500E 00	2.1834E 00	2.3209E 00	9.9062E-01	1.0090E 00
4.600E 00	4.2120E-01	5.4594E-01	9.9052E-01	1.0087E 00
4.700E 00	2.2553E-01	3.3870E-01	9.9052E-01	1.0087E 00
4.800E 00	-3.7002E-01	-2.6737E-01	9.9071E-01	1.0093E 00
4.900E 00	-1.2549E-02	8.0537E-02	9.9093E-01	1.0099E 00
5.000E 00	5.8809E-01	6.7248E-01	9.9080E-01	1.0095E 00

VARIABLE	MINIMUM	MAXIMUM
ØRE1	9.5288E-03	1.4142E 00
ØRE2	[illegible]	[illegible]

P1=15, S/N=1/2

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
1.9848E-01	-8.3592E-01	-1.2037E 00	-8.0152E-01
2.0988E-01	-7.9102E-01	-1.2429E 00	-7.9012E-01
2.0491E-01	-7.7995E-01	-1.2483E 00	-7.9509E-01
2.0944E-01	-7.8545E-01	-1.2473E 00	-7.9056E-01
1.9401E-01	-7.7017E-01	-1.2454E 00	-8.0599E-01
1.7653E-01	-7.5315E-01	-1.2391E 00	-8.2347E-01
1.4544E-01	-7.1939E-01	-1.2179E 00	-8.5456E-01
1.1134E-01	-6.7423E-01	-1.1807E 00	-8.8866E-01
8.5371E-02	-6.2375E-01	-1.1292E 00	-9.1463E-01
7.0683E-02	-5.6598E-01	-1.0608E 00	-9.2932E-01
7.3339E-02	-5.0238E-01	-9.7542E-01	-9.2666E-01
9.3852E-02	-4.4402E-01	-8.8878E-01	-9.0615E-01
1.2848E-01	-3.9177E-01	-8.0453E-01	-8.7152E-01
1.7528E-01	-3.4368E-01	-7.2138E-01	-8.2472E-01
2.3254E-01	-2.9853E-01	-6.3870E-01	-7.6746E-01
2.9340E-01	-2.5953E-01	-5.6356E-01	-7.0660E-01
3.6059E-01	-2.2299E-01	-4.9004E-01	-6.3941E-01
4.2945E-01	-1.9030E-01	-4.2168E-01	-5.7055E-01
4.9110E-01	-1.6411E-01	-3.6509E-01	-5.0890E-01
5.5470E-01	-1.3951E-01	-3.1038E-01	-4.4530E-01
6.1565E-01	-1.1779E-01	-2.6082E-01	-3.8435E-01
6.7064E-01	-9.9477E-02	-2.1810E-01	-3.2936E-01
7.1743E-01	-8.4854E-02	-1.8327E-01	-2.8257E-01
7.6036E-01	-7.2112E-02	-1.5239E-01	-2.3964E-01
7.9952E-01	-6.1023E-02	-1.2508E-01	-2.0048E-01
8.3517E-01	-5.1326E-02	-1.0086E-01	-1.6483E-01
8.6131E-01	-4.4466E-02	-8.3516E-02	-1.3869E-01
8.8558E-01	-3.8312E-02	-6.7770E-02	-1.1442E-01
9.0481E-01	-3.3565E-02	-5.5503E-02	-9.5187E-02
9.2413E-01	-2.8925E-02	-4.3400E-02	-7.5874E-02
9.4159E-01	-2.4826E-02	-3.2617E-02	-5.8412E-02
9.5739E-01	-2.1181E-02	-2.2968E-02	-4.2614E-02
9.6728E-01	-1.8946E-02	-1.7007E-02	-3.2723E-02
9.7684E-01	-1.6818E-02	-1.1298E-02	-2.3155E-02
9.8256E-01	-1.5564E-02	-7.9170E-03	-1.7442E-02
9.9138E-01	-1.3654E-02	-2.7440E-03	-8.6218E-03
9.9503E-01	-1.2873E-02	-6.1684E-04	-4.9695E-03
9.9993E-01	-1.1837E-02	2.2135E-03	-7.4811E-05
1.0038E 00	-1.1025E-02	4.4390E-03	3.7982E-03
1.0044E 00	-1.0892E-02	4.8050E-03	4.4410E-03
1.0071E 00	-1.0335E-02	6.3411E-03	7.1410E-03
1.0072E 00	-1.0328E-02	6.3638E-03	7.1835E-03
1.0104E 00	-9.6717E-03	8.1810E-03	1.0404E-02
1.0114E 00	-9.4706E-03	8.7404E-03	1.1399E-02
1.0119E 00	-9.3754E-03	9.0064E-03	1.1876E-02
1.0114E 00	-9.4757E-03	8.7287E-03	1.1381E-02
1.0114E 00	-9.4770E-03	8.7262E-03	1.1378E-02
1.0123E 00	-9.2887E-03	9.2531E-03	1.2327E-02
1.0134E 00	-9.0698E-03	9.8661E-03	1.3432E-02
1.0128E 00	-9.1974E-03	9.5097E-03	1.2789E-02

SAME SYSTEM,P1=17,P3=1.73,S/N=1/5

TIME	Y	FMT	MR1	MR2
0.	2.0298E 00	3.8935E 00	0.	0.
1.000E-01	3.0420E 00	3.7584E 00	1.6543E-01	-2.0480E-01
2.000E-01	1.1794E 00	1.3465E 00	2.1347E-01	-2.4696E-01
3.000E-01	3.5107E-01	3.4019E-01	2.2422E-01	-2.5340E-01
4.000E-01	-1.4000E-01	-1.1070E-01	2.2264E-01	-2.5286E-01
5.000E-01	1.1779E 00	1.3637E 00	2.2837E-01	-2.5174E-01
6.000E-01	-1.5589E 00	-1.1660E 00	2.5169E-01	-2.4243E-01
7.000E-01	-8.1703E-01	-2.0774E-01	2.8827E-01	-2.1993E-01
8.000E-01	-1.4086E 00	-5.9769E-01	3.3252E-01	-1.8311E-01
9.000E-01	-3.5572E 00	-2.5726E 00	3.8740E-01	-1.2765E-01
1.000E 00	1.0483E 00	2.1729E 00	4.4615E-01	-5.8174E-02
1.100E 00	7.6696E-01	1.9970E 00	5.0705E-01	2.3488E-02
1.200E 00	7.2821E-01	2.0307E 00	5.6715E-01	1.1268E-01
1.300E 00	-2.0826E-01	1.1370E 00	6.1985E-01	1.9762E-01
1.400E 00	-1.5512E 00	-1.8892E-01	6.7004E-01	2.8436E-01
1.500E 00	2.6865E 00	4.0442E 00	7.1505E-01	3.6679E-01
1.600E 00	1.6225E 00	2.9581E 00	7.5509E-01	4.4402E-01
1.700E 00	3.0016E 00	4.3012E 00	7.9041E-01	5.1512E-01
1.800E 00	-5.8859E-01	6.6461E-01	8.2394E-01	5.8527E-01
1.900E 00	-3.9665E 00	-2.7675E 00	8.5118E-01	6.4412E-01
2.000E 00	9.3937E-01	2.0791E 00	8.7446E-01	6.9589E-01
2.100E 00	-8.0953E-01	2.6757E-01	8.9584E-01	7.4467E-01
2.200E 00	1.8828E 00	2.8957E 00	9.1295E-01	7.8459E-01
2.300E 00	1.5763E 00	2.5247E 00	9.2705E-01	8.1818E-01
2.400E 00	1.5356E 00	2.4204E 00	9.3915E-01	8.4751E-01
2.500E 00	-1.2164E-01	7.0105E-01	9.4861E-01	8.7083E-01
2.600E 00	4.0556E-01	1.1683E 00	9.5674E-01	8.9113E-01
2.700E 00	1.3385E 00	2.0438E 00	9.6320E-01	9.0748E-01
2.800E 00	-1.0985E 00	-4.4780E-01	9.6888E-01	9.2199E-01
2.900E 00	2.4014E 00	3.0006E 00	9.7449E-01	9.3648E-01
3.000E 00	-1.3341E 00	-7.8343E-01	9.7825E-01	9.4631E-01
3.100E 00	-8.8528E-01	-3.7992E-01	9.8227E-01	9.5687E-01
3.200E 00	-7.7593E-01	-3.1285E-01	9.8540E-01	9.6517E-01
3.300E 00	8.7143E-01	1.2952E 00	9.8815E-01	9.7248E-01
3.400E 00	4.4988E 00	4.8862E 00	9.8994E-01	9.7730E-01
3.500E 00	5.8306E-01	9.3686E-01	9.9182E-01	9.8237E-01
3.600E 00	1.6475E 00	1.9703E 00	9.9360E-01	9.8719E-01
3.700E 00	-2.7221E-01	2.2066E-02	9.9577E-01	9.9310E-01
3.800E 00	-8.7325E-01	-6.0517E-01	9.9580E-01	9.9319E-01
3.900E 00	-1.6189E 00	-1.3748E 00	9.9660E-01	9.9538E-01
4.000E 00	-1.3979E 00	-1.1759E 00	9.9773E-01	9.9849E-01
4.100E 00	-3.1827E 00	-2.9808E 00	9.9817E-01	9.9972E-01
4.200E 00	-2.0811E 00	-1.8976E 00	9.9827E-01	1.0000E 00
4.300E 00	-1.0074E 00	-8.4067E-01	9.9871E-01	1.0012E 00
4.400E 00	-1.4364E 00	-1.2850E 00	9.9868E-01	1.0011E 00
4.500E 00	1.4386E 00	1.5761E 00	9.9874E-01	1.0013E 00
4.600E 00	2.4447E 00	2.5694E 00	9.9890E-01	1.0018E 00
4.700E 00	4.0602E-03	1.1723E-01	9.9941E-01	1.0032E 00
4.800E 00	-1.5212E 00	-1.4185E 00	9.9966E-01	1.0039E 00
4.900E 00	2.8182E 00	2.9113E 00	9.9960E-01	1.0037E 00
5.000E 00	-1.4929E 00	-1.4085E 00	9.9976E-01	1.0041E 00

VARIABLE	MINIMUM	MAXIMUM
ORF1	9.5288E-03	1.4142E 00
ORF2	-2.0000E 00	6.6667E-01

PI=1.7, S/N =1/5

MR3	ERR1	ERR2	ERR3
0.	-1.0000E 00	-1.0000E 00	-1.0000E 00
1.9840E-01	-8.3456E-01	-1.2048E 00	-8.0160E-01
2.1118E-01	-7.8653E-01	-1.2470E 00	-7.8882E-01
2.0798E-01	-7.7578E-01	-1.2534E 00	-7.9202E-01
2.0909E-01	-7.7736E-01	-1.2529E 00	-7.9091E-01
2.0317E-01	-7.7163E-01	-1.2517E 00	-7.9683E-01
1.7930E-01	-7.4831E-01	-1.2424E 00	-8.2070E-01
1.4531E-01	-7.1173E-01	-1.2199E 00	-8.5469E-01
1.1229E-01	-6.6748E-01	-1.1831E 00	-8.8771E-01
8.3359E-02	-6.1260E-01	-1.1277E 00	-9.1664E-01
6.8169E-02	-5.5385E-01	-1.0582E 00	-9.3183E-01
7.0592E-02	-4.9295E-01	-9.7651E-01	-9.2941E-01
9.1634E-02	-4.3285E-01	-8.8732E-01	-9.0837E-01
1.2640E-01	-3.8015E-01	-8.0238E-01	-8.7360E-01
1.7516E-01	-3.2996E-01	-7.1564E-01	-8.2484E-01
2.3228E-01	-2.8495E-01	-6.3321E-01	-7.6772E-01
2.9502E-01	-2.4491E-01	-5.5598E-01	-7.0498E-01
3.6003E-01	-2.0959E-01	-4.8488E-01	-6.3997E-01
4.3080E-01	-1.7606E-01	-4.1473E-01	-5.6920E-01
4.9488E-01	-1.4882E-01	-3.5588E-01	-5.0512E-01
5.5507E-01	-1.2554E-01	-3.0411E-01	-4.4493E-01
6.1496E-01	-1.0416E-01	-2.5533E-01	-3.8504E-01
6.6638E-01	-8.7049E-02	-2.1541E-01	-3.3362E-01
7.1148E-01	-7.2948E-02	-1.8182E-01	-2.8852E-01
7.5227E-01	-6.0853E-02	-1.5249E-01	-2.4773E-01
7.8572E-01	-5.1386E-02	-1.2917E-01	-2.1428E-01
8.1562E-01	-4.3263E-02	-1.0887E-01	-1.8438E-01
8.4025E-01	-3.6795E-02	-9.2521E-02	-1.5975E-01
8.6258E-01	-3.1120E-02	-7.8007E-02	-1.3742E-01
8.8531E-01	-2.5514E-02	-6.3516E-02	-1.1469E-01
9.0097E-01	-2.1748E-02	-5.3692E-02	-9.9026E-02
9.1805E-01	-1.7733E-02	-4.3135E-02	-8.1946E-02
9.3164E-01	-1.4598E-02	-3.4834E-02	-6.8357E-02
9.4378E-01	-1.1855E-02	-2.7517E-02	-5.6222E-02
9.5186E-01	-1.0059E-02	-2.2696E-02	-4.8140E-02
9.6042E-01	-8.1805E-03	-1.7631E-02	-3.9580E-02
9.6864E-01	-6.4021E-03	-1.2810E-02	-3.1356E-02
9.7879E-01	-4.2320E-03	-6.9041E-03	-2.1212E-02
9.7896E-01	-4.1996E-03	-6.8095E-03	-2.1044E-02
9.8277E-01	-3.3998E-03	-4.6151E-03	-1.7229E-02
9.8819E-01	-2.2726E-03	-1.5147E-03	-1.1807E-02
9.9037E-01	-1.8255E-03	-2.7872E-04	-9.6335E-03
9.9085E-01	-1.7270E-03	-3.1218E-06	-9.1452E-03
9.9299E-01	-1.2909E-03	1.2051E-03	-7.0054E-03
9.9287E-01	-1.3193E-03	1.1296E-03	-7.1339E-03
9.9317E-01	-1.2593E-03	1.2983E-03	-6.8309E-03
9.9399E-01	-1.0957E-03	1.7565E-03	-6.0070E-03
9.9652E-01	-5.9211E-04	3.1630E-03	-3.4809E-03
9.9778E-01	-3.4172E-04	3.8637E-03	-2.2184E-03
9.9750E-01	-3.9815E-04	3.7075E-03	-2.4976E-03
9.9829E-01	-2.4255E-04	4.1443E-03	-1.7069E-03

APPENDIX B

OBTAINING THE ORTHOGONAL SET

There exist several procedures for obtaining an orthogonal or orthonormal set. The Gram-Schmidt procedure is described and used to construct a polynomial set which is orthonormal over the interval $0 \leq t \leq 1$.

If a finite or infinite linearly independent set is given as

$$(e_1, e_2, \ldots e_k, \ldots) \tag{B-1}$$

it is possible to construct an orthonormal set

$$(\phi_1, \phi_2, \ldots \phi_k, \ldots) . \tag{B-2}$$

This method yields ϕ_k as a linear combination of only the first k elements, or

$$\phi_k = \sum_{j=1}^{k} a_{kj} e_j \tag{B-3}$$

First ϕ_1 is formed as

$$\phi_1 = e_1/\{e_1, e_1\}^{1/2} \tag{B-4}$$

where

$$\{e_1, e_1\} = \int_a^b |g_1(t)|^2 dt \tag{B-5}$$

and $a \leq t \leq b$ is the orthogonality interval.

Then one forms g_2 as

$$g_2 = e_2 - \{e_2, \phi_1\}\phi_1 \tag{B-6}$$

$$\phi_2 = g_2/\{g_2, g_2\}^{1/2} \tag{B-7}$$

The general form is then

$$g_k = e_k - \sum_{i=1}^{k-1} \{e_k, \phi_i\}\phi_i \tag{B-8}$$

$$\phi_k = g_k/\{g_k, g_k\}^{\frac{1}{2}} \tag{B-9}$$

The orthonormal polynomial set is then constructed. If a set

$$\{t, t^2, \ldots , t^n, \ldots\} \tag{B-10}$$

is given, and it is linearly independent because

$$a_1 t + a_2 t^2 + \ldots + a_n t^n = 0$$

for all t if, and only if, $a_1 = a_2 = \ldots = a_n = 0$, then ϕ_1 is given by

$$\phi_1 = \frac{e}{\{e_1, e_1\}^{\frac{1}{2}}} = \frac{t}{[\int_0^1 t^2 dt]^{\frac{1}{2}}} = \sqrt{3}\, t \tag{B-12}$$

and g_2 is given by

$$g_2 = e_2 - \{e_2, \phi_1\}\phi_1 \tag{B-13}$$

$$\{e_2, \phi_1\} = \int_0^1 t^2(\sqrt{3}\, t)dt = \frac{\sqrt{3}}{4} \tag{B-14}$$

$$g_2 = t^2 - \frac{3}{4}t \tag{B-15}$$

so that

$$\phi_2 = \frac{t^2 - \frac{3}{4}t}{[\int_0^1 (t^4 - \frac{3}{2}t^3 + \frac{9}{16}t^2)dt]^{\frac{1}{2}}} = \sqrt{5}(4t^2 - 3t) \tag{B-16}$$

The scaling value which makes the set orthonormal is given by

$$K_n \sqrt{2n + 1} \quad \text{(B-17)}$$

and the sum of the coefficients of each polynomial equals one. Then one can formulate ϕ_4 and g_4 as

$$\phi_4 = K_4\sqrt{9}\, g_4 \quad \text{(B-18)}$$

$$g_4 = e_4 - \sum_{i=1}^{3} \{e_4, \phi_i\}\phi_i \quad \text{(B-19)}$$

or

$$\phi_4 = \frac{K_4\sqrt{9}}{56} (56t^4 - 105t^3 + 60t^2 - 10t) \quad \text{(B-20)}$$

so that $K_4 = 56$ and

$$\phi_4 = \sqrt{9}\, (56t^4 - 105t^3 + 60t^2 - 10t) \quad \text{(B-21)}$$

which is the correct result. Thus the resulting four polynomials are

$$\phi_1 = \sqrt{3}\, t \quad \text{(B-22)}$$

$$\phi_2 = \sqrt{5}\, (4t^2 - 3t) \quad \text{(B-23)}$$

$$\phi_3 = \sqrt{7}\, (15t^3 - 20t^2 + 6t) \quad \text{(B-24)}$$

$$\phi_4 = \sqrt{9}\, (56t^4 - 105t^3 + 60t^2 - 10t) \quad \text{(B-25)}$$

APPENDIX C

Computer Programs

This appendix contains several computer programs which greatly assist the users in evaluating the worth of any set of orthogonal functions for use as the signal set for an orthogonal multiplexing system. The programs are written in DSL-90, a digital simulation language which is a part of the IBM share library. Instruction manuals are available from the IBM Corporation. DSL-90 is a non-procedural language which is composed of subroutines which can be closely related to analog computer blocks, and thus is ideal for use in simulating many physical systems. The DSL-90 language is based on Fortran IV and is very easy to use. The notation throughout most of the programs follows that shown in Figure C-1 and therefore it is possible to follow the flow of the programs without being familiar with the language. To use these programs to assist in the evaluation of a signal set it is only necessary to change the orthogonal functions in the program to the desired set and change the system parameters. In this collection each program will be preceded by a description of the purpose of the program and how the program can be adapted to another set of functions. The printout on page 107 is the DSL-90 deck which must go before the main program deck.

DSL-90 DECK


```
$ID                                  010   05000   S031    TOM WILLIAMS
$REWIND         SYSCK1
$REWIND         SYSLB3
$EXECUTE        IBJOB
$IBJOB          MAP,FIOCS
$IEDIT          SYSLB3,SRCH
$IBLDR CKSTOP
$IBLDR CONTIN
$IBLDR FINISH
$IBLDR INTEG
$IBLDR JIGSAW
$IBLDR NAME
$IBLDR OUTIN
$IBLDR OUTPUT
$IBLDR RDWRMX
$IBLDR SCAN
$IBLDR SORT
$IBLDR STORE
$IBLDR TRANSL
$IEDIT
$DATA
$IEDIT          SYSLB3,SRCH
$IBLDR MAIN
$IBLDR CENTRL
$IEDIT

          000025
```

Program 1

Title: Periodic Function Frequency Distribution

The Real Exponential Set

This program is used to calculate the Fourier coefficients of each orthogonal function and the composite waveform. The program notation is readily identified because of the similarity to the notation used in many texts.

$$A_0 = \frac{1}{T}\int_0^T f(t)\,dt \tag{C-1}$$

$$A_n = \frac{2}{T}\int_0^T f(t)\ \cos(n\omega_0 t)\,dt \tag{C-2}$$

$$B_n = \frac{2}{T}\int_0^T f(t)\ \sin(n\omega_0 t)\,dt \tag{C-3}$$

$$C_n = \sqrt{A_n^2 + B_n^2} \tag{C-4}$$

Thus for example:

AN1 = A_n of ORF1

AN2 = A_n of ORF2

ANT = A_n of FMT=ORF1 + ORF2 + ORF3

This program yields the information necessary to determine signal bandwidth in order to compare the bandwidth with that of other signals or to use to design a channel filter. In order to use this program to calculate the bandwidth of another signal set it is necessary to change ORF1, ORF2, ORF3, and change the coefficient values and the period T on the param card.

```
TITLE PERIODIC FUNCTION FREQUENCY DISTRIBUTION
TITLE THE REAL EXPONENTIAL SET
      ORF1=C11*EXP(-TIME)
      ORF2=C21*EXP(-TIME)+C22*EXP(-2.*TIME)
      ORF3=C31*EXP(-TIME)+C32*EXP(-2.*TIME)+C33*EXP(-3.*TIME)
CONST C11=1.4142,C21=4.,C22=-6.,C31=7.3485,C32=-29.39,C33=24.5,T=5.
      FMT=ORF1+ORF2+ORF3
      SIG1=ORF1*(SIN(N*6.28*TIME/T))
      SIG2=ORF2*(SIN(N*6.28*TIME/T))
      SIG3=ORF3*(SIN(N*6.28*TIME/T))
      SFMT=FMT*(SIN(N*6.28*TIME/T))
      CIG1=ORF1*(COS(N*6.28*TIME/T))
      CIG2=ORF2*(COS(N*6.28*TIME/T))
      CIG3=ORF3*(COS(N*6.28*TIME/T))
      CFMT=FMT*(COS(N*6.28*TIME/T))
      INTS1=INTGRL(0.,SIG1)
      BN1=(2.*INTS1)/T
      INTS2=INTGRL(0.,SIG2)
      BN2=(2.*INTS2)/T
      INTS3=INTGRL(0.,SIG3)
      BN3=(2.*INTS3)/T
      INTS4=INTGRL(0.,SFMT)
      BNT=(2.*INTS4)/T
      INTC1=INTGRL(0.,CIG1)
      AN1=(2.*INTC1)/T
      INTC2=INTGRL(0.,CIG2)
      AN2=(2.*INTC2)/T
      INTC3=INTGRL(0.,CIG3)
      AN3=(2.*INTC3)/T
      INTC4=INTGRL(0.,CFMT)
      ANT=(2.*INTC4)/T
      A01=(INTGRL(0.,ORF1))/T
      A02=(INTGRL(0.,ORF2))/T
      A03=(INTGRL(0.,ORF3))/T
      A0T=(INTGRL(0.,FMT))/T
      CN1=SQRT(AN1**2.+BN1**2.)
      CN2=SQRT(AN2**2.+BN2**2.)
      CN3=SQRT(AN3**2.+BN3**2.)
      CNT=SQRT(ANT**2.+BNT**2.)
 CONTRL FINTIM=5.,DELT=.01
 PARAM N=1.
 PRINT 1.,AN1,BN1,CN1,AN2,BN2,CN2,AN3,BN3,CN3,...
      ANT,BNT,CNT,A01,A02,A03,A0T
 INTEG MILNE
 END
 TITLE N=2
 PARAM N=2.
 END
 TITLE N=3
 PARAM N=3.
 END
 TITLE N=4
 PARAM N=4.
 END
 TITLE N=5
 PARAM N=5.
```

```
END
TITLE N=6
PARAM N=6.
END
TITLE N=7
PARAM N=7.
END
TITLE N=8
PARAM N=8.
END
TITLE N=9
PARAM N=9.
END
TITLE N=10
PARAM N=10.
END
TITLE N=11
PARAM N=11.
END
TITLE N=12
PARAM N=12.
END
TITLE N=13
PARAM N=13.
END
TITLE N=14
PARAM N=14.
END
TITLE N=15
PARAM N=15.
END
STOP
1      END OF FILE

         000089
```

Program 2

Title: Periodic Function Power Spectrum Zero Initial Value Polynomials and Gaussian Product

This program is similar to the previous one in that the power spectral density is calculated for an orthogonal set. Both C and C squared are printed for several values of N. This program is written to also evaluate the effect of multiplying the orthogonal set by another function which is in this case the Gaussian function.

$$\int_0^T [\underbrace{(FMT)P(t)}_{\text{PRODUCT FUNCTION EVALUATED}}][(ORF1)P^{-1}(t)]dt \qquad (C\text{-}5)$$

The advantages are reduced peak-to-average power requirements and reduced bandwidth. The value of the Fourier coefficients for the product function are CM and CSQM, or C modified and C^2 modified. This program provides a print-out of the values of FMT and FMTM, the product or modified function. In order to use this program to evaluate other functions it is necessary to substitute the desired orthogonal functions and the improvement factor function $p(t) = y_1$ for the present set, and change the values on the param card.

```
TITLE PERIODIC FUNCTION POWER SPECTRUM
TITLE ZIV POLYNOMIALS AND GAUSSIAN PRODUCT
      ORF1=(SQRT(3.))*(TIME-K)
      ORF2=(SQRT(5.))*((4.*((TIME-K)**2.))-(3.*(TIME-K)))
      ORF3=(SQRT(7.))*((15.*((TIME-K)**3.))-(20.*...
      ((TIME-K)**2.))+(6.*(TIME-K)))
      FMT=(ORF1+ORF2+ORF3)
      FMTM=FMT*Y1
       Y1=EXP(K1*(TIME**2.))
      FUNA=FMT*(COS(6.28*N*TIME/T))
      INTA=INTGRL(0.,FUNA)
      FUNB=FMT*(SIN(6.28*N*TIME/T))
      INTB=INTGRL(0.,FUNB)
      A0=(INTGRL(0.,FMT))/T
      AN=(2.*INTA)/T
      BN=(2.*INTA)/T
      CSQRD=((AN)**2.)+((BN)**2.)
      C=SQRT(CSQRD)
      FUNAM=FMTM*(COS(6.28*N*TIME/T))
      FUNBM=FMTM*(SIN(6.28*N*TIME/T))
      A0M=(INTGRL(0.,FMTM))/T
      INTAM=INTGRL(0.,FUNAM)
      INTBM=INTGRL(0.,FUNBM)
      ANM=(2.*INTAM)/T
      BNM=(2.*INTBM)/T
      CSQM=((ANM)**2.)+((BNM)**2.)
      CM=SQRT(CSQM)
 INTEG MILNE
 PRINT .05,A0,C,CSQRD,A0M,CM,CSQM,FMT,FMTM,ORF3
 PARAM N=1.,K=0.,K1=-1.,T=1.
 CONTRLFINTIM=1.,DELT=.001
 END
 TITLE N=2
 PARAM N=2.
 END
 TITLE N=3
 PARAM N=3.
 END
 TITLE N=4
 PARAM N=4.
 END
 TITLE N=5
 PARAM N=5.
 END
 TITLE N=6
 PARAM N=6.
 END
 TITLE N=7
 PARAM N=7.
 END
 TITLE N=8
 PARAM N=8.
 END
 STOP
```

Program 3

Steady State

Polynomial Signals and Channel Filter

The purpose of this program is to determine the distortion caused by a filter inserted in the channel to restrict signal energy to a certain pass-band. The program goes through several cycles and more can easily be added using the Title, Contin, and Param cards as examples. A more complex filter can be used (for example, an n stage Butterworth, etc.) to test any design. DSL-90 contains complex functions commands which allow the evaluation of any filter design. The notation for this program is standard and it is necessary to modify the "param", orthogonal functions, and possibly the "contrl" cards to use another set of functions.

```
$IEDIT          SYSLB3,SRCH
$IBLDR MAIN
$IBLDR CENTRL
$IEDIT
TITLE STEADY STATE
TITLE POLYNOMIAL SIGNALS AND CHANNEL FILTER
TITLE RC=.05
        ORF1=(SQRT(3.))*(TIME-K)
        ORF2=(SQRT(5.))*((4.*((TIME-K)**2.))-(3.*(TIME-K)))
        ORF3=(SQRT(7.))*((15.*((TIME-K)**3.))-(20.*...
        ((TIME-K)**2.))+(6.*(TIME-K)))
        FMT=M1*ORF1+M2*ORF2+M3*ORF3
        FMST=REALPL(0.,RC,FMT)
        SIG1=FMST*ORF1
        SIG2=FMST*ORF2
        SIG3=FMST*ORF3
        INT1=INTGRL(0.,SIG1)
        INT2=INTGRL(0.,SIG2)
        INT3=INTGRL(0.,SIG3)
        FMT2=C1*FMT
        FMST2=REALPL(0.,RC,FMT2)
        SIG12=FMST2*ORF1
        SIG22=FMST2*ORF2
        SIG32=FMST2*ORF3
        INT12=INTGRL(0.,SIG12)
        INT22=INTGRL(0.,SIG22)
        INT32=INTGRL(0.,SIG32)
        MR1=INT1-INT12
        MR2=INT2-INT22
        MR3=INT3-INT32
        ERR1=MR1-M1
        ERR2=MR2-M2
        ERR3=MR3-M3
CONST M1=1.,M2=1.,M3=1.
PARAM K=0.,C1=0.,RC=.05
CONTRL FINTIM=1.,DELT=.001
INTEG MILNE
PRINT .02,ERR1,ERR2,ERR3,FMT,FMST,INT3,INT32,INT2,INT22
END
TITLE CYCLE TWO
CONTIN
CONTRL FINTIM=2.,DELT=.001
PARAM K=1.,C1=1.
END
TITLE CYCLE THREE
CONTRL FINTIM=3.,DELT=.001
CONTIN
PARAM K=2.,C1=1.
END
TITLE RC=.005, CYCLE ONE
PARAM K=0.,C1=0.,RC=.005
CONTRL FINTIM=1.,DELT=.001
END
TITLE CYCLE TWO
CONTRL FINTIM=2.,DELT=.001
```

```
CONTIN
PARAM K=1.,C1=1.
END
TITLE CYCLE THREE
CONTIN
CONTRL FINTIM=3.,DELT=.001
PARAM K=2.,C1=1.
END
TITLE CYCLE FOUR
CONTIN
CONTRL FINTIM=4.,DELT=.001
PARAM K=3.,C1=1.
END
STOP
```

Program 4

Polynomial Functions/Synch Error

This program is used to determine the distortion caused by synchronization error in the receiver of the orthomux system. Time delay (TD) is varied from 0 to .3 seconds of a 1 second normalized period. The polynomial functions and system parameters can be changed to evaluate any other set, it will also be necessary to change the bandwidth for the new set and possibly the "control" time. For the particular set under consideration the print out runs for 1.5 seconds. The receiver period is from t=TD to t=TD+1. for each cycle, so the receiver output for each cycle should be read from the print out at t=TD+1.

```
TITLE POLYNOMIAL FUNCTIONS / SYNCH ERROR / BW=1
TITLE TIME DELAY EQUAL TO ZERO
      ORF1=(SQRT(3.))*(TIME-K)
      ORF2=(SQRT(5.))*((4.*((TIME-K)**2.))-(3.*(TIME-K)))
      ORF3=(SQRT(7.))*((15.*((TIME-K)**3.))-(20.*...
      ((TIME-K)**2.))+(6.*(TIME-K)))
      ORF12=(SQRT(3.))*(TIME-TD)
      ORF22=(SQRT(5.))*((4.*((TIME-TD)**2.))-(3.*(TIME-TD)))
      ORF32=(SQRT(7.))*((15.*((TIME-TD)**3.))-(20.*...
      ((TIME-TD)**2.))+(6.*(TIME-TD)))
      EMT=M1*ORF1+M2*ORF2+M3*ORF3
      EMST=REALPL(0.,BW,EMT)
      ST1=STEP(TD)
      DEL1=ORF12*ST1
      DEL2=ORF22*ST1
      DEL3=ORF32*ST1
      SIG1=EMST*DEL1
      SIG2=EMST*DEL2
      SIG3=EMST*DEL3
      INT1=INTGRL(0.,SIG1)
      INT2=INTGRL(0.,SIG2)
      INT3=INTGRL(0.,SIG3)
      ERR1=M1-INT1
      ERR2=M2-INT2
      ERR3=M3-INT3
 INTEG MILNE
 CONTRL FINTIM=1.5,DELT=.001
 PARAM BW=.053,K=0.,TD=0.,M1=1.,M2=1.,M3=1.
 PRINT .01,ERR1,ERR2,ERR3,ORF1,DEL1,ORF2,DEL2,ORF3,DEL3
 END
 TITLE TD=.05
 PARAM TD=.05
 END
 TITLE TD=.1
 PARAM TD=.1
 END
 TITLE TD=.15
 PARAM TD=.15
 END
 TITLE TD=.2
 PARAM TD=.2
 END
 TITLE TD=.25
 PARAM TD=.25
 END
 TITLE TD=.3
 PARAM TD=.3
 END
 STOP
 '      END OF FILE

           000050
```

Program 5

Title: Distortion Due to Amplitude Limiting

This program is used to calculate the distortion caused by limiting the maximum swing of the composite function FMT (often called clipping) to specific values. In order to use this program with other functions it is necessary to change the orthogonal functions (ORF1, etc.) and the "contrl" and "Param" cards.

```
TITLE DISTORTION DUE TO AMPLITUDE LIMITING
TITLE NO LIMITING
      ORF1=(SQRT(3.))*(TIME-K)
      ORF2=(SQRT(5.))*((4.*((TIME-K)**2.))-(3.*(TIME-K)))
      ORF3=(SQRT(7.))*((15.*((TIME-K)**3.))-(20.*...
      ((TIME-K)**2.))+(6.*(TIME-K)))
      FMT=ORF1*M1+ORF2*M2+ORF3*M3
      FMST=LIMIT(P1,P2,FMT)
      SIG1=FMST*ORF1
      SIG2=FMST*ORF2
      SIG3=FMST*ORF3
      MR1=INTGRL(0.,SIG1)
      MR2=INTGRL(0.,SIG2)
      MR3=INTGRL(0.,SIG3)
      ERR1=MR1-M1
      ERR2=MR2-M2
      ERR3=MR3-M3
 INTEG MILNE
 CONTRL FINTIM=1.,DELT=.001
 PRINT .02,FMST,ERR1,ERR2,ERR3,ORF1,ORF2,ORF3,FMT
 PARAM M1=1.,M2=1.,M3=1.,P1=-100.,P2=100.,K=0.
 RANGE ORF1,ORF2,ORF3,FMT
 END
 TITLE LIMITING AT .9 OF PEAK VALUE
 PARAM P1=-5.95,P2=5.95
 END
 TITLE LIMITING AT .8 OF PEAK VALUE
 PARAM P1=-5.3,P2=5.3
 END
 TITLE LIMITING AT .7 OF PEAK VALUE
 PARAM P1=-4.66,P2=4.66
 END
 TITLE LIMITING AT .6 OF PEAK VALUE
 PARAM P1=-3.99,P2=3.99
 END
 TITLE LIMITING AT .5 OF PEAK VALUE
 PARAM P1=-3.33,P2=3.33
 END
 STOP
 *      END OF FILE

            000040
```

Program 6

Othomux System with Additive Gaussian Noise

This program is written in order to insert random numbers into the channel with a Gaussian amplitude distribution and these values are added to the signal value. As would be expected, the orthomux system is not affected much by this type interference, but it is an interesting use of the digital computer. As before, the program is changed by modifying the "param," "contrl," "const" and orthogonal function cards.

```
TITLE ORTHOMUX SYSTEM WITH ADDITIVE GUASSIAN NOISE
TITLE POLYNOMIAL SET
      ORF1=(SQRT(3.))*(TIME-K)
      ORF2=(SQRT(5.))*((4.*((TIME-K)**2.))-(3.*(TIME-K)))
      ORF3=(SQRT(7.))*((15.*((TIME-K)**3.))-(20.*...
      ((TIME-K)**2.))+(6.*(TIME-K)))
      EMT=M1*ORF1+M2*ORF2+M3*ORF3
      EMTY=EMT+Y
      Y=NORMAL(P1,P2,P3)
      SIG1=EMTY*ORF1
      MR1=INTGRL(0.,SIG1)
      SIG2=EMTY*ORF2
      MR2=INTGRL(0.,SIG2)
      SIG3=EMTY*ORF3
      MR3=INTGRL(0.,SIG3)
      ERR1=MR1-M1
      ERR2=MR2-M2
      ERR3=MR3-M3
CONTRL FINTIM=1.,DELT=.001
CONST K=0.
PARAM M1=1.,M2=1.,M3=1.,P1=1.,P2=0.,P3=.245
RANGE Y,EMT,EMTY
INTEG MILNE
PRINT .01,ERR1,ERR2,ERR3,Y,EMT,EMTY
END
TITLE S/N=1
PARAM P1=3.,P2=0.,P3=1.73
END
TITLE S/N=1/5
PARAM P1=5.,P2=0,P3=3.87
END
TITLE S/N=1/10
PARAM P1=7.,P2=0,P3=5.48
END
STOP
1      END OF FILE

          000036
```

APPENDIX D

PROPERTIES OF THE PHASE ERROR OF THE OUTPUT OF A SECOND ORDER PHASE LOCKED LOOP

In many practical situations it is desired to detect a pure sinusoidal waveform in the presence of white Gaussian noise and other interfering signals. The best known practical technique for doing this is to use a second order phase lock tracking loop. The tracking loop is a closed loop servo system whose operation on the signal is analogous to that of a narrow band-pass filter. The theory of operation of such loops is thoroughly discussed in Gardner and Kent (1967) and Viterbi (1966). The statistics of the output phase error of a loop θ_ε, where θ_ε is the difference in phase between the sinusoidal signal being observed and the loop's estimate of the signal's phase is not known exactly. Viterbi (1963) has approximated the probability density function of θ_ε by

$$p(\theta_\varepsilon) = \frac{\exp(\alpha\cos\theta_\varepsilon)}{2\pi I_0(\alpha)} \qquad -\pi<\theta_\varepsilon<+\pi \tag{D-1}$$

where α is the signal-to-noise ratio in the bandwidth of the loop and I_0 is the zeroth order modified Bessel function. Recently published results (Charles, 1966) of experimental tests have shown Viterbi's formulation to be very accurate. Viterbi (1966) has calculated the mean and variance of θ_ε however, in the study of the performance of receivers with noisy phase references, it is also necessary to know the mean and higher order moments of $\cos\theta_\varepsilon$.

The calculation of the moments of $\cos\theta_\varepsilon$ is greatly simplified if the probability density function of θ_ε is expressed in series form by making use of the following identities from the theory of Bessel functions (Gradshteyn and Rhyzik, 1965)

$$\exp(jz\cos\theta) = J_0(Z)+2\sum_{k=1}^{\infty} j^k J_k(Z)\cos k\theta \tag{D-2}$$

and

$$I_n(Z) = i^{-n}J_n(Z) \tag{D-3}$$

where $i = \sqrt{-1}$, and I_n and J_n are the ordinary and modified Bessel functions respectively of order n. Using the above identities in Equation (D-1) yields after some manipulation

$$p(\theta_\varepsilon) = \frac{1}{2\pi I_0(\alpha)} \{I_0(\alpha)+2\sum_{k=1}^{\infty} I_k(\alpha)\cos k\theta_\varepsilon\} . \tag{D-4}$$

The mean of $\cos\theta_\varepsilon$, $E\{\cos\theta_\varepsilon\}$ is

$$E\{\cos\theta_\varepsilon\} = \int_{-\pi}^{\pi} \cos\theta_\varepsilon p(\theta_\varepsilon)d\theta_\varepsilon \tag{D-5}$$

$$= \frac{1}{2\pi I_0(\alpha)} \int_{-\pi}^{\pi} \cos\theta_\varepsilon I_0(\alpha)d\theta_\varepsilon$$

$$+ \frac{1}{\pi I_0(\alpha)} \int_{-\pi}^{\pi}\sum_{k=1}^{\infty} \cos\theta_\varepsilon \cos k\theta_\varepsilon I_k(\alpha)d\theta_\varepsilon$$

which can be integrated to yield

$$E\{\cos\theta_\varepsilon\} = \frac{I_1(\alpha)}{I_0(\alpha)} . \tag{D-6}$$

The second moment of $\cos\theta_\varepsilon$ is

$$E\{\cos^2\theta_\varepsilon\} = \int_{-\pi}^{\pi} \cos^2\theta_\varepsilon p(\theta_\varepsilon)d\theta_\varepsilon \tag{D-7}$$

$$= \frac{1}{2\pi I_0(\alpha)} \int_{-\pi}^{\pi} \cos^2\theta_\varepsilon I_0(\alpha)d\theta_\varepsilon$$

$$+ \frac{1}{\pi I_0(\alpha)} \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} \cos^2\theta_\varepsilon \cos k\theta_\varepsilon I_k(\alpha)d\theta_\varepsilon$$

$$= \tfrac{1}{2} + \frac{1}{2\pi I_0(\alpha)} \int_{-\pi}^{\pi} \cos^2 2\theta_\varepsilon I_0(\alpha)d\theta_\varepsilon$$

$$= \tfrac{1}{2} \left(1+ \frac{I_2(\alpha)}{I_0(\alpha)}\right) .$$

The variance of $\cos\theta_\varepsilon$, $\sigma_{\cos\theta_\varepsilon}$, using Equations (D-6) and (D-7) is seen to be

$$\sigma_{\cos\theta_\varepsilon} = \sqrt{\tfrac{1}{2}\left(1+\frac{I_2(\alpha)}{I_0(\alpha)}\right) - \left(\frac{I_1(\alpha)}{I_0(\alpha)}\right)^2} \quad . \tag{D-7}$$

APPENDIX E

THE PROBABILITY DENSITY FUNCTION OF A PHASE SHIFT KEYED SIGNAL WITH A RANDOM PHASE IN THE PRESENCE OF WHITE GAUSSIAN NOISE

Consider a communication system where the transmitter inserts one of two equally likely waveforms $s_1(t)$ or $s_2(t)$ into a channel during the period (t_0, t_0+T). Let the channel shifts the carrier phase by an unknown amount $\tilde{\theta}$. In addition, Gaussian noise, $n(t)$, of known covariance is added to the signal. In order to determine the optimum receiver structure for this system it is necessary to compute

$$p(\vec{R}|\tilde{\theta}) \tag{E-1}$$

the probability density function of the received signal given a phase shift $\tilde{\theta}$.

According to the law of total probability, Equation (E-1) can be written as

$$p(\vec{R}|\tilde{\theta}) = p(s_1)p(\vec{R}|\tilde{\theta},s_1) + p(s_2)P(\vec{R}|\tilde{\theta},s_2) \tag{E-2}$$

where $p(s_1)$ is the probability of choosing $s_1(t)$ and $p(\vec{R}|\tilde{\theta},s_1)$ is the probability distribution of $\vec{R}$ given that $s_1(t)$ was chosen and that there is a phase shift θ. Since $\vec{s}_1$ and $\vec{s}_2$ are assumed equally likely, Equation (E-2) can be written as

$$p(\vec{R}|\tilde{\theta}) = \tfrac{1}{2} \sum_{i=1}^{2} p(\vec{R}|\tilde{\theta},s_i) \tag{E-3}$$

Assume that it is known that $s_i=s_1$, $\tilde{\theta}=\theta$ and that the receiver obtains r samples of the received signal in the period $(t_0, t_0+r\Delta t)$ where $r\Delta t=T$. Now let

$$\vec{R} = [\, R(t_0+\Delta t), R(t_0+2\Delta t), \ldots, R(t_0+r\Delta t) \,] \tag{E-4}$$

$$\vec{N} = [\, n(t_0+\Delta t), n(t_0+2\Delta t), \ldots, n(t_0+r\Delta t) \,]$$

$$\vec{S} = [\, \sqrt{2E}\sin(\omega(t_0+\Delta t)+\theta), \sqrt{2E}\sin(\omega(t_0+2\Delta t)+\theta)\ldots,$$

$$\ldots\sqrt{2E}\sin(\omega(t_0+r\Delta t)+\theta) \,] .$$

If the noise samples are assumed to be jointly Gaussian with zero mean, the joint probability density function of the noise samples can be written as

$$p(\vec{N}) = \frac{1}{(2\pi)^{\frac{k}{2}}|\Phi_{kk}|^{\frac{1}{2}}} \exp-\{\tfrac{1}{2}\vec{N}\Phi_{kk}^{-1}\vec{N}^T\} \tag{E-5}$$

where $\vec{N}^T$ denotes the transpose of the vector $\vec{N}$ and Φ_{kk} is the covariance matrix of the noise samples, that is

$$\Phi_{kk} = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1k} \\ \vdots & & \vdots \\ \Phi_{k1} & \cdots & \Phi_{kk} \end{bmatrix} \tag{E-6}$$

where $\Phi_{ij} = E\{n(t+i\Delta t)n(t+j\Delta t)\}$
and where $E\{x\}$ denotes the expected value of x. Since

$$\vec{R} = \vec{S} + \vec{N} \tag{E-7}$$

where $\vec{S}$ is known it follows that $p(\vec{R}|\vec{S})$ is jointly Gaussian with mean $\vec{S}$ and covariance Φ_{kk}. That is

$$p(\vec{R}|\vec{S}) = \frac{1}{(2\pi)^{\frac{k}{2}}|\Phi_{kk}|^{\frac{1}{2}}} \exp-\tfrac{1}{2}\{[\vec{R}-\vec{S}]\,\Phi_{kk}^{-1}[\vec{R}-\vec{S}]^T\} \tag{E-8}$$

Equation (E-8) can be rewritten as

$$p(\vec{R}|\vec{S}) = C \exp\{\vec{S}\Phi_{kk}^{-1}\ [\vec{R}=\tfrac{1}{2}\vec{S}]^T\} \tag{E-9}$$

where C is independent of θ

Let

$$\Phi_{kk}^{-1}[\vec{R}-\tfrac{1}{2}\vec{S}]^T = \vec{G}^T\Delta t \tag{E-10}$$

where

$$\vec{G} = [g(t_0+\Delta t), g(t_0+2\Delta t) \ldots g(t_0+r\Delta t)]. \tag{E-11}$$

Then

$$p(\vec{R}|\vec{S}) = C \exp \vec{S}\, \vec{G}^T \Delta t \tag{E-12}$$

or

$$p(\vec{R}|\vec{S}) = C \exp \sum_{j=1}^{k} \sqrt{2E} \sin(\omega(t_0+j\Delta t)+\theta) g(t_0+j\Delta t)\Delta t \tag{E-13}$$

and

$$R(t_0+j\Delta t) - \sqrt{E/2} \sin(\omega(t_0+j\Delta t)+\theta) = \sum_{k=1}^{r} \Phi_{jk} g(t_0+k\Delta t)\Delta t \tag{E-14}$$

If the sampling interval Δt becomes arbitrarily small and r becomes arbitrarily large, so that the samples become dense in the interval and if the limits of the summations of Equations (E-13) and (E-14) converge, then these Equations converge to

$$p(\vec{R}|\vec{S}) = C \exp \int_{t_0}^{t_0+T} \sqrt{2E} \sin(\omega t+\theta) g(t) d\Delta t \tag{E-15}$$

and

$$R(t) - \sqrt{E/2} \sin(\omega t+\theta) = \int_{t_0}^{t_0+T} \Phi(t-x) g(x) dx \quad . \tag{E-16}$$

If the noise is assumed to be white then

$$\Phi(t-x) = \delta(t-x) N_0/2 \tag{E-17}$$

where N_0 is the one sided noise spectral density of the noise. Equation (E-17) then can be written as

$$R(t) - \sqrt{E/2} \sin(\omega t+\theta) = N_0/2\; g(t) \quad . \tag{E-18}$$

Substitute Equation (E-18) into (E-15) to obtain

$$p(\vec{R}|\vec{S}) = C \exp 2/N_0 \int_{t_0}^{t_0+T} |R(t)-\sqrt{E/2}\,\sin(\omega t+\theta)|\sqrt{2E}\,\sin(\omega t+\theta)dt \quad \text{(E-19)}$$

which can be rewritten as

$$p(\vec{R}|\vec{S}) = C \exp\{2/N_0 \int_{t_0}^{t_0+T} \sqrt{2E}\,R(t)\sin(\omega t+\theta)dt \quad \text{(E-20)}$$
$$- 2E/N_0 \int_{t_0}^{t_0+T} \sin^2(\omega t+\theta)dt\} \quad .$$

The last term of Equation (E-20) may be integrated to yield

$$\int_{t_0}^{t_0+T} \sin^2(\omega t+\theta)dt = T/2 - \frac{\sin 2(\omega(t_0+T)+\theta)-\sin 2(\omega t_0+\theta)}{\omega} \quad \text{(E-21)}$$

If $\omega >> 1$ the last term of Equation (E-21) is independent of θ and Equation (E-20) may be written as

$$p(\vec{R}|\vec{S}) = C^1 \exp \frac{2\sqrt{2E}}{N_0} \int_{t_0}^{t_0+T} R(t)\sin(\omega t+\theta)dt \quad \text{(E-22)}$$
$$= p(\vec{R}|\vec{\theta},S_1)$$

where C^1 is independent of θ.
Likewise

$$p(\vec{R}|\tilde{\theta},\vec{S}_2) = C^1 \exp -\frac{2\sqrt{2E}}{N_0} \int_{t_0}^{t_0+T} R(t)\sin(\omega t+\theta)dt\} \quad . \quad \text{(E-23)}$$

Substituting Equations (E-22) and (E-23) into Equation (E-2) yields

$$p(\vec{R}|\tilde{\theta}) = C^{11} \cosh \frac{2\sqrt{2E}}{N_0} \int_{t_0}^{t_0+T} R(t)\sin(\omega t+\theta)dt \quad \text{(E-24)}$$

where C^{11} is independent of θ.