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ABSTRACT

Certain problems in obtaining accurate altitude measurements over the ocean surface from a satellite pulsed-radar altimeter are considered. Particular attention is directed toward eliminating sources of bias errors that have the same order of magnitude as the ultimate desired resolution accuracy. Both narrow-beam and gated wide-beam radars are investigated. The discussion includes criteria for: selecting the radar beam width and pulse repetition frequency, geometrical correction for the beam's spherical wavefront, altitude estimation equations and analysis of estimation performance.

I. INTRODUCTION

The possibility of using a satellite in orbit about the earth to obtain accurate altitude measurements has been proposed in recent years by several authors[1,2,3] interested in the geophysical and oceanographic sciences. Such measurements are potentially useful for determining a better model of the earth's gravitational potential, for obtaining more accurate knowledge of satellite orbits, and for learning more about the characteristic features of the earth's surface. Altitude measurements above the surfaces of the oceans are considered especially useful. In the absence of tides, ocean currents, and wind waves, the oceans' surfaces conform to an equipotential surface defined as the geoid. Knowledge of satellite altitude above the geoid is useful for determining both the geoidal and orbital features. Although the presence of storm winds, tides, and currents distorts the features of interest to geodesists, they cause variations in ocean level which are of interest to oceanographers in determining the dynamic behavior of the seas.

Pulsed radar altimeter systems have been proposed [2,3] for making altitude measurements. With respect to altimeter design, the emphasis of these investigations is upon antenna and transmitter power level requirements, sources of uncertainty, and measurement accuracy requirements. The resulting conclusions that are relevant to this discussion are: (1) High receiver signal to noise ratios (10db) are achievable with modest transmitter power levels (a few microwatts to a few watts average power depending upon the radar beam dimensions); (2) To obtain measurements having the required resolution properties at the ocean's surface, one requires either a narrow radar beam (a few milliradians for a satellite altitude of 600 km) or accurate gating of the initial portion of each return pulse from a wide beam (a few degrees at 600 km altitude); (3) Ultimate altitude resolution (after filtering out high frequency noise in the sequence of altitude measurements) on the order of one foot or better is required in order to observe the desired ocean surface variations.

In the references mentioned above there has been no detailed discussion of the processing techniques used to obtain altitude estimates from each return pulse. It is implied that the leading edge of the return provides the estimate; if this be the case, the processing technique requires little elaboration. However, a potential disadvantage of the leading edge tracker in this application is that the resulting estimate may contain a bias of the same order as the ultimate resolution desired. One possible source of bias is the fact that reflections from wave tops arrive at the receiver first; hence the time of arrival of the return signal is affected by the sea state. This question is discussed briefly by Greenwood, et. al.[2] Another characteristic of the leading edge tracker is that only a small portion of the radar return signal is analyzed for useful information. Essentially the reflection from only a point is utilized. More efficient estimation procedures might be developed which process the entire return pulse to obtain an altitude estimate from an element of sea surface having non-zero area. On the other hand, if information from more than one point on the surface within the radar beam is to be processed, an appropriate definition of altitude must be established. It is the purpose of this report to discuss these questions.

The emphasis of this discussion is on those altimeter design features related to the processing of the radar return signal. Particular attention is directed toward eliminating sources of bias errors larger than the required post filtering resolution. The subjective contents of the following sections are: (1) A definition of satellite altitude appropriate for processing the pulsed radar return signal is established; (2) Criteria are developed for selecting the pulse repetition rate and the radar beam width; (3) A geometrical correction for the radar beam's spherical wavefront is derived; (4) An estimation for satellite altitude is derived; (5) The estimation error is analyzed; (6) Application of a gated wide-angle beam is considered.

The next five sections deal with the case where the entire radar return signal from a narrow beam is to be processed. Subsequently, modifications of the narrow beam techniques are suggested for the case of the gated wide beam proposed by Greenwood, et. al. [2] Because the hardware requirements of the narrow and wide beam radars differ markedly, one design may be distinctly preferable to another in a given experiment. Hence, it is desirable to have a signal processing technique that is adaptable to either.

II. A MODEL FOR THE RADAR ALTIMETER

The function of a radar altimeter can be interpreted as that of a sampler taking discrete measurements of altitude above the earth's surface. Each sample consists of a pulse of reflected energy that is processed to yield a measurement or estimate of altitude. As the satellite passes over the ocean, it views the continuously changing surface topography along its track. The surface features are a combination of waves that are time varying with respect to the earth, e.g. caused by winds and tides, and the stationary or standing waves caused by ocean currents and variations in the earth's gravitational potential.

To obtain a model for the altitude measuring process, assume that the ocean surface illuminated by the radar beam is stationary during each radar pulse. The situation is illustrated by the one-dimensional cross-sectional view in Fig. 1. A general model for the altitude measurement is that at time t_i, a sample h_i is generated by some functional operation $F(\ )$ upon the ocean area included within the radar beam. This is denoted by

$$h_i = F[h(x,y)]; \quad (x-x_i)^2 + (y-y_i)^2 \leq R \qquad (1)$$

where $h(x,y)$ denotes the altitude as a function of datum coordinates, x and y, and R is the radius of the radar beam cross-sectional area.

Assume that it is desired to detect variations in the ocean's surface having spatial periods greater than an amount D in both x and y coordinates. This implies that variations

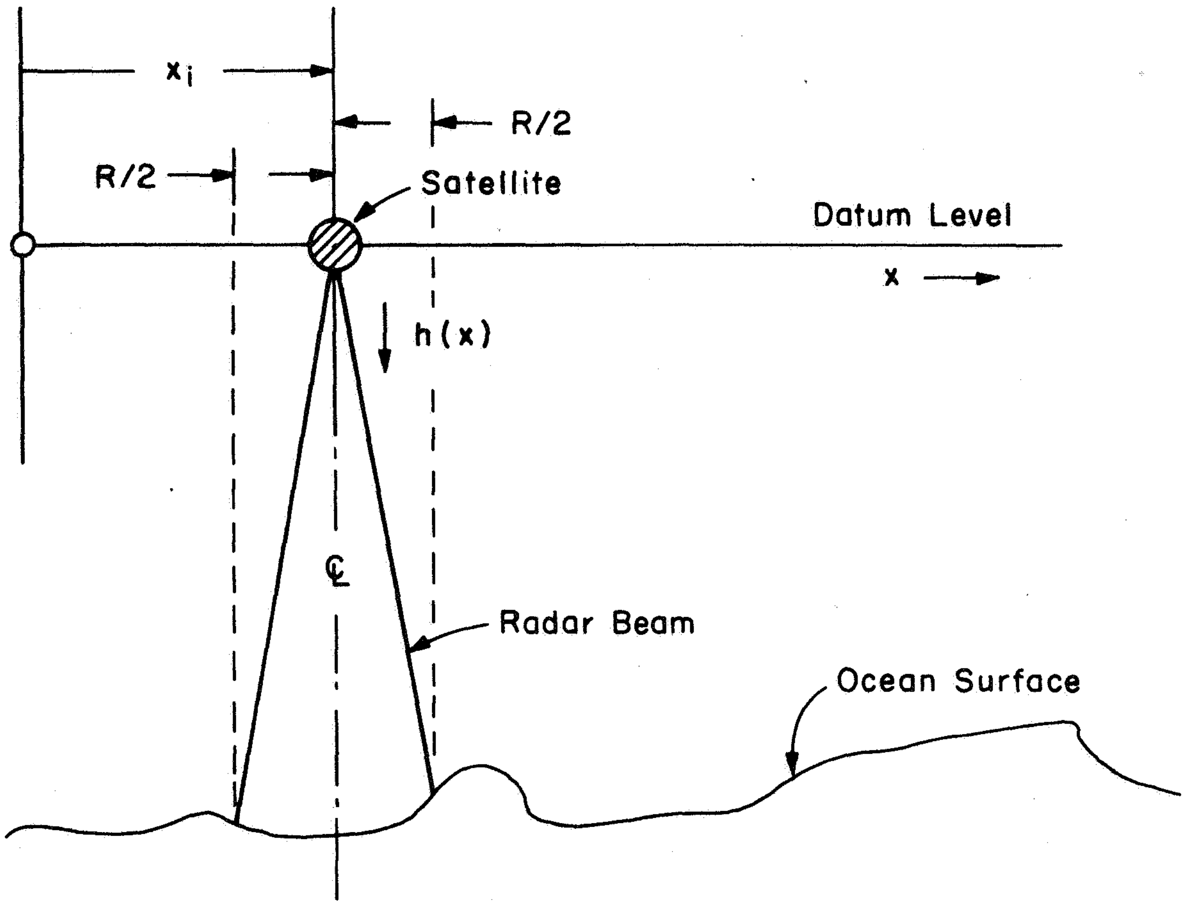

Figure 1 One-dimensional illustration of radar beam configuration.

with smaller periods are to be eliminated by an appropriate filter. Now suppose that h_i in Eq. (1) is simply a point sample

$$h_i = h(x_i,y_i)$$

We know from the sampling theorem [4] that no low frequency information is lost in the sampling operation if the sampling frequency is constant and equal to or greater than twice the highest frequency to be retained. However, if the sampling frequency is lower than twice the highest frequency to be eliminated, the unwanted high frequency variations are converted to low frequency variations having the same amplitude; the latter effect is known as aliasing. Therefore, if the sampling process is to be perfect in the sense that there is no net loss of information, the sampling frequency must be twice the highest frequency present in the ocean's surface variation.

Now because the radar beam has non-zero width and the ocean surface is irregular, it is not possible to sample the satellite's altitude uniformly over the datum plane. Even if uniform point sampling were possible, the sampling frequency required to preserve information is impractical to achieve. We desire to devise a radar signal processing technique that is a reasonable approximation to the perfect point sampler. This task is complicated by the fact that a given radar return siganl received as a function of one independent variable, time, does not map into a unique function h(x,y) within the radar beam because the latter is a function of two independent variables. Furthermore, if the reduction of the radar return signal to a number representing altitude is to be accomplished on board the satellite, the processing technique must be reasonably simple to implement. Both of these factors motivate the following heuristic arguments for the receiver design.

Assume, for the moment, that the radar beam width R at the ocean surface is much smaller than the smallest spatial period of interest. With this condition, at a given location of the beam in the x-y plane, the desired component of satellite altitude is nearly constant over the beam area. Now if the unwanted surface variations are of sufficiently high frequency, a nearly optimal altitude estimate for many estimation criteria is simply the average height of the satellite over the beam area. With altitude measured below the x-y plane in Fig. 1, the altitude estimate is given by

$$\bar{h}_i = \frac{1}{A} \iint\limits_{A(x_i,y_i)} h(x,y)\,dxdy \tag{2}$$

where A denotes the beam area and $A(x_i,y_i)$ denotes the region of the x-y plane enclosed by the beam area.

It is proposed here that $\bar{h}_i$ yields a useful estimate of altitude even when the conditions under which it is optimal do not hold. Furthermore, it is an estimate that can be obtained approximately from the radar return signal with a reasonably simple processing scheme. In succeeding sections the performance of this estimator is analyzed and a method of implementing it with the radar return signal is described.

III. PERFORMANCE ANALYSIS OF ALTITUDE ESTIMATOR

An analysis of the altitude estimate provided by Eq. (2) is facilitated if one axis, say x, of the x-y plane is taken along the satellite's track. Furthermore, it is mathematically convenient to assume that the radar beam has a square cross-sectional area with side length R so that $A(x_i,y_i)$ is a square element of ocean area. With these assumptions and the definition

$$\bar{h}_y(x) \triangleq \frac{1}{R}\int_{-R/2}^{R/2} h(x,y)dy \tag{3}$$

Eq. (2) becomes

$$\bar{h}_i = \frac{1}{R}\int_{x_i-R}^{x_i} \bar{h}_y(x)dx \tag{4}$$

Now assume that a sequence $\{\bar{h}_i\}$ of measurements generated by Eq. (4) is to be made with a uniform interval X between measurements. This sequence can be thought of as generated by sampling the function

$$\bar{h}(x) \triangleq \frac{1}{R}\int_{x-R}^{x} \bar{h}_y(x)dx \tag{5}$$

at regular intervals X. A block diagram illustrating the evolution of $\{\bar{h}_i\}$ is given in Fig.2 using imaginary frequency operator

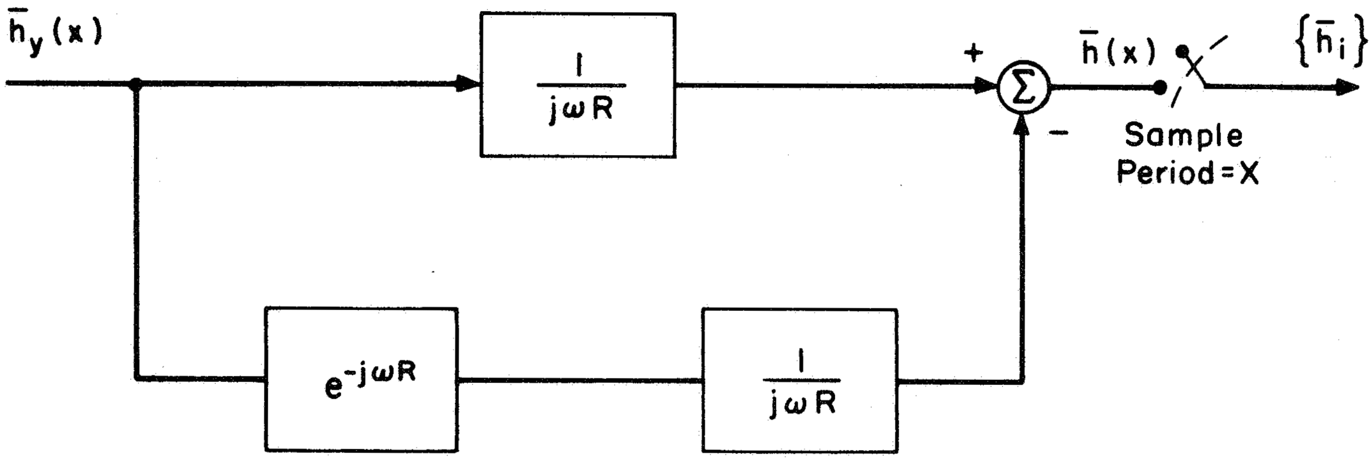

Figure 2 Generation of $\{\bar{h}_i\}$ from $\bar{h}_y(x)$.

notation with

$\omega = 2\pi x$

Recall that our objective is to obtain altimeter measurements that introduce as little distortion as possible: we desire to closely approach "perfect" sampling, in the sense defined previously. The actual performance obtained from the processing technique represented by Fig. 2 can be evaluated by comparison of the frequency spectra of $\{\overline{h}_i\}$ and $\overline{h}_y(x)$. Denoting spectra by capital letters and regarding $\{h_i\}$ as a sequence of impulses with the notation

$\overline{h}^*(x) \triangleq \{h_i\}$

we can relate $\overline{H}(j\omega)^*$ to $\overline{H}_y(j\omega)$ by [4]

$$\overline{H}(j\omega)^* = \frac{1}{X} \sum_{n=-\infty}^{\infty} \overline{H}(j\omega + nj\omega_X) \qquad (6)$$

$$\overline{H}(j\omega) = \left|\frac{1}{Rj\omega} (1 - e^{-j\omega R})\right|^2 \overline{H}_y(j\omega) \qquad (7)$$

where

$$\omega_X = \frac{2\pi}{X}$$

Equations (6) and (7) are derived from well known results in sampling theory, assuming $\overline{h}_y(x)$ is a stationary random process.

Under the constraint that the radar altimeter operate as illustrated in Fig. 2, it is easily determined from Eqs. (6) and (7) what choices of the parameters, X and R, lead to approximately perfect sampling. First, suppose that the desired portion of the spectrum of $\overline{H}_y(j\omega)$ lies within the interval

$$-2\pi/D \leqq \omega \leqq 2\pi/D \triangleq \omega_D$$

Define

$$G(j\omega) \triangleq \frac{1}{j\omega R}(1 - e^{-j\omega R}) \tag{8}$$

It follows that

$$|G(j\omega)|^2 = \left[\frac{\text{Sin}(\frac{\omega R}{2})}{\omega R/2}\right]^2 \tag{9}$$

Because

$$\overline{H}(j\omega) = |G(j\omega)|^2\ \overline{H}_y(j\omega)$$

it follows from Eq. (9) that the distortion of the desired portion of $\overline{H}_y(j\omega)$ caused by $G(j\omega)$ is determined by the beam dimension R. If

$$R\omega_D << 2\pi$$

the distortion is held to a small level. In practice one may require

$$R\omega_D = 2\pi K_D$$

where K_D is a design parameter chosen according to the acceptable departure of $|G(j\omega)|^2$ from the value, one, for $|\omega| \leqq \omega_D$. In terms of the spatial period D,

$$\frac{R}{D} = K_D << 1 \tag{10}$$

Typically, one may be interested in ocean surface variations for which D = 50-100 miles. Radar beam widths at the ocean surface less than ten miles appear possible[3] so that condition (10) can be achieved.

Our next task is to determine the value of X. From the adverse effects of aliasing, we infer that the sampling frequency should equal twice the highest significant frequency contained in $\overline{H}(j\omega)$. Because

$$\lim_{\omega \to \infty} |G(j\omega)|^2 = \frac{2}{R^2\omega^2}$$

the effect of $G(j\omega)$ is to suppress the highest frequencies in $\overline{H}_y(j\omega)$. For a given choice of R, assume that there is some frequency, ω_S, for which the quantity $1/R\omega_S$ is sufficiently small such that

$$\overline{H}(j\omega) \cong 0 \,; \quad |\omega| \geqq \omega_S$$

The quantity ω_S may be specified by a design parameter K_S according to

$$\frac{1}{R\omega_S} = K_S \quad << 1 \tag{11}$$

The sampling frequency should be twice the value of ω_S so that by Eq. (11),

$$X = \pi R K_S \tag{12}$$

Equations (10) and (12) provide simple design criteria for picking altimeter parameters, R and X, on the basis of "distortion" parameters K_D and K_S. Both K_D and K_S should be sufficiently small to achieve acceptable sampling fidelity. However, their lowest possible values are limited by the allowable antenna size and the available radar power, both of which increase as K_D and K_S are made small. The quantitative relationship is roughly indicated by

$$\text{Antenna Diameter} \sim \frac{h}{R} = \frac{h}{DK_D}$$

$$\text{Average Power} \sim \frac{R^2}{X} = \frac{DK_D}{\pi K_S}$$

where h is an approximate value of the satellite's altitude.

We have also noted that $G(j\omega)$ attenuated the high frequencies in $\bar{H}_y(j\omega)$. This is desirable behavior because we presume the high frequencies are to be eliminated from the ocean surface spectrum by appropriate filtering of the sequence $\{\bar{h}_i\}$.

This discussion has presented an evaluation of the estimate defined in Eq. (2) in terms of the resulting distortion of the desired ocean spectrum. The next task is to determine in what sense the radar return can be processed to obtain this estimate. The remainder of this report is concerned only with the details of obtaining the sequence $\{\bar{h}_i\}$ from radar return echoes. The problem of processing the sequence to obtain the desired portion of the sea surface spectrum should be amenable to standard filtering methods and is not treated here.

IV. THE GEOMETRY OF THE RADAR BEAM

Before considering the details of radar signal processing, we examine the effect of the radar beam geometry upon the altitude measurement. This subject has been discussed by others [2,3] from a somewhat different point of view.

The situation is as depicted in Fig. 3. The satellite transmits radiation confined to a beam half-width α having a spherical wavefront.* As is shown in the next section, the reflected return is processed to give a measure of the average distance, $\tilde{h}$, from the satellite of the ocean's surface contained within the radar beam. What is desired is the average altitude, $\bar{h}$, of the satellite measured from a plane centered at the satellite and perpendicular to the center line of the beam. In this section, a relationship between $\tilde{h}$ and $\bar{h}$ is derived.

In terms of the quantities defined in Fig. 3, $\tilde{h}$ is given approximately by

$$\tilde{h} \cong \frac{1}{\tilde{A}} \int_{-\alpha}^{\alpha} \int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} r^3(\theta,\phi)\cos\theta \, d\theta \, d\phi \tag{14}$$

where α is taken to be a small angle and $\tilde{A}$ is approximately

$$\tilde{A} \cong \int_{-\alpha}^{\alpha} \int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} h^2 \cos\theta \, d\theta \, d\phi \tag{15}$$

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* Assume the energy density in the beam is uniform in θ and ϕ; otherwise the subsequent discussion must be modified.

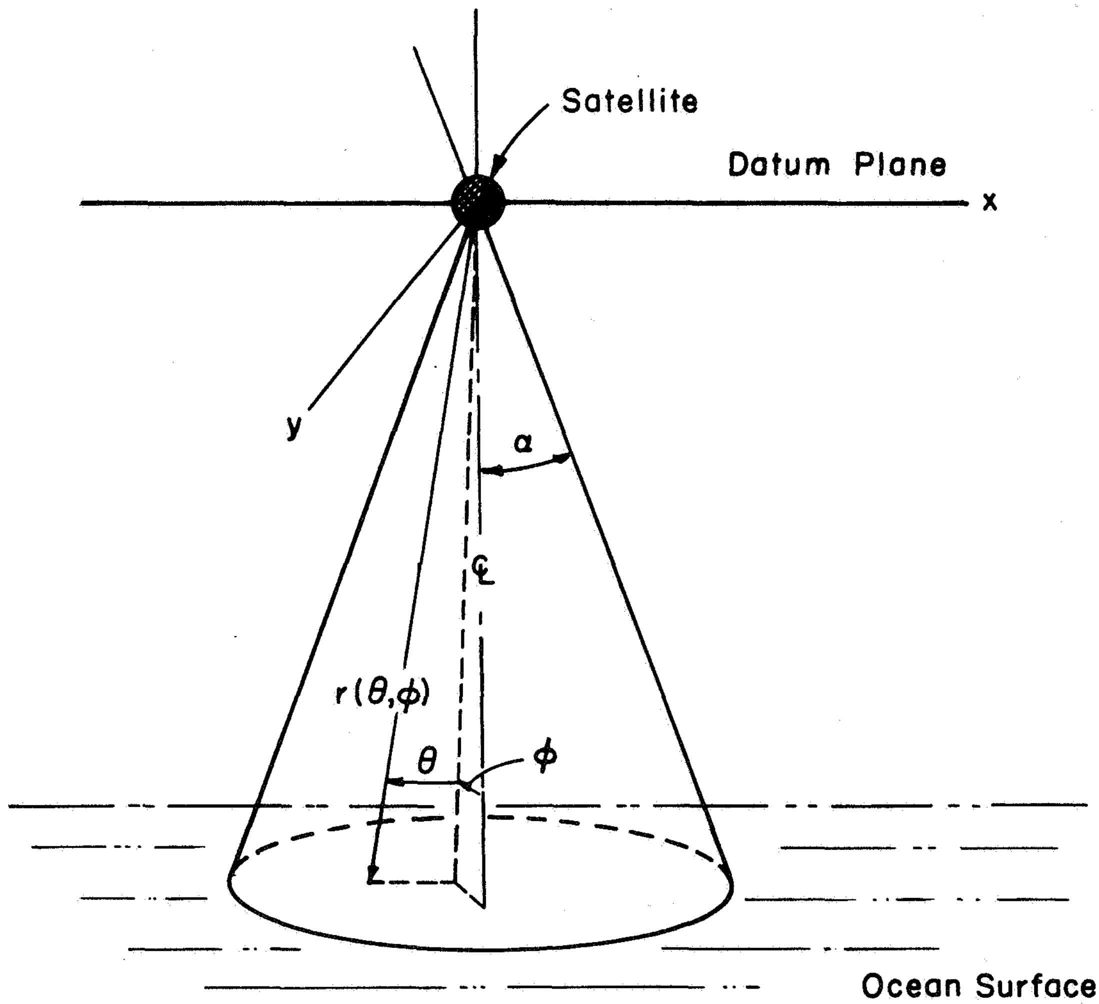

Figure 3 Satellite beam geometry

Equation (15) is useful in subsequent derivations; the resulting accuracy in $\tilde{A}$ is about one part in $1/\varepsilon_{\tilde{A}}$ where *

$$\varepsilon_{\tilde{A}} = \frac{4\alpha^2}{\tilde{A}} [\Delta r_{max}]^2 \tag{16}$$

with

$$\Delta r(\theta,\phi) \triangleq r(\theta,\phi) - \tilde{h} \tag{17}$$

$$\Delta r_{max} \triangleq \max_{\theta,\phi} \{\Delta r(\theta,\phi)\}$$

The desired average altitude defined in Eq. (2) is given by

$$\bar{h} = \frac{1}{A} \iint_A h(x,y)dxdy \tag{18}$$

where h(x,y) is the distance to the ocean surface along a normal to the x-y plane in Fig. 3 and A is the projection of $\tilde{A}$ onto the x-y plane,

$$A = \iint_A dxdy \tag{19}$$

Now introduce the transformation equations

$$h(x,y) = r(\theta,\phi)\cos\theta\cos\phi$$

$$y = r(\theta,\phi)\sin\theta$$

$$x = r(\theta,\phi)\cos\theta\sin\phi$$

* See Appendix A for the derivation of Eqs. (16) and (17).

Using these relations with Eq. (17) to change variables of integration in Eq. (18) and (19), one obtains

$$\bar{h} = \frac{\displaystyle\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} r^2\cos\theta\cos\phi[r\cos^2\theta\cos\phi+\Delta r_\phi\sin\phi+\Delta r_\theta\cos\theta\cos\phi\sin\theta]d\theta d\phi}{\displaystyle\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [r^2\cos^2\theta\cos\phi+\Delta r_\phi r\sin\phi+\Delta r_\theta r\cos\theta\cos\phi\sin\theta]d\theta d\phi}$$

where $r \triangleq r(\theta,\phi)$ and

$$\Delta r_\theta = \frac{\partial r(\theta,\phi)}{\partial\theta}$$

$$\Delta r_\phi = \frac{\partial r(\theta,\phi)}{\partial\phi}$$

Linearization of $\bar{h}$ about $\tilde{h}$, as defined in Eqs. (14) and (15), produces

$$\bar{h} \cong \tilde{h} - \frac{\tilde{h}^3}{\tilde{A}}\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}}\left(\frac{\theta^2}{2} + \frac{\phi^2}{2}\right)d\theta d\phi \qquad (21)$$

or

$$\bar{h} \cong \tilde{h} - \frac{\tilde{h}^3}{\tilde{A}}\cdot\frac{\pi\alpha^4}{4} \cong \tilde{h}[1-\frac{\alpha^2}{4}] \qquad (22)$$

Equation (22) is the desired expression for $\bar{h}$ in terms of $\tilde{h}$; it has the important property that knowledge of the details of of $r(\theta,\phi)$ is not required when h is given.

The accuracy in $\bar{h}$ computed by Eq. (22) is of the order of one part in $1/\varepsilon_{\bar{h}}$ where*

$$\varepsilon_{\bar{h}} = \frac{1}{\tilde{A}} \int_{-\alpha}^{\alpha} \int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} \phi \Delta r \Delta r_{\phi} d\theta d\phi \qquad (23)$$

An estimate of the magnitude of $\varepsilon_{\bar{h}}$ for typical ocean surface variations is obtained by letting

$$\Delta r(\theta,\phi) = -\Delta r_{max} \cos k\phi$$
$$\Delta r_{\phi} = k \Delta r_{max} \sin k\phi \qquad (24)$$

For large and small values of k,

$$\varepsilon_{\bar{h}} \sim \frac{\Delta r_{max}{}^2}{\tilde{A}} \alpha^2 \qquad k\alpha >> \pi/2$$
$$\qquad (25)$$
$$\varepsilon_{\bar{h}} \sim \frac{\Delta r^2_{max}}{\tilde{A}} k^2 \alpha^4 \qquad k\alpha << \pi/2$$

In the configuration considered by Frey, et. al.[3], it is concluded that an extremely narrow radar beam ($\alpha \sim 10^{-3}$ radians) is required to limit the error between the actual range

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* See Appendix A for derivation of Eq. (23).

measurement and the desired altitude measurement. In cases of practical interest using the processing technique considered here, the error in $\bar{h}$, computed from Eq. (22), is far below the ultimate accuracy required. Thus it is concluded that there is no need to use such a narrow beam solely for the purpose of having $\bar{h} \cong \tilde{h}$ when correction by Eq. (22) is possible. Of course, a narrow beam is desirable to obtain the desired altitude resolution for the reasons indicated in the preceding sections.

V. PROCESSING THE RADAR RETURN SIGNAL

In order to determine a useful technique for processing the radar return signal, first consider the signal caused by reflection from an "idealized" ocean surface corrupted by additive gaussian white noise, $n(t)$, which is independent of the signal with statistics

$$E\{n(t)\} = 0$$

$$E\{n(t_1)n(t_2)\} = \sigma_n^2 \, \delta(t_2 - t_1)$$

Imagine that the ocean surface is a two-dimensional staircase composed of equal step heights, Δh. A one-dimensional representation is given in Fig. 4 with $h(\theta,\phi)$ interpreted as the radial distance from the satellite. Let the transmitted signal, $s(t)$ be

$$s(t) = f(t)\cos\omega_c t \tag{26}$$

where ω_c is an appropriate carrier frequency and $f(t)$ is amplitude modulation.* Assuming the <u>total</u> surface area ΔA_i at a particular range h_i (see the shaded section in Fig. 4) to be a rough scatterer which has scattering properties that are independent of those for the surface area at any other range, one possible model[2] for the radar return is

$$r(t) = \sum_{k}^{k+N} f(t - \tau_i - \frac{\Delta\tau}{2}) \, [a_i \cos\omega_c t + b_i \sin\omega_c t] + n(t) \tag{27}$$

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*Conceivably the use of both amplitude and frequency modulation might be beneficial if short pulse durations are impractical. Only amplitude modulation is considered here.

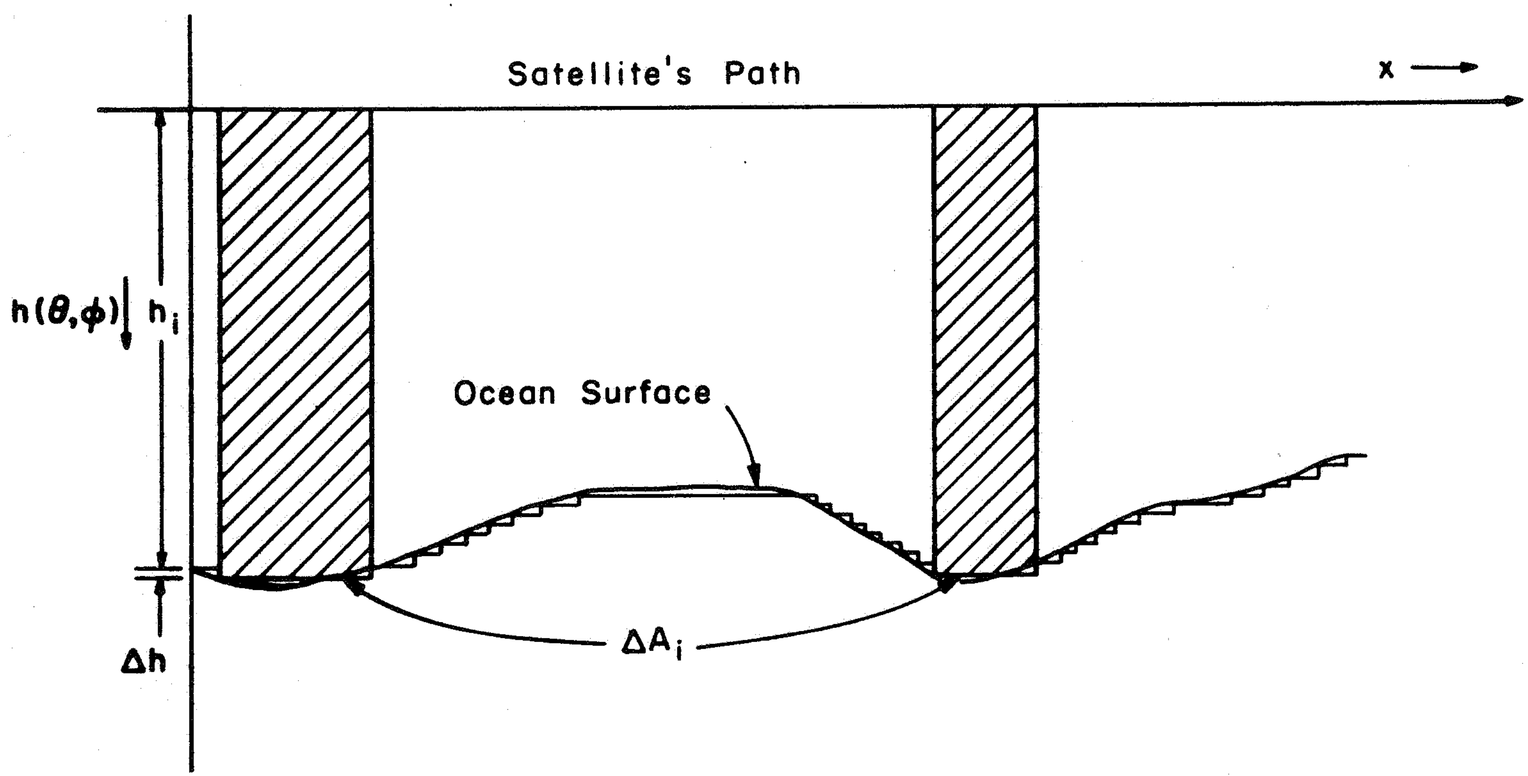

Figure 4 Staircase representation of ocean surface.

The a_i and b_i are independent, zero-mean, gaussian random variables with

$$E\{a_i^2\} = E\{b_i^2\} = \frac{1}{2}\rho_i \Delta A_i \tag{28}$$

where ρ_i is a reflection coefficient and

$$\tau_i \triangleq \frac{2h_i}{c}$$
$$\Delta\tau \triangleq \frac{2\Delta h}{c} \tag{29}$$

It is assumed in summing over the index i that the ocean surface lies within a known altitude band

$$h_k \leq h_i \leq h_{k+N}$$

Any doppler shift caused by motion of the ocean surface with respect to the satellite is neglected. Therefore, r(t) is a zero-mean gaussian random process.

Now with the assumption that

$$f(t) = 0; \ t < 0; \ t \geq \frac{2\Delta h}{c} \tag{30}$$

it is clear that Eq. (27) becomes

$$r(t) = f(t-\tau_i-\tfrac{\Delta\tau}{2})[a_i\cos\omega_c t+b_i\sin\omega_c t] + n(t); \quad \begin{cases} t \leq \tau_{i+1} + \dfrac{\Delta\tau}{2} \\ t \geq \tau_i + \dfrac{\Delta\tau}{2} \end{cases}$$

Recall that the quantity to be measured or estimated is the average range given by Eq. (18). This is determined from

$\tilde{h}$ in Eq. (14) and the transformation equation (22). In terms of the quantities in Fig. 4, $\tilde{h}$ is given by

$$\tilde{h} = \frac{\sum h_i \Delta A_i}{\sum \Delta A_i} \tag{32}$$

Because of the statistical dependence of the radar return upon ΔA_i, only an estimate $\hat{\tilde{h}}$ of $\tilde{h}$ can be obtained. To determine $\hat{h}$, define the quantities

$$\begin{aligned} V &= \sum h_i \Delta A_i \\ A &= \sum \Delta A_i \end{aligned} \tag{33}$$

and their estimates V and A. It is our intention to compute h as

$$\hat{h} = \frac{\hat{V}}{\hat{A}} \tag{34}$$

This is an optimal estimation procedure when the estimation criterion is that of maximum liklihood.[5]

Maximum liklihood estimates $\hat{V}$ and $\hat{A}$ are determined in terms of the corresponding estimates $\Delta\hat{A}_i$ of ΔA_i. With the model for the radar return given by Eq. (31), the contributions to r(t) from each ΔA_i are orthogonal in time. Hence the maximum liklihood estimate of ΔA_i is defined[5] by the relations*

$$\Delta\hat{A}_i = \frac{4}{\rho_i E_t^2}\left[r_{a_i}^2 + r_{b_i}^2 - E_t^2\sigma_n^2\right] \tag{35}$$

$$E_t = \int_0^{2\Delta h/c} f^2(t)\,dt \tag{36}$$

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* The contributions of double frequency terms in Eqs. (37) and (38) are ignored.

$$r_{a_i} = \int_{\tau_k}^{\tau_{k+N+1}} r(t)f(t-\tau_i-\frac{\Delta\tau}{2})\cos\omega_c t dt \tag{37}$$

$$r_{b_i} = \int_{\tau_k}^{\tau_{k+N+1}} r(t)f(t-\tau_i-\frac{\Delta\tau}{2})\sin\omega_c t dt \tag{38}$$

and

$$\begin{aligned} \hat{V} &= \sum h_i \Delta\hat{A}_i \\ A &= \sum \Delta\hat{A}_i \end{aligned} \tag{39}$$

The estimate of $\tilde{h}$ is computed according to Eq. (34).

The important statistical parameters of $\hat{\tilde{h}}$--its mean and variance--are calculated from a linearized analysis of the effect of the errors in $\hat{V}$ and $\hat{A}$. Neglecting the effect of the noise, n(t),

$$E\ \{\hat{\tilde{h}}\} = \tilde{h} \tag{40}$$

$$E\ \{(\tilde{h}-\hat{\tilde{h}})^2\} = \sum_{k}^{k+N} \frac{(\tilde{h} - h_i)^2\ \Delta A_i{}^2}{A^2} \tag{41}$$

The reason for neglecting n(t) in these calculations is that the return signal to noise ratio is quite large in this application so that estimation errors caused by the presence of n(t) are small compared with those attributed to the statistical nature of the reflection process. In particular, if there are N levels in Fig. 4 having equal areas ΔA, Eq. (41) becomes

$$E\{(\tilde{h}-\hat{h})^2\} = \frac{2\Delta h^2}{N^2} \sum_{n=1}^{\frac{N-1}{2}} n^2 \tag{42}$$

From the fact that Δh is inversely proportional to N, it is clear that

$$\lim_{N\to\infty} E\{(\tilde{h}-\hat{h})^2\} = 0$$

However this limiting behavior occurs because the return signal in Eq.(31) is assumed to consist of pulses that are disjoint in time; this condition cannot be practically maintained in the limit as $\Delta h \to 0$. In the remainder of this section we develop a signal model that conforms more closely to the actual radar return and propose a processing technique, motivated by the preceding discussion, to obtain an unbiased estimate of $\tilde{h}$ that is not necessarily statistically optimal. A high premium is not placed upon obtaining an optimum estimate because the signal to noise ratio at the radar receiver is quite large in this application.

A desirable simplification in the subsequent discussion is the assumption that ρ_i in Eqs. (28) and (35) is independent of i. It is well known that the gross scattering properties of the sea surface are a function of the sea state, i.e., surface turbulence. It might also be conjectured that the scattering from surface area at any particular altitude is a function of altitude because of a possible variation of average sea surface slope with altitude.[3] If such be the case, quantitative information about the variation of ρ_i is required. In the absence of such information we take $\rho_i = \rho$, which is altitude independent; this assumption materially simplifies the subsequent discussion.

The form of the estimation equations (35)-(39) suggests a useful estimation procedure when Eqs. (27) and (31) do not hold. Returning to the expression for r(t) in Eq. (27) which applies for a general f(t), require that f(t) be zero outside some interval;

$$f(t) \neq 0 \text{ only if } t \geqq 0 \text{ or } t \leqq t_f$$

Define

$$\begin{aligned} r_c(\lambda) &= \int_T r(t)f(t-\lambda) \cos\omega_c t dt \\ r_s(\lambda) &= \int_T r(t)f(t-\lambda) \sin\omega_c t dt \end{aligned} \qquad (43)$$

where T is a connected region for which it is known, a proiri that

$$r(t) - n(t) \equiv 0 \text{ if } t \notin T$$

Compute

$$\begin{aligned} A_c &= \int_L r_c^2(\lambda)d\lambda \\ A_s &= \int_L r_s^2(\lambda)d\lambda \end{aligned} \qquad (44)$$

and

$$\begin{aligned} V_c &= \frac{c}{2}\int_L \lambda r_c^2(\lambda)d\lambda \\ V_s &= \frac{c}{2}\int \lambda r_s^2(\lambda)d\lambda \end{aligned} \qquad (45)$$

where L is a connected region such that L contains T and L contains $(t - t_f)$ for every t in T. Upon substitution from Eq. (27) into Eqs. (43)-(45), it is readily verified that

$$r_c(\lambda) = \frac{1}{2}\sum_{i=k}^{k+N} g(\lambda-\tau_i-\frac{\Delta\tau}{2})a_i + \omega_c(\lambda) \tag{46}$$

$$r_s(\lambda) = \frac{1}{2}\sum_{i=k}^{k+N} g(\lambda-\tau_i-\frac{\Delta\tau}{2})b_i + \omega_s(\lambda)$$

$$\omega_c(\lambda) = \int_T f(t-\lambda)\cos\omega_c t\, n(t)dt \tag{47}$$

$$\omega_s(\lambda) = \int_T f(t-\lambda)\sin\omega_c t\, n(t)dt$$

where

$$g(\lambda) \triangleq \int_{-\infty}^{\infty} f(x)f(x+\lambda)dx \tag{48}$$

It follows that

$$E\{A_c\} = E\{A_s\} = \frac{\rho GA}{8} + \frac{F\sigma^2{}_n}{2} \tag{49}$$

$$E\{V_c\} = E\{V_s\} = \frac{c}{8}\left\{ \int_{-\infty}^{\infty} xg^2(x)dx\, \frac{\rho A}{2} + \frac{\rho}{2}\int_{-\infty}^{\infty} g^2(x)dx \sum_{k}^{k+N}(\tau_i+\frac{\Delta\tau}{2})\Delta A_i + 2\sigma^2{}_n H \right\} \tag{50}$$

where G, F, and H are given by

$$H \triangleq \int_L \lambda d\lambda \int_T f^2(u-\lambda)du;$$

$$G \triangleq \int_{-\infty}^{\infty} g^2(x)dx;$$

$$F \triangleq \int_L d\lambda \int_T f^2(u-\lambda)\ du \tag{51}$$

Now referring to Eqs. (49) and (50), we observe that by defining the estimates

$$\hat{A} \triangleq \frac{4}{\rho G}[A_c + A_s - F\sigma^2{}_n] \tag{52}$$

$$\hat{V} \triangleq \frac{V_s+V_c-\frac{c\sigma^2{}_n H}{2}}{\frac{\rho}{4} G} - \hat{A}\ \frac{c\int_{-\infty}^{\infty} \tau g^2(\tau)d\tau}{2G} \tag{53}$$

then

$$E\{\hat{A}\} = A$$

$$E\{\hat{V}\} = V$$

If $\hat{\tilde{h}}$ is computed as in Eq. (34), it follows that

$$\hat{\tilde{h}} = \frac{V_s+V_c-\frac{c\sigma^2{}_n H}{2}}{A_s+A_c-F\sigma^2{}_n} - \frac{c}{2G}\int_{-\infty}^{\infty} xg^2(x)dx \tag{54}$$

Observe that $\hat{h}$ is determined without knowledge of ρ; hence the sea state need not be known.

The calculations and operations required to obtain $\hat{\tilde{h}}$ from Eq. (54) are summarized in the block diagram of Fig. 5. The blocks denoted LPF symbolize a low pass filter which represents rejection of the double frequency terms in the outputs of the multipliers. The block labeled $f(-t)$ denotes a linear filter whose impulse response is $f(-t)$. Since $f(t) = 0$ for $t < 0$, the filter is unrealizable; however, it is realizable with a delay of t_f seconds.

Our next task is to determine the properties of the estimation error.

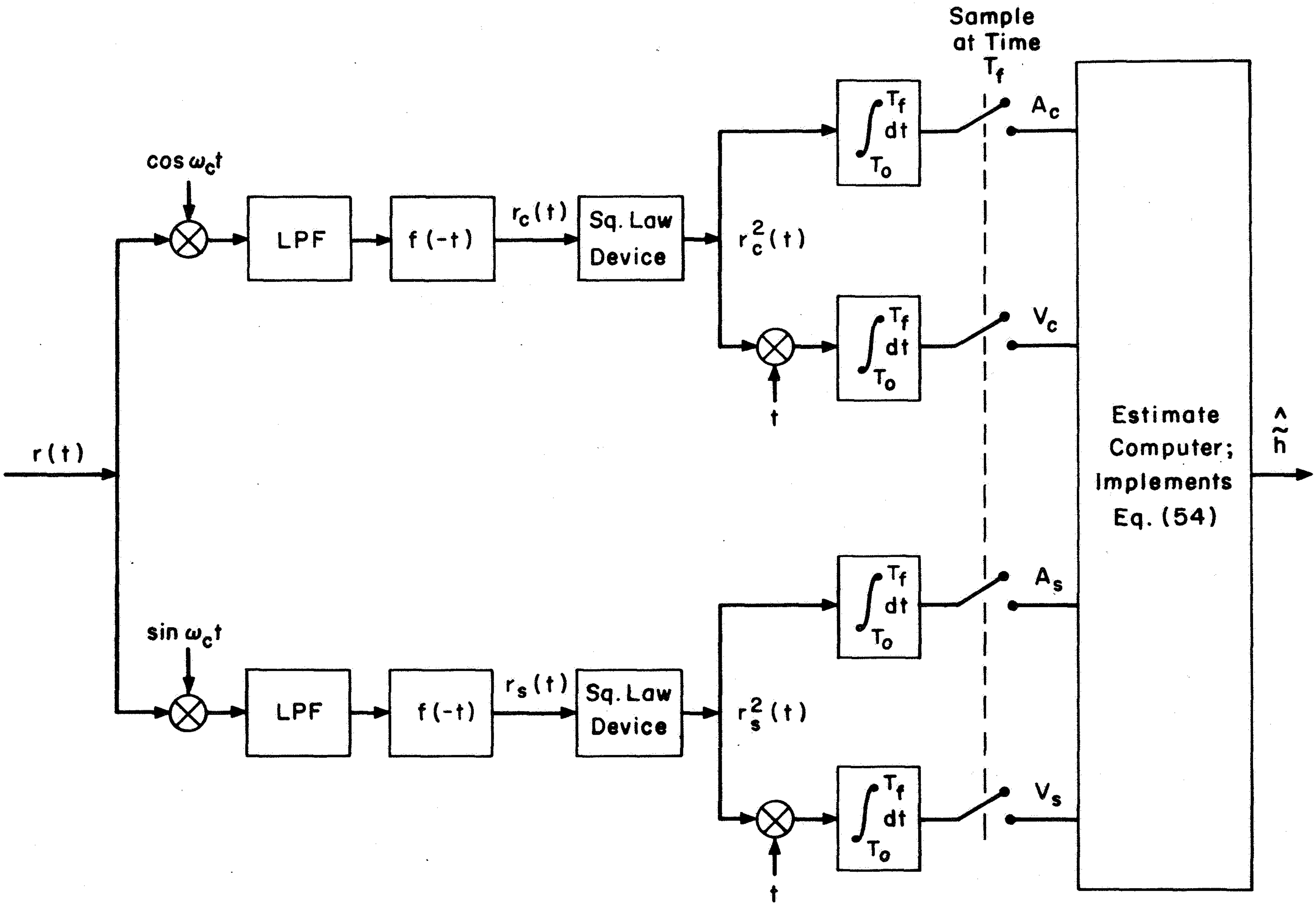

Figure 5 Radar return signal processing technique.

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VI. ESTIMATION ERRORS

Because the random variables--V_s, V_c, A_s, A_c--determine $\hat{h}$, their errors determine the inaccuracy in the altitude estimate. To facilitate the error analysis, expressions are developed for these quantities which are useful in the limit as $N \to \infty$ and $\Delta\tau \to 0$ in Eq. (27).

Consider $r_c(\lambda)$ and $r_s(\lambda)$ given by Eq. (46). Define the gaussian random processes

$$
\begin{aligned}
a(\tau) &\triangleq \sum_{i=k}^{k+N} a_i \delta\left(\tau-\tau_i-\frac{\Delta\tau}{2}\right) \\
b(\tau) &\triangleq \sum_{i=k}^{k+N} b_i \delta\left(\tau-\tau_i-\frac{\Delta\tau}{2}\right)
\end{aligned}
\tag{55}
$$

where $\delta(\ )$ is the Dirac delta function. Substitution of Eq. (55) into Eq. (46) produces

$$
\begin{aligned}
r_c(\lambda) &= \frac{1}{2}\int_T g(\lambda-\tau)a(\tau)d\tau \\
r_s(\lambda) &= \frac{1}{2}\int_T g(\lambda-\tau)b(\tau)d\tau
\end{aligned}
\tag{56}
$$

Equations (56) are to be substituted into Eqs. (44) and (45).

To shorten notation, define

$$c_1 \triangleq c\sigma_n{}^2 H/2$$

$$c_2 \triangleq F\sigma_n{}^2$$

$$c_3 \triangleq (c/2G)\int_{-\infty}^{\infty} \tau g^2(\tau)d\tau$$

so that the altitude estimate can be written as

$$\hat{\tilde{h}} = \frac{V_s + V_c - c_1}{A_s + A_c - c_2} - c_3 \tag{58}$$

and define

$$\begin{aligned} V_s &\triangleq E\{V_s\} + \Delta V_s \\ V_c &\triangleq E\{V_c\} + \Delta V_c \\ A_s &\triangleq E\{A_s\} + \Delta A_s \\ A_c &\triangleq E\{A_c\} + \Delta A_c \end{aligned} \tag{59}$$

Inspection of Eqs. (52)-(54) reveals that

$$\tilde{h} = \frac{E\{V_s + V_c\} - c_1}{E\{A_s + A_c\} - c_2} - c_3 \tag{60}$$

With altitude error defined as

$$\varepsilon_{\tilde{h}} = \hat{\tilde{h}} - \tilde{h} \tag{61}$$

it follows from the definitions (59) that

$$\varepsilon_{\tilde{h}} = \frac{\Delta V_s + \Delta V_c - (\tilde{h} + c_3)(\Delta A_s + \Delta A_c)}{E\{A_s + A_c\} + \Delta A_s + \Delta A_c - c_2} \tag{62}$$

The simplest analysis of $\varepsilon_{\tilde{h}}$ is obtained by assuming that the quantity $(\Delta A_s + \Delta A_c)$ in the denominator of Eq. (62) is small (on the average) compared with $(E\{A_s + A_c\} - c_2)$ so that a linearization is valid. Hence, subject to the condition,

$$E\{(\Delta A_s + \Delta A_c)^2\} << [E\{A_s + A_c\} - c_2]^2 \tag{63}$$

one can verify that

$$E\{\varepsilon_{\tilde{h}}\} \cong 0 \tag{64}$$

$$E\{\varepsilon_{\tilde{h}}^{2}\} \cong \frac{E\{[\Delta V_s+\Delta V_c-(\tilde{h}+c_3)(\Delta A_s+\Delta A_c)]^2\}}{[E\{A_s+A_c\}-c_2]^2} \tag{65}$$

To evaluate Eq. (65) assume that a(t) and b(t) in Eq. (55) satisfy

$$E\{a(t)a(u)\} = E\{b(t)b(u)\} = k(u)\delta(t-u) \tag{66}$$

where

$$\int_T k(u)du = \frac{\rho A}{2} \tag{67}$$

This is the conventional model[6] for a rough radar reflector whose dimensions are large compared with a wavelength.* With the aid of Eqs. (56) and (66) and considerable manipulation (using the fact that a(t) and b(t) are gaussian random processes having identical statistics), Eq. (65) becomes

$$E\{\varepsilon_{\tilde{h}}^{2}\} = \frac{16c^2}{\rho^2 G^2 A^2}\int_L\int_L tuR^2(t,u)dtdu + \frac{c^2}{4}\left(\frac{4}{\rho GA}\right)^4\left\{\frac{\rho A}{2}\int_{-\infty}^{\infty} xg^2(x)dx + \right.$$

$$\left. G\int_{-\infty}^{\infty}\tau k(\tau)d\tau\right\}^2\int_L\int_L R^2(t,u)dtdu - c^2\left(\frac{4}{\rho GA}\right)^3\left\{\frac{\rho A}{2}\int_{-\infty}^{\infty} xg^2(x)dx + \right.$$

$$\left. G\int_{-\infty}^{\infty}\tau k(\tau)d\tau\right\}\int_L\int_L tR^2(t,u)dtdu \tag{68}$$

---

*For the piecewise constant surface represented by Fig.4,

$$k(u) = \frac{\rho}{2}\sum \Delta A_i\delta(u-\tau_i-\frac{\Delta\tau}{2})$$

where

$$R(t,u) = \frac{1}{4}\int_T [g(t-y)g(u-y)k(y) + 2\sigma_n^2 f(y-t)f(y-u)]dy \tag{69}$$

There is no conceptual difficulty in determining $E\{\varepsilon_{\tilde{h}}^2\}$ for any choice of signal modulation, f(t), and any area distribution, k(t). However, primarily because k(t) and f(t) are non-zero only in finite intervals, the integrations required in Eqs. (68) and (69) are laborious to evaluate analytically and calculation by digital computer is probably more expedient. In this discussion an analytic expression for the mean squared error is desirable to aid in understanding how it is affected by various important parameters. Such a relation can be obtained if the finite intervals are approximated with functions f(t) and k(t) that approach zero asymptotically.

Suppose f(t) is given by*

$$f(t) = Fe^{-\alpha(t-t_0)^2} \tag{70}$$

where F and α are constants selected to meet energy and pulse duration constraints. Similarly assume that

$$k(t) = Be^{-\beta(t-t_1)^2} \tag{71}$$

subject to the constraint

$$\int_{-\infty}^{\infty} k(t)dt = B\sqrt{\frac{\pi}{\beta}} \triangleq \frac{\rho}{2} A \tag{72}$$

---

*The analytical invariance of gaussian pulses under convolution motivates these choices of f(t) and k(t). Note that α in Eqs.(70)-(76) has no relationship with the radar beam half-width in Fig.3.

The situation is as represented in Fig. 6. Let the region T be bounded by T_o and T_f and assume

$$\alpha \gg \frac{1}{T_f - T_o} \tag{73}$$

so that k(t) is approximately contained entirely within T. Note that k(t) in Fig. 6 represents the ocean surface area distribution as a function of delay time; It is not the return echo.

With the above assumptions the mean squared error, as determined by Eq. (68) is

$$E\{\varepsilon_{\tilde{h}}^2\} = \frac{c^2}{64\beta}\sqrt{\frac{1}{1+\alpha/\beta}}\left(\frac{3+\alpha/\beta}{1+\alpha/2\beta}\right) + \sqrt{\frac{4}{3}}\ \frac{\sigma_n^2 c^2}{\rho A F^2 \sqrt{\frac{\pi}{2}}}\left\{\frac{1}{\beta} + \frac{1}{3\alpha}\right\} +$$

$$\frac{4}{3}\ \frac{\sigma_n^4 c^2}{\rho^2 F^4 \left(\frac{\pi}{2\alpha}\right)^{3/2} A^2}\left(\Delta_1^3 + \Delta_2^3\right) \tag{74}$$

where Δ_1 and Δ_2 are defined in Fig. 6. The right hand side of Eq. (74) is thought of as consisting of two parts. The first term is caused by the statistical nature of scattering from the ocean's surface; the second and third terms represent the influence of the additive noise, n(t). Because the noise power is relatively small in this application, we examine the behavior of Eq. (74) when $\sigma_n^2 = 0$. A graph of $E\{\varepsilon_{\tilde{h}}^2\}$ as a function of α with β as a parameter is given in Fig. 7. Increasing values of α and β correspond to decreasing pulse "widths" for f(t) and k(t). Given a value of β, which determines the

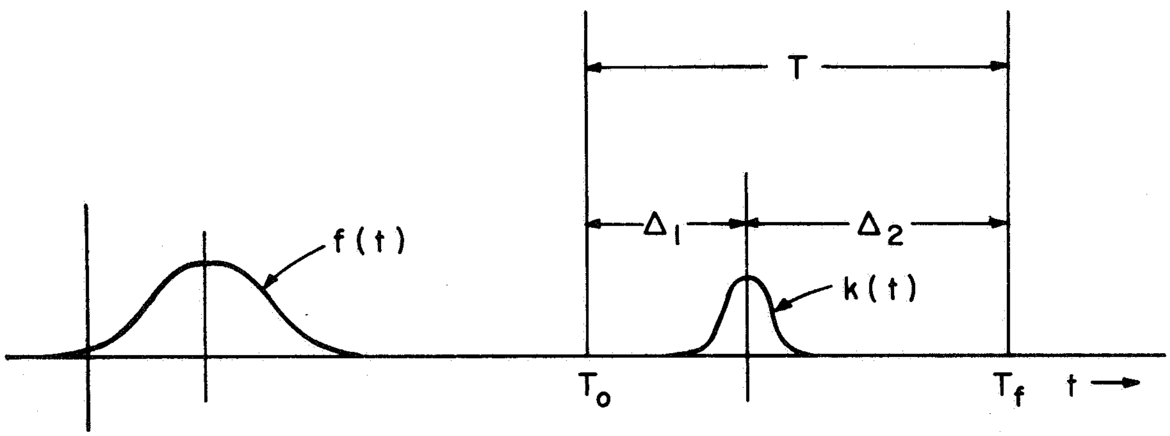

Figure 6 Qualitative form of f(t) and k(t) for illustrative example.

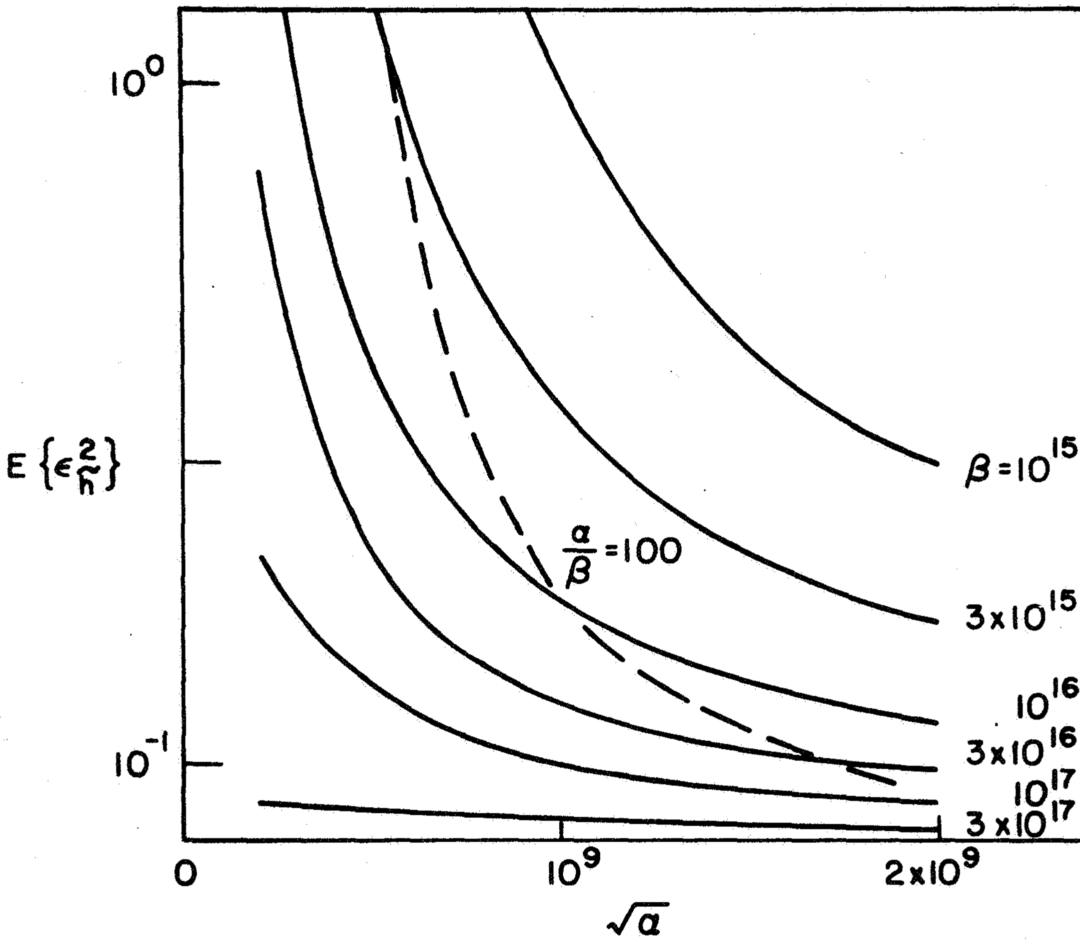

Figure 7 Mean square error vs $\sqrt{\alpha}$ for $\sigma_n^2=0$.

spread of $k(t)$, $E\{\varepsilon_{\tilde{h}}^2\}$ decreases with increasing α. This behavior is explained by the fact that as $f(t)$ becomes narrower, the return signal is approximately a large number of orthogonal (in time) reflections. We observed similar properties for the "staircase" ocean surface discussed in the preceding section. Given a value of α, the mean square error decreases with increasing β because the spreading of the return signal is reduced as the ocean surface becomes concentrated at one altitude. For large values of (α/β)

$$E\{\varepsilon_{\tilde{h}}^2\} \cong \frac{c^2}{32\beta}\sqrt{\frac{\beta}{\alpha}}\ ; \quad \alpha/\beta >> 1; \quad \sigma_n^2 = 0 \qquad (75)$$

Thinking in terms of the ocean area distributed over 10 feet in altitude and a one nanosecond duration for $f(t)$, take $\alpha = 4 \times 10^{18}/\text{sec.}^2$, $\beta = 10^{16}/\text{sec.}^2$, $c^2 = 10^{18}\text{ft.}^2/\text{sec.}^2$ For these values Eq. (75) yields

$$E\{\varepsilon_{\tilde{h}}^2\} = 0.16 \text{ feet}^2$$

Recall that the validity of this analysis is subject to (63). For $\sigma_n^2 = 0$, this condition takes the form

$$\frac{[E\{A_s + A_c\} - c_2]^2}{E\{[\Delta A_s + \Delta A_c]^2\}} = \frac{1}{2}\sqrt{1+\frac{\alpha}{\beta}} >> 1; \quad \sigma_n^2 = 0 \qquad (76)$$

which is satisfied for large (α/β). For small (α/β), the ocean surface becomes flat and Eq. (58) yields

$$\lim_{\beta\to\infty} \hat{h} = \frac{\tilde{h}\hat{A}}{\tilde{A}} = \tilde{h} \text{ with probability 1.}$$

The event that the denominator of the right hand side of Eq. (58) is zero, rendering $\hat{h}$ undefined, has probability zero.

VII. THE GATED WIDE-BEAM RADAR ALTIMETER

The preceding sections discuss the subject of processing the entire return from a narrow-beam radar altimeter. This section treats the problem of obtaining an altitude estimate from only a portion of the return signal for a wide beam radar. The wide beam has the advantages of requiring a relatively smaller antenna and requiring less pointing accuracy than a narrow beam. Hence an appropriate processing technique for the former may be desirable in some applications.

The physical situation for a wide-beam is illustrated in Fig. 8. The energy in the center of the cone strikes the ocean surface appreciably before the energy near its boundary. For the reasons cited in section III, only the return from a relatively small portion of the sea surface is desired. Hence it is proposed that the return signal be truncated to eliminate all reflections except those from the surface area near the beam's center.

Assume that the return signal is truncated at time T so that only reflections from objects within a distance, cT/2, are to be considered; the propagated position of the radar beam at time T/2 is indicated in Fig. 8. The receiver processes only the portion of the return signal occurring at $t \leq T$ as indicated in Fig. 9. Using the notation, a(t) and b(t), introduced in Eq. (55), the return signal r(t) (analogous to Eq. (27) for the discrete case) has the form

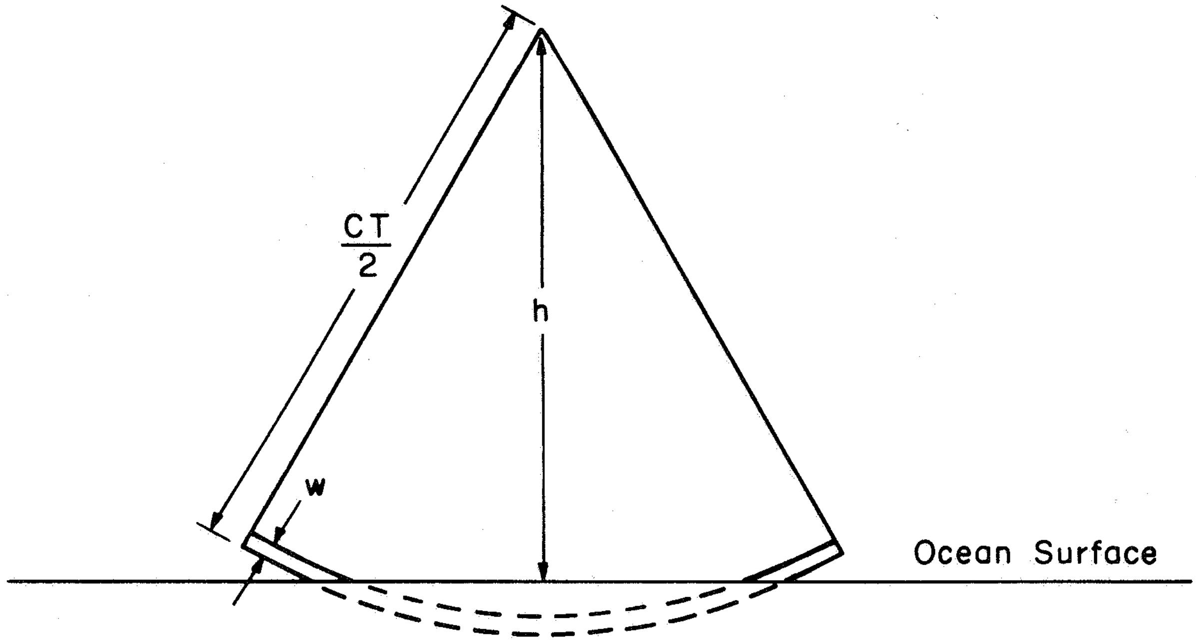

Figure 8 Wide - beam geometry.

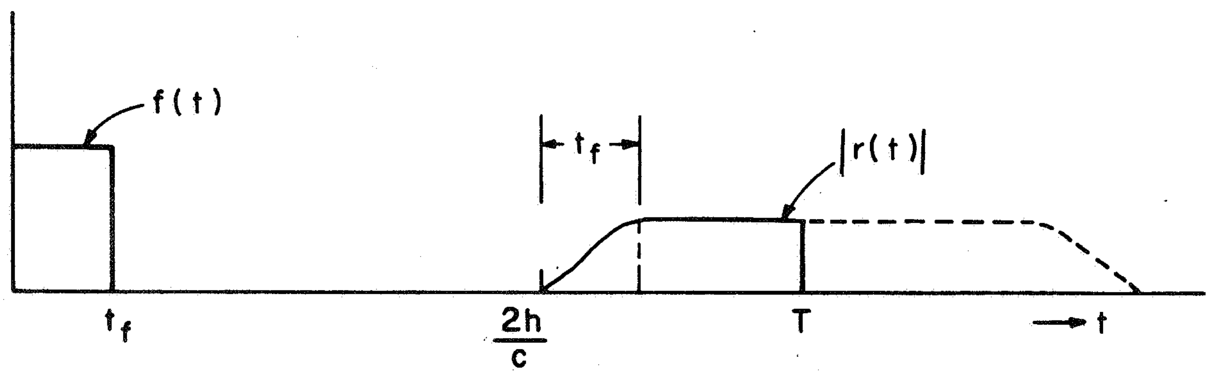

Figure 9 Illustration of truncated radar return signal.

$$r(t) = \begin{cases} \int_0^t f(t-\tau)[a(t)\cos\omega_c t + b(\tau)\sin\omega_c t]d\tau + n(t); & t \leq T \\ 0 \; ; \quad t > T \end{cases} \qquad (77)$$

In order to apply the processing technique represented by Eq. (58), it is desired that the return have the form $r_d(t)$ given by

$$r_d(t) = \int_0^t f(t-\tau)[a_d(\tau)\cos\omega_c t + b_d(\tau)\sin\omega_c t]d\tau + n_d(t); \quad t \geq 0 \qquad (78)$$

$$a_d(t) = \begin{cases} a(t); & t \leq T \\ 0 \; ; & t > T \end{cases}$$

$$b_d(t) = \begin{cases} b(t); & t \leq T \\ 0 \; ; & t > T \end{cases}$$

where $n_d(t)$ is unbiased noise. Note that

$$r(t) = r_d(t); \quad t \leq T$$

That is, $r_d(t)$ should contain the total reflection from the desired element of sea surface area; $r(t)$ in Eq. (77) does not have this property. If $\hat{h}$ is computed according to Eq. (58) for $r(t)$, the result is generally biased. This is qualitatively evident from the shape of the received pulse in Fig. 9; it lacks symmetry because of the rise of the leading edge. This bias is referred to here as truncation error.

An immediate question is whether there are circumstances in which $r(t) \cong r_d(t)$ for all t. The answer is obtained qualitatively by determining the bias for a particular case. It is intuitively clear that the amount of error is related to the width, w, of the wavefront in Fig. 8. A rough analysis for the case when f(t) is a rectangular pulse and the sea surface is normal to the center line of the radar beam indicates that

$$E\{\hat{h} - \tilde{h}\} \sim -\frac{\omega}{4} \tag{79}$$

Thus the truncation error for this case is negligible compared with a desired resolution of one foot if $\omega << 4$ feet, which implies that the duration of f(t) should be much less than four nanoseconds. Pulse durations of one or two nanoseconds at X-band frequencies have been proposed as feasible within the current state of the art.[2]

Another approach to this problem is to permit relatively long transmitted pulses and accurately compute the bias resulting from Eq. (58) for a "mean" sea state, say a flat sea, and then subtract the bias from $\hat{h}$. Because the value of the truncation error is a function of the shape of the ocean surface, this procedure can only be justified if a mean sea state can be defined. Such a definition implies some sort of a-priori knowledge about the probability distribution of the shape of k(t). With the mean bias error subtracted out of the altitude estimate, the remaining truncation error is caused by the variation in sea state about the mean and can be considered an additional source of estimation error. A procedure for computing the truncation error for a given sea state is outlined in Appendix B.

A third solution to the problem of eliminating truncation error is motivated by our previous assumption that the effect of $n(t)$ on the estimation error is small compared with the influence of other sources. If $n(t)$ were identically zero in Eq. (77), one could imagine processing $r(t)$ to recover $a(t)$ and $b(t)$ which are subsequently "filtered" to obtain $r_d(t)$ exactly. If $n(t)$ is negligible, one may try the same technique and analyze its performance. An equivalent procedure described here is to operate upon $f(t)$ to generate $r_{d_c}(t)$ and $r_{d_s}(t)$, corresponding to Eq. (56), for $r_d(t)$.

To facilitate the discussion, consider first the "quadrature amplitudes,"

$$q_c(t) \triangleq \begin{cases} \frac{1}{2}\int_0^t f(t-\tau)a(\tau)d\tau; & t \leq T \\ 0 \quad ; & t > T \end{cases} \tag{80}$$

$$q_{d_c}(t) \triangleq \frac{1}{2}\int_0^t f(t-\tau)a_d(\tau)d\tau; \quad t \geq 0 \tag{81}$$

After one determines how $q_{d_c}(t)$ can be obtained from $q_c(t)$, the method for obtaining $r_{d_c}(t)$ and $r_{d_s}(t)$ from $r(t)$ is obvious.

Using the notation $f^{-1}(t)$ to represent an inverse convolution (which is readily derived from the inverse fourier transform of $f(t)$), one can immediately derive a method for obtaining $q_{d_c}(t)$ by comparing Eqs. (80) and (81). This is indicated in Fig. 10 where the construction of both $q_c(t)$ and $q_{d_c}(t)$ are illustrated. The gate operation is a multiplication of the output of $f^{-1}(t)$ by a function that has the value, one, on the interval $0 \leq t \leq T$ and zero elsewhere. The blocks labled $f(t)$

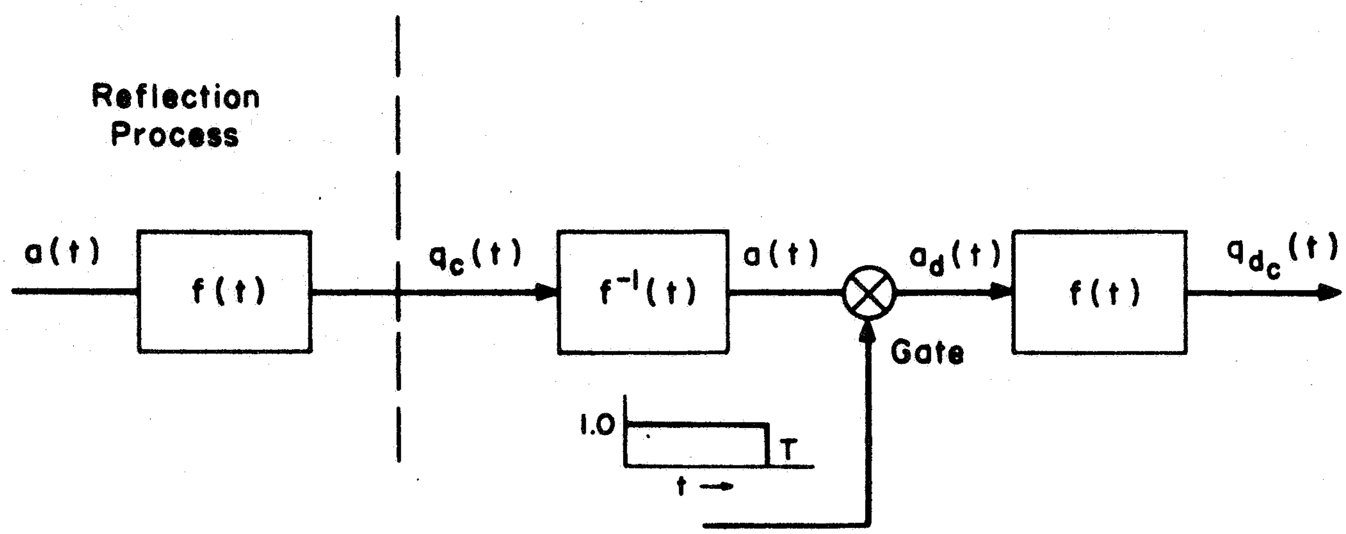

Figure 10 Procedure for generating $q_{d_c}(t)$ from $q_c(t)$.

and $f^{-1}(t)$ represent convolution operations on their respective inputs.

With Fig. 10 in mind, it is apparent that $r_{d_c}(t)$ and $r_{d_s}(t)$ are generated from r(t) by the operations shown in Fig. 11. These quantities play the role of $r_c(t)$ and $r_s(t)$ in Fig. 5 and are processed in the same manner. The presence of the low-pass filters symbolizes the neglect of trigonometric functions of twice the carrier frequency, as in Fig. 5.

The analysis of the performance of the estimate obtained by combining the operations represented in Figs. 5 and 11 is similar to that given in section VI. One difference is the fact that the characteristics of the noise components of $r_{d_c}(t)$ and $r_{d_s}(t)$ are not the same in the interval $t > T$ as they are in the interval $t \leq T$. This is explained by the fact that the gate in Fig. 11 cuts off the signal at time T and $r_{d_c}(t)$ and $r_{d_s}(t)$ exhibit a transient behavior. The signal component of this transient is just that required to duplicate the effect of processing only the signal component of $r_d(t)$ in Eq. (78) by the method of section V. However, the noise component of the transient is produced by the presence of n(t) in Eq. (77). Because the system in Fig. 11 is designed without regard for optimality in the presence of n(t), the noise component in the output for $t > T$ may or may not be excessive. The characteristics of $f^{-1}(t)$ are directly related to this question. Because this filter must be implemented, it is desirable that its output be bounded for bounded inputs; otherwise instability may cause saturation of physical

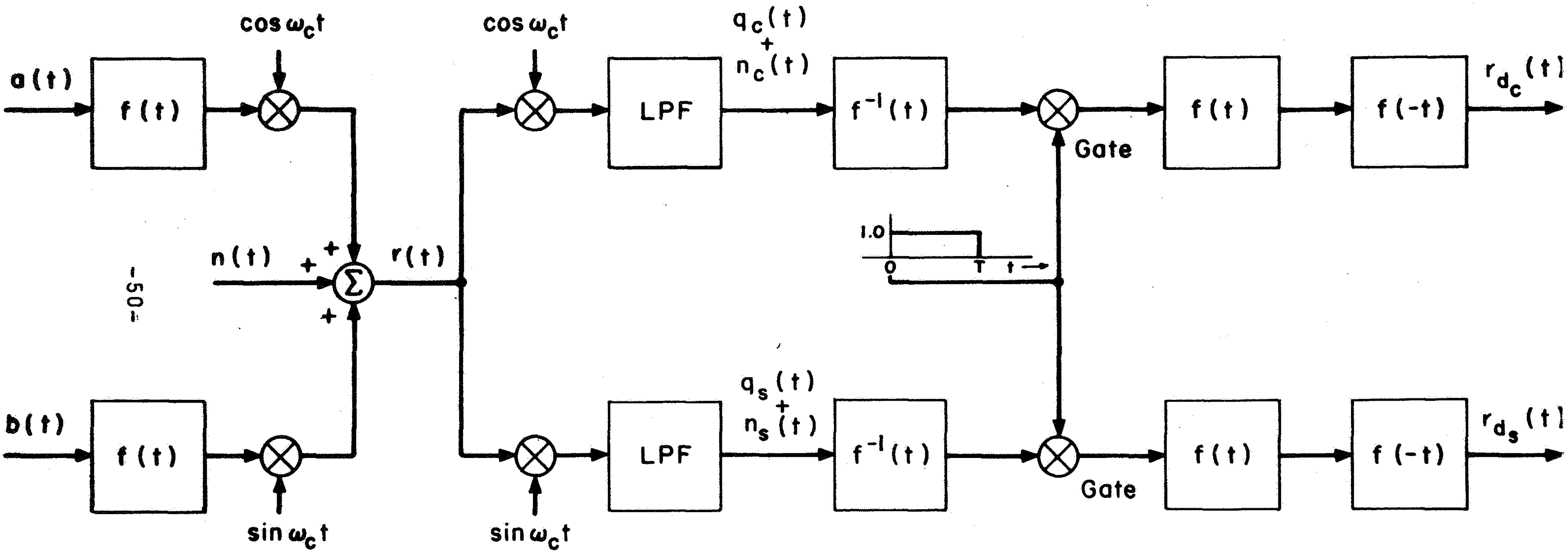

Figure 11 Procedure for generating $r_{d_c}(t)$ and $r_{d_s}(t)$ from $r(t)$

components in the presence of noise.

The intent of this section is to demonstrate that the processing techniques described in section V can be adapted to a gated wide-beam radar under certain conditions. Three suggested approaches are: (1) Use a short transmitted pulse to minimize the effect of truncation error, (2) Subtract a mean truncation error from the altitude estimate, (3) Eliminate the source of truncation error by suitable signal processing. These methods are listed in order of increasing complexity required for the signal processing system. Regardless of which technique is used, the objective is to eliminate systematic bias error introduced by truncating the radar return signal.

SUMMARY AND CONCLUSIONS

This report discusses methods for eliminating sources of bias errors in satellite altitude measurements made by a pulsed radar altimeter. In sections II and III, altitude is appropriately defined for a radar beam having non-zero width, and design criteria for beam width and pulse repetition frequency are suggested by Eqs. (10) and (12). This definition of altitude has the feature that effects of unwanted high frequency variations of the ocean's surface occurring within the beam area are partially eliminated from the return signal. Methods of estimating altitude are presented in sections V and VII for narrow beam and gated wide beam radars and are summarized in Figs. 5 and 11. A performance analysis given in section VI indicates that the effects of random scattering on estimation errors are small if the transmitted pulse is very short. In section IV it is shown that the altitude measurement obtained in terms of the radar's spherical coordinate system can be accurately modified according to Eq. (22) to obtain the corresponding measurement in a cartesian coordinate system.

The hardware requirements for processing each radar return pulse by the methods suggested here are more complex than those necessary for a leading edge tracker. However, the required mathematical operations--integration, filtering, squaring--are within the capabilities current radar system design. The processing time for each pulse is inherently small because few digital computations are required. The complexity of the system is a result of the need to perform certain averaging operations on the return pulse to prevent bias errors of the same order of magnitude as the ultimate resolution desired.

APPENDIX A

1. Accuracy Analysis for Average Radar Range Equations

The correct expression for $\tilde{A}$ in Eq. (14) is

$$\tilde{A} = \int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} r^2(\theta,\phi)d\theta d\phi \tag{A-1}$$

It is proved here that with $\tilde{h}$ substituted for $r(\theta,\phi)$ in Eq. (A-1), as indicated in Eq. (15), the resulting accuracy in computing A is given by Eqs. (16) and (17).

Substitution of Eq. (A-1) into Eq. (14) produces

$$\tilde{h} = \frac{\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [\tilde{h}^3 + g(\Delta r)]\cos\theta d\theta d\phi}{\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [\tilde{h}^2 + f(\Delta r)]\cos\theta d\theta d\phi} \tag{A-2}$$

where

$$r = \tilde{h} + \Delta r(\theta,\phi)$$

$$g(\Delta r) = 3\tilde{h}^2\Delta r + 3\tilde{h}\Delta r^2 + \Delta r^3 \tag{A-3}$$

$$f(\Delta r) = 2\tilde{h}\Delta r + \Delta r^2$$

For notational convenience, the functional dependence of Δr upon θ and ϕ is understood. Regarding the effect of $f(\Delta r)$ as small compared with $\tilde{h}^2$ in the denominator of Eq. (A-2), write $\tilde{h}$ in expanded form as

$$\tilde{h} = \left[\tilde{h} + \frac{I[g(\Delta r)]}{I[1]\tilde{h}^2}\right]\left[1 - \frac{I[f(\Delta r)]}{I[1]\tilde{h}^2} + \left\{\frac{I[f(\Delta r)]}{I[1]\tilde{h}^2}\right\}^2 - \ldots\right] \tag{A-4}$$

where

$$I[\ ] \triangleq \int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [\quad]\cos\theta\, d\theta\, d\phi \tag{A-5}$$

Expansion of Eq. (A-4) to second order in Δr produces

$$I[\Delta r] - \frac{2}{\tilde{h}I[1]}\{I[\Delta r]\}^2 + \frac{2}{\tilde{h}} I[\Delta r^2] \cong 0 \tag{A-6}$$

From the physical requirement that Δr be small, the only acceptable solution to Eq. (A-6) is

$$I[\Delta r] \cong -\frac{2}{\tilde{h}} I[\Delta r^2] > -\frac{2}{\tilde{h}}\Delta r_{max}{}^2\alpha^2 \tag{A-7}$$

where Δr_{max} is defined in Eq. (17).

Now observe that

$$\tilde{A} = I[\tilde{h}^2 + 2\tilde{h}\Delta r + \Delta r^2] \cong I[\tilde{h}^2] + 2I[\tilde{h}\Delta r] \tag{A-8}$$

But from Eq. (A-7)

$$|2I[\tilde{h}\Delta r]| \leq \frac{4\Delta r_{max}{}^2\alpha^2}{\tilde{h}} \tag{A-9}$$

Hence

$$\tilde{A} = I[\tilde{h}^2] = \tilde{h}^2\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} \cos\theta\, d\theta\, d\phi \tag{A-10}$$

to the order accuracy stated in Eq. (16).

2. Accuracy Analysis for Average Radar Altitude Equation

To determine the accuracy of $\bar{h}$ in Eq. (20), first expand the trigonometric functions in numerator and denominator to second order in θ and ϕ. Next expand $\bar{h}$ to first order about the quantities

$$\tilde{V} \triangleq I[r^3]$$
$$\tilde{A} \triangleq I[r^2] \qquad \text{(A-11)}$$

The result is

$$\bar{h} \cong \frac{1}{\tilde{A}}[\tilde{V} + \Delta\tilde{V} - \frac{\tilde{V}\Delta\tilde{A}}{\tilde{A}}] \qquad \text{(A-12)}$$

where

$$\Delta\tilde{A} \triangleq \int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [-r^2(\frac{\theta^2}{2}+\frac{\theta^2}{2})+r\phi\Delta r_\phi+r\theta\Delta r_\theta]d\theta d\phi$$

$$\text{(A-13)}$$

$$\Delta\tilde{V} \triangleq \int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}} [-r^3(\theta^2+\phi^2)+r^2\phi\Delta r_\phi]d\theta d\phi$$

With

$$\tilde{h} \triangleq \frac{\tilde{V}}{\tilde{A}}$$

substitute for r from Eq. (A-3) into Eq. (A-13) so that Eq. (A-12) becomes

$$\bar{h} \cong \tilde{h}-\frac{\tilde{h}^3}{\tilde{A}}\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}}[\frac{\theta^2}{2}+\frac{\phi^2}{2}]d\theta d\phi + \frac{\tilde{h}}{\tilde{A}}\int_{-\alpha}^{\alpha}\int_{-\sqrt{\alpha^2-\phi^2}}^{\sqrt{\alpha^2-\phi^2}}[\Delta r\Delta r_\phi\phi+\Delta r\Delta r_\theta\theta]d\theta d\phi \qquad \text{(A-14)}$$

The last term on the right of Eq. (A-14) represents the error in $\bar{h}$ computed from Eq. (21). Assuming the terms, $\Delta r \Delta r_{\phi} \phi$ and $\Delta r \Delta r_{\theta} \theta$, are the same order of magnitude, the error is as given in Eq. (23).

APPENDIX B

Bias Computation for a Gated Wide-Beam Radar

To determine the effect of processing r(t) in Eq. (77) according to Eq. (58), define $r_c(\lambda)$ as in Eq. (56) and

$$r_{d_c}(\lambda) \triangleq \int_T r_d(t) f(t-\lambda)\cos\omega_c t\,dt \tag{B-1}$$

Using Eq. (77) and (78) for r(t) and $r_d(t)$ in Eq. (77) and (B-1), it follows that

$$\begin{aligned} r_c(\lambda) &= r_{d_c}(\lambda) \quad : \quad \lambda \leqq T_c - t_f \\ r_c(\lambda) &= r_{d_c}(\lambda) - \int_{T_c}^{\lambda+t_f} f(t-\lambda) r_d(t)\cos\omega_c t\,dt; \; T_c - t_f \leqq \lambda \leqq T_c \\ r_c(\lambda) &= r_{d_c}(\lambda) - r_{d_c}(\lambda) = 0; T_c \leqq \lambda \end{aligned} \tag{B-2}$$

Defining A_c as in Eq. (44) and A_{d_c} by

$$A_{d_c} \triangleq \int_L r_{d_c}{}^2(\lambda), \tag{B-3}$$

one determines from Eq. (B-2) that for a given sea state

$$E\{A_c\} = E\{A_{d_c}\} + E\left\{\int_{T_c-t_f}^{T_c} d\lambda \left[\left(\int_{T_c}^{\lambda+t_f} f(t-\lambda) r_d(t)\cos\omega_c t\,dt\right)^2 - 2r_{d_c}(\lambda)\int_{T_c}^{\lambda+t_f} f(t-\lambda) r_d(t)\cos\omega_c t\,dt\right] - \int_{T_c}^{T+t_f} d\lambda\, r_{d_c}(\lambda)^2\right\} \tag{B-4}$$

Defining $r_{d_s}(\lambda)$ and A_{d_s} in a similar fashion with $\cos\omega_c t$ in Eq.

(B-1) replaced by $\sin\omega_c t$, one can compute $E\{A_s\}$. As in section V, we neglect the contribution of double frequency terms to these expressions so that

$$
\begin{aligned}
E\{A_c\} &= E\{A_s\} \\
E\{A_{d_c}\} &= E\{A_{d_s}\}
\end{aligned}
\tag{B-5}
$$

An analogous argument applied to V_c with

$$V_{d_c} \triangleq \frac{c}{2}\int \lambda r_{d_c}{}^2(\lambda) \tag{B-6}$$

produces

$$
\begin{aligned}
E\{V_c\} = E\{V_{d_c}\} + \frac{c}{2}E\Bigg\{\int_{T_c-t_f}^{T_c} d\lambda\cdot\lambda\Bigg[\Bigg(\int_{T_c}^{\lambda+t_f} f(t-\lambda)r_d(t)\cos\omega_c t\,dt\Bigg)^2 - \\
2r_{d_c}(\lambda)\int_{T_c}^{\lambda+t_f} f(t-\lambda)r_d(t)\cos\omega_c t\,dt\Bigg] - \int_{T_c}^{T_c+t_f} d\lambda\cdot\lambda\cdot r_{d_c}(\lambda)^2\Bigg\}
\end{aligned}
\tag{B-7}
$$

and

$$
\begin{aligned}
E\{V_c\} &= E\{V_s\} \\
E\{V_{d_c}\} &= E\{V_{d_c}\}
\end{aligned}
\tag{B-8}
$$

Now the quantities $E\{V_{d_c}\}$and $E\{A_{d_c}\}$ are known from Eqs. (49) and (50). Hence only the second expectation on the right hand sides of Eqs. (B-4) and (B-7) results in a bias error in $\hat{h}$. If we define

$$E\{A_c\} = E\{A_{d_c}\} + B_A$$

$$E\{V_{d_c}\} + E\{V_{d_c}\} + B_V$$

where identification of B_A and B_V with terms in Eqs. (B-4) and (B-7) is obvious, the estimation error in Eq. (59) becomes

$$\varepsilon_{\tilde{h}} = \frac{\Delta V_{d_s} + \Delta V_{d_c} + 2B_V - (\tilde{h} + c_3)(\Delta A_s + \Delta A_c + 2B_A)}{E\{A_{d_s} + A_{d_c}\} + \Delta A_{d_s} + \Delta A_{d_c} + 2B_A - c_2} \qquad \text{(B-9)}$$

Linearization of Eq. (B-9) produces

$$E\{\varepsilon_{\tilde{h}}\} \cong \frac{2B_V - (\tilde{h} + c_3) \cdot 2B_A}{E\{A_{d_s} + A_{d_c}\} + 2B_A - c_2} = E\{\hat{h} - \tilde{h}\} \qquad \text{(B-10)}$$

As was stated in section VII, the values of B_V and B_A depend upon the sea state determined by the area distribution, k(t). If a "mean" functional form of k(t) (such as that for a flat sea surface perpendicular to the radar beam's center line) is hypothesized, one can subtract the corresponding mean values of B_A and B_V from A_s, A_c, V_s, and V_c. A rough evaluation of $\varepsilon_{\tilde{h}}$ for the horizontal sea surface leads to Eq. (79).

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