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Abstract

Base flow theories are considered to be divided into four main groups, namely semi-empirical theories, those based on the Chapman-Korst model, integral methods, and multimethod base flow theories which attempt to take the dynamics of the recirculating flow into account. Each of these categories is considered in detail, important theoretical contributions being summarized, criticized and evaluated. A new model for the reattachment process is introduced, and it is inferred that the Chapman-Korst reattachment criterion is not a good approximation for turbulent or hypersonic flows, and that the reattachment process is independent of conditions downstream of the rear stagnation point. For these reasons it is inferred that reattachment has not yet been analyzed satisfactorily. A new interpretation of the variation of base pressure with Reynolds number is introduced. The expansion process at the trailing edge and its effect on the separating boundary layer is considered extremely important and not yet well understood. It is concluded that, although the more sophisticated theories are invaluable for extending the knowledge of base flow, for practical purposes the original Korst and Chapman theories with their extensions and some modifications are the most useful.

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S Y M B O L S

English Alphabet:

A	=	cross sectional area of body
A	=	see eq. (3.9.28)
$A(x^S)$	=	Lees-Reeves shape parameter
a	=	speed of sound
a_s	=	speed of sound at stagnation conditions
B	=	see Eq. (4.3.21)
C_p	=	$\frac{P}{\frac{1}{2}\rho_\infty V_\infty^2}$ = coefficient of pressure
C_D	=	$\frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 A}$ = coefficient of drag
C	=	$\frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}$ = Crocco Number
$\bar{C}$	=	$\frac{\rho\mu}{\rho_e\mu_e}$ = Chapman-Rubesin const.
c	=	length of body
D	=	drag force
D()	=	symbolic representation of functional form denominator
DSL	=	Dividing streamline
E	=	see eq. (3.9.27)
$E(S^*, \phi)$	=	see eq. (5.1.14)
erft	=	$\frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$ = error function
$F(\phi)$	=	Non-dimensional stream function expressed as a function of ϕ .
F^*	=	see Eq. (5.1.9)
$f(\eta)$	=	Non-dimensional stream function expressed as a function of ϕ .
G_i	=	see eq. (3.1.19)

$g(H)$	= see eq. (3.9.38)
H	= δ^*/δ^{**} = shape factor
$\overline{H}$	= $\delta^{**s}/\delta_t^{*s}$
H_0	= total enthalpy
H_{0c}	= total enthalpy in recirculation region
h	= step height, half width or radius of base.
I	= see eq. (4.1.2)
I_1	= see eq. (3.9.21)
I_2	= see eq. (3.9.22)
i	= momentum
J	= see eq. (4.2.6)
$\tilde{J}$	= see eq. (5.1.17)
K	= const eq. (2.1)
$\overline{K}$	= const eq. (2.1)
$\tilde{K}$	= see eq. (5.1.18)
k	= Crocco-Lees mixing coefficient eq. (4.1.15)
k_1, k_t	= parametric constants eqs. (2.9, 2.10)
L	= reference length
$\tilde{L}$	= see eq. (5.1.19)
l_i	= a typical length scale of boundary geometry.
M	= Mach number
m	= $\frac{\gamma-1}{2} M^2$
$\overline{m}$	= $\int_0^\delta \rho u dy$
N	= reattachment parameter see eq. (3.7.1)
$N()$	= symbolic representation of functional form of numerator
$N_1()$	= symbolic representation of functional form of numerator

$N_2()$	=	symbolic representation of functional form of numerator
P, p	=	pressure
P_0, p_0	=	total pressure
Pr	=	Prandtl number
Q, q	=	mass flow
q	=	dynamic pressure
R	=	Reattachment point
R	=	eq. (4.2.11)
Re	=	Reynolds number
RSP	=	rear stagnation point
$r(x)$	=	radial distance of DSL from base centerline.
S	=	separation point
S	=	see eq. (4.2.8)
$\overline{S}$	=	see eq. (5.1.4)
S^*	=	see eq. (5.1.8)
T	=	temperature
T_0	=	total temperature
$\overline{T}$	=	see eq. (4.2.10)
u	=	longitudinal component of velocity
V_{p_∞}	=	velocity corresponding to a pressure equal to free stream value
v	=	transverse component of velocity
$W(S^*, \phi)$	=	see eq. (5.1.14)
$w_k(\phi^D)$	=	weighting function
$w = \frac{u}{a_s}$	=	reduced velocity
x	=	reference coordinate, see eq. (3.1.8)
x	=	constant of proportionality, see eq. (3.9.3)
x	=	longitudinal coordinate
x_v	=	longitudinal coordinate measured from virtual origin
x'	=	displacement of virtual origin from S .

Y = reference coordinate see eq. (3.1.8)

$\bar{Y}$ = see eq. (5.1.4)

y = transverse coordinate

z = $\left(\frac{\partial \phi^D}{\partial y^D} \right) x^D$

Greek Alphabet:

β	= Falkner-Skan index
γ	= ratio of specific heats
δ	= boundary layer thickness
δ^*	$= \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy$ = displacement thickness
δ^{**}	$= \int_0^\delta \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e}\right) dy$ = momentum thickness
δ^{***}	$= \int_0^\delta \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u^2}{u_e^2}\right) dy$ = energy thickness
δ_t^*	= see eq. (4.2.7)
δ_{sl}	= thickness of mixing layer above $u = 0$.
Δ	= Characteristic boundary layer thickness see eq. (3.6.1)
$\bar{\Delta}$	= sublayer thickness
ϵ	= eddy viscosity
ϵ	= non-dimensional parameter, see eq. (2.11)
η	= $\frac{\sigma y}{x}$ = half-jet similarity coordinate
η	= see eq. (3.1.16)
η_p	= see eq. (3.1.15)
$\Phi(\eta)$	= Crocco-Lees universal function, see eq. (4.1.19)
ϕ	= $\frac{u}{u_e}$ = non-dimensional velocity
$\mathcal{H}$	= $\frac{u_\alpha}{u_e}$ = Crocco-Lees parameter
Λ	= transformation parameter, see eq. (3.9.2)
λ	= constant parameter, eq. (2.6)

λ'	= constant parameter, eq. (2.8)
μ	= coefficient of viscosity
ν	= Prandtl-Meyer variable
Ω	= nondimensional thermal energy
ω_α	= see eq. (4.1.7)
ω_e	= see eq. (4.1.8)
σ	= jet spreading parameter
θ	= flow inclination angle
τ	= shear stress

$$\xi = \int_0^{x/\delta_{2,0}} \frac{\epsilon}{\delta_{2,0} u_{2,e}} dt = \text{see eq. (3.2.11)}$$

Sub and Superscripts:

() _a	= conditions at C.S.A. (a) see Fig. 6
() _α	= average quantity across viscous region
() _b	= conditions immediately behind base of body
() _{b1}	= conditions in flow being injected into recirculation region.
() _c	= conditions on constant velocity streamline
() _{Cr}	= conditions at a singular or critical point
() _C	= conditions on base centerline
() _d	= conditions on DSL
() _{d1}	= conditions on DSL at beginning of recompression.
() ^D	= Dorodnitsyn transformation quantity, see eq. (4.3.12)
() _e	= conditions in inviscid external flow
() _i	= conditions on an arbitrarily chosen streamline
() _j	= conditions on jet axis
() _j	= conditions at a join
() ₁	= conditions at beginning of recompression
() ^M	= Mangler transformation, see eq. (3.9.44)

- ()_{rm} = conditions on line of maximum reversed flow.
- ()^s = Stewartson transformation, see eq. (4.3.11)
- ()^t = transformation, see eq. (3.9.2)
- ()^T = Mager transformation

$$dx^T = \frac{\rho}{\rho_0} \frac{a_e}{a_0} \frac{\mu}{\mu_0} dx$$

$$dy^T = \frac{a_e}{a_0} \frac{\rho}{\rho_0} dy$$

- ()_∞ = conditions in free stream
- ()₀ = conditions at trailing edge
- ()₁ = see Fig. 6
- ()₂ = see Fig. 6
- ()_{2'} = see Fig. 6
- ()₃ = see Fig. 6
- ()' = First derivative unless otherwise indicated

1. Introduction

The aim of this report is to present a comprehensive review of the principle methods and theories available for analyzing the base flow field and related problems. A statement of the overall problem can be written as follows:

How can the values for the fluid properties behind a two-dimensional or axi-symmetric blunt based body traveling at supersonic speeds be calculated?

What is the nature of the flow mechanisms in this region?

The base flow problem is one of a class of problems which involve the interaction of an external hyperbolic region with a region where viscous effects are important. One of the main difficulties lies with the sensitivity of hyperbolic regions to slight changes of the geometry of their boundaries.

It is important to distinguish, at this point, between two main classes of theories in this field, that is, between base pressure theories and base flow field theories. The first of these merely attempts the determination of certain select fluid properties, primarily the pressure acting on the base, the so-called base pressure; whilst, the latter seeks a complete understanding of the entire flow field, attempting to take all of the interactions into account.

Base pressure theories can be subdivided into:

a) Semi-Empirical theories.

- b) Chapman-Korst model theories. These usually envisage the flow field as consisting of free shear layers, originating from the trailing edge, which develop in a constant pressure region, and then pass into a region of confluence, the "throat", where the free shear layers merge and continue downstream; and as including a recirculation region, which is required to keep the flow field in equilibrium. See Fig. 1. The interaction between the recirculation region and free shear layer is largely ignored and the interaction between the free shear layer and external hyperbolic region is usually determined by analogy with asymptotic half-jet. The boundary layer equations are assumed to be valid only for the free shear layer.

Base flow field theories can also be subdivided into two groups:

- c) Integral methods. These would be analagous to the Karman-Polhausen and other methods which are used for attached boundary layers. However, in the case of the base flow field the pressure gradient is not predetermined, and is determined by a free interaction relation. The boundary layer equations are usually assumed to be valid over all of the flow field that does not include the hyperbolic region.

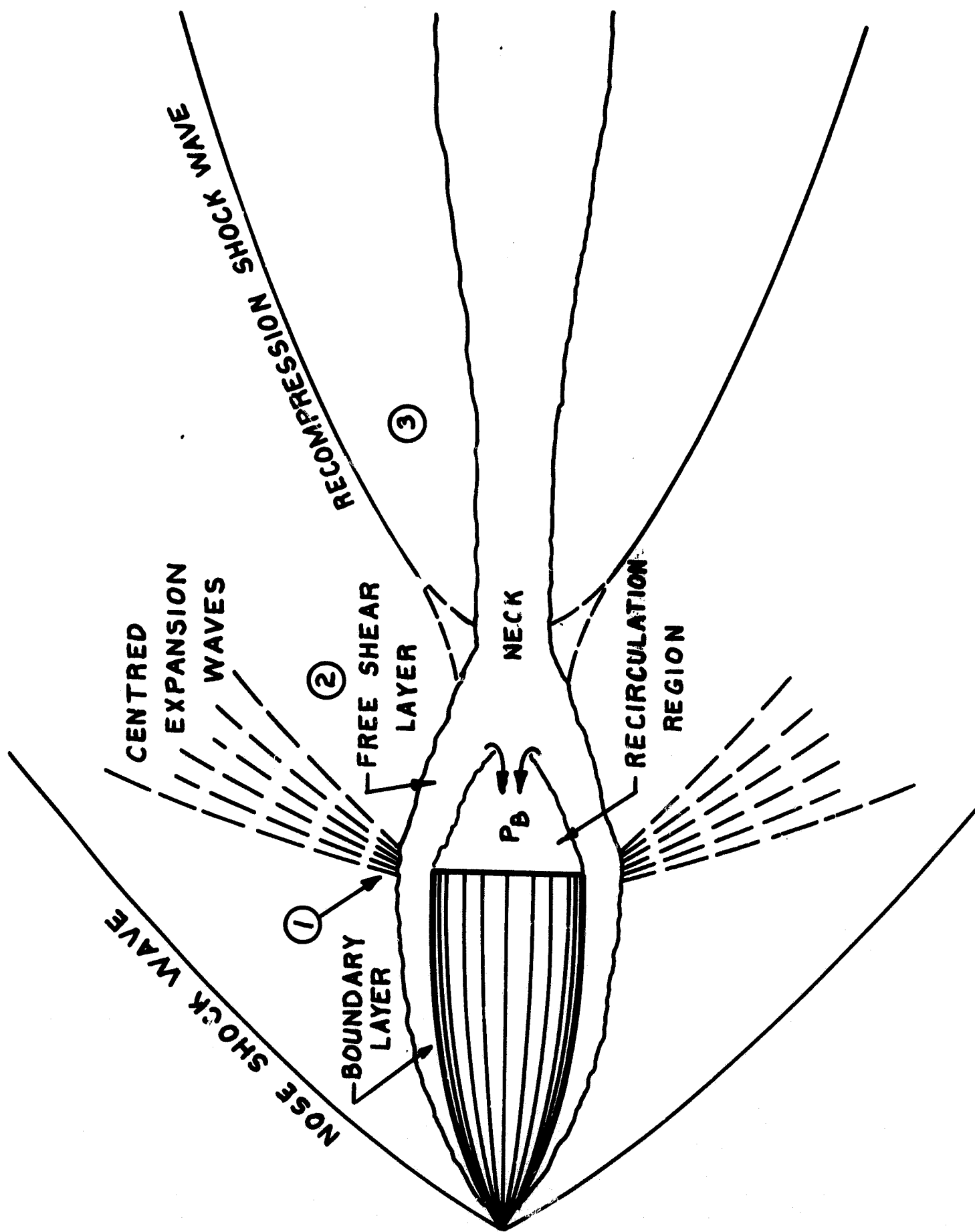


FIGURE 1. THE FLOW FIELD ABOUT A PROJECTILE.

- d) Multi-method theories. These theories are invariably extremely complicated and divide the flow field into regions each of which interact with one another, and thus the boundary conditions in each of the regions are interdependent.

The following pages contain a review of the theories and methods available in each of the four categories, with special emphasis on the case where the boundary layer at the trailing edge is turbulent. References to experimental results are introduced where they complement the particular theoretical methods under consideration rather than under a separate heading. The literature on certain related problems such as the expansion of a supersonic boundary layer around the trailing edge, turbulent reattachment, free shear layer profiles etc., is also reviewed.

2. Semi-Empirical Theories

It appears that the first contribution to supersonic base pressure theories was published by Gabeaud¹. This turbulence theory of residual resistance, envisaged a cavity behind the projectile where ring vortices were created and carried downstream. By means of an electro-magnetic analogy, Gabeaud arrived at the following formula for the base pressure

$$p_{\infty} - p_b = K \cdot \frac{1}{4} \cdot \rho_{p_{\infty}} \cdot V_{p_{\infty}}^2 \quad (2.1)$$

where $K = 1$ for supersonic speeds. $\rho_{p_{\infty}}$, $V_{p_{\infty}}$ are density and velocity corresponding to the pressure at the free stream value, Gabeaud assumed that after passing through the nose shock wave, the air expanded until $P = P_{\infty}$ again. Gabeaud subsequently improved his theory^{2,3,4}, and in its final form, the expression for the base pressure coefficient becomes

$$C_{p_b} = \frac{2}{\gamma M^2} \cdot \frac{2\gamma M^2 - (\gamma - 1)}{\gamma + 1} \left(\frac{2}{\gamma + 1} \right)^{1.4} \frac{1}{M^{2.8}} - \frac{2}{\gamma M^2} \quad (2.2)$$

In 1932, von Karman and Moore⁵ derived a semi-empirical theory which was obtained from the following expression for the contribution of base pressure to the drag of a low speed

projectile,

$$D_b = C_{D_b} \cdot \frac{\rho V_\infty^2}{2} \cdot A \cong 0.2 \frac{\rho V_\infty^2}{2} \cdot A \quad (2.3)$$

where $C_{D_b} = 0.2$ was obtained experimentally. Therefore, it was argued that the upper limit to the base pressure was given by,

$$(p_\infty - p)_{\min} \cong \lambda \cdot \frac{\rho V^2}{2} \quad (2.4)$$

and the lower limit would be attained, when a vacuum existed behind the projectile,

$$(p_\infty - p)_{\max} = \text{Const.} = 1 \text{ atmos.} \quad (2.5)$$

Thus to establish a transition between these limiting cases, von Karman and Moore wrote,

$$\int_{p_b}^{p_\infty} \frac{dp}{\rho} = \frac{\lambda V^2}{2} \quad (2.6)$$

using the same value for λ , as at low speeds i.e. $\lambda = 0.2$. From eq. (2.6), with the assumption of adiabatic expansion, was obtained the formula,

$$\frac{p_b}{p_\infty} = \left[1 - \frac{\gamma-1}{2} \lambda M^2 \right]^{\frac{\gamma-1}{\gamma}} \quad (2.7)$$

As late as 1950, several formulae for base drag were proposed depending on Mach number only, these were mainly determined from ballistic tests^{6,7,8,9,10}. Hill¹⁰ proposed an improvement to eq. (2.7), for the range $2 < M < 6$, which was written,

$$C_{p_b} = \frac{p_b - p_i}{q_i} \quad (2.8)$$

$$C_{p_b} = \lambda' \frac{2}{\gamma M^2} \left(1 - \frac{\gamma-1}{2} \lambda M^2 \right)^{\frac{\gamma}{\gamma-1}} - \frac{2}{\gamma M^2}$$

where λ' and λ are arbitrary constants, to be determined by experimental data. Hill used $\lambda = 0.055$ and $\lambda' = 0.5394$.

With the advent of tests in closed circuit wind tunnels having variable stagnation pressures, it was demonstrated that the base pressure also depended on the Reynolds number. The shortcomings of the original formulae were first pointed out by Cope¹¹ and Hankins¹².

Chapman¹³ made an analysis of the base pressure in an inviscid fluid, both for two-dimensional and axially-symmetric flow. He showed that for two-dimensional flow, and also for the flow over a body of revolution with a cylindrical sting attached to the base, there are an infinite number of possible solutions. For the particular case of a body having no sting attached, he showed that only one solution is possible in an inviscid flow, but it corresponds to zero base drag. Accordingly, he concluded that a strictly inviscid fluid theory cannot be satisfactory for practical applications.

Chapman then proceeded to formulate a semi-empirical theory based on the following assumptions:

- 1) The base pressure coefficient C'_{p_b} depends only on the type of boundary layer flow, the Mach number M_1 , and the dimensionless boundary layer thickness δ_1/h which exists just upstream of the base.
- 2) At a given M_1 , the difference $(C_{p_b}^* - C'_{p_b})$, due to the effects of viscosity, is proportional to the dimensionless boundary-layer thickness δ_1/h .

These assumptions led to the equations,

$$C_{p_b} = C_{p_b}^* \left(1 - \frac{k_l}{\sqrt{Re}} \cdot \frac{C}{h} \right) (1 + \epsilon) + C_{p_{b1}} \quad (2.9)$$

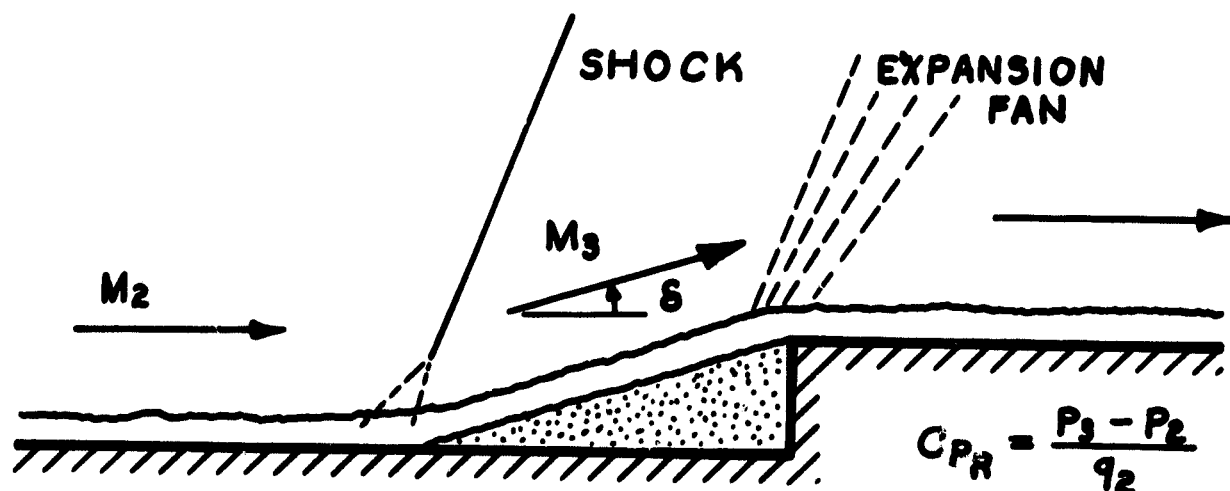
for laminar boundary layer flow, and

$$C_{p_b} = C_{p_b}^* \left(1 - \frac{k_t}{(Re)^{1/5}} \cdot \frac{C}{h} \right) (1 + \epsilon) + C_{p_{b1}} \quad (2.10)$$

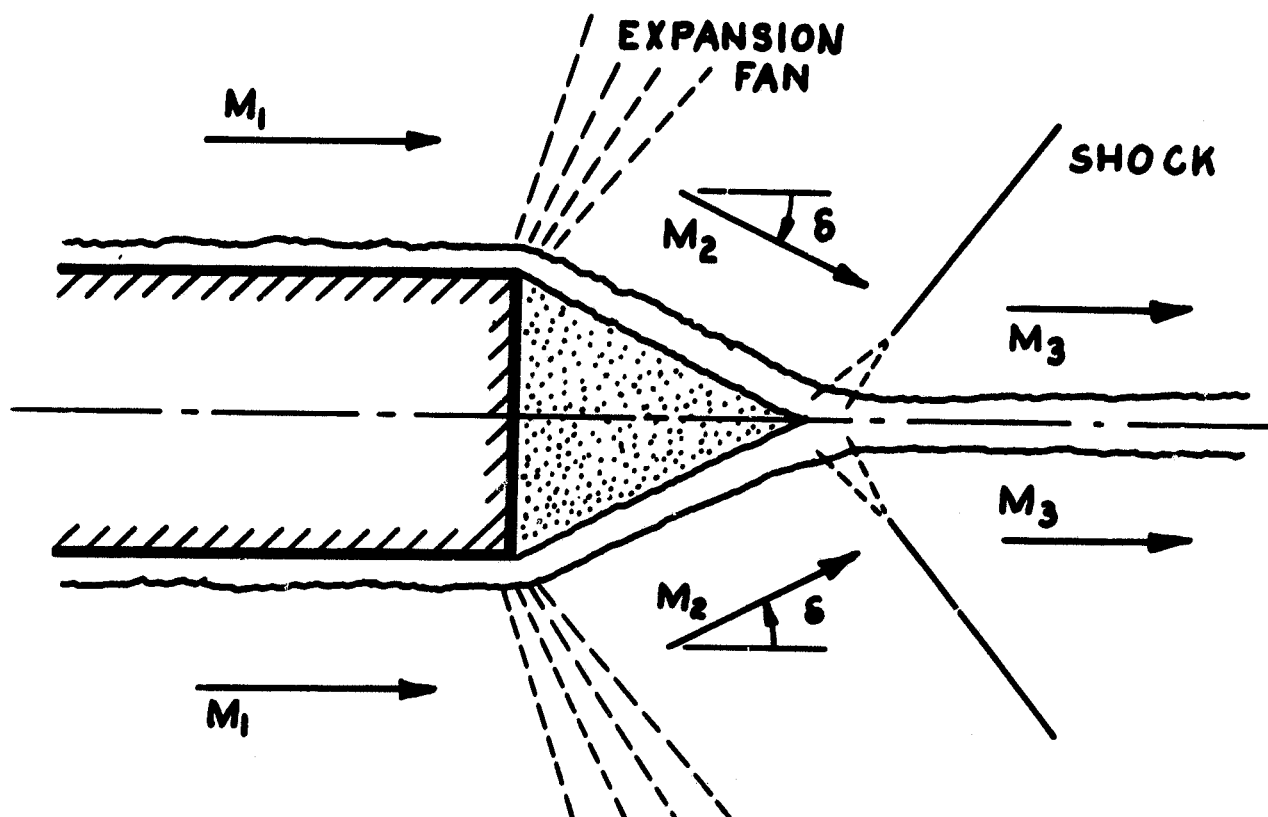
for turbulent boundary layer flow. These equations provide no information as to the dependence of P_b^* and k_l or k_t on Mach number. These quantities were to be evaluated by experiments. ϵ was defined by the equation

$$C_{p_b} = C'_{p_b} (1 + \epsilon) + C_{p_1} \quad (2.11)$$

and ϵ and C_{p_1} are virtually zero for two-dimensional profiles.



(a). SKETCH OF SEPARATION PHENOMENA ON A FLAT PLATE CAUSED BY A FORWARD-FACING STEP.



(b). SKETCH OF PHENOMENA BEHIND A TWO-DIMENSIONAL BASE.

FIGURE 2. ANALOGY BETWEEN FLOW PHENOMENA FOR FORWARD-FACING STEP AND TWO-DIMENSIONAL BASE.

The analogy between the base pressure problem and determination of peak pressure rise associated with the step induced separation of a boundary layer (see Fig. 2) is the basis for Love's semi-empirical theory¹⁴. Love argued that there seemed little reason to believe that the peak-pressure-rise coefficient C_{p_r} in the first case would differ by a significant amount from the second. On the evidence of data presented by Donaldson et al.¹⁵, and on tests carried out by himself, Love came to the conclusion that C_{p_r} experiences almost no variation with Reynold's number when the boundary layer is turbulent, provided the step height is at least several times the boundary layer thickness. Hence, on the basis of the above analogy, the experimental values of C_{p_R} as a function of M_2 may be used to find $\frac{p_3}{p_2}$ across the shock and, thereby, the value of δ may be calculated. Once δ is known, M_1 is determined, and the variation of C_{p_b} with M , may be computed from the variation of M_2 with M_1 .

The latest contribution to the list of semi-empirical theories is that of de Krasinski¹⁶ who uses Love's values as the limit as $\delta_{0,2}^{**} \rightarrow 0$. According to de Krasinski's data, base pressures increase with momentum thickness up to $\delta_{0,2/h}^{**} = 0.02$, after which they are essentially constant. By plotting the relation $\Delta(p_b/p_i) / \Delta(\delta_{0,2/h}^{**})$ against the Mach number of the free stream a linear fraction resulted. Taking $\delta_{0,2}^{**} = 0.02$, he obtained the following relation

$$\Delta\left(\frac{p_b}{p_i}\right) = (2.19 M_\infty - 1.4) \Delta\left(\frac{\delta_{0,3}^{**}}{h}\right) \quad (2.12)$$

which allows the computation of base pressures above the lowest estimated by Love. (loc. cit.).

Obviously, the usefulness of semi-empirical theories is almost non-existent where an understanding of the mechanics of base flow is being sought. However, since all of the more rigorous base pressure theories involve considerable time and effort to calculate values for base pressures, semi-empirical theories do serve as a means of obtaining approximate values quickly. Love's theory appears to be the most suitable, and it does not seem worthwhile to apply de Krasinski's correction. A comparison between the various theories cited above and selected experimental data is shown in Figs. 3 and 4.

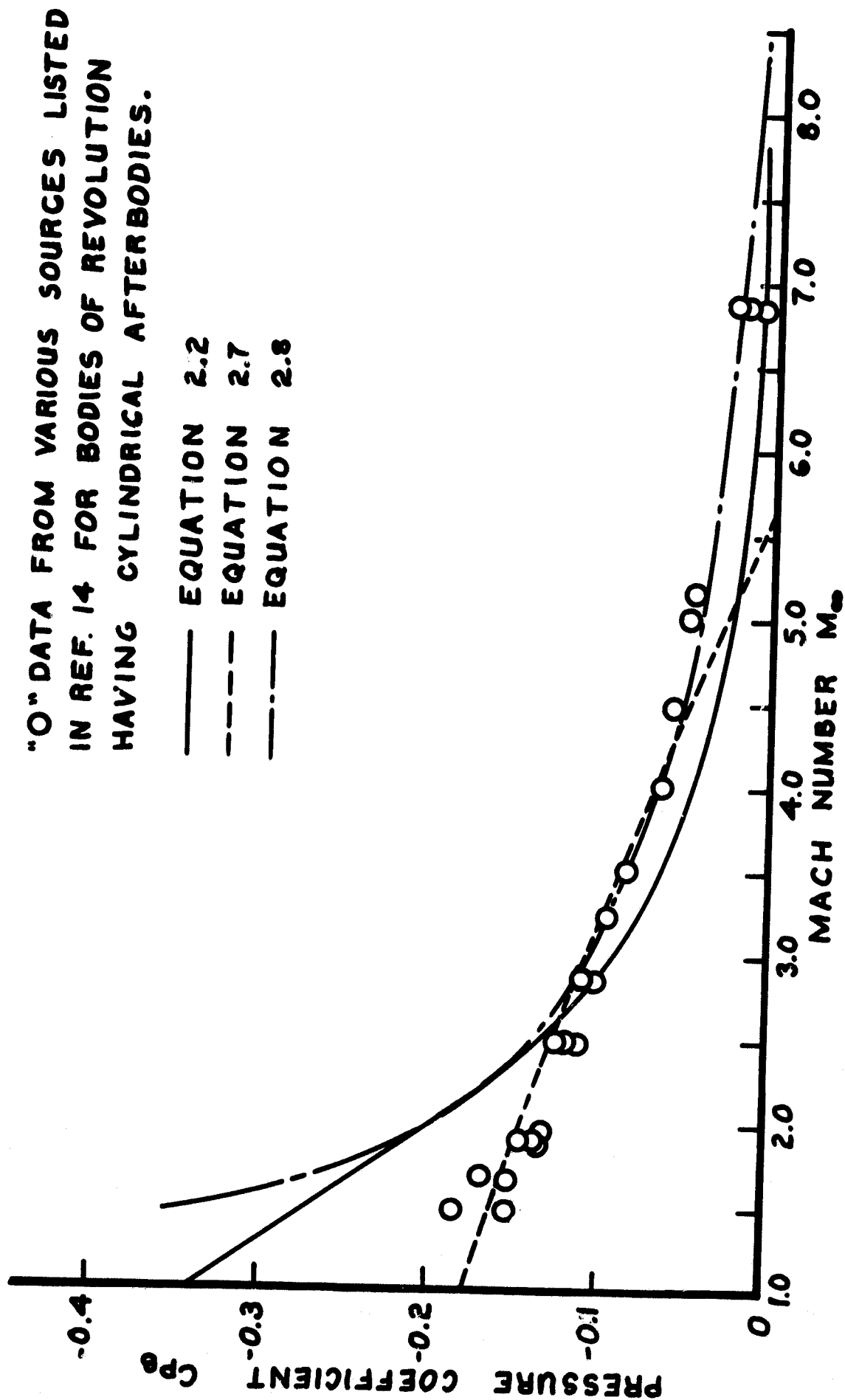


FIGURE 3. COMPARISON OF SEMI-EMPIRICAL THEORIES.

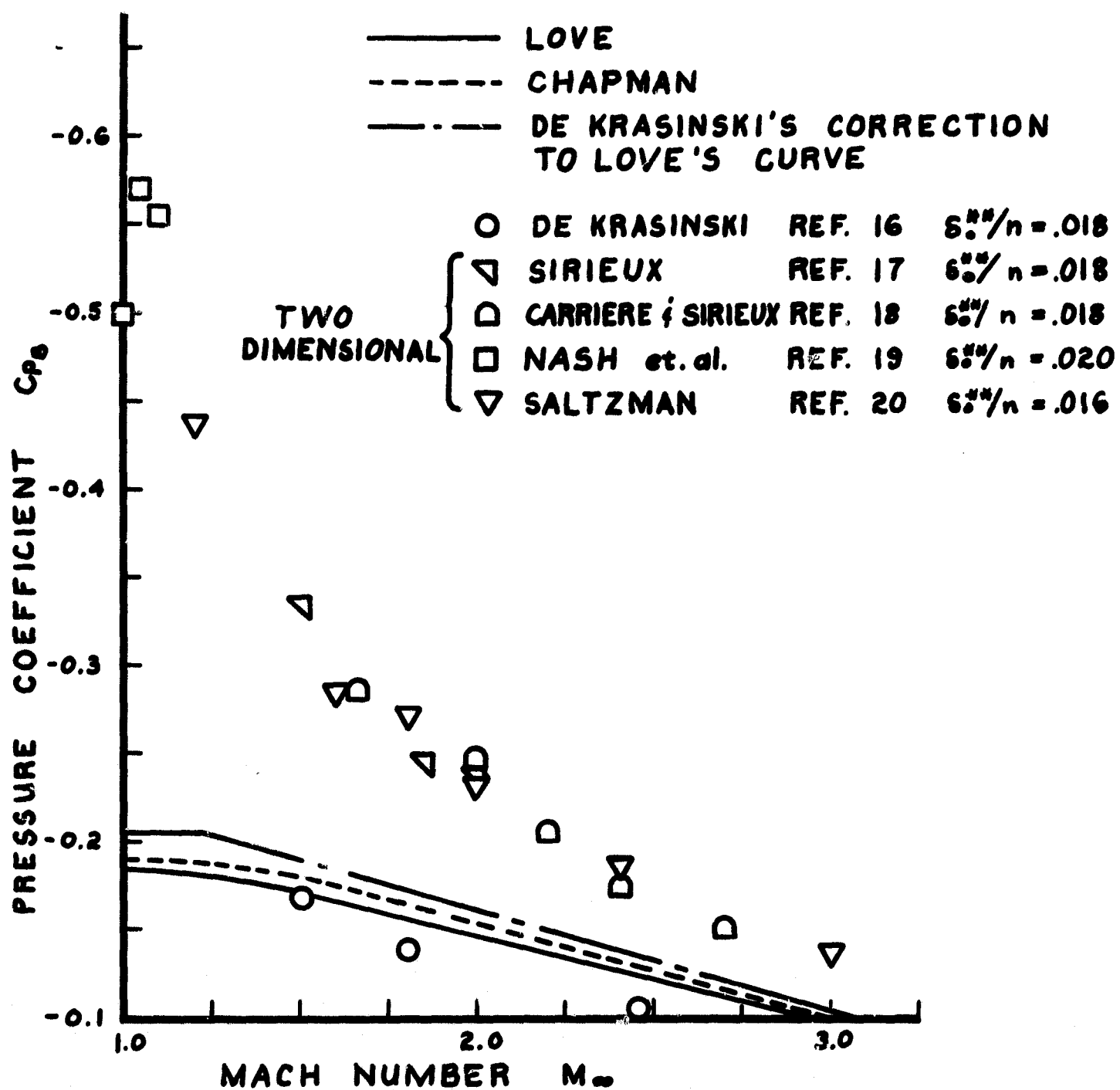


FIGURE 4. COMPARISON OF SEMI-EMPIRICAL THEORIES

3. Chapman-Körst Flow Model Theories

The qualitative basis for this group of theories was first adumbrated by Chapman¹³, later¹⁷ he elaborated on these ideas. Chapman envisaged the effects of viscosity on base flow, in the following manner. Fig. 5 is a sketch of the qualitative flow characteristics, as assumed by Chapman. The flow along the first expansion wave starts with a non-uniform distribution of Mach number M_1 , pressure, and with a boundary layer thickness $\delta_1, 0$. A fan of expansion waves originate near the trailing edge because the base pressure P_b is lower than P_1 . An important feature of this qualitative flow model, is the assumption that because of the existence of the "dead-air" region, the pressure along streamline BC is approximately constant, and that the separated boundary layer experiences boundary conditions of the same nature as those existing at the boundaries of a supersonic half-jet issuing into ambient air.

The base pressure was considered to attain an equilibrium value by the following process. Suppose, initially, P_b is only a little less than P_1 , then this would result in the free shear layer being very long and consequently scavenging large amounts of air from the "dead-air" region, thereby causing the value of P_b to decrease, thus causing the turning angle to increase. However, this process cannot continue indefinitely because as the base pressure drops, the pressure rise, to be negotiated by the free shear

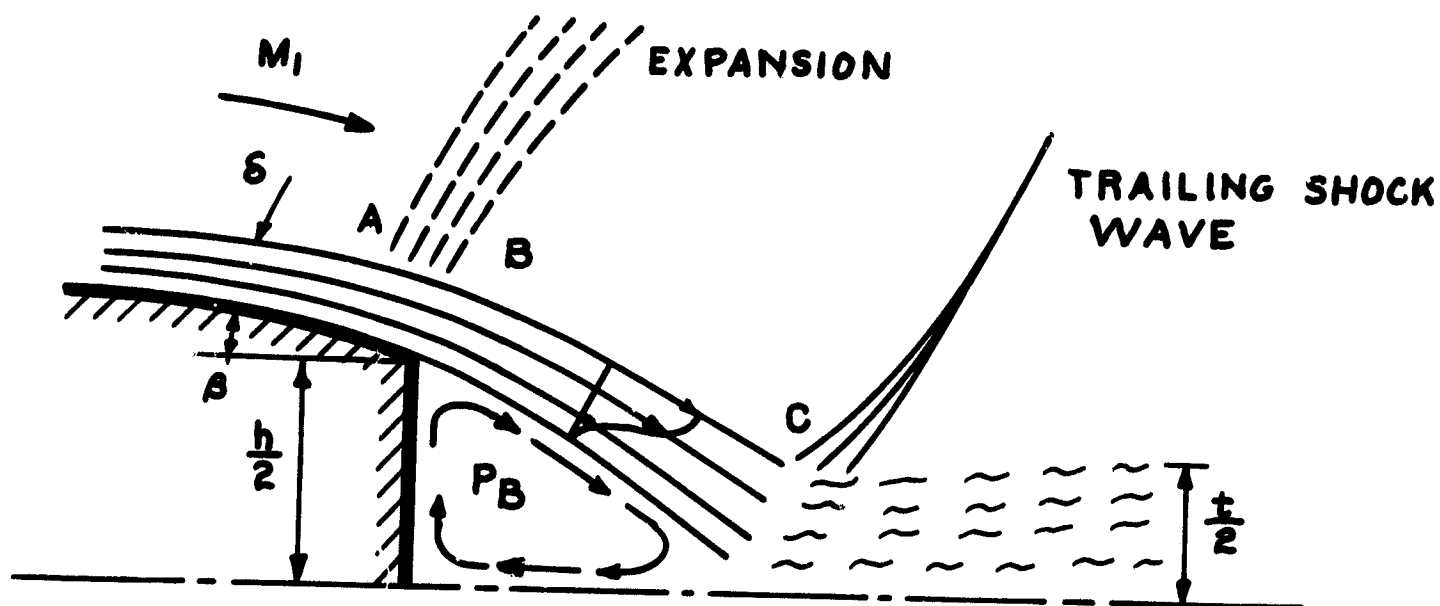


FIGURE 5. QUALITATIVE FLOW CHARACTERISTICS AS ENVISAGED BY CHAPMAN.

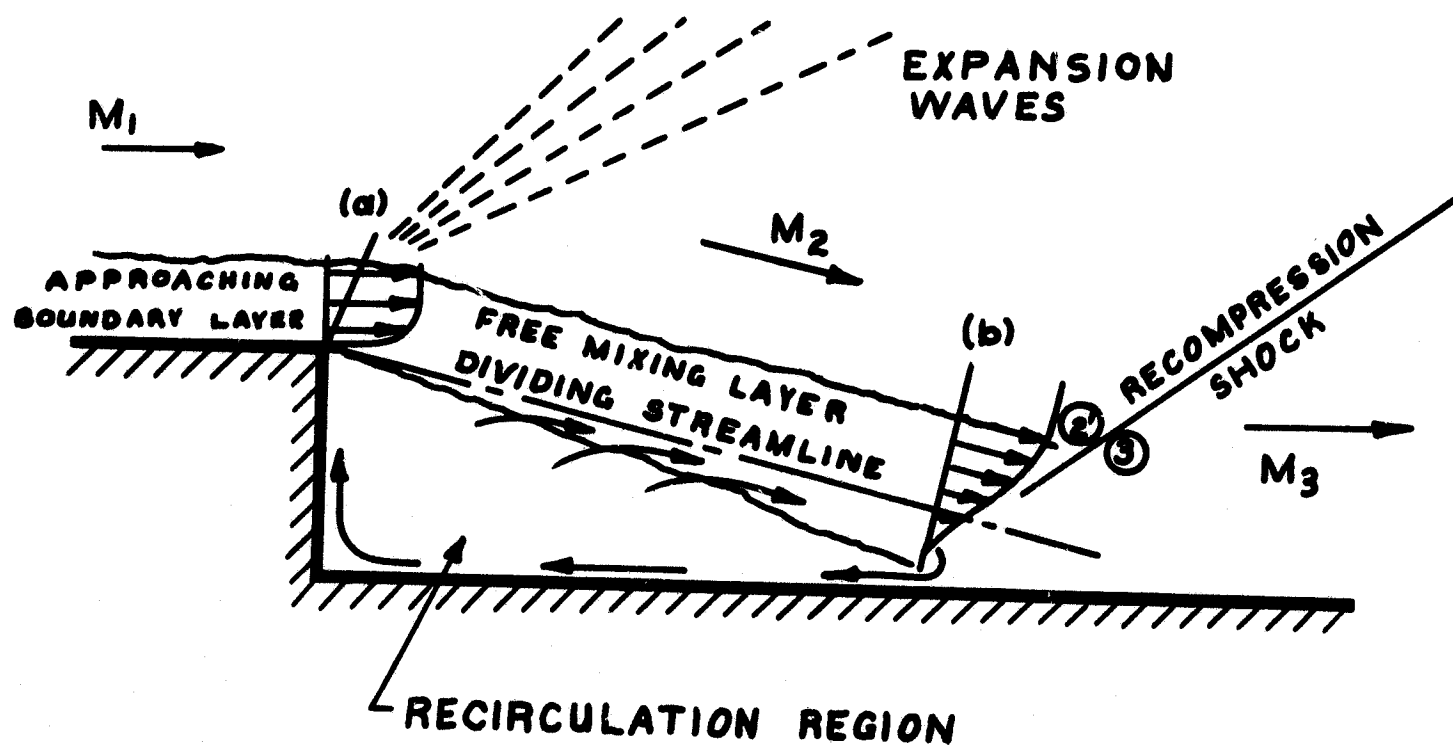


FIGURE 6. CHAPMAN - KORST FLOW MODEL

layer at C, increases. Hence, the point C moves in towards the base until it reaches a position where the amount of air scavenged by the free shear layer is exactly balanced by the quantity of air reversed by the pressure rise at C. Thus the concept of the "dead air" region as a recirculation region which is responsible for maintaining equilibrium, was born. Subsequently in this report, this idea will be known as the "dividing streamline concept", it is the cornerstone of all theories based on the Chapman-Korst flow model. The term "Chapman-Korst flow model" will be used to refer to cases where the assumption of a free shear layer with supersonic-jet-type boundary conditions is made together with the dividing streamline concept.

Apart from an expression developed by Kurzweg¹⁸ which was based on the assumption that the velocity of the air which is reduced in the boundary layer has to be restored after separation, the following section deals with what appears to be the first fully theoretical treatment on the base flow problem.

3.1 Korst's Theory

Although the qualitative formulation of the Chapman-Korst flow model originated with Chapman¹³, it was first exploited to obtain quantitative results in a base pressure theory published by Korst in 1955^{19,20,21}. Hence, the commonly used title "Chapman-Korst flow model".

Fig. 6 illustrates the flow mechanism model assumed by Korst. The theory was based on the following five propositions.

(i) The flow in the free mixing layer or dissipative shear flow region has its static pressure impressed by the adjacent nearly uniform free stream,

$$\frac{P_{2i}}{P_{2e}} = 1 \quad (3.1.1)$$

(ii) The pressure in the recirculation region corresponds to the Prandtl-Meyer relation for the expansion of the free stream around the corner, i.e.,

$$\frac{P_2}{P_1} = \frac{P_2}{P_1} (M_{1e}, \vartheta_{2e} - \vartheta_{1e}) \quad (3.1.2)$$

and

$$M_{2e} = M_{2e} (M_{1e}, \vartheta_{2e} - \vartheta_{1e}) \quad (3.1.3)$$

Also for a given boundary layer configuration, $\phi_{1,0}(\zeta_{1,0})$ at trailing edge, the velocity distribution at the trailing edge immediately after undergoing expansion may be expressed formally by,

$$\phi_{2,0}(\zeta) = \phi_{2,0}[\phi_{1,0}(\zeta_{1,0}), M_{1e}, P_2/P_1] \quad (3.1.4)$$

and

$$\frac{\delta_{2,0}}{\delta_{1,0}} = \frac{\delta_{2,0}}{\delta_{1,0}} [\phi_{1,0}(\zeta_1), M_{1e}, P_2/P_1] \quad (3.1.5)$$

(iii) As the mixing process between cross-sections (a) and (b) occur at constant pressure and as long as the mixing layer thickness is small compared with any significant length in the wake region such as step height, the results of the two-dimensional compressible jet mixing theory²¹ are applicable.

Briefly, this jet mixing theory is as follows:

The shear layer was assumed to develop as if the external stream were mixing with air at rest, at a constant pressure, thus the governing momentum equation,

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y} \quad (3.1.6)$$

Korst introduced a "reference system" of coordinates (X,Y) as an orthogonal and generally curvilinear system of coordinates which follows the jet boundary of the corresponding inviscid jet. This was defined as the hypothetical frictionless jet which has the same approach Mach number, M_{1e} expands through the same pressure ratio, $\frac{P_2}{P_1}$, and is influenced by the same geometry of boundaries $\sum_n \frac{l_i}{L}$, as the actual viscous jet. The angular deviation of the reference system of coordinates from the uniform flow approaching cross section (a) is

$$\Delta \vartheta_a = \vartheta_a - \vartheta_{1e} = \Delta \vartheta_a (M_{1e}, P_2/P_1, \sum_n \frac{l_i}{L}) \quad (3.1.7)$$

where, $\Delta \vartheta_a$ is determined by the method of characteristics or

some other method.

Korst then linearized the above momentum equation by introducing the assumption of small perturbations, i.e. he assumed there were only small differences in velocity and temperature between the external stream and the mixing layer. Obviously, this did not correspond to the case of mixing of an external supersonic flow with a stationary fluid. Thus, to compensate a posteriori for this discrepancy, Korst introduced an "intrinsic system" of coordinates (x,y) which is displaced in the direction of the Y axis from the "reference system" by an amount $Y_m(X)$. The angle between the two coordinate systems is small, so that

$$X \cong x$$

and

(3.1.8)

$$Y = y - y_m(x) \quad ; \quad y_m(0) = 0$$

$y_m(X)$ is such as to satisfy the conservation of momentum.

The linearization of eq. (3.1.6) results in a form of the heat equation,

$$\frac{\partial U}{\partial X} = \frac{\epsilon}{U_{2e}} \frac{\partial^2 U}{\partial Y^2} \quad (3.1.9)$$

introduction of a similarity parameter

$$\zeta = \frac{y}{\delta_{2,0}} \quad (3.1.10)$$

$$\xi = \int_0^{x/\delta_{2,0}} \frac{\epsilon}{\delta_{2,0} u_{2,e}} dt \quad (3.1.11)$$

reduced the problem to the following formulation

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial^2 \phi}{\partial \zeta^2} \quad (3.1.12)$$

$$\begin{aligned} \phi(0, \zeta) &= 0 & \text{for } -\infty < \zeta < 0 \\ \phi(0, \zeta) &= \phi_{2,0}(\zeta) & \text{for } 0 < \zeta < 1 \\ \phi(0, \zeta) &= 1 & \text{for } 1 < \zeta < +\infty \\ \phi(\xi, -\infty) &\longrightarrow 0 & \text{for } \xi > 0 \\ \phi(\xi, +\infty) &\longrightarrow 1 & \text{for } \xi > 0 \end{aligned} \quad (3.1.13)$$

The approximate solution was given as

$$\begin{aligned} \phi &= \phi[\phi_{2,0}(\zeta), \eta_p; \eta] \\ &= \frac{1}{2} [1 + \operatorname{erf}(\eta - \eta_p)] + \frac{1}{\sqrt{\pi}} \int_{\eta - \eta_p}^{\eta} \phi_2\left(\frac{\eta - t}{\eta_p}\right) e^{-t^2} dt \end{aligned} \quad (3.1.14)$$

where

$$\eta_P = \frac{1}{2\sqrt{\xi}} \quad (3.1.15)$$

$$\eta = \xi \eta_P \quad (3.1.16)$$

Korst restricted the theory to adiabatic wakes originally although an extension to non-adiabatic wakes was made later²³. For convenience the Mach number was replaced by the Crocco number, a non-dimensional velocity. The base pressure problem could now be generalized in three equations: a) The displacement η_m of the intrinsic system of coordinates with respect to the reference system must be such that the momentum flux entering at the separation point is equal to the momentum flux at any station downstream, this was expressed as,

$$\begin{aligned} & \eta_m[\phi_{2,0}(\xi), \eta_P, C_{2e}] \\ &= \eta_{ref} - \eta_P + (1 - C_{2e}^2) \left[\eta_P \int_0^1 \frac{\phi_{2,0}^2 d\xi}{1 - C_{2e}^2 \phi_{2,0}^2} - \int_{-\infty}^{\eta_{ref}} \frac{\phi_2^2}{1 - C_{2e}^2 \phi^2} d\eta \right] \end{aligned} \quad (3.1.17)$$

(b) The location, η_i , of an arbitrary streamline ψ , is given by setting the mass flow between ψ and the dividing streamline ψ_d , equal to the mass flux in the mixing layer initially at S plus the

mass entrained from the external stream, using eq. (3.1.17) this can be expressed as,

$$\int_{\eta_i}^{\eta_{ref}} \frac{\phi d\eta}{1 - C_{2e}^2 \phi^2} + \frac{\eta_P G_i}{\delta_{2,0} \rho_{2e} u_{2e} (1 - C_{2e}^2)} \quad (3.1.18)$$

$$= \int_{-\infty}^{\eta_{ref}} \frac{\phi^2 d\eta}{1 - C_{2e}^2 \phi^2} + \eta_P \int_0^1 \frac{\phi_{2,0} (1 - \phi_{2,0})}{1 - C_{2e}^2 \phi_{2,0}^2} d\zeta$$

where

$$G_i = \int_{y_j}^{y_i} \rho u dt \quad (3.1.19)$$

(c) The location η_d of the dividing streamline which separates the fluid of the approaching stream from the fluid in the recirculation region, was specified by $G_d = 0$, (or $G_d = q_{ble}$ for case of base bleed).

$$\int_{\eta_d}^{\eta_{ref}} \frac{\phi d\eta}{1 - C_{2e}^2 \phi^2} = \int_{-\infty}^{\eta_{ref}} \frac{\phi^2 d\eta}{1 - C_{2e}^2 \phi^2} + \eta_P \int_0^1 \frac{\phi_{2,0} (1 - \phi_{2,0})}{1 - C_{2e}^2 \phi_{2,0}^2} d\zeta \quad (3.1.20)$$

(IV) The pressure rise at recompression was assumed to be given by oblique shock relations.

$$\frac{P_3}{P_2} = \frac{P_3}{P_2} (M_{2e}, \vartheta_{2'e} - \vartheta_{3e}) \quad (3.1.21)$$

(V) The static pressure ratio at recompression was set equal to the total pressure on the dividing streamline at station (b), i.e.

$$\frac{P_{02d}}{P_2} = \frac{P_b}{P_2} = \left(1 + \frac{\gamma-1}{2} M_{2d}^2 \right)^{\gamma/(\gamma-1)} \quad (3.1.22)$$

Thus the base pressure can be found from eqs. (3.1.14, 17, 18, 20, 21, and 22) when $\delta_{2,0}$, $\phi_{2,0}$ and η_P are given.

The case of base bleed could also be covered in theory by setting $G_d = q_{bl}$ and specifying its value). Unfortunately, Korst et al. could find no way of dealing with the case of finite boundary layer thickness, without resorting to experimental data. Accordingly, they restricted the theory to the special case of zero approaching boundary layer.

The result of this restriction was, $\frac{\delta_{2,0}}{x} \rightarrow 0$,
 $\frac{x}{\delta_{2,0}} \rightarrow \infty$, so $\xi \rightarrow \infty$ while $\eta_P = \frac{1}{2\sqrt{\xi}} \rightarrow 0$
 and $\eta = \sigma \frac{y}{x}$

Hence, eqs. (3.1.14, 17, 18, 20) become after modification to the case of non-adiabatic wakes²³, and finite bleed flux,

$$\phi = \frac{1}{2} (1 + \operatorname{erf} \eta) \quad (3.1.23)$$

$$\eta_m = \eta_{erf} - (1 - C_{2e}^2) \int_{-\infty}^{\eta_{erf}} \frac{\phi^2 d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} \quad (3.1.24)$$

$$\frac{G_d \delta}{\chi(1 - C_{2e}^2) \rho_{2e} U_{2e}} = \int_{-\infty}^{\eta_d} \frac{\phi d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} - \int_{-\infty}^{\eta_j} \frac{\phi d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} \quad (3.1.25)$$

$$\int_{-\infty}^{\eta_j} \frac{\phi d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} = \int_{-\infty}^{\infty} \frac{\phi(1 - \phi) d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2}$$

And the total rate of energy transmitted to the wake was written

$$\Omega = \Omega_e + \Omega_d \quad (3.1.26)$$

$$= \frac{\chi}{\delta} \rho_{2e} U_{2e} C_p T_{0e} (1 - C_{2e}^2) \left[\int_{\eta_j}^{\infty} \frac{\phi^2 (1 - \frac{T_0}{T_e}) d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} + \int_{\eta_j}^{\eta_d} \frac{\frac{T_0}{T_{0e}} \phi d\eta}{\frac{T_0}{T_{0e}} - C_{2e}^2 \phi^2} \right]$$

From empirical information, Korst and Tripp²⁴ expressed the jet-spreading or similarity parameter, δ with the linear equation

$$\bar{C} = 12 + 2.758 M_{2e} \quad (3.1.27)$$

Korst's theory is, in principle, extremely versatile, easily modified to cover a broad range of applications, e.g. mass-bleeding effects, heat addition, two-jet interference etc. However, it does have severe limitations namely, it applies only to two-dimensional turbulent wakes and only gives an estimate of the lowest possible base pressure, since no way was found to include the effects of the initial boundary layer analytically.

3.2 Kirk's Theory

About the same time as Korst, Kirk²⁵ also developed a base pressure theory, although it was not published until 1959. Unlike Korst's theory (Chap. 3.1), this theory, which deals with two dimensional turbulent base flow, relates the base pressure to the initial boundary layer conditions rather than to conditions behind the recompression shock. Kirk uses Tollmien's solution²⁶ for the incompressible turbulent half-jet, concluding on comparison with Abramovich's extension²⁷ that the effects of compressibility could be neglected. This solution tends to underestimate base pressures at higher Mach numbers for the limiting case when $\delta_{2,0} \rightarrow 0$.

The main feature of Tollmien's half-jet model is the existence of constant velocity lines (see Fig. 7). When the approaching

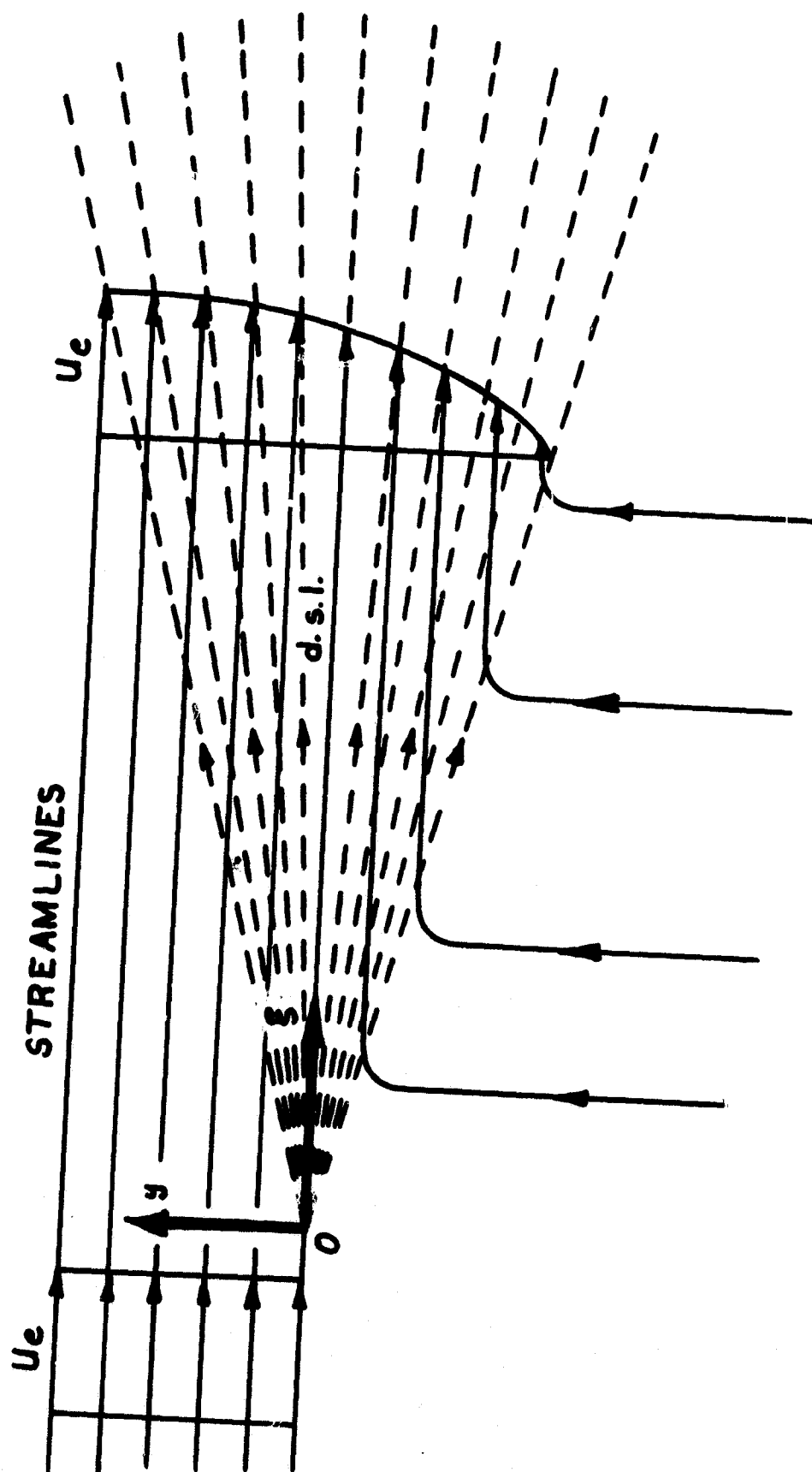


FIGURE 7. TOLLMIEEN'S TURBULENT HALF-JET MODEL AS USED BY KIRK

boundary layer thickness is negligible, $\delta_{2,0} \rightarrow 0$, the only streamline coinciding with a constant velocity line is the dividing streamline. For this case Kirk obtained an expression for base pressure identical to that of Chapman (eq. 3.2.6), except for the numerical value of ϕ_d .

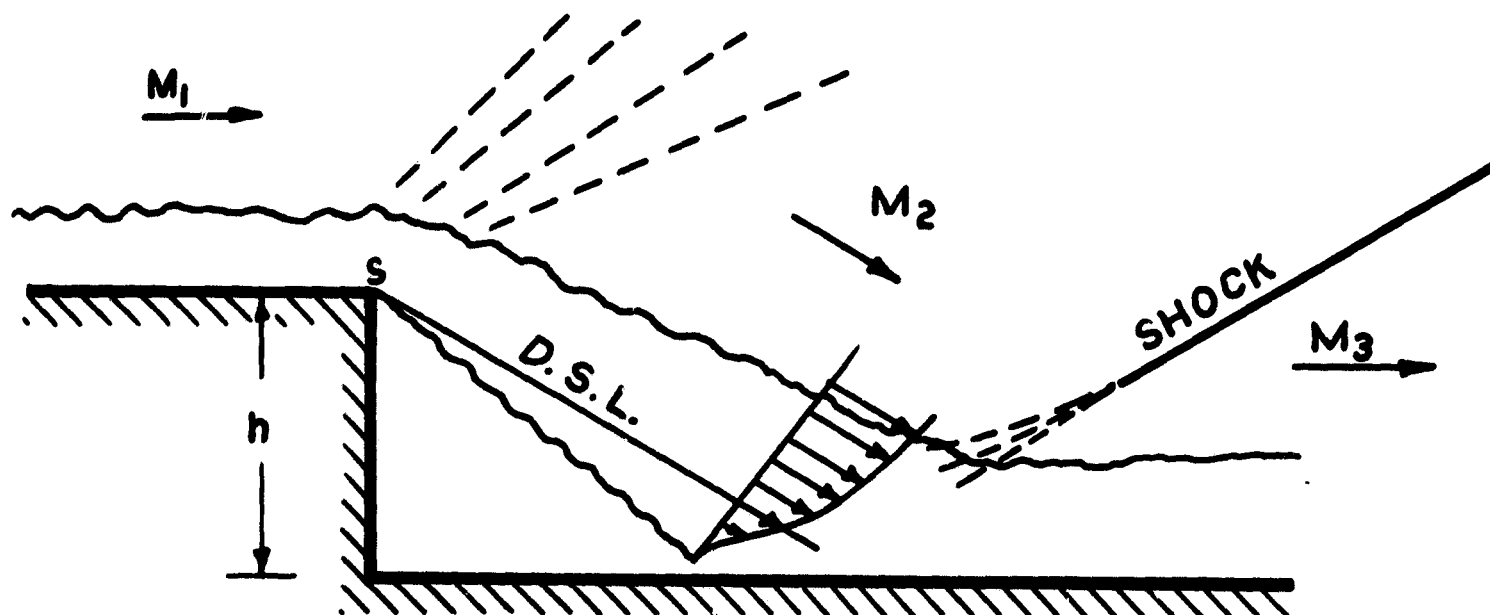
The most important innovation introduced by Kirk is the idea of substituting an asymptotic half-jet, with a displaced origin, for the real mixing layer and in this way allowing for the effect of the oncoming boundary layer, (see Fig. 12).

In order that this equivalent asymptotic half-jet generates velocity profiles downstream from S which are identical with those of the real mixing layer, it is necessary to require that the equivalent half-jet represent a total momentum at S equal to that of the boundary layer at separation and after expansion.

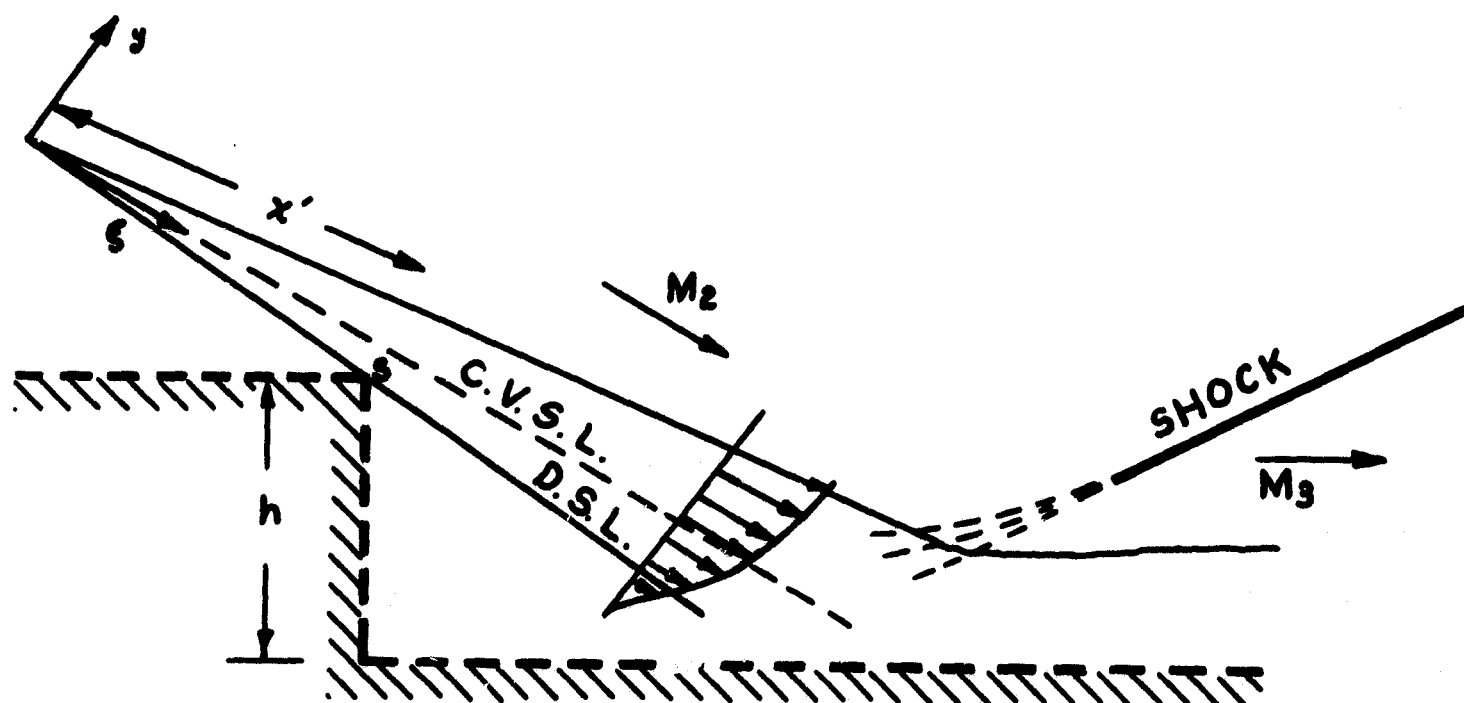
For the case of finite oncoming boundary layer thickness, the dividing streamline and constant velocity streamline are no longer coincident. The mass flow between the constant velocity streamline and an arbitrary streamline is given by,

$$Q = \int_y^{y_c} \rho u dy \quad (3.2.1)$$

Hence, upon introducing a similarity profile, the following expression is obtained for the non-dimensional stream function $f(\eta)$,



(a) REAL MIXING LAYER



(b) EQUIVALENT ASYMPTOTIC FREE JET

FIGURE 8. VIRTUAL ORIGIN CONCEPT

$$\frac{Q}{\rho_{e2} u_{e2} \frac{x_v}{\sigma}} = \int_{\eta}^{\eta_c} \frac{\rho u}{\rho_{e2} u_{e2}} d\eta = f(\eta) \quad (3.2.2)$$

where $\eta = \frac{\sigma y}{x_v}$

σ is a similarity parameter.

But from Tollmien's similar solution can be obtained $\phi = \phi(\eta)$, hence it is possible to set $f(\eta) = F(\phi)$. Thus, the lower edge of the mixing layer is defined by,

$$\frac{Q}{\rho_{e2} u_{e2} \frac{x_v}{\sigma}} = \int_{\eta_b(0)}^{\eta_c(\phi_c)} \frac{\rho u}{\rho_{e2} u_{e2}} d\eta = F(0) \quad (3.2.3)$$

and hence at S,

$$\frac{Q}{\rho_{e2} u_{e2} \frac{x'_v}{\sigma}} = \int_{\eta_s(0)}^{\eta_c(\phi_c)} \frac{\rho u}{\rho_{e2} u_{e2}} d\eta = F(0) \quad (3.2.4)$$

If at the end of the mixing region, the velocity on the dividing streamline reaches ϕ_{d_1} one obtains

$$\frac{Q}{\rho_e U_{e2} \frac{(x_2 + x')}{\sigma}} = F(\phi_{d2}) \quad (3.2.5)$$

then eliminating, the displacement of the virtual origin, x' , between (3.2.5) and (3.2.4), yields

$$\frac{Q}{\rho_e U_{e2}} = \frac{F(\phi_{d2}) F(0)}{F(0) - F(\phi_{d2})} \cdot \frac{x_2}{\sigma} \quad (3.2.6)$$

Finally it can be recognized from eq. (3.2.1) and the definition of momentum thickness that

$$\frac{Q}{\rho_e U_{e2}} = \delta_{2,0}^{**} = \frac{F(\phi_{d2}) F(0)}{F(0) - F(\phi_{d2})} \cdot \frac{x_2}{\sigma} \quad (3.2.7)$$

Since $F(\phi)$ is fully defined by (3.2.3), the function

$$F(\phi_{d2}) F(0) / [F(0) - F(\phi_{d2})]$$

can be determined as a function of ϕ_{d1} . x_1 can be estimated, by assuming a Prandtl-Meyer expansion at S, to be

$$x_2 = h \operatorname{cosec} \vartheta_{2e} \quad (3.2.8)$$

σ is found by comparing calculated velocity profiles with measured ones. Kirk assumed $\sigma = 15$.

It is, of course, necessary in order to relate the base pressure to conditions in the oncoming boundary layer, to estimate $\delta_{1,0}^{**}$ in terms of $\delta_{2,0}^{**}$. Briefly, Kirk obtained the ensuing relationship by assuming that at the base corner immediately after the expansion the separated boundary layer behaves similarly to one expanding along a wall, deducing that,

$$\frac{\delta_{2,0}^{**}}{\delta_{1,0}^{**}} = 1 + \left[\frac{\sqrt{M_{e1}^2 - 1}}{1} + \frac{1 + \left(\frac{\delta_{1,0}^*}{\delta_{1,0}^{**}} \right)}{\sqrt{M_{e1}^2 - 1}} \right] \theta_{2e} \quad (3.2.9)$$

However, the various methods extant for relating conditions immediately upstream of expansion to those immediately downstream, will be reviewed in detail in a subsequent chapter.

It is perhaps appropriate at this juncture to point out the advantage of the constant velocity streamline as a "jet" axis, primarily this is due to the ease with which ϕ_c can be calculated, given an expression for the velocity profile in the half-jet. Secondly, the case of base bleed can be dealt with, simply by distinguishing between the dividing streamline and the separation streamline (i.e. the streamline that emanates from the separation point S) and by setting the bleed mass flux equal to the mass flow between the two streamlines.

Even though the case of finite approaching boundary layer has been dealt with by this theory, it is restricted to thin

approaching boundary layer thickness.

3.3 Chapman's Theory

Following the successful application of the Chapman-Korst flow model by Korst^{19,20,21}, to the case of negligible approaching boundary layer and fully turbulent mixing layer, Chapman²⁸ carried out a similar analysis for the case of fully laminar mixing layer (transition occurring downstream of R in Figs. 7 and 8) with zero initial boundary layer.

The principle difference recognized by Chapman between the laminar and turbulent cases was in the rate of growth of the mixing layer, this being many times greater for the latter. Apart from using results obtained for the laminar mixing of a compressible fluid²⁹, Chapman followed essentially the same path as Korst, although since he began with a less general case his theoretical development was a good deal simpler. A brief exposition of Chapman's theory follows. Assuming the flow along the dividing streamline compresses isentropically to P_3 at the stagnation point R, yields the expression,

$$\frac{P_2}{P_3} = \left(1 + \frac{\gamma-1}{2} M_d^2 \right)^{\frac{\gamma}{\gamma-1}} \quad (3.3.1)$$

For expediency, it is necessary to relate M_d and M_{e2} . Assuming

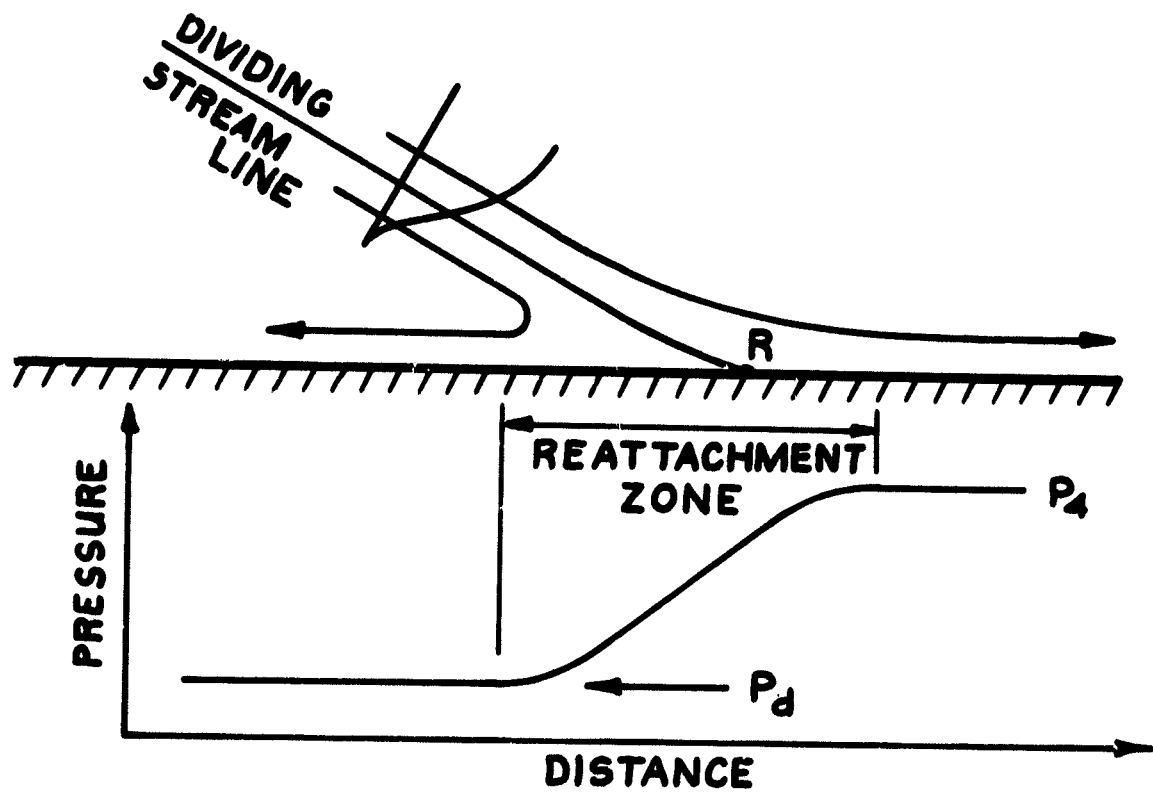


FIGURE 9. REATTACHMENT

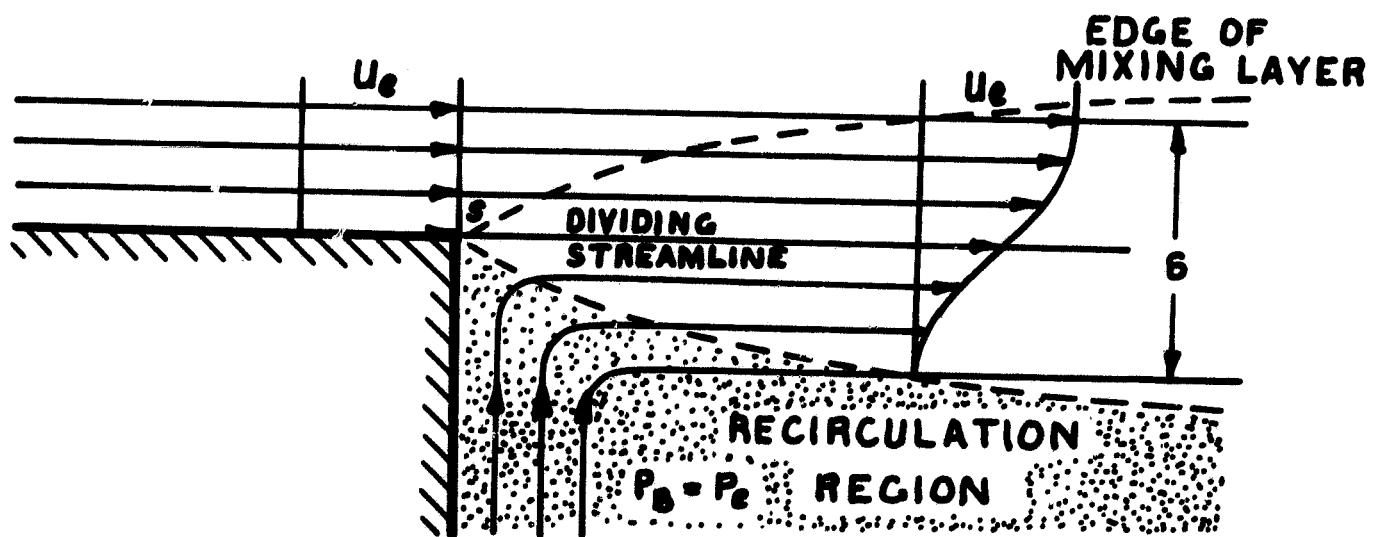


FIGURE 10. MIXING REGION

the Prandtl number in the mixing layer is unity, and that temperature in the recirculation region, T_b , is constant and equal to the recovery temperature (T_{t2e} for $P_r = 1$), Busemann's integral for a perfect gas yields,

$$M_d^2 = \frac{\phi_d^2 Me_2^2}{1 + \frac{\gamma-1}{2} Me_2^2 (1-\phi_d)^2} \quad (3.3.2)$$

combining the above two equations, gives

$$\frac{P_2}{P_3} = \left[\frac{1 + (1-\phi_d^2) \frac{\gamma-1}{2} Me_2^2}{1 + \frac{\gamma-1}{2} Me_2^2} \right]^{\frac{\gamma}{\gamma-1}} \quad (3.3.3)$$

by using the results of an analysis of the laminar mixing of a compressible fluid²⁹, it transpires that $\phi_d = 0.587$ for all Me_2 and is also independent of Reynold's number. And the body shape affects P_d only through P_3 (although for $\delta_{0,1} = 0$ there can be no forebody, in theory). Chapman assumed that, as the recompression shock wave did not form near the viscous layer, then the flow at the edge of the mixing layer recompresses isentropically, thus

$$\frac{P_3}{P_2} = \left(\frac{1 + \frac{\gamma-1}{2} Me_2^2}{1 + \frac{\gamma-1}{2} Me_3^2} \right)^{\frac{\gamma}{\gamma-1}} \quad (3.3.4)$$

this combined with eq. (3.3.3) gives

$$Me_3^2 = (1 - \phi_d^2) Me_2^2 \quad (3.2.5)$$

Thus, eqs. (3.3.3) and (3.3.5) provide an explicit equation for the base pressure

$$\frac{P_2}{P_3} = \left[\frac{1 + \frac{\gamma-1}{2} Me_3^2}{1 + \frac{\gamma-1}{2} \frac{Me_3^2}{(1-\phi_d)^2}} \right]^{\frac{\gamma}{\gamma-1}} \quad (3.3.6)$$

Where Me_3 is usually taken to be identical with the free stream Mach number, M_∞ .

Like Korst, Chapman considered this equation for the base pressure to represent a lower limit.

3.4 Charwat's Modified Model of the Supersonic Laminar Wake

Charwat³⁰ deemed it necessary to modify Chapman's theory in order that the decrease of the base pressure from the constant "Laminar plateau" as the Reynold's number becomes smaller could be explained theoretically (see Fig. 11). He argued as follows. The mass reversed at recompression, i.e. mass scavenged, increases as $Re^{-1/2}$, and the return velocity U_r increases accordingly, and that the dissipation per unit breadth of two-dimensional wake in reversed flow varies qualitatively as follows: (see Fig. 12)

$$\begin{aligned} \phi &= \mu \left(\frac{dU_r}{dy} \right)^2 l, h \sim \mu \left(\frac{U_r}{h} \right)^2 l, h \\ &\sim \mu U_{e2} \left(\frac{1}{Re_c} \right) \left(\frac{cx^2}{h^3} \right) \end{aligned} \quad (3.4.1)$$

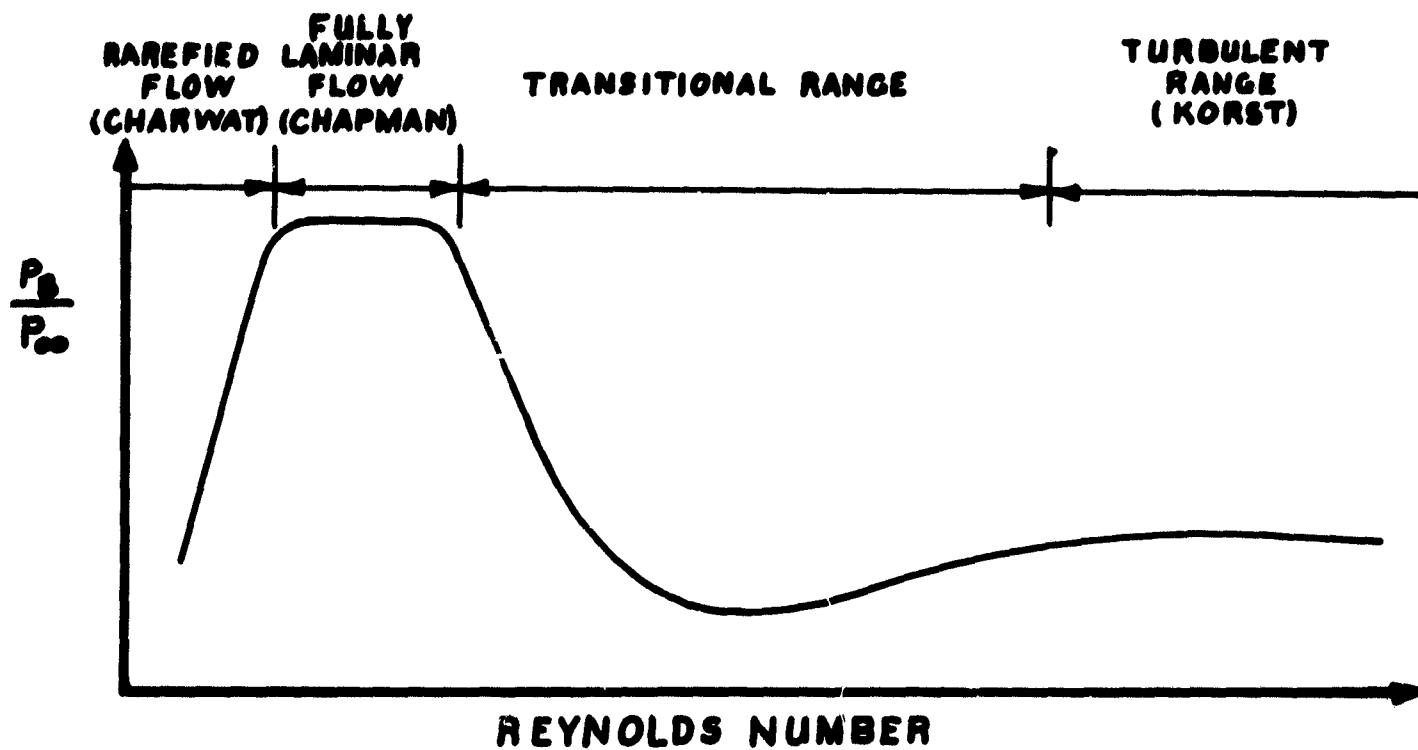


FIGURE 11. BASE PRESSURE RATIO AS A FUNCTION OF REYNOLDS NUMBER.

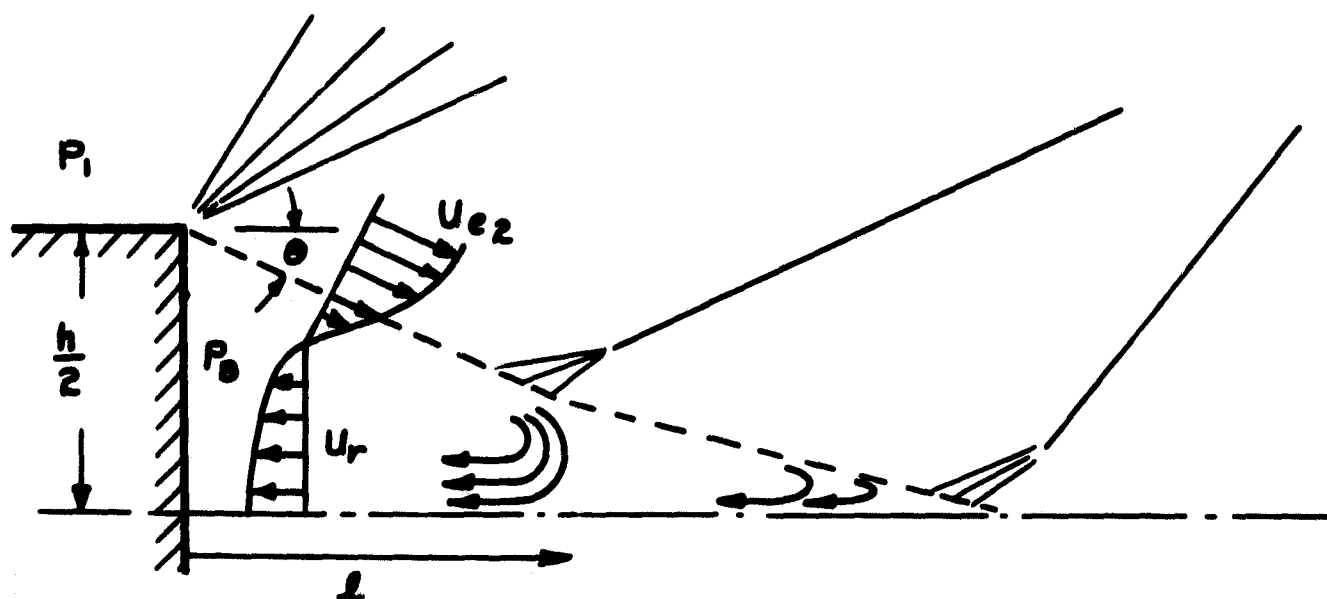


FIGURE 12. CHARWAT'S MODEL. MULTI-SHOCK RECOMPRESSION

where the mass scavenged by mixing layer is represented by $(U_r h)$. It was assumed that the similar asymptotic laminar shear layer velocity profile was still valid. Charwat thought of eq. (3.3.1) as representing the minimum dissipation possible, since it would increase with turbulence in the recirculation region, etc. He represented the work done on an isobaric wake by friction at the dividing streamline by the expression

$$\tau_{LX} \sim \mu U_{e2}^2 \sqrt{Re_c} \left. \frac{d\phi}{d\left(\frac{y}{x\sqrt{Re_c}}\right)} \right|_{y=0} \phi_d \quad (3.4.2)$$

where the velocity gradient and velocity at dividing streamline is obtained from Chapman's solution, hence setting the above two equations equal, it follows approximately that with $l_1 \sim x$

$$\frac{x}{h} = \csc \theta \sim Re_c^{\frac{1}{2}} \quad (3.4.3)$$

Thus, as the Reynold's number grows less, the length of the shear layer decreases, but the expansion angle θ increases resulting in lower base pressure. But then, Charwat reasoned, the recompression dynamics postulated by Chapman are no longer tenable because the shock causes a pressure rise which is larger than that required to reverse the scavenged flow. However, if the recompression occurred through the intermediary of several shocks, then a direct modification of Chapman's analysis gives,

$$\frac{P_2}{P_3} = \left[\frac{1 + \frac{\gamma-1}{2} M_{e_3}^2}{1 + \frac{\gamma-1}{2} \frac{M_{e_3}^2}{(1-\phi_2^2)^\eta}} \right]^{\frac{\gamma}{\gamma-1}} \quad (3.4.4)$$

where η is the number of discrete recompression steps.

3.5 Carriere and Sirieix Work

Carriere and Sirieix in a series of papers^{30,31,32,33,34} dealing with both experimental and analytical studies, developed an approach for analyzing free turbulent mixing which was similar to Korst's. They drew an analogy between the case of base bleed and that of non-negligible approaching boundary layer.

Briefly, for conservation of mass in Fig. 13, it is necessary that

$$q_{bl} + \int_0^{y_e} \rho u dy = \int_{y_d(x)}^{y_e'} \rho u dy \quad (3.5.1)$$

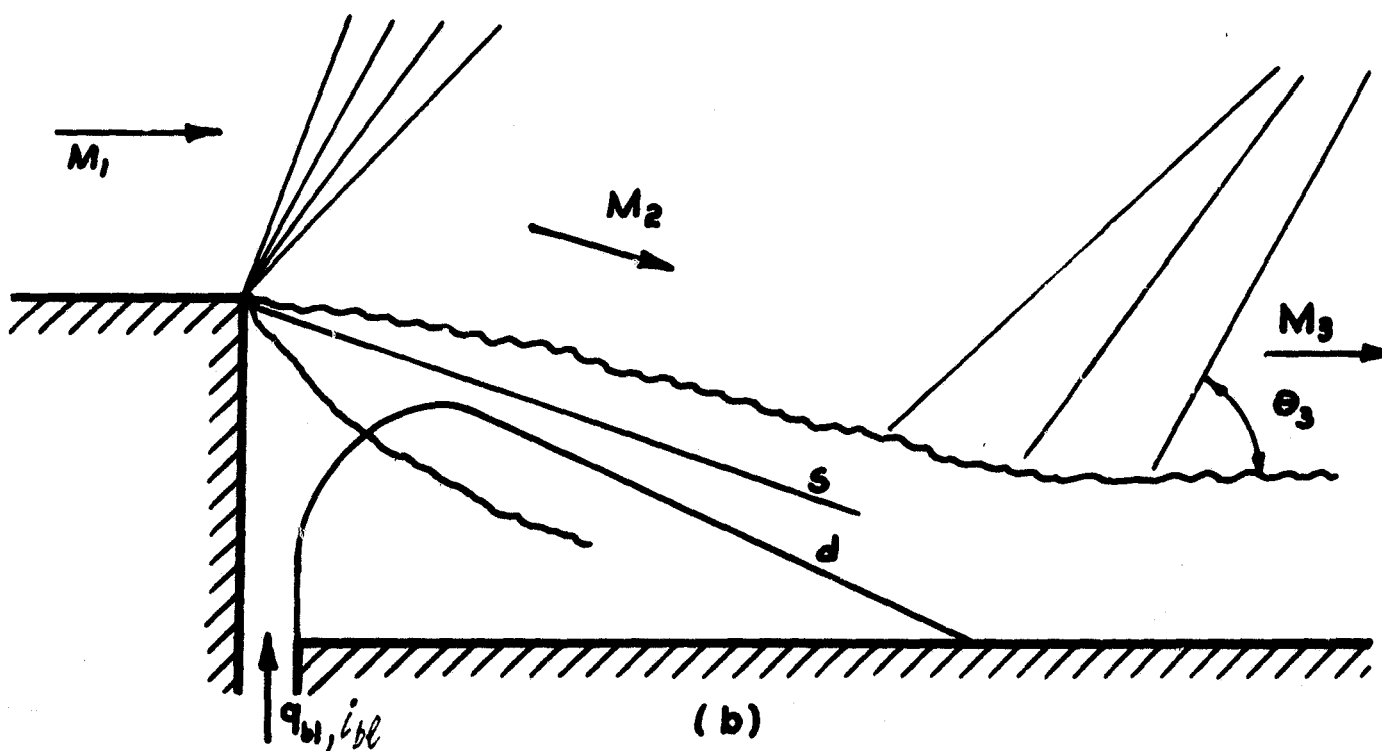
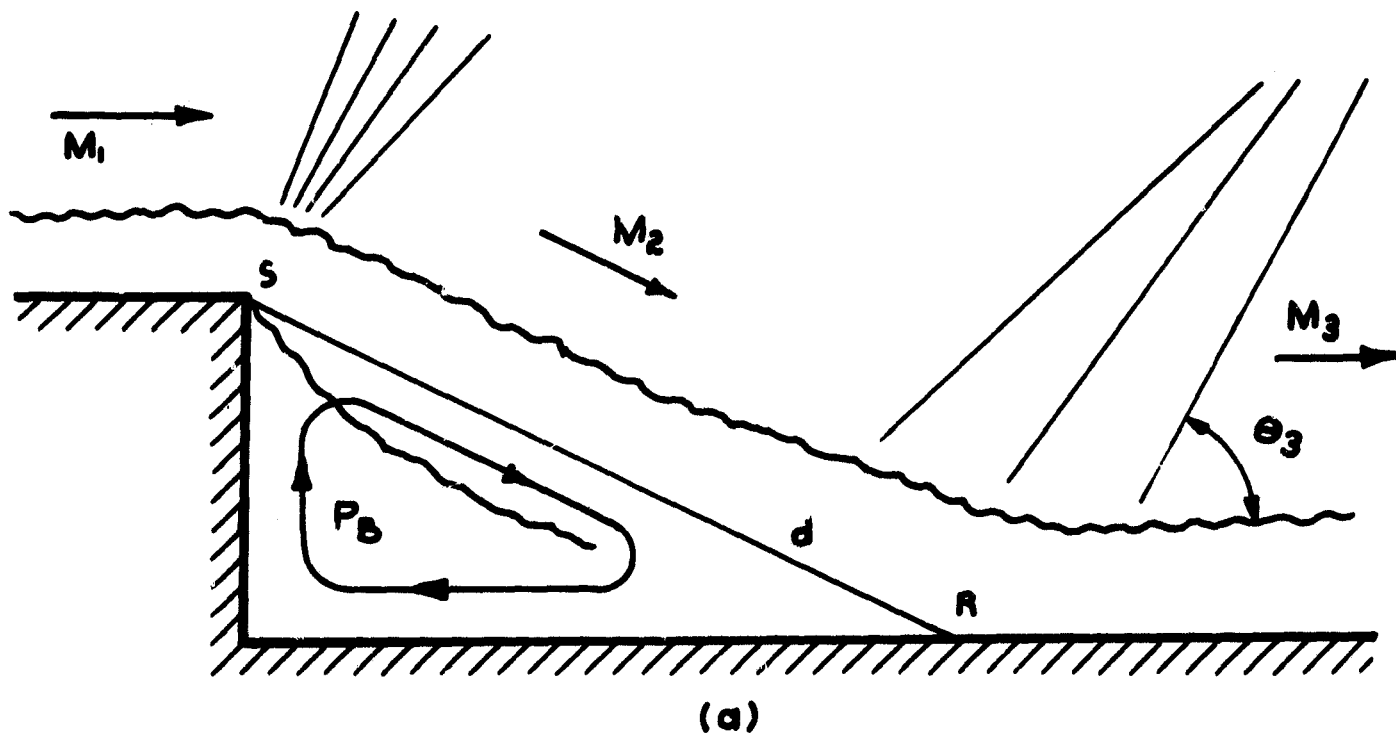
or on introducing non-dimensional variables

$$\frac{\sigma}{x} \frac{q_{bl}}{\rho_{e_2} u_{e_2}} + \frac{\sigma}{x} \int_0^{y_e} \frac{\rho u}{\rho_{e_2} u_{e_2}} dy = \int_{\eta_{d_2}}^{\eta_{e'}} \frac{\rho}{\rho_{e_2}} \phi_2 dy \quad (3.5.2)$$

At the origin, the momentum entering the mixing layer per unit breadth and non-dimensionalized is written

$$\frac{i_{bl}}{\rho_1 u_1^2} + \int_0^{y_e} \frac{\rho u^2}{\rho_{e_2} u_{e_2}^2} dy$$

hence with the assumption of isobaric mixing, conservation of



NOTE: (a) IS EQUIVALENT TO (b).

FIGURE 13. CARRIERE-SIRIEIX ANALOGY

momentum leads to the equation

$$\frac{\delta}{x} \frac{U_{b1}}{\rho_1 U_1^2} + \frac{\delta}{x} \int_0^{y_e} \frac{\rho U^2}{\rho_{e2} U_{e2}^2} dy = \int_{-\infty}^{\eta_{e'}} \frac{\rho}{\rho_{e2}} \phi_2^2 d\eta \quad (3.5.3)$$

Together with the definition of momentum thickness, eqs. (3.5.2) and (3.5.3) combine, yielding,

$$\int_{-\infty}^{\eta_d} \frac{\rho}{\rho_{e2}} \phi_2 d\eta = \frac{\delta}{x} \left(\frac{\rho_{b1} U_{b1}^2}{\rho_{e2} U_{e2}^2} - \frac{\rho_{b1} U_{b1}}{\rho_{e2} U_{e2}} - \delta_{2,0}^{**} \right) + \int_{-\infty}^{\infty} \frac{\rho}{\rho_{e2}} \phi_2 (1 - \phi_2) d\eta \quad (3.5.4)$$

for the determination of the dividing streamline location. Effectively, the approaching boundary layer is regarded as equivalent to a non-dimensional base bleed of $\delta_{2,0}^{**}$.

Carriere and Sirieix wrote the reattachment criterion in terms of the angle the external flow undergoes at recompression,

$$\vartheta_3 = \vartheta_3(Me_2, C_q) \quad (3.5.5)$$

for a small value of C_q , the equivalent mass flow coefficient, it was reasoned that ϑ_2 could be determined by perturbing the ideal case as studied by Korst²⁰. Thus ϑ_3 could be approximated by the linear form,

$$\vartheta_3(Me_2, C_q) = \vartheta_3^0(Me_2) + C_q \frac{\partial \vartheta_3^0(Me_2)}{\partial C_q} \quad (3.5.6)$$

where $\theta_3^0(Me_2)$ and its derivative could be obtained by applying the Chapman-Korst reattachment criterion (i.e. reattachment occurs at the summit of a pressure rise equal to the total pressure on the dividing streamline at the end of the mixing process), and C_g would be given by the equality,

$$C_g = \frac{f_{01} U_{b1}}{f_{e2} U_{e2}} \left(1 - \frac{U_{b1}}{U_{e2}} \right) + \frac{\delta_{2,0}^{**}}{h} \quad (3.5.7)$$

obtained from consideration of (3.5.4).

It was found from an experimental study that this reattachment criterion was not satisfactory in that although the Korst theory predicted $\frac{\partial \theta_3^0}{\partial C_g}$ reasonably accurately, the values predicted for $\theta_3^0(Me_2)$ were consistently too low. Later³⁵ a correction was proposed of the form,

$$\theta_3 = \theta_{3\text{KORST}} - 0.005 \left(\frac{\partial \theta_3^0}{\partial C_g} \right)_{\text{KORST}} \quad (3.5.8)$$

It should be noted that the method of allowing for the initial boundary layer used by Carriere and Sirieix is equivalent to that of Kirk's²⁵ as is demonstrated by a detailed analysis³⁴.

3.6 Karashima's method for calculating the effect of the initial boundary layer.

The theory published by Karashima³⁶ for base pressures was inspired to some extent by Crocco and Lees³⁷ but was based, essentially on the Chapman-Korst model. (Note: Crocco and Lees theory will be discussed in section 4.1).

In essence, Karashima allowed for the effect of the initial boundary layer, by distinguishing between the half-jet axis, which separated the flow originally in the boundary layer from that which was later entrained from the external flow; and the dividing streamline, which separates the recirculating flow from the remainder in the shear layer. Obviously, in the case of zero initial boundary layer and zero bleed flux, the two streamlines coincide.

The 'jet axis' streamline was defined, as

$$\int_{y_1/\Delta_{2,0}}^{\delta/\Delta_{2,0}} \frac{\rho}{\rho_{2e}} \phi \, d\left(\frac{y}{\Delta_{2,0}}\right) = \frac{\dot{m}}{\rho_{2e} U_{2e} \Delta_{2,0}} \quad (3.6.1)$$

where $\Delta_{2,0}$ is a characteristic thickness of the boundary layer immediately after separation.

Using the Crocco-Lees mixing coefficient $k(x)$, defined as

$$\frac{d\dot{m}}{dx} = k(x; Re_c, Me_2) \rho_{2e} U_{2e} \quad (3.6.2)$$

$$\dot{m} = \dot{m}(0; Re_c, Me_2) \quad (3.6.3)$$

eq. (3.6.1) can be written

$$\frac{x}{\Delta_{2,0}\delta} \int_{\eta_j}^{\eta_e} \frac{\rho}{\rho_{e2}} \phi(\eta) d\eta = \frac{1}{\Delta} \int_0^x k(t) dt \quad (3.6.4)$$

k was assumed to be constant, resulting in the above integral being expressed as,

$$\frac{1}{\delta} [f(\eta_e) - f(\eta_j)] = k \quad (3.6.5)$$

The dividing streamline, it was argued, must be positioned in such a way that the mass of fluid existing between it and the 'jet axis' streamline equals that existing in the boundary layer at separation. Hence,

$$\int_{y_d}^{y_j} \rho u dy = \rho_{e2} u_{e2} \Delta_{2,0} \quad (3.6.6)$$

or

$$\frac{x}{\Delta\delta} \{f(\eta_j) - f(\eta_d)\} = 1 \quad (3.6.7)$$

Eliminating δ between eq. (3.6.5) and (3.6.7) leads to

$$K \times \frac{f(\eta_i) - f(\eta_d)}{f(\eta_e) - f(\eta_j)} = \Delta_{2,0} \quad (3.6.8)$$

After applying conservation of mass

$$\rho_{e2} u_{e2} \Delta_{2,0} = \rho_{e1} u_{e1} (\delta_{1,0} - \delta_{1,0}^*) \quad (3.6.9)$$

Gortler's³⁸ first approximation to the solution for the incompressible asymptotic half-jet was assumed a sufficiently accurate description of the compressible shear layer.

That is the velocity was given by

$$\phi = \frac{1}{2} [1 + \operatorname{erf} \eta] \quad (3.6.10)$$

Thus, this would mean that x in eq. (3.6.9) would have to originate from the virtual origin of some equivalent asymptotic shear layer, as in Kirk's theory in section 3.4 (compare eq. 3.6.9 to 3.2.7) to ensure the conservation of momentum, Karashima failed to recognize this.

Another peculiar feature of the theory was the assumption of laminar profiles at separation, transition occurring at separation, followed by fully developed turbulent profiles in the free shear layer. This is clearly an unlikely situation. However, this assumption lead to the following equation, connecting conditions

at the end of mixing to those in the initial boundary layer,

$$\frac{C}{h} \frac{1}{\sqrt{Re_c}} = \frac{0.1482}{h} \frac{M_2}{M_1} \left(\frac{5+M_1^2}{5+M_2^2} \right)^{\frac{1}{2}} \csc \theta_2 \frac{f(\eta_j) - f(\eta_{dr})}{f(\eta_e) - f(\eta_j)} \quad (3.6.11)$$

Consequently, Karashima's theory predicted base pressures in this functional form

$$P_b = P_b \left(M_1, \frac{C}{h} \frac{1}{\sqrt{Re_c}} \right) \quad (3.6.12)$$

In line with Chapman's correlation in refs. 17 and 28 and the predictions of Crocco and Lees³⁷.

3.7 Nash's Theory

In most respects, the theory for turbulent base pressures proposed by Nash³⁹, is simply an improvement on methods introduced by Kirk. However, in at least one important respect it is radically different. Like Carriere and Sirieix, Nash recognized that the Chapman-Korst reattachment criterion was invalid for a turbulent mixing layer. To rectify this, he introduced an additional parameter, (see Fig. 14).

$$N = \frac{P_r - P_2}{P_1 - P_2} \quad (3.7.1)$$

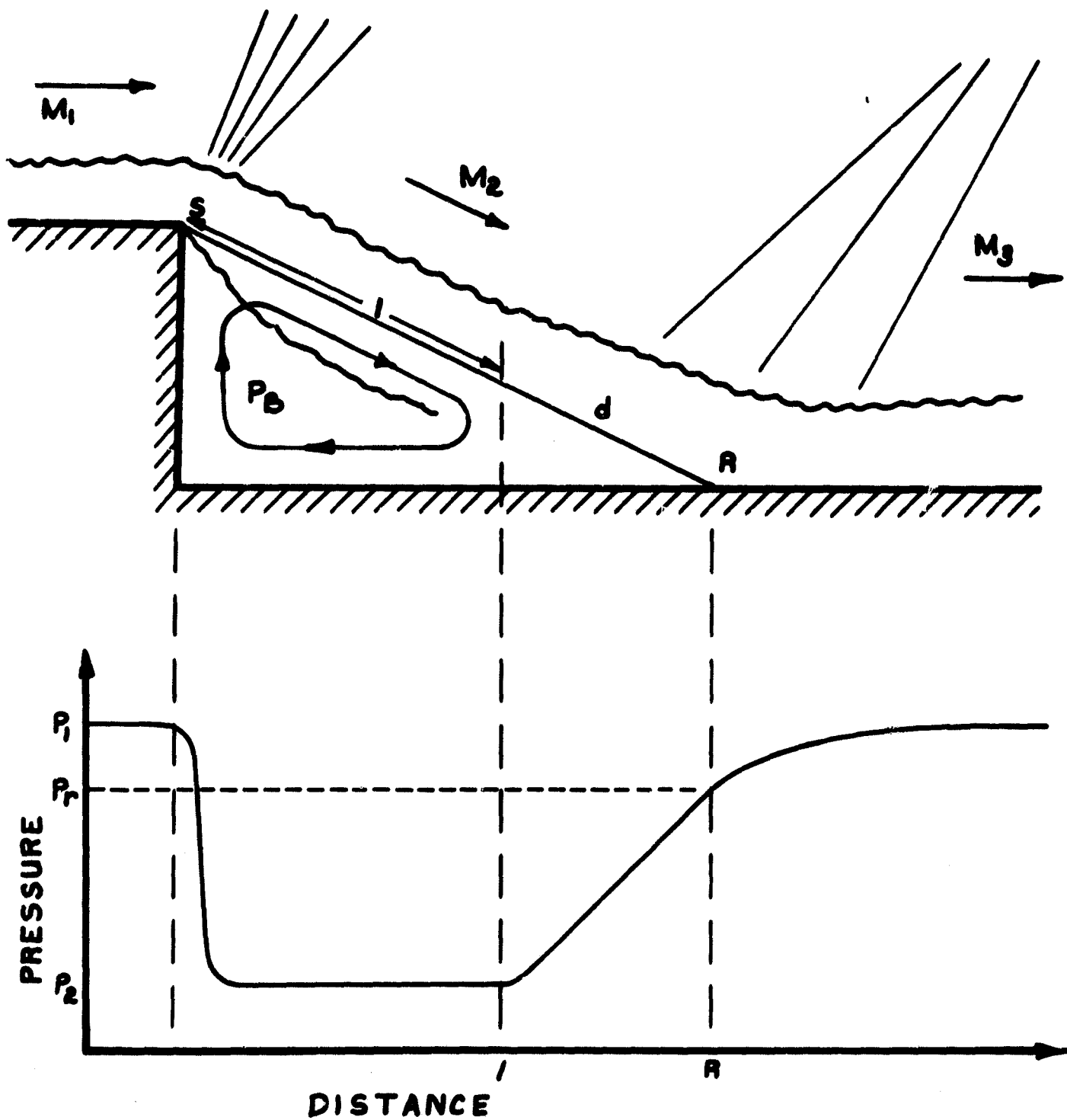


FIGURE 14. PRESSURE RISE TO REATTACHMENT

On the basis of the experimental evidence, then available, it was proposed that for supersonic flow, $N = 0.35$. Later, it was recognized⁴⁰ that N varied with $\delta_{2,0}^{**}$ and M_2 .

Nash used a result, for the velocity profile in the mixing layer, originally put forward by Gortler³⁸ for the incompressible case, i.e.

$$\phi = \frac{u}{u_e} = \frac{1}{2} (1 + \operatorname{erf} \frac{\sigma y}{x}) \quad (3.7.2)$$

where σ was selected to give the best fit with compressible data, the linear interpolation of Korst and Tripp²⁴ was used, viz.

$$\sigma = 12 (1 + 0.23 M_{e2}) \quad (3.7.3)$$

The mass flux between the separation streamline and any adjacent streamline was derived similarly to the expressions in section (3.2) and expressed in terms of functions which relate to the velocity profile of the asymptotic turbulent mixing layer, and the momentum thickness $\delta_{2,0}^{**}$

$$\frac{\psi_s - \psi}{\rho_{e2} u_{e2}} = x F(\phi_d) - \delta_{2,0}^{**} \left\{ 1 - \frac{F(\phi_d)}{F(0)} \right\} \quad (3.7.4)$$

By using the following definition for the constant velocity streamline,

$$u_e \int_{y_c}^{y_e} \rho u dy = \int_{-\infty}^{y_e} \rho u^2 dy \quad (3.7.5)$$

and eq. (3.7.2), the constant velocity ϕ_c was evaluated to be

$$\phi_c = 0.348 + 0.018 M_{e2} \quad (3.7.6)$$

Hence, the non-dimensional streamfunction $F(\phi)$ can be expressed by

$$F(\phi) = \frac{1}{\sigma} \int_{\eta}^{\eta_c} \frac{\rho}{\rho_{e2}} \phi d\eta \quad (3.7.7)$$

Making a change of variable from η to ϕ

$$F(\phi) = \frac{1}{\sigma} \int_{\phi}^{\phi_c} \frac{\rho}{\rho_{e2}} \phi \frac{d\eta}{d\phi} d\phi \quad (3.7.8)$$

A further change of variable was made from ϕ to ρ/ρ_{e2} ,

whence

$$F(\phi) = \frac{\sqrt{\pi}}{(\gamma-1)\sigma M_{e2}^2} \int_{\rho/\rho_{e2}}^{\rho_c/\rho_{e2}} \frac{d(\rho/\rho_{e2})}{\rho/\rho_{e2}} \quad (3.7.9)$$

$$= \frac{\sqrt{\pi}}{(\gamma-1)\sigma M_{e2}^2} \log\left(\frac{\rho_c}{\rho}\right) \quad (3.7.10)$$

where use was made of the well known relation,

$$\frac{\rho}{\rho_{e2}} = \left[1 + \frac{\gamma-1}{2} M_{e2}^2 (1-\phi^2) \right]^{-1} \quad (3.7.11)$$

to obtain

$$\frac{p_c}{p} = \frac{1 + \frac{\gamma-1}{2} M_{e2}^2 (1 - \phi^2)}{1 + \frac{\gamma-1}{2} M_{e2}^2 (1 - \phi_c^2)} \quad (3.7.12)$$

With the additional assumption that the recompression process is isentropic along streamlines Nash obtained,

$$\begin{aligned} q_{bl} &= \psi_s - \psi_d \\ &= p_{e2} U_{e2} \frac{h F(\phi_d)}{\sin(\theta_2 - \theta_1)} - p_{e2} U_{e2} \delta_{2,0}^{**} \left\{ 1 - \frac{F(\phi_d)}{F(0)} \right\} \end{aligned} \quad (3.7.13)$$

where

$$F(\phi_d) = \frac{\sqrt{\pi}}{(\gamma-1) \delta M_{e2}^2} \log \left\{ \frac{p_c}{p_b} \left(\frac{p_2}{p_r} \right)^{\frac{\gamma-1}{\gamma}} \right\} \quad (3.7.14)$$

and

$$\frac{F(\phi_d)}{F(0)} = 1 - \frac{\log \left(\frac{p_r}{p_b} \right)^{\frac{\gamma-1}{\gamma}}}{\log \left(\frac{p_c}{p_b} \right)} \quad (3.7.15)$$

For the case of zero bleed, the separation streamline and dividing streamline (or reattaching streamline) are coincident and $q_{bl} = 0$.

The base pressures are calculated by assuming a range of values for p_2 and ascertaining from eq. (3.7.13) what value of

the boundary-layer momentum thickness is necessary to support a particular value of base pressure.

The following relation, connecting boundary layer momentum thickness before and after the expansion, was used

$$\frac{\rho_{e2} u_{e2} \delta_{2,0}^{**}}{\rho_{e1} u_{e1} \delta_{1,0}^{**}} = \frac{Me_1^2}{Me_2^2} \quad (3.7.16)$$

This will be discussed in a separate chapter.

Nash's theory is restricted to the two-dimensional case with thin approaching boundary layer and no heat transfer effects.

3.8 Cooke's Theory

In ref. 57, Cooke presents theoretical methods for the determination of base pressure behind rear-ward facing steps with both laminar and turbulent mixing layers. The following are the salient features of his treatment which differ from previous work.

1) Lock's⁹⁸ velocity profiles were used for the laminar mixing layer.. Lock used Pohlhausen methods to obtain approximate solutions, taking sine functions for his velocity profiles,

$$\phi_U = \phi_d + (1 - \phi_d) \sin \frac{\pi}{2} \eta_U, \quad \eta = \frac{y}{\delta_U} \quad (3.8.1)$$

for the part above the dividing streamline, and

$$\phi_L = \phi_d - \phi_d \sin \frac{\pi}{2} \eta_L, \quad \eta_L = \frac{-y}{\delta_L} \quad (3.8.2)$$

for the part below the dividing streamline, where $y = 0$. δ_u and δ_l are "thicknesses" in the Pohlhausen sense of the upper and lower mixing regions. ϕ_d is, here, a function of x approaching 0.587 as x tends to infinity.

2) On the basis of the examination of pictures of the flow, the beginning of the pressure rise and end of mixing layer development was taken to be where the edges of the mixing region (if continued) would meet the Wall. See Fig. 15. This is in contrast to previous investigations which assumed $x = h \operatorname{cosec} (\theta_{2e} - \theta_{1e})$.

3) It was assumed that reattachment took place half-way up the pressure rise at recompression, i.e. $N = 0.5$, for both laminar and turbulent case.

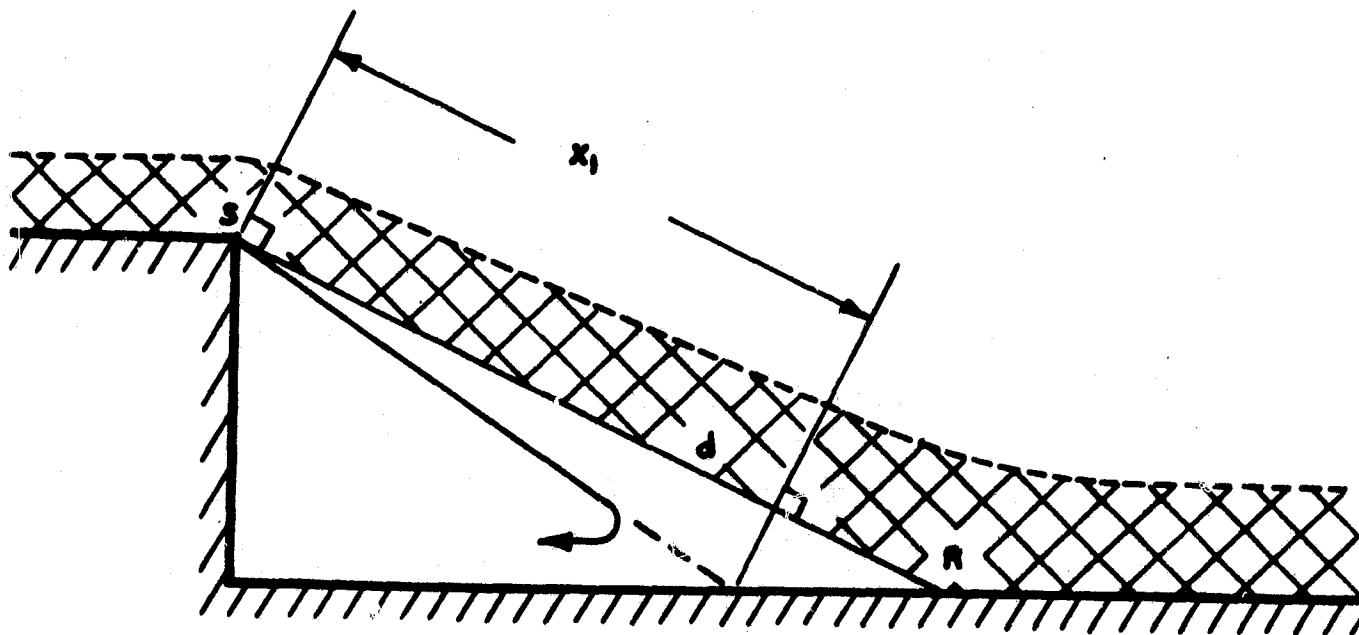


FIGURE 15. ESTIMATION OF THE LENGTH OF MIXING LAYER

3.9 MacDonald's Theory

MacDonald⁴¹ introduced a base pressure theory, for the case of turbulent mixing layer behind a step, which is radically different in many respects from preceding theoretical contributions. This being the case, his methods will be dealt with in some detail, and divided into subsections. This analysis was carried out with the understanding that both initial Mach number and boundary-layer thickness were known. A base pressure was assumed and the analysis developed by tracing the boundary layer development from upstream of the base to the end of the pressure rise. MacDonald assumed the same expression as Nash³⁹ (eq. 3.7.16) to allow for the effect of the initial expansion, but in the analysis of the other sections of the flow field he introduces important new ideas.

a. The Free Shear Layer

MacDonald argued that the various velocity profiles developed to date, in particular, the first approximation to Gortler's solution³⁸,

$$\phi = \frac{1}{2}(1 + \operatorname{erf} \eta) \quad (3.9.1)$$

were strictly speaking only solutions to the incompressible equations. However, if transformation relations, in the style of Mager⁴² and Coles⁴³ were introduced, the compressible turbulent equations of motion could be reduced to their incompressible equivalents. From Cole's discussion, MacDonald inferred that if the coefficient of eddy viscosity, $\mathcal{E}$, is independent of y in the

transformed plane (i.e. incompressible plane), then the same transformation is obtained as for the laminar case provided that a certain relationship holds between the shear stresses in compressible and incompressible turbulent shear flow.

The transformation was defined by the relationship,

$$\psi^t = \psi, \quad x^t = \Lambda x, \quad y^t = \int_0^y \left(\frac{\rho}{\rho_s} \right) dy \quad (3.9.2)$$

whence it can easily be shown that

$$u = u^t, \quad \phi = \phi^t \quad (3.9.2a)$$

Λ is a parameter dependent on Mach number, and having the following form

$$\Lambda = \frac{\chi}{\sigma/\sigma^t} \quad (3.9.3)$$

This relation was deduced from the fact that $\eta = \eta^t$ and from certain experimental evidence. In ref. 48, the following equation is given,

$$\frac{\sigma}{\sigma^t} = \frac{(1 + \frac{\gamma-1}{2} M_{e_2}^2)(1 + 0.035 M_{e_2}^2)}{(1 + 0.004 M_{e_2}^4)} \quad (3.9.4)$$

On the basis of fitting velocity profiles to experimental data the following was obtained

$$\Lambda = (1 + \frac{\gamma-1}{2} M_{e_2}^2)^{-1} \quad (3.9.5)$$

In eq. (3.9.2), ρ_s , the density at the inner edge of the shear layer, was chosen as a reference so that at the inner edge of the shear layer, the transformation becomes an identity transformation. On the assumption that the total energy in the shear layer is constant, there resulted,

$$\frac{\rho}{\rho_s} = \frac{1}{1 - C^2 \phi^2} \quad (3.9.6)$$

where

$$C^2 = \frac{\frac{\gamma-1}{2} M_{e_2}^2}{1 + \frac{\gamma-1}{2} M_{e_2}^2} \quad (3.9.7)$$

These transformation relations yielded various results, two of the most important are for the non-dimensional stream function,

$$F(\phi) = \frac{\int_{y_s^t}^{y^t} \rho_s u^t dy^t}{\rho_s u_{e_2}^t \delta_{2,0}^{***t}} \quad (3.9.8)$$

and

$$\frac{\delta_{2,0}^{**}}{X} = \frac{\delta_{2,0}^{***t}}{X^t} \left(1 + \frac{\gamma-1}{2} M_{e_2}^2 \right)^{-2} \quad (3.9.9)$$

Using methods analagous to Nash³⁹ (section 3.6) a velocity profile can be found up to the start of the pressure rise. The development length, x_d , of the shear layer being taken as,

$$x_d = h \operatorname{cosec}(\vartheta_2 + \Delta\vartheta) \quad (3.9.10)$$

where $\Delta\vartheta$ is the angular spread of the jet. From definition it was deduced that σ is inversely proportional to the rate of spread of the asymptotic jet, hence, since 15° is the jet spread for the incompressible case,

$$\Delta\vartheta = 15^\circ \times \left(1 + \frac{\gamma-1}{2} M_{e_2}^2\right)^{-1} \quad (3.9.11)$$

b. The Reattachme .t Region

The main purpose of this part of the analysis was to obtain expressions in terms of known quantities for the boundary layer thickness, displacement and momentum thicknesses at reattachment. Initially, MacDonald used the assumption that the total pressure along the dividing streamline equals the pressure rise to reattach-ment, hence

$$\frac{P_r}{P_2} = \left(1 - c^2 \phi_d^2\right)^{\frac{\gamma}{\gamma-1}} \quad (3.9.12)$$

Using isentropic flow relations along a streamline, according to Bernoulli

$$U_{2'}^2 = U_2^2 \left\{ 1 + \frac{2}{(\gamma-1)M_2^2} \left[1 - \left(\frac{P_r}{P_2}\right)^{\frac{\gamma-1}{\gamma}} \right] \right\} \quad (3.9.13)$$

whence

$$\frac{P_2' u_2'^2}{P_2 u_2^2} = \left(\frac{P_r}{P_2} \right)^{\frac{1}{r}} \left\{ 1 - \frac{2}{(r-1)M_2^2} \left[\left(\frac{P_e}{P_2} \right)^{\frac{r-1}{r}} - 1 \right] \right\} \quad (3.9.14)$$

Using conservation of mass, that is $P_2 u_2 \delta y_2 = P_2' u_2' \delta y_2'$, he obtained

$$\delta_2' = \int_d^e dy_2' = \int_d^e \left(\frac{P_r}{P_2} \right)^{\frac{1}{r}} \left\{ 1 + \frac{1}{(r-1)M_2^2} \left[\left(\frac{P_r}{P_2} \right)^{\frac{r-1}{r}} - 1 \right] \right\} dy_2 \quad (3.9.15)$$

where

$$\frac{1}{M_2^2} = \frac{1}{M_{e2}^2 (1 - c^2)} \left(\frac{1}{\phi^2} - c^2 \right) \quad (3.9.16)$$

and e has the usual meaning that is the point where $\phi = 0.999$.

MACDONALD'S CONTROL VOLUME

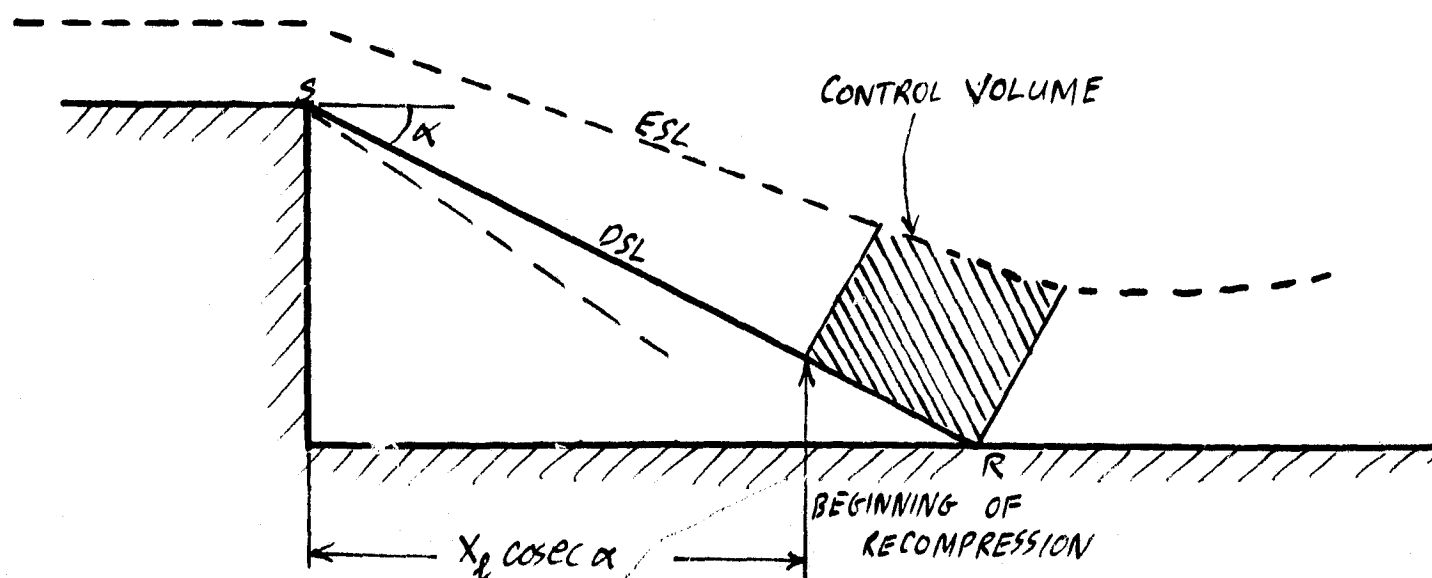


Figure 16

MacDonald then introduced a control volume, see Fig. 16. Carrying out a momentum balance on this control volume yielded,

$$\begin{aligned} \int_d^e p_2 u_2^2 dy_2 - \int_d^e p_2' u_2'^2 dy_2' \\ = (\delta_2' p_2' - \delta_2 p_2) - \int_{x_2}^{x_r} u_e \frac{dm}{dx} dx + \int_{x_2}^{x_r} \tau_d dx \end{aligned} \quad (3.9.17)$$

He neglected the last two terms on the grounds that they were typically of an order less than the pressure terms. Hence,

$$\int_d^e p_2 u_2^2 dy_2 - \int_d^e p_2' u_2'^2 dy_2' = (\delta_2' p_2' - \delta_2 p_2) \quad (3.9.18)$$

making the same approximation in the case of a mass balance gave,

$$\int_d^e p_2 u_2 dy_2 = \int_d^e p_2' u_2' dy_2' \quad (3.9.19)$$

By using the transformation relations, the above expressions (3.9.18 and 19) and Kirk's virtual origin concept, the following relationships were obtained,

$$\frac{p_{e2}}{p_s} (1 - C^2) \frac{\sigma^t}{x_v^t} \delta_2 = (\eta_{e2}^t - \eta_{d2}^t) - C^2 I_2^t \quad (3.9.20)$$

$$\frac{p_{e_2}}{p_s} \frac{\sigma^t}{x_u^t} I_1(\delta_2) = I_1^t \quad (3.9.21)$$

$$\frac{p_{e_2}}{p_s} \frac{U^t}{x_u^t} I_2(\delta_2) = I_2^t \quad (3.9.22)$$

where

$$I_1^t = \int_{\eta_{d_2}^t}^{\eta_{e_2}^t} \phi d\eta^t \quad (3.9.23)$$

$$I_2^t = \int_{\eta_{d_2}^t}^{\eta_{e_2}^t} \phi^2 d\eta^t \quad (3.9.24)$$

and

$$\frac{p_{e_2}}{p_s} \frac{\sigma^t}{x_{u_2}^t} \delta_{2'}^{**} = \frac{p_{e_2}}{p_s} \frac{\sigma^t}{x_{u_2}^t} \delta_2 - \frac{I_1^t}{\left(\frac{p_e}{p_s}\right)^{\frac{1}{r}} [1 - E(1 - C^2)]} \quad (3.9.25)$$

$$\frac{\rho_{e2}}{\rho_3} \frac{\sigma^t}{x_e^t} \delta_2^{**} = \frac{I_1^t}{\left(\frac{\rho}{\rho_2}\right)^{\frac{2}{r}} [1-E(1-C^2)]} - \frac{I_2^t - A}{\left(\frac{\rho}{\rho_2}\right)^{\frac{2}{r}} [1-2E(1-C^2)]} \quad (3.9.26)$$

where

$$E = \frac{\left(\frac{\rho}{\rho_2}\right)^{\frac{r-1}{r}} - 1}{(\gamma-1) M_{e2}^2 (1-C^2)} \quad (3.9.27)$$

and

$$A = \frac{\rho_{e2}}{\rho_3} \frac{\sigma^t}{x_{u2}^t} \frac{\delta_2' P_r - \delta_2 P_2}{\rho_{e2} u_{e2}^2} \quad (3.9.28)$$

This completed the solution up to the reattachment point.

c. The Rehabilitation Region

MacDonald found that application of the preceding analysis predicted that the shape parameter δ^*/δ^{**} would increase during rehabilitation. This result was at variance with experimentally measured decreases in the shape parameter.

This apparently anomalous behavior between theory and experiment was explained by reference to the fact that in the previous section, no restrictions were placed on the development of the displacement thickness, which was free to vary in accordance

with predictions, simply by altering the position of the dividing streamline.

However, downstream of reattachment the dividing streamline is replaced by a fixed wall, and because of this it was argued that the assumption of negligible mass addition was violated. The reason for this was suggested to be that, immediately after reattachment, the fact that the fluid is now flowing parallel to the wall is not instantaneously transmitted to the free stream. Thus, for some distance, the free stream continues to feed the boundary layer with mass and momentum, resulting finally in a profile for which the shape parameter decreases with increase in pressure. Rather than estimating this mixing rate, an empirical correction to the existing theory was obtained, using the result of Squire and Young⁴⁴ which correlates the change in momentum thickness during the rehabilitation of a wake in incompressible flow.

$$\frac{\delta_3^{**}}{\delta_{2'}^{**}} = \left(\frac{U_{e2'}}{U_{e3}} \right)^{H_{2'} + 2 - \frac{(H_{2'} + H_3 - 2)(H_{2'} - H_3)}{2(H_{2'} - 1)}} \quad \text{--9.29)}$$

MacDonald justified using a wake recompression law and applying it to the case of boundary layer recompression, by appealing to the fact that the length of the recompression region is of the same order of magnitude as the boundary layer thickness, and thus

the neglected shear stress, which is in any case initially zero, would not have much affect on the value of the momentum thickness.

From Eq. (3.9.29) using the transformation relations in ref. 42 was obtained,

$$\begin{aligned} & \log\left(\frac{\delta_3^{**T}}{\delta_{2'}^{**T}}\right) \\ &= (H_{2'})^T + 2 - \left\{ \frac{(H_{2'}^T + H_3^T - 2)(H_{2'}^T - H_3^T)}{2(H_{2'}^T - 1)} \right\} \times \log\left(\frac{M_{2'}}{M_3}\right) \end{aligned} \quad (3.9.30)$$

where

$$\delta^{**} = \delta^{**T} \left(1 + \frac{\gamma-1}{2} M_e^2\right)^3 \quad (3.9.31)$$

$$H = \left(1 + \frac{\gamma-1}{2} M_e^2\right) H^T + \frac{\gamma-1}{2} M_e^2 \quad (3.9.32)$$

Noting that

$$1 + \frac{\gamma-1}{2} M_{e2'}^2 = \left(\frac{P_2'}{P_3}\right)^{-\frac{\gamma-1}{\gamma}} \left(1 + \frac{\gamma-1}{2} M_{e1}^2\right) \quad (3.9.33)$$

The momentum and moment of momentum equations in the transformed plane, neglecting the wall shear stress, are given by

$$\frac{d\delta^{**T}}{\delta^{**T}} + (2 + H^T) \frac{dM_e}{M_e} = 0 \quad (3.9.34)$$

$$dH^T = - \frac{dMe}{2Me} H^T (H^{T2} - 1)(H^T + 1) \quad (3.9.35)$$

The integration of these equations gave the momentum thickness and shape parameter at the end of the pressure rise. Eq. (3.9.35) was expected to give the wrong answer, as explained above, thus the following correction was proposed

$$dH_{actual}^T = \lambda dH_{theory}^T \quad (3.9.36)$$

where λ was to be determined empirically.

Thus the solution of eq. (3.9.35) was given as

$$\left(\frac{Me_3}{Me_2} \right)^{\frac{1}{\lambda}} = \frac{g(H_3^T)}{g(H_2^T)} \quad (3.9.37)$$

$$g(H^T) = \frac{H^{T2}}{(H^{T2} - 1)^{\frac{1}{2}} (H^T + 1)} \exp\left(\frac{1}{H^T + 1}\right) \quad (3.9.38)$$

A case for the "real" momentum thickness and the "theoretical" momentum thickness being equal was presented above, thus

$$d\delta_{act}^{**} = d\delta_{th}^{**}$$

or

$$d\delta_{act}^{**T} = d\delta_{th}^{**T}$$

Thus substituting (3.9.37) into eq. (3.9.34) and integrating yielded,

$$\left(\frac{\delta_{3act}^{**T}}{\delta_{2act}^{**T}} \right)^{\frac{1}{\lambda}} = \frac{G(H_{3act}^T)}{G(H_{2act}^T)} \quad (3.9.39)$$

where,

$$G(H^T) = \frac{(H^{T2} - 1)^{\frac{3}{2}} (H^T + 1)}{H^{T4}} \exp\left(-\frac{1}{H^T + 1}\right) \quad (3.9.40)$$

Thus eqs. (3.9.37 - 40) provided a complete solution to the problem in terms of λ . By adopting various values of λ , eqs. (3.9.38 - 40) provided values of the final shape parameter and momentum thickness, given the initial conditions and final Mach number. Using these given conditions and the predicted value of the final shape parameter, the Squire and Young formula (eq. 3.9.29) gave another estimate of the final momentum thickness. It was found that with $\lambda = -1$ good agreement could be obtained between the two estimates. Thus, MacDonald arrived at the simple result that for a reattaching boundary layer the rate of change of shape parameter with pressure rise is merely minus that which occurs in a detaching boundary layer. This result is supported by experimental evidence of Mueller and Robertson⁴⁶.

In order to calculate base pressures, a further empirical relation was required for H^T , that of Maskell⁴⁷ was used, viz.

$$H^T = 1.754 - 0.149 \log_{10} R_{\delta^{**T}}^T + 0.0105 [\log_{10} R_{\delta^{**T}}^T]^2 \quad (3.9.41)$$

where from Mager transformation⁴²

$$R_{\delta^{**T}}^T = (1 + \frac{\gamma-1}{2} M_e^2)^{-0.76} R_{\delta^{**}} \quad (3.9.42)$$

Given the initial Mach number M_e , and boundary-layer momentum thickness just upstream of the base, the base pressure was obtained by following this procedure:

(i) Assume a base pressure $\frac{p_2}{p_1}$ and, using Prandtl-Meyer flow relationships, find M_{e2} , ρ_{e2}/ρ_s , c^2 and ϑ_2 .

(ii) Find $\delta_{2,0}^{**}$ from eq. (3.7.16) and $\Delta\vartheta$ from eq. (3.9.11): thus, from eq. (3.9.10) obtain $\delta_{2,0}^{**} / x_l$

(iii) Using (3.9.9) obtain $\frac{\delta_{2,0}^t}{x^t}$ and hence ϕ_{d_l}

(iv) From (3.9.1) obtain $\eta_{d_2}^t$ and hence I_1^t, I_2^t .

(v) From (3.9.6) obtain p_2'/p_2 .

(vi) From (3.9.2), (3.9.3), (3.9.4) and expression from Kirk for origin shift (viz. $x_{vl}^t = 30 \delta^{**t} + x_l^t$) obtain x_{vl}^t .

(vii) From (3.9.20), (3.9.25) and (3.9.26) obtain $\delta_{2,1}$, $\delta_{2,1}^*$, $\delta_{2,1}^{**}$, and $H_{2,1}$.

(viii) Obtain M_2' from (3.9.33) hence H_2^T , from (3.9.32) and δ_2^{**} from (3.9.31).

(ix) Use isentropic relationships to give $M_3 = M_1$; so with $\lambda = -1$, obtain H_3^T from (3.9.37 and 38) and δ_3^{**} from (3.9.39 and 40).

(x) Compare H_3^T with value given by (3.9.42) and so iterate to find $\frac{p_2}{p_1}$.

MacDonald⁴⁸ extended this theory to include the case of the axisymmetric step. This was achieved by introducing the Mangler⁴⁹ transformation relations which transform the axisymmetric equations of motion into their two-dimensional equivalents. The transformation relations were written,

$$x^M = h^2 \int_0^x \frac{r^2(s)}{\ell^2} ds \quad (3.9.44)$$

$$y^M = h \frac{r(x)}{\ell} \quad (3.9.45)$$

In order that this transformation be valid for the case of turbulent free shear layers, it was necessary for the eddy viscosity, ϵ , to be the same at corresponding points in the real and transformed flow. Since ϵ is practically independent of y in both two-dimensional and axisymmetric free shear flows, this reduced to

showing that corresponding x stations the value was the same. Assuming that δ grows linearly with x at the same rate in both planes, there resulted

$$\frac{\epsilon}{\epsilon^M} = \frac{\delta}{\delta^M} = \frac{x}{x^M} = 1$$

i.e.

$$\frac{\delta}{x} = \frac{\delta^M}{x^M}$$

(3.9.46)

MacDonald demonstrated that provided the radius of the free shear layer does not diminish by more than a factor of 2, then this required relationship was satisfied. This condition may eliminate the possibility of using this theory for predicting pressures behind axisymmetric bases as opposed to steps. Moreover the theory was restricted to the case of thin approaching boundary layers and no provision was made for any heat transfer effects.

3.10 The Stratford-Townsend-MacDonald Two Layer Reattachment Model

MacDonald⁵¹ applied a separation flow model introduced by Stratford⁵² and improved by Townsend^{53,54} to the case of reattachment after a compression corner. He alluded in ref. 50 to the obvious possibility of applying this model to the problem of base pressure, thus extending the theory in refs. 41 and 48 (section 3.8), which is only valid for small boundary layer thickness, to cover the cases of moderate and large boundary thickness.

The basic principles and assumptions on which the model is

constructed are as follows.

The reattaching shear layer is assumed to be split into two layers. Reattachment is considered to be fairly abrupt, so that along streamlines of the mean flow in the outer part of the shear layer isentropic recompression can be assumed. In the inner layer, however, the effect of the pressure gradient is countered by the shear gradient.

In the subsequent brief analysis, it is tacitly assumed that what is shown to be true for the incompressible flow case can, by means of a suitable transformation, be applied to the compressible case.

If the lateral velocity component is neglected, a reasonable assumption for the case being considered, the momentum equation for a turbulent shear layer can be written,

$$\frac{\partial}{\partial x} \left(P + \frac{1}{2} \rho u^2 \right) = \frac{\partial \tau}{\partial y} \quad (3.10.1)$$

Considering now the inner layer, Stratford showed that, for small y and $\frac{1}{2} \rho u^2 \approx 0$ integration of eq. (3.10.1) leads to

$$\tau \approx \tau_d + y \left(\frac{dP}{dx} \right) \quad (3.10.2)$$

Thus, if $\tau_d = 0$ as at the reattachment point,

$$\tau \cong y \left(\frac{dP}{dx} \right) \quad (3.10.3)$$

This approximate expression is compared with the standard theory which gives, when $\frac{\partial u}{\partial y}$ is positive,

$$\tau = \rho K^2 y^2 \left(\frac{\partial u}{\partial y} \right)^2 \quad (y \text{ small}) \quad (3.10.4)$$

Thus τ is eliminated between eqs. (3.10.3) and (3.10.4) resulting, after an integration, in an expression for the velocity in the inner layer at reattachment, viz.

$$\frac{1}{2} \rho u^2 = \frac{2}{K_0^2} \frac{dP}{dx} y \quad (3.10.5)$$

where

$$K_0 \cong 0.40$$

To deal with the outer layer, Stratford assumed that the Reynold's stress remains constant along the streamlines; and that for the determination of velocity the actual shear layer could be set equal to an equivalent ideal shear layer. This equivalent ideal shear layer was one which had developed at constant pressure over the same distance, and thus the velocity could be determined from

$$\phi = \frac{1}{2} [1 + \operatorname{erf} \eta] \quad (3.10.6)$$

or some similar convenient expression. Thus, after making these assumptions, it can be deduced from eq. (3.10.1) that

$$(\frac{1}{2} \rho U^2)_{2'} = (\frac{1}{2} \rho U^2)_2 - (P_r - P_2) \quad (3.10.7)$$

which is an expression for the velocity in the outer layer.

It remains to join the two solutions. Townsend pointed out that the join is specified by continuity of velocity, shear stress and mass flux, yielding respectively these equalities,

$$\frac{1}{2} \rho U_j^2 - (P_r - P_2) = \frac{2}{K_0^2} \frac{dP}{dx} y_j \quad (3.10.8)$$

$$\tau_j = (\rho \epsilon \frac{\partial U}{\partial y})_j = y_j \frac{dP}{dx} \quad (3.10.9)$$

$$\int_0^y \rho U dy = \frac{2}{3} \rho \left(\frac{4}{\rho K_0^2} \frac{dP}{dx} \right)^{\frac{1}{2}} y_j^{\frac{3}{2}} \quad (3.10.10)$$

Evidently, if the velocity and shear stress distribution of the ideal shear layer are known from eq. (3.10.6) or its equivalent, the trio of equations above give a zero wall stress condition as a function of C_{p_r} and dC_{p_r}/dx where $C_{p_r} = \frac{P_r - P_2}{\frac{1}{2} \rho u_{e2}^2}$. Hence, if an

assumption concerning dC_{p_r}/dx were made e.g. that it is linear, eqs. (3.10.8), (3.10.9) and (3.10.10) could be solved for reattachment pressure.

This is done by MacDonald who continues on to determine the shape factor and attendant parameters required for the application of the methods introduced in refs. 39 and 46.

3.11 Robert's Theory

Essentially Roberts method⁵⁵ for calculating base pressures is simply a refinement of previously introduced techniques. The main points of interest are: a) He introduces a much more accurate expression for relating momentum thickness before and after expansion, viz.

$$\frac{p_{e2} u_{e2} \delta_{2,0}^{**}}{p_{e1} u_{e1} \delta_{1,0}^{**}} = \frac{\int_0^1 \frac{\phi_1^n d\phi_1}{(1 - C_{e,1}^2 \phi_1^2)} - \int_0^1 \frac{\phi_1^n \left[\left(\frac{p_2}{p_1} \right)^{\frac{r-1}{r}} (C_{e,1}^2 \phi_1^2 - 1) + 1 \right]^{\frac{3}{2}} d\phi_1}{(1 - C_{e,1}^2 \phi_1^2) C_{e,1}}}{\int_0^1 \frac{\phi_1^n (1 - \phi_1)}{(1 - C_{e,1}^2 \phi_1^2)} d\phi_1} \quad (3.11.1)$$

This will be discussed in an ensuing chapter. b) He used Crane's⁵⁶ solution for the turbulent half-jet. Crane used Gortler's solution as a starting point, and concluded that, due to compensating effects, the form of the velocity profile is essentially unchanged

by compressibility. c) Robert's approach to the reattachment problem based on the method of Reshotko and Tucker⁴⁵, and is very similar to that of MacDonald. The principle difference is that instead of using eq. (3.10.41) viz

$$H_3^T = 1.754 - 0.149 \log_{10} Re_{\delta^{**T}}^T + 0.0105 [\log_{10} Re_{\delta^{**T}}^T]^2$$

as a reattachment criterion, he uses

$$\frac{Me_3}{Me_2'} = 0.799 + 0.1560 Me_2 - 0.08237 Me_2^2 + 0.009564 Me_2^2 \quad (3.10.2)$$

An empirical expression obtained from data on shock induced separation.

This theory suffers from the same limitations as MacDonald's.

3.12 Tagirov's Reattachment Criterion

Tagirov⁵⁷ presents an interestingly different reattachment criterion, in an attempt to correct Korst's theory. The method is based on the experimentally observed and theoretically predicted³⁷

'neck' of the wake, that is the point where the total thickness

δ of the viscous wake is a minimum (see Fig. 1). Tagirov assumed that this minimum and the reattachment point coincided.

He introduced Crocco and Lees³⁷ mixing equation

$$\frac{dm}{dx} = \frac{k p_{e2} U_{e2}}{\cos(\vartheta_2 - \vartheta_1)} \quad (3.12.1)$$

then for the boundary of the mixing zone

$$\frac{d\delta}{dx} = \frac{k - \sin(\vartheta_2 - \vartheta_1)}{\cos(\vartheta_2 - \vartheta_1)} \quad (3.12.2)$$

assuming k remains constant, then if the reattachment point is characterized by a minimum in δ ,

$$\left(\frac{d\delta}{dx}\right)_R = 0 ; \quad k_R = k = \sin(\vartheta_2 - \vartheta_1) \quad (3.12.3)$$

Tagirov used an equation similar to eq. (3.1.24) to determine k and bring it in line with the Korst theory. The angle ϑ_2 , was then determined from eq. (3.12.3), and the pressure p_r corresponding to this angle was determined by the Prandtl-Meyer relationship.

3.13 Other Theoretical Contributions

The theoretical contributions reviewed in the preceding pages is far from being an exhaustive list. Only those theories introducing important new concepts, or having historical importance have been reviewed in detail. Many theoretical contributions are primarily oriented towards the application of Korst-Chapman principles to practical problems connected with rocket powered launch

vehicles etc.^{59,60,61,62,63,64}.

Owing to the nature of the Chapman-Korst flow model, it is evident that any solution to the problems of free laminar mixing and free turbulent mixing are applicable to the calculation of base pressures. Some of the more important are Denison and Baum's solution⁶⁵, which is applied to a base flow theory reviewed in Section 5.1, and those of Toba⁶⁶ and Myskova⁶⁷ in laminar flow. The first two^{65,66} analyze the problem of laminar mixing with an initial boundary layer profile, the last one⁶⁷ uses the method of Korst on the laminar mixing problem. In the turbulent case, the analysis of Crane⁵⁶, and Channapragada⁶⁸ have been widely referenced. A recent contribution by Jacques and Gailly⁶⁹ is of interest and will be mentioned later. They also suggested that the recompression process along the dividing streamline be regarded as a polytropic process, fixing the index at about 1.3 from comparison with theory.

Some investigators have attempted to modify the Chapman-Korst model so that it takes the recirculating flow into consideration. Lamb and Lenzo⁷⁰ represented the recirculating flow as a jet directed towards the base; and Carpenter⁷¹ tried a similar approach. Korst⁷² suggested the effect of the recirculating flow be likened to a parallel stream of low speed air mixing with a high speed stream.

3.14 Discussion

Although the base flow theories that are based on integral methods and on combined methods of analysis have yet to be reviewed, it is felt that a major discussion should follow the review of the Chapman-Korst model theories. The reasoning behind this decision is that since the other base flow models are invariably extremely complicated mathematically, a practical interest in them would only result if the Chapman-Korst model was thought to give an incomplete or incorrect picture of the flow field, in addition, most experimental studies have been oriented toward criticising the Chapman-Korst model.

It is probably advisable at this juncture to redefine in a more rigorous fashion, exactly what is meant by a Chapman-Korst model theory. The essential assumption underlying all theories in this group is that the base flow field properties are dependent predominantly on the processes of turbulent and laminar mixing and on the reattachment of the free mixing layer. Another way of saying the same thing, is to say that the only significant effect of the recirculating flow is to act as a means of satisfying the conservation of mass at reattachment; that, although the concept of recirculating flow is essential to keep the flow field in equilibrium, for all practical purposes the fluid immediately behind the base may be considered "dead air". If in the course of analysis, the velocity profile of the

recirculating fluid is determined, or the effect the recirculating fluid has on the boundary conditions of the mixing process and vice-versa is generated by some interactive process, then the analysis does not belong in the ranks of Chapman-Korst model theories. In short, Chapman-Korst model theories do not allow for any dynamic effects the recirculating flow may have.

This discussion will take the form of a list of questions or topics followed by examination of the experimental evidence available relevant to the particular question raised.

3.14.1 If the free stream is steady, does it follow that the flow behind a base or step is steady?

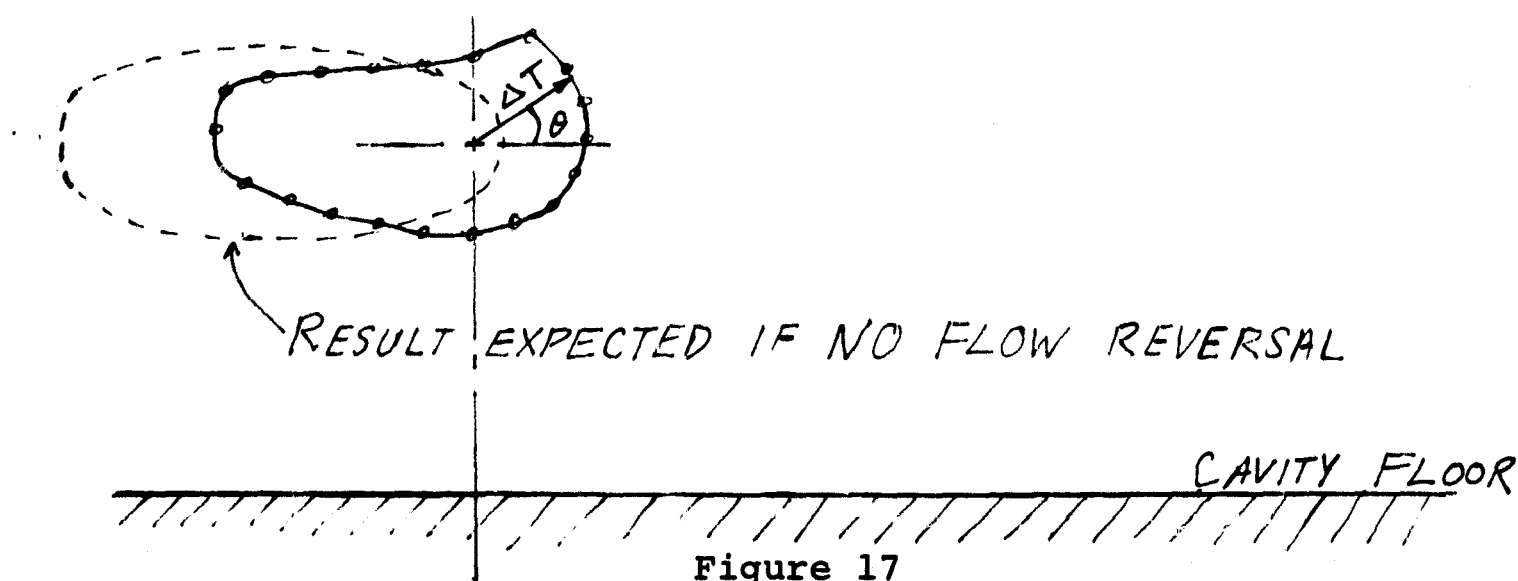
This is a basic question, since all base flow theories presently available assume the answer is positive. It is well known that at low speeds the wake behind bluff bodies is unsteady, but it is generally assumed that a supersonic free stream has a very strong steadying influence. This assumption is at variance with some of the experimental evidence available.

In an experimental study of the flow over a forward facing step, Chapman²⁸ et al. encountered extreme unsteadiness when the flow was transitional, although it was steady for the laminar or turbulent cases. Subsequently, in experiments specifically designed to detect unsteadiness, Kistler⁷³ found that associated with a turbulent supersonic separated flow over forward facing steps are large time-dependent forces, and that based on his data

the forces can be resolved into two components viz. a low frequency buffeting caused by changes in the geometry of the separated flow, and a wide-band fluctuation apparently originating in the free shear layer. The experiments were carried out at $M = 3.01$ and 4.54 .

Charwat et al.⁸⁵ undertook polar heat diffusion experiments in the course of an experimental study of cavity flows, results like the one shown below

POLAR HEAT DIFFUSION TRAVERSE
AT A DISTANCE FROM THE CAVITY FLOOR



strongly indicated the existence of periodic reversals in the cavity flow. However, other investigators found no evidence of unsteadiness in cavity flows.

The steadiness of the reattachment process has been questioned by Bogdonoff^{74,75,76} who observed movement of the

separation and reattachment points over distances of an order of two or three boundary layer thicknesses in laminar flow and of half a boundary layer thickness in the turbulent case. However, Stollery⁷⁵ has made tests using spark photographs, Fastax cinefilms and heat transfer gauges, with a response time of one microsecond; none of which show any significant unsteadiness.

Few base flow experiments have been conducted with tests for steadiness as a prime purpose, but de Krasinski^{16,75,77,78} found small aperiodic pulsations about one or two per second in the recirculation region, and also observed rather larger frequencies of about 4000 c/s during hot-wire measurements.

It appears quite likely that step induced separation and certain cavity flows have an unsteady character. However, it is probably safe to assume that if the unsteady character of base flows were such as to involve a significant amount of energy, then detailed hot-wire studies like that of Lewis and Behrens⁷⁹ would have shown this.

3.14.2 The Chapman-Korst model postulates the existence of a recirculating flow, can it be detected experimentally?

Charwat and Yakura⁸⁰ detected recirculation qualitatively in an experiment on a base covered with pH indicator. They found that if ammonia was injected into the wake downstream of the throat, then the indicator remained unchanged. However, if the

ammonia was injected upstream of the throat then a color change resulted. Similar positive qualitative evidence was furnished by experiments with soot by Panov and Shvets⁸¹. De Kraninski^{16,77} employed small balsa wood arrows to probe the base flow, and found a well defined reversed flow which was steady for laminar and transitional cases. These experiments seem to put the existence of a reversed flow beyond doubt. However, Smith⁸² recently made a curious discovery in the course of an experimental study on the laminar and transitional flow over a rearward facing step at Mach numbers between 2.5 and 5.0. He found that if tufts of yarn were attached to the downstream wall, the ones upstream of reattachment showed no motion at all, as if the air in this region were truly dead air.*

Some investigators have measured the recirculating velocity quantitatively, this will be dealt with later.

3.14.3 Existence of recirculating flow is a necessary but not sufficient condition for the validity of the dividing streamline concept, it is also required that the inner near wake remains closed with respect to mass transfer. Does experimental evidence support this?

Again this is a fundamental question since the dividing streamline hypothesis is made by every base flow theory.

* Roshko & Thomke may have observed a similar phenomenon in the course of locating the reattachment point using an orifice dam technique.

Bandrinarayan⁸³ attempted to show mass conservation by measurements of the mass flow in the shear layers, but his results were not conclusive. Charwat et al⁸⁴ measured recirculating velocity in a cavity and found that it was not sufficiently high to ensure mass conservation, also that injected ammonia and smoke were immediately found downstream and that when the injection was interrupted, the cavity quickly emptied of smoke. However, this should not be treated as strictly relevant evidence for the question of closure of wakes.

De Krasinski¹⁶ carried out tests with lamp black powder. He mounted his axially symmetric model on a greasy, sharp-edged sheet of metal, overlapping on both sides of the base and supporting the model from behind. When lamp black powder was introduced into the recirculation region, traces of escape from this region were observed; first at the throat and later close to the base. It was also observed that even when the base flow was completely laminar, particles introduced in the recirculation region quickly disappeared. It should be noted, however, that with the addition of the metal sheet, the flow field behind the model would not correspond to axisymmetric base flow, in fact, several complicating three dimensional effects would be present. Even without these three dimensional effects the evidence would not be so damning as at first sight, for two reasons. First, separated flows are extremely sensitive to changes in the external supersonic flow, with every small disturbance in the free stream

the recirculating flow will readjust possibly ejecting fluid, also the injection of foreign particles in itself acts as a perturbation on this sensitive region. Secondly, since lamp black, smoke particles, etc. are heavier than air particles, they suffer greater centrifugal forces and consequently would be among the first particles ejected; because of this it is quite probable that lamp black tests exaggerate considerably any "escape" phenomenon. Besides, if transition occurs at the throat, turbulent mixing would explain the traces of escape appearing at the throat.

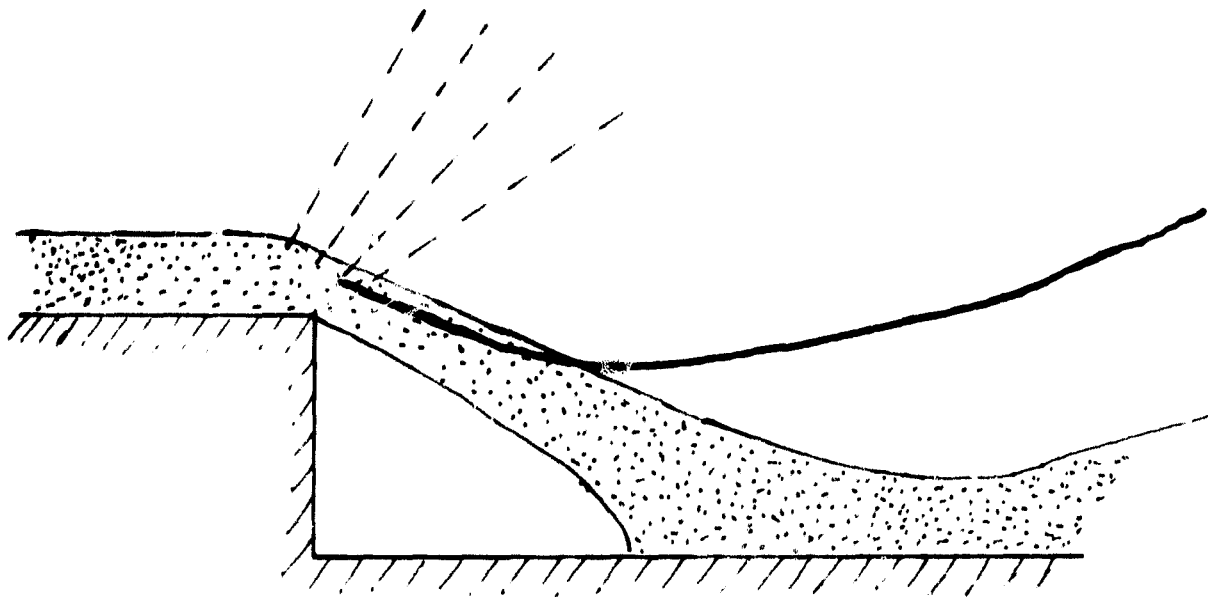
3.14.4 Chapman-Korst model theories usually assume that the mixing process up to recompression is isobaric, does experimental evidence support this?

Even when the initial boundary layer is thick and the free stream Mach number high, the transverse pressure gradient across the mixing zone is essentially constant, although some investigators have reported considerable transverse pressure gradients at the throat^{16,86}. However, the existence of a separation shock wave tends to complicate things since at high Mach numbers accompanied by high Reynold's number the separation shock tends to be immersed in the mixing layer⁸⁷. The separation shock will be discussed in detail in Chapter 6.

A slight but perceptible longitudinal pressure drop before recompression has been observed by some investigators^{16,83,88}

De Krasinski¹⁶ suggested it may be due to the recirculating flow recompressing as it approaches the base. Another suggestion is that in the axisymmetric case the mixing layer follows a slightly curved path like the corresponding inviscid free streamline. Some investigators⁸⁸ have displayed Schlieren photographs with calculated free streamline superimposed to support this claim. If this latter explanation were correct, the slight pressure drop would not exist in the two-dimensional case. Actually most of the measurements showing this effect were made on axisymmetric models.

At Mach numbers above about 5.0, the constant pressure region seems to shorten drastically and eventually disappear, particularly for step models^{88,89}. This effect is strikingly illustrated by Shang and Korkeg's measurements⁸⁹ on step flow at $M = 5.0$ to 7.5 . Smith's transitional two dimensional step, $M = 5.0$, results show a constant pressure mixing zone of 4 to 5 step heights in extent, in contrast to Roshko and Thomke's⁸⁸ turbulent axisymmetric step results at the same Mach number which show no appreciable constant pressure mixing zone. The explanation for these results probably lies with the behavior of the separation shock; as the Mach number increases the separation and recompression shock tend to merge into one continuous shock wave.⁸⁷



3.14.5 Since it appears that for moderate Mach numbers, at least, the mixing layer develops at a constant pressure, what are the possible sources of error when using an isobaric mixing theory to calculate base flow?

There are four main factors that could produce appreciable errors.

- a) The solution used is too approximate even if the conditions, for which isobaric mixing theory is valid, exist.
- b) The assumptions made concerning the reattachment process are invalid.
- c) The dynamics of the recirculating flow cannot be neglected.
- d) The distortion suffered by the boundary layer during the expansion process is too great to be neglected.

These topics will now be considered, in turn:

- a) Most of the isobaric mixing solutions commonly used in base pressure theories such as Chapman's²⁹, Korst's²², Gortler's³⁸ are asymptotic similar solutions, i.e., they assume that the mixing process is so far developed for the effects of an initial velocity profile to be completely neglected.

For laminar mixing layers, the variable $\left(\frac{x}{\delta_{2,0}^{**}}\right)^2 \frac{1}{Re_x}$ must be of an order of 500 for Chapman's solution to be strictly valid⁹⁰. However, the turbulent shear layer develops much more rapidly. Sirieix et al.^{91,92} and Bradshaw^{93,94} found that $x/\delta_{2,0}^{**}$ must be of an order of a 1000 before the mixing layer becomes fully developed. Sirieix et al. (loc. cit.) and Maydew and Reed⁹⁵ found that the virtual origin shift to be well predicted by Kirk's method, but Bradshaw^{93,94} reports that the origin shift is strongly dependent on Reynold's number and may, under some conditions, actually become negative, he suggests that this might explain the controversy over compressibility effects. Apparently compressibility, axisymmetry, and even pressure gradient have very little effect on the form of the velocity profile⁹², the effects of compressibility being given by a change in the jet spreading rate σ , whose value is best predicted by Channapragada⁶⁸. Crane's solution⁵⁶ appears

to be a better fit for the velocity profiles than Korst's theory²². Korst's solution has been criticized³⁹ because it does not reduce to Gortler's³⁸ exact solution in the incompressible limit, however, an analysis⁶⁹ which makes no simplifying assumptions shows that while this may be true, Korst's value for ϕ becomes almost identical with the exact value, after $M = 3.0$. Because of this, it is concluded that the adoption of Korst profiles when $\frac{\delta^{**}}{x} \leq 0.001$ does not constitute a serious error in itself.

- b) Reattachment process. To discuss this topic it is helpful to introduce the control volume ABRL:

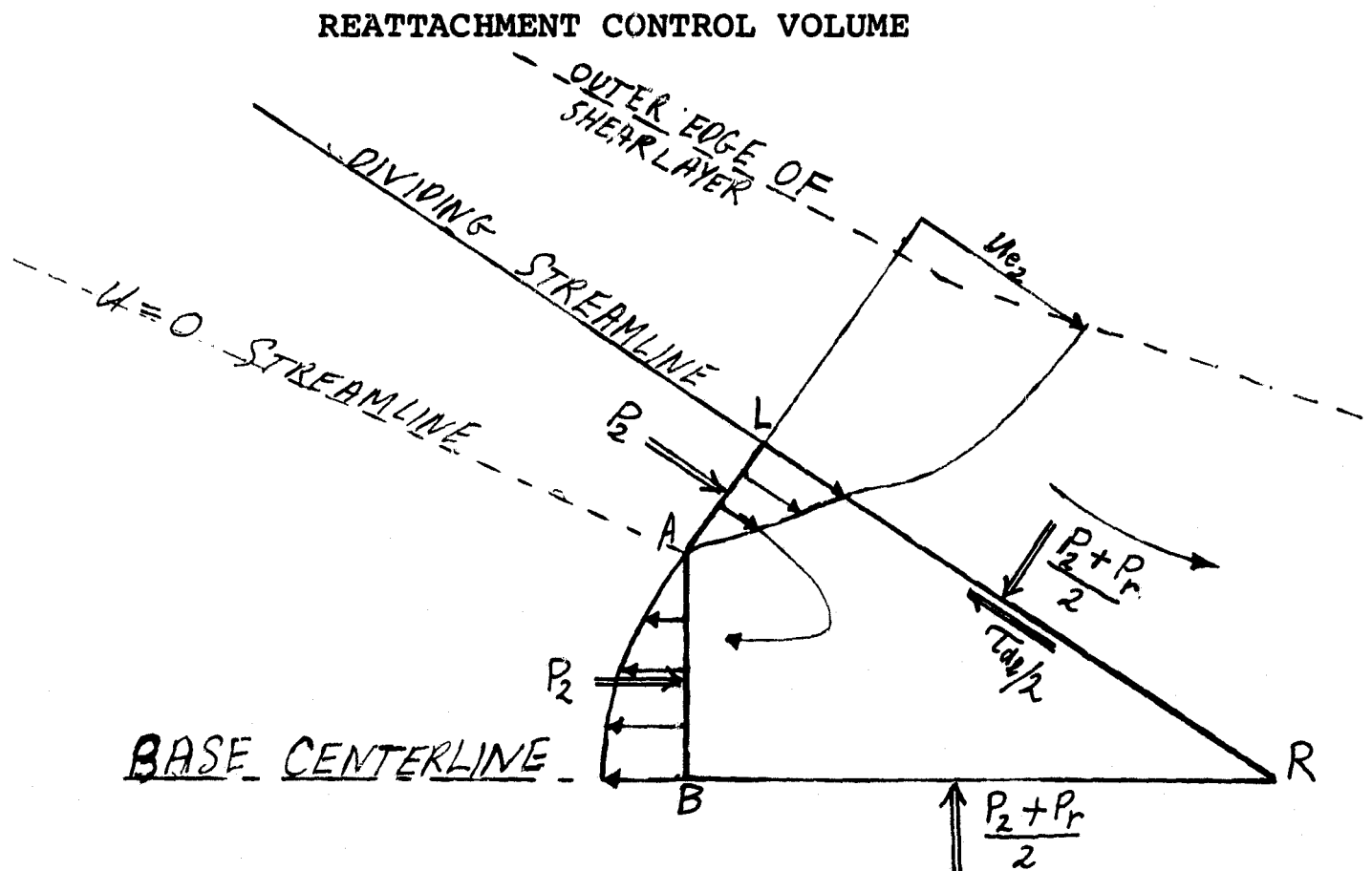


Figure 18

Since the pressure rise to reattachment has been well established as linear, the pressure acting on side LR can be considered as half the pressure rise, in addition, it is assumed that the dividing streamline is straight and that the shear stress acting on LR is half that acting at L, the beginning of recompression. These assumptions appear reasonable for reattachment behind 2-D bases and the above model should provide a reasonable basis for criticism of reattachment theories.

If it is assumed that an isobaric mixing theory furnishes values for ϕ_{d_L} , τ_L , momentum mass and kinetic energy crossing AL, then for a given M_1 , there are six unknowns $\frac{P_2}{P_1}$, $\frac{P_r}{P_2}$, length AB, mass, momentum and kinetic energy crossing AB, to be determined from four main equations viz. Conservation of mass, momentum along LR and perpendicular to LR, and kinetic energy. Hence it can be seen that two additional assumptions or relations are required for a complete solution.

Chapman and Korst's reattachment criterion satisfies the conservation of mass, neglects the momentum and kinetic energy crossing AB, and assumes an isentropic pressure rise from L to R, which implies that for practical purposes.

$\tau_{d_L} = 0$. This amounts to assuming that

$$\frac{\tau_{du}}{\frac{1}{2} \rho_2 u_2^2} \ll \frac{\frac{P_r}{P_2} - 1}{\frac{1}{2} \gamma M_{e_2}^2}$$

This is a reasonable assumption in the laminar case, but in turbulent mixing layers using an error function profile

$$\frac{\tau_{du}}{\frac{1}{2} \rho_2 u_2^2} \sim 0.08 \quad \text{at } M_2 = 2.0$$

and

$$\frac{\frac{P_r}{P_2} - 1}{\frac{1}{2} \gamma M_{e_2}^2} \sim 0.5 \text{ or less.}$$

Clearly, this is not a good assumption. There is at this time ample evidence to show that reattachment does not occur at the apex of the recompression rise for the turbulent case; but does, approximately, for the laminar case. Interpreting the experimental results for turbulent reattachment with the aid of the schematic representation in Fig. 18, it is hypothesized that the action of the shear stress along the dividing streamline prevents reattachment from occurring at the summit of recompression. It now remains to explain how the fluid immediately above the dividing streamline, apparently without sufficient kinetic energy, manages to surmount the full pressure rise. The answer to this probably lies with the greatly increased mixing rate occurring at recompression, thus energy and momentum are entrained from the external stream to help the fluid in the viscous layer withstand the recompression process. It is also apparent from

Figure 18 that the reattachment process is independent of flow conditions downstream of R. This deduction is strongly supported by the experimental work of Sirieix, Carriere et al.^{92,96,97}.

How do the improved reattachment criteria look in the light of the deductions stated above? Those based on the reattachment parameter $N = \frac{p_r - p_2}{p_1 - p_2}$ introduced by Nash³⁹ are basically sound since N does not depend on conditions downstream of reattachment. However, N has been observed to vary a great deal and should decrease in value as M_{e2} increases. Nash's³⁹ average value, 0.35, appears to be much too low in general, Jacques-Gailly's⁶⁹ value, 0.66 is a much better estimate. In addition, to assume an isentropic recompression process after allowing that the Korst reattachment criterion is incorrect is inconsistent. Methods based on an analogy with separating boundary layers also appear to be basically sound providing they do not require that the reattachment process be dependent on the conditions downstream of reattachment.

Thus it appears that the process of turbulent reattachment has not yet been adequately analyzed, and that this could well be a source of great error in base pressure calculations.

The four conservation equations for the control volume in Fig. 18, could be solved if two suitable auxiliary assumptions are made, e.g. approximate the pressure rise along the dividing streamline by a polytropic process, let $AL = AB$, or assume a form for the reversed flow velocity profile, e.g. parabolic $u = a + bx + cx^2$, where a, b, c can be determined by certain known requirements. When the problem is set up in this way a clear difference can be seen between the two-dimensional and axisymmetric cases and between base and step reattachment. Sirieix, Carriere et al. (loc. cit.) have observed a marked difference in behavior between two-dimensional and axisymmetric reattachment; and with experiments using splitter plates, Hama⁸⁷ has observed small (up to 20%) quantitative differences between step and base reattachment.

It should be pointed out that step reattachment introduces another unknown, namely the mean shear stress on BR. This additional shear stress may have the effect of tending to cancel that on the dividing streamline.

- c) If the dynamic effects of the recirculating flow are important, they should significantly affect the pressure, temperature and heat transfer distribution across the base, and, in addition, a considerable recirculating velocity should be observed.

In a sense the Chapman-Korst model is self-contradictory; on one hand, it postulates dividing streamline velocities of

up to half the external velocity at, say, $M_{e_2} = 5.0$, and on the other hand it says that when the fluid below this streamline, with a not inconsiderable maximum velocity, is reversed towards the base, its effect on the base flow is negligible.

What level of reversed flow velocity has been measured experimentally?

Scott and Eckert⁹⁹ measured reverse flow velocities in excess of 400 ft./sec. During the course of experiments on magnetically suspended models Sirieix and Delery⁹⁷ found that at $M_\infty = 1.92$, the reversed velocity was about 20% the external stream whilst at $M_\infty = 4.00$ it increased to 33%. Thus the effect increases with Mach number, this result can presumably be interpreted as evidence that the rate of dissipation of kinetic energy due to viscous effects during reversal decreases with Mach number. In contrast to this result, de Krasinski¹⁶ shows measured reverse velocity profiles, like the ones shown below,

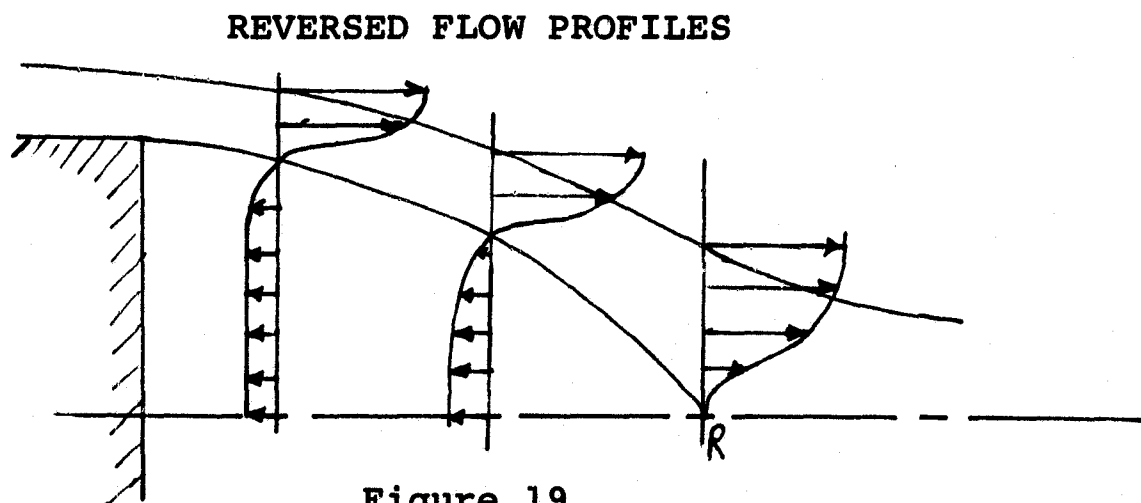


Figure 19

which indicate that the inverted Mach number reaches about 0.5 to 0.6 for turbulent mixing layers and about 0.2 to 0.3 for the laminar and transitional cases, irregardless of the external Mach number. That the level of reversed flow velocity is higher in the turbulent cases than in the equivalent transitional or laminar cases, is a completely contradictory result, to say the least, if Chapman's figure²⁹ of 0.587 for ϕ_{d_1} is roughly correct, since this is considerably larger than the corresponding turbulent values. However, if the laminar mixing layer was not close to being fully developed at the beginning of recompression, then ϕ_{d_1} could be as low as 0.2 or even less*. Moreover, apart from this, the additional factor of greatly increased viscous stresses in the throat in the transitional case can also be offered as a possible explanation of this apparently anomalous experimental observation.

According to the Chapman-Korst model theory, the heat transfer to the base is dominated by the heat transfer processes in the mixing layer, and the temperature in the recirculation region is essentially constant and equal to the stagnation temperature of the mixing layer. Also according

* A calculation of $\xi = (x_{dl}/\delta_{2,0}^{**}) Re_{x_{dl}}^{-1}$ was carried out for de Krasinski's purely laminar case, its value was approximately 2.0 which would make $\phi_{dl} \cong 0.30$ or possibly less. See comments on (d).

to the Chapman-Korst model, the base pressure increases if there is heat addition to the recirculating region, or decreases if the base is cooled^{100,101}.

Powers et al.¹⁰² and Rabinowicz¹⁰³ both found that their data was in general agreement with the Chapman-Korst model. Measurements of heat transfer rate distributions over a hemispherical base made by Francis and Davey¹⁰⁴ show a near constant distribution for laminar flow and the probable presence of a distribution which increases towards the base centre for the turbulent case. However, Scott & Eckert⁹⁹ found that for turbulent mixing layers, the recirculation zone temperature deviated only slightly from free stream total temperature; whereas in the laminar case it approached the base wall temperature, and thus, they concluded, in contradistinction to Chapman-Korst model theories, that the shear layers play only a minor role in the laminar base heat transfer process, the dominating influence being a thin boundary layer on the base wall. Also, in contradiction to the predictions of the Chapman-Korst model, Smith⁸² found that cooling the base wall had no apparent effect at all.

Another way in which the dynamic effects of the recirculating fluid could be manifested is in the existence of transverse pressure gradients on the base wall. Significant

transverse gradients were observed by Dewey⁹⁰ during experiments on the hypersonic wake of blunt bodies, a typical result being approximately of the form shown below,

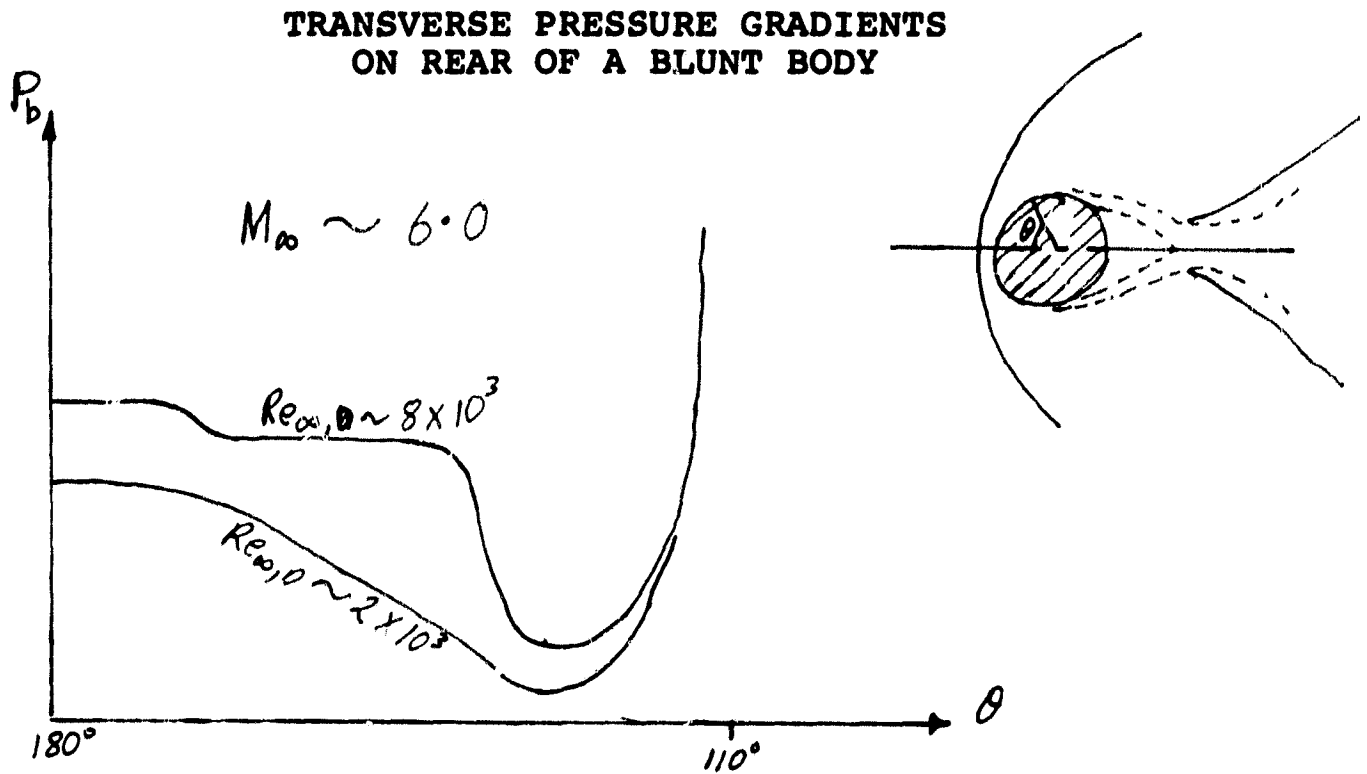
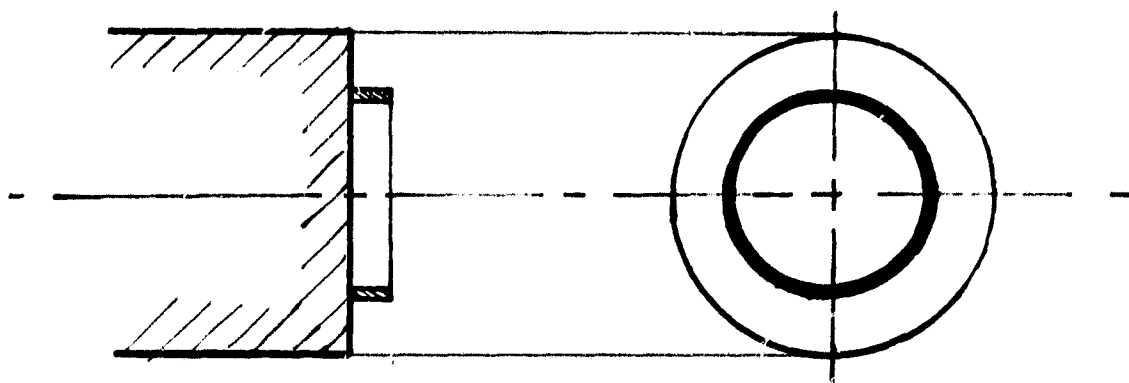


Figure 20

The plateau around $\theta = 180^\circ$ in the higher Reynolds number case suggests a stagnating reversed flow, however, this measurement was made at Reynolds numbers in the rarified flow range. The contention, that the existence of a well defined transverse pressure gradient is confined to the lower Reynold's number range, is strongly supported by Cassanto's¹⁰⁵ results. In experiments on cones at $M \approx 20$, he found severe radial pressure gradients at $Re_L = 5.0 \times 10^5$ but at $Re_L = 7.5 \times 10^6$ these had decreased to near negligible proportions.

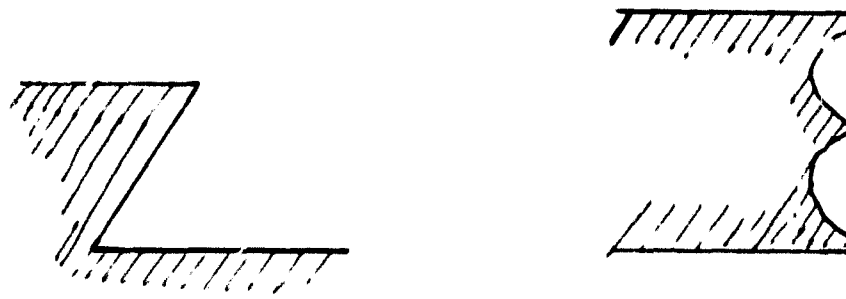
Although no investigators* have reported significant transverse pressure gradients at moderate Mach numbers and higher Reynold's numbers, de Krasinski^{16,77} carried out an interesting experiment with an object to detect dynamic effects of the recirculating flow at the base. He constructed a model with an annular modification to the base as shown below



When pressure measurements were taken, a definite drop in pressure was observed between a point just inside the annules and one just outside.

If the recirculating flow had an effect of any importance on the level of base pressure, it would be reasonable to assume that changes in the geometry of the recirculation region would result in changes in base pressure. Tests carried out by de Krasinski^{17,77}, Smith^{82,106}, Hastings¹⁰⁷ and others failed to detect any significant difference between steps and bases with modifications like

* Hama⁸⁷ did observe transverse pressure gradients, however, it is reasonably certain that these can be ascribed to the effects of the boundary layer expanding around the corner. See Section 6.



and similar configurations and those without. Indeed Hama⁸⁷ found that Schlieren photographs indicated practically identical flow patterns for the stops with and without over-hand, i.e. in comparing



Probably the only conclusion on the structure of the recirculation flow field, that can be drawn from the presently available literature, is that a great deal of contradictory and apparently anomalous experimental results exist, and that the flow processes concerned are far from being well understood.

- d) The expansion process and its effect on the boundary layer will be discussed in Section 6. It suffices to point out, here, the considerable effect the initial velocity profile has on the growth of the mixing layer, particularly in the laminar case⁹⁰.

AFFECT OF INITIAL VELOCITY PROFILE ON THE DEVELOPMENT OF A LAMINAR MIXING LAYER

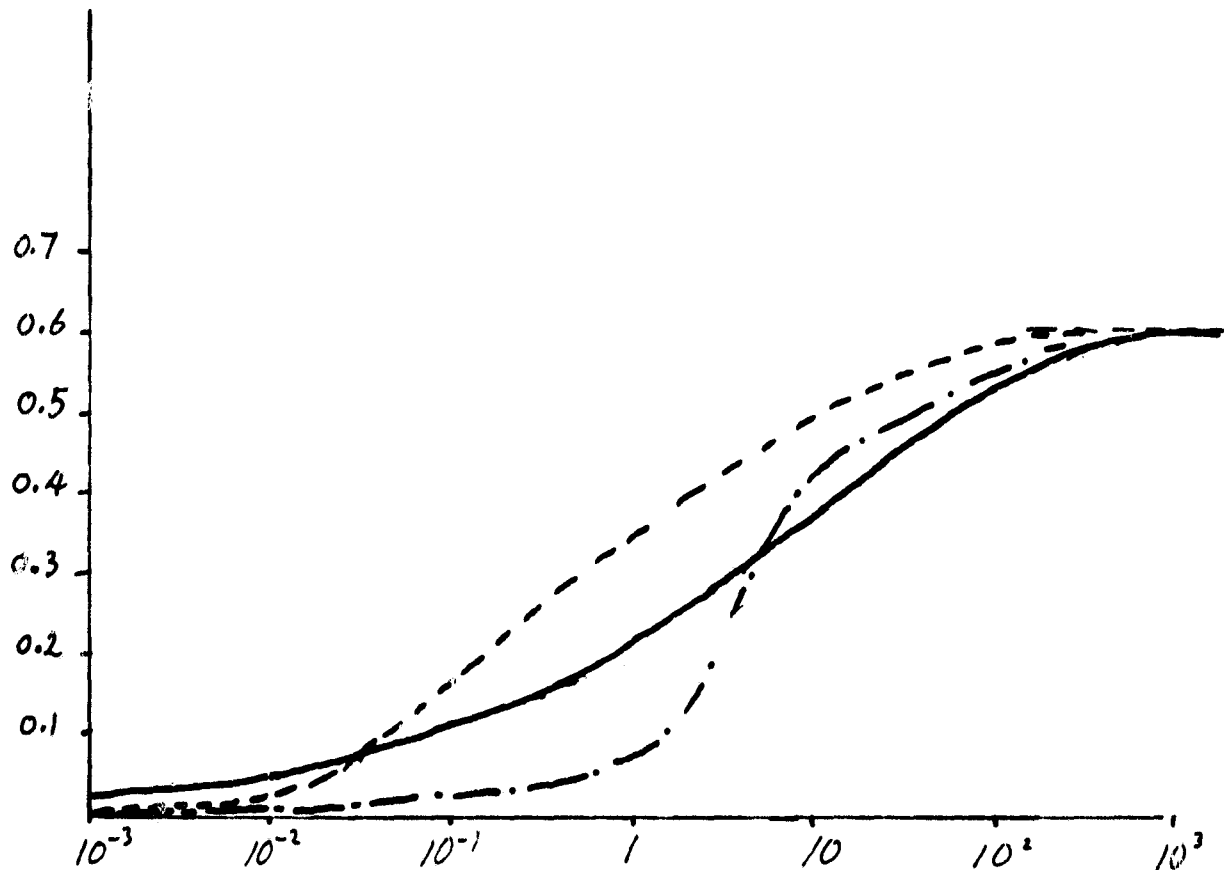


Figure 21

In addition to this is the interaction between the separation shock and the mixing layer and other related effects.

3.14.6 How does the base pressure vary with Reynold's number and why?

The following sketch shows the main features of the variation of base pressure ratio with Reynold's number

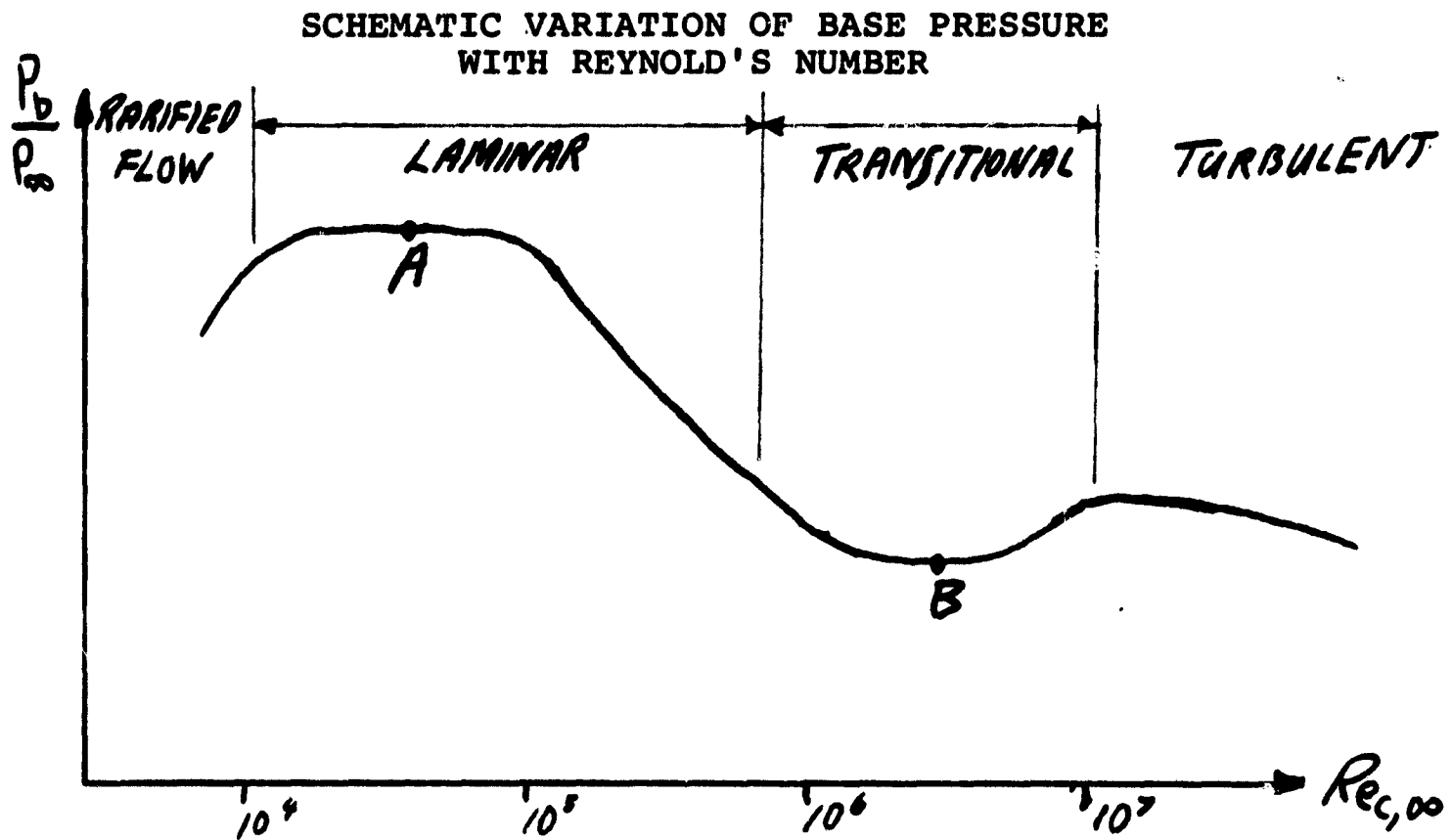


Figure 22

The first qualitative explanation of this curve was made in terms of the Crocco-Lees theory³⁷ (see Section 4.1). The initial increase of P_b/P_∞ with Re_L was attributed to the high value of the mixing coefficient k in low density flow. Experiments conducted by Kavanau^{108,109} in the slip flow regime tended to confirm this interpretation. The maximum in the $P_b/P_\infty - Re_{c,\infty}$ curve occurs when the entire flow field lies in the laminar regime where k has its smallest value. There it is supposed that transition moves upstream from the throat, an approximately tenfold increase in k occurring at transition, thus bringing about a decrease in base pressure ratio with $Re_{c,\infty}$. In this way a minimum B is achieved in the curve corresponding to transition

occurring at the trailing edge. $Re_{c,\infty}$ is increased still further; the transition moves upstream on the body and the consequent thickening of the boundary layer causes an increase in P_b/P_∞ . Finally, however, as $Re_{c,\infty} \rightarrow \infty$, the turbulent boundary layer thickness tends to zero, and P_b/P_∞ decreases to an asymptotic limit. The experiments of Holder and Gadd¹¹⁰ tend to support the preceding point of view, except they found in general that the minimum in the curve does not coincide with transition at the trailing edge.

SCHEMATIC REPRESENTATION OF THE CROCCO-LEES THEORY INTERPRETATION

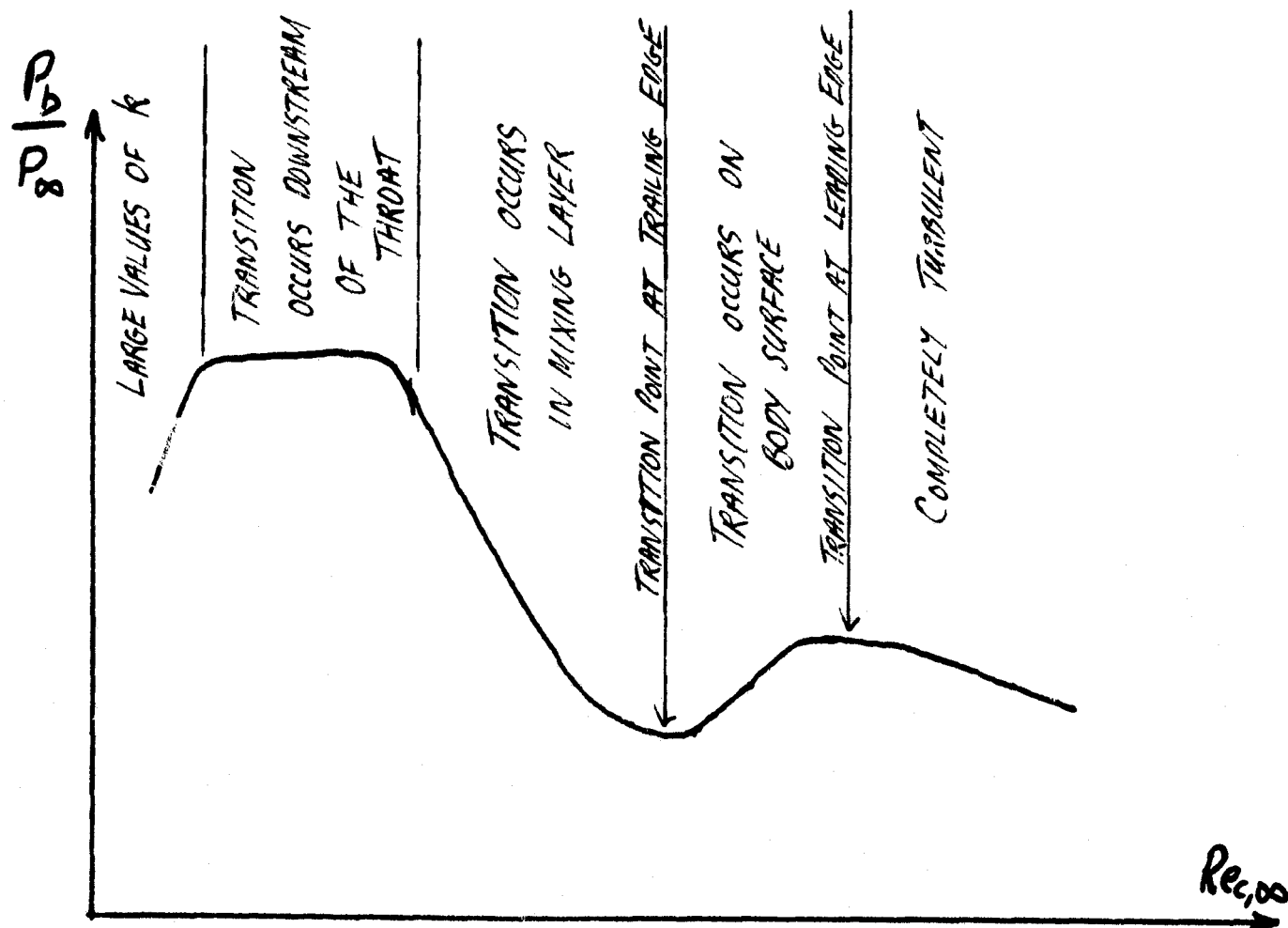


Figure 23

The curve shown in Fig. 22 can also be explained with reference to the Chapman-Korst model. Charwat's modified model³⁰ (see Section 8.4) serves to explain the rarified flow portion of the curve. Chapman²⁸ offered the following explanation for the behavior of the curve in the transition region. Suppose the transition point was suddenly moved slightly upstream from the throat. Then, the effect on the level of the mass of air, m_s , scavenged from the "dead-air" would only be slight, since most of the scavenging still occurs when the mixing layer is laminar. But the presence of eddies in the throat increases the energy transferred from the low speed position of the mixing layer and thus the mass of air reversed, m_r , decreases to m_{r_2} . Hence, in order to maintain equilibrium the throat has to move inward, and the pressure decrease.

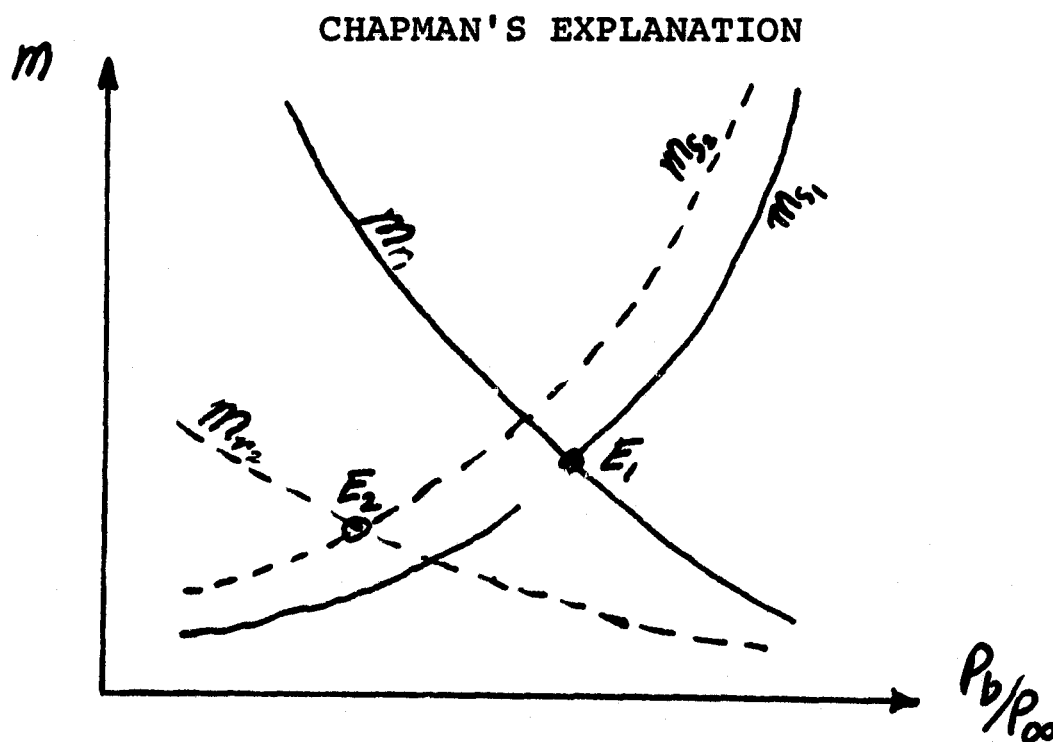


Figure 24

It is not clear how to interpret the slight rise after the minimum in terms of the Chapman explanation.

Other investigators^{82,111} prefer a somewhat different schematic interpretation of the curve in Fig. 22.

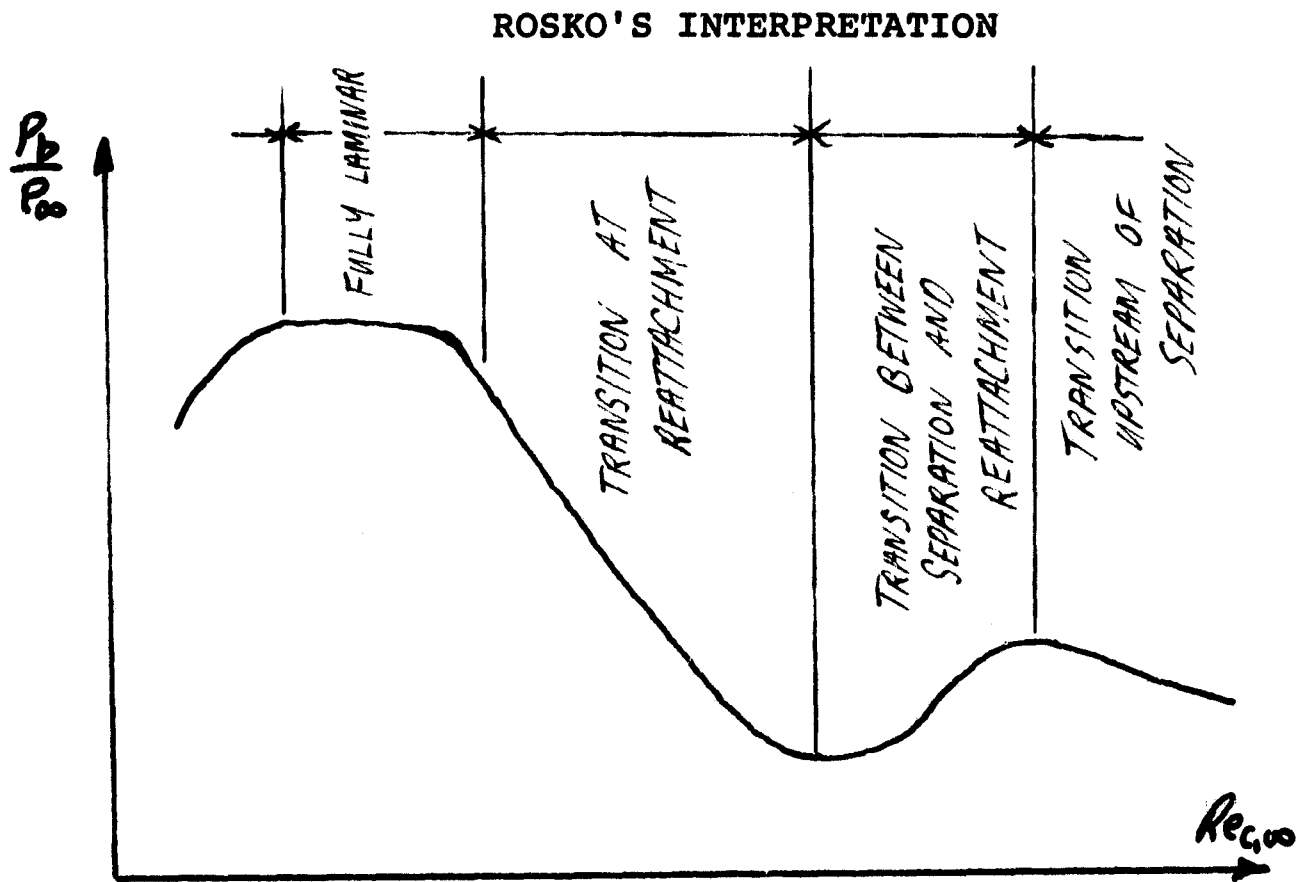


Figure 25

An objection can be raised concerning transition in the mixing layer. First, some time ago Bogdonoff⁷⁴ demonstrated that at $M = 2.95$ the mixing layer has a laminar aspect between A and B in Fig. 22. In fact, the only piece of evidence available in the literature supporting the contention that transition can occur in the mixing layer is a rather doubtful Schlieren

photograph¹⁶. Secondly, if transition did occur in the mixing layer, this would presumably result in the recirculating flow being turbulent and since recompression and recirculation are highly unlikely to cause relaminarization; this would, in consequence, also result in turbulent air being scavenged by a laminar mixing layer approaching transition. Owing to this effect, it is suggested that transition jumps from the throat to the trailing edge as $Re_{c,\infty}$ increases.

An alternative explanation for the small increase of P_b/P_∞ with $Re_{c,\infty}$ after point B in Fig. 22 is now offered. It has been shown theoretically^{111,112} and confirmed experimentally¹¹³ that shear layers tend to be more stable at higher Mach numbers. Therefore, it is suggested that it is possible for transition to occur just prior to separation and for the expansion process to cause relaminarization^{114,115}. The small increase in P_b/P_∞ with $Re_{c,\infty}$ would then correspond to the situation where the boundary layer is turbulent or transitional and the mixing layer is laminar. This hypothesis is supported to some extent by the hot wire studies of Lewis and Behrens⁷⁹. They found that, if the boundary layer was made turbulent by tripping, the shear flow resulting from the corner expansion had a definitely laminar inner layer with a reverse flow that was both laminar and steady. Whether or not corresponding results can be observed for naturally turbulent boundary layers is not, as yet, known.

The interpretation of the $P_b/P_\infty - Re_c$ curve that is tentatively advanced here, can be represented schematically as,

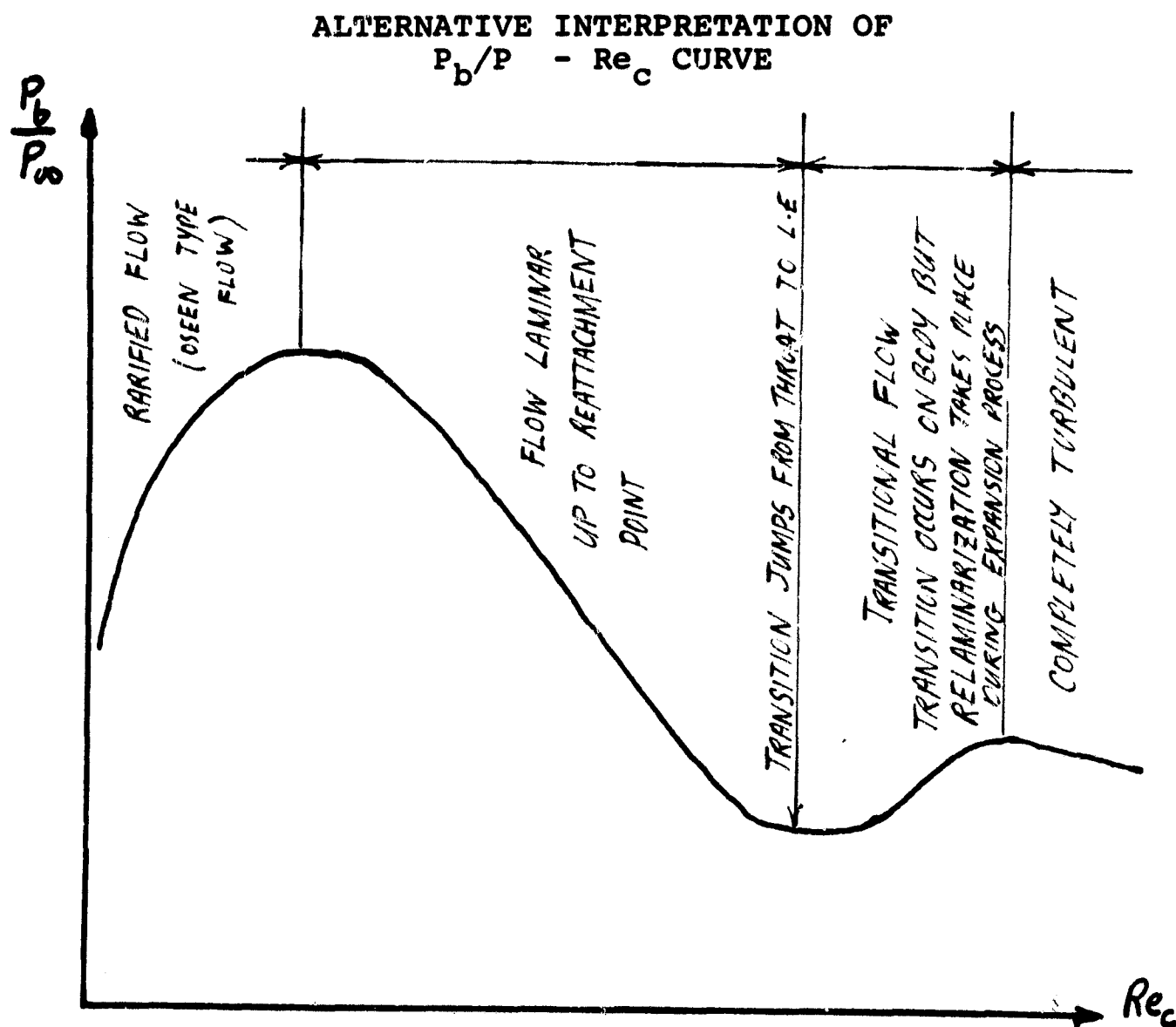


Figure 26

3.14.7 How does the base pressure vary with initial boundary layer momentum thickness?

This is a fairly important question since Nash³⁹ criticized Korst's theory on the basis that it only appeared to accurately

predict the base pressure for the limiting case of $\delta_{2,0} \rightarrow 0$ owing to the fact that a rapid increase in P_b/P_∞ with momentum thickness occurs initially, and thus although Korst's value was too high, it was close to the value for very thin but non-zero boundary layers. It is also an important question because apart from Nash's theory³⁹ those of McDonald⁴¹ and Roberts⁵⁵ predict an initial steep rise of base pressure with momentum thickness.

This initial step increase in base pressure is well evidenced by Hastings¹⁰⁷ experimental results. However, these results are disputed by Roshko and Thomke⁸⁸ who suggested that altering the step height to change the ratio of momentum thickness to step height may have a different effect than altering the momentum thickness and keeping the step height constant. However, Roshko and Thomke's results consist of only two or three points whereas Hastings results are very complete. Moreover, there are strong theoretical grounds for believing that Nash's contention is correct.

A comparison between various theoretical curves and some experimental results is shown in Fig. 27.

Although there is no conclusive experimental evidence that the Chapman-Korst model is not a good approximation for the base

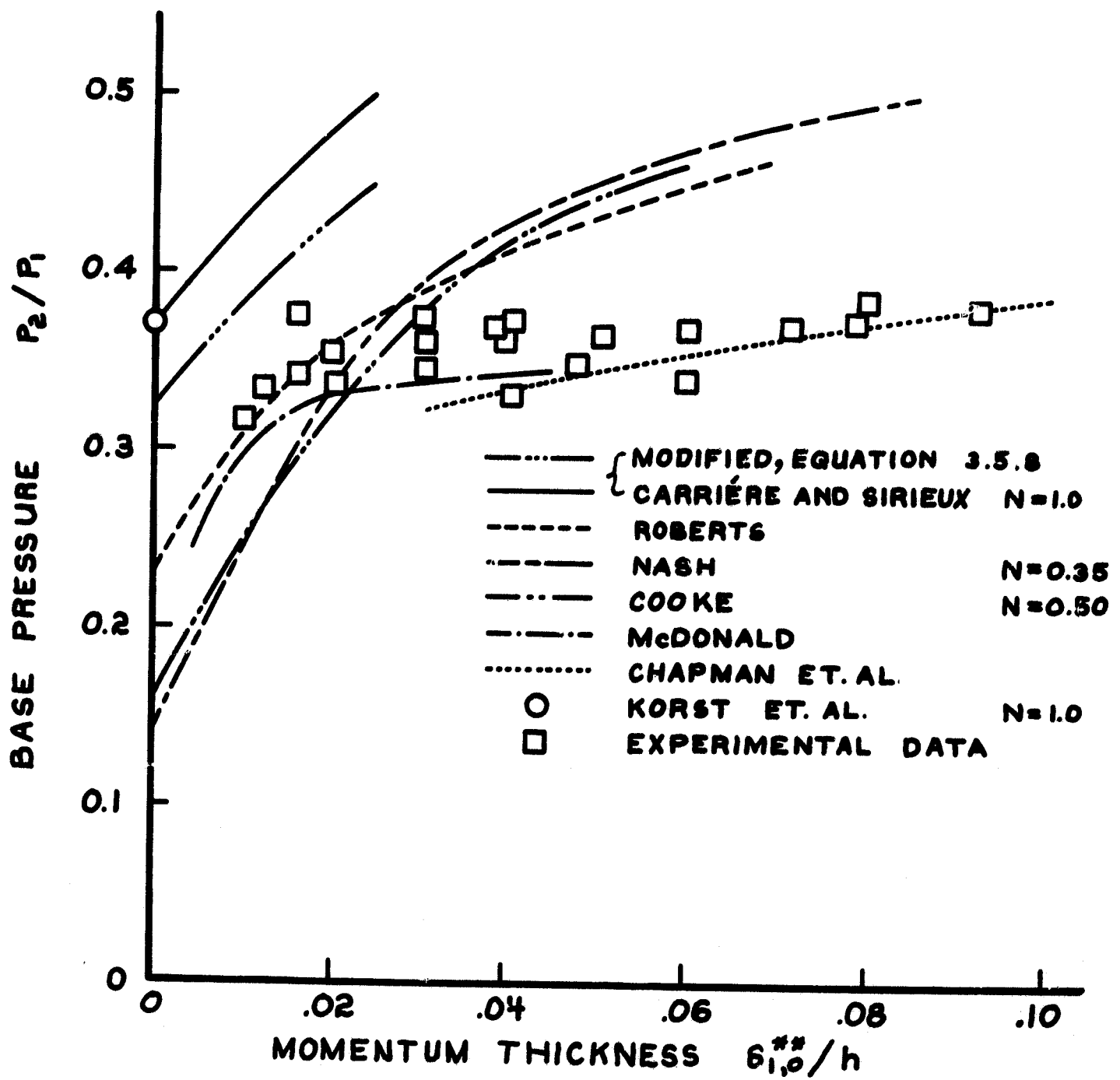


FIGURE 27. VARIATION OF BASE PRESSURE

flow, there is considerable doubt concerning the accuracy of its predictions especially for hypersonic and turbulent flows. Moreover, certain important details, for example, the separation shock wave, cannot be analyzed in terms of Chapman-Korst model theories. The integral methods reviewed in Chapter 4 and the theories reviewed in Chapter 5 provide an important means of checking the assumptions of the Chapman-Korst model.

4. Integral Methods

This chapter contains a review of integral methods of solving the boundary layer equations in the case of separated flows and their application to the determination of base flow. These theories differ from those of Chap. 5, which integrate the full Navier-Stokes equation over part of the viscous region, in that they assume the boundary layer equations are valid over the entire viscous region.

The main advantage of integral methods is that their use enables the entire base flow region, including the approaching boundary layer, to be analyzed within the framework of a single theory. The use of integral methods to calculate attached boundary layers has been widespread for many years, however, until comparatively recently it was not possible to integrate past the separation point because of the existence of a singularity. This singularity ceases to be much of a problem if the assumption of free interaction (rather than a prescribed pressure distribution) is made just upstream of separation. See Fig. 28 below.

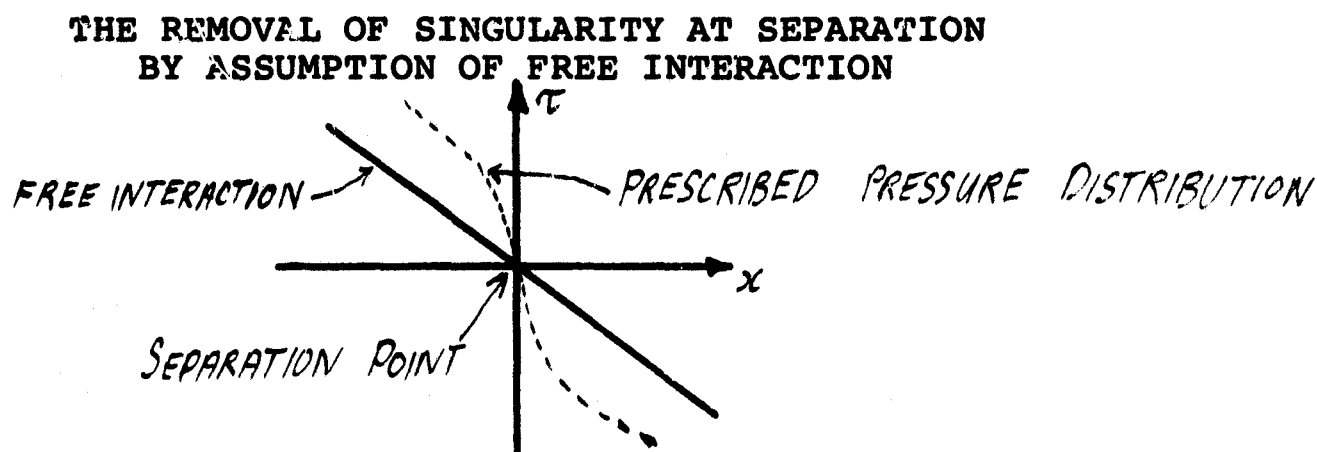


Figure 28

Free interaction means that the local interaction between the boundary layer and the outer flow depends solely on local quantities and is independent of the phenomenon causing separation. This assumption endows the problem with a parabolic nature in keeping with the parabolic boundary layer equations. The usual procedure for allowing for free interaction in supersonic flow is to increase the body coordinates by the amount of the displacement thickness and to assume that the pressure distribution with free interaction coincides with that on the augmented body in inviscid flow.

It should be pointed out that only those integral methods which have been applied to calculating base flows are reviewed in the following pages.

4.1 Crocco-Lees Theory³⁷

This theory was developed to analyze the class of viscous flows for which a static pressure is not a given datum of the problem, but is determined by the interaction between an external inviscid flow and an inner viscous region. It is essentially a one parameter method based on the momentum integral equation analogous to the Karman-Pohlhausen method. Existing methods of this type suffered from certain defects as far as application to separated flows is concerned. The chief shortcoming is that the shape

parameter does not change when the pressure gradient is zero, this is inadmissible for separated flows because regions of zero pressure gradient may exist where the velocity profiles are certainly not similar; coinciding neither with Blasius flow nor with the Stewartson limiting reversed flow profile.¹¹⁶ In addition, since the geometry of the separated flow is not usually known a priori, a polynomial expansion in terms of η is inappropriate. Apart from these considerations, it is also desirable that the parameter be invariant under a Stewartson transformation,¹¹⁷ so that results from incompressible theory could be used.

With these provisos, the theory was developed as follows.

The initial assumptions made were:

1. The boundary layer equations are valid over the entire viscous region.
2. Pressure gradients across the direction of flow are neglected.
3. The flow angles relative to the axis of flow are small.
4. The gradients of viscous stresses in the flow direction are negligible compared with static pressure gradients in the flow direction.
5. Steady flow.
6. External flow is plane supersonic and isentropic, thus the Prandtl-Meyer relation is valid.
7. $Pr = 1$
8. $\mu \sim T$

Note that δ is differently defined from the Chapman-Korst model theories, and no reverse flow is postulated.

The governing equations can be written:

1. Momentum equation for viscous region in x direction,

$$\frac{dI}{dx} = u_e \frac{d\bar{m}}{dx} - \delta \frac{dP}{dx} - \tau_w \quad (4.1.1)$$

where

$$I = \int_0^{\delta} \rho u^2 dy \quad (4.1.2)$$

$$\bar{m} = \int_0^{\delta} \rho u dy = \rho_{\alpha} u_{\alpha} \delta = \frac{P u_{\alpha}}{R T_{\alpha}} \quad (4.1.3)$$

The mean density, ρ_{α} , and mean temperature T_{α} are introduced without any particular physical significance. The average velocity is given by,

$$u_{\alpha} = \frac{I}{\bar{m}} \quad (4.1.4)$$

2. A mixing equation governing the rate at which mass is transferred to the internal flow, see Fig. 30.

$$\frac{d\bar{m}}{dx} = \rho_e u_e \left(\frac{d\delta}{dx} - \theta \right) = \rho_e u_e \frac{d}{dx} (\delta - \delta^*) \quad (4.1.5)$$

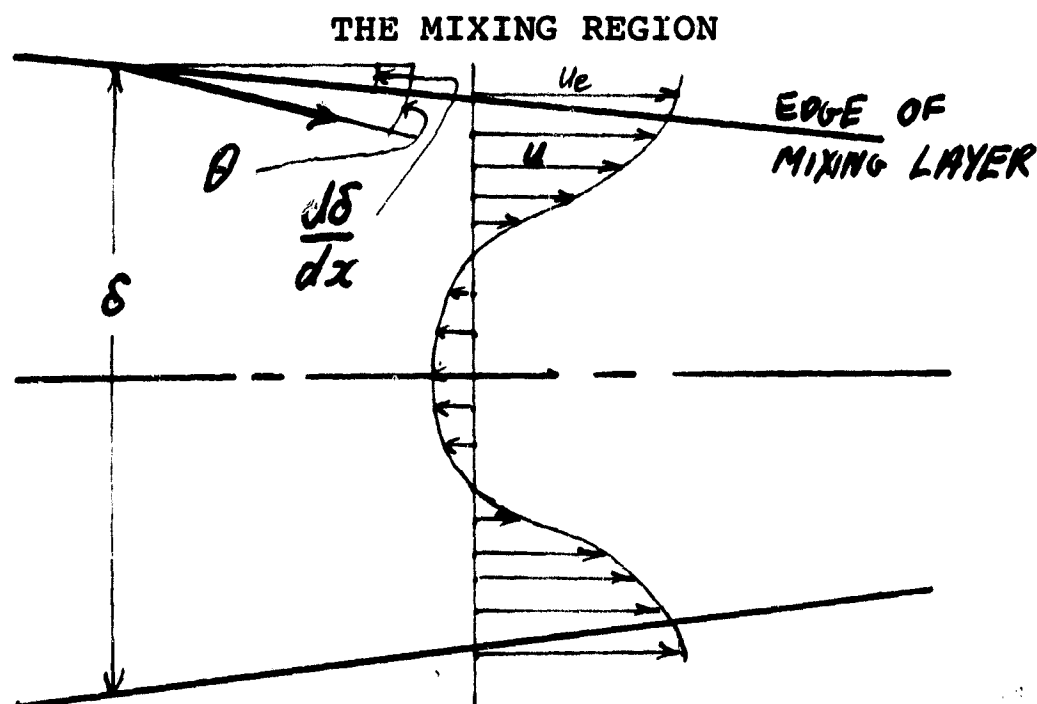


Figure 30

3. The Bernoulli equation which is written in terms of the reduced velocity,

$$W = \frac{u}{a_s} \quad (4.1.6)$$

and the auxiliary parameters,

$$w_\alpha = \frac{T_\alpha}{T_s} \frac{1}{\gamma W_\alpha} = \left(1 - \frac{\frac{\gamma-1}{2} u_\alpha^2}{\gamma R T_s}\right) \frac{1}{\gamma W_\alpha} = \frac{1 - \frac{\gamma-1}{2} W_\alpha^2}{\gamma W_\alpha} \quad (4.1.7)$$

$$w_e = \frac{T_e}{T_s} \frac{1}{\gamma W_e} = \frac{1 - \frac{\gamma-1}{2} W_e^2}{\gamma W_e} \quad (4.1.8)$$

Thus Bernoulli's equation becomes,

$$\left(\frac{dP}{P}\right)_e = \frac{-\rho_e u_e du_e}{P} = \frac{-u_e du_e}{R T_e} = \frac{-\gamma T_s W_e dw_e}{T_e} = \frac{dw_e}{w_e} \quad (4.1.9)$$

4. The Prandtl-Meyer equation

$$\theta_e = \theta(W_e) \quad (4.1.10)$$

Recognizing that,

$$\bar{m} a_s = \rho_\alpha u_\alpha \delta a_s = \frac{P \delta}{w_\alpha} \quad (4.1.11)$$

$$\rho_e u_e a_s \delta = \frac{P \delta}{w_e} \quad (4.1.12)$$

and setting $\tau_w = 0$ for separated flow, it can be seen that the problem is essentially the solution of a system of four simultaneous partial differential equations,

$$\left. \begin{aligned} \frac{d}{dx}(\bar{m} a_s \chi W_e) &= W_e \frac{dm}{dx} - \delta \frac{dP}{dx} && \text{MOM.} \\ \frac{d}{dx}(\bar{m} a_s) &= \rho_e u_e \theta_s \left(\frac{d\delta}{dx} - \theta \right) && \text{MIXING} \\ \left(\frac{dP}{P} \right)_e &= - \frac{dW_e}{W_e} && \text{BERNOULLI} \\ \theta_e &= \theta(W_e) && \text{PRANDTL-MEYER} \end{aligned} \right\} \quad (4.1.13)$$

Eqs. (4.1.13) involve six unknowns viz.

$\delta, \bar{m} a_s, W_e, \chi, \theta_s$ & $\chi = \frac{u_\chi}{u_e}$, so clearly two additional equations are required to find a solution. Recognizing that the relation between P and θ can be eliminated between the Bernoulli and Prandtl-Meyer equations, the two additional equations can be expressed formally as,

$$\left. \begin{aligned} F_1 &= F_1(\chi, W_e, \bar{m} a_s, \delta) \\ F_2 &= F_2(\chi, W_e, \bar{m} a_s, \delta) \end{aligned} \right\} \quad (4.1.14)$$

The first relation (4.1.14) was sought by means of mixing coefficient k , which was considered as empirical data, it was assumed constant for either turbulent or laminar mixing, increasing abruptly

approximately tenfold at transition.

$$k = \frac{d\delta}{dx} - \rho \quad (4.1.15)$$

The second relation was sought in the guise of "universal curve" i.e. in the form of some quantity that would have a unique value for each value of the shape parameter. Accordingly with this in mind, $\mathcal{H} = \frac{u_x}{u_e}$ was chosen as the parameter. It is readily apparent that it satisfies the requirements previously stated.

From basic definitions

$$\mathcal{H} = \frac{u_x}{u_e} = \frac{I}{\bar{m} u_e} = \frac{\delta - \delta^* - \delta^{**}}{\delta - \delta^*} \quad (4.1.16)$$

$$\frac{T_x}{T_e} = \frac{\rho_e}{\rho_x} = \frac{\delta}{\delta - \delta^*} \mathcal{H} \quad (4.1.17)$$

Introducing the Stewartson transformations to relate the compressible flow properties to their incompressible equivalents (noting $\mathcal{H}$ remains invariant) it was found that

$$\frac{T_x}{T_s} = \mathcal{H} \frac{\delta^s}{\delta^s - \delta^{*s}} - \frac{\gamma - 1}{2} W_x^2 \quad (4.1.18)$$

where superscript s denotes Stewartson plane. The universal function was then defined as

$$\Phi(\eta) = \frac{\delta^s}{\delta^s - \delta^{*s}} \cdot \frac{1}{\eta} - 1 = \frac{\delta^{*s} + \delta^{***s}}{\delta^s - \delta^{*s} - \delta^{***s}} \quad (4.1.19)$$

The form of $\Phi(\eta)$ was obtained from known solutions to the incompressible boundary layer equations e.g. Blasius, Falkner-Skan etc., as shown below.

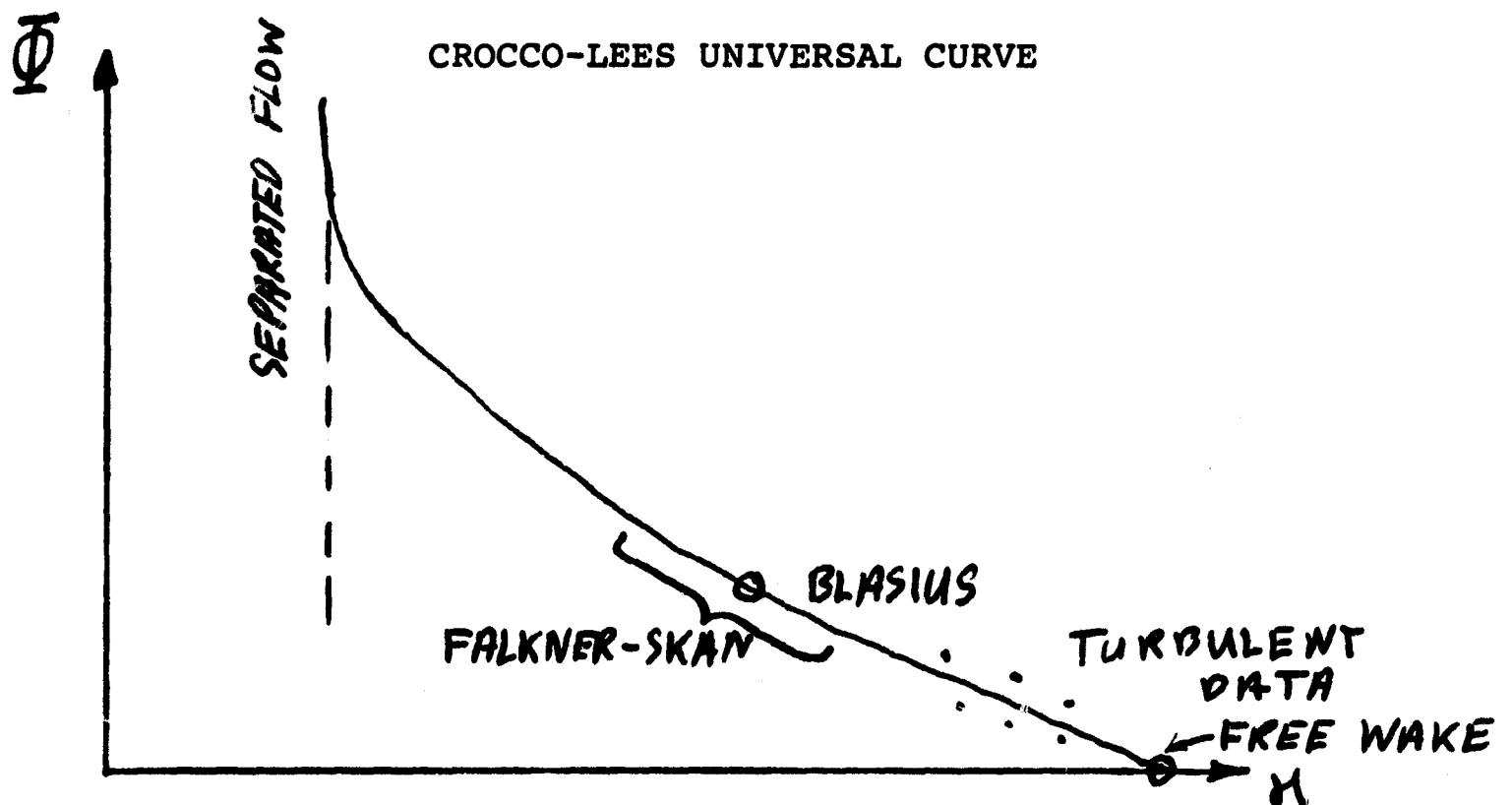


Figure 31

Hence, given η , $\Phi(\eta)$ would be determined from the universal curve, and thus

$$\omega_\alpha = \frac{\eta}{\gamma We} \left[\Phi(\eta) + 1 - \frac{\gamma-1}{2} We^2 \right] \quad (4.1.20)$$

With the help of these two additional relations (4.1.15) and (4.1.20), the system of equations (4.1.13) can be reduced to an ordinary first-order non-linear differential equation with W_e as the dependent variable and χ as the independent variable. This can be represented symbolically as,

$$\frac{dW_e}{d\chi} = \frac{N(W_e, \chi)}{D(W_e, \chi)}. \quad (4.1.21)$$

This equation can be integrated numerically. It may at first sight seem anomalous to solve the equations (4.1.13) for a quantity representing external velocity, but actually the physical quantities of immediate interest such as the flow angle, and therefore the geometry, and the pressure p_e , temperature T_e can be obtained using the Prandtl-Meyer relation and the isentropic relations.

An important feature of the solution, $W_e - \chi$, plane is the existence of a singular or critical point, a saddle point, where $N(W_e, \chi) = 0$ and $D(W_e, \chi) = 0$. Because of the form of the universal curve (Fig. 31), and owing to the properties of saddle points, it transpires that in order for a solution curve in $W_e - \chi$ space to pass from a point corresponding to separated flow to one corresponding to a physically meaningful flow far downstream, i.e. a free wake profile, the curve must pass through the singular point. This means that the singular point plays a very important role in

the solution of the base flow problem using the Crocco-Lees method. If a value of the dependent variable is specified at the singular point, the other variables can be obtained from the equations.

$$\left. \begin{aligned} N(We_{cr}, x_{cr}) &= 0 \\ D(We_{cr}, x_{cr}) &= 0 \end{aligned} \right\} \quad (4.1.22)$$

In this way a unique solution can be determined by finding the initial conditions which allow the solution curve to pass through the singular point. Thus this is an iterative process and an inverse method of solution.

By means of this theory, Crocco and Lees were able to predict, with considerable success, the variation of base pressure with Reynold's number (see sub-section 3.14.6). However, another aspect of the theory led to the controversial implication that the base pressure ratio depended in part on the total wake thickness at separation. This resulted from the peculiar definition of δ in this theory (see Fig. 29), and was disputed by several investigators, notably Chapman,²⁸ who devised a series of experiments involving the use of triangular insets (see Fig. 32) which indicated fairly conclusively that the initial wake thickness acted only as a convenient measure of mixing layer development length and had no intrinsic significance in itself.

CHAPMAN's EXPERIMENT
(Various sizes of insets were employed)

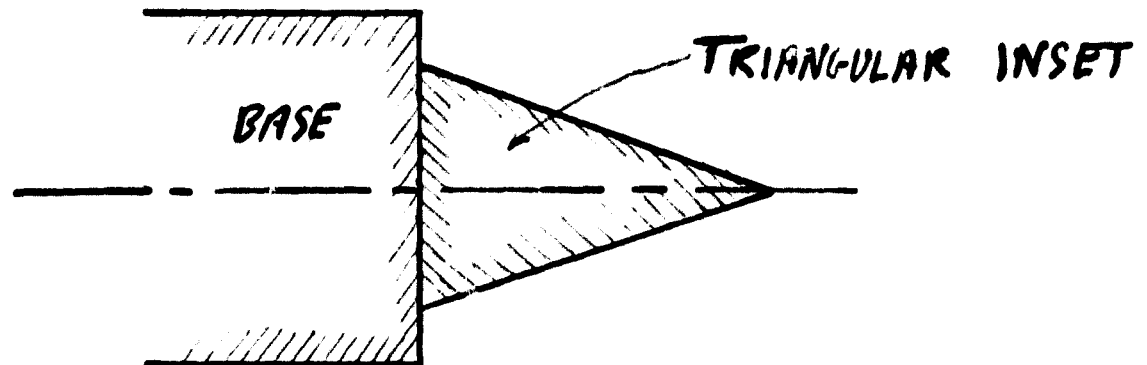


Figure 32

The main practical objections to the theory were that reliable experimental data for the mixing parameter, k , were required, and, secondly, that the position of the universal curve and value taken from data for k , both depended strongly on the definition adopted for δ . (i.e. whether $u_\delta/u_e = 0.99$ or 0.95 etc.). However, Glick^{118,119} showed that improved correlation could be obtained for the determination of k , and that the sensitivity of the original method to the definition of δ could be overcome when a self consistent definition of its magnitude was used. He suggested the universal curve be given in the analytic form

$$\Phi = \frac{2(1-\mathcal{H})}{2\mathcal{H}+1} \quad (4.1.23)$$

Notwithstanding this, de Krasinski¹⁶ carried out extensive tests which indicate that it is highly improbable that an investigator,

possessing no prior knowledge of the form of the curve, could deduce it from experimental data. He also extended the theory to axi-symmetric flows¹²⁰. Rom¹²¹ used the theory to determine the pressure behind rearward facing steps and blunt trailing edges. His formulae for base pressure show a quantitative difference between steps and bases in contrast to any available method based on the Chapman-Korst model.

4.2 Lees-Reeves Theory

Lees and Reeves¹²² developed a theory for supersonic separated and reattaching flow, which could be described as a second generation Crocco-Lees theory. It was a distinct improvement on the original theory, in that it eliminated the empirically determined mixing coefficient, k , by introducing an additional equation namely, the first moment of momentum equation; and that the universal curve was replaced by another artifice with sounder theoretical grounds. In addition, the new theory can be generalized to the non-adiabatic case without much difficulty, unlike the Crocco-Lees theory.

The new theory is still essentially a one-parameter integral method. The chief disadvantage of these methods is that although in any limited set of circumstances a specific one-parameter family of velocity profiles might be quite accurate, this accuracy may only extend over a relatively narrow range of flow conditions. Lees and Reeves overcome this difficulty to a certain extent by tailoring the parameter to suit the needs of a local flow. To illustrate this, the flow field behind a cylinder is divided into various regions, (see Fig. 33).

SCHEMATIC REPRESENTATION OF NEAR-WAKE INTERACTION REGIONS OF A BLUNT BODY

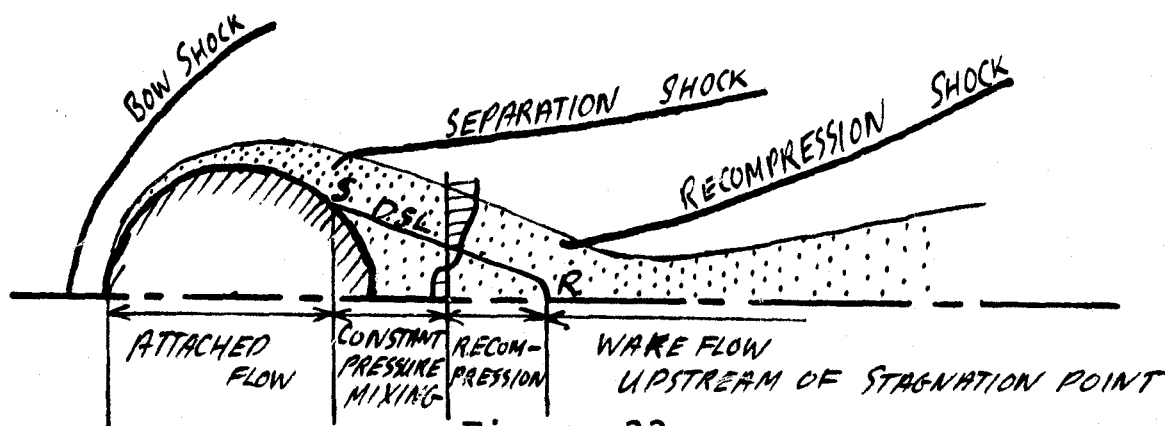


Figure 33

1. Attached Flow. Here the shape parameter $A(x^s)$ is taken as proportional to the wall, shear stress in Stewartson coordinates,

$$A(x^s) = \left[\frac{\partial \phi^s}{\partial (y^s/\delta^s)} \right]_{y^s=0} \quad (4.2.1)$$

2. Immediately after separation, there is a constant pressure mixing region, here $A(x^s)$ is taken proportional to the position of the zero velocity streamline,

$$A(x^s) = \left(\frac{y^s}{\delta^s} \right)_{u^s=0} \quad (4.2.2)$$

3. Following the constant pressure region is one where a strong viscous inviscid interaction occurs with recompression, in order to satisfy conservation of mass at reattachment (Chapman-Korst criterion), the parameter is changed to one proportional to the velocity on the dividing streamline,

$$A(x^s) = \phi_{\psi=0}^s : 0.54 < A(x^s) < 0 \quad (4.2.3)$$

4. After the rear stagnation point, the parameter is taken to be simply the non-dimensional velocity on the centerline of the wake,

$$A(x^s) = \phi_{\xi}^s : 0 < A(x^s) < 0.75 \quad (4.2.4)$$

Similar governing equations as before are used, i.e. the momentum equation (4.1.1), mixing (4.1.5) and the Prandtl-Meyer relation

(4.1.10). In addition, the first moment of momentum equation is introduced,

$$J \frac{d\delta_t^{*s}}{dx^s} + \delta_t^{*s} \frac{dJ}{d\bar{H}} \frac{d\bar{H}}{dx^s} + (3J + 2S_w \bar{T}) \frac{\delta_t^{*s}}{M_e} \frac{dM_e}{dx^s} = \frac{v_\infty R}{a_\infty M_e \delta_t^{*s}} \quad (4.2.5)$$

where

$$J = \frac{\delta^{***s}}{\delta_t^{*s}} \quad (4.2.6)$$

$$\delta_t^{*s} = \delta_t^{*s} + \int_0^{\delta^s} S dy^s \quad (4.2.7)$$

$$S = \frac{h_s}{h_{se}} - 1 \quad (4.2.8)$$

$$\bar{H} = \frac{\delta^{***s}}{\delta_t^{*s}} \quad (4.2.9)$$

$$\bar{T} = \frac{1}{S_w \delta_t^{*s}} \int_0^{\delta^s} \frac{u^s}{u_e^s} S dy^s \quad (4.2.10)$$

$$R = \frac{2\delta_t^{*s}}{(u_e^s)^2} \int_0^{\delta^s} \frac{du^s}{dy^s} dy^s \quad (4.2.11)$$

Eq. (4.2.5) was obtained by multiplying the momentum equation by u and integrating across the boundary layer. The other equations were written in a somewhat different form from (4.1.1), (4.1.5), and (4.1.10) but essentially they are the same.

The artifice referred to earlier is the use of families of

known similar solutions to determine values for δ^{*s} , δ^{**s} , δ^{***s} etc. for a given $A(x^s)$. For example, in the constant pressure region the lower branch Stewartson similarity solutions ($\beta = 0$)¹¹⁶ are used, given a value for $A(x^s) = \left(\frac{y^s}{\delta^s}\right)_{u^s=0}$, the appropriate similar solution is ascertained and the integral quantities determined. In the recompression region the variable β solutions are used, although it is important to point out that the one-to-one correspondence between any given profile and the associated β is not retained. Appropriate families of similar solutions are used in the other regions.

By means of this artifice the governing equations can be reduced to a system of three first order non-linear differential equations of the form,

$$\left. \begin{aligned} \frac{\delta^{*s}}{M_e} \frac{dM_e}{dx^s} &= \frac{1}{Re_{\delta^{*s}}} \frac{N_1(A(x^s), M_e)}{D(A(x^s), M_e)} \\ \delta^{*s} \frac{d\bar{H}(A(x^s))}{dx^s} &= \frac{1}{Re_{\delta^{*s}}} \frac{N_2(A(x^s), M_e)}{D(A(x^s), M_e)} \\ \frac{d\delta^{*s}}{dx^s} &= - \frac{1}{Re_{\delta^{*s}}} \frac{\bar{K}(A(x^s)) N_1 + N_2}{D} \end{aligned} \right\} (4.1.12)$$

Like the old Crocco-Lees theory, a singular point exists in the solution plane, or rather a line of saddle points¹²³, when

N_1 , N_2 and D are zero. (See Fig. 34).

SKETCH OF THE INTERSECTION OF N_1 , N_2 and D

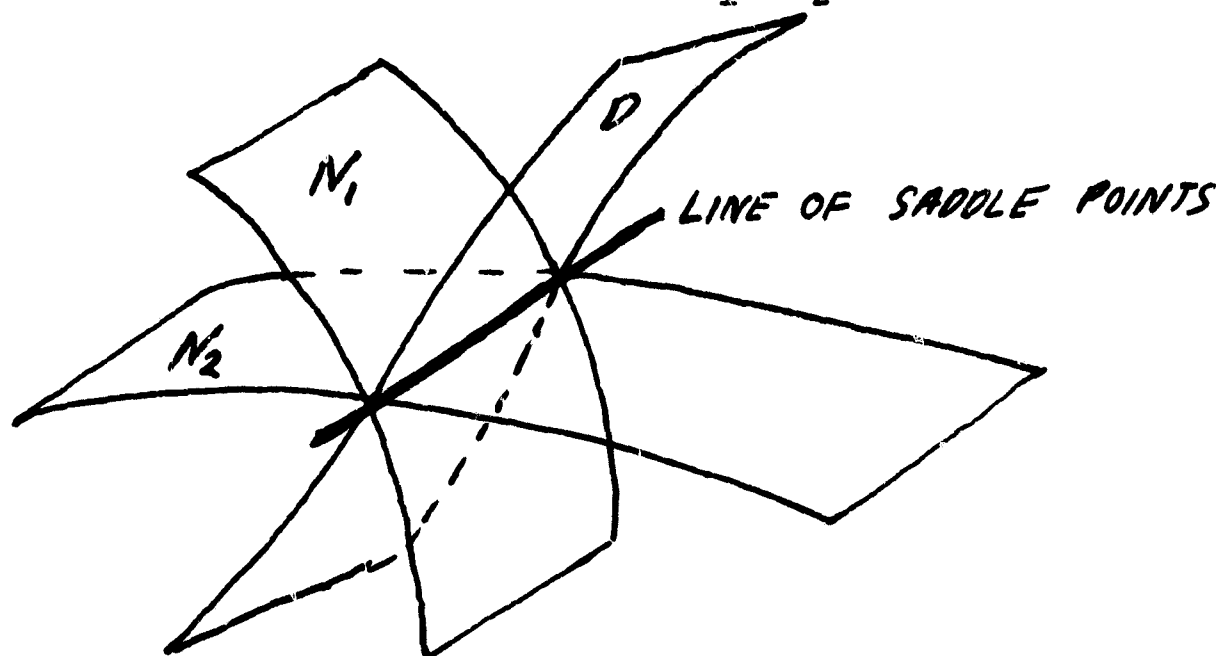


Figure 34

As before, the solution curve must pass from points corresponding to separated flow through the critical line to achieve physically meaningful solutions in the wake after the rear stagnation point, or in Lees-Reeves parlance, reattachment is a subcritical-supercritical process. The line of critical points always lies downstream of the rear stagnation point (in contrast to the original Crocco-Lees theory where it could be either up or downstream).

Once again in order to find a unique solution, the conditions at the critical point are specified, and the initial conditions allowing the solution curve to reach these specified values is sought. There is an added complication since the constant pressure solution

must be joined to the recompression solution at a location which cannot be specified exactly a priori. The requirements to be satisfied at this juncture are the continuity of M_e (or θ_e), $\phi_{\psi}^s = 0$ and δ_{sl} (where δ_{sl} is the thickness of the mixing layer above $u = 0$). By matching δ_{sl} or alternatively the mass flow above $u = 0$, it was assured that the mass flux above the dividing streamline is continuous and moreover that the mass scavenged by the mixing process is equal to the mass flow reversed by recompression. The values of $\phi_{\psi}^s = 0 = A(x^s)$, M_e and δ_{sl} uniquely determine the overall wake thickness δ^s , and δ^{*s} . Suppose the location of the separation point on the given body is known a priori and that the boundary layer thickness at separation is determined. The unknown quantity is the constant flow angle θ in the mixing region, from Fig. 35.

DETERMINATION OF θ FOR CONSTANT PRESSURE REGION

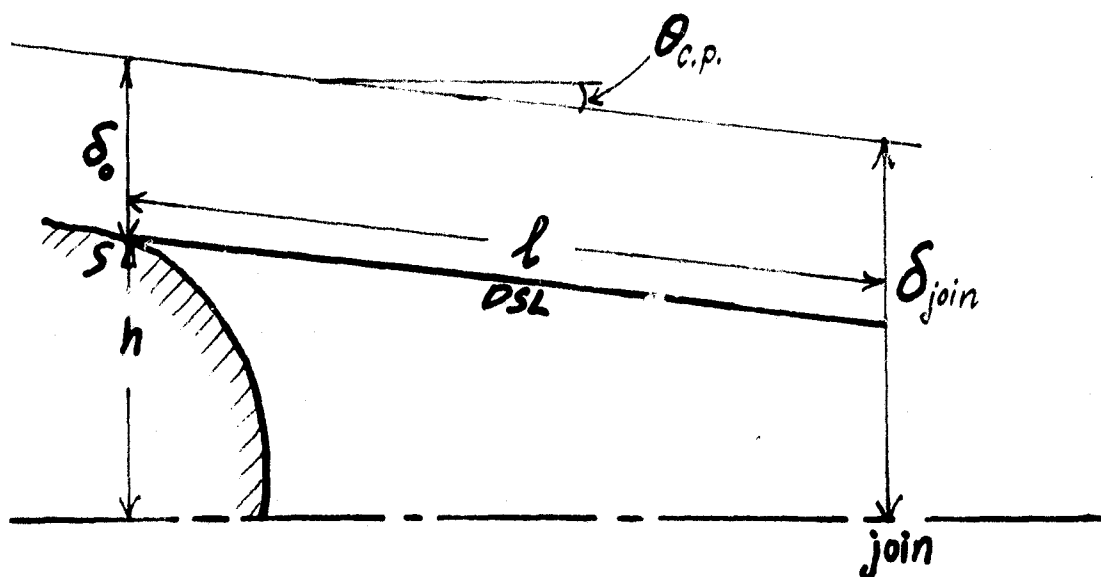


Figure 35

$$\theta_{c.p.} = \frac{\delta_0 + h - \delta_{join}}{l} = \frac{h + \Delta\delta}{l} = \frac{h}{l} \left(1 + \frac{\Delta\delta}{\delta_0} \frac{\delta_0}{h} \right) \quad (4.2.13)$$

If $\phi_{\psi=0}$ is specified at the join, the quantities l/h and $\delta_t^{*s}/(\delta_t^{*s})_0$ are determinable, and $\Delta\delta/\delta_0$ can be calculated. From Eq. (4.2.13), $\theta_{c.p.}$ is determined for every $(\phi_{\psi=0}^s)_{join}$. But for a given initial $(\phi_{\psi=0}^s)_{join}$ and $(\delta^{*s})_{join}$ there is only one possible value of θ which permits the recompression solution to pass smoothly through the critical line. Clearly, an iterative method is required in which a series of $(\phi_{\psi=0}^s)_{join}$ values are chosen until $\theta_{c.p.} = \theta_{recompression}$. Unless some sort of double iteration process is employed (there are usually problems of convergence with these processes), an inverse method of solution must be used, in which $(\phi_{\psi=0}^s)_{join}$, $(\delta^{*s})_{join}$ and Re_{c*} are specified, found from recompression solution and then r/l and δ_0/r determined using Eq. (4.2.13) etc.

Grange, Klineberg and Lees¹²⁴ extended the theory to cover cases where the separation point need not be specified in advance. An interesting feature of the complete solution plane is the possibility of the existence of another singular line in the flow field. When it is present this second singular line occurs in the attached flow region. If the attached flow does go supercritical i.e. pass through a singular point (for instance this happens at $\theta = 118^\circ$ on a circular cylinder) a strange situation exists, for in order for the flow to separate $A(x)$ must be zero and therefore the flow must

become subcritical again, but there is no way that a supercritical flow can pass continuously into a subcritical one. Therefore, a supercritical-subcritical jump was postulated and analyzed²⁴.

Non-adiabatic flows can be treated by a fairly straightforward extension¹²⁵ of the theory summarized above. The Stewartson¹¹⁶ similarity solutions are replaced by those of Cohen and Reskotko¹²⁶. A parameter, S_w , additional to $A(x^S)$ is introduced, but because the enthalpy profiles are assumed linked to the velocity profile parameter $A(x^S)$, no separate energy equation is required.

The method could be readily extended to axisymmetric flows, but in this case the Prandtl-Meyer relationship would have to be replaced by an individual characteristic calculation.

Alber¹²⁷ extended the theoretical methods outlined above to the case of supersonic turbulent base flows. In the course of this, he introduced a flow model differing to some extent from the traditional Chapman-Korst model. (See Fig. 36).

FLOW MODEL USED BY ALBER

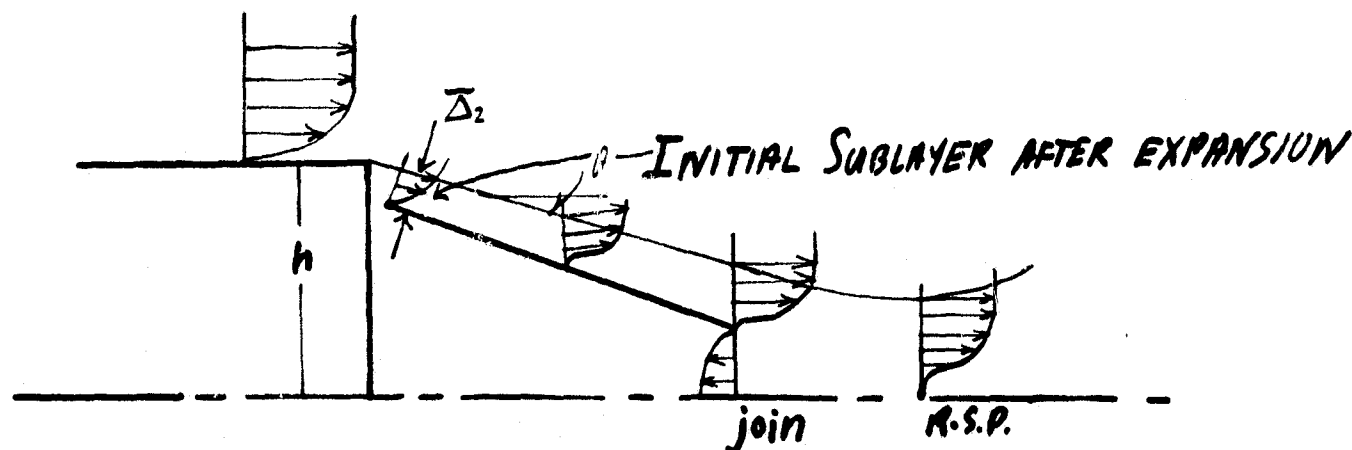


Figure 36

Alber assumes that when the boundary layer undergoes the expansion process, a new viscous sublayer is formed immediately downstream, approximately 10-20% of the thickness of the original layer. This sublayer then grows downstream of the corner as a free mixing layer. While the thickness of the total initial layer may have increased 50% due to the corner expansion, it is asserted that the length scale of the problem is determined only by the inner layer or sublayer and that the outer portion of the expanded boundary layer merely acts as a weak inviscid field of external vorticity. As will be seen in Chap. 6, this coincides with the theoretical findings of Weinbaum and is well supported by the experimental data of Lewis and Behrens.⁷⁹

The relationship between the "before" and "after" sublayer thicknesses $\bar{\Delta}_1$, and $\bar{\Delta}_2$, is established by applying mass and momentum conservation to a rather crude control volume, using the shear stress distribution calculated by Mellor and Gibson¹²⁸ for a highly accelerating equilibrium turbulent boundary layer.

In addition to the above, the kinematic viscosity ν , in the transformed dissipation integral $\int_0^{\delta} \tau \left(\frac{\partial u}{\partial y} \right) dy$, was replaced by a transformed eddy viscosity of the form,

$$\tilde{\epsilon} = \left(\frac{\rho_{ref}}{\rho_e} \right) K_0 u_e \delta^{**s} \quad (4.2.14)$$

The constant K_θ was determined by incompressible correlation. ρ_{ref} was selected as $\rho_{u=0}$. The Stewartson transformation was modified slightly, i.e.

$$dx^s = \frac{\rho_e a_e}{\rho_\infty a_\infty} dx \quad \text{laminar} \quad (4.2.15)$$

$$dx^s = \frac{a_e \rho_e}{a_\infty \rho_\infty} dx \quad \text{turbulent} \quad (4.2.16)$$

There are two major aspects of the Lees-Reeves and Crocco-Lees type theories that can be criticized on theoretical grounds:

1. Essentially, one is dealing with a one-parameter family of velocity profiles. These families cannot be expected to have general applicability, although in any limited set of circumstances a specific one-parameter family might be quite accurate, as the methods summarized above show quite adequately, but it must be verified a posteriori that this is true in each new instance. Makofski¹²⁹ has shown that the use of a two parameter velocity profile would improve the accuracy of prediction but there are problems with tricky double iterations to overcome.
2. The inverse method of solution is inconvenient in practical problems. Moreover, integrating upstream from a join could

perhaps be questioned on grounds of uniqueness. Suppose the integration proceeds upstream from an arbitrary profile, there is no absolute assurance that the upstream profile even exists. This point becomes particularly important when a "supercritical-subcritical" jump becomes necessary in order for the flow to separate. In addition, as Ai¹²³ discovered in a detailed study of the singular points in the laminar wake, there exists a minimum free stream Reynold's number, for a given Mach number, below which no solution can be obtained with the Reeves-Lees method.

4.3 The Method of Integral Relations

The difficulties mentioned at the end of the previous section can be largely overcome when the method of integral relations is used. For this reason and because it is a method that is capable of wide application (e.g. it can, in theory, be used to solve the full Navier-Stokes equations), it should prove to be a distinct improvement on the previous integral methods used on separated flows.

The method of integral relations, or the GKD (Galerkin-Kantorovich-Dorodnitsyn) method as it is sometimes called,¹³⁰ was first introduced to fluid dynamics by Dorodnitsyn¹³¹. The method can be described in general terms as follows.

Denoting a given partial differential equation simply as,

$$F[u(x,y)] = 0 \quad (4.3.1)$$

with the appropriate boundary conditions. Consider an approximate solution of the form,

$$u(x,y) \cong \bar{u}_n(x,y) = \sum_{j=1}^n a_j(x) \phi_j(x,y) \quad (4.3.2)$$

where the assumed functions $\phi_j(x,y)$ satisfy the given y-boundary conditions for all values of the parameters $a_j(x)$. Further, the functions $\phi_j(x,y)$ are taken to be the first n members of a complete set of functions, which can be found and which guarantee the convergence of the approximate solution. That is to say, in the limit as n approaches infinity, the approximating function $\bar{u}_n(x,y)$ equals $u(x,y)$.¹³⁰ In practice it is usually unnecessary to go beyond $n = 4$.

$$\lim_{n \rightarrow \infty} \bar{u}_n(x,y) = \lim_{n \rightarrow \infty} \sum_{j=1}^n a_j(x) \phi_j(x,y) = u(x,y) \quad (4.3.3)$$

The parameters $a_j(x)$ are determined by requiring that

$$\int_{y_1}^{y_2} H_j(y) F[\bar{u}_n(x,y)] dy = 0 ; j=1, 2, \dots, n \quad (4.3.4)$$

where the weighting functions, $H_j(y)$ are any set of linearly independent functions. Eq. (4.3.4) represents a set of n ordinary differential equations which will be non-linear when the operator $F[u(x,y)]$ is non-linear.

Several investigators have applied this procedure to a variety of problems notably, Belotserkovskii¹³³ to the blunt body problem; Pavlovskii¹³² to compressible boundary layers; Iiu¹³⁴ has included the effects of fluid injection or suction; Nielsen et.al.¹³⁵ and Holt¹³⁶ have analyzed separated flows at concave corners; and recently Crawford¹³⁷ has applied the method to the analysis of the supersonic laminar flow over a rearward facing step, his theoretical methods will be summarized in the ensuing paragraphs.

SEPARATED FLOW DOWNSTREAM OF A BACKWARD FACING STEP

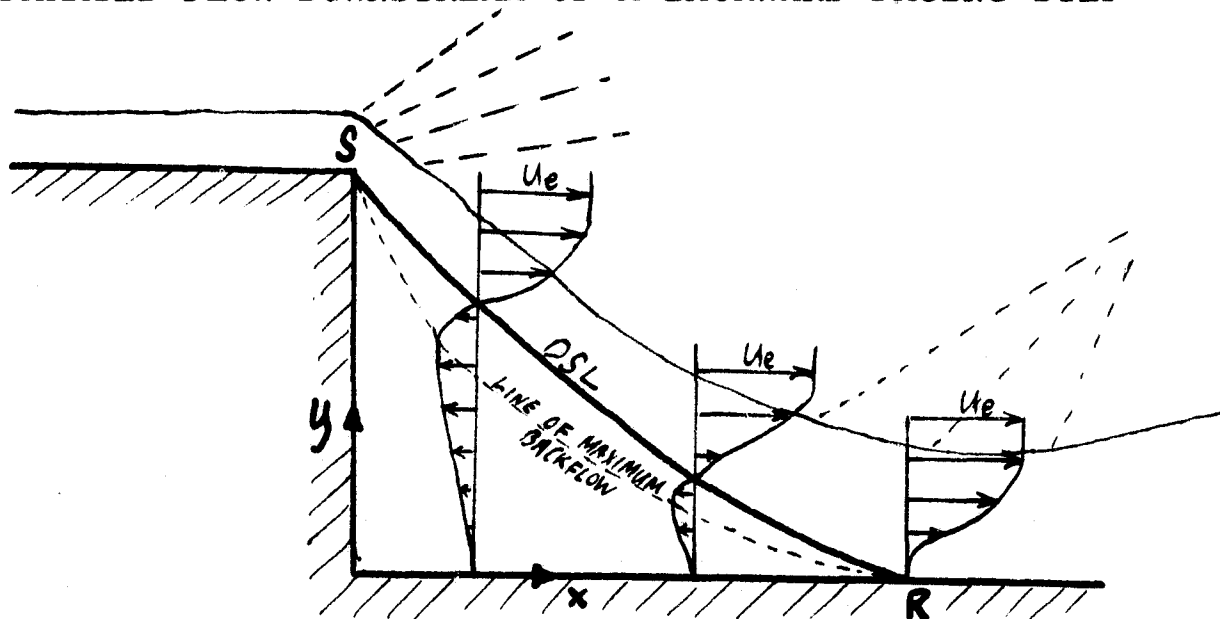


Figure 37

The flow is assumed to be adiabatic with $Pr = 1$; under these conditions the Busemann integral can be used to eliminate the

energy equation. Then the governing equations can be written

$$\frac{d}{dx}(\rho u) + \frac{d}{dy}(\rho v) = 0 \quad (4.3.5)$$

$$\rho u \frac{du}{dx} + \rho v \frac{dv}{dy} = \rho_e u_e \frac{du_e}{dx} + \frac{d}{dy} \left(\mu \frac{du}{dy} \right) \quad (4.3.6)$$

$$P = \rho R T \quad (4.3.7)$$

$$\frac{T}{T_e} = 1 + m(1 - \phi^2) \quad (4.3.8)$$

$$\text{where } m = \frac{\gamma - 1}{2} M^2$$

with boundary conditions,

At

$$\left. \begin{array}{ll} y = 0 & u = v = 0, \frac{dT}{dy} = 0 \\ y \rightarrow \infty & u \rightarrow u_e, T \rightarrow T_e \end{array} \right\} \quad (4.3.9)$$

In addition an interaction relationship is required, this is written

$$\frac{d\delta^*}{dx} = \tan \theta \quad (4.3.10)$$

where θ is given by the Prandtl-Meyer relationship.

The equations (4.3.5) and (4.3.6) are now transformed to equivalent incompressible form using the Stewartson¹¹⁷ transformations, defined by

$$\left. \begin{aligned} x^s &= \int \frac{\rho_e a_e}{\rho, a,} dx \\ y^s &= \left(\frac{T_w \mu_1}{T, \mu_w} \right)^{1/2} \frac{\rho_e a_e}{\rho, a,} \int \frac{\rho}{\rho_e} dy \\ u^s &= \partial \psi / \partial y^s, \quad v^s = -\partial \psi / \partial x^s \end{aligned} \right\} (4.3.11)$$

The equations are transformed once again using a Dorodnitsyn¹³¹ transformation, defined by,

$$\left. \begin{aligned} x^0 &= \int \frac{u_e^s dx^s}{u_i^s h} \\ y^0 &= \frac{u_e^s}{u_i^s} \left(\frac{u_i^s h}{\nu_1} \right)^{1/2} \frac{y^s}{h} \\ \phi^0 &= \frac{u_i^s}{u_e^s} = \frac{\partial \psi}{\partial y^0}; \quad \frac{u^0}{u_e^0} = -\frac{\partial \psi}{\partial x^0} \end{aligned} \right\} (4.3.12)$$

this essentially has the effect of making the independent variables of the same order of magnitude.

Then the transformed continuity equation is multiplied by a weighting function $w_k(\phi^0)$ and the transformed momentum equation by $w_k'(\phi^0)$. The two equations obtained are then added and their sum integrated with respect to y^0 from zero to infinity. In this way the following equation is obtained,

$$\begin{aligned} & \int_0^\infty \frac{d}{dx^0} [\phi^0 w_k(\phi^0)] dy^0 + \frac{v^0}{u_e^0} w_k(\phi^0) \Big|_0^\infty \\ &= \frac{1}{2m} \frac{dm}{dx^0} \int_0^\infty (1 - \phi^{02}) w_k'(\phi^0) dy^0 + w_k'(\phi^0) \frac{d\phi^0}{dy^0} \Big|_0^\infty - \int_0^\infty w_k''(\phi^0) \left(\frac{d\phi^0}{dy^0} \right)^2 dy^0 \end{aligned} \quad (4.3.13)$$

It is now necessary to transform the independent variables (x^0, y^0) to (x^0, ϕ^0) , this is an advantage in that the independent variables now both have a finite range, also it is a step that is strongly suggested by the form of Eq. (4.3.13). Providing that $W(\phi^0)$ is picked in such a way that $\phi^0 W(\phi^0)$ and $\frac{d}{dx^0} [\phi^0 W_k(\phi^0)]$ are continuous, the Eq. (4.3.13) can now be written

$$\frac{d}{dx^0} \int \phi^0 W_k(\phi^0) \frac{d\phi^0}{z} = \frac{1}{2m} \frac{dm}{dx^0} \int_0^1 (1-\phi^{0^2}) W_k'(\phi^0) \frac{d\phi^0}{z} - \int_0^1 W_k''(\phi^0) z d\phi^0 - W_k'(\phi^0) z_w \quad (4.3.14)$$

where

$$z = \left(\frac{d\phi^0}{dy^0} \right)_{x^0} \quad (4.3.15)$$

4.3.14 is the basic equation of the analysis.

The next step is to determine a suitable form of the function $z(\phi^0)$, this form is largely dictated by the behaviour of ϕ^0 in the neighborhood of certain crucial values of y^0 .

For instance as y^0 approaches infinity, ϕ must exponentially approach unity, therefore

$$\left. \begin{aligned} \phi &\longrightarrow 1 - \lambda_1 e^{-y^0/c_1} \\ \frac{d\phi}{dy} &\longrightarrow \frac{\lambda_1}{c_1} e^{-y^0/c_1} \\ z &\longrightarrow \frac{1}{c_1} (1 - \phi^0) \text{ as } \phi^0 \rightarrow 1 \end{aligned} \right\} \quad (4.3.16)$$

In a similar fashion, if it is assumed that the velocity profile acts in a locally parabolic fashion near the point of maximum reversed velocity,

$$\left. \begin{aligned} \phi^o &\longrightarrow -\phi_{rm}^o + \lambda_2 \frac{(y^o - y_{rm}^o)^2}{C_2^2} \\ \frac{d\phi^o}{dy^o} &\longrightarrow 2\lambda_2 \frac{(y^o - y_{rm}^o)}{C_2^2} \\ z &\longrightarrow \pm 2 \frac{\lambda_2^{1/2}}{C_2} (\phi^o + \phi_{rm}^o)^{1/2} \end{aligned} \right\} (4.3.17)$$

Thus it can be seen that if z takes the following form, conditions (4.3.16) and (4.3.17) will be satisfied,

$$\left. \begin{aligned} -\phi_{rm}^o &\leq \phi^o < 1 \quad \text{AND} \quad z > 0 \\ z &= (\phi^o + \phi_{rm}^o)^{1/2} (1 - \phi^o) \sum_{j=1}^n \beta_j \phi^j \\ z^{-1} &= (\phi^o + \phi_{rm}^o)^{-1/2} (1 - \phi^o)^{-1} \sum_{j=1}^n \gamma_j \phi^j \end{aligned} \right\} (4.3.18)$$

the coefficients $\beta_j(x^o)$ and $\gamma_j(x^o)$ are related by matching z and $\frac{1}{z^{-1}}$ at $n + 1$ values of the velocity between $-\phi_{rm}^o$ and 1. Similarly, for the inner profile,

$$\left. \begin{aligned} 0 \geq \phi^o \geq -\phi_{rm}^o \quad \text{AND} \quad z < 0 \\ z &= -(\phi^o + \phi_{rm}^o)^{1/2} \sum_{j=1}^m \delta_j \phi^j \\ z^{-1} &= -(\phi^o + \phi_{rm}^o)^{-1/2} \sum_{j=1}^m \epsilon_j \phi^j \end{aligned} \right\} (4.3.19)$$

Crawford only carried out the analysis for $n = m = 0$, hence

$$z = z(x^o, \phi^o) = \begin{cases} \frac{1}{C} (\phi^o + \phi_{rm}^o)^{1/2} (1 - \phi) & \begin{matrix} z > 0 \\ -\phi_{rm}^o \leq \phi^o < 1 \end{matrix} \end{cases} \quad (4.3.20)$$

$$\begin{cases} -\frac{B}{C} (\phi^o + \phi_{rm}^o)^{1/2} & \begin{matrix} z < 0 \\ -\phi_{rm}^o \leq \phi^o \leq 0 \end{matrix} \end{cases} \quad (4.3.21)$$

It is necessary that $\delta^* \phi^0 / \delta y^{02}$ be discontinuous at $z = 0$, so as to effect a change in velocity profile.

Now the velocity profile can be obtained in terms of m, B, C and ϕ_{rm}^0 from

$$y = \int_0^y ds = \int_0^u \frac{dy}{dy^s} \frac{dy^s}{dy^0} \frac{d\phi}{z} \quad (4.3.22)$$

and Eqs. (4.3.19) and (4.3.20). Formally,

$$y_{ou} = y_{ou}(m, B, C, \phi_{rm}^0); z > 0 \quad (4.3.23)$$

$$y_{in} = y_{in}(m, B, C, \phi_{rm}^0); z < 0$$

Similarly, the displacement thickness can be obtained,

$$\frac{\delta^*}{h} = \frac{\delta^*}{h}(m, B, C, \phi_{rm}^0) \quad (4.3.24)$$

the velocity on the dividing streamline from

$$\int_0^{y_{dl}} \left(\frac{\rho u}{\rho_e u_e} \right) dy = 0 \quad \text{is found.}$$

$$\phi_{dl} = \phi_{dl}(m, B, C, \phi_{rm}^0) \quad (4.3.25)$$

and the skin friction coefficient at the wall

$$C_{fw} = C_{fw}(m, B, C, \phi_{rm}^o) \quad (4.3.26)$$

Similarly, if the velocity profile in the reattached region is considered to be an "outer profile", then versions of Eqs. (4.3.23), 24, 26) can be determined for this region.

So essentially at this point in the analysis, if the appropriate forms of z are substituted into the basic equation, Eq. (4.3.14), the problem reduces to one of determining the behaviour of four variables ϕ_{rm} , B , C and m with respect to x^o . Then evidently four equations are required. Three of these are obtained by choosing different forms of the weighting function $W_k(\phi^o)$. A fourth equation is provided by a suitable form of the interaction equation (4.3.10).

If the form

$$W_k(\phi^o) = (1 - \phi^o)^k \quad k=1,2,3 \quad (4.3.27)$$

is chosen the integrations are simple and the four equations can be represented symbolically in matrix form as

$$\sum_{j=1}^4 a_{ij} D_j = b_j \quad i=1 \dots 4 \quad (4.3.28)$$

where the variables have been changed to

$$D_1 = \dot{\alpha} , D_2 = \frac{\dot{B}}{B} , D_3 = \frac{\dot{C}}{C} , D_4 = \frac{\dot{m}}{m}$$

where the dot represents differentiation with respect to x^D .

Actually with some rearrangement and appealing to Eq. (4.3.21), the system of equations (4.3.28) can be reduced to the system

$$\sum_{j=1}^3 \bar{a}_{ij} D_j = \bar{b}_i \quad i = 1, 2, 3 \quad (4.3.29)$$

where $D_1 = \dot{\alpha}$, $D_2 = \frac{\dot{C}}{C}$, $D_3 = \frac{\dot{m}}{m}$ in the separated flow region.

Similarly in the reattached flow region, the system of equations can be represented symbolically, as

$$\sum_{j=1}^3 \tilde{a}_{ij} D_j = \tilde{b}_i \quad i = 1, 2, 3 \quad (4.3.30)$$

In general, the coefficients a_{ij} , $\bar{a}_{ij}$, $\tilde{a}_{ij}$ and the right hand sides b_j , $\bar{b}_i$ and $\tilde{b}_j$ depend on ϕ_{rm} , B, C, and m.

In order to start the integration of the differential equations it is necessary to obtain initial values of ϕ_{rm} , B, C and m. These initial values must be such that as the step is approached from a downstream direction, the dividing streamline will leave the corner at $y = h$, $\phi_{rm}^0 \rightarrow 0$ and δ^* goes to some constant $\delta_0^* \geq h$.

For the case studied by Chapman, $\delta^*/h \rightarrow 1$ it is found from Eq. (4.3.24) that if $\phi_{rm}^0 \rightarrow 0$ and $y_0/h \rightarrow 1$ this implies $B \sim \alpha^{1/2}$. $C \sim \alpha$ while $m \rightarrow m_0 = \text{const.}$ Thus since $\frac{\alpha^{1/2}}{B} \gg 1$, only the leading term in each of the coefficients of the governing equations (4.3.28) need be considered.

Consideration of these coefficients is found to imply that if m goes to infinity at the corner then so does the pressure gradient. Clearly this is physically unreasonable, so the only alternative appears to be that the interaction equation (4.3.10) is invalid in the neighborhood of the corner.

Owing to this, the interaction equation was rejected, $\frac{\dot{m}}{m} = 0$, and the first three equations of (4.3.28) were analyzed for a constant pressure region. However, it proved necessary to eliminate yet another equation because three linearly independent equations for two unknowns were obtained. Hence to accomplish this elimination $\phi_{d\ell}$ was set constant and $B = B(\phi_{rm}^0)$ determined from the appropriate version of Eq. (4.3.24). Finally the two remaining "moment" equations were analyzed yielding a value of 0.670 for $\phi_{d\ell}$ (c.f. Chapman's exact value 0.587) and initial conditions for ϕ_{rm}^0 and C .

A similar but more complicated procedure is followed in the case of finite initial boundary layer displacement thickness.

In order to integrate through the reattachment point, the

behaviour of the variables at this point must be analyzed. Here the maximum reverse flow velocity ϕ_{rm}^D , the dividing streamline position y_{dl}/h , the shear stress at the wall all go to zero, and the displacement thickness δ^*/h is to be finite.

From

$$\left. \begin{array}{l} y_{rm}/h \longrightarrow 0 \\ \delta^*/h \longrightarrow \text{CONST} \end{array} \right\} \text{AS } \phi_{rm}^D \longrightarrow 0$$

it can be deduced that C is finite and $(\phi_{rm}^D)^{1/2}/B \rightarrow 0$ as $\phi_{rm}^D \rightarrow 0$.

As before the leading terms in the matrix coefficients a_{ij} of Eq. (4.3.28) are examined. This leads to descriptions of the behaviour of ϕ_{rm}^D , C and m near R, viz.

$$\left. \begin{array}{l} \phi_{rm}^D = \phi_{rm_0}^D x^D \\ C = C_0 + C_+ x^D \\ m = m_0 + m_+ x^D \end{array} \right\} \quad (4.3.31)$$

But it proves impossible to describe the behaviour of B correctly. The equations admit solutions of $B \sim (\phi_{rm}^D)^{\pm 1/2}$, but in each of these cases $(\phi_{rm}^D)^{1/2}/B$ and $(\phi_{rm}^D)^{1/2}B$ do not approach zero simultaneously, as $\phi_{rm}^D \rightarrow 0$, as required. However, if the second derivative of ϕ^D is matched at $\phi^D = -\phi_{rm}^D$ i.e. at $z = 0$, then

$$B = 1 + \phi_{rm}^D \quad (4.3.32)$$

and so $B \rightarrow 1$ as $\phi_{rm}^0 \rightarrow 0$. Thus in order to avoid singular behaviour at the reattachment point, it has to be assumed that Eq. (4.3.32) holds in the neighborhood of R. Then, when ϕ_{rm}^0 is large enough the governing equations (4.3.29) can be used. Unfortunately, this is equivalent to solving the problem inversely where the displacement thickness, Mach number and location of the reattachment point is specified and the location of the step and stepheight calculated.

Crawford called the first two "moment" equations, the interaction equation and Eq. (4.3.32), the "B-switch" equations.

The behaviour of the reattached flow equations near the reattachment point, being analyzed, turns out to be similar to that of the "B-switch" equations except for a sign change in ϕ_{rm}^D .

Ideally the numerical procedure for a complete solution was envisaged as follows. Knowing the initial conditions before the corner, M_1 , and Re_h , an initial turning angle is guessed, and the integration downstream begun with the interaction equation neglected initially (constant pressure region). Then the interaction equation is "switched on" so that the inviscid external flow can interact with the growth of the viscous layer as R is approached. Continuing to integrate downstream, the separated flow equations are replaced by reattached flow equations at R.

Far downstream the Mach number should asymptotically approach the initial value, if the initial turning angle was correctly guessed.

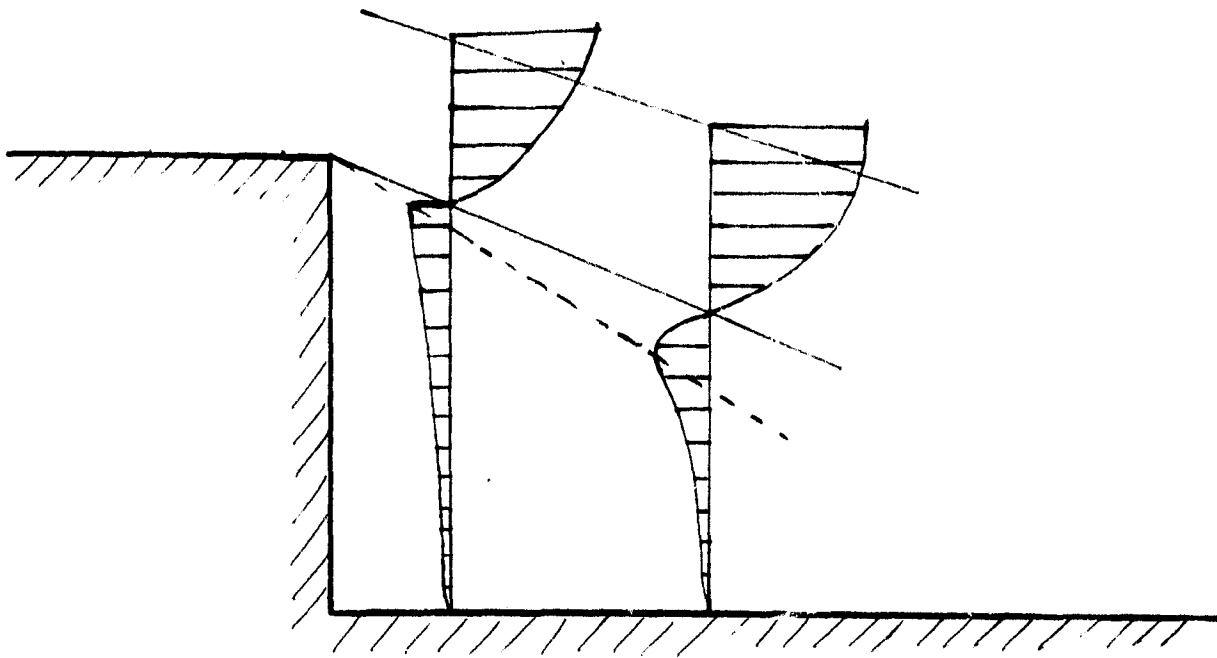
However, due to the difficulties encountered at reattachment, this procedure cannot be followed. Since the difficulties arose because of a discontinuity in the second derivative of velocity, Crawford suggested that if z taken to have the form,

$$\bar{z} = \begin{cases} \frac{1}{C} (\phi^0 + \phi_{rm}^0)^{1/2} [1 + B(\phi^0 + \phi_{rm}^0)^{1/2}] (1 - \phi); & z > 0 \\ -\frac{1}{C} (\phi^0 + \phi_{rm}^0)^{1/2} (1 + \phi_{rm}^0) & ; z < 0 \end{cases} \quad (4.3.33)$$

it will embody the essential features necessary to obtain a constant pressure flow near a sharp corner and a separated type profile in the reattachment region, without introducing a discontinuity in the second derivative, which was introduced in the analysis summarized above in order to change the shape of the velocity profiles. In this way the difficulties at the reattachment point would be avoided.

Two interesting results were obtained from this analysis:

(i) the locus of the point of maximum reversed flow was found to be linear in a constant pressure region. This would result in velocity profiles similar to those shown below.



This may explain the anomalous results obtained by Smith (see subsection 3.14.2).

(ii) the shear stress and pressure was found to vary linearly up to reattachment. This supports the assumption made in sub-section (3.14.5-6) concerning the reattachment process.

4.4 Discussion

It would seem appropriate at this point to briefly mention some of the experimental work connected with the predictions of the integral theories summarized in the preceding sections. Since the inception of the Crocco-Lees theory measurements have been made of its characteristic parameters, in particular Badrinarayan⁸³ and de Krasinski¹⁶ measured the mixing coefficient and the universal curve, demonstrating on the whole a need for a more sophisticated theory. This was supplied by the Reeves-Lees theory which has received, as yet, little independent confirmation, although mention should be made of the measurements of the variation of the parameters $\bar{H} = \frac{\delta^{**s}}{\delta_t^{*s}}$ and $J = \frac{\delta^{***s}}{\delta_t^{*s}}$ with the shape parameter $A(x^S)$, carried out by Sirieix and Delery⁹⁷. Good agreement was found between these measurements and the predictions of the Lees-Reeves theory. In addition, the pressure distribution along the base centerline predicted by Alber¹²⁷ agrees well with the experimental results available^{83,88,107,138}. It should be pointed out, however, that Alber's¹²⁷ predictions for base pressure (20-30% too high) are no more accurate than some Chapman-Korst model theories.

The most characteristic feature of the integral theories in sections 4.1 and 4.2 is the singular point just downstream of the rear stagnation point. The existence of this point has been

well confirmed experimentally especially by the work of Sirieix et.al.^{35,92,97}. That is to say, it has been observed that, within certain limits, the reattachment process is entirely independent of the behaviour of the flow downstream of a point just downstream of the reattachment point. However, this phenomenon can be explained in other ways without resorting to integral theories. For instance, this behaviour is also predicted by the "control volume" interpretation of the reattachment process introduced in subsection 3.14.5(b). Actually, it would be expected from consideration of the boundary layer equations themselves, because owing to their parabolic nature the equations would not allow the transmission of signals upstream in the reattached flow. This brings up a criticism of Crawford's¹³⁷ theory which in its present formulation uses the recovery of the wake, to the initial pressure, downstream of reattachment, as a criterion for the correct choice of initial conditions. This is not appropriate in view of what has been stated above.

An important question that can be raised in connection with integral theories is to what extent the validity of boundary layer equations limit the applicability of these theories?

Some investigators have taken the existence of transverse pressure gradients as an indication of the invalidity of boundary layer theory; actually, it is sufficient that $\partial^2 P / \partial n \partial x = 0$. Hence

it is possible for the boundary layer equations to be valid in regions of non-zero transverse pressure gradient. This does, however, lead to a further criticism of theories using fixed axes, i.e. take the abscissa as the base centerline and ordinate as the base wall, then it is probable that transverse pressure gradients of some magnitude would artificially be introduced at recompression.

Evidently the assumption of very small transverse velocities i.e. $\frac{v}{u} \ll 1$ is violated in, at least, part of the recirculation region; thus using the boundary layer equations here is tantamount to neglecting important dynamic effects at the base wall. Secondly, the assumption $\delta/L \ll 1$ is invalid except for very slender bodies. These objections are even more important for turbulent separated flows, since there is no reason in this case to assume $\frac{\delta}{L} \rightarrow 0$ as $Re_L \rightarrow \infty$, rather it is more plausible that $\frac{\delta}{L} \rightarrow \text{constant}$ for $Re_L \rightarrow \infty$. For example, in a wake flow if the Reynold's number is defined based on the eddy viscosity instead of ordinary viscosity, the Reynold's number tends to be constant of an order of 10.

The interaction between the boundary layer and the external supersonic inviscid flow during the approach to a rearward facing step is another instance where the boundary layer equations do not strictly apply.

UPSTREAM INFLUENCE OF A STEP

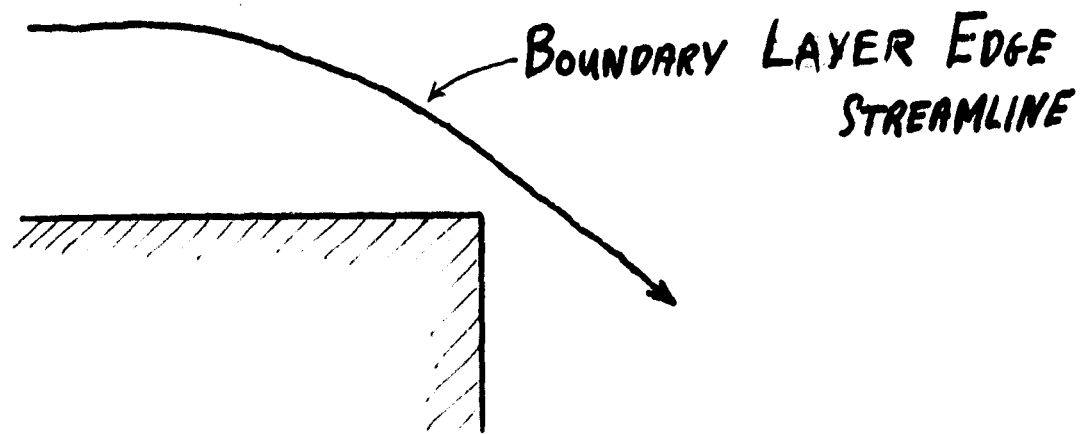


Figure 38

A feature of such interactions, which is apparently anomalous if boundary layer theory is valid, is the fairly extensive upstream influence of the disturbance as is shown in Fig. 38. Since, as mentioned before, the boundary layer equations are parabolic, there is no direct mechanism by which the presence of the step downstream can be transmitted upstream. Nevertheless, the theories summarized in this chapter have used the boundary layer equations to describe this phenomenon and the related one of reattachment, with, at least, some success.

Baum¹³⁹ offers an explanation for this. He considers that, in the case of an interacting boundary layer, the neutrally stable solution corresponds to a normal, attached boundary layer. A perturbation in initial conditions from those producing the neutrally stable solution results in one of two types of downstream behaviour. Either the pressure starts to increase and the

shear stress at the wall decreases until the boundary layer separates or the pressure decreases at an accelerating rate causing streamlines to turn inward as shown in Fig. 38.

What is lacking, when the boundary layer equations are used, is a means of coupling the perturbation, determining the downstream behaviour, with the position of the step. This coupling must take place by means of higher order terms in the Navier-Stokes equations which are not present in the boundary layer formulation and which would introduce a direct mechanism of upstream communication if included.

The problem of the flow approaching a rearward facing step is solved to a large extent by methods summarized in Chapters 5 and 6. The reattachment process, however, has not yet been fully analyzed.

5. Multimethod Base Flow Theories

It is apparent that the integral methods surveyed in the last chapter do not represent an improvement over Chapman-Korst theories as far as taking proper account of the dynamics of the recirculation region is concerned. The theoretical methods reviewed in this chapter do attempt to correct this situation. Apart from possible use in direct application to base flow calculations, these theories provide a means of ascertaining quantitatively the effect the recirculating flow has on the base flow properties.

The earlier theoretical studies of Baum et.al. belong strictly speaking in the ranks of Chapman-Korst model theories, but are included in this chapter for the sake of continuity.

5.1 The Studies of Baum et.al.

The starting point of this base flow theory is a constant pressure mixing solution for the case of finite initial boundary layer thickness. This was introduced in ref. 65.

The governing equations,

$$\frac{d}{dx}(\rho u r_o^k) + \frac{d}{dy}(\rho v r_o^k) = 0 \quad \text{Cont. (5.1.1)}$$

$$\rho \left(u \frac{du}{dx} + v \frac{du}{dy} \right) = \frac{d}{dy} \left(\mu \frac{du}{dy} \right) \quad \text{Mom.} \quad (5.1.2)$$

$$\rho \left(u \frac{dH_0}{dx} + v \frac{dH_0}{dy} \right) = \frac{d}{dy} \left(\frac{\mu}{Pr} \frac{dH_0}{dy} \right) + \frac{d}{dy} \left[\frac{\mu}{Pr} (Pr-1) \frac{d^2 H_0}{dy^2} \right] \quad \text{Energy} \quad (5.1.3)$$

are first transformed into equivalent incompressible form using Howarth type transformations,

$$\left. \begin{aligned} \bar{Y} &= \rho_e u_e r_0^k \int_0^y \left(\frac{\rho}{\rho_e} \right) dy \\ \bar{S} &= \int_0^x \bar{C} \rho_e u_e \mu_e r_0^{2k} dx \end{aligned} \right\} \quad (5.1.4)$$

where $\bar{C}$ is the Chapman-Rubesin constant

$$\bar{C} = \frac{\rho \mu}{\rho_e \mu_e} \quad (5.1.5)$$

The equations are then expressed in Crocco coordinates making use of the usual definition of stream function. Crocco coordinates are introduced to make the range of both independent variables finite to carry out numerical integration. The equations now have the following form,

$$\phi \frac{\partial F^*}{\partial S^*} = F^{*2} \frac{\partial^2 F^*}{\partial \phi^2} \quad \text{Mom.} \quad (5.1.6)$$

$$\phi \frac{\partial H_0}{\partial S^*} = F^{*2} \frac{\partial^2 H_0}{\partial \phi^2} \quad \text{Energy} \quad (5.1.7)$$

where

$$S^* = \frac{\bar{S}}{\bar{S}_w} \quad (5.1.8)$$

and

$$F^* = \frac{\partial \phi}{\partial Y} \bar{S}_w^{1/2} \quad (5.1.9)$$

with the boundary conditions

$$F^*(S^*, 0) = 0 \quad (5.1.10)$$

$$H_0(S^*, 0) = H_{0e} \quad (5.1.11)$$

and at outer edge,

$$F^*(S^*, 1) = 0 \quad (5.1.12)$$

$$H_0(S^*, 1) = H_{0e} \quad (5.1.13)$$

The unknown constant recirculation region enthalpy H_{0e} is a boundary condition, and because of this the equations are rearranged so that the numerical solution can be carried out without knowing the quantity H_{0e} . Since Eq. (5.1.6) and (5.1.7)

is linear in H_0 , H_0 can be replaced by linear combinations of functions of S^* and ϕ with H_{0e} appearing as a multiplier, thus

$$H_0 - H_{0e} = (H_w - H_{0e})W(S^*, \phi) + (H_{0i} - H_{0e})E(S^*, \phi) \quad (5.1.14)$$

$W(S^*, \phi)$ and $E(S^*, \phi)$ have to satisfy Eq. (5.1.7) replacing H_0 and also satisfying initial and boundary conditions.

Difficulty is found in starting the numerical solution of Eqs. (5.1.6) and (5.1.7) because a singularity exists at the separation point. The difficulty was overcome by expanding F^* as a power series, valid near the separation point, and solving the resulting differential equation.¹⁴⁰

Originally, the initial profiles were taken to be those of Blasius with the Crocco integral relation for the total enthalpy, however, Baum¹⁴¹ subsequently introduced a method for calculating the effect of the sharp turn at separation. The boundary layer profiles were assumed to distort according to an isentropic expansion to base pressure along streamlines.

The unknown recirculation enthalpy H_{0e} is determined from an overall energy balance on the following control volume,

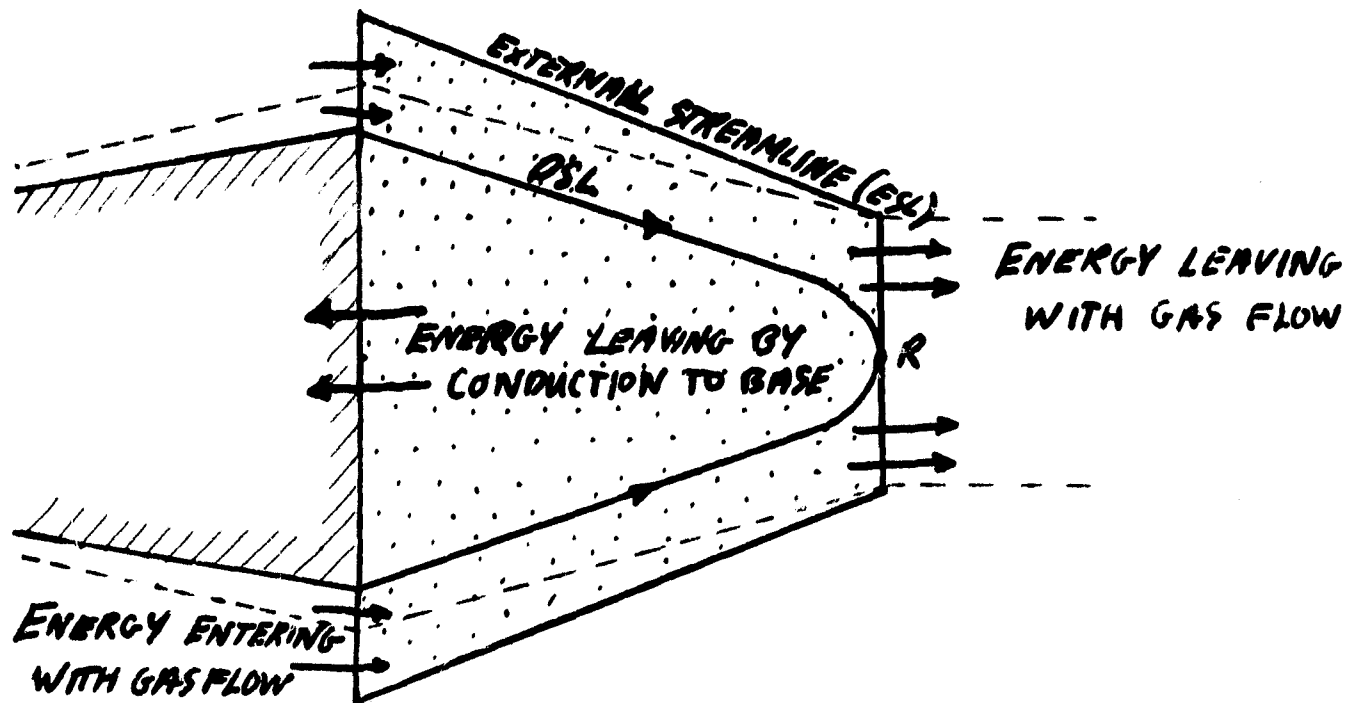


Figure 39

This results in the equation,

$$\left[- \int_{DSL}^{ESL} (H_{oe} - H_b) \rho u (2\pi r_o)^k dy \right]_{Body}$$

Energy entering from body boundary layer

(5.1.15)

$$= \left[- \int_{DSL}^{ESL} (H_{oe} - H_b) \rho u (2\pi r_o)^k dy \right]_{NECK} + Q_b$$

Energy leaving through neck

Energy leaving by conduction and radiation to base

If this equation is written in terms of the coordinate system used in Eqs. (5.1.6) and (5.1.7), it reduces to

$$H_{oc} = H_{oe} - \frac{(H_{oe} - H_{ow})(\tilde{K} - \tilde{J}) + \left[\frac{Q_b}{(2\pi)^k} - H_{oe} \gamma_d \right] S_w^{-1/2}}{\tilde{L}} \quad (5.1.16)$$

where

$$\tilde{J}(S_{de}^*) \equiv \left[\int_{\phi_d}^{\phi} \frac{\phi W}{F^*} d\phi \right]_{NECK} \quad (5.1.17)$$

$$\tilde{K} \equiv \left[\int_0^1 \frac{\phi(1-\phi)}{F^*} d\phi \right]_{\text{BODY}} \quad (5.1.18)$$

$$\tilde{L}(S_{d_b}^*) \equiv \left[\int_{\phi_h}^1 \frac{\phi E}{F^*} d\phi \right]_{\text{NECK}} \quad (5.1.19)$$

To match the shear layer solution to the recirculating flow and external flow conditions, it is necessary to specify the body shape, free-stream conditions, body surface enthalpy H_{0W} and the base enthalpy H_b or if Q_b is assumed negligible H_b need not be specified. The outer flow is given by either the Prandtl-Meyer relationship or a characteristic solution. The Chapman-Korst reattachment criterion is used to determine the correct level of total pressure on the dividing streamline.

Solutions obtained by finite difference methods are presented tabularly for the case of $Q_b = 0$ in refs. 141, 142, 143. An extension¹⁴⁴ to the case of base bleed is fairly simple and is presented in the references quoted.

A method for determining the base heat transfer has been developed^{145,146}. Essentially, the Chapman type model is retained to make the shear layer calculations and the matching with the inviscid flow carried out as above. Then it is assumed that the recirculating flow is like a low speed potential flow emanating

from a source, and that a stagnation-point-type boundary layer exists on the base wall. Thus this boundary layer provides a thermal resistance between the base wall at enthalpy H_b and the recirculating flow at enthalpy H_{oc} .

SOURCE ANALOGY TO RECIRCULATING FLOW

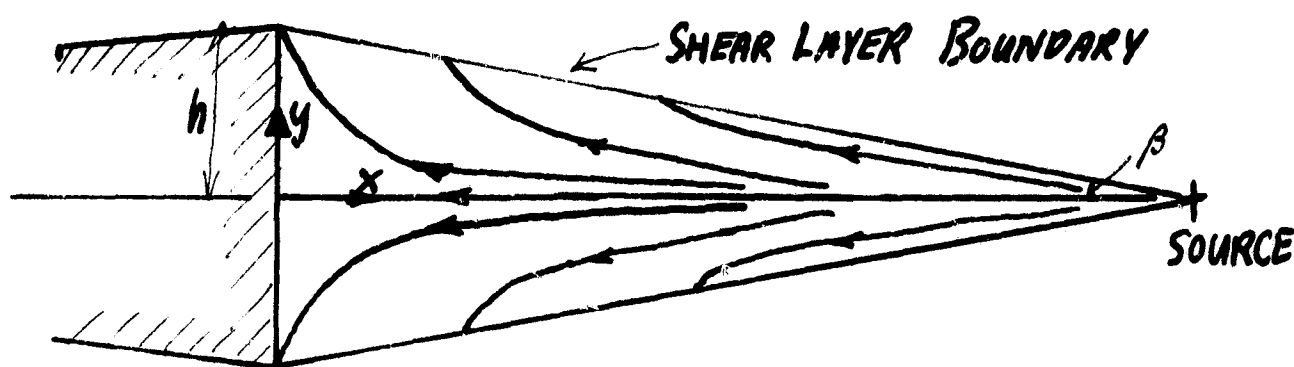


Figure 40

The problem to be solved now is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{k}{y} \frac{\partial \psi}{\partial y} = 0 \quad (5.1.20)$$

$\psi = 0$ on the base wall,

$\psi = 0$ on the centerline,

$\psi = \psi_c$ on the shear layer.

ψ_c is taken from the shear layer solution, and the subsequent solution to Eq. (5.1.20) then provides outer boundary conditions for the base wall boundary layer calculation, and hence an estimate of the base heat transfer. The shear layer/inviscid flow

matching is then repeated, until all the features of the base flow match with the exception of the tangential velocity along the inner edge of the shear layer. Since only normal velocity components can be specified in the potential flow this last quantity cannot be matched satisfactorily.

Calculations performed using this model find that at moderate Reynold's number, the base heat transfer is a negligible portion of the total body heat transfer and therefore not important in Eq. (5.1.16). Whether or not this is a result of the simplifications introduced remains to be seen.

Baum¹⁴⁷ has recently developed an implicit finite difference method to study the interaction of a laminar viscous wake with a supersonic external inviscid flow downstream of the rear stagnation point, but the method has not yet been extended to cover the base flow region.

5.2 The Studies of Weiss

It was suggested some time ago by Batchelor¹⁴⁸ that the recirculation region be represented by model consisting of an inner core of inviscid vorticity surrounded by the mixing layers and a thin boundary layer on the base wall. While at the other end of the scale, analysis had been carried out assuming a Stoke's type flow¹⁴⁹, Weiss¹⁵⁰ carried out an order of magnitude

study to determine the characteristic Reynold's number of the recirculating flow, $R_{e_{h,b}}$ and thus the thickness of the base wall boundary layer. He concluded that for $R_{e_{c,\infty}} < 10^6$ the recirculating flow is fully viscous with no inviscid core, and, on the basis of Weinbaum's¹⁵¹ study, that for $R_{e_{c,\infty}} < 10^5$ a Stoke's type solution is appropriate for the recirculation region.

Carrying the order of magnitude study to other parts of the flow field resulted in the formulation of a new model for base flow theories^{150,152}. First, it is assumed that an inviscid expansion of the body boundary layer to base pressure occurs, resulting in a vortical flow outside the dividing streamline. Then a new shear layer of thickness, δ_e , grows into this vortical flow. Boundary layer theory is valid in this new shear layer which separates an outer inviscid vortical flow from an inner viscous region of closed streamline flow.

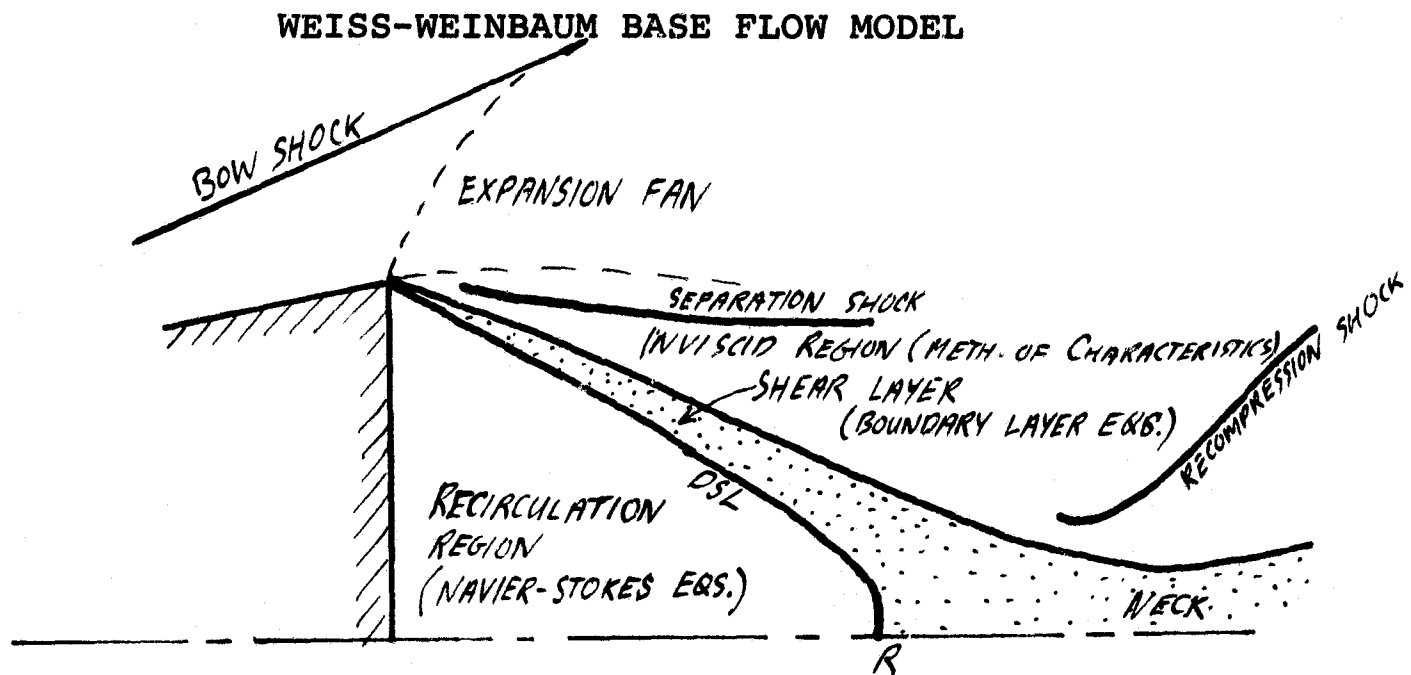


Figure 41

If it is assumed that recirculation region could be approximated by a Stoke's type flow, the shear stress on the inside of the recirculation region would be of the form,

$$\tau_b = K \frac{\mu_d u_d}{h} \quad (5.2.1)$$

K is a numerical constant and obtained by a Stoke's analysis¹⁵³.

In a hypersonic boundary layer, the velocity profile is approximately linear, hence

$$\tau_d \doteq \frac{\mu_d (u_e - u_d)}{\delta_\varepsilon} \quad (5.2.2)$$

Since the shear stress must be continuous across the dividing streamline,

$$\tau_d \doteq \tau_b \quad ; \quad \phi_{dl} = \left(1 + K \frac{\delta_\varepsilon}{h}\right)^{-1} \quad (5.2.3)$$

If it is assumed that recompression along the dividing streamline occurs isentropically, the base pressure is given by

$$\frac{P_b}{P_\infty} = \bar{N} \left[1 + \frac{\gamma-1}{2} M_{dl}^2\right]^{-\frac{\gamma}{\gamma-1}} \quad (5.2.4)$$

where

$$M_{dl}^2 = \left(\frac{u_{dl}}{u_e}\right)^2 \left(\frac{T_e}{T_{dl}}\right) M_e^2$$

and $\bar{N} < 1$ (Chapman and Baum theories assume $N = 1$).

Hence Eq. (5.2.3) shows that $\phi \sim Re_{c,\infty}^{1/2}$ causing P_b/P_∞ to vary inversely with Reynolds number from Eq. (5.2.4).

On the basis of this preliminary analysis, Weiss concluded that the variation of base pressure with Reynolds number in laminar flow can be divided into three regimes. At low $Re_{c,\infty}$ the base region is Stoke's-like with P_b/P_∞ increasing until $Re_{c,\infty} \sim 10^4$. Then the viscous flow above the dividing streamline takes on a boundary layer character causing P_b/P_∞ to decrease as $Re_{c,\infty}$ increases. Finally at $Re_{c,\infty} > 10^6$ an inviscid recirculating core forms and the base pressure tends towards being independent of Reynolds number. This conclusion supports the explanations offered in subsection 3.14.6, and also shows that the Reynolds number based on base height could be an important length scale in base flows.

Later Weiss^{154,155,156,157} refined his theoretical approach and obtained complete numerical solutions of the entire base flow. He divided the flow field into three regions external, vortical inviscid flow; shear layer; and viscous recirculation region; as shown in Fig. 41. Each of these regions are dependent on one another in that the shear layer connects the two other regions and supplies, and is supplied in turn, boundary conditions.

The expansion process at the base edge and its effect on the separating boundary layer will be dealt with in the next

chapter, suffice to say Weiss employed the methods developed by Weinbaum^{155,156}. The relatively small portion of the boundary layer which is initially subsonic is neglected, and the solution of the inviscid equation of motion is carried out by the method of characteristics, the shock relations are added when characteristics of the same family intersect, and in this way a procedure is included to trace the separation and recompression shock waves through a vortical non-adiabatic field. This inviscid solution has two major applications, first, it describes the properties of a large part of the near wake; secondly, boundary conditions for the shear layer, originating at the trailing edge, are obtained.

The shear layer is analyzed using the linearized Oseen-type boundary layer equations,

$$c \frac{\partial \phi}{\partial \bar{x}} = -\bar{\Phi} + \frac{\partial^2 \phi}{\partial \bar{y}^2} \quad (5.2.5)$$

$$c \frac{\partial T_0}{\partial \bar{x}} = \frac{\partial^2 T_0}{\partial \bar{y}^2} \quad (5.2.6)$$

where

$$\bar{x} = \frac{x u_e}{v_e} \quad (5.2.7)$$

$$\bar{y} = \left(\frac{u_e}{v_e} \right) \int_0^y \left(\frac{\rho}{\rho_e} \right) dy \quad (5.2.8)$$

$$\text{AND} \quad \Phi = \frac{\partial P}{\partial \bar{x}} \frac{\rho_d}{\rho_e} \frac{1}{\rho_d u_e^2} \quad (5.2.9)$$

The coordinates x, y are parallel and perpendicular to the dividing streamline. The vortical flow profiles after the trailing edge written in suitable form are used as initial conditions. The boundary conditions on the dividing streamline,

$$\left. \begin{aligned} u &= u(x, 0) \quad 0 \leq x \leq l \\ T_0 &= T_0(x, 0) \quad 0 \leq x \leq l \end{aligned} \right\} \quad (5.2.10)$$

are obtained from the recirculation region solution, and are expressed in the form of a Fourier series, in this way the boundary conditions $u(x, \infty) = T_0(x, \infty) = 1$ will be automatically satisfied.

To solve the system (5.2.5), (5.2.6) and (5.2.10) Laplace transforms are used; let the solutions be formally written as,

$$\left. \begin{aligned} u &= u[x, y; u(x, 0)] \\ T_0 &= T_0[x, y; T_0(x, 0)] \end{aligned} \right\} \quad (5.2.11)$$

It turns out that the boundary conditions are satisfied independent of the value of the linearizing constant C in Eqs. (5.2.5) and (5.2.6). Because of this C can be chosen so as the solutions

(5.2.11) approximate most accurately the solutions to the exact nonlinear boundary layer equations on the dividing streamline ($y = 0$), since this is where accuracy is most crucial.

When $u(x,y)$ and $T_0(x,y)$ are determined the effect of the shear layer on the boundary conditions of the external inviscid flow may be calculated. The displacement thickness of the viscous sublayer is defined as

$$\Delta\delta_e^* = \delta^*(x) - \delta^*(0) \quad (5.2.12)$$

where displacement thickness δ^* is defined in the usual way except the integration is carried out from 0 to ∞ , and $\delta^*(0)$ is the value of the integral carried out in the inviscid vortical flow at $x = 0$.

The recirculation region is handled by integrating the full Navier-Stokes equations over the region. The incompressible equations are considered sufficient and the equations written as two second order partial differential equations for the diffusion of vorticity ζ and the definition of vorticity, respectively,

$$Re_h \left[\frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{\zeta}}{\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial \bar{\zeta}}{\partial \bar{x}} \right] = \nabla^2 \bar{\zeta} \quad (5.2.13)$$

$$\bar{\zeta} = \nabla^2 \bar{\psi} \quad (5.2.14)$$

where the equations have been non-dimensionalized as follows

$$\left. \begin{aligned} \bar{u} &= \frac{u}{q_0} , \quad \bar{v} = \frac{v}{q_0} \\ \bar{x} &= \frac{x}{h} , \quad \bar{y} = \frac{y}{h} \\ Re_h &= \frac{\rho_0 h}{\mu} , \quad \bar{\psi} = \frac{\psi}{q_0 h} , \quad \bar{\xi} = \frac{\xi h}{q_0} \end{aligned} \right\} \quad (5.2.15)$$

the coordinate system (x,y) is now parallel to the base center-line and base wall, and q_0 is the maximum velocity on the dividing streamline.

The boundary conditions for these equations are

$$\left. \begin{aligned} \bar{\psi} &= 0 \quad \text{AT } x=0, y=0, y_d(x) \\ \bar{\xi} &= 0, \quad \text{AT } y=0; \quad \bar{\xi} = \partial^2 \bar{\psi} / \partial x^2 \quad \text{AT } x=0 \\ \bar{\xi} &= \bar{\xi}(x) \quad \text{AT } y_d(x) \end{aligned} \right\} \quad (5.2.16)$$

This last is obtained by differentiating Eq. (5.2.11).

Since low Mach numbers are expected in the recirculation region, it is assumed that Re_h is a slowly varying function of x,y and that the density and velocity fields are weakly coupled. Thus a first step in an iterative procedure could be to specify constant density and temperature over the entire region and solve Eqs. (5.2.13) and (5.2.14), then to determine temperature, density and Re_h variations from an energy equation and the equation of state.

The integration is carried out using finite difference

techniques. A complete base flow solution is arrived at when the properties become continuous across the boundaries between the three regions and when the pressure recompresses back to the ambient pressure. To find this solution is a formidable task requiring many iterations, so Weiss begins the search for a unique base pressure by finding that pressure for which the dividing streamline is straight, since it is assumed that the unique base pressure is close to this value.

The calculation of base flow at the higher Reynolds numbers in the laminar regime is not practicable owing to the small mesh size required for stable solutions; and at very low Reynolds numbers the flow model used is unrealistic.

5.3 Discussion

The use of the Chapman-Korst reattachment criterion to determine the level of total pressure on the dividing streamline as a means of arriving at a unique base pressure can be criticized as before; so can the use of recovery of the flow downstream of reattachment to free-stream pressure as a uniqueness criterion. Actually if Eq. (5.2.2) is taken as a reasonable approximation for τ_d , it can be seen that at hypersonic speeds the shear stress can no longer be neglected, compared to the pressure, during the reattachment process. Thus the Chapman-Korst reattachment criterion is probably as bad an assumption

for hypersonic laminar flow as it is for turbulent flows at moderate speeds.

If the conclusion made by Weiss about the existence of an inviscid core in the recirculation region is reasonable, then its safe to say that Baum and Kings' source flow model (see Fig. 40) is at best applicable when $Re_c > 10^6$. So the conclusion that the base heat transfer is a negligible fraction of the total body heat transfer may only apply at higher Reynolds number.

Owing to the difficulties mentioned at the end of section 5.2, only measurements at hypersonic speeds can be compared to the theoretical predictions. Accurate results in this regime are not numerous. However, the measurements of Muntz and Softley^{160,161} confirm some aspects of Weiss' theory particularly the predicted behaviour of base pressure with Reynolds number; also the predicted dividing streamline passes precisely through the stagnation point location inferred by Todisco and Pallone¹⁵⁹ from their total temperature data. Also, Muntz and Softley results for the ratio of total base heat transfer to total cone heat transfer correspond very convincingly with the predictions of King¹⁴⁵. This supports the conjecture that base heat transfer is a very small fraction of the total (not more than 0.006 in Muntz and Softley results).

6. The Effect of the Rapid Expansion at the Trailing Edge

The interaction between the separating boundary layer and the centered expansion waves at the trailing edge has lately been recognized as having an important effect on the base flow. Owing to the reasonably large amount of independent investigation devoted to this topic, and since it is treated rather differently from other aspects of base flow, it was decided to devote a separate chapter to this subject.

The earlier theoretical methods of Kirk²⁵, Carriere and Sirieix²⁶, Nash³⁹, and Roberts⁵⁵ are dealt with in the first section; they are relatively uncomplicated simply assuming an isentropic expansion process along streamlines. However, these investigators either ignored or overlooked the presence of a separation shock. This phenomenon appears to have been first observed by Charwat and Yakura⁸⁰, but its effect was generally regarded as unimportant until Hama^{87,162} established the converse. Thus subsequent theoretical treatments of boundary layer/expansion fan interaction are much more sophisticated and take the separation shock into account.

6.1 Methods Based on the Prandtl-Mayer Relationship

The theoretical methods in this group are almost exclusively concerned with the prediction of momentum thickness after

the corner expansion in terms of given upstream conditions. Except for the equation due to Kirk²⁵ which is only really valid in the case of small turning angles, they are essentially based on the same concept, and only differ in the degree of approximation etc.

The separating boundary layer is regarded as analagous to one negotiating a corner on a wall, an isentropic expansion process along each streamline is assumed, and the viscous effects are ignored except insofar as they generate an initial profile. Roberts' method will now be briefly summarized.

A section CD is chosen so that the pressure P , is uniform and the streamlines are parallel.

ROBERT'S SCHEMATIC FOR THE CORNER EXPANSION

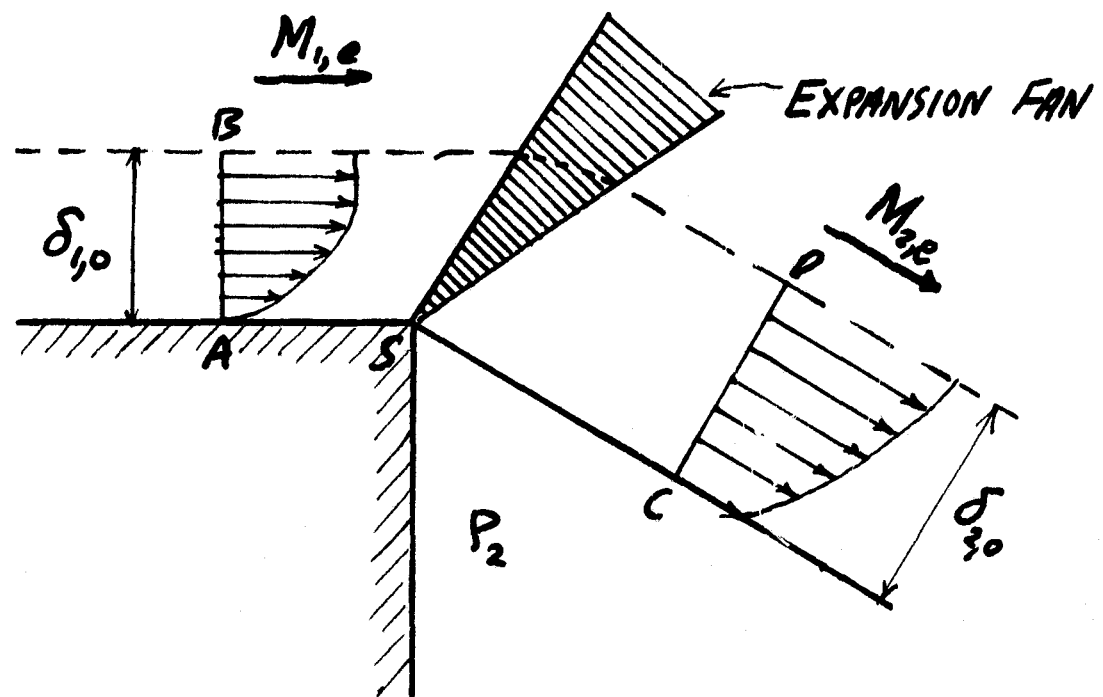


Figure 42

then from applying mass continuity along any streamline between AB and CD,

$$\rho_1 u_1 dy_1 = \rho_2 u_2 dy_2 \quad (6.1.1)$$

and from the usual definition of momentum thickness combined with Eq. (6.1.1) is obtained,

$$\frac{\rho_{e2} u_{e2}}{\rho_{e1} u_{e1}} \delta_{2,0}^{**} = \int_0^{\delta_{1,0}} \frac{\rho_1 u_1}{\rho_{e1} u_{e1}} dy_1 - \int_0^{\delta_{1,0}} \frac{\rho_1 u_1^2}{\rho_{e1} u_{e1}^2} dy_1 \quad (6.1.2)$$

combining this equation with the assumption of homenergetic flow i.e. total temperature is conserved across the boundary layer or

$$\frac{\rho}{\rho_e} = \frac{1 - C_e^2}{1 - C_e^2 \phi^2} \quad (6.1.3)$$

and isentropic flow along streamlines

$$\left(\frac{P_2}{P_1} \right)^{\frac{\gamma-1}{\gamma}} = \frac{1 - C_{2e}^2}{1 - C_{1e}^2} = K \quad (6.1.4)$$

leads to

$$\frac{\rho_{e2} u_{e2} \delta_{2,0}^{**}}{\rho_{e1} u_{e1} (1 - C_{e1}^2)} = \int_0^{\delta_{1,0}} \frac{\phi_1 dy_1}{1 - C_{e1}^2 \phi_1^2} - \int_0^{\delta_{1,0}} \frac{\phi_1 [K C_{e1}^2 \phi_1^2 - (K-1)]^{1/2}}{(1 - C_{e1}^2 \phi_1^2) C_{e1}} dy_1 \quad (6.1.5)$$

By representing the initial boundary layer profile by a power law; i.e.

$$\frac{y_1}{\delta_1} = \phi_1^n \quad (6.1.6)$$

and some manipulation Eq. (6.1.5) reduces to

$$\frac{\rho_e u_{e2} \delta_{2,0}^{**}}{\rho_e u_{e1} \delta_{1,0}^{**}} = \frac{\int_0^1 \frac{\phi_1^n d\phi_1}{1 - C_{e1}^2 \phi_1^2} - \int_0^1 \frac{\phi_1^n [K C_{e1}^2 \phi_1^2 - (\kappa - 1)]^{1/2} d\phi_1}{(1 - C_{e1}^2 \phi_1^2) C_{e1}}}{\int_0^1 \frac{\phi_1^n (1 - \phi_1)}{(1 - C_{e1}^2 \phi_1^2)} d\phi_1} \quad (6.1.7)$$

by the ignoring terms of order

$$\frac{1 - \frac{M_{e1}^2}{M_{e2}^2}}{2} \left(2 - \frac{\delta_{2,0}^{***}}{\delta_{1,0}^{**}} \right) \quad \text{compared}$$

to unity, Nash arrived at the much simpler expression,

$$\frac{\rho_e u_{e2} \delta_{2,0}^{**}}{\rho_e u_{e1} \delta_{1,0}^{**}} = \frac{M_{e1}^2}{M_{e2}^2} \quad (6.1.8)$$

Carriere and Sirieix in a similar fashion obtained,

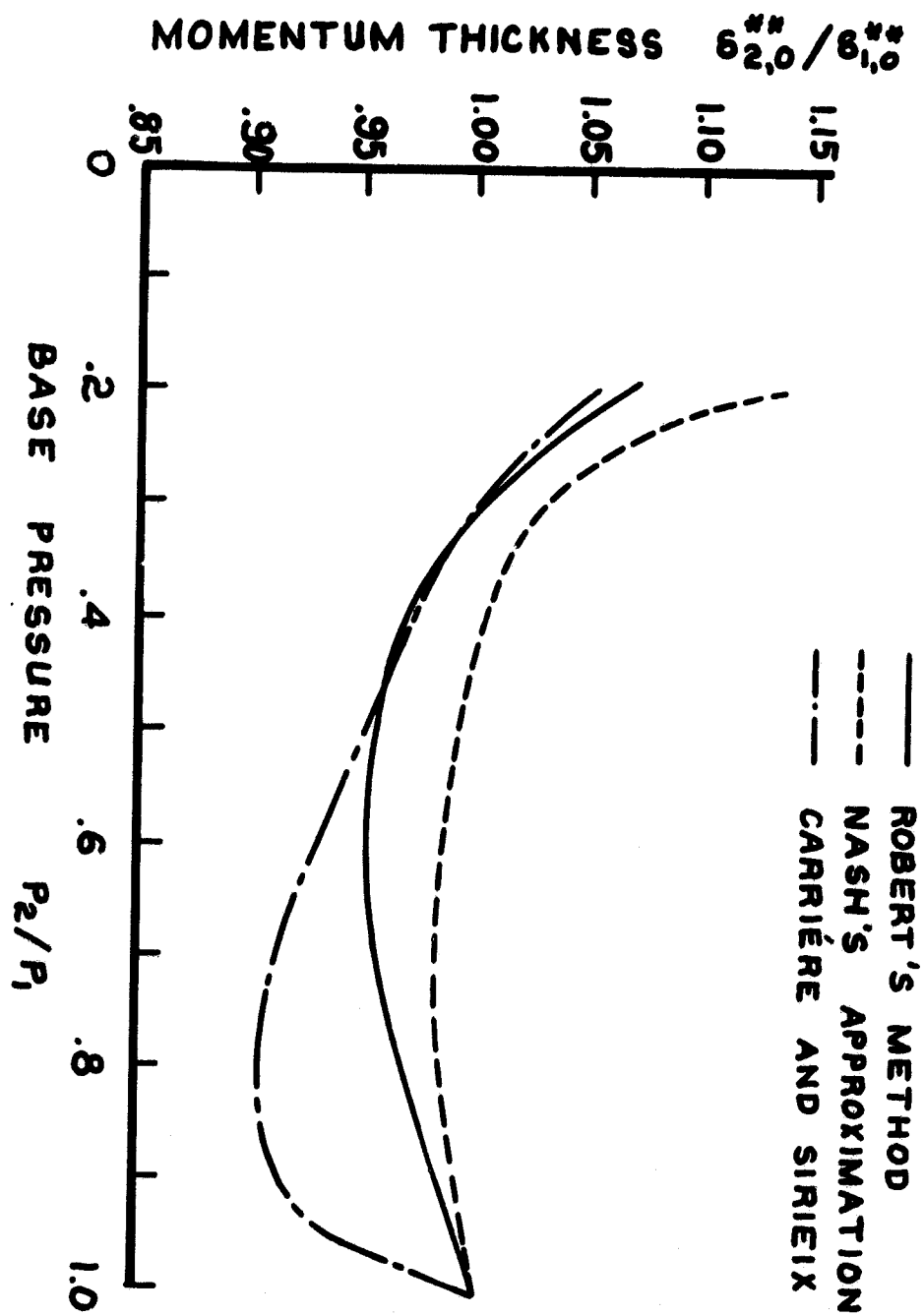
$$\frac{\rho_e u_{e2} \delta_{2,0}^{**}}{\rho_e u_{e1} \delta_{1,0}^{**}} = 1 - \frac{1}{2} \left(\frac{M_{e2}^2}{M_{e1}^2} - 1 \right)^{1/2} \quad (6.1.9)$$

and for completeness, Kirk's result is listed,

$$\frac{\delta_{2,0}^{**}}{\delta_{1,0}^{**}} = 1 + \left(\sqrt{M_{e1}^2 - 1} - \frac{1 + \frac{\delta_{1,0}^*}{\delta_{1,0}^{**}}}{\sqrt{M_{e1}^2 - 1}} \right) \theta_2 \quad (6.1.10)$$

Fig. 43 shows a comparison between the four expressions.

FIGURE 43. EFFECT OF ABRUPT EXPANSION ON BOUNDARY LAYER
MOMENTUM THICKNESS.



6.2 Weinbaum-Weiss Interaction Model

Weinbaum¹⁵⁸ carried out an order of magnitude analysis of a supersonic boundary layer undergoing separation and expansion at a corner. He concluded that the viscous forces in the supersonic part of the boundary layer are very much smaller than the inertia and pressure forces, and that the gross behaviour of the boundary layer could be predicted by considering the process to be inviscid, neglecting the subsonic part of the boundary layer. Under these conditions pressure signals are transmitted along Mach lines which are highly curved in the lower velocity part of the boundary layer due to the strong entropy gradient. Therefore, because of this, the flow in the upper part of the boundary layer is subjected to a much weaker expansion process than that in the lower, and thus a streamline near the outer edge turns through a smaller angle than one below. This creates a large normal pressure gradient which the fluid is unable to support and therefore a separation shock wave is formed to recompress the flow by varying amounts. Another consequence of this unequal turning process is that after the expansion the streamlines have become highly divergent and the flow loses its boundary layer character, becoming an inviscid highly vortical flow with a new viscous sublayer growing by mixing.

WEINBAUM-WEISS INTERACTION MODEL

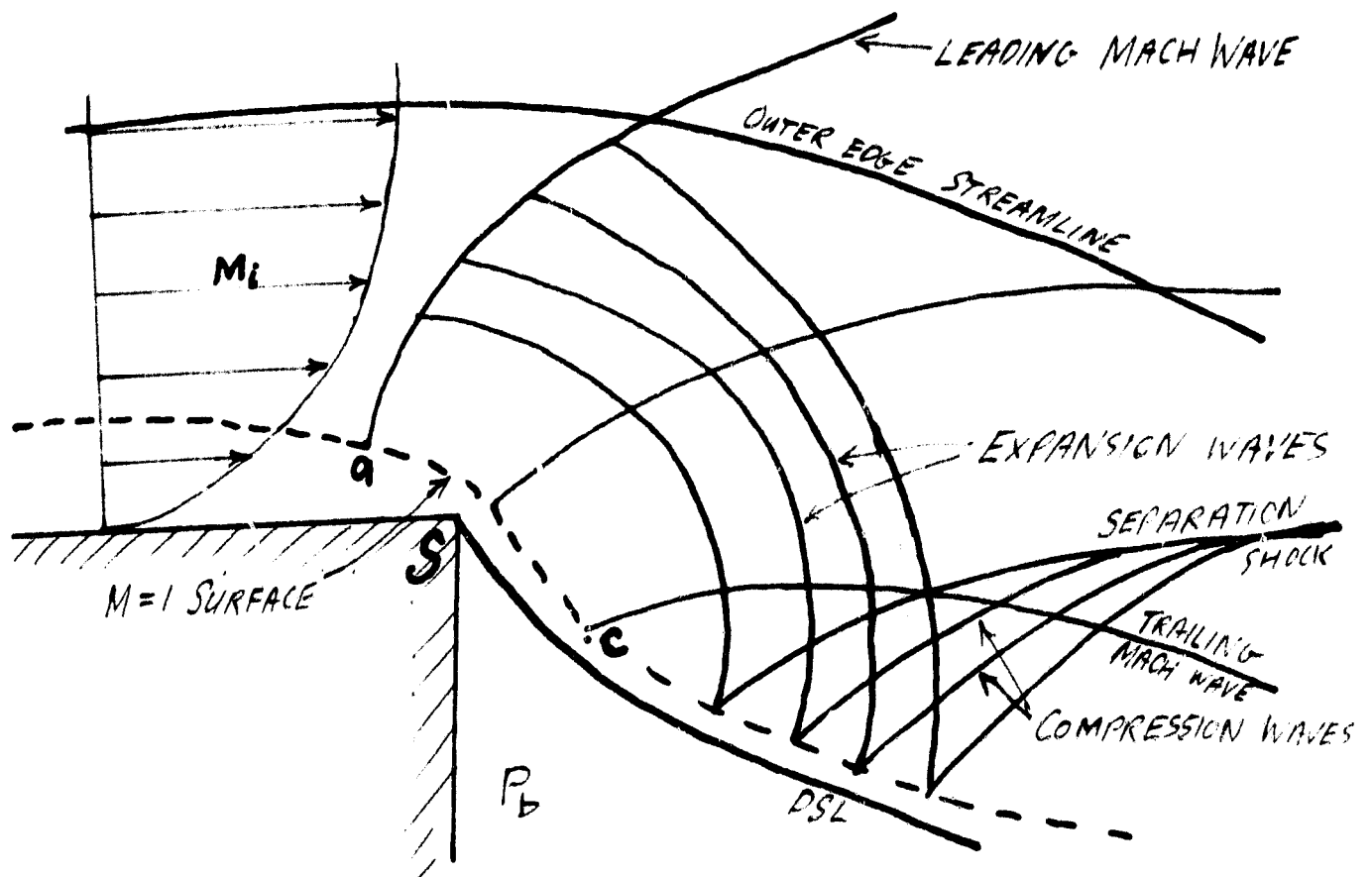


Figure 44

Actually as a Mach wave is propagated outward through the vortical boundary layer it is partly reflected and partly transmitted. Weinbaum¹⁴⁸ idealized this situation by considering the primary waves to be nonlinear and the reflected waves to be linear. He found from analysis of this idealization that the reflected wave changes sign at $M_i = 2^{\frac{1}{2}}$, being an expansion wave when $M_i > 2^{\frac{1}{2}}$ and a compression when $M < 2^{\frac{1}{2}}$. Thus in Fig. 44, the separation shock wave is shown as a reflection of the secondary waves emanating from the primary waves where $M_i > 2^{\frac{1}{2}}$.

A simpler model for practical purposes is obtained by collapsing points a and c in Fig. 44 into one point, and using the

method of characteristics for rotational flow, starting the numerical procedure with velocity and entropy profiles¹⁶³ of the upstream boundary layer. This simplified model has enjoyed considerable success,^{154,155,156,157} but it does have certain limitations: 1) the subsonic part of the boundary layer is neglected requiring the assumption that the dividing streamline separates parallel to the wall; 2) the velocity and temperature profiles of the subsonic portion are not calculated and thus initial conditions for the shear layer are not provided, and; 3) the wave interaction in the supersonic rotational flow due to the expansion upstream is neglected.

Weiss^{164,165} attempted to remove these limitations by introducing a stream tube model for the subsonic part of the boundary layer. In this model the flow in the subsonic layer is assumed to be inviscid and rotational, normal pressure gradients are neglected, and the pressure is assumed to approach the base pressure monotonically. The subsonic layer is subdivided into a number of streamtubes (say about 10), and the governing equations are written:

- 1) Continuity of flow direction at interface between sub- and supersonic flow

$$-\frac{d\Delta}{dx} = v - v'(0) \quad (6.2.1)$$

2)

$$\Delta = \sum_{j=1}^n \Delta_j \quad (6.2.2)$$

3) Area relationship for isentropic flow along a streamtube,

$$\frac{\Delta_j}{\Delta_j(0)} = \frac{M_j(0)}{M_j} \left\{ \frac{1 + \frac{\gamma-1}{2} M_j^2}{1 + \frac{\gamma-1}{2} M_j^2(0)} \right\}^{\frac{\gamma+1}{2(\gamma-1)}} \quad (6.2.3)$$

The Prandtl-Meyer relationship is introduced into Eq. (6.2.1), resulting in the following equation,

$$-\frac{d\left(\frac{P}{P_{0e}}\right)}{d\left(\frac{x}{\Delta(0)}\right)} = \frac{[\nu - \nu(0)]}{\sum_{j=1}^n \frac{d\left(\frac{\Delta_j}{\Delta_j(0)}\right)}{dM_j} \frac{dM_j}{d\left(\frac{P}{P_{0e}}\right)}} \quad (6.2.4)$$

$\frac{dM_j}{dP_j}$ and $d\left(\frac{\Delta_j}{\Delta_j(0)}\right)/dM_j$ are obtained from Eq. (6.2.3) and the isentropic flow equation for pressure can be solved with an initial Mach number profile is given, and an initial pressure perturbation

$$P_{PERT.} = \alpha P(0) \quad (6.2.5)$$

is introduced. α is taken to be about 0.99.

The location of the corner is self determining since its presence is necessary to prevent an incompatibility arising between the supersonic and subsonic portions of the boundary layer.

Weiss suggests a coupled stream-tube/rotational characteristics model should prove a sufficiently accurate representation of the corner expansion flow.

6.3 Baum's Interaction Model

The fact, that methods based on the boundary layer approximation seem to predict quite successfully the behaviour of a boundary layer approaching separation, is rather surprising, because owing to the parabolic nature of the equations no mechanism exists for the transmission of signals upstream in the boundary layer formulation. Baum¹³⁹ recognized this anomaly and suggested that the approaching boundary layer be thought of as being in a state of very sensitive neutral stability. The neutrally stable boundary layer then experiences some very weak perturbations emanating from the corner or whatever. These perturbations are propagated by some mechanism which is described by higher order terms of the Navier-Stokes equation that are omitted in the boundary layer equations, and although infinitesimal are sufficiently strong to cause instability. Two types of behaviour are now possible; either the pressure starts to increase and the shear stress at the wall decreases until the boundary layer separates (like separation induced by a forward facing step), or the pressure decreases at an accelerating rate causing the streamlines to turn inward toward the wall (separation after a rearward facing step).

If an initial perturbation is assumed the subsequent behaviour of the boundary layer is, in many cases, quite well described by the boundary layer equations, however, what is missing in this formulation is a means of coupling the perturbation which determines the downstream behaviour with the downstream obstacle in the flow (e.g. a step). Although there is no way to obtain this formally without going to higher order terms, Baum¹³⁹ got around the problem by systematically altering the initial perturbation until the resulting solution coincides with the physical location of the obstacle thus giving the problem an eigenvalue character.

Briefly, Baum uses as his governing equations, the boundary layer momentum, energy and continuity equations together with the inviscid transverse momentum equation to allow for normal pressure gradients (this is replaced by an assumption of zero normal pressure gradient for the subsonic part of the boundary layer) and the Prandtl-Meyer relation. In each of these four equations terms are included to allow for the effect of local wall curvature. The equations are transformed to von Mises coordinates and cast in finite difference form.

At a distance L from the leading edge a perturbation of arbitrary size is chosen and the coupled viscous layer-inviscid flow equations are solved by marching downstream. One of the two types of unstable behaviour will result, a perturbation

corresponding to the other type of unstable behaviour is then found, and an iterative procedure followed until a sufficiently good approximation to the neutral solution is found. This procedure seems to work well for rounded trailing edges even predicting the presence of a separation shock, and also with modification it predicts the behaviour of the flow approaching a sharp corner.

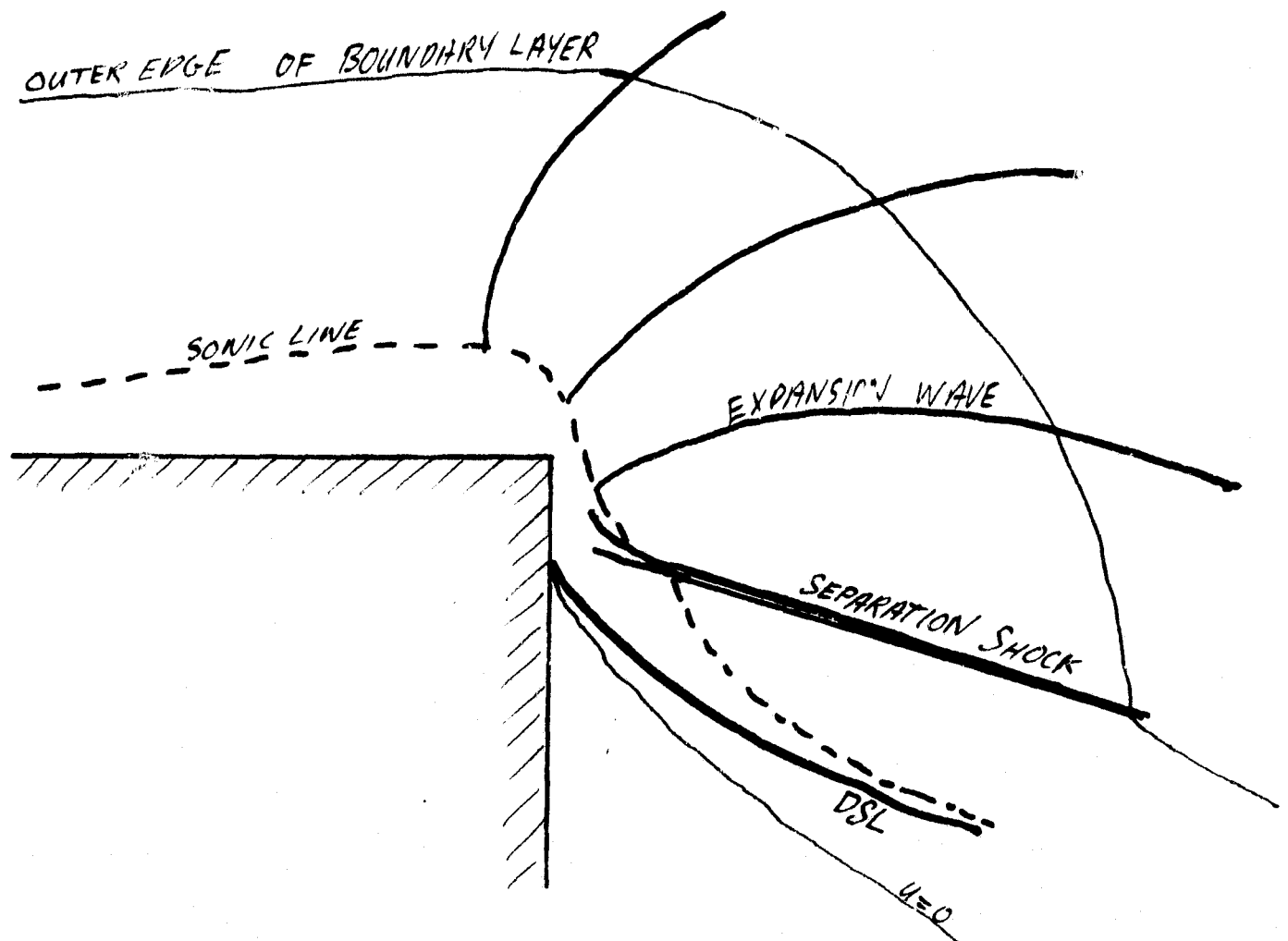
6.4 Discussion

Although the effects discussed in the last two sections are more marked at higher Mach numbers, they are by no means unimportant at moderate supersonic speeds. For instance Murthy and Hammitt¹⁶⁶, studying an $M_\infty = 1.88$ turbulent boundary layer flow over rearward facing wedges, found discrepancies of as much as 20% from the predictions of the Prandtl-Meyer theory for the outer flow. Also the extensive studies of Hama showed among other things that the separation or lip shock had substantial strength at all the Mach numbers tested (2.0 to 4.5), and that the expansion process may well be directly or indirectly responsible for many inexplicable features of the flow e.g. Charwat-Yakura⁸⁰ slip lines, and pressure "overshoot" in the recompression process.

The use of the method of characteristics to predict the flow behaviour in the expansion process as suggested by Weinbaum and

Weiss produces very good results. Some investigators^{97,155} have shown the characteristic solution superimposed on a Schlieren photograph, the position of the separation shock is invariably accurately predicted. The hot wire study of Lewis and Behrens⁷⁹ also provides convincing evidence in support of the Weiss-Weinbaum base flow model.

In recent experimental studies a curious feature of the expansion process has been discovered. It appears that the boundary layer does not separate from the corner but from a point a small distance below the corner. This is shown schematically below,



This means that the flow behaviour near separation on a sharp corner is not essentially different from that near a rounded corner.

Donaldson¹⁶⁷ has pointed out that the flow pattern, near the separation point on the base wall indicated by his measurements, is very similar to one of Weinbaum's¹⁶⁸ Stokes flow solutions for flow around a sharp corner. This suggests that there is a small inner layer where viscous forces predominate to such an extent that separation is delayed until after the corner.

Although the careful experimental studies of Hama⁸⁷ and Donaldson¹⁶⁷ appear to put beyond doubt the fact that separation occurs below the corner rather than at it, two puzzling points remain to be cleared up. First, how is it that the simplified procedure based on the method of characteristics predicts the position of the lip shock so accurately? Secondly, why is there practically no difference between the flow patterns for separation at a sharp corner and that at a overhanging corner? (see sub-section 3.14.5(d)).

7. Conclusion

For practical purposes the theoretical methods reviewed in Chaps. 4 and 5 are too complicated and not sufficiently accurate to be worth using. However, they provide a valuable means for extending knowledge of the base flow and for checking the accuracy of the simpler models.

It is recommended that Chapman's or Baum's solution for the development of a free shear layer, together with the Chapman-Korst reattachment criterion, be used for laminar base pressure calculations. In the case of turbulent base flows, it is suggested that Korst's method be employed, with equivalent bleed concept to account for initial boundary layer, and with the correction (Eq. 3.5.8)* proposed by Carriere and Sirieix to allow for the inaccuracy of the Chapman-Korst reattachment criterion. The Korst method is suggested, not because it is the most accurate, but because it has been generalized to include the non-adiabatic²³ case, the axisymmetric case^{59,169}, the case of interferring unequal shear layers²⁴, the case of bleed flux, and the case of parallel jets⁷², etc. Also these extensions are accompanied by auxiliary tables of the various integral quantities involved making calculations comparatively simple and rapid.

* This is equivalent to an additional bleed flux given by

$$\frac{\rho_0 U_{bl}^2}{\rho_2 U_2^2} = -0.005.$$

At the present time, the method of integral relations appears to offer the most promising road to progress, because problems with uniqueness and other snags that arise with the Lees-Reeves theoretical methods can be eliminated. In addition, it is possible to make the solution as accurate as the boundary layer equations themselves, unlike one-parameter methods, however sophisticated. There is no reason why the method of integral relations cannot be extended to turbulent flows, in fact, this has recently been accomplished¹⁶⁹. There is also one other advantage and that is that this is a general method for solving differential equations and can, in principle, be used to solve both the outer inviscid flow or the full Navier-Stokes equations.

The Weiss theoretical approach has proved to be an important advance in base pressure theories, and provides a means, although a very cumbersome one, of investigating base flows. However, it seems unlikely at the present time that it could be extended to cover higher Reynolds numbers and the turbulent case.

Since this report has primarily emphasized the theoretical side of the base flow problem, perhaps it would be appropriate to end by listing a few of the outstanding experimental studies.

Hastings¹⁰⁷

Two-dimensional turbulent flow over rearward facing steps. $M = 2.5$ to 3.5 . Pressure distributions for various values of $\delta_{1,0}^{**}/h$.

Roshko and Thomke⁸⁸

Axisymmetric, turbulent rearward facing step flow. M from 2 to 4 . Pressure distribution for various values of $\delta_{1,0}^{**}/h$.

de Krasinski^{16,77,78}

Axisymmetric, turbulent and laminar base flow. $M = 1.48$ and 2.55 . Enormous amount of detailed data.

Smith^{82,106}

Two-dimensional, laminar and transitional, rearward facing step flow. $M = 2.5$, 3.5 and 5.0 . Heat transfer and pressure distribution.

Carriere, Sirieix
et. al^{92,96,97}

Detailed studies of the reattachment process, and also in ref. 97, study of near wake flow field.

Hama⁸⁷

Detailed study of the expansion process and the separation shock.

Lewis and Behrens⁷⁹

Detailed hot wire study of wedge near wake at $M = 4$.

Batt & Kubota¹⁷⁰

Detailed investigation of laminar wedge near wakes at $M = 6$.

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