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A SIMPLIFIED STATISTICAL DESCRIPTION OF TURBULENT CHEMICALLY REACTING FLOWS (COUETTE FLOW)

Paul M. Chung

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ABSTRACT

A simplified turbulence model has been constructed which describes the statistical behavior of the fluid elements which comprise the larger eddies, and which contain the chemically reactive species. The theory was then employed, as a primary test, to analyze the turbulent Couette flow of a chemically inert fluid. In the analysis, it was shown that the present theory as applied to a chemically inert fluid is self-containing up to the dissipation function. An experimental value of dissipation function for a pipe flow, for lack of the same for a Couette flow, was adapted and used in the analysis. A comparison of the present theoretical results with the available experimental data on Couette flow showed a satisfactory agreement between the two. A simple relationship has been derived between the dissipation function and the other variables of the theory which would render the present theory completely self-containing for the chemically inert single-component flow.

I. INTRODUCTION

Turbulence is one of the classical subjects which have been studied for years by many people. Most of the existing works can be divided into two broad categories. The first category is comprised of those studies which analyzed the structure and dynamics of various idealized turbulence fields. The idealizations were necessary because of the extremely complicated nature of the subject. The major contribution of these studies was to give a better description, and therefore a better understanding, of the phenomenon of turbulence itself rather than to supply the engineers with the solutions of various turbulent flow problems of practical interest. There are voluminous publications belonging to this category.*

In contrast to this, the other category of works expended its major effort on deriving various relationships which can be employed to quantitatively study the real turbulent flow problems of engineering interest. By necessity, these works had to heavily rely upon empiricism and phenomenological arguments guided by experimental data. Because of the empiricism, the usefulness of these theories is severely limited. As it is with the first category, there is an abundance of publications belonging to this phenomenological category.

The problem of "turbulent wake observables" perhaps has done more to bring these two categories of works together than any other engineering problem of recent years. When it was first noticed that the radar return from an under-densed plasma comprising a turbulent wake was a function of the mean-square-fluctuation of the electron concentration, it became

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*Due to lack of space, these publications are not listed herein.

clear that the conventional phenomenological theories were no longer adequate even if one were to accept the inherent empiricism. It became inevitable that at least certain aspects of the statistical approach be brought into the analysis of real flow problems such as the problem of the turbulent wake. The works of Proudian and Feldman,[1] Lin,[2,3] Sutton,[4] and Gibson[5] represent the earlier endeavor to incorporate portions of the statistical concept into the turbulent wake analysis in a tractable manner. We shall be discussing the analyses of Lin[2,3] and Sutton[4] along with a few other pertinent works in the next section.

The turbulent wake is in fact only one out of the class of chemically reacting turbulent shear flow problems that defies the solution either by the conventional phenomenological approaches or by the classical statistical turbulence theories. The problems of chemically reacting turbulent boundary layers and turbulent combustion are a few of the problems also belonging to this class.

In the present paper, a tractable statistical model of chemically reacting incompressible turbulent shear flows is formulated. Particular effort is made to eliminate the shortcomings of the conventional phenomenological theories pertaining to the turbulent fluctuation, turbulent transport, and the chemical reaction. The theory is then, as a primary test, applied to a chemically inert turbulent Couette flow. The result is compared with available experimental data. In view of the complexities of the problem, a considerable sacrifice of statistical rigor is made in favor of tractability.

II. PERTINENT PREVIOUS WORKS

There have recently developed several simplified theories since the

earlier works of references 1 through 5 mentioned in section I which attempted, as does the present theory, to incorporate various statistical concepts into the real shear flow problems. The works of Harlow and Nakayama,[6] and Nee and Kovasznay[7] are the typical ones. The approaches of all these analyses[1-7] are basically different from the present one.* Moreover, most of these works are not intended for chemically reacting flows. However, all these works, including the present one, have one general purpose in common: that is to eliminate, as much as possible, the empiricism involved in the conventional phenomenological theories by the use of a certain statistical concept. We shall in this section present a very brief description of the analyses of Lin,[2,3] Sutton,[4] Harlow and Nakayama,[6] and Nee and Kovasznay.[7]

Lin[2,3] likened the turbulent transport unto the mixing of two classes of fluid elements possessing two diametrically different dissipation characteristics. With this assumption, Lin formulated a set of tractable governing equations for a turbulent wake and solved it. Lin made a valuable contribution in that he was one of the first to pose the possibility of incorporating the statistical concepts into a real flow problem. The major shortcoming of this work[2,3] is that the relationship between the governing equations he developed and those rigorously derived from the Navier-Stokes equation cannot be made clear. Hence, the degree of overall approximation involved cannot be assessed.

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*The main difference between these analyses and the present study will be discussed in the next section under Turbulence Model.

Sutton[4] essentially followed the rather classical approaches previously employed by such as Deissler.[8,9] Here, one first considers that the mean profiles are all given a priori. Then one applies the closure to the classical turbulence equations at the lowest order by discarding all correlation terms that are of higher order than the second order. In addition to these, Sutton assumed that the turbulent transport is expressible by the Boussinesq relationship[10] which says that it is equal to certain turbulent diffusivity times the mean gradient. Furthermore, the dissipation function was also assumed to be known a priori. The equations are then self-containing and amenable to a solution.

Harlow and Nakayama[6] made the classical Reynolds stress equation determinate by introducing several transport flux approximations. These flux approximations were, like Sutton's[4] work, basically patterned after the Boussinesq form. In the approximate expressions, the dissipation scale became an important variable. A transport equation for this dissipation scale was formulated, mainly from a physical argument, and was analyzed along with the Reynold's stress equation.

Nee and Kovasznay[7] formulated a transport equation for a total viscosity again from a physical argument similar to that employed by Harlow and Nakayama.[6] The total viscosity was defined as the quantity which when multiplied by the local mean velocity gradient yields the total stress due to both molecular and eddy viscosities. This equation, along with the classical turbulent momentum equation, was then analyzed.

III. FORMULATION OF THEORY

General Concept

The similarity between the behavior of molecules according to the

nonequilibrium kinetic theory and that of fluid elements in turbulence fields has been recognized from the days of Prandtl.[10] Although there exist certain differences between the kinetic theory and the statistical turbulence theory as was pointed out by Taylor,[10] one can exploit the various fundamental similarities underlying the two statistical phenomena, especially in view of the substantial progress that has been made in kinetic theory during the past decade.

Most of the classical statistical analyses of turbulence begin with the various moment equations derived from the Navier-Stokes equation. Each order averaged quantity such as the Reynold's stress, the triple correlation function, and etc., appearing in these moment equations must then be considered as an independent unknown variable. The equations are, hence, intractable for real flow problems unless one is willing to discard certain terms and replace certain other terms by known relationships argued from physical grounds as Sutton[4] has done.

In order to circumvent this difficulty, we shall develop a governing equation for the lowest order distribution function of the random fluid elements which can be analyzed for the real flow problems. Formulation of turbulence equations in terms of a distribution function is not new in itself, and such formulations derived from the Navier-Stokes equation[11;12] are just as intractable as the more familiar form of turbulence equations.

In the present study, we shall first divorce ourselves from Navier-Stokes equation like Lin[2,3] has done. We shall then show that certain similarities between the Brownian motion and turbulence can be exploited in such a manner that a modified form of Fokker-Planck equation can be derived to describe the salient characteristics of the turbulent chem-

ically reacting flows. The consistency of this equation in view of the rigorous statistical turbulence equations is then argued by obtaining various moment equations from this equation, and comparing them with the corresponding order moment equations derived from the Navier-Stokes equation.

For more detailed description of the concept and the equations to be derived below, the reader is referred to a previous report[13] written by this author.

Turbulence Model

One basic factor that is common in all the previous works mentioned in section II is that the turbulent transport phenomena are described in analogy to the molecular transport in a continuum laminar flow.* Thus, all important diffusions were assumed to take place at the rates proportional to the local gradients of the average quantities.

Most of the transport phenomena, however, associated with a turbulent motion are due to the larger eddies with long preferred memories. The transport of various properties by these eddies is somewhat analogous to the kinetic-theory description of the molecular transport in a rarefied gas flow where the transport rate is not necessarily determined by the local mean gradient. In addition to the fact that the larger eddies are mainly responsible for the transport, the fluctuation comprising such as the turbulence energy is mostly due to these same eddies. In the present study, it is assumed that the turbulent fluctuation and transport are

*The only exception to this perhaps is Lin's work.[2,3]

entirely determined by the fluid elements which comprise the larger eddies. The roles of the smaller eddies are considered to be those of degenerating the long preferred memories of the larger eddies, and of causing the dissipation of the turbulence energy through the interactions with the larger eddies.

No clear-cut distinction, of course, can be made between the larger and the smaller eddies. However, it is known[10,14] that there exists a statistical seperation between the larger eddies with long preferred memories and the smaller random eddies across a sort of inertial subrange, when the Reynolds number is sufficiently high. Confining our analysis to the cases of high Reynolds numbers, we first assume that the interaction between the various eddies can be represented by the interaction between the larger eddies with long characteristic times and the smaller random eddies with short characteristic times.

The above-described model of interaction is then analogous to the interaction between the giant molecules with long characteristic times and memories and the small equilibrium molecules with short characteristic times in the classical Brownian motion. We, therefore, express the change of the momentum of a fluid element which contains a transferable property, such as a chemical species, and is a part of a larger eddy by the following Langevin's stochastic equation.[15,16]

$$\frac{\partial u_i}{\partial t} = - \beta(x_j)(u_i - u_{oi}) + A_i(t) + K_i(x_j, u_j) \qquad (1)^*$$

In the present description of turbulence, the first two terms on the

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*We employ the Cartesian tensor notation. All the symbols are defined in the Nomenclature.

right-hand side of Eq. (1) represent the decay rate of the preferred momentum of the fluid element, which is a part of the larger eddy, due to the momentum interaction between the larger and the smaller eddies. The momentum decay rate is, in analogy to the Brownian motion, divided into the systematic portion, $\beta(u_i - u_{oi})$, and the rapidly fluctuating portion of the smaller eddies, $A_i(t)$. $1/\beta(x_j)$ is the characteristic relaxation time of the larger eddies due to the momentum interaction.

The last term K_i is interpreted in the present turbulence model as the force experienced by the fluid element due to the molecular viscosity and the average pressure gradient. The rate at which the momentum of the fluid element changes because of the molecular viscosity is assumed to be given by the value of $\nu \frac{\partial^2 u_j}{\partial x_m \partial x_m}$ observed in that fluid element. The quantity $\nu \frac{\partial^2 u_j}{\partial x_m \partial x_m}$ is considered herein as one of the properties of the fluid element. Thus, we write,

$$K_i = \nu \frac{\partial^2 u_i}{\partial x_m \partial x_m} - \frac{1}{\rho} \frac{\partial p_o}{\partial x_i} \tag{2}$$

Distribution Function

We define a distribution function $f(x_i, u_i, t)$ such that $f(x_i, u_i, t)\ \vec{dx}\ \vec{du}$ is the occupancy probability of the fluid element in the phase cell $\vec{dx}\ \vec{du}$.* Next, we define $n(x_i, u_i, t)$ as the concentration of a transferrable quantity, such as a chemically reactive specie, per unit mass of the fluid element containing this quantity. We assume in the present analysis that the fluid is incompressible and that the transferrable quantity is a passive scalar quantity, which means that its effect on the turbulence itself is negligible.

With the above definitions of f and n, we now define the α-th order

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*$\vec{dx}\ \vec{du} = dx_1 dx_2 dx_3 du_1 du_2 du_3$.

distribution function of the transferrable quantity as,

$$F^{(\alpha)}\ (x_i,u_i,t) = n^{\alpha}(x_i,u_i,t)\ f(x_i,u_i,t) \tag{3}$$

where α is a constant which is equal to or greater than zero. The probability average of α power of n, $\langle n^{\alpha} \rangle$, is then,

$$\langle n^{\alpha} \rangle = \int F^{(\alpha)}\ \overrightarrow{du} \tag{4}$$

which, with a suitable ergodic argument for a steady state system[14,17] (stationary random system with respect to time), is the same as the time average of n^{α}.

Transition Probability

We have previously exploited the analogy between the present turbulence model and the Brownian motion. Brownian motion is a Markov process whose general treatment can be found elsewhere.[15,16] The portions of the development which can be found in these references are slighted in the following analysis.

Let Δt be a time interval which lies between the two characteristic times of the larger and the smaller eddies. Such time interval exists because the statistical separation of the larger and the smaller eddies has been considered to exist. We then let $\tau(x_j - u_j\Delta t,\ u_j - \Delta u_j;\ \Delta u_j)$ be the transition probability of the fluid element containing the chemical specie between the two phase points $(x_j - u_j\Delta t,\ u_j - \Delta u_j)$ and $(x_j,\ u_j)$ connected by the path of the fluid element. This transition probability is now determined from the stochastic solution of Eq. (1) to satisfy the boundary condition that, as $1/\beta \to 0$, $F^{(1)}$ must approach Maxwellian at the average turbulence energy $1/3\ \langle U_k U_k \rangle$. In a steady state flow system,

$1/\beta \to 0$ implies that the characteristic turbulence decay time is much shorter than other characteristic times of the system.

In the classical Brownian motion, the boundary condition applied to the Langevin equation is that the Brownian particles should approach Maxwellian at the temperature (average energy) of the small particles. In the present study, wherein the turbulence energy is contained in the larger eddies only, the boundary condition of approaching Maxwellian at the average turbulence energy implies the following. The effect of the momentum interaction between the larger and the smaller eddies represented by the first two terms on the right-hand side of Eq. (1) is to degenerate the preferred memories of the fluid elements comprising the larger eddies toward the random state, at the constant average energy of these fluid elements. The last term of Eq. (1) then contains the dissipation of the momentum out of these fluid elements by molecular viscosity.

The stochastic solution of Eq. (1) is obtained in a manner similar to that obtained in reference 15, though the present boundary condition is a bit different from that employed there, as

$$\tau(u_i;\ \Delta u_i) = \frac{1}{\left(\frac{4}{3}\pi \langle U_k U_k \rangle \beta \Delta t\right)^{3/2}} \exp$$

$$- \left\{ \left[\Delta u_i + (\beta u_i - K_i)\Delta t\right]^2 \Big/ \left(\frac{4}{3} \langle U_k U_k \rangle \beta \Delta t\right) \right\} \qquad (5)$$

Governing Equation for $F^{(1)}$

The governing equation for the distribution function $F^{(1)}$ is formulated herein, which is the Fokker-Planck equation adapted and modified for the present purpose. The governing equation for $f(= F^{(0)})$ is then

readily obtained as the degenerate case of the equation for $F^{(1)}$ by setting n = 1.

Since we are considering only one chemically reactive specie mixed with an incompressible chemically inert fluid, the only appropriate chemical reaction is that given below,

$$\left(\frac{\partial n}{\partial t}\right)_{chem} = \gamma\, n^{\alpha} \tag{6}$$

where γ is a rate constant which may be either positive or negative.

We assume that the molecular diffusion of n into and out of the fluid element takes place in accordance with the value of $D\,\frac{\partial^2 n}{\partial x_m \partial x_m}$ observed in that fluid element.

The governing integral equation for $F^{(1)}$ is now constructed as,

$$F^{(1)}(x_i,u_i,t+\Delta t) = \int_{\infty}^{\infty}\Big[F^{(1)}(x_i - u_i\Delta t,\ u_i - \Delta u_i,t)$$

$$+ \gamma F^{(\alpha)}(x_i - u_i\Delta t,u_i - \Delta u_i,t)\cdot\Delta t$$

$$+ D\left(\frac{\partial^2 n}{\partial x_m \partial x_m}\right)_{x_i-u_i\Delta t,u_i-\Delta u_i,t}\cdot f(x_i - u_i\Delta t,u_i - \Delta u_i,t)\cdot\Delta t\Big]$$

$$\tau(x_i - u_i\Delta t,\ u_i - \Delta u_i;\ \Delta u_i)\ d(\vec{\Delta u}) \tag{7}$$

In the above equation the left-hand side is, of course, the probability of finding the chemical specie at the phase point (x_i,u_i) at time $t + \Delta t$. The quantity within the square bracket on the right represents the probability of finding the specie at the neighboring phase point at time t, plus the probabilities that the specie concentration contained in the fluid element will be changed by the chemical reaction and the molecular

diffusion during the time interval Δt which the fluid element takes for the transition between the two phase points. On the right-hand side of Eq. (7), then, the above-mentioned quantity is multiplied by the transition probability, and is integrated over all the neighboring points accessible to the point (x_i, u_i) during the time Δt.

Eq. (7) is transformed into a differential equation by expanding the various functions in Taylor series and by discarding the terms of order $(\Delta t)^2$ and higher. Then after a considerable manipulation (see reference 13) there results, with the use of Eqs. (2) and (5),

$$u_j \frac{\partial F^{(1)}}{\partial x_j} + \frac{\partial}{\partial u_j}\left[F^{(1)}\left(\nu \frac{\partial^2 u_i}{\partial x_m \partial x_m} - \frac{1}{\rho}\frac{\partial p_o}{\partial x_i}\right)\right]$$

$$= \beta\left[\frac{\partial}{\partial U_j}\left(F^{(1)} U_j\right) + \frac{\langle U_k U_k\rangle}{3}\frac{\partial^2 F^{(1)}}{\partial U_k \partial U_k}\right] + D f \frac{\partial^2 n}{\partial x_m \partial x_m} + \gamma F^{(\alpha)} \qquad (8)$$

The above equation is the fundamental equation for the present purpose which governs the distribution function $F^{(1)}$.

It is readily seen that as $1/\beta \to 0$, the above equation gives the following limiting solution which was to be expected from the boundary condition applied on Eq. (1).

$$F^{(1)}(x_i, u_i) = \frac{n(x_i)}{\left[\frac{2}{3}\pi \langle U_k U_k\rangle\right]^{3/2}} \exp - \left(\frac{U_k U_k}{\frac{2}{3}\langle U_k U_k\rangle}\right) \qquad (9)$$

It can be shown,[13] in a manner similar to the Chapman-Enskog expansion in kinetic theory,[18] that as $1/\beta$ is slightly increased from zero and as $F^{(1)}$ is perturbed slightly from the Maxwellian distribution of Eq. (9), the transport rates of momentum and chemical specie become proportional

to the mean velocity and concentration gradients, respectively. In this near-equilibrium (near-isotropic) limit, therefore, the Boussinesq relationship is valid. In a typical shear flow, however, $1/\beta$ is not sufficiently small to ensure the validity of the Boussinesq relationship.

Eq. (8) is not determinate because the quantities $\frac{\partial^2 u_i}{\partial x_m \partial x_m}$ and $\frac{\partial^2 n}{\partial x_m \partial x_m}$ are not known. Since these quantities have been assumed to be properties of the fluid elements, either these must be considered as additional independent variables for the distribution function or one must be able to relate them to other properties which have been already described.

In the subsequent analysis of chemically inert Couette flow, Eq. (8) will be analyzed by an approximate method which utilizes the moment equations derived from that equation. Out of those terms appearing in the moment equations which are generated from the term $\frac{\partial^2 u_i}{\partial x_m \partial x_m}$, only one term is important and is unknown.* It is the dissipation function. We shall analyze the problem first by employing the experimentally measured value of the dissipation function. Later, a theoretical approximation of the dissipation function will be given in the light of the solution obtained.

Moment Equations

The generalized moment equation will be first derived from Eq. (8). Several particular order moment equations will then be deduced from the generalized moment equation. The reason of deriving these moment equations is twofold. The first is that the particular order moment equations derived will be compared with the corresponding moment equations classically derived from the Navier-Stokes equation. From this comparison, we

*In that problem, $n = 1$ and $\partial^2 n/\partial x_m \partial x_m = 0$.

shall be able to estalish a certain consistency of the present model in the light of the rigorous classical approach. Secondly, as we mentioned in the preceding subsection, we shall be utilizing these moment equations for the solution of the Couette flow.

We let $Q(x_i,u_i)$ be a generalized property function, and multiply Eq. (8) through by this function. Then, after some manipulation similar to that employed in deriving the Enskog's generalized moment equation from the Boltzmann equation,[18] we obtain:

$$\frac{\partial}{\partial x_k}\int (u_{ok}+U_k)Q\,F^{(1)}d\vec{U}-\int (u_{ok}+U_k)\,\frac{\partial Q}{\partial x_k}\,d\vec{U}$$

$$+\int\frac{\partial Q}{\partial U_m}\left[(u_{ok}+U_k)\,\frac{\partial u_{om}}{\partial x_k}\right]F^{(1)}d\vec{U}$$

$$-\int\frac{\partial Q}{\partial U_k}\left[\nu\,\frac{\partial^2(u_{ok}+U_k)}{\partial x_m\partial x_m}-\frac{1}{\rho}\,\frac{\partial p_o}{\partial x_k}\right]F^{(1)}d\vec{U}$$

$$=-\beta\left[\int U_k\,\frac{\partial Q}{\partial U_k}\,F^{(1)}d\vec{U}+\frac{\langle U_kU_k\rangle}{3}\int\frac{\partial Q}{\partial U_k}\,\frac{\partial F^{(1)}}{\partial U_k}\,d\vec{U}\right]$$

$$+\gamma\int F^{(\alpha)}\,Q\,d\vec{U}+D\int f\,Q\,\frac{\partial^2 n}{\partial x_m\partial x_m}\,d\vec{U} \qquad (10)$$

Now, the above equation can be first reduced to the generalized moment equation for $F^{(0)}=f$ by setting $n=1$. Then, as we substitute $Q=1$ and $Q=u_i$ successively in Eq. (10), there result the continuity and the momentum equations which are identical to the corresponding equations derived from the Navier-Stokes equation. Next, we derive the following stress tensor equation by substituting $Q=(u_{oi}+U_i)(u_{oj}+U_j)$ in Eq. (10).

$$\frac{\partial}{\partial x_k}\left[u_{oi}u_{oj}u_{ok} + u_{oj}\langle U_iU_k\rangle + u_{oi}\langle U_jU_k\rangle + u_{ok}\langle U_iU_j\rangle + \langle U_iU_jU_k\rangle\right]$$

$$- \nu \int f\left[(u_{oj} + U_j)\frac{\partial(u_{oi} + U_i)}{\partial x_m \partial x_m} + (u_{oi} + U_i)\frac{\partial(u_{oj} + U_j)}{\partial x_m \partial x_m}\right] d\vec{U}$$

$$+ \frac{1}{\rho}\int f\left[(u_{oj} + U_j)\frac{\partial p_o}{\partial x_i} + (u_{oi} + U_i)\frac{\partial p_o}{\partial x_j}\right] d\vec{U}$$

$$= -\beta\left\{2\langle U_iU_j\rangle + \frac{\langle U_kU_k\rangle}{3}\int\left[(u_{oj} + U_j)\frac{\partial f}{\partial U_i} + (u_{oi} + U_i)\frac{\partial f}{\partial U_j}\right] d\vec{U}\right\} \qquad (11)$$

The above equation can be put into a more convenient form in the following manner. We first return to the momentum equation derived from Eq. (10) by setting $Q = u_i$, which is identical to the momentum equation derived from Navier-Stokes equation and is not shown here. We multiply that equation through by u_{oj}. We then rewrite the momentum equation in terms of u_{oj} instead of u_{oi} and multiply it through by u_{oi}. Adding these two equations and subtracting the resulting equation from Eq. (11) gives the following equation.

$$\langle U_iU_k\rangle \frac{\partial u_{oj}}{\partial x_k} + \langle U_jU_k\rangle \frac{\partial u_{oi}}{\partial x_k} + u_{ok}\frac{\partial}{\partial x_k}\langle U_iU_j\rangle$$

$$= -\frac{\partial}{\partial x_k}\langle U_iU_jU_k\rangle - 2\beta\left[\langle U_iU_j\rangle - \frac{1}{3}\delta_{ij}\langle U_kU_k\rangle\right]$$

$$+ \nu\left[\frac{\partial^2\langle U_iU_j\rangle}{\partial x_m \partial x_m} - 2\left\langle\frac{\partial U_i}{\partial x_m}\frac{\partial U_j}{\partial x_m}\right\rangle\right] \qquad (12)^*$$

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*Here we assumed that the present probability average is the same as the time average and, hence, we replaced $\left\langle\frac{\partial^2 U_iU_j}{\partial x_m \partial x_m}\right\rangle$ by $\frac{\partial^2\langle U_iU_j\rangle}{\partial x_m \partial x_m}$.

The turbulence energy equation is derived by a contraction of Eq. (12) as,

$$u_{ok} \frac{\partial}{\partial x_k} \langle U_i U_i \rangle + 2 \langle U_i U_k \rangle \frac{\partial u_{oi}}{\partial x_k} = - \frac{\partial}{\partial x_k} \langle U_k U_i U_i \rangle + \nu \left[\frac{\partial^2 \langle U_i U_i \rangle}{\partial x_m \partial x_m} - 2 \left\langle \frac{\partial U_i}{\partial x_m} \frac{\partial U_i}{\partial x_m} \right\rangle \right] \tag{13}$$

Similar moment equations for the chemical specie n can be derived from Eq. (10). However, we shall include them in a future paper for lack of space here.

It has been already stated that the continuity and momentum equations obtained from Eq. (10) by substituting $Q = 1$ and $Q = u_i$, respectively, are identical to the corresponding equations derived from the Navier-Stokes equation. A comparison of Eq. (12) with the corresponding stress equation derived from the Navier-Stokes equation* shows that the two are identical if,

$$2\beta \left[\langle U_i U_j \rangle - \frac{1}{3} \delta_{ij} \langle U_k U_k \rangle \right] = \frac{1}{\rho} \left[\frac{\partial}{\partial x_i} \langle p U_j \rangle + \frac{\partial}{\partial x_j} \langle p U_i \rangle \right] - \frac{1}{\rho} \left\langle p \left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \right) \right\rangle \tag{14}$$

A contraction of Eq. (14) gives,

$$\frac{\partial}{\partial x_k} \langle p U_k \rangle = 0 \tag{15}$$

Implications of Eqs. (14) and (15) are as follows. First, the present theory neglects the first term on the right-hand side of Eq. (14) as compared to the second term; and secondly, the present equation when compared to the classical stress equation says that,

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*See, for instance, Eq. (4-3) of reference 10.

$$2\beta\ [\langle U_i U_j\rangle - \frac{1}{3}\,\delta_{ij}\ \langle U_k U_k\rangle] = -\frac{1}{\rho}\left\langle p\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right)\right\rangle \tag{16}$$

The approximation of Eq. (15) probably is caused by the assumption of constant β made in the Langevin's equation, Eq. (1). We shall rectify this approximation partially by letting β be a function of $\langle U_k U_k\rangle$ in the subsequent analysis.

Eq. (16) is identical to the relationship between the stress and the pressure-velocity-gradient correlation function derived by Rotta[10] if we let,

$$\beta = \frac{\Lambda}{2\Lambda}\ \langle U_k U_k\rangle \tag{17}$$

Furthermore, the expression on the left-hand side of Eq. (16) was derived from the present turbulence model to describe the decay of the preferred memories of the larger eddies toward the randomized state. Reference 10, at the same time, gives a rather detailed physical explanation showing that the right-hand side of Eq. (16) is the term which describes precisely the same decay of the preferred memories of the eddies in the classical turbulence stress equation. Hence Eq. (16) is essentially a physical identity.

Now, we have shown that the present moment equation, Eq. (10), is consistent with the classically derived moment equations, at least up through the stress equations, except for the approximation that it neglects the first term as compared to the second term on the right-hand side of Eq. (14).

Finally, we shall, in the following, employ the value of β as defined by Eq. (17). It is because, aside from the fact that Eq. (17) makes Eq.

(16) identical to the Rotta's[10] expression, the decay rate of the larger eddies, ß, is well known[10,14] to be expressible by an expression such as Eq. (17).

This concludes the formulation of the general theory. We shall now apply the theory, as a first test, to the simplest flow configuration of Couette flow.

IV. COUETTE FLOW OF A CHEMICALLY INERT FLUID

Formulation

We consider a turbulent Couette flow of a chemically inert fluid, as shown in Figure 1.

Because of the reason given in the sentences following Eq. (9), we shall analyze Eq. (8) for this problem by a moment method that has been previously employed in solving the Boltzmann equation for highly non-equilibrium (kinetic theory-wise) rarefied gas flows or shock structures. Out of the numerous methods available, we shall employ the moment method first used by Liu and Lees[19] in the solution of a rarefied Couette flow. It is recognized at the outset that this is not the best suited one for the present turbulent flow problem. This method is chosen, however, because of its simplicity. Since the detailed discussion of the method is given in reference 19, we shall in the following apply the method to the present problem without elaboration.

We first approximate the distribution function f by the following two-stream half Maxwellian functions.

$$f = f_1 + f_2 \tag{18}$$

where

$$f_1 = \frac{1}{(\frac{2}{3}\pi E_1)^{3/2}} \exp - \left[\frac{(u - u_{o1})^2 + V^2 + W^2}{2E_1/3}\right], \quad \text{for } V > 0 \qquad (19)$$

$$f_2 = \frac{1}{(\frac{2}{3}\pi E_2)^{3/2}} \exp - \left[\frac{(u - u_{o2})^2 + V^2 + W^2}{2E_2/3}\right], \quad \text{for } V < 0 \qquad (20)$$

and f_1 and f_2 are zero for $V < 0$ and $V > 0$, respectively. In Eqs. (19) and (20), E_1, E_2, u_{o1}, and u_{o2} are the unknown functions of y to be determined.

With the above distribution function, the salient mean quantities are defined in terms of the functions E_1, E_2, u_{o1}, and u_{o2} as,

$$u_o = \langle u \rangle = \int_{-\infty}^{\infty} dU \int_{-\infty}^{\infty} dV \int_{0}^{\infty} dW\, f_1\, u + \int_{-\infty}^{\infty} dU \int_{-\infty}^{0} dV \int_{-\infty}^{\infty} dW\, f_2\, u = (u_{o1} + u_{o2})/2 \qquad (21)$$

$$\langle UV \rangle = \frac{1}{(6\pi)^{\frac{1}{2}}} \left[- \left[\frac{u_{o1} + u_{o2}}{2}\right] (E_1^{\frac{1}{2}} - E_2^{\frac{1}{2}}) + (u_{o1} E_1^{\frac{1}{2}} - u_{o2} E_2^{\frac{1}{2}}) \right] \qquad (22)$$

$$\langle U_k U_k \rangle = \left[\frac{u_{o1} - u_{o2}}{2}\right]^2 + \left[\frac{1}{2} - \frac{1}{6\pi}\right](E_1 + E_2) + \frac{1}{3\pi}(E_1 E_2)^{\frac{1}{2}} \qquad (23)$$

We now employ the first four moment equations derived in the preceding section from the generalized moment equation, Eq. (10). These are the continuity equation, the momentum equation, and Eqs. (12) and (13).

These equations are first simplified by recognizing the fact that $v_o = w_o = \frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$ for Couette flow. The various average quantities of the equations are then evaluated in terms of E_1, E_2, u_{oi}, and u_{o2} by the use of Eqs. (18) through (20). The resulting continuity equation is immediately integrated with the boundary condition of $v = 0$ at the wall to give,

$$E_1 = E_2 = E \qquad (24)$$

The remaining three equations are now manipulated and nondimensionalized to give,

$$\frac{1}{Re}\frac{d^2\phi}{dY^2} = \left(\frac{2}{3\pi}\right)^{\frac{1}{2}}\frac{d(\psi\theta)}{dY} \tag{25}$$

$$\frac{1}{Re}\left(\frac{6}{\pi}\right)^{\frac{1}{2}}\frac{d^2(\psi\theta)}{dY^2} = 2\left(\frac{6}{\pi}\right)^{\frac{1}{2}}\left(\frac{\psi^2}{4}+\theta^2\right)^{\frac{1}{2}}(\psi\theta) + \theta^2\frac{d\phi}{dY} \tag{26}$$

$$\frac{1}{Re}\left(\frac{3\pi}{8}\right)^{\frac{1}{2}}\frac{d^2(\psi'^2+4\theta^2)}{dY^2} = (\psi\theta)\frac{d\phi}{dY} + 2(6\pi)^{\frac{1}{2}}\frac{L}{u_{oc}^3}\varepsilon' \tag{27}$$

where

$$\phi = (u_{o1}+u_{o2})/u_{oc}\ , \qquad \psi = (u_{o1}-u_{o2})/u_{oc}\ ,$$

$$\theta = E^{\frac{1}{2}}/u_{oc}\ , \qquad Y = y/L\ , \qquad Re = u_{oc}L/\nu\ , \tag{28}$$

$$\varepsilon' = \nu\left\langle\frac{\partial U_k}{\partial x_m}\frac{\partial U_k}{\partial x_m}\right\rangle$$

In deriving (26), it was assumed that $\nu\left\langle\frac{\partial U}{\partial x_m}\frac{\partial V}{\partial x_m}\right\rangle$ << $\beta\langle UV\rangle$. This assumption implies that the effect of molecular viscosity in degenerating the preferred memories of the larger eddies is negligible as compared to that of the momentum interaction between the eddies.

Also, in Eq. (26), the expression of Eq. (17) was employed for β with $A = 1$ and $\Lambda = \frac{1}{2}L$. This particular value of Λ was chosen because of the following reason. The quantity Λ is the characteristic scale of the larger eddies which is probably of order y measured from the wall reaching the maximum value of L at the midpoint between the two plates. For simplicity, it was decided to set Λ equal to an average value of L/2 for the entire flow region.

The only unspecified quantity in Eqs. (25) through (27) is the dis-

sipation function* ε' appearing in Eq. (27). As it has been already stated, following Eq. (9), we shall employ an experimentally measured value of ε' for a pipe flow in the solution of Eqs. (25) through (27). At the end of the present paper, however, we shall advance an approximate theory which predicts the dissipation function.

The dissipation function measured by Laufer for a pipe flow given in reference 10 is used because no published dissipation function is available for a Couette flow, and also because the pipe flow is perhaps more similar to Couette flow than any other flow configuration. The following modification, however, was first made on the value of the dissipation function measured in the pipe flow so that it will more closely describe the dissipation in the Couette flow.

In the Couette flow, the mean velocity at the midpoint u_{oc} is the average mean velocity of the flow field between the two plates. The mean velocity at the center of the pipe flow, on the other hand, is the maximum velocity but is not the average mean. We assumed that the dissipation in the Couette flow normalized with u_{oc} and L is equal to that in the pipe flow normalized with the average mean velocity, u_{ave}, and D/2. Relationships were needed between the friction velocity, maximum velocity, and the average mean velocity of the pipe flow in order to translate the dissipation data of Laufer. These relationships are found in reference 20, and the eighth power mean velocity profile for the pipe flow was employed in utilizing these relationships.

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*To be precise, ε' is not exactly equal to dissipation. It is, however, sufficiently close to it, and ε' is usually measured to represent the dissipation.[10]

According to reference 10, out of the two Reynolds numbers at which the measurement of the dissipation function was made, the most reliable measurement is that which was made at the Reynolds number, based on the maximum velocity and the pipe radius, of 2.5×10^4. At the same time, the most complete experimental values of mean velocity profile for Couette flow are presented in reference 21 at $Re = 10^4$, with which we would compare the present theoretical results subsequently. It was decided, therefore, to solve the governing equations, Eqs. (25) through (27), for $Re = 10^4$, with an interpolated pipe dissipation value.

In accordance with the discussions presented in the preceding two paragraphs, we obtained the following expression for the dissipation function for Couette flow at $Re = 10^4$.

$$\frac{\varepsilon' L}{u_{oc}^3} = \begin{cases} \dfrac{.0004}{Y}, & \text{for } 0 < Y \leq 1 \\ \dfrac{.0004}{2 - Y}, & \text{for } 1 \leq Y < 2 \end{cases} \tag{29}$$

The above expression was chosen because it is simple, yet it approximates the experimental value closely for all Y except for $Y \to 0$ (and $Y \to 2$) and $Y = 1$. On a physical ground, it is obvious that ε' should decrease near the wall, and should become zero at the wall. Also, at $Y = 1$, $d\varepsilon'/dY$ must be zero from symmetry condition. The fact that Eq. (29) fails to describe the dissipation correctly at these isolated regions is not critical in the present analysis, and these points will be discussed further subsequently.

The boundary conditions for Eqs. (25), (26), and (27) are as follows.

At $Y = 0$,

$$\phi = \psi\theta = \psi^2 = 0 \tag{30}$$

and at Y = 2,

$$\phi = 4 , \quad \psi 0 = \psi^2 = 0 \tag{31}$$

An inspection of Eqs. (25) through (27) shows that these equations exhibit a singular behavior at large values of Re. We shall therefore exploit this singular behavior and solve the governing equations by the method* of "inner and outer" expansions in the following.

Manipulation of Equations

The relative orders of magnitude of the various terms comprising the governing equations can be made clear through the following manipulations.

Eq. (25) is first integrated once. The resulting equation is then combined with Eqs. (26) and (27) to give,

$$\frac{1}{Re}\left[\frac{d^3\phi}{dY^3} - \frac{\pi}{2}\frac{d\phi}{dY}\left(\frac{d\phi}{dY} - C_1\right)^2/\psi^2\right]$$

$$- 2\left[\frac{\psi^2}{2} + \frac{3\pi}{2}\left(\frac{1}{Re}\right)^2\left(\frac{d\phi}{dY} - C_1\right)^2/\psi^2\right|^{\frac{1}{2}}\left(\frac{d\phi}{dY} - C_1\right) = 0 \tag{32}$$

$$\frac{1}{Re}\frac{d^2}{dY^2}\left[\psi^2 + 6\pi\left(\frac{1}{Re}\right)^2\left(\frac{d\phi}{dY} - C_1\right)^2/\psi^2\right] - 2\left(\frac{1}{Re}\right)\frac{d\phi}{dY}\left(\frac{d\phi}{dY} - C_1\right)$$

$$- 8\left(\frac{\varepsilon' L}{u_{oc}^3}\right) = 0 \tag{33}$$

where C_1 is the constant of integration of Eq. (25). We have now reduced Eqs. (25) through (27) to the two equations, Eqs. (32) and (33). The coupled Eqs. (32) and (33) govern ϕ and ψ^2. θ is then given by Eq. (25). Eq. (25) converts the boundary condition $\psi\theta = 0$ of Eqs. (30) and (31) into the following condition at Y = 0 and Y = 2.

$$\left(\frac{d\phi}{dY}\right)_w = C_1 \tag{34}$$

*See reference (22) for the general method.

It is known from experimental data[21,23] on surface skin friction that $(d\phi/dY)_w$ is of much greater order of magnitude than one, and, also, it is related to Reynolds number. We therefore first let

$$C_1 = \left(\frac{d\phi}{dY}\right)_w = C_2/(1/Re)^q \tag{35}$$

where C_2 is a constant of order one. We also let

$$\psi^2 = (1/Re)^m \Psi \tag{36}$$

with Ψ considered as order one. The exponents m and q, which are positive, will be determined subsequently.

Eqs. (35) and (36) transform Eqs. (32) and (33), along with Eq. (25), to,

$$\left(\frac{1}{Re}\right)^{m+2n} \frac{d^3\phi}{dY^3} - \frac{\pi}{2}\left[\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY} - C_2\right]^2 \frac{d\phi}{dY}/\Psi$$

$$- 2\left(\frac{1}{Re}\right)^{\frac{3m}{2}+n-1}\left[\frac{\pi}{4} + \frac{3\pi}{2}\left(\frac{1}{Re}\right)^{2(1-m-n)} C_2^2/\Psi\right]^{\frac{1}{2}}\left[\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY} - C_2\right] = 0 \tag{37}$$

$$\left(\frac{1}{Re}\right)^{m+n} \frac{d^2}{dY^2}\left\{\Psi + 6\pi\left(\frac{1}{Re}\right)^{2(1-m-n)}\left[\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY} - C_2\right]^2/\Psi\right\}$$

$$- 2\frac{d\phi}{dY}\left[\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY} - C_2\right] - 8\left(\frac{1}{Re}\right)^{n-1}\left(\frac{\varepsilon' L}{u_{oc}^3}\right) = 0 \tag{38}$$

$$\Psi^{\frac{1}{2}}\theta = \left(\frac{3\pi}{2}\right)^{\frac{1}{2}}\left(\frac{1}{Re}\right)^{1-\frac{m}{2}-n}\left[\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY} - C_2\right] \tag{39}$$

Boundary conditions for the above equations are those given by Eqs. (30), (31), and (34).

The coefficients of the highest order differential terms of Eqs. (37) through (39) are very small. This indicates that there exists an "inner"

region (see references 22 and 24) near the wall. These highest order terms describe the laminar transport, and this "inner" region, only within which the laminar transport is important, would correspond to the conventional "laminar sublayer." The method of the "inner and outer" expansions[22,24] (singular perturbation) is well suited for solution of such a problem. It is pointed out here at the outset, however, that Eq. (38) with $\varepsilon' L/u_{oc}^3$ given by Eq. (29) is valid for $Re = 10^4$ only. Hence, the full use of the method, resulting for instance in the universal values for m and q, is not possible here since such analysis requires the functional relationship between the dissipation function and the Reynolds number.

In the following analysis by the singular perturbation, Re is simply a large number equal to 10^4, and the values of m and q are chosen so as only to render the salient terms of the inner and outer regions order one for $Re = 10^4$.

Outer Solution; Fully Turbulent Region

After a study of various terms of Eqs. (37) through (39), it was decided to let $m = 1/4$, and $q = 1/2$. We shall, in the outer region, consistently neglect the terms of order $(1/Re)^{1/2}$ and higher. Up to $O(1/Re)^{1/2}$, Eqs. (37) through (39) become,

$$\frac{\pi}{2}\frac{C_2}{\Psi}\frac{d\phi}{dY} = 2(10)^{1/2}\left[\frac{\Psi}{4} + \frac{3\pi}{2}(10)^{-2}C_2^2/\Psi\right]^{1/2} \tag{40}$$

$$\frac{d\phi}{dY} = \begin{cases} \left(\frac{.33}{2C_2}\right)\frac{1}{Y} & \text{for } 0 < Y \leq 1 \\ \left(\frac{.33}{2C_2}\right)\frac{1}{(2-Y)}, & \text{for } 1 \leq Y < 2 \end{cases} \tag{41}$$

$$\Psi^{1/2}\theta = -\left(\frac{3\pi}{2}\right)^{1/2}(10)^{-3/2}C_2 \tag{42}$$

From the symmetry which exists between $Y < 1$ and $Y > 1$, the two boundary conditions on ϕ given in Eqs. (30) and (31) imply that $\phi(1) = 2$. Eq. (41) is now integrated and after applying the boundary condition $\phi(1) = 2$, there results,

$$\phi = \begin{cases} \dfrac{0.33}{2C_1} \ln Y + 2, & \text{for } Y \leq 1 \\ \dfrac{0.33}{2C_2} \ln (2 - Y) + 2, & \text{for } Y \geq 1 \end{cases} \tag{43}$$

Eq. (41) is then substituted into Eq. (40) for $d\phi/dY$ and there results for Ψ,

$$\begin{aligned} &\Psi^3 + 6\pi(10^{-2})\, C_2^2\, \Psi - \frac{(0.33\pi)^2}{160} \frac{1}{Y^2} = 0 && \text{for } Y \leq 1 \\ &\Psi^3 + 6\pi(10^{-2})\, C_2^2\, \Psi - \frac{(0.33\pi)^2}{160} \frac{1}{(2 - Y)^2} = 0, && \text{for } Y \geq 1 \end{aligned} \tag{44}$$

Standard solution for linear cubic equations is of course available, and, therefore, Ψ described by Eqs. (44) is considered to have been obtained. With ϕ and Ψ known, Eq. (42) immediately gives 0.

Eqs. (43) and (44) show that ϕ is antisymmetric and Ψ is symmetric about $Y = 1$, as was expected. In addition to this, we should expect that $d^2\phi/dY^2 = d\Psi/dY = 0$ at $Y = 1$ on physical grounds, which are not satisfied by Eqs. (43) and (44). One can readily see, however, that this discrepancy is precisely due to the fact that we approximated the dissipation function by Eq. (29) which, though symmetric, does not give the correct physical behavior of $d(\varepsilon' L/u_{oc}^3)/dY = 0$ at $Y = 1$. This point was explained earlier following Eq. (29). However, the values of $d^2\phi/dY^2$ and $d\Psi/dY$ at $Y = 1$ given by Eqs. (43) and (44) are negligibly small.

The outer solution has now been obtained up to the unknown constant

C_2 . This constant which represents the surface shear (see Eq. (35)) is determined through a matching with the inner solution which is to be described below.

Inner Solution; Laminar Viscous Region

It was mentioned following Eq. (29) that this equation approximating the dissipation should fail near the wall. In fact, the dissipation should decrease toward zero through the inner region. In the inner region, the laminar viscous phenomena predominate over the turbulent phenomena including the dissipation. It is assumed in the following that the dissipation term whose correct value is not known is of smaller order of magnitude than other terms of the governing equations in the inner region.

Now for near Y = 0, we stretch the inner region by defining an inner independent variable by,

$$\eta = Y/(1/Re)^{\frac{1}{2}} \tag{45}$$

Eqs. (37) and (38) become up to $O(1/Re)^{\frac{1}{2}}$, with the use of Eq. (45), as

$$\frac{d^3\phi}{d\eta^3} - 10^{-\frac{1}{2}}\,\psi^{\frac{1}{2}}\left(\frac{d\phi}{d\eta} - C_2\right) - 5\pi\,\frac{d\phi}{d\eta}\left(\frac{d\phi}{d\eta} - C_2\right)^2/\psi = 0 \tag{46}$$

$$\frac{d^2\psi}{d\eta^2} = 20\,\frac{d\phi}{d\eta}\left(\frac{d\phi}{d\eta} - C_2\right) \tag{47}$$

Eq. (39) is unchanged except that $\left(\frac{1}{Re}\right)^n \frac{d\phi}{dY}$ becomes $\frac{d\phi}{d\eta}$.

Boundary conditions for the above equations at the wall are those given by Eqs. (30) and (35). The other set of boundary conditions are that ϕ and Ψ should asymptotically match to those given by the outer solutions, Eqs. (43) and (44), respectively, as $\eta \to \infty$.

Remembering that C_2 appears both in the outer solutions and in the inner equations, the particular outer and inner solutions satisfying all the boundary conditions are obtained in the following manner. The outer solutions and Eqs. (46) and (47) are first made explicit by guessing a value of C_2. Eqs. (46) and (47) are then integrated by the use of a computer to satisfy Eqs. (30) and the asymptotic conditions at $\eta \to \infty$. $(d\phi/d\eta)_w$ derived from the integration is then compared with the guessed value of C_2 to see if Eq. (35) is satisfied. If. Eq. (35) is not satisfied, a new value of C_2 is guessed and the above-mentioned process is repeated. This iteration on C_2 finally resulted in a matching set of solutions. Near Y = 2, the inner region can be stretched, solved, and matched to the outer solutions, Eqs. (43) and (44), in the same manner as we have done for near Y = 0.

Couette Flow Results

From the solutions of Eqs. (25) through (27) described in the preceding subsections, the profiles of the mean velocity, Reynolds stress, and the turbulence energy have been computed by the use of Eqs. (21) through (23). These quantities are plotted in Figure 2. The stretched inner profile (laminar sublayer) is shown on Figure 3.

The available experimental data for Couette flow are summarized in a paper by Robertson.[21] All the published measurements were on the mean velocity profile and the surface shear, and no experimental data are available for the Reynolds stress and the turbulence energy.

Figure 2 shows that the velocity profile obtained in the present analysis compares very satisfactorily with the experimental values. The coefficient of skin friction, C_f, was computed from the present solution

for Re = 10^4 as .498. This value is within a percent or two of that measured in the experiment of Robertson.[21] Since the approximate dissipation function deduced from the experimental data of pipe flow was employed in the present analysis, this close agreement of C_f with that of Robertson is rather fortuitous.

V. APPROXIMATION OF DISSIPATION FUNCTION

The dynamics of turbulence leading to dissipation, and the relationship between the turbulence energy spectrum and the dissipation, have been studied quite thoroughly for various idealized turbulence fields (see, for instance, references 10 and 14). These, however, do not immediately produce the quantitative dissipation functions needed in shear flow analyses such as the present one.

We shall, in this section, formulate a simple relationship between the dissipation function, more precisely the ε', and the other physical variables of the present theory. The formulation is based on a rather crude physical model. However, it will be shown that this relationship approximates the dissipations in the Couette, pipe, and the boundary layer flows quite closely.

We consider that the vorticities responsible for dissipation are confined within the sheets, whose thickness is of the order of the Taylor microscale λ_T, surrounding the larger eddies. Such a general model has been proposed previously[25] in connection with certain other turbulence studies. These sheets are being stretched by the non-uniform mean velocities in shear flows. If we consider that the average scale of the larger eddies is proportional to the distance y measured from a wall where

$u_{ow} = 0$, then the average velocity with which these sheets are being stretched is the local mean velocity u_o. Now, we assume that these sheets are analogous to the laminar free-shear-layers[26] formed by the stretching velocity u_o and the characteristic length y. Then the thickness* of the shear layer, which is now assumed to be equivalent to λ_T, is given by[20,26]

$$\lambda_T = B\left(\frac{\nu y}{u_o}\right)^{1/2} \tag{50}$$

where $4 \lesssim B \lesssim 8$. Then we write after reference 25,

$$\varepsilon' = \nu \frac{\langle U_k U_k \rangle}{\lambda_T^2} = \frac{\langle U_k U_k \rangle u_o}{B^2 y} \tag{51}$$

The dissipation functions computed by the above equation are compared with the experimental values in Figure 4.

First, we see that Eq. (51) with B = 5 gives ε' which agrees very closely with the experimental value employed for the Couette flow. In computing ε' by Eq. (51) for the Couette flow, the values of $\langle U_k U_k \rangle$ and u_o obtained ~~the~~ in the present analysis were utilized. Hence, if we had used Eq. (51), instead of the modified experimental value of pipe flow, for ε' in Eqs. (25) through (27), we would have obtained essentially the same solutions shown on Figures 2 and 3.

The dissipation function for the boundary layer flow, without mean pressure gradient, and the pipe flow are also computed by Eq. (51), and are compared with the corresponding experimental values given in reference 10. For these computations, B was chosen as 7, and the experimental

*A suitable thickness such as the displacement thickness of the laminar shear layer.

values of $\langle U_k U_k \rangle$ and u_o are used. The agreements between Eq. (51) and the experimental data are again satisfactory.

VI. CONCLUDING REMARKS

A simplified turbulence model has been constructed which describes the statistical behavior of the fluid elements that comprise the larger eddies, and which contain the chemically reactive species. A modified Fokker-Planck equation has been derived and adapted to govern the distribution function of the fluid elements and, hence, of the chemical species. First, four order moment equations were derived from the present theory, and they were shown to be identical up to a term to those derived from the Navier-Stokes equation, thus establishing the basic consistency of the present theory.

The thoery was then employed, as a primary test, to analyze the turbulent Couette flow of a chemically inert fluid. In the analysis, it was shown that the present theory as applied to a chemically inert fluid is self-containing up to the dissipation function. An experimental value of dissipation function for a pipe flow, for lack of the same for a Couette flow, was adapted and used in the analysis. A comparison of the present theoretical results with the available experimental data on Couette flow showed a satisfactory agreement between the two.

A simple relationship has been derived between the dissipation function and the other variables of the theory which would render the present theory completely self-containing for the chemically inert single-component flow. Though this relationship is based on a rather crude physical model, it was shown that this relationship predicts the dissipation for the Couette, boundary layer, and the pipe flows quite satisfactorily.

Basically, the present theory concentrated on describing the behavior of the larger eddies as affecting the turbulent fluctuation and transport phenomena. The effect of the smaller eddies, aside from their role in degenerating the larger eddies through the Brownian type interactions, was slighted. For instance, the dissipation which is due to these smaller eddies was not described by the theory itself but was considered to be given in the original development. We have demonstrated, however, through the comparison of the moment equations and the successful solution of the Couette flow, that the present theory can be applied to shear flow problems in a tractable manner to produce various important statistical quantities. Also, we have shown that the roles of the smaller eddies, such as the dissipation, which we had slighted, can be estimated without excessive difficulty, because these basically depend on the local properties.

The present theory should be improved by describing the behavior of the smaller eddies simultaneously with the description of the larger eddies.

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NOMENCLATURE

A	constant of order one
C_1, C_2	constants of integration defined by Eqs. (34) and (35)
D	diffusion coefficient or diameter of pipe
E	function defined by Eq. (24)
E_1, E_2	functions defining f_1 and f_2, respectively
$F^{(\alpha)}$	α-th order distribution function defined by Eq. (2)
f	distribution function of fluid elements
f_1, f_2	functions defined by Eqs. (19) and (20), respectively
K_i	function defined by Eq. (2)
L	half width of Couette flow
n	mass fraction of a chemical specie
P	pressure
Q	property of fluid elements
Re	Reynolds number, $u_{oc}L/\nu$
t	time
U_i	velocity vector relative to average velocity
u_i	absolute velocity vector
U	relative velocity in x-direction
u	absolute velocity in x-direction
u_{o1}, u_{o2}	functions defining f_1 and f_2, respectively
V, W	relative velocities in y- and z-directions, respectively
v, w	absolute velocities in y- and z-directions, respectively
x, y, z	directions shown on Figure 1
x_i	position vector
Y	y/L (y is measured from a wall in Couette flow)

$1/\beta$	characteristic decay time of larger eddies
γ	chemical reaction rate constant
δ	boundary layer thickness
δ_{ij}	Kronecker delta
ϵ'	function defined by Eqs. (28)
η	inner variable defined by Eq. (45)
θ	function defined by Eqs. (28)
Λ	characteristic scale of larger eddies
ν	kinematic viscosity
τ	transition probability defined by Eq. (5)
ϕ	function defined by Eqs. (28)
Ψ,ψ	functions defined by Eqs. (36) and (28), respectively
$<Q>$	$\int f\ Q\ d\vec{u}$

Subscripts

c	midpoint between two plates in Couette flow
i,j,k,m	Cartesian tensor indices
o	averaged value
w	wall

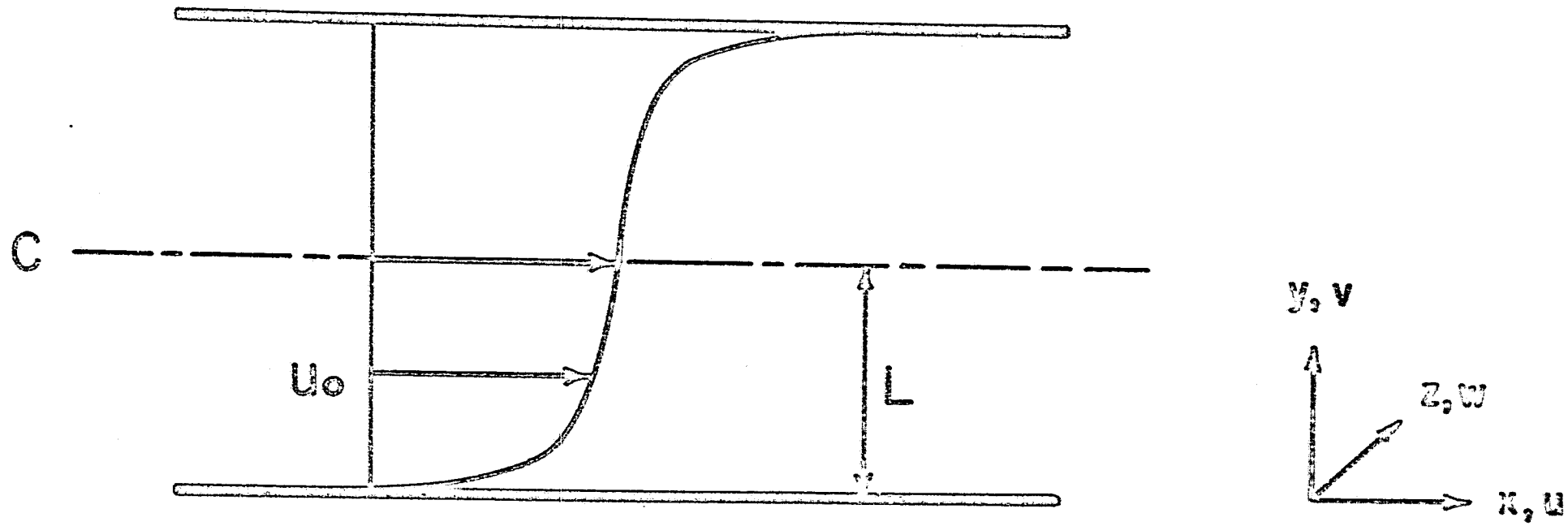

FIG. 1 Couette Flow Configuration

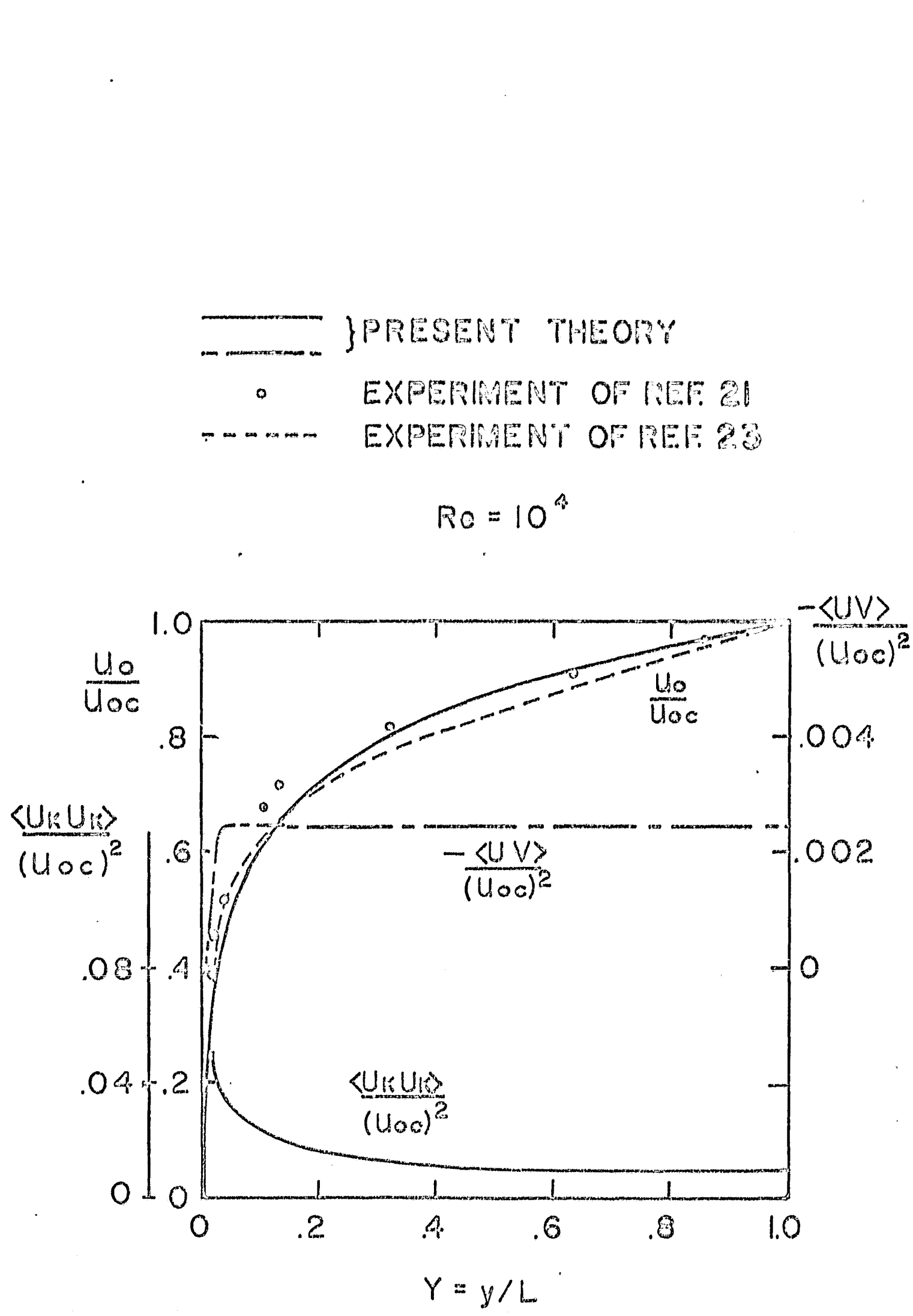

FIG. 2 Couette Flow Solution and Experimental Results

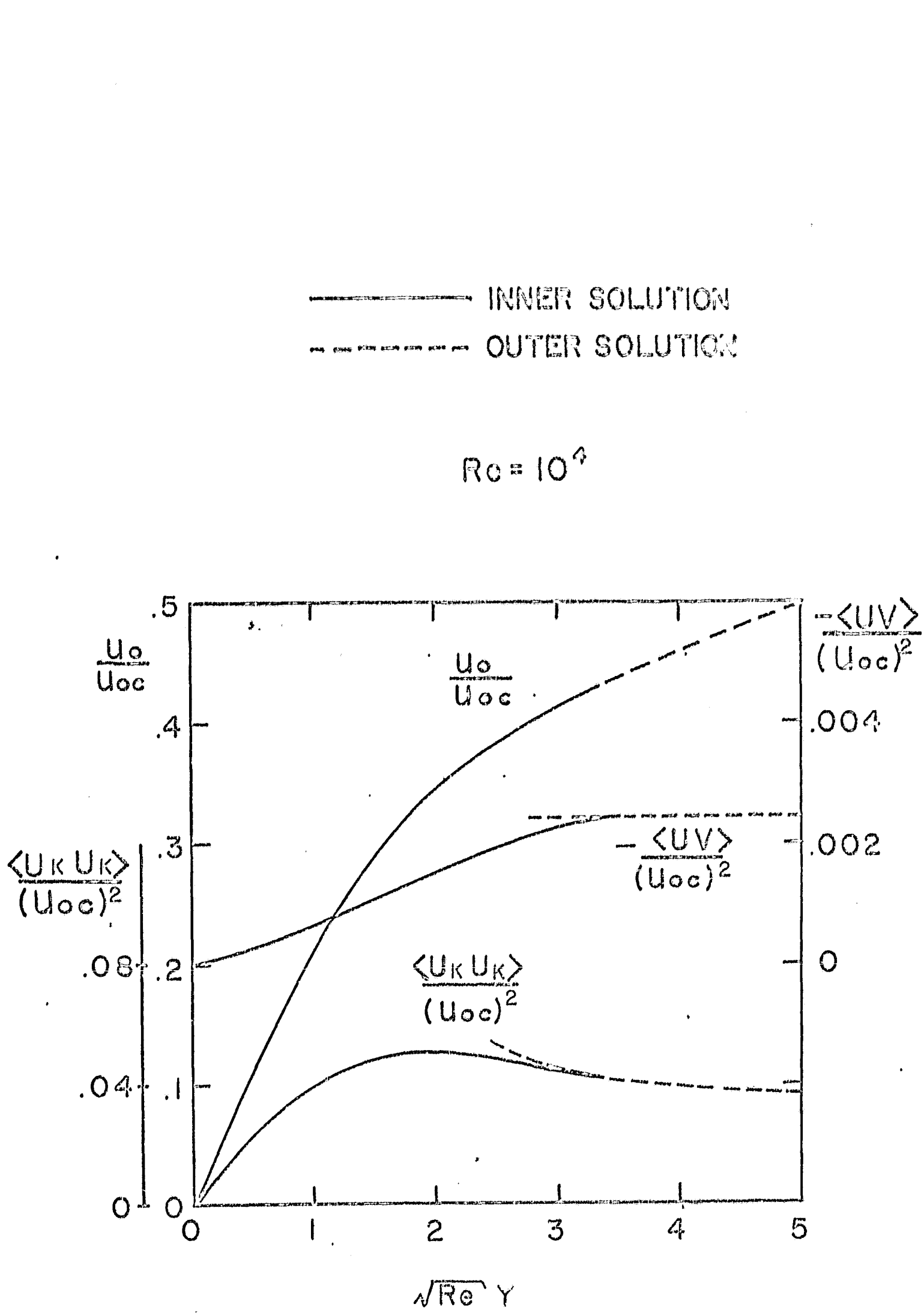

FIG. 3 Inner Solutions for Couette Flow

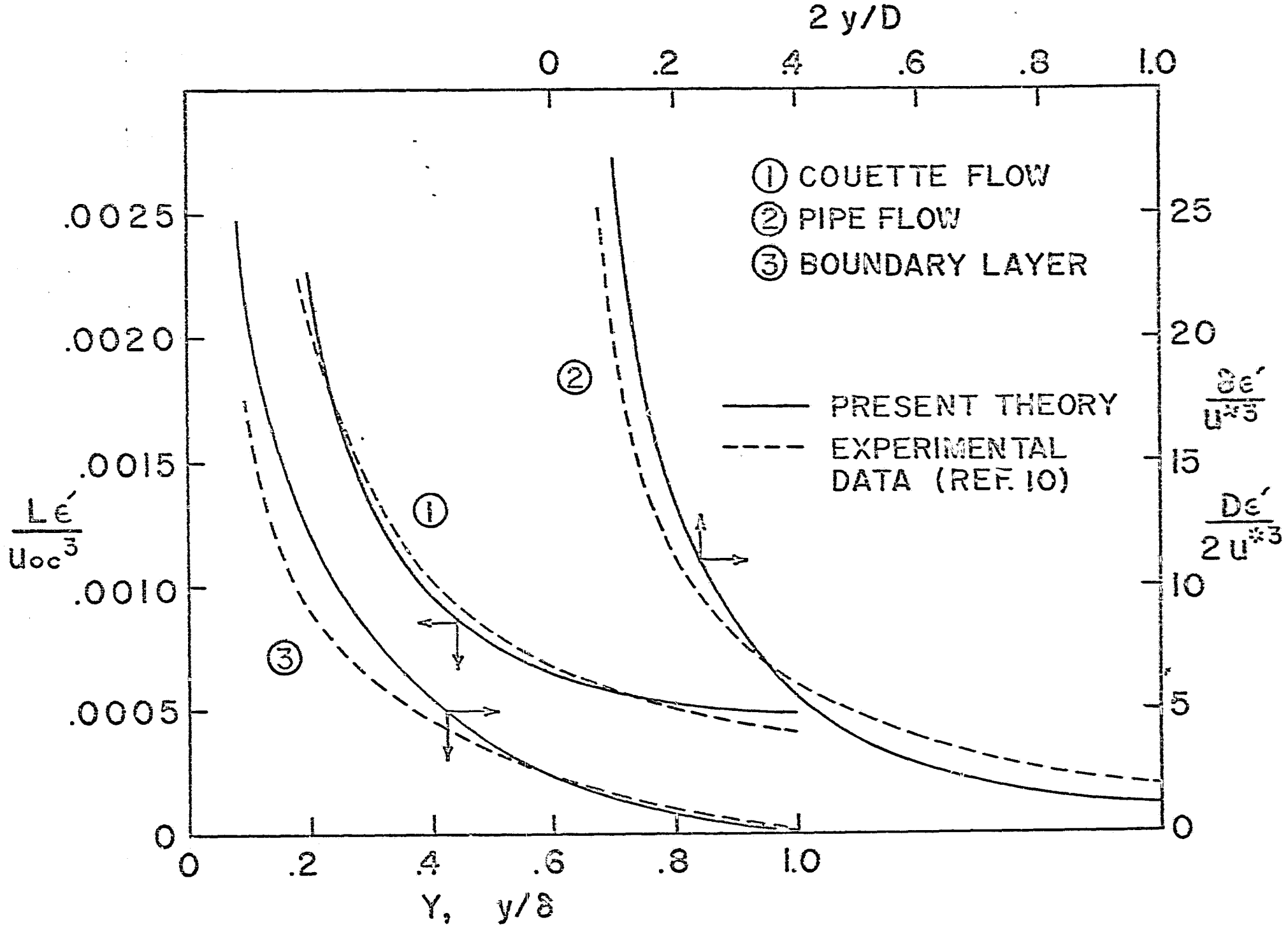

FIG. 4 Dissipation Functions (u^* = friction velocity)