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HIGH-POWER KINETICS OF A TWO-DIMENSIONAL CIRCULATING-LIQUID-METAL-FUEL REACTOR

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ABSTRACT

A two-dimensional model of a single channel unreflected circulating fuel reactor is used to investigate the effect of a radial temperature distribution on the neutron kinetics. A slug flow model with no delayed neutrons is used. The effects of a nonuniform radial heat source distribution and radial heat transfer are taken into account.

HIGH-POWER KINETICS OF A TWO-DIMENSIONAL CIRCULATING-LIQUID-METAL-FUEL REACTOR *

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SUMMARY

A two-dimensional model of a single channel unreflected circulating fuel reactor is used to investigate the effect of a radial temperature distribution on the neutron kinetics. A slug flow model with constant fluid properties describes the heat transfer and fluid flow. A one-neutron velocity model with no delayed neutrons is used. The effects of a nonuniform radial heat source distribution and radial heat transfer are taken into account. The results show that, in many practical cases, radial heat transfer has little effect on the kinetics. Furthermore, if changes in multiplication are not too large, the flux and radial temperature distribution maintain their steady-state shapes throughout the transient. For certain combinations of the physical parameters, resonances can occur in the time dependent flux and temperature. These resonances can be predicted by either a line reactor model or a two-dimensional reactor model; however, a two-dimensional model is necessary to predict the value of the steady-state flux and temperature following a transient.

INTRODUCTION

Since the early days of reactor technology, liquid-metal-fuel reactors have been the source of much speculation and research. Although this type of reactor was first suggested in 1941 (unpublished data by H. Halban and L. Kowarski mentioned in ref. 1), it received little attention until about 1947. At that time a program was started at Brookhaven National Laboratory to develop a Liquid-Metal-Fuel Reactor (LMFR). Basic research on the LMFR was continued until about 1957. During this time, a number of design studies were conducted (refs. 2 to 4) which indicated that this reactor concept was attractive economically. A number of LMFR experiments were planned in order to prove that the concept was technically feasible. The investigation was discontinued in 1957.

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Since the LMFR is economically attractive, it is likely that it will be investigated again for use as a ground based power reactor. Furthermore, such a reactor could prove useful for space power generation since it could be operated as a relatively compact fast reactor. While there are no current programs to construct such reactors, there are a number of interesting research problems which can be done independent of such programs.

Before discussing the particular problem of interest, it will be useful to describe two possible LMFR concepts. The first concept is that of a single fluid thermal reactor. The fissionable material is dissolved in a single fluid carrier (uranium in bismuth is one possibility) which circulates inside a closed loop. In the reactor portion of the loop, there is a block of graphite (or other) moderator material with cylindrical fuel passages in it. A neutron reflector surrounds this portion of the loop. The fuel solution flows through the reactor where it is heated and then through an external heat exchanger. The second LMFR concept is similar to the first except that no moderator or reflector is present. Such a reactor would operate in the intermediate or fast regions of the neutron spectrum. The reactor portion of the loop would be slightly larger in radius than the rest of the loop to reduce the neutron losses due to leakage. Nearly all other LMFR designs are variations of these two concepts.

All previous research on the LMFR has dealt with the thermal reactor concept. In these studies, many authors have used an idealized model to study the reactor physics. This model consists of a single channel loop in which the cross-sectional area is constant throughout and the unreflected reactor is specified to be in some fixed portion of the loop. This model permits the study of certain aspects of the reactor physics without tying the problem to a specific system. Since it is the intent of this report to discuss a reactor physics problem which is common to both fast and thermal reactors, the single channel model will be employed.

A reactor using a circulating fluid fuel has a number of advantages over one with stationary fuel elements. First, the structure of the reactor is relatively simple, that is, the fuel can be cooled in an external heat exchanger separate from the reactor core. Hence, the nuclear requirements in the core and the heat transfer requirements in the heat exchanger need not both be satisfied at the same location. This permits high specific power design since, for example, materials such as tungsten (which has a high neutron capture cross section) can be used in the heat exchanger without affecting the nuclear requirements of the reactor. Fuel handling, fuel reprocessing, and waste disposal are easier in circulating fuel reactors than in stationary fuel reactors. Another advantage of circulating fuel reactors is that fission products can be removed continuously. The removal of poisons (neutron absorbers) improves the neutron economy and permits higher fuel burnup. Consequently, less radioactive material is present in the reactor so that the potential hazard is decreased. One of the most advantageous features of a fluid fuel reactor is its inherent safety and ease of control. A liquid fuel which expands on heating results in a negative temperature coefficient of reactivity. Since the rate of expansion is limited only by the speed of sound in the liquid, this effect is essentially instantaneous and tends to make the reactor self-regulating. Furthermore, a circulating fluid fuel reactor can be controlled by adjusting the fuel concentration at the reactor inlet.

Circulating fluid fuel reactors also have several disadvantages relative to stationary solid fuel reactors. Changes in the fuel density or concentration can bring about fluctuations in the reactivity; this may occur, for example, if the local temperature changes in the fuel. Another disadvantage is that some delayed neutrons are lost from the core. This shortens the time between neutron generations and reduces the effectiveness of external control systems. Furthermore, there is considerable induced activity in pumps and heat exchangers. The induced activity produces radiation levels which require development of remote maintenance techniques. One of the biggest problems involved in all types of fluid fuel reactors is that of corrosion and erosion of structural materials by the fuel solution.

Some of these disadvantages will be eliminated as experience is gained in handling liquid metals and in designing remote maintenance equipment. On the other hand, the problems of reactivity fluctuations and loss of delayed neutrons are inherent in circulating fuel reactors; thus it must be assumed that these reactors can be designed to operate efficiently and safely in spite of these disadvantages. A large portion of the literature on circulating fuel reactors and the LMFR deals with these two problems. One may classify these as reactor kinetics problems since they involve the time-dependent behavior of the flux after some change occurs in the system. The reactor kinetics will now be considered in more detail.

When the number of neutrons produced in a reactor changes from generation to generation, the neutron flux will change and a transient is said to occur. This situation is described by saying that a change in the multiplication or in the reactivity has occurred. After a change in multiplication in any reactor which has a negative temperature coefficient of reactivity, the neutron flux must attain a new steady state after some average temperature has been reached. It is often found, however, that the temperature rise and neutron flux can be out of phase so that flux and temperature oscillations can occur. Local temperature variations can have a large effect on the multiplication and, therefore, the spatial temperature distribution in the reactor must be known.

In a circulating fuel reactor, the time history of the neutron flux after a change in multiplication depends not only on the temperature, but also on the fuel velocity distribution; this is true for two reasons. First, the velocity and temperature distributions are directly related through the energy balance in the reactor. Second, the fuel velocity affects the neutron kinetics because of the loss of delayed neutron emitters. If, for example, the reactor is operating at steady state with a certain fraction of the delayed neutrons being lost to the system, then a change in the fuel velocity will change this fraction and a steady state will no longer exist.

The fuel velocity in a circulating fuel reactor depends, in general, on the desired outlet temperature. For example, with a specified outlet temperature and a fixed inlet temperature, low velocities are necessary if the reactor power is low and high velocities if the power is high. While there is a complete spectrum of velocities which could be encountered in a circulating fuel reactor, there are many practical situations in which one of these two extremes (high or low power) exist. In each of these extremes, there are a number of simplifying circumstances which permit a rather clear exposition of the reactor kinetics.

In a low power case, temperature changes in the system may be small. In this case, temperature effects on reactivity can be ignored and the kinetic effect of delayed neutrons can be studied alone. Fleck (refs. 5 and 6) has described a single channel circulating fuel reactor operating at low power by means of several linear partial differential equations. Expressions for the time dependent neutron flux and delayed neutron precursor distribution are developed and the in-hour equation for the circulating fuel reactor is obtained. The fuel is assumed to move in slug flow, and only axial variations in the flux and precursor distribution are considered. Expressions for the in-hour equation for a circulating fuel reactor have also been developed by Ergen (refs. 7 and 8) and Wolfe (ref. 9). The low power kinetics of circulating fuel reactors have been further discussed by MacPhee (ref. 10) and LoSurdo (ref. 11). The results of these studies show that the change in multiplication due to a change in fuel velocity is proportional to the square of the multiplication necessary to maintain the system at steady state, the delayed neutron fraction, and the fuel transit time across the reactor; it is inversely proportional to the velocity and the fuel transit time around the entire loop.

When a circulating fuel reactor operates at high power, the fuel velocities encountered are usually high enough so that a large fraction of the delayed neutrons is lost from the system. In this case it is often assumed that the reactor must operate on prompt neutrons alone. A reactor which is prompt critical has a period which is in the order of microseconds to milliseconds. A period this short may preclude the use of external control systems; thus, the temperature coefficient of reactivity is the chief stabilizing factor in the system. A number of authors have studied the high-power kinetics of a circulating fuel reactor; in most cases a single channel model was used and all delayed neutrons were assumed lost to the reactor. Some of these analyses predict the time behavior of the flux and temperature as a function of axial position. A slug fluid flow model is assumed, along with constant fluid properties and constant flow rate. Ergen (refs. 12 and 13), Weinberg (ref. 14), Welton (refs. 15 and 16), Thompson (ref. 17), and Nohel (ref. 18) have formulated the problem in terms of equations involving integrals over the past history of the neutron flux. This leads to a set of equations in terms of the time variable alone. Fleck (ref. 19) has started with the partial differential equations for the neutron and energy balance. He has shown that, by expanding the flux in a time-dependent Fourier series, the resulting equations for the time-dependent coefficients are simply the relations used by the authors mentioned previously. The results of these analyses show that the negative temperature coefficient of reactivity is usually sufficient to stabilize the reactor after a step change in multiplication. However, if the reactor power, temperature coefficient, and the fuel transit time across the reactor are related in a certain manner, large oscillations of the flux and temperature can occur.

The previous discussion indicates that a number of simplifying assumptions have been used in the study of the kinetics of circulating fuel reactors. Unfortunately, the conditions for these assumptions to be valid are neither quantitiatively defined nor universally met. In past studies of the high power kinetics of the LMFR, the radial flux and temperature distributions have been assigned through a heuristic argument. Consider, for example, reference 19 where the radial flux is assumed to be exactly the same as the steady-state flux in a bare cylindrical reactor but the radial temperature distribution is assumed to be a constant. This will be referred to as a line reactor model. These assumptions are motivated by the physical situation which is often encountered in a circulating fuel reactor. From a mathematical point of view, these assumptions are inconsistent since a radial flux distribution implies a radial heat source distribution; hence, the radial temperature distribution should not be a constant. Therefore, in the interest of obtaining a mathematically consistent solution and as a direct extension of the work of reference 19, a study of the high-power kinetics of a cylindrical circulating-fuel reactor has been made in which the radial terms in both the flux and temperature equations are retained; that is, a two-dimensional model has been used.

SYMBOL LIST

a	length of reactor, cm
$\bar{a}$	$p_1 a/h$, dimensionless
$\vec{a}_z$	unit vector in z-direction
B^2	geometric buckling, B_{11}^2, cm^{-2}
B_c^2	physical buckling, $(\nu\Sigma_f p_{th} g_{th} - \Sigma_a)/D$, cm^{-2}
B_{nk}^2	geometric buckling of order n, k, cm^{-2}
C_1	constant

c_l	$\int_0^a Z_l(z)Z_l(z)\,dz$, cm
c_p	heat capacity of fuel solution, $J/(g)(^oC)$
D	neutron diffusion coefficient, cm
E	energy release per fission, J
F	arbitrary function of r and z
F_{ij}	$P_L(B^2 - B_{ij})/B^2 + \delta k_0$, dimensionless
f	arbitrary function of time
f_1, f_2	constants
G	$\gamma M_{111}(\rho c_p)^{-1}K_{111}(v_0x)e^{-\alpha\xi_1^2 x}\,u(\tau_a - x)$, cm^3/J
$\dot{G}$	derivative of G with respect to t, $cm^3/(J)(sec)$
G^i_{nlm}	$\int_0^{\tau_a} K_{nlm}(v_0x)e^{-\alpha\xi_i^2 x}\,dx$, sec
g_{th}	fast nonleakage probability
h^2	P_L/B^2, cm^2
I	$2/\pi \int_0^\infty S(x)\sin(\Omega x)\,dx$, dimensionless
J_0	zero-order Bessel function
J_1	first-order Bessel function
K_{nlm}	$1/c_l\left[\int_{v_0x}^a Z_n(z)Z_l(z)Z_m(z - v_0x)\right]dz$, dimensionless
K_1	$\gamma Q_0 M_{111}$, cm^2
K^2	$T_{1ex}/(3 - T_{1ex})$, dimensionless
k	finite medium multiplication
k_∞	infinite medium multiplication
L	diffusion length, cm
l_{th}	finite medium lifetime for neutrons with speed v_k, sec
l_0	infinite medium lifetime for neutrons with speed v_k, sec

M_{ijk} $2/\left\{R^2\left[J_1(\nu_k)\right]^2\right\} \int_0^R r\mathcal{R}_i(r)\mathcal{R}_j(r)\mathcal{R}_k(r)\,dr$, dimensionless

N integer ≥ 3

P reactor power density, $E\Sigma_f\Phi$, J/(cm^3)(sec)

$\dot{P}$ derivative of P with respect to t, J/(cm^3)(sec^2)

P_L thermal leakage probability, $1 - P_{NL}$

P_{NL} thermal nonleakage probability, $(1 + L^2B^2)^{-1}$

P_0 initial reactor power density, $E\Sigma_f\Phi_0$, J/(cm^3)(sec)

p Laplace variable, sec^{-1}

p_{th} resonance escape probability

p_1^2 $\delta k_0 - h^2\epsilon_1^2$

Q function of time, $\ln(P/P_0)$, dimensionless

Q_0 $E\Sigma_f(\rho c_p)^{-1}$, (cm^2)($^{\circ}$C)

$\ddot{Q}$ second derivative of Q with respect to t, sec^{-2}

R reactor radius, cm

$\mathcal{R}_j(r)$ $J_0(\xi_j r)$, dimensionless

r radial coordinate, cm

S $-\dot{G}(t)/G(0)$, sec^{-1}

T_0 inlet temperature, $^{\circ}$C

T_1 $\gamma_1 \upsilon/p_1^2$, dimensionless

T_{1ex} T_1 when $\upsilon = \upsilon_{ex}$, dimensionless

t time, sec

t_1, t_2 arbitrary time, sec

$u(t - x)$ unit step function, $\begin{cases} 0 & \text{where } t < x \\ 1 & \text{where } t > x \end{cases}$

v_k neutron speed, cm/sec

v_0 velocity of fuel solution, cm/sec

x dummy integration variable

y $d\upsilon/dz$, $^{\circ}$C/cm

$Z_n(z)$	$\sin(\epsilon_n z)$
z	axial coordinate, cm
$\bar{z}$	$p_1 z/h$, dimensionless
α	thermal diffusivity, $\kappa(\rho c_p)^{-1}$, cm^2/sec
β	Φ_1^2/Φ_1^1, dimensionless
γ	negative of temperature coefficient of multiplication, $^{\circ}C^{-1}$
γ_1	γM_{111}, $^{\circ}C^{-1}$
$\delta(t-x)$	Dirac delta function defined by $f(t) = \int_{t_1}^{t_2} f(x)\,\delta(t-x)\,dx$ where $t_1 \leq t \leq t_2$
δ_{ij}	Kronecker delta, $\begin{cases} 0 & \text{where } i \neq j \\ 1 & \text{where } i = j \end{cases}$
δk	change in multiplication, $k - 1$
δk_0	initial change in multiplication
ϵ_n	$n\pi/a$ (where $n = 1, 2, \ldots$), cm^{-1}
ζ_1	$J_1(\nu_1)/J_1(\nu_2)$, dimensionless
θ	temperature rise above inlet temperature, $^{\circ}C$
θ_{line}	equivalent to θ_0 but developed from line reactor model of circulating fuel reactor, $^{\circ}C$
θ_{max}	maximum temperature reached during excursion, $^{\circ}C$
θ_R	wall temperature rise, $^{\circ}C$
θ_0	initial and/or steady-state temperature rise, $^{\circ}C$
θ_{2-d}	equivalent to θ_0, $^{\circ}C$
κ	sum of molecular and turbulent conductivities of fuel solution, $J/(cm)(sec)(^{\circ}C)$
ν	number of neutrons produced per fission
ν_i	solution of $J_0(\nu_i) = 0$
ξ_i	radial buckling, ν_i/R, cm^{-1}
ρ	density of fuel solution, g/cm^3
Σ_a	macroscopic absorption cross section, cm^{-1}
Σ_f	macroscopic fission cross section, cm^{-1}
τ_a	transit time for fuel across reactor, sec

υ function of axial coordinate, ^{o}C

υ_{ex} υ evaluated at $z = a$, ^{o}C

$\Phi(t)$ flux amplitude equivalent to Φ_1^1, $[(cm^2)(sec)]^{-1}$

$\dot{\Phi}(t)$ derivative of $\Phi(t)$ with respect to t, $[(cm)(sec)]^{-2}$

$\Phi(0)$ initial flux amplitude, $[(cm^2)(sec)]^{-1}$

Φ_{max} maximum flux amplitude during excursion, $[(cm^2)(sec)]^{-1}$

Φ_0 steady-state flux amplitude, $[(cm^2)(sec)]^{-1}$

Φ_l^k l,k coefficient of flux in Fourier-Bessel expansion, $[(cm^2)(sec)]^{-1}$

$\dot{\Phi}_l^k$ derivative of Φ_l^k with respect to t, $[(cm)(sec)]^{-2}$

$\Phi_{1,\,1\text{-term}}^1$ solution for Φ_1^1 when one-term approximation is used, $[(cm^2)(sec)]^{-1}$

$\Phi_{1,\,2\text{-term}}^1$ solution for Φ_1^1 when one-axial-term, two-radial-term approximation is used, $[(cm^2)(sec)]^{-1}$

φ neutron flux, $[(cm^2)(sec)]^{-1}$

φ_0 initial neutron flux, $[(cm^2)(sec)]^{-1}$

$\varphi_s(z)$ steady state axial flux, $[(cm^2)(sec)]^{-1}$

Ψ $\tau_a \alpha\left(\xi_1^2 - \xi_2^2\right)/2\pi$, dimensionless

Ω^2 $G(0)P_0/l_{th}$, sec^{-2}

PHYSICAL MODEL

The reactor consists of a right circular cylinder with radius R and length a with fuel moving from left to right as shown in figure 1. The radial coordinate is r; the axial coordinate is z, and symmetry about z is assumed.

The fuel enters the reactor in a uniform thermodynamic state, that is, the external heat exchanger is assumed capable of maintaining a constant temperature at the reactor inlet. The temperature of the walls is assumed, in general, to be a specified function of the axial coordinate and time. If the flow is turbulent, the turbulent conductivity is taken as constant; in this case, time averaged quantities are used in the energy equation (ref. 20). Radial molecular conduction and turbulent heat transfer are therefore represented by the same term in the energy equation. An internal heat source is present in the fuel because of fission heating, the heat source being taken as proportional to the neutron flux.

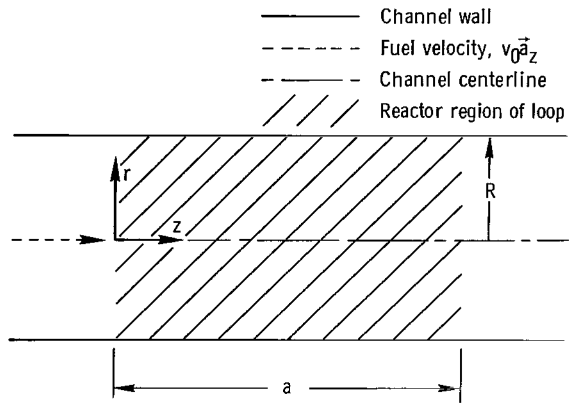

Figure 1. - Geometrical model of circulating fuel reactor; fuel temperature is T_0 at $z = 0$.

The reactor is an entrance region as far as the fluid flow is concerned. Because of the low Prandtl number of liquid metals, the fuel velocity can be considered constant over a cross section of the reactor. Thus, the assumption of slug flow seems reasonable. In the present discussion, the effects of free convection cells, viscous dissipation and axial conduction as compared to axial convection will be ignored. It is assumed that turbulence does not affect the neutron kinetics except through the energy equation. Also, all physical parameters involved in either the heat transfer or fluid flow are assumed to remain constant throughout the problem.

A one-neutron velocity model is employed. It is assumed that all delayed neutron emitters are swept out of the reactor before decaying so that the reactor operates on prompt neutrons alone. The neutron flux is taken to be zero at the entrance, the exit, and the walls of the reactor. While changes in density are assumed negligible when considering the heat transfer and fluid flow problem, it is assumed that changes in density can cause significant changes in the multiplication. These changes occur simultaneously with changes in temperature.

The reactor is initially operating at some power when a change in the multiplication is made; the change is uniform and instantaneous at all points in the reactor. The behavior of the neutron flux and fuel temperature after this change has been made will be studied.

ANALYSIS

Basis Equations

The time dependent diffusion equation for one-velocity neutrons can be written (ref. 21)

$$-D\nabla^2\varphi(\vec{r},t) = \nu\Sigma_f p_{th} g_{th}\varphi(\vec{r},t) - \frac{1}{v_k}\frac{\partial\varphi(\vec{r},t)}{\partial t} - \Sigma_a\varphi(\vec{r},t) \tag{1}$$

where $\varphi(\vec{r},t)$ is the neutron flux and all other symbols are defined in the list of symbols. Since symmetry about z has been assumed, the flux is a function of only r, z, and t. The flux is zero at the reactor extremities, symmetric about the axis, and is a specified function of position at $t = 0$. Thus

$$\varphi(r,0,t) = 0 \tag{2a}$$

$$\varphi(r,a,t) = 0 \tag{2b}$$

$$\varphi(R,z,t) = 0 \tag{2c}$$

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=0} = 0 \tag{2d}$$

$$\varphi(r,z,0) = \varphi_0(r,z) \tag{2e}$$

Dividing equation (1) by Σ_a and defining $l_0 \equiv (v_k\Sigma_a)^{-1}$, $L^2 \equiv D\Sigma_a^{-1}$, and $k_\infty = \nu\Sigma_f p_{th}\Sigma_a^{-1}$ result in

$$l_0\frac{\partial\varphi}{\partial t} = (k_\infty g_{th} - 1)\varphi + L^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2}\right] \tag{3}$$

If $P_{NL} \equiv (1 + L^2B^2)^{-1}$, $P_L \equiv 1 - P_{NL}$, $k \equiv k_\infty g_{th}P_{NL}$, and $l_{th} \equiv l_0P_{NL}$, equation (3) becomes

$$l_{th}\frac{\partial\varphi}{\partial t} = (k - P_{NL})\varphi + P_LB^{-2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2}\right] \tag{4}$$

where $B^2 = (\nu_1/R)^2 + (\pi/a)^2$. Let k, the multiplication, be written as $k = 1 + \delta k$ where δk represents the deviation from equilibrium. Then the time-dependent diffusion equation finally becomes

$$l_{th} \frac{\partial \varphi}{\partial t} = \delta k \varphi + P_L B^{-2} \left[B^2 \varphi + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{\partial^2 \varphi}{\partial z^2} \right] \tag{5}$$

All of the parameters which multiply the flux and its derivatives in equation (5) vary with temperature. However, it is assumed that only variations in δk have a significant effect on the reactor kinetics. To understand the conditions under which this assumption is valid, it is necessary to consider one solution of equation (5). Assume all parameters in the equation are constant; then the solution is $\varphi(r, z, t) = F(r, z)e^{\delta kt/l_{th}}$ where $F(r, z)$ is some function of the spatial coordinates. $F(r, z)$ arises directly from the terms in brackets on the right side of equation (5). Note that only the ratio $\delta k/l_{th}$ is involved in the transient part of the solution for the flux. Next the parameters in equation (5) are permitted to vary slightly with temperature. The function $F(r, z)$ is assumed to change very little under these circumstances relative to the change in the exponential. Hence, it seems reasonable to assume that temperature changes in those parameters involved in the transient solution for the flux will be more important than temperature changes in parameters involved in the spatial part of the solution. If this is true, only small quantities of second order are neglected by assuming that the temperature enters equation (5) only through δk. While the functional dependence of δk on temperature is ordinarily quite complicated, it is usually possible to use a linear variation of δk with temperature as an adequate representation. Hence, δk is assumed to have the form (see ref. 21, pp. 308 to 327)

$$\delta k = \delta k_0 - \gamma \theta(r, z, t) \tag{6}$$

where δk_0 is an initial change in multiplication made instantaneously and homogeneously throughout the reactor and γ, the negative of the temperature coefficient, is assumed constant throughout the transient (γ is defined by the relation $\gamma \equiv - \partial k/\partial \theta$). Note that $\theta(r, z, t)$ is the temperature rise above inlet temperature at the spatial location and time at which the change in multiplication occurs.

Using equation (6) in equation (5) results in one equation which relates the flux and the local temperature rise

$$l_{th} \frac{\partial \varphi}{\partial t} = (\delta k_0 + P_L)\varphi - \gamma \theta \varphi + P_L B^{-2} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{\partial^2 \varphi}{\partial z^2} \right] \tag{7}$$

A second expression relating φ and θ can be obtained from the Navier-Stokes equations. For an incompressible fluid in slug flow, the continuity and momentum equations become ρ = constant and $\vec{v} = v_0 \vec{a}_z$, respectively. The energy equation becomes (ref. 20)

$$\rho c_p \left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}\right)\theta(r, z, t) = E\Sigma_f \varphi(r, z, t) + \kappa \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right) \tag{8}$$

where axial conduction has been ignored. Equation (8) is simply an energy balance; $E\Sigma_f \varphi$ represents the fission heating, and κ is the sum of the molecular and turbulent (if any) conductivities. In the case of turbulent flow, the temperature and heat transfer parameters are assumed to be time averaged quantities.

The temperature is measured with respect to the inlet temperature which is constant. The temperature rise is symmetric about the z-axis, a specified function of axial position and time at the wall and is a specified function of position at $t = 0$. Therefore,

$$\theta(r, 0, t) = 0 \tag{9a}$$

$$\left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0 \tag{9b}$$

$$\theta(R, z, t) = \theta_R(z, t) \tag{9c}$$

$$\theta(r, z, 0) = \theta_0(r, z) \tag{9d}$$

Define $Q_0 \equiv E\Sigma_f(\rho c_p)^{-1}$ and $\alpha \equiv \kappa(\rho c_p)^{-1}$; then equation (8) may be written

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}\right)\theta = Q_0 \varphi + \alpha \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right) \tag{10}$$

Equations (7) and (10) are a coupled set of partial differential equations; together they describe the kinetics of a LMFR which behaves according to the assumed physical model.

Fourier-Bessel Form of the Basic Equations

Equation (7) is a second-order nonlinear partial differential equation, the nonlinearity being a result of the temperature coupling. The solution to such an equation depends on the magnitude of the nonlinear term when compared to the other terms in the equation. If equation (7) is used to describe a high power LMFR in which a reasonably large initial change in reactivity has occurred, then the equation is highly nonlinear. On the other hand, for a low power LMFR or for small perturbations about equilibrium in the high power case, the temperature rise may be small and the nonlinearity is correspondingly small.

By a qualitative discussion of equation (7), a great deal of insight into the kinetic behavior of a reactor can be obtained. For example, if the temperature rise is taken to be zero, the equation is linear and the neutron flux is given by a product of two space functions and an exponential in time. The period of such a system is proportional to $1/\delta k$ (ref. 21). If the temperature is permitted to rise in time, however, it can be seen that eventually the initial change in multiplication will be exactly cancelled by the temperature term. At this point, the reactor period is infinite or, in other words, the flux no longer rises. This would occur if the temperature were coupled to the flux only through equation (7).

Actually, the flux and temperature are also coupled through equation (10). If equation (10) were solved for the flux and the result inserted in equation (7), a single equation for the temperature would be obtained. One of the terms in this equation would involve the second derivative of the temperature with respect to time. Hence, one might suspect that oscillatory solutions for the temperature are possible. If this were true, then flux oscillations would also occur. Any study of the reactor kinetics must therefore determine whether such oscillations are possible and, if they are possible, whether the oscillations are bounded and are damped. It has been previously shown (ref. 19) that, for a line reactor model, such oscillations can exist and are bounded and damped. It is the purpose of the present work to investigate the effect of radial temperature gradients and heat transfer on the behavior of these oscillations. To do this, the coupled equations (7) and (10) must be solved.

Two possible methods present themselves for solving these equations. A direct numerical solution by finite difference techniques is one possibility; this method would give little insight into the physics of the problem since each reactor system would be a special case. The second method and the one which will be used here is to solve equation (10) for $\theta(r, z, t)$ by analytical methods and use the resulting expression in equation (7). The flux is then expanded in a time-dependent Fourier-Bessel series and the important parameters are determined.

An expression for $\theta(r, z, t)$ in terms of $\varphi(r, z, t)$ can be obtained in the following manner. It is shown in appendix A that the application of a finite Hankel transform (ref. 22) with respect to r, followed by a Laplace transform (ref. 22) with respect to t to equation (10) results in an ordinary first-order differential equation for the transformed temperature. After solving the differential equation, the temperature is retransformed with the result

$$\theta(r, z, t) = \frac{2}{R^2} \sum_i \bar{\theta}(\xi_i, z, t) \frac{J_0(r \xi_i)}{\left[J_i(\nu_i)\right]^2} \tag{11}$$

where

$$\bar{\theta}(\xi_i, z, t) = Q_0 \int_0^{z/v_0} \bar{\varphi}(\xi_i, z - v_0 x, t - x) e^{-\alpha \xi_i^2 x} u(t-x)\, dx$$

$$+ \int_0^{z/v_0} \bar{\theta}_0(\xi_i, z - v_0 x) e^{-\alpha \xi_i^2 x} \delta(t-x) u(t-x)\, dx$$

$$+ \alpha \xi_i R J_i(\nu_i) \int_0^{z/v_0} \theta_R(z - v_0 x, t - x) e^{-\alpha \xi_i^2 x} u(t-x)\, dx \qquad (12)$$

Here, $\bar{\varphi}(\xi_i, z, t)$ is the finite Hankel transform of the flux φ, $\bar{\theta}_0(\xi_i, z)$ is the finite Hankel transform of the initial temperature distribution, and $\theta_R(z, t)$ is the specified wall temperature. Equation (11) can now be inserted into equation (7) to yield a single equation involving the flux.

Assume that the flux can be expanded in a time dependent Fourier-Bessel series of the form

$$\varphi(r, z, t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \Phi_n^j(t) Z_n(z) \mathcal{R}_j(r) \qquad (13)$$

where $\Phi_n^j(t)$ is a time-dependent flux amplitude; $Z_n(z) = \sin(\epsilon_n z)$, $\epsilon_n = n\pi/a$; and $\mathcal{R}_j(r) = J_0(\xi_j r)$, $\xi_j = \nu_j/R$, ν_j is defined by the equation $J_0(\nu_j) = 0$. The form of the series expansion for the flux is suggested by the steady-state solution for the flux in a cylindrical reactor with no temperature coupling, that is, $\varphi(r, z, 0) = \Phi(0) Z_1(z) \mathcal{R}_1(r)$ (see appendix D).

It is shown in appendix B that, when equations (11) and (13) are used in equation (7), then the usual methods of Fourier analysis lead to the following coupled set of equations for the coefficients $\Phi_l^k(t)$:

$$
\begin{aligned}
l_{th}\dot{\Phi}_l^k(t) = {} & \left[\delta k_0 + P_L B^{-2}(B^2 - B_{nk}^2)\right] \Phi_l^k(t) \\
& - \gamma \sum_{\substack{n,j\\m,i}} M_{ijk}\Phi_n^j(t) \int_0^t dx\, e^{-\alpha \xi_i^2 x}\left[Q_0\Phi_m^i(t-x)K_{nlm}(v_0x)\right. \\
& \left. + \frac{2\alpha\xi_i}{c_l R J_1(\nu_i)} \int_{v_0x}^a \theta_R(z-v_0x, t-x)Z_l(z)Z_n(z)\, dz\right] \\
& - \gamma \sum_{\substack{n,j\\m,i}} M_{ijk} \frac{2e^{-\alpha\xi_i^2 t}\Phi_n^j(t)}{c_l R^2\left[J_1(\nu_i)\right]^2} \int_{v_0t}^a Z_l(z)Z_n(z)\bar{\theta}_0(\xi_i, z-v_0t)\, dz \qquad 0 \le t \le \tau_a \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
l_{th}\dot{\Phi}_l^k(t) = {} & \left[\delta k_0 + P_L B^{-2}(B^2 - B_{nk}^2)\right] \Phi_l^k(t) \\
& - \gamma \sum_{\substack{n,j\\m,i}} M_{ijk}\Phi_n^j(t) \int_0^{\tau_a} dx\, e^{-\alpha \xi_i^2 x}\left[Q_0\Phi_m^i(t-x)K_{nlm}(v_0x)\right. \\
& \left. + \frac{2\alpha\xi_i}{c_l R J_1(\nu_i)} \int_{v_0x}^a \theta_R(z-v_0x, t-x)Z_l(z)Z_n(z)\, dz\right] \qquad t > \tau_a \qquad (15)
\end{aligned}
$$

Here,

$$
K_{nlm}(v_0x) = \frac{1}{c_l} \int_{v_0x}^a Z_n(z)Z_l(z)Z_m(z-v_0x)\, dz
$$

and

TABLE I. - FIRST FEW VALUES OF $K_{nlm}(v_0x)$ AND M_{ijk}

n	l	m	$K_{nlm}(v_0x)$
1	1	1	$\frac{1}{\pi}\left(1 + \frac{4}{3}\cos\frac{\pi x}{a} + \frac{1}{3}\cos\frac{2\pi x}{a}\right)$
1	2	1	$\frac{1}{2\pi}\left[-\frac{1}{4}\cos\frac{\pi x}{a} - \pi\left(1 - \frac{x}{a}\right)\sin\frac{\pi x}{a} + \frac{1}{4}\cos\frac{3\pi x}{a}\right]$
1	1	2	$\frac{1}{2\pi}\left[1 - \cos\frac{2\pi x}{a} + \pi\left(1 - \frac{x}{a}\right)\sin\frac{2\pi x}{a}\right]$
2	1	1	$\frac{1}{2\pi}\left[-\frac{1}{4}\cos\frac{\pi x}{a} - \pi\left(1 - \frac{x}{a}\right)\sin\frac{\pi x}{a} + \frac{1}{4}\cos\frac{3\pi x}{a}\right]$
1	2	2	$\frac{1}{2\pi}\left(\frac{4}{3}\cos\frac{\pi x}{a} + \frac{32}{15}\cos\frac{2\pi x}{a} + \frac{4}{5}\cos\frac{3\pi x}{a}\right)$
2	1	2	$\frac{1}{2\pi}\left(\frac{4}{3}\cos\frac{\pi x}{a} + \frac{32}{15}\cos\frac{2\pi x}{a} + \frac{4}{5}\cos\frac{3\pi x}{a}\right)$
2	2	1	$\frac{1}{\pi}\left(1 + \frac{16}{15}\cos\frac{\pi x}{a} + \frac{1}{15}\cos\frac{4\pi x}{a}\right)$
2	2	2	$\frac{1}{2\pi}\left(1 - \frac{4}{3}\cos\frac{2\pi x}{a} + \frac{1}{3}\cos\frac{4\pi x}{a}\right)$

where $K_{nlm}(v_0x) = \frac{1}{c_l}\int_{v_0x}^{a} Z_n(z)Z_l(z)Z_m(z - v_0x)\,dz$

i	j	k	M_{ijk}*
1	1	1	0.37550
1	2	1	.06953
1	1	2	-.10606
2	1	1	.06953
1	2	2	-.21405
2	1	2	-.21405
2	2	1	.14225
2	2	2	-.04393

where $M_{ijk} = \frac{2}{R^2\left[J_1(\nu_k)\right]^2}\int_0^R r\mathcal{R}_i(r)\mathcal{R}_j(r)\mathcal{R}_k(r)\,dr$

*Values of $\int_0^1 xJ_0(\nu_i x)J_0(\nu_j x)J_0(\nu_k x)\,dx$ were obtained from ref. 23.

$$M_{ijk} = \frac{2}{R^2\left[J_1(\nu_k)\right]^2} \int_0^R r\mathcal{R}_i(r)\mathcal{R}_j(r)\mathcal{R}_k(r)\,dr$$

Table I lists the values of the first few K_{nlm} and M_{ijk}. Equations (15) and (16) are an infinite coupled set of nonlinear integro-differential equations. It will be shown that in many cases of practical interest this set can be approximated by a single term.

VALIDITY OF ONE-TERM APPROXIMATION

The assumption is now made that $\theta_R(z,t)$ is zero for all z and t. With this assumption in mind, an argument is given in appendix C to the effect that, under certain conditions, the first term in the set (eq. (14) and (15)) will represent the kinetics problem extremely well. The basis of the argument is that the steady-state flux and temperature distributions obtained by taking either a single term in the set, two terms, or the exact steady-state solution when the conductivity is zero all yield the same result. This implies that the second term in the series for the flux is an extremely small perturbation on the first term and all other terms are smaller than the second.[1] It should be noted that a one-term approximation implies that the spatial flux distribution and the radial temperature distribution do not change during the transient.

The conditions for a one term approximation to be valid are $\delta k_0 << P_L \epsilon_1^2 B^{-2}$, $\delta k_0 << \left|P_L(\xi_1^2 - \xi_2^2)B^{-2}\right|$, and $\alpha \xi_1^2 \tau_a << 1$. Then the kinetic equations reduce to

$$l_{th}\dot{\Phi}(t) = \delta k_0 \Phi(t) - \gamma\Phi(t)M_{111} \int_0^t dx\, e^{-\alpha\xi_1^2 x} Q_0 K_{111}(v_0 x)\Phi(t-x)$$

$$- \gamma M_{111} \frac{2e^{-\alpha\xi_1^2 t}\Phi(t)}{c_1 R^2\left[J_1(\nu_1)\right]^2} \int_{v_0 t}^a \left[Z_1(z)\right]^2 \bar{\theta}_0(\xi_1, z - v_0 t)\,dz \qquad 0 \le t \le \tau_a \qquad (16)$$

---

[1]Note that the results obtained in appendix C make this statement plausible. In order to further investigate this statement, a numerical solution to the one-term equations was compared to a numerical solution of the exact equations given in ref. 19 for a line reactor. The two methods were very nearly the same in their results.

$$l_{th}\dot{\Phi}(t) = \delta k_0 \Phi(t) - \gamma M_{111}\Phi(t) Q_0 \int_0^{\tau_a} dx\, e^{-\alpha \xi_1^2 x} \Phi(t-x) K_{111}(v_0 x) \qquad t \geq \tau_a \tag{17}$$

where the subscripts on $\Phi(t)$ have been dropped and the fact that $\overline{\theta}_0(\xi_i, z)$ contains the term δ_{i1} has been anticipated. This is demonstrated in appendix D where the initial steady-state temperature distribution is derived.

The kinetic behavior of the temperature is described by the equations

$$\theta(r,z,t) = \frac{2}{R^2\left[J_1(\nu_1)\right]^2}\overline{\theta}(\xi_1, z, t) J_0(r\xi_1) \tag{18a}$$

$$\overline{\theta}(\xi_1, z, t) = Q_0 \int_0^t \overline{\varphi}(\xi_1, z-v_0 x, t-x) e^{-\alpha \xi_1^2 x}\, dx + \overline{\theta}_0(\xi_1, z-v_0 t) e^{-\alpha \xi_1^2 t} \qquad 0 \leq t \leq \tau_a,\ t \leq z/v_0 \tag{18b}$$

$$\overline{\theta}(\xi_1, z, t) = Q_0 \int_0^{z/v_0} \overline{\varphi}(\xi_1, z-v_0 x, t-x) e^{-\alpha \xi_1^2 x}\, dx \qquad 0 \leq t \leq \tau_a,\ t > z/v_0, \text{ or } t > \tau_a \tag{18c}$$

STABILITY OF SYSTEM

A reactor system is said to be stable if a departure from an equilibrium state ultimately results in a return to equilibrium. Linear stability implies that the disturbance which causes the departure from equilibrium is arbitrarily small, while nonlinear stability implies that the departure from equilibrium is large enough for nonlinear effects to become important.

Consider the kinetic equation for the flux after a time greater than τ_a (eq. 17); this equation describes the flux after a long period of time so that its stability should guarantee the stability of the system. Following Welton (ref. 16), define $P(t) = E\Sigma_f \Phi(t)$ and $G(x) \equiv \gamma M_{111}(\rho c_p)^{-1} K_{111}(v_0 x) e^{-\alpha \xi_1^2 x} u(\tau_a - x)$; then equation (17) can be written as

$$l_{th}\dot{P} = P\left[\delta k_0 - \int_0^\infty G(x)P(t-x)\,dx\right] \tag{19}$$

Using the definitions $P(t) = P_0 e^{Q(t)}$ and $S(x) = -\dot{G}(x)/G(0)$ in equation (19) and differentiating the result with respect to time give

$$\ddot{Q} + \Omega^2(e^Q - 1) = \Omega^2 \int_0^\infty dx\, S(x)\left[e^{Q(t-x)} - 1\right] \tag{20}$$

where $\Omega^2 \equiv G(0)P_0/l_{th}$ and the fact that $\int_0^\infty S(x)\,dx = 1$ is used. When the right side of equation (20) is zero, the resulting equation is that of a one-dimensional, nonlinear oscillator with no driving force or damping. The motion of such a system is strictly periodic. As a matter of fact, for small Q (i.e., for linear power changes) and with the right side of equation (20) equal to zero, the result is $\ddot{Q} + \Omega^2 Q = 0$ which shows explicitly that Ω^2 plays the role of a frequency squared. This equation also shows that, for linear deviations from a steady state, it may be possible to obtain purely sinusoidal motion of the system.

The right side of equation (20) acts like a hysteresis effect and, under suitable conditions, will lead to damping. It should be noted that if delayed neutrons are present, an additional term appears on the right side of equation (20) which always leads to damping (ref. 13). Welton has shown[2] that, even in the absence of delayed neutrons, equation (20) is never unstable if

$$\frac{2}{\pi}\int_0^\infty S(x)\sin(\Omega x)\,dx \geq 0 \tag{21}$$

for all possible values of Ω. It is shown in appendix E that, if $\alpha\xi_1^2\tau_a$ is small enough to permit the approximation $e^{-\alpha\xi_1^2\tau_a} \simeq 1$, then S(x) fulfills the condition (eq. (21)) provided that $\Omega \leq 3\pi/\tau_a$. For larger values of Ω, the left side of equation (21) oscillates between positive and negative values. Hence, it is concluded that the system is damped at least for values of $\Omega \leq 3\pi/\tau_a$.

[2]The procedure followed by Welton is to obtain the Lagrangian corresponding to the motion described by equation (20) from which the Hamiltonian is obtained. Now the Hamiltonian corresponds to the total energy excess above equilibrium. If the Hamiltonian is positive definite and bounded, then so is the energy thus implying conservative motion, and unconditional stability is therefore guaranteed. For many reactor systems this condition is actually too strong.

Fleck (ref. 19) has argued that equation (19) is bounded and is certainly damped in the case of nonlinear disturbances but that purely oscillatory motion can occur in the case of linear disturbances for certain values of Ω.[3] Specifically, those values of Ω which make the integral $\int_0^\infty G(x) \cos (\Omega x) dx$ equal to zero result in the flux and temperature being completely out of phase. It is shown in appendix E that this is the same integral used in Welton's stability criterion. For the particular G(x) used in this report, the "resonance" values of Ω, that is, those values of Ω which make $\int_0^\infty G(x) \cos (\Omega x) dx$ equal to zero, are given by $\Omega = N\pi/\tau_a$ where $N = 3, 4, 5, \ldots$ In most practical cases, it will be found that $\Omega \leq 3\pi/\tau_a$; when the equality holds, the fuel temperature rise will be so large that many of the previous assumptions about various parameters being constant will no longer be valid. Using the definition of Ω and the fact that the steady-state flux amplitude is given by $\Phi_0 = \pi\delta k_0(\gamma Q_0 M_{111}\tau_a)^{-1}$ (see appendix C, eq. (C-10)) shows that resonances occur when

$$\frac{\tau_a \delta k_0}{l_{th}} = \frac{3N^2\pi^2}{8} \qquad N = 3, 4, 5, \ldots \tag{22}$$

It is concluded that the system will be stable for the case of high-power kinetics for all disturbances which are nonlinear. However, damping may be quite slow near one of the resonances described by equation (22). It is interesting to note that equation (22) is identical for either the line reactor or a two-dimensional reactor model and is independent of the radial distribution of internal heat sources.

NUMERICAL RESULTS

In order to compare the present work with the line reactor model and to verify the analytical results, a few numerical examples were run on a digital computer using the computer program Mimic (ref. 24). The one-term equations (eqs. (16), (17), and (18)) were used along with the parameters given in table II; these parameters are similar to those given in reference 19 for a Brookhaven National Laboratory U-Bi LMFR design. With the density and heat capacity held constant, the thermal conductivity was varied

[3] Fleck has assumed that the boundedness and nonperiodicity of solutions to equation (19) implies that the solution damps. By expanding the flux in a Fourier series and examining the individual components he is able to show that periodic motion is possible only for linear disturbances of the system and that these motions occur when $\int_0^\infty G(x) \cos (\Omega x) dx = 0$.

TABLE II. - PARAMETERS USED IN NUMERICAL CALCULATIONS

Length of reactor, a, cm	170.031
Reactor radius, R, cm	92
Negative of temperature coefficient of reactivity, γ, $^{\circ}C^{-1}$	5×10^{-5}
Diffusion length squared, L^2, cm^2	229
Geometric buckling, B^2, cm^{-2}	1.026×10^{-3}
Sum of molecular and turbulent conductivities of fuel solution, κ, J/(cm)(sec)($^{\circ}$C)	≥ 0, ≤ 0.155
Heat capacity of fuel solution, c_p, J/(g)($^{\circ}$C))	0.149
Density of fuel solution, ρ, g/cm^3	10
Thermal diffusivity, α, cm^2/sec	≥ 0, ≤ 0.104
Transit time for fuel across reactor, τ_a, sec	1.0
Finite medium lifetime for neutrons with speed v_k, l_{th}, sec	6.56×10^{-4}
Initial flux amplitude, $\Phi(0)$, $[(cm^2)(sec)]^{-1}$	1.0×10^{15}

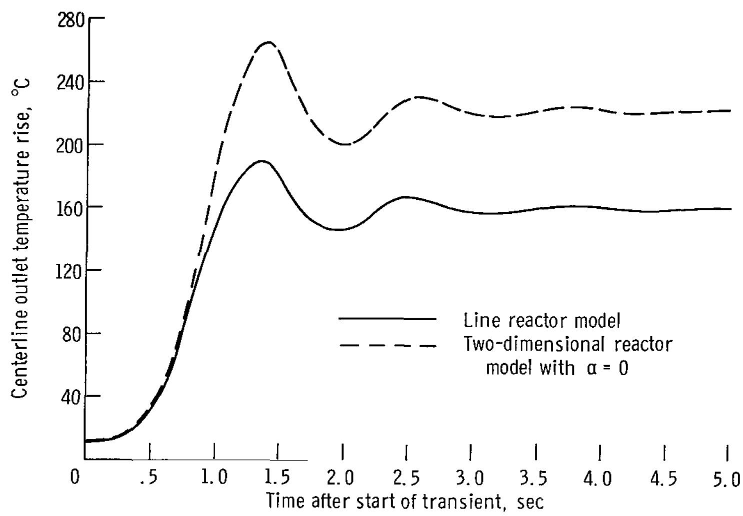

Figure 2. - Comparison of line reactor and two-dimensional reactor model with $\alpha = 0$.

between zero and 0.155 joule per centimeter per second per $^{\circ}$C to study its effect on the kinetic equations. The one-term equations for a line reactor were also programmed and the results compared to the two-dimensional reactor.

Figure 2 shows a plot of the centerline outlet temperature as a function of time for the line reactor and for the two-dimensional reactor with the thermal conductivity equal to zero. The figure indicates that the temperature history behaves nearly the same in each case but that the absolute values of the temperature rises are quite different. To see why the difference in magnitude exists, the analytic expressions for the steady-state outlet temperature and the steady-state flux amplitudes for the line reactor and the two-dimensional reactor can be compared. These are

$$\theta_{line}(0,a) = \frac{2Q_0\Phi_0\tau_a}{\pi} = \frac{2\delta k_0}{\gamma}; \qquad (\Phi_0)_{line} = \frac{\delta k_0 \pi}{\gamma Q_0 \tau_a}$$

and when $\alpha = 0$

$$\theta_{2\text{-}d}(0,a) = \frac{2\delta k_0 J_1(\nu_1)}{\gamma M_{111}}; \qquad (\Phi_0)_{2\text{-}d} = \frac{\delta k_0 \pi J_1(\nu_1)}{\gamma Q_0 \tau_a M_{111}}$$

It is seen that the radial distribution of heat sources, which gives rise to the M_{111} term, causes the difference in magnitude in temperature. The same result is obviously true for the flux. It is concluded, therefore, that a change in the radial internal heat source distribution will alter the magnitude of the temperature in the reactor but it will not disturb the kinetic behavior of the reactor significantly.

Figures 3, 4, and 5 are plots of the flux amplitude and axial outlet temperature for several values of δk_0. These correspond to "values of N" of 0.632, 1.28, and 3.22 where N is computed from equation (22). These "values of N" indicate how close the system is to one of the predicted resonances. In figure 3 the system is practically in phase and the flux and temperature simply approach their steady-state values along a smooth curve. In figure 4 the system is closer to its first resonance and the flux and temperature both oscillate about their steady-state value before damping. Note that the temperature oscillations are less severe than the flux oscillations. This effect is more obvious in figure 5 where the system is very near its first resonance.

The condition for a resonance was obtained by ignoring the effects of thermal conductivity. Figures 3, 4, and 5 were obtained from the two-dimensional reactor kinetics

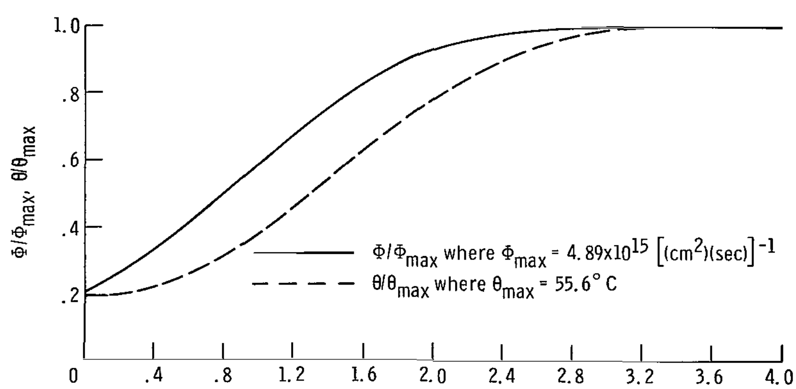

Figure 3. - Flux amplitude and centerline outlet temperature for "N" = 0.632; δk_0 = 0.001.

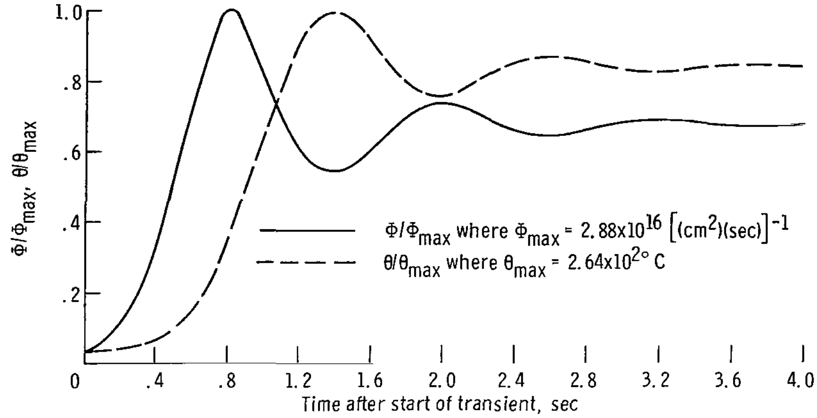

Figure 4. - Flux amplitude and centerline outlet temperature for "N" = 1.28; δk_0 = 0.004.

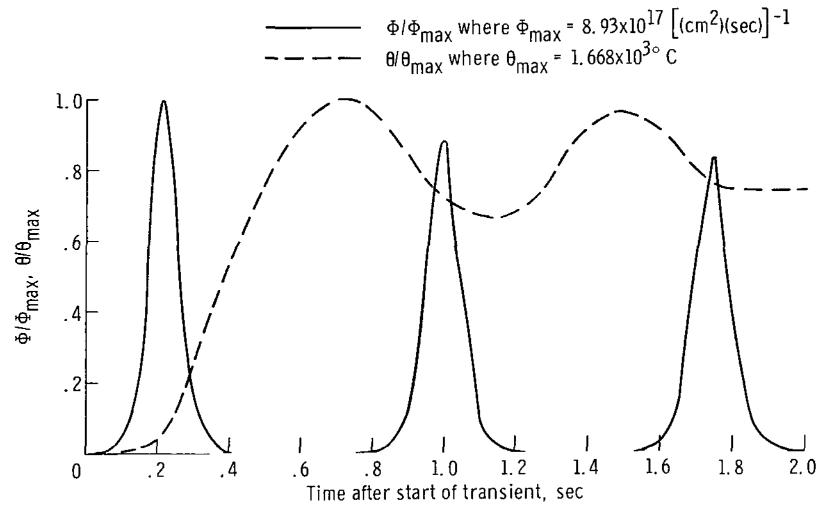

Figure 5. - Flux amplitude and centerline outlet temperature for "N" = 3.22; δk_0 = 0.025.

equations with $\kappa = 0.155$ joule per centimeter per second per $^{\mathrm{o}}$C. It is concluded, therefore, that the position of the resonances of the system is not affected by radial heat transfer provided, at least, that $\alpha\xi_1^2\tau_a << 1$. Three runs were made with the conditions used in figure 4 while the thermal conductivity was given values of 0.0, 1.55×10^{-3}, and 1.55×10^{-1} joule per centimeter per second per $^{\mathrm{o}}$C. (This is equivalent to holding κ constant and varying R in the term $\alpha\xi_1^2\tau_a$.) The results showed that the kinetics were nearly identical, the only change being a difference of 0.002^{o} C out of about 222^{o} C in the outlet temperature between the lowest and highest values of κ. Hence, it is expected that, in many cases of practical interest, radial heat transfer can be ignored.

SUMMARY OF RESULTS

The following conclusions can be drawn about the high-power kinetics of a single channel unreflected circulating fuel reactor with a nonuniform radial heat source distribution and radial heat transfer permitted. These conclusions are valid provided the one-neutron velocity diffusion equation is a valid description of the system and a slug fluid flow model is adequate.

If a step change in multiplication is made homogeneously throughout the reactor and is such that $\delta k_0 << P_L \epsilon_1^2 B^{-2}$, $\delta k_0 << P_L |\xi_1^2 - \xi_2^2| B^{-2}$, and $\alpha \xi_1^2 \tau_a << 1$ where δk_0 is the change in multiplication, P_L is the thermal leakage probability, B^2 is the buckling, ξ_1 and ξ_2 are the first and second modes of the radial buckling, α is the thermal diffusivity, and τ_a is the transit time across the reactor, then the following hold:

1. The neutron flux will essentially maintain its initial steady-state radial distri- - tion throughout the ensuing transient.

2. The temperature will essentially maintain its initial steady-state radial distribution throughout the transient, provided the wall temperature rise is zero.

3. A number of oscillations in the flux and temperature are possible; however, a new steady state will be reached. These oscillations will be bounded but will have a higher frequency and will be weakly damped when a certain relation exists between the parameters of the problem, that is, $\tau_a \delta k_0 l_{th}^{-1} = 3N^2\pi^2/8$ where $N = 3, 4, 5, \ldots$ and l_{th} is the neutron lifetime.

4. A line reactor model gives an adequate description of the flux and temperature time histories, but the two-dimensional model must be used to obtain the magnitude of the flux and temperature.

5. In many cases of practical interest, radial heat transfer can be ignored.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, July 3, 1968,
122-29-03-01-22.

APPENDIX A

SOLUTION OF ENERGY EQUATION IN TERMS OF THE FLUX

Consider the energy equation in the following form:

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}\right) \theta = Q_0 \varphi + \alpha \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right) \tag{10}$$

Apply a finite Hankel transform (ref. 22) to this equation letting

$$\overline{\theta}(\xi_i, z, t) \equiv \int_0^R r\theta(r, z, t) J_0(r\xi_i)\, dr \tag{A1}$$

where ξ_i satisfies $J_0(r\xi_i) = 0$. Assume that the flux goes to zero at the walls of the reactor so that

$$\overline{\varphi}(\xi_i, z, t) \equiv \int_0^R r\varphi(r, z, t) J_0(r\xi_i)\, dr \tag{A2}$$

is the finite Hankel transform of the flux. Furthermore, since $\partial\theta/\partial r\big|_{r=0} = 0$ and $\theta(R, z, t) = \theta_R(z, t)$,

$$\int_0^R r\left[\frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \theta}{\partial r}\right)\right] J_0(\xi_i r)\, dr = \alpha R \xi_i \theta_R(z, t) J_1(\xi_i R) - \alpha \xi_1^2 \overline{\theta}(\xi_i, z, t)$$

To apply a finite Hankel transformation to equation (10), multiply by $rJ_0(r\xi_i)$ and integratefrom zero to R. Then applying the previous definitions to the result gives

$$\left(\frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial z}\right) \overline{\theta} = Q_0 \overline{\varphi} - \alpha \xi_1^2 \overline{\theta} + \alpha R \xi_i \theta_R(z, t) J_1(\xi_i R) \tag{A3}$$

The transformed boundary conditions are

$$\overline{\theta}(\xi_i, z, 0) = \overline{\theta}_0(\xi_i, z) \tag{A4a}$$

$$\overline{\theta}(\xi_i, 0, t) = 0 \tag{A4b}$$

To solve equation (A3) subject to the boundary conditions (eqs. (A4)), apply a Laplace transformation (ref. 22) to equation (A3). Define

$$\bar{\bar{\varphi}}(\xi_i, z, p) = \int_0^{\infty} e^{-pt}\bar{\varphi}(\xi_i, z, t)\, dt \tag{A5a}$$

and

$$\bar{\bar{\theta}}(\xi_i, z, p) = \int_0^{\infty} e^{-pt}\bar{\theta}(\xi_i, z, t)\, dt \tag{A5b}$$

noting that

$$\int_0^{\infty} e^{-pt}\frac{\partial\bar{\theta}}{\partial t}\, dt = -\bar{\theta}_0(\xi_i, z) + p\bar{\bar{\theta}}(\xi_i, z, p)$$

To apply the transform to equation (A3), multiply by e^{-pt} and integrate over t from zero to infinity; then employing the previous definitions gives

$$\frac{d\bar{\bar{\theta}}}{dz} + \left(\frac{p + \alpha\xi_1^2}{v_0}\right)\bar{\bar{\theta}} = \frac{Q_0}{v_0}\bar{\bar{\varphi}} + \frac{\bar{\theta}_0(\xi_i, z)}{v_0} + \frac{\alpha\xi_i R J_1(\xi_i R)}{v_0}\bar{\bar{\theta}}_R(z, p) \tag{A6}$$

The solution to this equation is simply

$$\begin{aligned}\bar{\bar{\theta}}(\xi_i, z, p) = e^{-(p+\alpha\xi_i^2)z/v_0}\Bigg[&\int_0^z \frac{Q_0}{v_0}\bar{\bar{\varphi}}(\xi_i, y, p)e^{(p+\alpha\xi_i^2)y/v_0}\, dy \\ &+ \int_0^z \frac{1}{v_0}\bar{\theta}_0(\xi_i, y)e^{(p+\alpha\xi_i^2)y/v_0}\, dy + \int_0^z \frac{\alpha\xi_i R J_1(\xi_i R)}{v_0} \\ &\times \bar{\bar{\theta}}_R(y, p)e^{(p+\alpha\xi_i^2)y/v_0}\, dy + \bar{\bar{\theta}}(\xi_i, 0, p)\Bigg]\end{aligned} \tag{A7}$$

Equation (A4) shows that $\bar{\bar{\theta}}(\xi_i, 0, p) = 0$. Equation (A7) may be cast into a slightly different form by changing variables twice, first to y/v_0 and then to $x = (z/v_0) - (y/v_0)$. The result is

$$\bar{\bar{\theta}}(\xi_i, z, p) = Q_0 \int_0^{z/v_0} \bar{\bar{\varphi}}(\xi_i, z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\, dx$$

$$+ \int_0^{z/v_0} \bar{\theta}_0(\xi_i, z-v_0x)e^{-(p+\alpha\xi_i^2)x}\, dx$$

$$+ \alpha\xi_i RJ_1(\xi_i R) \int_0^{z/v_0} \bar{\bar{\theta}}_R(z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\, dx \qquad \text{(A8)}$$

In order to find $\theta(r, z, t)$, $\bar{\bar{\theta}}(\xi_i, z, p)$ must be inverted. Now the inverse of the Laplace transformation is simply a contour integration over the variable p. Denote this integration by the symbol $\mathscr{L}^{-1}$. Assuming that the $\mathscr{L}^{-1}$ operation may be interchanged with the integration over x in equation (A8) it can be seen that

$$\mathscr{L}^{-1}\left[\bar{\bar{\theta}}(\xi_i, z, p)\right] = Q_0 \int_0^{z/v_0} \mathscr{L}^{-1}\left[\bar{\bar{\varphi}}(\xi_i, z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\right] dx$$

$$+ \int_0^{z/v_0} \mathscr{L}^{-1}\left[\bar{\theta}_0(\xi_i, z-v_0x)e^{-(p+\alpha\xi_i^2)x}\right] dx$$

$$+ \alpha\xi_i RJ_1(\xi_i R) \int_0^{z/v_0} \mathscr{L}^{-1}\left[\bar{\bar{\theta}}_R(z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\right] dx \qquad \text{(A9)}$$

Now $\mathscr{L}^{-1}\left[\bar{\bar{\theta}}(\xi_i, z, p)\right] = \bar{\theta}(\xi_i, z, t)$ and $\mathscr{L}^{-1}\left[\bar{\bar{\varphi}}(\xi_i, z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\right] = \bar{\varphi}(\xi_i, z-v_0x, t-x)$ $\times e^{-\alpha\xi_i^2 x}$ $u(t-x)$ where $u(t-x)$ is the unit step function defined by

$$u(t-x) = \begin{cases} 0 & t < x \\ 1 & t > x \end{cases} \tag{A10}$$

Also note that $\mathscr{L}^{-1}\left[\overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2x}e^{-px}\right] = \overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2x}\delta(t-x)u(t-x)$ where $\delta(t-x)$ is the Dirac delta function and

$$\mathscr{L}^{-1}\left[\overline{\overline{\theta}}_R(z-v_0x, p)e^{-(p+\alpha\xi_i^2)x}\right] = \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2x}u(t-x)$$

Using the previous relations reveals that the application of the inverse of the Laplace transform to equation (A8) yields

$$\overline{\theta}(\xi_i, z, t) = Q_0\int_0^{z/v_0}\overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2x}u(t-x)\,dx$$

$$+\int_0^{z/v_0}\overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2x}\delta(t-x)u(t-x)\,dx$$

$$+\alpha\xi_iRJ_1(\xi_iR)\int_0^{z/v_0}\theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2x}u(t-x)\,dx \tag{A11}$$

Let τ_a be the time it takes for the fuel to traverse the reactor, that is, τ_a is the transit time across the reactor. Consider the time interval $0 \le t \le \tau_a$ for axial positions in the reactor where $t \le z/v_0$. Then the unit function $u(t-x)$ is zero for values of x greater than t and $\overline{\theta}(\xi_i, z, t)$ becomes

$$\overline{\theta}(\xi_i, z, t) = Q_0 \int_0^t \overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx$$

$$+ \int_0^t \overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2 x}\,\delta(t-x)\,dx$$

$$+ \alpha\xi_i R J_1(\xi_i R) \int_0^t \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx \qquad \begin{array}{l} 0 \le t \le \tau_a \\ t \le z/v_0 \end{array}$$

Now by definition, $\int_{t_1}^{t_2} f(x)\delta(t-x)\,dx = f(t)$ provided $t_1 \le t \le t_2$. Thus, the previous equation finally becomes

$$\overline{\theta}(\xi_i, z, t) = Q_0 \int_0^t \overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx + \overline{\theta}_0(\xi_i, z-v_0t)e^{-\alpha\xi_i^2 t}$$

$$+ \alpha\xi_i R J_1(\xi_i R) \int_0^t \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx \qquad \text{(A12)}$$

Next consider the case when $0 \le t \le \tau_a$ and $t > z/v_0$. In this case the unit function is unity over the entire range of the variable x. However, the integration over the delta function does not include t in this case so that the integral is zero. Thus

$$\overline{\theta}(\xi_i, z, t) = Q_0 \int_0^{z/v_0} \overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx$$

$$+ \alpha\xi_i R J_1(\xi_i R) \int_0^{z/v_0} \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\,dx \qquad \begin{array}{l} 0 \le t \le \tau_a, \\ t > z/v_0 \end{array} \qquad \text{(A13)}$$

When $t > \tau_a$, the unit function has value unity over the entire range of x and the integral over the delta function never contains t. Therefore $\overline{\theta}$ is exactly as in equation (A13), that is,

$$\overline{\theta}(\xi_i, z, t) = Q_0 \int_0^{z/v_0} \overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\, dx$$

$$+ \alpha\xi_i R J_1(\xi_i R) \int_0^{z/v_0} \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2 x}\, dx \qquad t > \tau_a \tag{A14}$$

To obtain $\theta(r, z, t)$, apply the inversion operation for the finite Hankel transformation, that is,

$$\theta(r, z, t) = \frac{2}{R^2} \sum_i \overline{\theta}(\xi_i, z, t) \frac{J_0(r\xi_i)}{\left[J_1(\nu_i)\right]^2} \tag{11}$$

where the appropriate relation for $\overline{\theta}(\xi_i, z, t)$ must be obtained from equations (A12), (A13), and (A14).

APPENDIX B

APPLICATION OF FOURIER-BESSEL EXPANSION TO THE FLUX

Consider the kinetic equation for neutrons in the form of equation (7), that is,

$$l_{th}\dot{\varphi} = (\delta k_0 + P_L)\varphi + P_L B^{-2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{\partial^2 \varphi}{\partial z^2}\right] - \gamma\theta\varphi \tag{B1}$$

From appendix A, it is known that

$$\theta(r, z, t) = \frac{2}{R^2}\sum_i \overline{\theta}(\xi_i, z, t)\frac{J_0(r\xi_i)}{\left[J_1(\nu_i)\right]^2} \tag{11}$$

and

$$\overline{\theta}(\xi_i, z, t) = Q_0\int_0^{z/v_0} \overline{\varphi}(\xi_i, z-v_0x, t-x)e^{-\alpha\xi_i^2 x}u(t-x)\,dx$$

$$+ \int_0^{z/v_0} \overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2 x}\delta(t-x)u(t-x)\,dx$$

$$+ \alpha\xi_i R J_1(\xi_i R)\int_0^{z/v_0} \theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2 x}u(t-x)\,dx$$

Assume that the flux can be expanded in the form $\varphi(r, z, t) = \sum_{n,j} \Phi_n^j(t)Z_n(z)\mathcal{R}_j(r)$ where $Z_n(z) = \sin(\epsilon_n z)$, $\mathcal{R}_j(r) = J_0(\beta_j r)$, $\epsilon_n = n\pi/a$, $\xi_j = \nu_j/R$, and $J_0(\nu_j) = 0.$ Then

$$\overline{\varphi}(\xi_i, z, t) = \int_0^R r\varphi(r, z, t) J_0(r\xi_i)\, dr$$

$$= \sum_{m,l} \Phi_m^l(t) Z_m(z) \int_0^R rJ_0(\xi_l r) J_0(\xi_i r)\, dr$$

$$= \frac{R^2}{2} \sum_{m,l} \Phi_m^l(t) Z_m(z) \left[J_1(\nu_l)\right]^2 \delta_{il}$$

$$= \frac{R^2}{2} \sum_m \Phi_m^i(t) Z_m(z) \left[J_1(\nu_i)\right]^2$$

Using the previous relations in equation (B1) results in

$$l_{th} \sum_{n,j} \dot{\Phi}_n^j(t) Z_n(z) \mathcal{R}_j(r) = (\delta k_0 + P_L) \sum_{n,j} \Phi_n^j(t) Z_n(z) \mathcal{R}_j(r)$$

$$+ P_L B^{-2} \sum_{n,j} \Phi_n^j(t) \left\{ Z_n(z) \frac{1}{r} \frac{d}{dr} \left[r \frac{d\mathcal{R}_j(r)}{dr} \right] + \mathcal{R}_j(r) \frac{d^2 Z_n(z)}{dz^2} \right\}$$

$$- \gamma \sum_{n,j} \Phi_n^j(t) Z_n(z) \mathcal{R}_j(r) \frac{2}{R^2} \frac{\mathcal{R}_i(r)}{\left[J_1(\nu_i)\right]^2} \left\{ \frac{Q_0 R^2 \left[J_1(\nu_i)\right]^2}{2} \int_0^{z/v_0} \Phi_m^i(t-x) \right.$$

$$\times Z_m(z - v_0 x) e^{-\alpha \xi_i^2 x} u(t-x)\, dx + \int_0^{z/v_0} \overline{\theta}(\xi_i, z - v_0 x) e^{-\alpha \xi_i^2 x} \delta(t-x) u(t-x)\, dx$$

$$\left. + \alpha \xi_i R J_1(\xi_i R) \int_0^{z/v_0} \theta_R(z - v_0 x, t-x) e^{-\alpha \xi_i^2 x} u(t-x)\, dx \right\} \tag{B2}$$

Now

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{R}_j}{dr}\right) = -\frac{\nu_j^2}{R^2}J_0\left(\frac{\nu_j r}{R}\right) = -\xi_j^2\mathcal{R}_j(r)$$

and $d^2Z_n(z)/dz^2 = \epsilon_n^2 Z_n(z)$. Define $B_{nj}^2 = \epsilon_n^2 + \xi_j^2$; then equation (B2) may be written as

$$\begin{aligned}
l_{th}\sum_{n,j}\dot{\Phi}_n^j(t)Z_n(z)\mathcal{R}_j(r) &= \sum_{n,j}\left[\delta k_0 + P_L B^{-2}\left(B^2 - B_{nj}^2\right)\right]\Phi_n^j(t)Z_n(z)\mathcal{R}_j(r) \\
&- \gamma\sum_{n,j}\Phi_n^j(t)Z_n(z)\mathcal{R}_j(r)\mathcal{R}_i(r)\left\{Q_0\int_0^{z/v_0}\Phi_m^i(t-x)Z_m(z-v_0x)e^{-\alpha\xi_i^2x}\right. \\
&\times u(t-x)\,dx + \frac{2}{R^2\left[J_1(\nu_i)\right]^2}\int_0^{z/v_0}\overline{\theta}(\xi_i, z-v_0x)e^{-\alpha\xi_i^2x}\delta(t-x)u(t-x)\,dx \\
&\left. + \frac{2\alpha\xi_i}{RJ_1(\nu_i)}\int_0^{z/v_0}\theta_R(z-v_0x, t-x)e^{-\alpha\xi_i^2x}u(t-x)\,dx\right\} \qquad \text{(B3)}
\end{aligned}$$

Multiply equation (B3) by $Z_l(z)$ and integrate from zero to a noting that $\int_0^a Z_l(z)Z_m(z)\,dz = c_m\delta_{l,m}$; then

$$l_{th} \sum_j \dot{\Phi}_l^j(t) c_l \Re_j(r) = \sum_j \left[\delta k_0 + P_L B^{-2}(B^2 - B_{nj}^2)\right] \Phi_l^j(t) c_l \Re_j(r)$$

$$- \gamma \sum_{\substack{n,j \\ m,i}} \int_0^a Z_l(z) Z_n(z) \Phi_n^j(t) \Re_j(r) \Re_i(r) \left\{ Q_0 \int_0^{z/v_0} \Phi_m^i(t-x) Z_m(z-v_0 x) \right.$$

$$\times e^{-\alpha \xi_i^2 x} u(t-x)\, dx\, dz + \frac{2}{R^2 \left[J_1(\nu_i)\right]^2} \int_0^{z/v_0} \overline{\theta}(\xi_i, z-v_0 x) e^{-\alpha \xi_i^2 x}$$

$$\left. \times\, \delta(t-x) u(t-x)\, dx + \frac{2\alpha \xi_i}{R J_1(\nu_i)} \int_0^{z/v_0} \theta_R(z-v_0 x, t-x) e^{-\alpha \xi_i^2 x} u(t-x)\, dx \right\} \tag{B4}$$

Now if the z and x integrations are interchanged, the result is

$$\int_0^a dz \int_0^{z/v_0} f(z) g(z-v_0 x)\, dx = \frac{1}{v_0} \int_0^a dz \int_0^z f(z) g(z-s)\, ds$$

$$= \frac{1}{v_0} \int_0^a ds \int_s^a f(z) g(z-s)\, dz$$

$$= \int_0^{a/v_0} dx \int_{v_0 x}^a f(z) g(z-v_0 x)\, dz$$

$$= \int_0^{\tau_a} dx \int_{v_0 x}^a f(z) g(z-v_0 x)\, dz$$

where f and g are arbitrary functions and $s = v_0 x$. Define

$$K_{nlm}(v_0 x) = \frac{1}{c_l} \int_{v_0 x}^a Z_n(z)\, Z_l(z) Z_m(z-v_0 x)\, dz \tag{B5}$$

Then equation (B4) becomes

$$
\begin{aligned}
l_{th} \sum_j \dot{\Phi}_l^j(t)\mathcal{R}_j(r) = {} & \sum_j \left[\delta k_0 + P_L B^{-2}(B^2 - B_{nj}^2)\right] \Phi_l^j(t)\mathcal{R}_j(r) \\
& - \gamma \sum_{\substack{n,j \\ m,i}} \Phi_n^j(t) \int_0^{\tau_a} dx\, \mathcal{R}_j(r)\mathcal{R}_i(r) \Bigg\{ Q_0 \Phi_m^i(t-x) K_{nlm}(v_0 x) e^{-\alpha \xi_i^2 x} u(t-x) \\
& + \frac{2}{c_l R^2 \left[J_1(\nu_i)\right]^2} \int_{v_0 x}^{a} \bar{\theta}_0(\xi_i, z - v_0 x) e^{-\alpha \xi_i^2 x} Z_l(z) Z_n(z) \delta(t-x) u(t-x)\, dz \\
& + \frac{2\alpha \xi_i}{c_l R J_1(\nu_i)} \int_{v_0 x}^{a} \theta_R(z - v_0 x, t-x) Z_l(z) Z_n(z) e^{-\alpha \xi_i^2 x} u(t-x)\, dz \Bigg\}
\end{aligned}
\qquad \text{(B6)}
$$

Next, multiply equation (B-6) by $r\mathcal{R}_k(r)$ and integrate from zero to R noting that

$$
\int_0^R r J_0(\xi_j r) J_0(\xi_i r)\, dr = \frac{R^2}{2} \left[J_1(\nu_j)\right]^2 \delta_{ij}
$$

The result is

$$l_{th}\dot{\Phi}_l^k(t)\,\frac{R^2}{2}\left[J_1(\nu_k)\right]^2 = \left[\delta k_0 + P_L B^{-2}(B^2 - B_{nk}^2)\right]\Phi_l^k(t)\,\frac{R^2}{2}\left[J_1(\nu_k)\right]^2$$

$$- \gamma \sum_{\substack{n,j\\ m,i}} \Phi_n^j(t) \int_0^{\tau_a} dx \int_0^R r\mathcal{R}_k(r)\mathcal{R}_j(r)\mathcal{R}_i(r)\,dr \left\{ Q_0\Phi_m^i(t-x)K_{nlm}(v_0x)e^{-\alpha\xi_i^2x}u(t-x)\right.$$

$$+ \frac{2}{c_l R^2\left[J_1(\nu_i)\right]^2}\int_{v_0x}^{a} Z_l(z)Z_n(z)\overline{\theta}_0(\xi_i, z-v_0x)e^{-\alpha\xi_i^2x}\delta(t-x)u(t-x)\,dz$$

$$\left. + \frac{2\alpha\xi_i}{c_l RJ_1(\nu_i)}\int_{v_0x}^{a}\theta_R(z-v_0x, t-x)Z_l(z)Z_n(z)e^{-\alpha\xi_i^2x}u(t-x)\,dz\right\} \qquad \text{(B7)}$$

Defining

$$M_{ijk} \equiv \frac{2}{R^2\left[J_1(\nu_k)\right]^2}\int_0^R r\mathcal{R}_k(r)\mathcal{R}_j(r)\mathcal{R}_i(r)\,dr$$

gives the desired equation for the coefficients $\Phi_l^k(t)$,

$$
\begin{aligned}
l_{th}\dot{\Phi}_l^k(t) = &\left[\delta k_0 + P_L B^{-2}(B^2 - B_{nk}^2)\right]\Phi_l^k(t) \\
&- \gamma \sum_{\substack{n,j \\ m,i}} \Phi_n^j(t) \int_0^{\tau_a} dx\, M_{ijk} \Bigg\{ Q_0 \Phi_m^i(t-x) K_{nlm}(v_0 x) e^{-\alpha \xi_i^2 x} u(t-x) \\
&+ \frac{2}{c_l R^2 \left[J_1(\nu_i)\right]^2} \int_{v_0 x}^{a} Z_l(z) Z_n(z) \overline{\theta}_0(\xi_i, z - v_0 x) e^{-\alpha \xi_i^2 x} \delta(t-x) u(t-x)\, dz \\
&+ \frac{2\alpha \xi_i}{c_l R J_1(\nu_i)} \int_{v_0 x}^{a} \theta_R(z - v_0 x, t-x) Z_l(z) Z_n(z) e^{-\alpha \xi_i^2 x} u(t-x)\, dz \Bigg\} \qquad \text{(B8)}
\end{aligned}
$$

APPENDIX C

VALIDITY OF ONE-TERM APPROXIMATION

To investigate the one-term approximation, the steady-state flux amplitude and the corresponding temperature rise at $r = 0$, $z = a$ (exit temperature) will be calculated. If the thermal conductivity is not too large, the axial and radial fluxes can be considered as independent quantities. Two cases will be investigated. First, the thermal conductivity will be set equal to zero and the exact solution to the flux and temperature equations will be obtained. The conditions under which $\sin \epsilon_1 z$ is an appropriate description of the axial flux will be found from this case. Second, assuming a single axial term is appropriate, a two-term radial approximation will be used in the coupled equations and the conditions under which the two-term flux and exit temperature agree with the one-term solutions will be investigated.

The basic equations are the following steady-state flux and temperature equations:

$$\frac{P_L}{B^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{\partial^2 \varphi}{\partial z^2}\right] + (\delta k_0 - \gamma\theta + P_L)\varphi = 0 \tag{C1}$$

$$\frac{\partial \theta}{\partial z} = \frac{Q_0}{v_0}\varphi + \frac{\alpha}{v_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) \tag{C2}$$

Case 1: $\alpha = 0$

In this case, it is assumed that the radial temperature follows the flux to obtain $\theta(r, z) = \mathcal{R}_1(r)\upsilon(z)$ and $\varphi(r, z) = \mathcal{R}_1(r)\varphi_s(z)$ where $\mathcal{R}_1(r) = J_0(\xi_1 r)$. Noting that $B^2 = \epsilon_1^2 + \xi_1^2$ and that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathcal{R}_1}{dr}\right) = -\xi_1^2\mathcal{R}_1(r)$$

means equations (C1) and (C2) may be written as

$$\frac{P_L}{B^2}\mathcal{R}_1\left(\frac{d^2\varphi_s}{dz^2} + \epsilon_1^2\varphi_s\right) + (\delta k_0 - \gamma\upsilon\mathcal{R}_1)\varphi_s\mathcal{R}_1 = 0$$

$$\frac{v_0}{Q_0}\frac{d\upsilon}{dz}\mathcal{R}_1 = \varphi_s\mathcal{R}_1$$

Multiplying these equations by $r\mathcal{R}_1(r)$ and integrating from zero to R yield (Note that $(R^2/2)\left[J_1(\nu_1)\right]^2 = \int_0^R r\mathcal{R}_1(r)\mathcal{R}_1(r)\,dr$ and $M_{111} = \int_0^R r\mathcal{R}_1(r)\mathcal{R}_1(r)\mathcal{R}_1(r)\,dr \times 2/R^2\left[J_1(\nu_1)\right]^2$.)

$$\frac{P_L}{B^2}\left(\frac{d^2\varphi_s}{dz^2} + \epsilon_1^2\varphi_s\right) + \delta k_0\varphi_s - \gamma\upsilon\varphi_s M_{111} = 0 \tag{C3}$$

and

$$\frac{v_0}{Q_0}\frac{d\upsilon}{dz} = \varphi_s \tag{C4}$$

Using equation (C4) in equation (C3) gives

$$\frac{d}{dz}\left(h^2\frac{d^2\upsilon}{dz^2} + p_1^2\upsilon - \frac{\gamma_1\upsilon^2}{2}\right) = 0 \tag{C5}$$

where $h^2 \equiv P_L/B^2$, $\gamma_1 \equiv \gamma M_{111}$, and $p_1^2 \equiv \delta k_0 + h^2\epsilon_1^2$. Integrating equation (C5) gives

$$h^2\frac{d^2\upsilon}{dz^2} + p_1^2\upsilon - \frac{\gamma_1}{2}\upsilon^2 = f_1 = \text{constant}$$

Let $y = d\upsilon/dz$ and $y\,dy/d\upsilon = d^2\upsilon/dz^2$; then the previous equation becomes

$$h^2 y\frac{dy}{d\upsilon} + p_1^2\upsilon - \frac{\gamma_1}{2}\upsilon^2 = f_1$$

Integrating this yields

$$h^2\frac{y^2}{2} = -\frac{p_1^2\upsilon^2}{2} + \frac{\gamma_1\upsilon^3}{6} + f_1\upsilon + f_2$$

where f_2 is a constant. Using the definition of y results in

$$\frac{h^2}{2}\left(\frac{d\upsilon}{dz}\right)^2 = -\frac{p_1^2\upsilon^2}{2} + \frac{\gamma_1\upsilon^3}{6} + f_1\upsilon + f_2 \tag{C6}$$

Now the axial temperature rise is a minimum and is equal to zero at the inlet and is a maximum at the outlet, so that $d\upsilon/dz = 0$ when $z = 0, a$ and $\upsilon = 0$ when $z = 0$. Thus, $f_2 = 0$ and

$$f_1 = \frac{p_1^2\upsilon_{ex}}{2} - \frac{\gamma_1\upsilon_{ex}^2}{6}$$

where $\upsilon_{ex} = \upsilon$ when $z = a$. Thus, equation (C6) finally becomes

$$\frac{d\upsilon}{dz} = \sqrt{-\frac{p_1^2}{h^2}\upsilon^2 + \frac{\gamma_1}{3h^2}\upsilon^3 + \left(\frac{p_1^2\upsilon_{ex}}{h^2} - \frac{\gamma_1\upsilon_{ex}^2}{3h^2}\right)\upsilon}$$

Let $T_1 = \gamma_1\upsilon/p_1^2$ and $\bar{z} = p_1 z/h$. Then

$$\bar{z} = \int_0^{T_1} \frac{dT_1}{\sqrt{-T_1^2 + \frac{T_1^3}{3} + \left(T_{1ex} - \frac{T_{1ex}^2}{3}\right)T_1}}$$

If $T_1/T_{1ex} = \sin^2 x_1$, the previous equation can be cast into the form

$$\bar{z} = 2\sqrt{\frac{3}{3 - T_{1ex}}}\int_0^{\sin^{-1}\sqrt{T_1/T_{1ex}}} \frac{dx_1}{\sqrt{1 - K^2\sin^2 x_1}}$$

where $K^2 \equiv T_{1ex}/(3 - T_{1ex})$. When $T_1 = T_{1ex}$, $\bar{z} = \bar{a}$ and

$$\bar{a} = 2\sqrt{\frac{3}{3 - T_{1ex}}} \int_0^{\pi/2} \frac{dx_1}{\sqrt{1 - K^2 \sin^2 x_1}} \tag{C7}$$

Equation (C7) contains a complete elliptic integral of the second kind; an approximate value of the integral can be obtained if $K^2 << 1$. Then

$$\left(1 - K^2 \sin^2 x_1\right)^{-1/2} \simeq 1 + \frac{K^2 \sin^2 x_1}{2}$$

and

$$\bar{a} = \pi \sqrt{\frac{3}{3 - T_{1ex}}} \left(1 + \frac{K^2}{4}\right) \tag{C8}$$

Equation (C8) must be solved for T_{1ex} when K^2 is small; the solution is $T_{1ex} \simeq 2\delta k_0/h^2\epsilon_1^2$ where it has also been assumed that $\delta k_0 << h^2\epsilon_1^2$. Using the definition of T_{1ex} results in the steady-state exit temperature

$$\upsilon_{ex} = \frac{2\delta k_0}{\gamma_1} \tag{C9}$$

Note that two things have been assumed: first, that $K^2 << 1$ and, second, that $\delta k_0 << h^2\epsilon_1^2$. These assumptions are quite good in most cases of practical interest. The first assumption holds provided that $\upsilon_{ex}\gamma_1 << 1$ which is obviously true if $\delta k_0 << 1$. This is essentially the same as the second assumption.

Now the one-term steady-state equation for $\alpha = 0$ and $t > \tau_a$ is simply (see eq. (17))

$$\Phi_0 = \frac{\delta k_0}{K_1 \int_0^{\tau_a} K_{111}(v_0 x)\, dx} \tag{C10}$$

where $K_1 = \gamma Q_0 M_{111}$ and $\int_0^{\tau_a} K_{111}(v_0 x)\, dx = \tau_a/\pi$. The steady-state temperature is given in appendix D as

$$\theta_0(r, z) = \frac{Q_0 \Phi_0 J_0(r\xi_1)}{\left[\left(\alpha\xi_1^2\right)^2 + (\pi/\tau_a)^2\right]}\left[\alpha\xi_1^2 \sin \epsilon_1 z - \frac{\pi}{\tau_a}\left(\cos \epsilon_1 z - e^{-\alpha\xi_1^2 z/v_0}\right)\right] \tag{D7}$$

which, when evaluated at $r = 0$, $z = a$ and using equation (C10) for Φ_0 yields

$$\theta_0(0, a) = \upsilon_{ex} = \frac{2\delta k_0}{\gamma_1} \tag{C11}$$

The equality of equations (C9) and (C11) indicates that a one-term axial approximation is adequate to describe the system provided that $\delta k_0 << P_L \epsilon_1^2/B^2$.

Case 2: $\alpha \neq 0$

The first two radial term equations for the steady-state flux amplitude are

$$0 = \Phi_1^1 \delta k_0 - Q_0\gamma\left[\left(\Phi_1^1\right)^2 M_{111}G_{111}^1 + \Phi_1^1 M_{121}G_{111}^2 \Phi_1^2 + \Phi_1^2\Phi_1^1 M_{211}G_{111}^1 + \Phi_1^2 M_{221}G_{111}^2\Phi_1^2\right] \tag{C12}$$

$$0 = \Phi_1^2\left[\frac{P_L}{B^2}\left(B^2 - B_{12}^2\right) + \delta k_0\right] - Q_0\gamma\left(\Phi_1^1 M_{112}G_{111}^1\Phi_1^1 + \Phi_1^1\Phi_1^2 M_{122}G_{111}^2 + \Phi_1^2\Phi_1^1 M_{212}G_{111}^1 + \Phi_1^2 M_{222}G_{111}^2\Phi_1^2\right) \tag{C13}$$

where

$$M_{ijk} = \frac{2}{R^2\left[J_1(\nu_k)\right]^2}\int_0^R r\mathcal{R}_i(r)\mathcal{R}_j(r)\mathcal{R}_k(r)\, dr$$

and

$$G_{nlm}^{i} = \int_0^{\tau_a} K_{nlm}(v_0 x) e^{-\alpha \xi_i^2 x}\, dx$$

Let

$$F_{12} = \frac{P_L}{B^2}\left(B^2 - B_{12}^2\right) + \delta k_0$$

multiply equations (C12) and (C13) by Φ_1^1/Φ_1^1 and define $\beta \equiv \Phi_1^2/\Phi_1^1$. Then

$$\frac{\delta k_0}{\gamma Q_0} = \Phi_1^1 M_{111} G_{111}^1 + \Phi_1^1 \beta M_{112}\left(G_{111}^1 + G_{111}^2\right) + \Phi_1^1 \beta^2 M_{112} G_{111}^2 \qquad \text{(C14a)}$$

$$\beta \frac{F_{12}}{Q_0} = \Phi_1^1 M_{112} G_{111}^1 + \Phi_1^1 \beta M_{112}\left(G_{111}^1 + G_{111}^2\right) + \Phi_1^1 \beta^2 M_{222} G_{111}^2 \qquad \text{(C14b)}$$

Note that, when $\beta = 0$, equation (C14a) yields the one-term approximation for Φ_1^1, that is,

$$\Phi = \Phi_{1,\,1\text{-term}}^1 = \frac{\delta k_0}{\gamma Q_0 M_{111} G_{111}^1} \qquad \text{(C15)}$$

Solving equations (C14) for Φ_1^1 and equating the results yield the following equation:

$$\beta^3\left(M_{122} G_{111}^2 F_{12}\right) + \beta^2\left[M_{112}\left(G_{111}^1 + G_{111}^2\right)F_{12} - M_{222} G_{111}^2 \delta k_0\right]$$
$$+ \beta\left[M_{111} G_{111}^1 F_{12} - M_{122}\left(G_{111}^1 + G_{111}^2\right)\delta k_0\right] - M_{112} G_{111}^1 \delta k_0 = 0 \qquad \text{(C16)}$$

where the common factor $(\gamma Q_0)^{-1}$ has been multiplied out. This cubic equation must now be solved for β. It can be shown by a straightforward calculation that, if $\delta k_0 << \left|P_L(\xi_1^2 - \xi_2^2)/B^2\right|$ and $\alpha \xi_1^2 \tau_a << 1$, then there is only one real root for β which is

$$\beta = 0.0005\left(1 + \frac{4}{3}\Psi\right) \qquad \text{(C17)}$$

where $\Psi = \tau_a \alpha(\xi_1^2 - \xi_2^2)/\pi^2 << 1$. Using equation (C17) in equation (C14a) gives

$$\Phi_{1,\,2\text{-term}}^1 = \frac{\delta k_0}{\gamma Q_0 M_{111} G_{111}^1} \frac{1}{1 + \beta \xi_1 \frac{M_{112}}{M_{111}} \left(1 + \frac{G_{111}^2}{G_{111}^1}\right)} \tag{C18}$$

where terms in β^2 have been neglected. Now

$$1 + \frac{G_{111}^2}{G_{111}^1} \simeq 2\left(1 - \frac{4}{3}\Psi\right)$$

and $M_{112}/M_{111} = 0.185\ (1/\xi_1)$ where $\xi_1 = J_1(\nu_1)/J_1(\nu_2)$. Thus

$$\Phi_{1,\,2\text{-term}}^1 = 0.9998 \frac{\delta k_0}{\gamma Q_0 M_{111} G_{111}^1} = 0.9998\ \Phi_{1,\,1\text{-term}}^1$$

and $\Phi_1^2 \leq 0.00049\ \Phi_{1,\,1\text{-term}}^1$. Thus, the one-term approximation is accurate to within 0.05 percent for Φ.

APPENDIX D

INITIAL AND STEADY-STATE FLUX AND TEMPERATURE DISTRIBUTIONS

The initial flux may be obtained from equation (7) with $\partial\varphi/\partial t = 0$ and $\delta k = 0$. Then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_0}{\partial r}\right) + \frac{\partial^2\varphi_0}{\partial z^2} + B_c^2\varphi_0 = 0 \tag{D1}$$

where $B_c^2 = (\nu\Sigma_f p_{th} g_{th} - \Sigma_a)/D$ is the physical buckling. The solution to the equation is

$$\varphi_0 = \Phi_0 \sin(\epsilon_1 z)J_0(\xi_1 r) \tag{D2}$$

which can be demonstrated by using equation (D2) in equation (D1). Here, Φ_0 is an amplitude which depends on the particular initial conditions in the reactor. Furthermore, $B_c^2 = B^2 = \epsilon_1^2 + \xi_1^2$ for this solution to hold.

The initial temperature distribution can be obtained from equation (10) with $\partial\theta/\partial t = 0$, that is,

$$\frac{\partial\theta_0}{\partial z} = \frac{Q_0}{v_0}\varphi_0 + \frac{\alpha}{v_0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta_0}{\partial r}\right) \tag{D3}$$

The boundary conditions are $\theta_0(r, 0) = 0$, $\theta_0(R, z) = 0$, and $\partial\theta_0/\partial r\,|_{r=0} = 0$. Then applying a finite Hankel transform (see appendix A) to equation (D3) results in

$$\frac{d\overline{\theta}_0}{dz} = \frac{Q_0\Phi_0}{v_0}\frac{R^2\left[J_1(\nu_i)\right]^2}{2}\delta_{i1}\sin\epsilon_1 z - \frac{\alpha\xi_i^2}{v_0}\overline{\theta}_0 \tag{D4}$$

where $\overline{\theta}_0(\xi_i, z)$ is the transformed temperature and equation (D2) is used for the flux. Since $\overline{\theta}_0(\xi_i, 0) = 0$ from the first boundary condition, the solution of equation (D4) is

$$\overline{\theta}_0(\xi_i, z) = e^{-\alpha\xi_i^2 z/v_0}\int_0^z \frac{Q_0\Phi_0}{v_0}\frac{R^2\left[J_1(\nu_i)\right]^2}{2}\delta_{i1}\sin\epsilon_1 y\, e^{\alpha\xi_i^2 y/v_0}\,dy \tag{D5}$$

Performing the integration in equation (D5) gives

$$\overline{\theta}_0(\xi_i, z) = \frac{Q_0 \Phi_0 R^2 \left[J_1(\nu_i)\right]^2 \delta_{i1}}{2\left[\left(\alpha \xi_i^2\right)^2 + (\pi/\tau_a)^2\right]} \left[\alpha \xi_i^2 \sin \epsilon_1 z - \frac{\pi}{\tau_a}\left(\cos \epsilon_1 z - e^{-\alpha \xi_i^2 z/v_0}\right)\right] \tag{D6}$$

Using the inversion formula finally yields

$$\theta_0(r, z) = \frac{2}{R^2} \sum_i \overline{\theta}_0(\xi_i, z) \frac{J_0(r \xi_i)}{\left[J_1(\nu_i)\right]^2}$$

where

$$\overline{\theta}_0(r, z) = \frac{Q_0 \Phi_0 J_0(r \xi_1)}{\left[\left(\alpha \xi_1^2\right)^2 + (\pi/\tau_a)^2\right]} \left[\alpha \xi_1^2 \sin \epsilon_1 z - \frac{\pi}{\tau_a}\left(\cos \epsilon_1 z - e^{-\alpha \xi_1^2 z/v_0}\right)\right] \tag{D7}$$

APPENDIX E

EVALUATION OF STABILITY CRITERION

The following integral will be investigated:

$$I = \frac{2}{\pi} \int_0^{\infty} S(x) \sin \Omega x \, dx \tag{E1}$$

where $S(x) = \dot{G}(x)/G(0)$ and $G(x) = \gamma M_{111}(\rho c_p)^{-1} K_{111}(v_0 x) e^{-\alpha \xi_1^2 x} u(\tau_a - x)$. Integrating by parts and using the fact that $G(\infty) = 0$ results in

$$I = \frac{2\Omega}{\pi G(0)} \int_0^{\infty} G(x) \cos \Omega x \, dx \tag{E2}$$

Using the definition of $G(x)$ gives

$$I = \frac{3\Omega}{4} \int_0^{\tau_a} K_{111}(v_0 x) e^{-\alpha \xi_1^2 x} \cos \Omega x \, dx \tag{E3}$$

Now when $\alpha \xi_1^2 \tau_a << 1$, the exponential term adds very little to the integral. For convenience, therefore, the exponential is excluded and

$$I \approx \frac{3\Omega}{4} \int_0^{\tau_a} K_{111}(v_0 x) \cos \Omega x \, dx \tag{E4}$$

Now

$$K_{111}(v_0 x) = \frac{1}{\pi} \left(1 + \frac{4}{3} \cos \frac{\pi x}{\tau_a} + \frac{1}{3} \cos \frac{2\pi x}{\tau_a}\right)$$

so that equation (E4) becomes

$$I = \frac{3}{\pi} \frac{\left(\frac{\pi}{\tau_a}\right)^4}{\left[\left(\frac{\pi}{\tau_a}\right)^2 - \Omega^2\right]\left[\left(\frac{2\pi}{\tau_a}\right)^2 - \Omega^2\right]} \sin \Omega \tau_a \tag{E5}$$

It follows immediately that $I \geq 0$ if $\Omega \leq 3\pi/\tau_a$. For values of $\Omega > 3\pi/\tau_a$, I oscillates with $\sin \Omega\tau_a$. Equation (E5) specifically precludes $\Omega = \pi/\tau_a$ or $\Omega = 2\pi/\tau_a$; when either of these values are used in equation (E4), the value of I is greater than zero.

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