RADIATIVE TRANSFER BY DOUBLING VERY THIN LAYERS

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ABSTRACT

A doubling principle first used by van de Hulst has been developed to provide rapid and accurate results for the problem of diffuse reflection from a plane parallel atmosphere. The method described here eliminates the need of numerically solving an equation of radiative transfer by beginning with a layer of such small optical thickness ($\tau \sim 2^{-25}$) that the initial scattering and transmission functions are given by the phase function. An application is made to spectral scattering by clouds in the near-infrared where the phase function is strongly peaked in the forward direction.

I. INTRODUCTION

The problem of radiative transfer in plane parallel atmospheres has been solved "exactly" only for very simple phase functions so that a numerical approach is required for most practical applications. van de Hulst was the first to note that considerable computing time could be saved if the transfer equation was solved only for a layer of small thickness τ_0 and this solution was then used to obtain solutions for layers of thickness $2\tau_0$, $4\tau_0$, etc. by a "doubling" procedure. Numerical verification of the efficacy of this method was made by van de Hulst and Grossman (1968) who used the Neumann series method (iteration in orders of scattering) to obtain solutions at τ_0. Then, after several doublings, they fit their numbers to asymptotic ($\tau \to \infty$) results.

In this paper we describe a computing method which gives fast and accurate results for all optical thicknesses. We begin with a layer of such small optical thickness that the scattering and transmission functions are given to high accuracy by the phase function for single scattering. This accuracy allows the doublings to be carried to large optical thicknesses and asymptotic solutions are not required. With this method it is not necessary to solve the usual transfer equation and the computing program is applicable to all phase functions.

II. DOUBLING EQUATIONS

In the "standard"problem of diffuse reflection from a planetary atmosphere considered by Chandrasekhar (1960) a parallel beam of radiation of net flux πF per unit area normal to itself is incident from the direction $(-\mu_0, \phi_0)$ on a plane parallel atmosphere of optical thickness τ. Now if the scattering function $S(\tau;\mu,\phi;\mu_0,\phi_0)$ and the transmission function $T(\tau;\mu,\phi;\mu_0,\phi_0)$ (defined by eq. [120], p.20, Chandrasekhar 1960) are known then $S(2\tau;\mu,\phi;\mu_0,\phi_0)$ and $T(2\tau;\mu,\phi;\mu_0,\phi_0)$ may be found by adding the ways in which diffuse radiation may escape from the double layer:

$$\frac{S(2\tau;\mu,\phi;\mu_0,\phi_0)F}{4\mu} = \frac{S(\tau;\mu,\phi;\mu_0,\phi_0)F}{4\mu} + e^{-\tau/\mu}\,\frac{1}{\mu}\Sigma_0(\tau;\mu,\phi;\mu_0,\phi_0)F\,e^{-\tau/\mu_0}$$

$$+ \frac{1}{4\pi\mu}\int_0^1\int_0^{2\pi} T(\tau;\mu,\phi;\mu',\phi')\,\frac{1}{4\mu'}\,\Sigma_0(\tau;\mu',\phi';\mu_0,\phi_0)d\mu' d\phi' F e^{-\tau/\mu_0} \tag{1}$$

$$+ e^{-\tau/\mu} \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} \Sigma_o(\tau;\mu,\phi;\mu',\phi') \frac{T(\tau;\mu',\phi';\mu_o,\phi_o)}{4\mu'} d\mu' d\phi' \ F$$

$$+ \frac{1}{4\pi\mu} \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} T(\tau;\mu,\phi;\mu'',\phi'') \frac{1}{4\pi\mu''} \Sigma_o(\tau;\mu'',\phi'';\mu',\phi')$$

$$\frac{T(\tau;\mu',\phi';\mu_o,\phi_o)}{4\mu'} d\mu' d\phi' d\mu'' d\phi'' \ F$$

where

$$\Sigma_o(\tau;\mu,\phi;\mu_o,\phi_o) \equiv \sum_{n=1,3,\ldots}^{\infty} S_n(\tau;\mu,\phi;\mu_o,\phi_o), \tag{2}$$

$$S_1(\tau;\mu,\phi;\mu_o,\phi_o) \equiv S(\tau;\mu,\phi;\mu_o,\phi_o), \tag{3}$$

and

$$S_n(\tau;\mu,\phi;\mu_o,\phi_o) \equiv \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} S(\tau;\mu,\phi;\mu',\phi') S_{n-1}(\tau;\mu',\phi';\mu_o,\phi_o) \frac{d\mu'}{\mu'} d\phi'. \tag{4}$$

The factor $F/4\mu$ is included in equation (1) to make clear the physical basis for that equation. The first term on the right side is the intensity of radiation diffusely scattered from the upper layer without interaction with the lower layer. The remaining terms are the radiation transmitted downward by the upper layer (diffusely or without interaction), scattered any number of times back and forth between the two layers, and finally transmitted upward by the upper layer. Similarly an equation for $T(2\tau;\mu,\phi;\mu_o,\phi_o)$ is

$$T(2\tau;\mu,\phi;\mu_o,\phi_o) = T(\tau;\mu,\phi;\mu_o,\phi_o)\, e^{-\tau/\mu_o} + e^{-\tau/\mu} T(\tau;\mu,\phi;\mu_o,\phi_o)$$

$$+ e^{-\tau/\mu}\, \Sigma_\varepsilon(\tau;\mu,\phi;\mu_o,\phi_o)\, e^{-\tau/\mu_o}$$

$$+ \frac{1}{4\pi} \int_0^1\!\!\int_0^{2\pi} T(\tau;\mu,\phi;\mu',\phi')\, T(\tau;\mu',\phi';\mu_o,\phi_o)\, \frac{d\mu'}{\mu'}\, d\phi' \qquad (5)$$

$$+ e^{-\tau/\mu_o}\, \frac{1}{4\pi} \int_0^1\!\!\int_0^{2\pi} T(\tau;\mu,\phi;\mu',\phi')\, \Sigma_\varepsilon(\tau;\mu',\phi';\mu_o,\phi_o)\, \frac{d\mu'}{\mu'}\, d\phi'$$

$$+ e^{-\tau/\mu}\, \frac{1}{4\pi} \int_0^1\!\!\int_0^{2\pi} \Sigma_\varepsilon(\tau;\mu,\phi;\mu',\phi')\, T(\tau;\mu',\phi';\mu_o,\phi_o)\, \frac{d\mu'}{\mu'}\, d\phi'$$

$$+ \frac{1}{16\pi^2} \int_0^1\!\!\int_0^{2\pi}\!\!\int_0^1\!\!\int_0^{2\pi} T(\tau;\mu,\phi;\mu'',\phi'')\, \Sigma_\varepsilon(\tau;\mu'',\phi'';\mu',\phi')$$

$$T(\tau;\mu',\phi';\mu_o,\phi_o)\, \frac{d\mu'}{\mu'}\, d\phi'\, \frac{d\mu''}{\mu''}\, d\phi''$$

where

$$\Sigma_\varepsilon(\tau;\mu,\phi;\mu_o,\phi_o) \equiv \sum_{n=2,4\ldots}^{\infty} S_n(\tau;\mu,\phi;\mu_o,\phi_o). \qquad (6)$$

The above equations are equivalent to those of van de Hulst (1963). We have further derived equations which include the effect of a reflecting planetary surface and internal sources (e.g., thermal emission) but we will omit these since they easily follow from the doubling principle.

III. INITIAL LAYER FOR COMPUTATIONS

The above equations provide the scattering and transmission functions for layers of increasing thickness provided those functions are known for any initial layer. However, accurate solutions of the radiative transfer equation are often difficult to obtain and inaccuracies can lead to an increased error after many doublings. But for very thin layers the radiation singly scattered is an approximate solution which is known for any given phase function. For $\tau << 1$ the ratio of multiply scattered radiation to singly scattered radiation is $\approx \tau$ and hence the relative error in S or T is only about 1 part in 10^7 at an initial optical thickness $\tau_0 = 2^{-25}$. Moreover, the above equations suggest that the relative error will not increase much during the doublings until τ approaches unity (since for $\tau << 1$, S and T are $<< 1$) and the numerical computations verify this. Consequently, if the computations are started at very small τ_0, then when τ reaches values ~ 1 the scattering and transmission functions are sufficiently accurate for the doubling to be carried to very thick layers.

The well known expression for the intensity of singly scattered radiation (Chandrasekhar 1960) gives us the following initial functions

$$S(\tau_0;\mu,\phi;\mu_0,\phi_0) \approx \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)^{-1}\left\{1-\exp\left[-\tau_0\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right]\right\}P(\mu,\phi;-\mu_0,\phi_0) \qquad (7)$$

$$T(\tau_o;\mu,\phi;\mu_o,\phi_o) = \left(\frac{1}{\mu} - \frac{1}{\mu_o}\right)^{-1}\left\{\exp(-\tau_o/\mu_o)-\exp(-\tau_o/\mu)\right\}P(-\mu,\phi;-\mu_o,\phi_o) \quad (8)$$

where $P(\mu,\phi;\mu_o,\phi_o)$ is the phase function normalized to ω_o, the albedo for single scattering. These equations provide the initial values for S and T but since the terms in brackets involve the differences of nearly equal numbers some of the accuracy would be lost on a computer; hence it is better to expand the exponentials in power series and keep terms through τ_o^2.

The sums (eq.[2] and eq.[6]) may be truncated at some $n=N\sim5$ with the omitted terms replaced by the geometric formula. This is a result of the fact that radiation scattered back and forth between the two layers tends toward an isotropic distribution after a number of such scatterings and hence the ratio of successive terms in the infinite series approaches a constant value (a conclusion already reached by van de Hulst (1963) for thick layers).

Considerable advantage is gained in computing time if the azimuth dependent functions are expanded in Fourier series in $\phi-\phi_o$. If the phase function may be expanded in cosines of the scattering angle then

$$S(\tau;\mu,\phi;\mu_o,\phi_o) = \sum_{m=0}^{\infty} S^m(\tau;\mu,\mu_o)\cos m(\phi-\phi_o) \quad (9)$$

where

$$S^m(\tau;\mu,\mu_o) = \frac{1}{(2-\delta_{0,m})}\int_0^{2\pi} S(\tau;\mu,\phi';\mu_o,\phi_o)\cos m(\phi'-\phi_o)\,d\phi' \tag{10}$$

and

$$\delta_{0,m} = 0 \text{ if } m \neq 0$$

$$= 1 \text{ if } m = 0$$

and similar expressions exist for T. From these it is easily verified that each component doubles independently, i.e., each pair S^m and T^m satisfies equations like (1) and (5) but the terms in the infinite series are given by

$$S_1^m(\tau;\mu,\mu_o) = S^m(\tau;\mu,\mu_o) \tag{11}$$

and

$$S_n^m(\tau;\mu,\mu_o) = \frac{1}{(4-2\delta_{0,m})}\int_0^1 S^m(\tau;\mu,\mu')\, S_{n-1}^m(\tau;\mu',\mu_o)\,\frac{d\mu'}{\mu'}\,. \tag{12}$$

The computational method is straightforward and efficient since it consists of evaluating sums and integrals which may be replaced by sums through Gauss quadrature. The same program is applicable to any phase function but for elongated functions an increased number of Gauss divisions must be used to maintain the accuracy. For finite τ the results check exactly to the 6 digit accuracy available for a few simple phase functions and the same is true when the results for $\tau = 2^{15}$ are compared to the "exact" values for $\tau = \infty$. In the azimuth dependent case with elongated phase functions the results check to the 4 digits of accuracy which is estimated for calculations by Hansen (1967), who used the invariant imbedding approach, and for calculations by Potter (1968), who used the Neumann series method.

Except for strongly peaked phase functions the computing time is negligible. For aerosol phase functions varying by 3 orders of magnitude from their peak value to their low value (requiring ~ 50 terms in the cos $m\phi$ expansion) the total computing time is 3 minutes on the IBM 360/95 for the scattering and transmission functions (and derived quantities) for every (not each) τ multiple of 2 from 2^{-25} to 2^{7} for 13 values of μ, 13 values of μ_0, and any reasonable number of values of $\phi-\phi_0$.

IV. REFLECTIVITY OF CLOUDS

To illustrate the utility of the computing method the diffuse reflection from ice clouds in the near-infrared has been computed. The phase functions including the albedo for single scattering, ω_0, were kindly computed by H. Cheyney from Mie scattering theory for spherical particles. The "cloud model" size distribution of particles (Deirmendjian, 1964), which has its maximum at diameter 8μ, was used and the optical constants were taken from Irvine and

Pollack (1968). Although natural ice crystals are highly nonspherical, it is often assumed that randomly oriented nonspherical particles may be approximated by spheres. The results for water clouds would be qualitatively the same as those for ice since the optical constants of water and ice are similar in the near-infrared; however, the corresponding features would be shifted slightly ($\lesssim .1\mu$) in wavelength and in absolute value.

The calculations were made at 20 wavelengths between $.95\mu$ and 3.6μ including each wavelength at which the tables of Irvine and Pollack show the absorbtivity to be an extremum. The computed values of $\omega_0(\lambda)$ are given in Table 1 along with the relative optical thickness of the cloud as a function of wavelength. Representative phase functions shown in Figure 1 illustrate the increasingly sharp forward scattering toward shorter wavelengths and the damping of backscattering at wavelengths ($\sim 3\mu$) where the absorption is large.

The spherical albedo as a function of wavelength is shown in Figure 2 for several optical thicknesses of the cloud. Although the scattering is not far from being conservative ($\bar{\omega}_0 \gtrsim .95$) for $\lambda < 2.7\mu$, it is obvious from the albedos for increasingly thick layers that multiple scattering greatly enhances the "absorption" features near 1.5 and 2.0μ; the minor features at 1.03 and 2.6μ, however, are just visible for thick layers. Since $\omega_0(\lambda)$ depends strongly on particle size it will be worthwhile to make calculations for other particle size distributions and to compare the results to measurements of atmospheric and laboratory clouds.

Angular distributions of the scattered light are shown in Figures 3-8 for the phase function for $\lambda=.95\mu$. In the calculations for those figures the scattering was assumed to be conservative ($\omega_o=1$); hence the results are approximately valid for clouds of either water or ice spheres for wavelengths from 4000Å to 10,000Å because the absorbtivity of both materials is negligible in that region and the real refractive indices are within the small range $1.30 \lesssim n_r \lesssim 1.34$. The effects of Rayleigh scattering and molecular absorption are not included. For wavelengths other than $.95\mu$ the results apply to a size distribution peaking at a diameter d such that $d/\lambda = 8/.95$. In the calculations for Figures 3-8 140 terms were employed in the cosine expansions and 40 μ divisions on the interval (0,1); the number of divisions was varied and the results suggest that the errors are $\lesssim$ 1% for all angles. The reflection function which has been graphed is defined by

$$R(\tau;\mu,\phi;\mu_o,\phi_o) = \frac{S(\tau;\mu,\phi;\mu_o,\phi_o)}{4\mu} . \tag{13}$$

Reflectivities are shown for thin ($\tau=1$), intermediate ($\tau=4$) and very thick ($\tau=128$) clouds for which cases the spherical albedos are 13.3, 34.0 and 93.5%, respectively. Thus the diffuse transmission is significant even for extremely dense clouds; this is due to the fact that the scattering is conservative and peaked in the forward direction.

For conservative scattering it is often assumed that thick clouds may be approximated by a Lambert reflector ($R(\tau;\mu,\phi;\mu_0,\phi_0) \propto \mu_0$) in which case the reflectivities would be horizontal lines in Figures 3-8. The graphs confirm a trend toward Lambertian reflection but Figures 3-5 indicate that even for τ=128 there is still a nonnegligible dependence of the reflectivity on μ, especially when the incident direction is near grazing ($\cos^{-1}\mu_0 \sim 90^\circ$). Figures 6-8 verify that much of the azimuth dependence of the reflectivity disappears for thick layers but this also does not hold for large incident angles.

In Figures 6-8 the intensity of light singly scattered from the cloud is also shown; these curves are only given for the layer of thickness τ=1 since the increase for thicker layers is small. Except for thin layers and near grazing incident and emergent angles it is apparent that the light singly scattered represents only a small fraction of the total reflected radiation, and it is for this reason that the features of the phase function are much diluted for thick layers.

The results of this section suggest that it may be possible to obtain a certain amount of information about the phase function of cloud particles from measurements of the angular distribution of reflected visual light, but since measurements near the grazing direction are difficult that approach may not be very practical except for thin clouds. However, for thick clouds the spectral features in the near infrared become strong and hence that wavelength region appears especially promising for cloud identification.

I am grateful to Dr. R. Jastrow, director of The Institute for Space Studies, for his hospitality. I would like to thank Professor S. Ueno who suggested the possibility of adding the intensities from the different layers in a multilayered medium; Professor Ueno pointed out an article by Hawkins (1961) and after finding Hawkins' equations to be incomplete I derived equations (1) and (5) which are equivalent to those of van de Hulst. I would like to thank H. Cheyney for supplying the phase functions, Professor van de Hulst for correspondence, and Drs. A. Arking, K. Grossman, J. Potter and R. Samuelson for discussions. This research was completed while I held an NRC-NASA Resident Research Associateship at the Institute for Space Studies.

TABLE 1

SINGLE PARTICLE ALBEDO AND RELATIVE CLOUD OPTICAL THICKNESS

λ	$\omega_o(\lambda)$	$\tau(\lambda)$	λ	$\omega_o(\lambda)$	$\tau(\lambda)$
.95	.99995	1.0000	2.30	.99424	1.0828
1.03	.99986	1.0057	2.50	.98153	1.1490
1.10	.99994	1.0096	2.55	.98172	1.1883
1.30	.99929	1.0223	2.60	.98402	1.2333
1.52	.97126	1.0350	2.70	.95297	1.3117
1.70	.99201	1.0445	2.80	.85250	1.2866
1.85	.99754	1.0520	3.00	.46917	.9704
1.90	.98574	1.0545	3.20	.49668	1.0919
2.00	.94775	1.0597	3.40	.64311	1.1132
2.15	.97887	1.0688	3.60	.83445	1.1477

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Figure Captions

Fig. 1. Phase functions normalized to unity at four representative wavelengths in the near infrared for a cloud of ice spheres. The spheres follow the "cloud model" size distribution which has its maximum at diameter 8μ. The inset on the right is an enlargement of the phase functions near the forward peak.

Fig. 2. Spherical albedo in the near infrared of a plane parallel cloud of ice spheres with the "cloud model" size distribution of particles.

Fig. 3. Reflection function versus $\theta = \cos^{-1}\mu$ for $\phi-\phi_o = 0^\circ$. The results are for the sharply peaked phase function typical of some clouds in the range 4000Å-10,000Å (the curve labeled λ=.95 in Fig. 1).

Fig. 4. The same as Figure 3 but for $\phi-\phi_o = 60^\circ$

Fig. 5. The same as Figure 3 but for $\phi-\phi_o = 180^\circ$

Fig. 6. Reflection function versus azimuth angle $\phi-\phi_o$ for $\theta_o \equiv \cos^{-1}\mu_o = 85^\circ$. The results are for the peaked phase function labeled $\lambda = .95\mu$ in Figure 1. The single scattering curves are for an optical thickness τ=1 but they are also approximately valid for thicker layers.

Fig. 7. The same as Figure 6 but for $\theta_o = 60^\circ$

Fig. 8. The same as Figure 6 but for $\theta_o = 30^\circ$

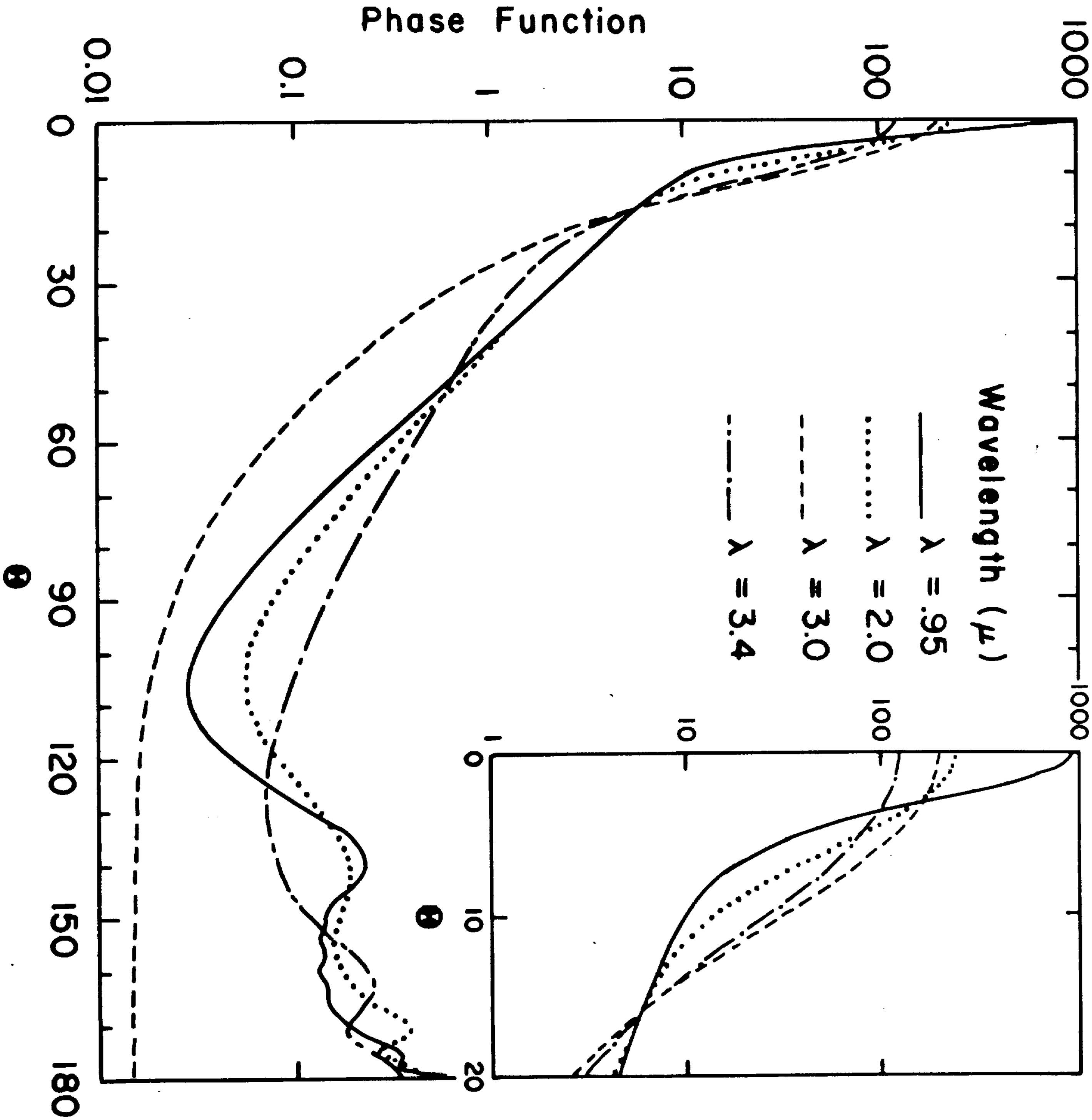
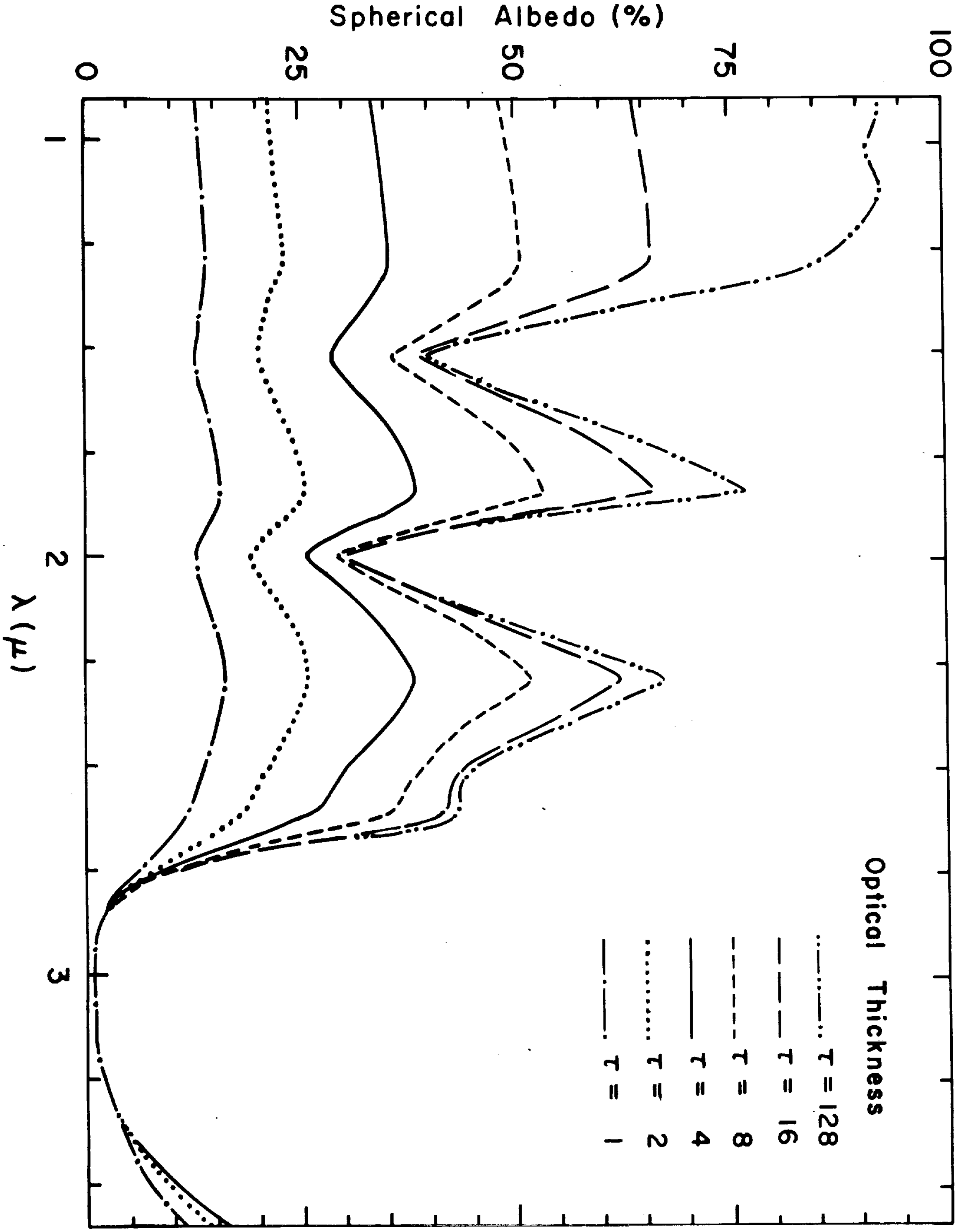
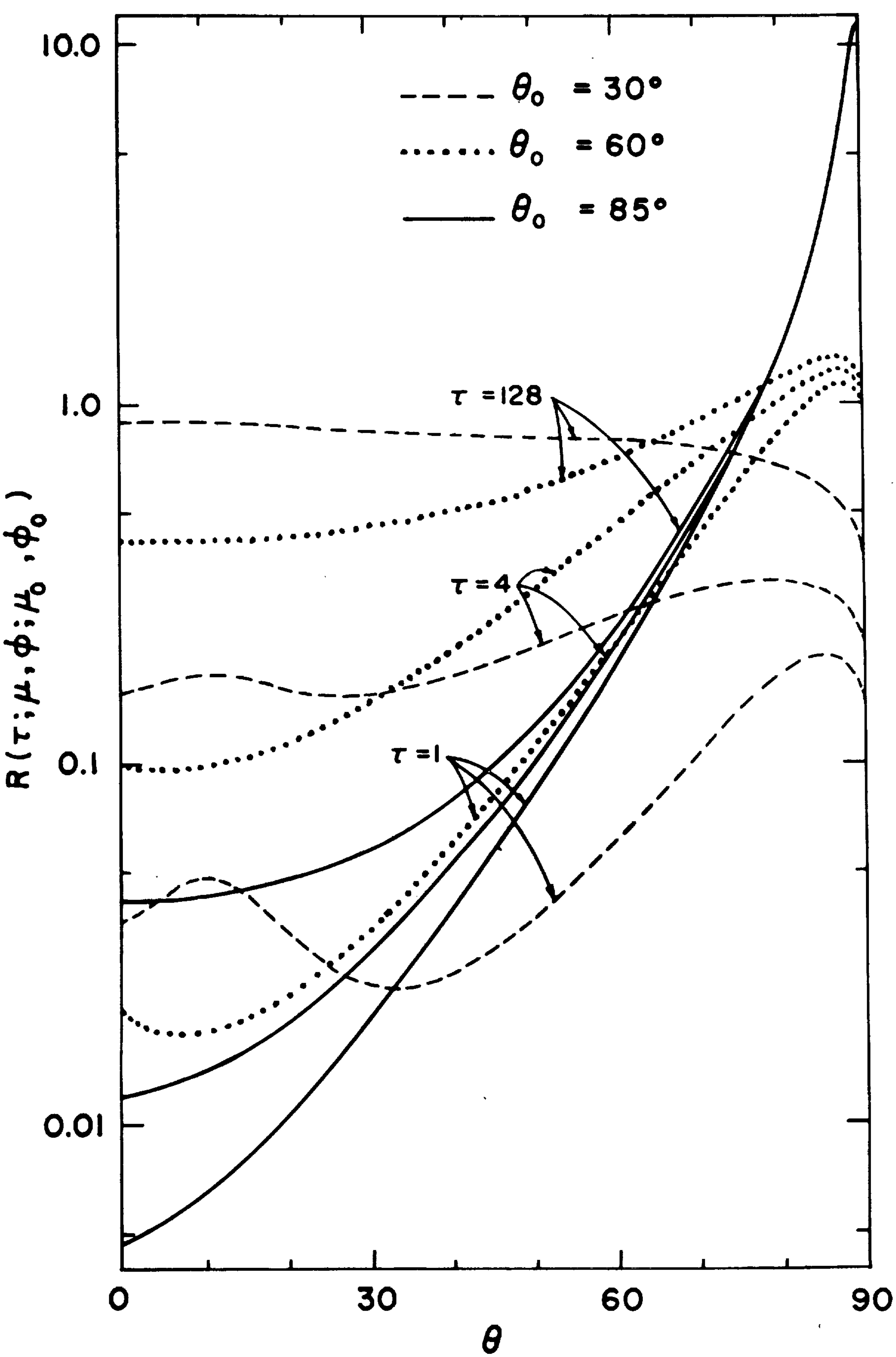

Phase Function
Wavelength (μ)
λ = .95
λ = 2.0
λ = 3.0
λ = 3.4
Θ
1000
100
10
1
0.1
0.01
0
30
60
90
120
150
180
20

Fig. 1



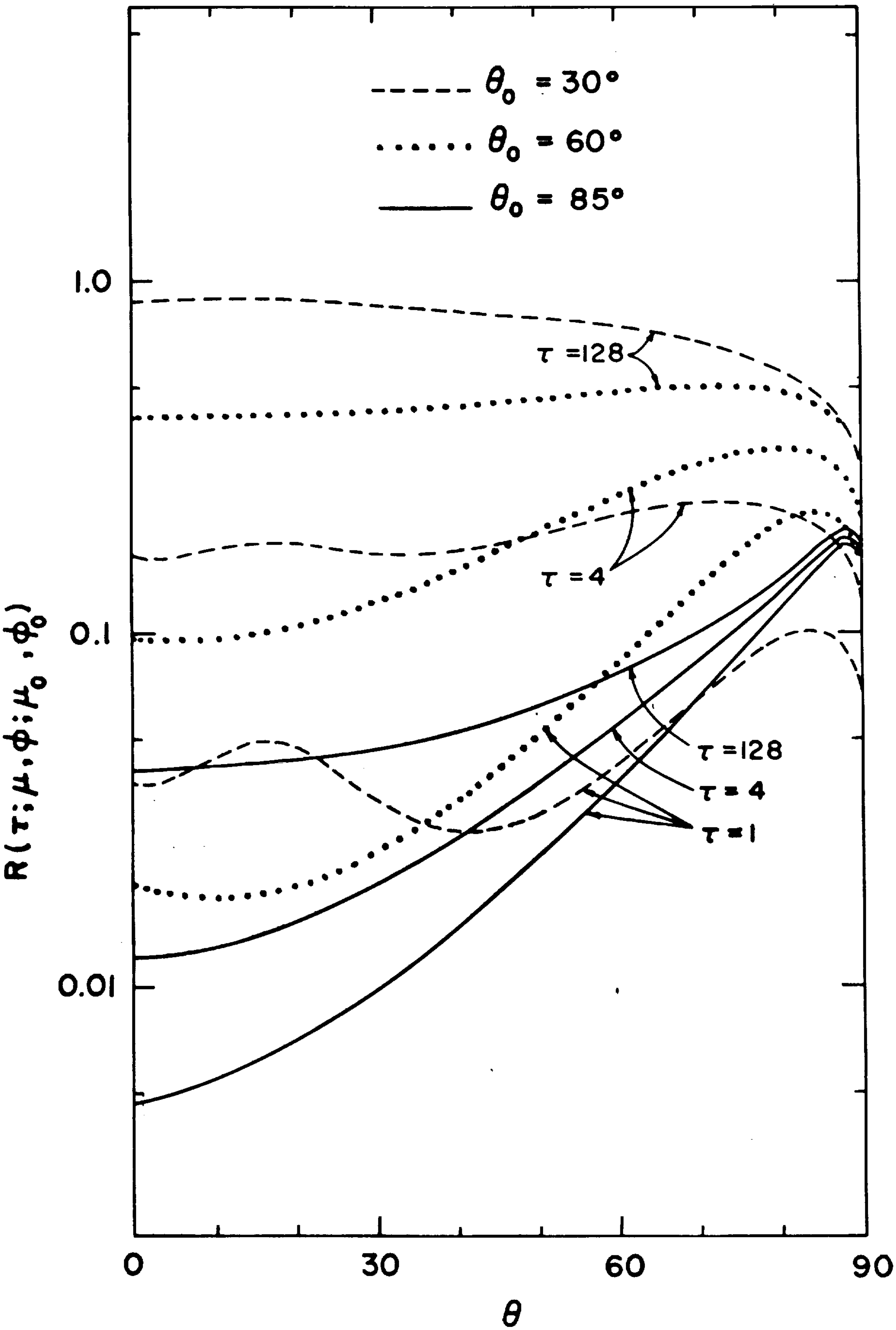
Spherical Albedo (%)
0
25
50
75
100
1
2
3
λ (μ)
Optical Thickness
τ = 128
τ = 16
τ = 8
τ = 4
τ = 2
τ = 1

Fig. 2



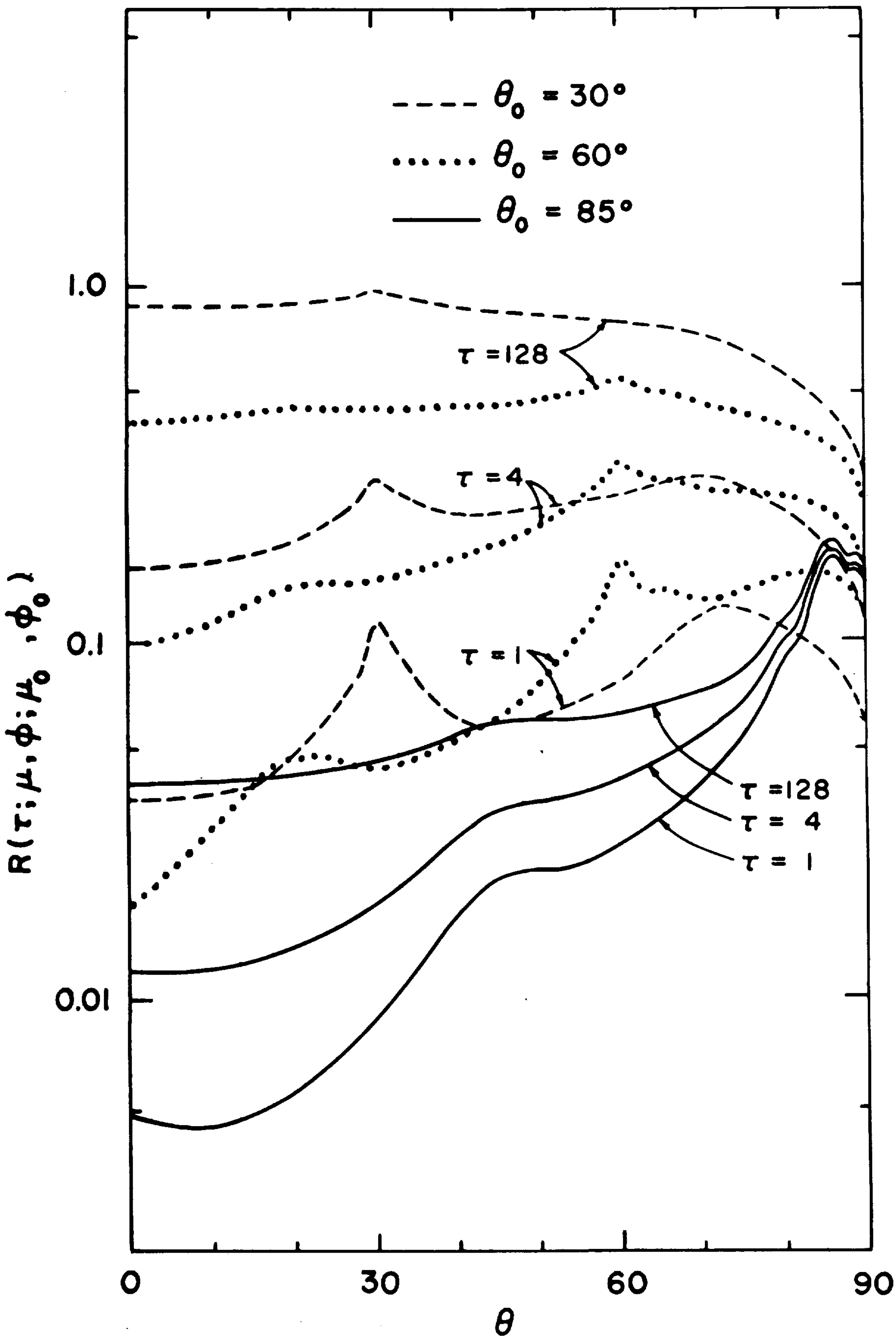
$\theta_0 = 30°$
$\theta_0 = 60°$
$\theta_0 = 85°$
$\tau = 128$
$\tau = 4$
$\tau = 1$
$R(\tau;\mu,\phi;\mu_0,\phi_0)$
10.0
1.0
0.1
0.01
0
30
60
90
θ

Fig. 3



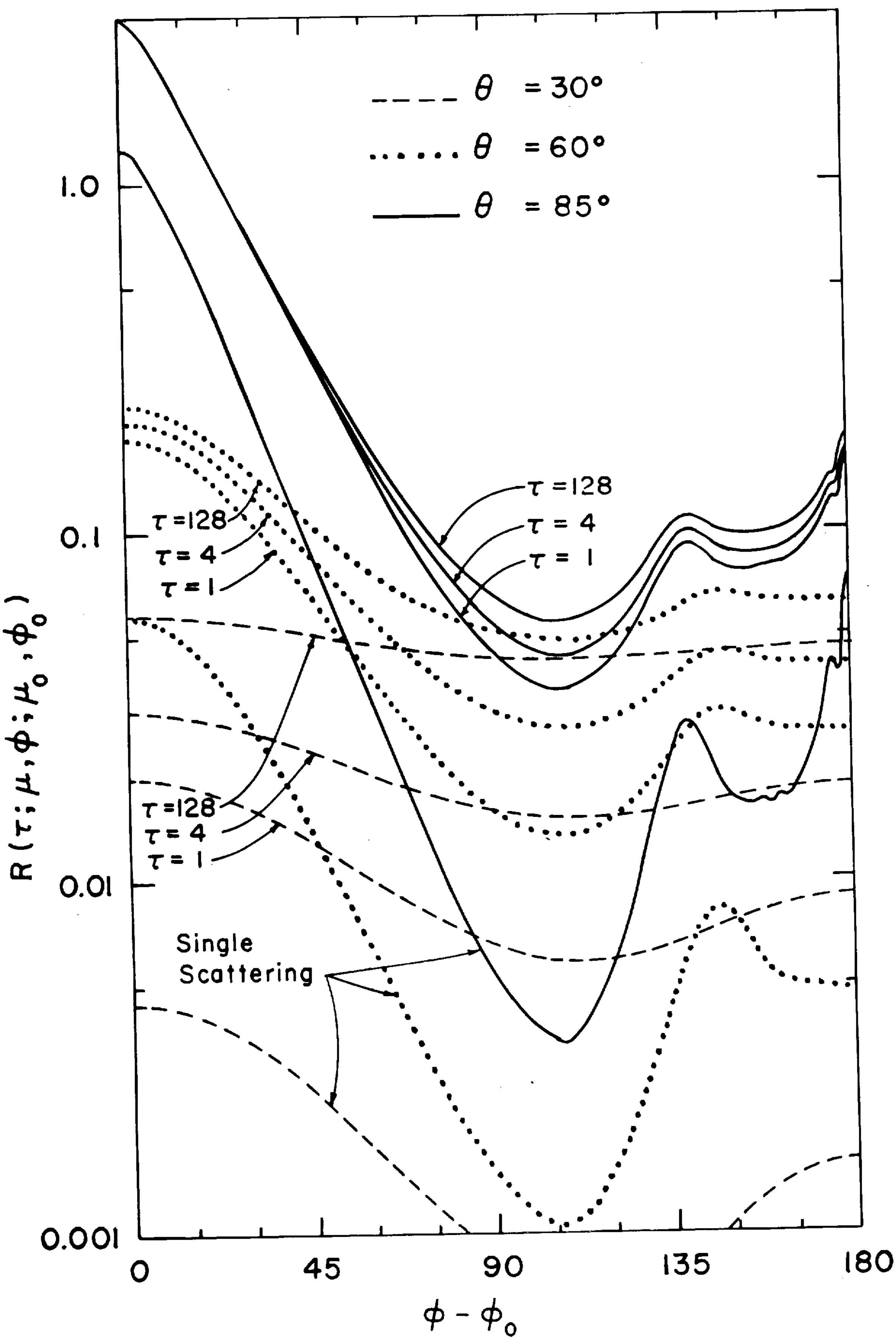
$\theta_0 = 30°$
$\theta_0 = 60°$
$\theta_0 = 85°$
1.0
0.1
0.01
$\tau = 128$
$\tau = 4$
$\tau = 128$
$\tau = 4$
$\tau = 1$
$R(\tau;\mu,\phi;\mu_0,\phi_0)$
0
30
60
90
θ

Fig. 4



θ0 = 30°
θ0 = 60°
θ0 = 85°
1.0
0.1
0.01
τ =128
τ =4
τ = 1
τ =128
τ = 4
τ = 1
R(τ; μ, φ; μ0 , φ0)
0
30
60
90
θ

Fig. 5



θ = 30°
θ = 60°
θ = 85°
1.0
0.1
0.01
0.001
τ =128
τ = 4
τ = 1
τ =128
τ = 4
τ = 1
τ =128
τ = 4
τ = 1
Single
Scattering
R(τ; μ, φ; μ₀, φ₀)
0
45
90
135
180
φ - φ₀

Fig. 6

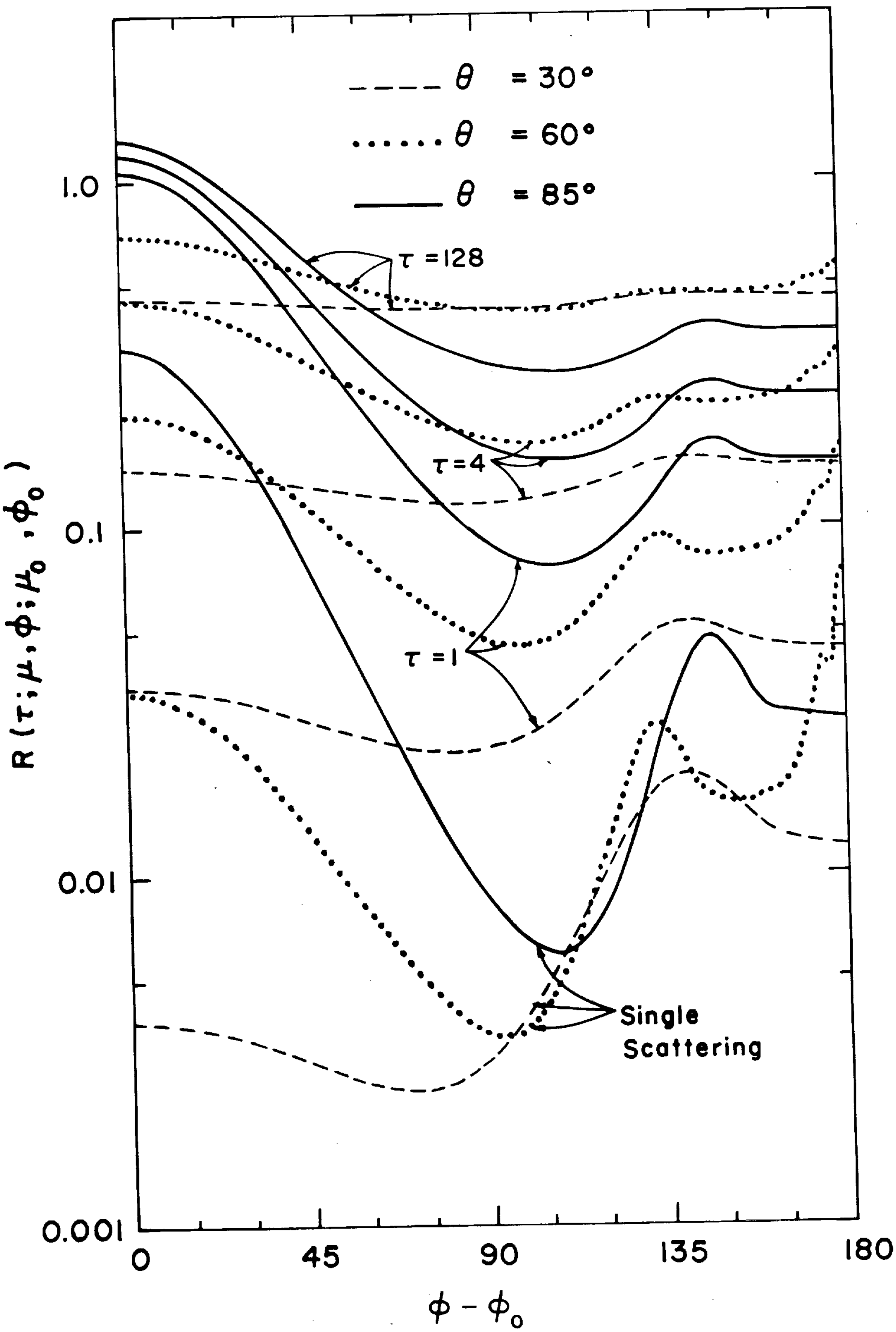
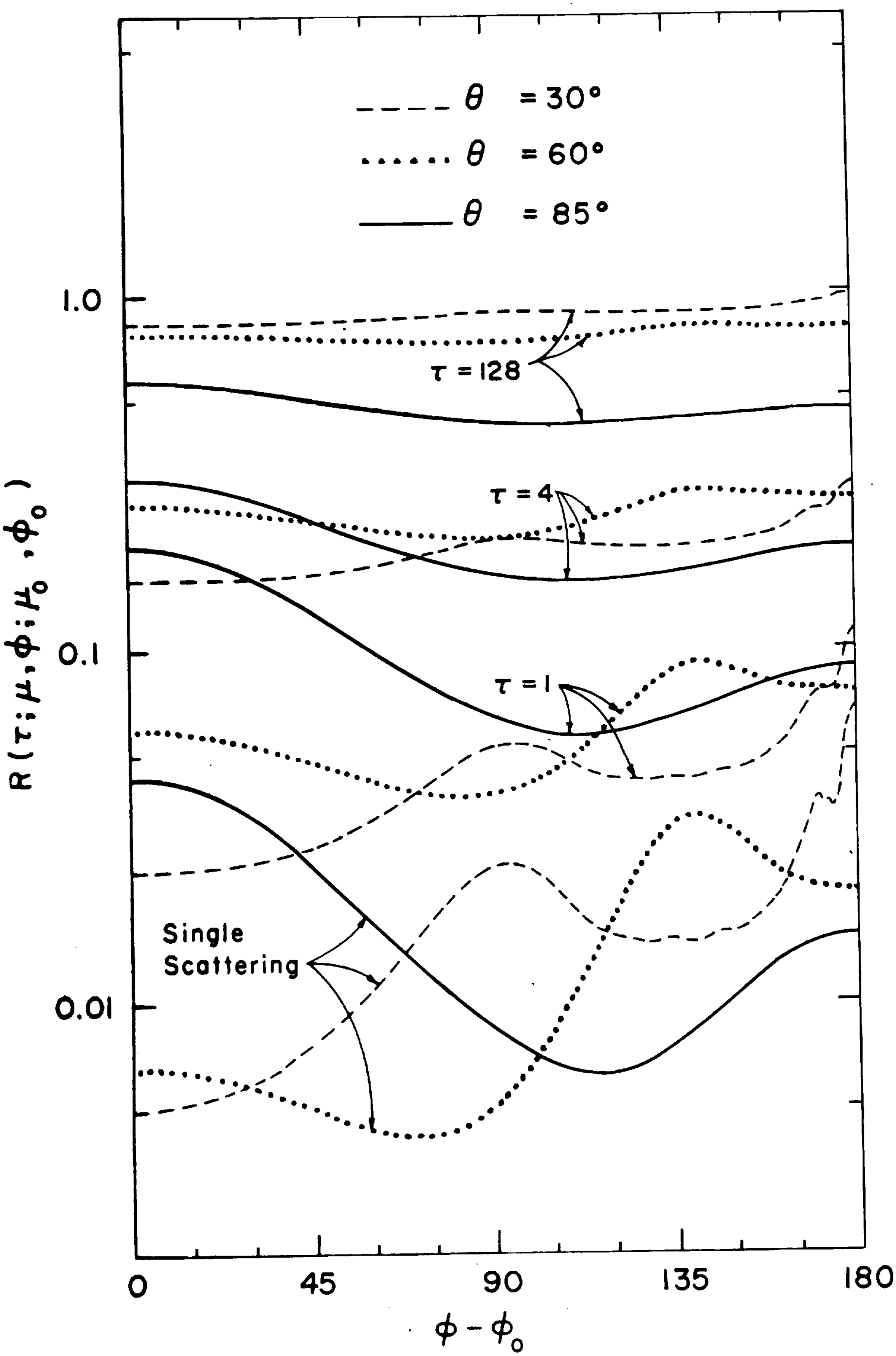

θ = 30°
θ = 60°
θ = 85°
τ = 128
τ = 4
τ = 1
Single Scattering
1.0
0.1
0.01
0.001
0
45
90
135
180
R (τ; μ, φ; μ0, φ0)
φ − φ0

Fig.7



θ = 30°
θ = 60°
θ = 85°
1.0
0.1
0.01
τ = 128
τ = 4
τ = 1
Single Scattering
$R(\tau;\mu,\phi;\mu_0,\phi_0)$
0
45
90
135
180
$\phi - \phi_0$

Fig. 8