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LENGTH OF WINDING ON A TORUS IN TWO IDEAL MODELS

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Lewis Research Center
Cleveland, Ohio

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ABSTRACT

An analysis of the relation of the total number of turns on a torus to the length of winding is presented for two models. One model is based on the assumption of circular turns; the other assumes close-packed windings. Results are presented for ratios of major to minor radii of the torus of 1.25 to 10 and for fractional filling of the winding window from 0 to 1.0. The curves presented enable the winding length for a particular number of turns to be easily determined.

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SUMMARY

An analysis of the relation of the total number of turns on a torus to the length of winding is presented for two models. One model is based on the assumption of circular turns, the other assumes close-packed windings. Results are presented for ratios of major to minor radii of the torus of 1.25 to 10 and for fractional filling of the winding window from 0 to 1.0. The curves presented enable the winding length for a particular number of turns to be easily determined.

INTRODUCTION

Design of electrical coils, such as might be used in some transformers or inductors, presents the problem of relating the total length of wire to the number of turns wound on the core. For the straight circular cylinder, and to a lesser extent for a rectangular cylinder, a single solution of a good approximation can be obtained. When the base core is a torus, additional considerations enter into the calculations because the base surface is now curved in two dimensions. The latter problem can be solved by using two different assumptions about the packing of the turns.

The first case is that of circular turns centered around the core cross section. Insulation between layers supports the turns, which are adjacent at the window surface of the torus but spaced apart on the outer periphery. The second case considered is that of close packing of windings, where the turns at the window surface are crowded on top of each other but allowed to sink between turns on the outer periphery. Both cases are examined for a variety of shapes of the torus defined by a minor radius a and a major radius b. The ratio b/a is varied, and the effect on the mean length of turn is determined. The mean length of turn is defined as the total length of winding L divided by the total number of turns N.

GENERAL FORMULATION

A sketch of the torus with minor radius a and major radius b is shown in figure 1. Two systems of plane polar coordinates are introduced. One system (ρ, θ) is centered in the plane of the window, and the other system (r, φ) is centered in the circular cross section. In general, the problem is to seek functions f which give the length of each turn and whose sum over all turns is the total length. It is always assumed that the winding is distributed uniformly in the θ direction, so that f depends only on ρ. There is, thus, no loss of generality in making the winding occupy the whole range $0 \leq \theta \leq 2\pi$. If α is the area of the window taken up by each turn and is assumed to be a constant, then the total length is

$$L = \frac{1}{\alpha} \int_{\rho_0}^{(b-a)} \int_0^{2\pi} f(\rho)\rho \, d\theta \, d\rho = \frac{2\pi}{\alpha} \int_{\rho_0}^{(b-a)} f(\rho)\rho \, d\rho$$

where ρ_0 is the radius of the unfilled part of the window, if any. (All symbols are defined in appendix A.) The discreet summation has been replaced by an integral under the assumption that the wire is sufficiently fine to permit this.

From this formulation it is immediately apparent that L can be proportional to the total number of turns N only if ρ_0 is kept fixed as N is varied, which in turn implies that for any given N the α has to be chosen to satisfy

$$N = \frac{\pi}{\alpha}\left[(b - a)^2 - \rho_0^2\right]$$

WINDINGS WITH CIRCULAR TURNS

In this case the problem is solved by assuming that all the turns are circles centered about the origin of the coordinate system (r, φ). Each turn is thus assumed essentially to close on itself and have the length

$$f(\rho) = 2\pi(b - \rho)$$

giving a total length of

$$L = \frac{2\pi^2}{3\alpha}\left[(b - a)^2(b + 2a) - \rho_0^2(3b - 2\rho_0)\right]$$

or

$$L = \frac{2\pi^2}{3\alpha}\left\{\frac{3}{\pi} b\alpha N - 2(b - a)^3 + 2\left[(b - a)^2 - \frac{\alpha N}{\pi}\right]^{3/2}\right\}$$

This circular-turns case corresponds to the physical situation where successive layers of wire are spaced apart by a constant thickness of insulation. The wires in the window are, in each layer, as close to each other as insulation permits but have open spaces between them on the outer periphery. Therefore, the density of packing of the wires varies with φ in this case. Finally, the particularly simple formula

$$L = \frac{2\pi}{3}(b + 2a)N$$

holds for a completely filled window.

WINDINGS WITH CONSTANT DENSITY OF TURNS

In a somewhat different approximation, assume that there are on the average an equal number of wires per unit area everywhere in the winding. Such a situation might arise if layer is wound on top of layer without any additional insulation except for that covering the wire. The turns then tend to crowd on top of each other in the window and sink between each other on the outer periphery. The very first layer has circular turns, but with increasing depth of winding the turns distort into some oval shape. Again, functions f are sought which give the length of each turn.

Figure 2 shows two representative noncircular turns separated by a small distance in depth; by symmetry, it is sufficient to deal with only half turns. Each turn is identified by its intercept z with the plane of the window, and the equation for a turn then is of the form $r = r(\varphi, z)$ (note that $z = r(0, z)$). At the point (r, φ) the distance between wires separated in z by the distance dz is denoted by dn. Since the turns are not circles, the radius vector subtends an angle ϵ with the normal to the curve representing the wire. By the "differential construction" in figure 2, it follows that

$$dn = \frac{\partial r}{\partial z} dz \cos \epsilon$$

and further, from the elements of polar coordinates,

$$\cos \epsilon = \frac{r}{\sqrt{\left(\frac{\partial r}{\partial \varphi}\right)^2 + r^2}} = \frac{1}{\sqrt{\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 + 1}}$$

If β now is the average area occupied per wire, then the element $(b - z)dz\, d\theta$ of window area must contain $\beta^{-1}(b - z)dz\, d\theta$ wires since the wires intercept the plane of the window at right angles. This group of wires occupies the same cross-sectional area dA at each angle φ, although dA becomes thinner and wider as φ increases from 0 to π. The area bounded by the planes θ and $\theta + d\theta$ and by the wires z and z + dz is, at any φ,

$$dA = (b - r \cos \varphi)d\theta\, dn$$

which must equal the area $(b - z)dz\, d\theta$. Equating these expressions and making the obvious cancellations and substitutions result in the partial differential equation

$$(b - z) = (b - r \cos \varphi)\ \frac{\partial r}{\partial z}\ \frac{1}{\sqrt{\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 + 1}}$$

for the function $r(\varphi, z)$. It is subject to the condition $r(\varphi, a) = a$, its variables being limited to $a \le z \le b$, $0 \le \varphi \le \pi$. This equation is nonlinear and does not appear to have a simple solution which satisfies this boundary condition.

Once $r(\varphi, z)$ has been determined L can be obtained by first writing down the arc length f for a single turn and then summing over the turns or, alternatively, by calculating the total volume of the winding V and dividing by β because $V = L\beta$. The first method does require a substitution from the differential equation and proceeds as follows:

$$f = 2\int_0^{\pi} \sqrt{\left(\frac{\partial r}{\partial \varphi}\right)^2 + r^2}\ d\varphi = 2\int_0^{\pi} \frac{(b - r \cos \varphi)}{(b - z)}\left(\frac{\partial r}{\partial z}\right) r\ d\varphi$$

Let Z be the maximum value of z (i.e., for the outermost layer of the winding).

$$L = \frac{1}{\beta}\int_0^{2\pi}\int_a^Z (b - z)f\,dz\,d\theta = \frac{2\pi}{\beta}\int_a^Z (b - z)f\,dz$$

$$= \frac{4\pi}{\beta}\int_a^Z\int_0^{\pi} (b - r\cos\varphi)\left(\frac{\partial r}{\partial z}\right) r\,d\varphi\,dz$$

$$= \frac{2\pi}{\beta}\int_0^{\pi} \left\{ b\left[r^2(\varphi, Z) - a^2\right] - \frac{2}{3}\left[r^3(\varphi, Z) - a^3\right]\cos\varphi \right\} d\varphi$$

An approximate solution, which gives an exact value for $r(\pi, z)$, can be derived by integrating the differential equation at constant φ.

$$\int_a^r (b - r\cos\varphi)dr = \int_a^z (b - z)\sqrt{\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 + 1}\;dz$$

or upon integration of the left side

$$r = \begin{cases} \dfrac{b}{\cos\varphi} - \dfrac{1}{\cos\varphi}\left[(b - a\cos\varphi)^2 - 2\cos\varphi\displaystyle\int_a^z (b - z)\sqrt{\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 + 1}\;dz\right]^{1/2} & \varphi \neq \dfrac{\pi}{2} \\[2em] a + \dfrac{1}{b}\displaystyle\int_a^z (b - z)\sqrt{\left(\frac{1}{r}\frac{\partial r}{\partial \varphi}\right)^2 + 1}\;dz & \varphi = \dfrac{\pi}{2} \end{cases}$$

If it is true that the angle ϵ remains small, then the quantity under the square-root sign can be approximated by 1, and the integration can be carried out to arrive at the approximate solution

$$r \approx \begin{cases} \dfrac{b}{\cos\varphi} - \dfrac{1}{\cos\varphi}\left\{(b - a\cos\varphi)^2 + \left[(b - z)^2 - (b - a)^2\right]\cos\varphi\right\}^{1/2} & \varphi \neq \dfrac{\pi}{2} \\[1.5em] a + \dfrac{1}{2b}\left[(b - a)^2 - (b - z)^2\right] & \varphi = \dfrac{\pi}{2} \end{cases}$$

Clearly, even the approximate solution is sufficiently complicated to require numerical methods to evaluate the integral for L.

CALCULATION OF L FOR CONSTANT-DENSITY-OF-TURNS WINDING

A direct accurate solution for the partial differential equation is obtained by applying a technique of successive numerical approximations to the equation in the integral form. The first steps in the process are to increment z from $z = a$ by a small amount dz, assume $\partial r/\partial \varphi = 0$, and find values of r for a set of some 100 points $(\varphi_n, a + dz)$. The values of r thereby obtained are used to estimate $\partial r/\partial \varphi$ at these points, and the process is repeated until convergence is obtained. After that, z is incremented further to $z + 2dz$ and the process is repeated, generating a family of solutions. These solutions then were used to find L and compared with the approximate solution. A few curves are presented in figure 3.

The results are presented in terms of the winding volume instead of length, the two being related by $V = L\beta$. Values of winding volumes are calculated for unit volume tori of b/a ranging from 1.25 to 10, which permits easy scaling of the winding volume to tori of other sizes. In figure 4 the volume of winding divided by the volume of the torus is plotted against the fraction of the window area filled, using the exact solutions.

DISCUSSION OF RESULTS

The curves in figure 4 are based on a direct numerical solution of the differential equation; however, the difference between the exact and approximate solutions is at most only a few percent, and often only a fraction of a percent.

When plotted linearly, the volume V against the fraction of window filled W are smooth curves, concave upward because of the increasing contribution per turn as W approaches 1. To find L from the curves of $V(W)$ divide by β.

$$L = \frac{V(W)}{\beta}$$

where $W = N\beta/\left[\pi(b - a)^2\right]$. The use of the curves of figure 4 to find the length of winding on a torus of arbitrary volume and shape is illustrated by a sample calculation in appendix B.

It is of interest now to compare the length of wire for the constant-density case, which produces a minimum-length winding, with the length found in the circular-turns case. The difference will be maximum whenever the window area is completely filled. The circular-turns case then has a mean length of turn

$$\left(\frac{L}{N}\right)_{circ} = \frac{2}{3}\pi(b + 2a)$$

and the constant-density case gives an

$$\left(\frac{L}{N}\right)_{min} = \frac{V(1)}{\pi(b - a)^2}$$

As expected, for small b/a the two types of windings have only a small difference in length, which increases quite rapidly as b/a exceeds 1. By inspection of figure 5, which compares the two cases quantitatively, it is evident that the ratio of $(L/N)_{circ}$ to $(L/N)_{min}$ starts at 1 and then gradually increases to about 1.25 at $b/a = 10$.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, July 8, 1968,
120-27-04-22.

APPENDIX A

SYMBOLS

- a — minor radius of torus
- b — major radius of torus
- f — length of one turn of winding
- L — total length of winding
- N — total number of turns
- n — distance normal to turn of wire
- r — radial coordinate
- V — volume of winding for constant-density-of-turns model
- v — volume of torus, $2\pi^2 ba^2$
- W — fraction of window area occupied by winding
- Z — value of z for outermost layer of winding
- z — defined by $z = r_{\varphi=0}$ for any given turn of wire
- α — area of window occupied by one turn of wire in circular-turns case
- β — area occupied by one wire in constant-density-of-turns case
- ϵ — angle subtended by radius r and normal n
- θ — angular coordinate
- ρ — radial coordinate
- ρ_0 — radius of unfilled part of window in circular-turns case
- φ — angular coordinate

APPENDIX B

USE OF CURVES OF FIGURE 4 TO FIND LENGTH OF WINDING

Suppose a given torus with the dimensions $a = 2$ centimeters and $b = 6$ centimeters is to be wound with $N = 1000$ turns of wire having an effective cross section β of 0.04 square centimeter. The volume of this torus is

$$v = 2\pi^2 ba^2 = 473\ \text{cm}^3$$

The fraction W of the window area filled by this winding is

$$W = \frac{N\beta}{\pi(b - a)^2} = \frac{10^3 \times 4 \times 10^{-2}}{\pi(6 - 2)^2} = 0.796$$

The ratio of the volume V(W) of winding to $2\pi^2 ba^2$ can now be read from the curve for $b/a = 3$ (fig. 4).

$$\frac{V(W)}{2\pi^2 ba^2} = \frac{V(0.796)}{473\ \text{cm}^3} = 1.4$$

which gives the volume of winding $V = 662$ cubic centimeters. The length L of the wire follows from

$$L = \frac{1}{\beta} V(W) = \frac{662\ \text{cm}^3}{0.04\ \text{cm}^2} = 166\ \text{meters}$$

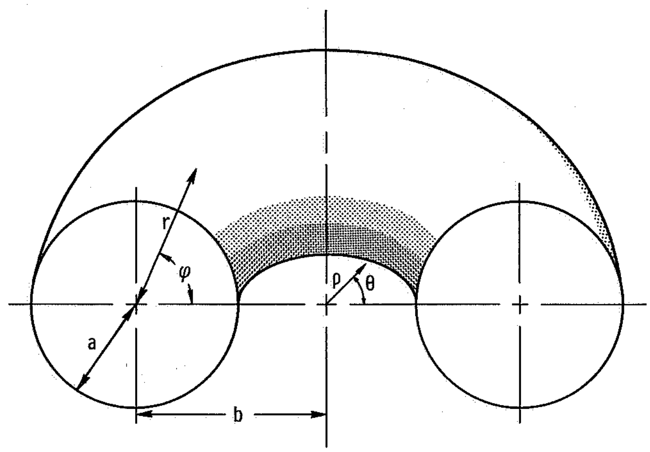

Figure 1. - Cross section of core showing origins of two coordinate systems.

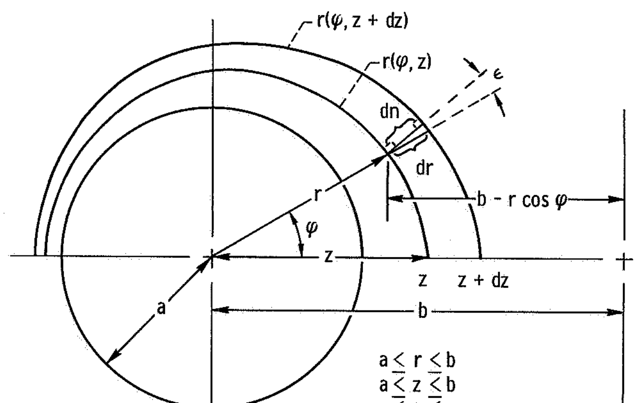

Figure 2. - Two closely spaced turns and differential construction for uniform-density winding on torus. (Because of symmetry only half of each turn is shown.)

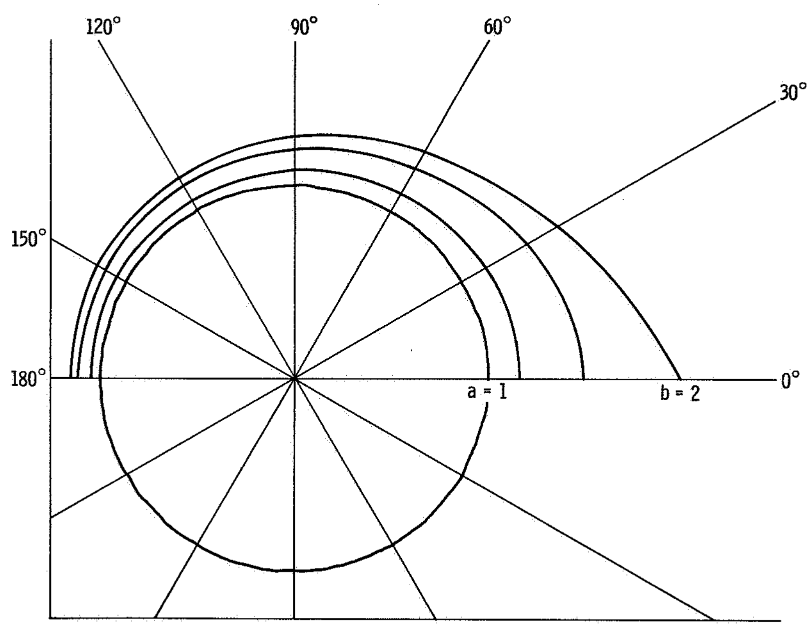

Figure 3. - Shape of constant-density winding on torus.

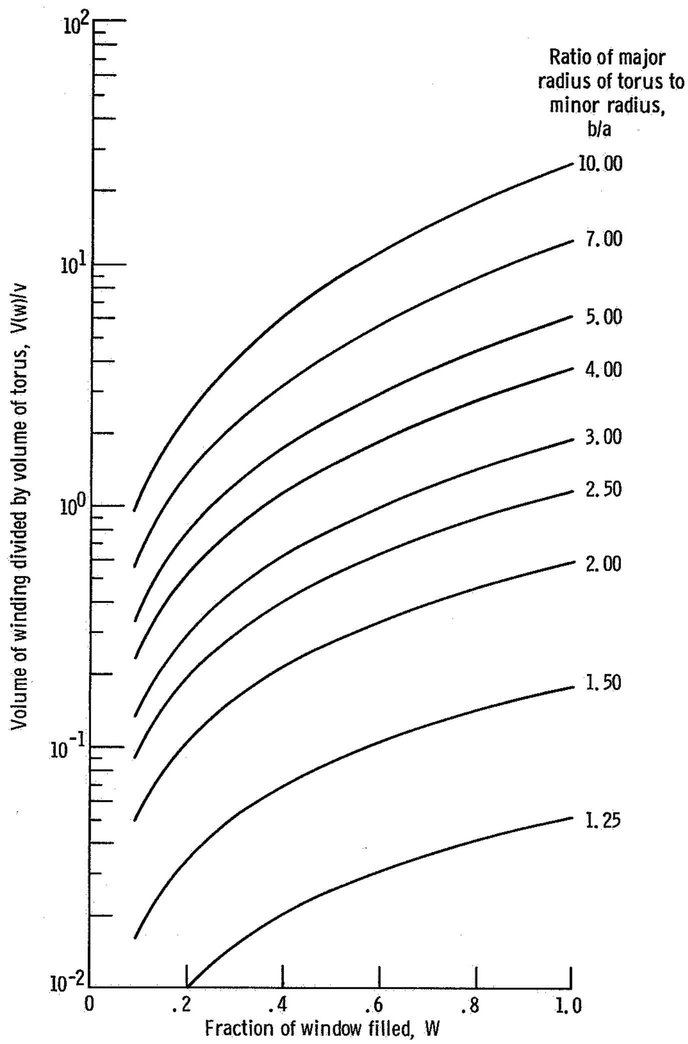

Figure 4. - Volume of winding divided by volume of torus as function of fraction of window filled for constant-density windings.

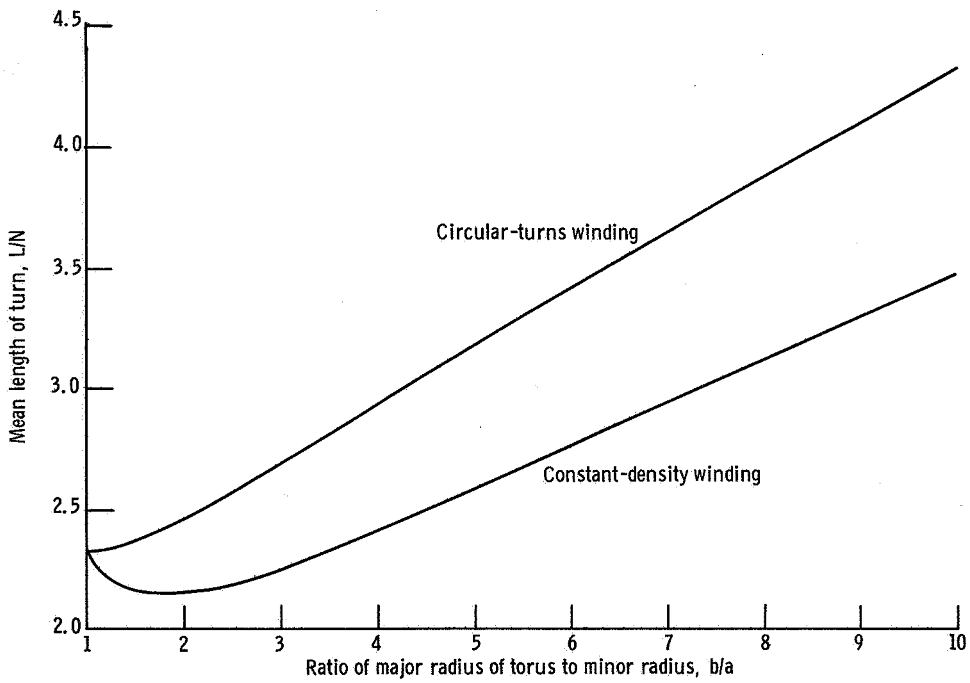

Figure 5. - Mean length of turn for unit volume tori having completely filled windows as function of ratio b/a.

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