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SOME NEW POWER SERIES INVERSIONS OF KEPLER'S EQUATION

E. R. LANCASTER

R. H. ESTES

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GODDARD SPACE FLIGHT CENTER

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LIST OF SYMBOLS

a	semi-major axis of orbit
e	eccentricity of orbit
E	eccentric anomaly of orbit
E_0	eccentric anomaly at time t_0
M	mean anomaly
r	magnitude of the orbital position vector
r_0	magnitude of the orbital position vector at time
$\dot{r}$	time derivative of r
$\dot{r}_0$	time derivative of r at time t_0
t	time
μ	gravitational constant times the mass of the attracting body
x	component of the position vector on an axis with origin at the primary body and positive direction toward pericenter
y	component of the position vector on an axis in the plane of motion 90° from the x-axis in the direction of motion

ABSTRACT

Recursion formulas are derived for the coefficients of three power series inverses of Kepler's equation:

1. a series in powers of time,
2. a series in powers of eccentricity,
3. a bivariate series in powers of two small parameters.

For each series a formula is given for the radius of convergence.

SOME NEW POWER SERIES INVERSIONS OF KEPLER'S EQUATION

THE CLASSICAL SERIES

Let E be defined as a function of e by Kepler's equation

$$E - e \sin E = M. \tag{1}$$

We wish to express E as a Maclaurin series. This is customarily done [1] by means of a Lagrange expansion. The procedure we follow is simpler and produces recursion formulas for the coefficients in the series. Moreover, the method simultaneously produces series for sin E and cos E.

Let

$$u = \sin E,$$

$$w = \cos E.$$

Then

$$u' = wE', \tag{2}$$

$$w' = -uE', \tag{3}$$

where the prime indicates differentiation with respect to e. From (1) we also have

$$E - e u = M. \tag{4}$$

Let

$$E = \sum_{i=0}^{\infty} a_i e^i, \tag{5}$$

$$u = \sum_{i=0}^{\infty} b_i \, e^i, \tag{6}$$

$$w = \sum_{i=0}^{\infty} c_i \, e^i. \tag{7}$$

Then

$$E' = \sum_{i=0}^{\infty} (i + 1) \, a_{i+1} \, e^i, \tag{8}$$

$$u' = \sum_{i=0}^{\infty} (i + 1) \, b_{i+1} \, e^i, \tag{9}$$

$$w' = \sum_{i=0}^{\infty} (i + 1) \, c_{i+1} \, e^i. \tag{10}$$

Substitute (5) through (10) in (2), (3), and (4) and equate coefficients of like powers of e to obtain

$$a_0 = M,$$

$$b_0 = \sin M,$$

$$c_0 = \cos M,$$

$$a_{i+1} = b_i, \; i \geq 0,$$

$$d_j = j \, a_j,$$

$$(i+1)\,b_{i+1} = \sum_{j=0}^{i} d_{j+1}\, c_{i-j}\,, \quad i \geq 0,$$

$$(i+1)\,c_{i+1} = \sum_{j=0}^{i} d_{j+1}\, b_{i-j}, \quad i \geq 0.$$

Having series for sin E and cos E, we can easily obtain series for r, x, and y from the following formulas.

$$r = a(1 - e \cos E),$$

$$x = a(\cos E - e),$$

$$y = (1 - e^2)^{1/2} \sin E.$$

Plummer [2] uses Lagrange's theorem to obtain the radius of convergence for these series. We have obtained the same result by the following simpler procedure.

The radius of convergence is equal to the smallest magnitude of the values of e associated with the singularities of the function E (e,M), which occur where E′ (e, M) is infinite. Since

$$E' = \frac{\sin E}{1 - e \cos E},$$

the singularities are determined by

$$1 - e \cos E = 0.$$

Substituting this in (1), we have

$$E - \tan E = M. \qquad (11)$$

Let $E = u + iv$ and $\overline{E} = u - iv$. Then

$$\tan E = \frac{\sin E \cos \overline{E}}{\sin E \cos \overline{E}} = \frac{\sin 2u + i \sinh 2v}{\cos 2u + \cosh 2v}. \tag{12}$$

Let $w = \frac{1}{2} |e|^2$. Then

$$w = \frac{1}{2} e \overline{e} = \frac{1}{\cos 2u + \cosh 2u}. \tag{13}$$

Substituting (12) in (11) and equating real and imaginary parts gives

$$u - w \sin 2u = M, \tag{14}$$

$$v - w \sinh 2v = 0. \tag{15}$$

Equations (13), (14), and (15) give u, v, and w as functions of M. We desire the values of M which minimize and maximize $|e|$ or, equivalently, w. We are thus led to set $dw/dM = 0$, obtaining

$$\frac{dv}{dM} \sinh 2v - \frac{du}{dM} \sin 2u = 0,$$

$$\frac{dv}{dM} (1 - 2w \cosh 2v) = 0,$$

$$\frac{du}{dM} (1 - 2w \cos 2u) = 1.$$

These equations imply $\sin 2u = 0$ or $\cos 2u = \pm 1$. Substituting this in (15), we obtain

$$v \cosh 2v = \sinh 2v \pm v,$$

which can be reduced to

$$\coth v = v \tag{16}$$

for the + sign and to

$$\tanh v = v \tag{17}$$

for the - sign. The solution of (17) is $v = 0$. The solution of (16) is approximately 1.20. Substituting these values in (13) we obtain $|e| = 1$ and $|e| = 0.663$ for the maximum and minimum radii of convergence.

A BIVARIATE SERIES

For M in equation (1) we have

$$M = (t - t_0)\,(\mu/a^3)^{1/2} + E_0 - e \sin E_0 .$$

Because E_0 in this expression is indeterminate for circular orbits, Kepler's equation is often written in the form [3]

$$\gamma - \alpha \sin \gamma + \beta\,(1 - \cos \gamma) = \tau, \tag{18}$$

where

$$\gamma = E - E_0,$$

$$\alpha = e \cos E_0 = 1 - \frac{r_0}{a},$$

$$\beta = e \sin E_0 = r_0\, \dot{r}_0/(\mu a)^{1/2},$$

$$\tau = (t - t_0)\,(\mu/a^3)^{1/2}.$$

Since α and β are small when e is small, it seems natural to seek a Maclaurin series in α and β.

Let

$$u = \sin \gamma,$$

$$w = \cos \gamma.$$

Then

$$\gamma - \alpha u - \beta w + \beta = \tau, \tag{19}$$

$$\frac{\partial u}{\partial \alpha} = w \frac{\partial \gamma}{\partial \alpha}, \tag{20}$$

$$\frac{\partial u}{\partial \beta} = w \frac{\partial \gamma}{\partial \beta}, \tag{21}$$

$$\frac{\partial w}{\partial \alpha} = - u \frac{\partial \gamma}{\partial \alpha}, \tag{22}$$

$$\frac{\partial w}{\partial \beta} = - u \frac{\partial \gamma}{\partial \beta}. \tag{23}$$

Let

$$\gamma = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \, \alpha^i \beta^j, \tag{24}$$

$$u = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} \, \alpha^i \beta^j, \tag{25}$$

$$w = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} \, \alpha^i \beta^j. \tag{26}$$

Substituting (24) through (26) into (19) through (23), using the formula derived in the appendix, and equating coefficients of like powers, we obtain the following recursion formulas.

$$a_{00} = \tau, \; b_{00} = \sin \tau, \; c_{00} = \cos \tau$$

$$a_{10} = b_{00}, \; a_{01} = c_{00} - 1$$

$$a_{i+1,0} = b_{i0}, \; a_{0,j+1} = c_{0j}$$

$$a_{i+1,j} = b_{ij} + c_{i+1,j-1}$$

$$a_{i,j+1} = b_{i-1,j+1} + c_{ij}$$

$$p_{mk} = (m + 1) \, a_{m+1,k}$$

$$q_{mk} = (k + 1) \, a_{m,k+1}$$

$$(i + 1) \, b_{i+1,j} = \sum_{k=0}^{j} \sum_{m=0}^{i} p_{mk} \, c_{i-m,j-k}$$

$$(j + 1) \, b_{i,j+1} = \sum_{k=0}^{j} \sum_{m=0}^{i} q_{mk} \, c_{i-m,j-k}$$

$$(i + 1) \, c_{i+1,j} = - \sum_{k=0}^{j} \sum_{m=0}^{i} p_{mk} \, b_{i-m,j-k}$$

$$(j + 1) \, c_{i,j+1} = - \sum_{k=0}^{j} \sum_{m=0}^{i} q_{mk} \, b_{i-m,j-k}$$

These series converge for

$$\alpha^2 + \beta^2 < 2w,$$

where w is found as for the classical series. When computing with these series the coefficients are found in sets such that for the nth set the sum of the subscripts on each coefficient is n.

A TIME SERIES INVERSION

If we consider γ as a function of τ in equation (18), we can again obtain a Maclaurin series inverse. Letting $u = \sin\gamma$ and $w = \cos\gamma$ as before, we have

$$\gamma - \alpha u - \beta w + \beta = \tau, \tag{27}$$

$$\dot{u} = w\,\dot{\gamma}, \tag{28}$$

$$\dot{w} = -u\,\dot{\gamma}, \tag{29}$$

where the dot indicates differentiation with respect to τ.

Let

$$\gamma = \sum_{i=0}^{\infty} a_i\,\tau^i, \tag{30}$$

$$u = \sum_{i=0}^{\infty} b_i\,\tau^i, \tag{31}$$

$$w = \sum_{i=0}^{\infty} c_i\,\tau^i. \tag{32}$$

Substituting equations (30) through (32) in equations (27) through (29) and equating coefficients of like powers of τ yields

$$c_0 = 1,\ b_0 = a_0 = c_1 = 0,$$

$$d = a/r_0,\ b_1 = a_1 = d$$

$$c_2 = -\frac{1}{2} a_1 b_1,\ b_2 = \beta\, d c_2,\ a_2 = b_2$$

$$\gamma_j = j\ a_j$$

$$c_{i+1} = -\left(\sum_{j=1}^{i} \gamma_j\, b_{i-j+1}\right)\Big/(i+1)$$

$$b_{i+1} = d\left[\beta c_{i+1} + \left(\sum_{j=1}^{i-1} \gamma_j\, c_{i-j+1}\right)\Big/(i+1)\right]$$

$$a_{i+1} = \alpha\, b_{i+1} + \beta\, c_{i+1}$$

It is not difficult to show that the radius of convergence for these series is

$$\tau = \left[\left(\arctan\frac{\beta}{\alpha} - \beta\right)^2 + \left(\operatorname{sech}^{-1} e - \sqrt{1 - e^2}\right)^2\right]^{1/2},$$

where

$$-\pi < \arctan\frac{\beta}{\alpha} \leq \pi.$$

COMPUTATIONAL CONSIDERATIONS

If one desires merely to compute updated values of E or γ and associated quantities, the following alterations in the algorithms reduce the amount of computation required.

For the classical series replace the equation $a_{i+1} = b_i,\ i \geq 0$ by

$$a_{i+1} = e\, b_i\,,\ i \geq 0$$

so that the resulting series then become

$$E = \sum_{i=0}^{\infty} a_i\,,$$

$$\sin E = \sum_{i=0}^{\infty} b_i ,$$

$$\cos E = \sum_{i=0}^{\infty} c_i .$$

For the bivariate series use

$$a_{10} = \alpha\, b_{00},\ a_{01} = \beta\,(c_{00} - 1)$$

$$a_{i+1,0} = \alpha\, b_{i0},\ a_{0,j+1} = \beta\, c_{0j}$$

$$a_{i+1,j} = \alpha\, b_{ij} + \beta\, c_{i+1,j-1}$$

$$a_{i,j+1} = \alpha\, b_{i-1,j+1} + \beta\, c_{ij}$$

The series then become

$$\gamma = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} ,$$

$$\sin \gamma = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} ,$$

$$\cos \gamma = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} .$$

Finally, for the time series we make only the change

$$b_1 = a_1 = \tau\, d,$$

and the series become

$$\gamma = \sum_{i=0}^{\infty} a_i \,,$$

$$\sin \gamma = \sum_{i=0}^{\infty} b_i \,,$$

$$\cos \gamma = \sum_{i=0}^{\infty} c_i \,.$$

APPENDIX

BIVARIATE CAUCHY PRODUCTS

The Cauchy product of the series

$$S_1 = \sum_{i=0}^{\infty} a_i z^i$$

and

$$S_2 = \sum_{i=0}^{\infty} b_i z^i$$

is defined as the series

$$S = \sum_{i=0}^{\infty} c_i z^i$$

where

$$c_i = \sum_{j=0}^{i} a_j b_{i-j} .$$

We extend this definition in a natural way to the product of two bivariate series. Let

$$S_1 = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_{ij} \alpha^i \beta^j = \sum_{j=0}^{\infty} A_j B^j , \ A_j = \sum_{i=0}^{\infty} a_{ij} \alpha^j ,$$

$$S_2 = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} b_{ij}\, \alpha^i \beta^j = \sum_{j=0}^{\infty} B_j \beta^j, \quad B_j = \sum_{i=0}^{\infty} b_{ij}\, \alpha^i .$$

Applying the above definition of a Cauchy product to these two series gives

$$S = S_1 S_2 = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} A_k B_{j-k} \right) \beta^j ,$$

$$A_k = \sum_{i=0}^{\infty} a_{ik}\, \alpha^i , \quad B_{j-k} = \sum_{i=0}^{\infty} b_{i,\, j-k}\, \alpha^i ,$$

$$A_k B_{j-k} = \sum_{i=0}^{\infty} \left(\sum_{m=0}^{i} a_{mk}\, b_{i-m,\, j-k} \right) \alpha^i ,$$

$$S = S_1 S_2 = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} c_{ij}\, \alpha^i \beta^j ,$$

$$c_{ij} = \sum_{k=0}^{j} \sum_{m=0}^{i} a_{mk}\, b_{i-m,\, j-k} .$$

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