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A Note on the First Passage Time Problem

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Preface

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Abstract

An analytical formulation of the first passage time problem for a linear single-degree-of-freedom vibratory system with a linear viscous damping, and subjected to either stationary or nonstationary white noise excitation, is obtained as a well posed initial-boundary value problem. The formulation is based on the Kolmogorov backward equation, rather than the Fokker-Planck equation. This note should provide a valuable insight to the problem and serve as an important foothold for the future study in which, it is hoped, an efficient numerical integration scheme can be found.

A Note on the First Passage Time Problem

I. Introduction

In the reliability analysis of engineering systems, it is often very important to obtain probabilistic information on the time T at which a random response process $X(t)$ of the system exits the safe domain of operation for the first time (first passage time). Although some approximate methods (Ref. 1) and bounding techniques have been developed (Refs. 2–4), even for most fundamental cases, such as a linear single-degree-of-freedom system with a linear viscous damping (for which the theory of the second-order Markov process can apply), the exact solution to this first passage time problem is not known at present.

In this note, an analytical formulation of the first passage time problem for a linear single-degree-of-freedom vibratory system with a linear viscous damping, subjected to either stationary or nonstationary white noise excitation, is obtained as a well posed initial-boundary value problem. This formulation is based on the Kolmogorov backward equation, rather than the Fokker-Planck equation (Ref. 5), since the latter (Kolmogorov forward formulation) presents an analytical ambiguity in specifying the boundary condition.

Because this point has never been noted and no consistent initial-boundary value formulation involving the first passage time of the system has been explicitly given, this note should provide a valuable insight to the problem and serve as an important foothold for the future study in which, it is hoped, an efficient method of numerical integration can be found, if not an analytical solution.

Consider a single-degree-of-freedom linear oscillator subject to the random excitation $g(t)$:

$$\ddot{X} + 2\zeta\omega\dot{X} + \omega^2 X = g(t) \tag{1}$$

where ζ and ω represent, respectively, the damping ratio and the undamped natural frequency of the oscillator. The excitation $g(t)$ can be a stationary white noise $n(t)$ or a nonstationary white noise $\psi(t)\,n(t)$ with $\psi(t)$ being a deterministic function of t.

Equation 1 is equivalent to a set of two equations

$$\dot{X} = Y$$

$$\dot{Y} + 2\zeta\omega Y + \omega^2 X = g(t) \tag{2}$$

It is well known (Ref. 5) that X and Y are jointly Markovian under the excitation $g(t)$ described above. In general, let

$$\mathbf{X}(t) = [X_1(t), X_2(t), \cdots, X_n(t)]$$

be an n-dimensional Markovian random vector process and

$$\mathbf{x} = [x_1, x_2, \cdots, x_n] \quad \text{and} \quad \overline{\mathbf{x}} = [\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_n]$$

be n-dimensional vectors. The transition probability density $f(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2)$ and the transition distribution function $F(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2)$ of the Markovian random vector process are by definition

$$\begin{aligned} f(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) &= F_{\overline{x}_1 \overline{x}_2 \ldots \overline{x}_n}(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) \\ F(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) &= P[\mathbf{X}(t_2) \leqq \overline{\mathbf{x}} | \mathbf{X}(t_1) = \mathbf{x}] \end{aligned} \tag{3}$$

where the subscript represents the partial differentiation and $P[E_1|E_2]$ is the conditional probability of the event E_1 given E_2.

It is shown (Refs. 6 and 7) that the transition probability density satisfies the generalized Kolmogorov backward equation

$$\begin{aligned} -f_{t_1}(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(\mathbf{x}, t_1) f_{x_i x_j}(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) \\ + \sum_{i=1}^{n} b_i(\mathbf{x}, t_1) f_{x_i}(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_2) \quad (t_2 > t_1) \end{aligned} \tag{4}$$

and

$$f(\mathbf{x}, t_1; \overline{\mathbf{x}}, t_1) = \prod_{i=1}^{n} \delta(x_i - \overline{x}_i) \tag{5}$$

where

$$\begin{aligned} a_{ij}(\mathbf{x}, t_1) &= \lim_{\Delta t_1 \to 0} \frac{1}{\Delta t_1} E[\Delta X_i(t_1) \Delta X_j(t_1) | \mathbf{X}(t_1) = \mathbf{x}] \\ b_i(\mathbf{x}, t_1) &= \lim_{\Delta t_1 \to 0} \frac{1}{\Delta t_1} E[\Delta X_i(t_1) | \mathbf{X}(t_1) = \mathbf{x}] \\ \Delta X_i(t_1) &= X_i(t_1 + \Delta t_1) - X_i(t_1) \end{aligned} \tag{6}$$

with E and $\delta(t)$ denoting the expectation and the Dirac delta function, respectively. Equation 5 simply states that no change of state may occur if the transition time is zero.

II. White Noise Excitation

A. Differential Equation and Initial Condition

Integrating Eq. 2 and using Eq. 6 with $X_1(t) = X(t)$ and $X_2(t) = Y(t)$ one can show that the Kolmogorov backward equation associated with the equation of motion (Eq. 2) is

$$\begin{aligned} f_\tau(x, y; \overline{x}, \overline{y}; \tau) = 2\zeta\omega^3\sigma^2 f_{yy}(x, y; \overline{x}, \overline{y}; \tau) \\ - (2\zeta\omega y + \omega^2 x) f_y(x, y; \overline{x}, \overline{y}; \tau) \\ + y f_x(x, y; \overline{x}, \overline{y}; \tau) \end{aligned} \tag{7}$$

where

$$\tau = t_2 - t_1$$

$$\sigma^2 = \pi s / 2\zeta\omega^3$$

and

$$\begin{aligned} f(x, y; \overline{x}, \overline{y}; \tau) &= f(x, y, t_1; \overline{x}, \overline{y}, t_2) \\ &= F_{\overline{x}\overline{y}}(x, y, t_1; \overline{x}, \overline{y}, t_2) \\ F(x, y; \overline{x}, \overline{y}; \tau) &= F(x, y, t_1; \overline{x}, \overline{y}, t_2) \\ &= P[X(t_2) \leqq \overline{x}, Y(t_2) \leqq \overline{y} | X(t_1) = x, Y(t_1) = y] \\ &= P[X(\tau) \leqq \overline{x}, Y(\tau) \leqq \overline{y} | X(0) = \overline{x}, Y(0) = \overline{y}] \end{aligned} \tag{8}$$

with s = the mean square spectral density function of $n(t)$.

Equation 5 reduces to

$$f(x, y; \overline{x}, \overline{y}; 0) = \delta(x - \overline{x}) \, \delta(y - \overline{y}) \tag{9}$$

B. Boundary Condition

For the two-sided barrier problem, where the safe domain D is defined by $-b < x < a$, $-\infty < y < \infty$, $(b > 0, a > 0)$, the boundary conditions are

$$\left.\begin{aligned} f(a, y \geqq 0; \overline{x}, \overline{y}; \tau) &= 0 \\ f(-b, y \leqq 0; \overline{x}, \overline{y}; \tau) &= 0 \\ f(x, y = \infty; \overline{x}, \overline{y}; \tau) &= 0 \\ f(x, y = -\infty; \overline{x}, \overline{y}; \tau) &= 0 \end{aligned} \qquad (\overline{x}, \overline{y}) \in D \right\} \tag{10}$$

It can be seen from Eq. 10 that $x = a$, $y \geqq 0$ and $x = -b$, $y \leqq 0$ are two mathematically, as well as physically meaningful, absorbing barriers in the phase plane. If either a or b is infinite, it is the one-side barrier problem.

C. Distribution Function of the First Passage Time T

The distribution function, $F(\tau; x, y)$, of the first passage time T given $X(0) = x$, $Y(0) = y$ is

$$F(\tau; x, y) = P[T \leqq \tau | X(0) = x, Y(0) = y]$$

$$= 1 - \int_{-b}^{a} \int_{-\infty}^{\infty} f(x, y; \bar{x}, \bar{y}; \tau)\, d\bar{y}\, d\bar{x} \tag{11}$$

In other words, $F(\tau; x, y)$ is the probability that the response process $X(t)$ exits the safe domain D before time τ, whereas it is at the state (x, y) at $t = 0$.

Integrating Eq. 7 and using Eq. 11, the governing differential equation is obtained. To obtain the initial and boundary conditions for the distribution function of the first passage time, Eqs. 9 and 10 are integrated and used with Eq. 11, as follows:

$$\left.\begin{aligned}
\frac{1}{2\pi} F_N(N; x, y) &= 2\zeta F_{yy}(N; x, y) \\
&\quad - (2\zeta y + x) F_y(N; x, y) \\
&\quad + y F_x(N; x, y) \qquad N > 0 \\
F(0; x, y) &= 0 \qquad (x, y) \in D \\
F(N; a/\sigma, y \geqq 0) &= 1 \\
F(N; -b/\sigma, y \leqq 0) &= 1 \\
F(N; x, \infty) &= 1 \\
F(N; x, -\infty) &= 1
\end{aligned}\right\} \tag{12}$$

where the following transformation has been made to nondimensionalize x, y and τ

$$x_1 = x/\sigma$$

$$y_1 = y/\sigma\omega \tag{13}$$

$N = \tau\omega/2\pi$ = number of cycles of undamped oscillation and x and y are written for x_1 and y_1 again. It is to be noted from Eq. 12 that when the barrier levels are measured in terms of σ, standard deviation of the response process $X(t)$, the distribution function $F(N; x, y)$ depends only on the damping ratio ζ for a given starting condition.

D. Stationary Starting Condition

If one is interested in the stationary starting condition for which there is an ensemble of values distributed according to the stationary response distribution, then the distribution function $F(N)$ of the first passage time N is

$$F(N) = \int_{-\infty}^{-b/\sigma} V(x)\, dx + \int_{a/\sigma}^{\infty} V(x)\, dx + \int_{-b/\sigma}^{a/\sigma} \int_{-\infty}^{\infty} F(N; x, y)\, V(x)\, V(y)\, dy\, dx \tag{14}$$

where

$$V(x) = (2\pi)^{-1/2} \exp(-x^2/2) \tag{15}$$

since it can be shown that $X(t)$ and $Y(t)$ are independent standardized gaussian processes under stationary starting conditions. The first two terms on the right hand side of Eq. 14 indicate the probability of initial failure.

III. Nonstationary Excitation

When the excitation $g(t)$ is a nonstationary white noise, $\psi(t)\, n(t)$, one can derive in a similar fashion the governing differential equation and specify the initial and boundary conditions for the first passage time. However, the transition probability density and the transition distribution function no longer depend on the time difference τ only, but on t_1 and t_2. The results are given as follows:

$$\left.\begin{aligned}
F_\tau(\tau; x, y, t_1) &= \pi S \psi^2 (t_2 - \tau) F_{yy}(\tau; x, y, t_1) \\
&\quad - (2\zeta\omega y + \omega^2 x) F_y(\tau; x, y, t_1) \\
&\quad + y F_x(\tau; x, y, t_1) \qquad \tau > 0 \\
F(0; x, y, t_1) &= 0 \qquad (x, y) \in D \\
F(\tau; a, y \geqq 0, t_1) &= 1 \\
F(\tau; -b, y \leqq 0, t_1) &= 1 \\
F(\tau; x, \infty, t_1) &= 1 \\
F(\tau; x, -\infty, t_1) &= 1
\end{aligned}\right\} \tag{16}$$

where

$$t_1 = t_2 - \tau$$

$$F(\tau; x, y, t_1) = P[T \leqq \tau | X(t_1) = x, Y(t_1) = y] \tag{17}$$

is the probability that the response process $X(t)$ exits the safe domain D before time τ has elapsed (or before time t_2 is reached) given $X(t_1) = x$, $Y(t_1) = y$ with $(x, y) \in D$.

IV. Conclusion

In conclusion, the differential equations in Eqs. 12 and 16 belong to a class known as the degenerate elliptic-parabolic equation, for which many results on existence, uniqueness, and regularity theory have been obtained (Refs. 8–10). It is suggested that effort be made to develop an efficient numerical scheme for the present problem.

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