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X-541-68-391  
PREPRINT

NASA TM X- 63369

# THE COSMIC GAMMA-RAY SPECTRUM FROM SECONDARY PARTICLE PRODUCTION IN THE METAGALAXY

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GPO PRICE \$ \_\_\_\_\_

CFSTI PRICE(S) \$ \_\_\_\_\_

Hard copy (HC) 3.00

Microfiche (MF) 1.65

# 653 July 65

OCTOBER 1968

GSFC

GODDARD SPACE FLIGHT CENTER  
GREENBELT, MARYLAND

FACILITY FORM 602

N 68-37754  
(ACCESSION NUMBER)

(THRU)

26  
(PAGES)

(CODE)

TMX 63369  
(NASA CR OR TMX OR AD NUMBER)

(CATEGORY)



X-641-68-391  
Preprint

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by

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THE COSMIC GAMMA-RAY SPECTRUM  
FROM SECONDARY PARTICLE PRODUCTION IN THE METAGALAXY

F. W. Stecker\*

ABSTRACT

The purpose of this paper is to discuss the form and intensity of the spectrum of cosmic gamma-rays resulting from the production and decay of neutral pi-mesons produced in metagalactic cosmic-ray p-p collisions. It is assumed that intergalactic space contains ionized hydrogen gas at a density of  $10^{-5} \text{ cm}^{-3}$  as is consistent with recent X-ray observations at 0.25 keV.

Using the Friedmann solution to the Einstein field equations of general relativity as a description of our expanding universe, a discussion is presented of the effects of red-shift and spatial curvature on the generation and distortion of the local gamma-ray spectrum from the decay of neutral pi-mesons. Numerical calculations are presented for the Einstein-de Sitter solution, which is found to be an adequate model for these calculations. Two models are presented to represent the possible flux of metagalactic cosmic-rays. In calculating metagalactic gamma-ray spectra, the effect of gamma-ray absorption at large red-shifts is taken into account.

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A discussion of the results is given. The results indicate that future gamma-ray experiments in the 1-100 MeV region may yield valuable information relating to cosmology, cosmogeny, and the metagalactic cosmic-ray flux. In particular, the metagalactic gamma-ray spectra predicted tend to peak near  $70 (1 + z_{\max})^{-1}$  MeV where  $z_{\max}$ , the maximum red-shift at which cosmic rays are produced, may correspond to the age of the universe at the epoch of galaxy formation.

# THE COSMIC GAMMA-RAY SPECTRUM FROM SECONDARY PARTICLE PRODUCTION IN THE METAGALAXY

## INTRODUCTION

In recent investigations (Stecker, 1967; Stecker, Tsuruta and Fazio, 1968) the author made use of recent accelerator and cosmic-ray data to determine the details of the cosmic gamma-ray spectrum from the secondary particles produced by cosmic-ray collisions in the galaxy. The purpose of this paper is to determine the cosmic gamma-ray spectrum from secondary particles produced by cosmic-ray collisions in the metagalaxy. This spectrum will differ from the galactic (or local) gamma-ray spectrum because most of the generating collisions take place at large distances where we are looking back to a time when the universe was more compact and collisions were more frequent. These "early" gamma-rays will be of lower energy due to the progressive red-shift of the general cosmic expansion. Although various estimates of the flux of these metagalactic gamma-rays have been made (Ginzburg and Syrovatskii, 1964a, b; Gould and Burbidge, 1965; Garmire and Kraushaar, 1965), none of these workers have taken cosmological factors into account in order to properly calculate a spectrum, as has been done in calculating Compton X-ray spectra (Cheng, 1967; Brecher and Morrison, 1967; Silk, 1968).

## THE COSMOLOGICAL EQUATIONS

For the purpose of these calculations, we may consider models of the universe which are both homogeneous and isotropic on a large scale. Such models can, in general, be described by the Robertson-Walker line element

$$ds^2 = c^2 dt^2 - d\ell^2 = c^2 dt - R^2(t) du^2. \quad (1)$$

Gamma-rays travel along geodesics such that  $ds^2 = 0$ . Assuming a gamma-ray is emitted at a time  $t_e$  in an interval  $\Delta t_e$  and received at time  $t_r$  in an interval  $\Delta t_r$ , it can be shown that the relation between red-shift and radius of the universe given by

$$\frac{R(t_r)}{R(t_e)} = 1 + z(t_e) \quad (2)$$

It follows from (2) that in a universe where most of the energy density is in the form of matter

$$\frac{n(t_e)}{n(t_r)} = (1 + z)^3 \quad (3)$$

$$\frac{T_\gamma(t_e)}{T_\gamma(t_r)} = (1 + z) \quad (4)$$



and

$$\frac{n_\gamma(t_e)}{n_\gamma(t_r)} = (1+z)^3 \quad (5)$$

where  $n(t)$ ,  $T_\gamma(t)$  and  $n_\gamma(t)$  are the average particle density of matter and temperature and photon density of cosmic blackbody radiation in the universe. (For a more detailed discussion of the cosmological relations, the reader is referred to some excellent articles by Sandage (1961a, b; 1962).)

We will hereafter designate local ( $z = 0$ ) quantities with a subscript zero. Let  $G_g(E_\gamma)$  be the gamma-ray spectrum generated by the galactic cosmic-ray spectrum,  $I_g(E_p)$  in traveling a unit particle length ( $1 \text{ cm}^{-1}$ ) through the intergalactic medium. (This spectrum is the same as the quantity  $I(E_\gamma)/\langle nL \rangle$  calculated by Stecker (1967) and Stecker, et. al. (1968).)

We now assume that some ubiquitous generating mechanism causes cosmic-rays to be produced with the same power law throughout the universe as observed at the earth, so that the metagalactic cosmic-ray spectrum differs only in absolute intensity from the galactic cosmic-ray spectrum. It follows that the form of the cosmic gamma-ray spectrum anywhere in the metagalaxy, when observed in the co-moving frame at that point, will be the same as the form of  $G_g(E_\gamma)$ .

We may then write down an expression for the integrated metagalactic gamma-ray flux in any direction as

$$I(E_\gamma) = \int_0^{\ell_{\max}} d\ell n(\ell) \frac{I(\ell)}{I_g} \frac{G_g(E_\gamma, \ell)}{(1+z(\ell))^4} e^{-\tau(E_\gamma, \ell)} \quad (6)$$

where the factor,  $(1+z)^4$ , takes into account the reduction in flux due to the time dilation and volume diminuation factor and  $e^{-\tau}$  represents absorption of gamma-rays along the line of sight,  $I_g$  is the galactic cosmic-ray flux and  $I(\ell)$  is the cosmic-ray flux at a distance  $\ell$ . Equation (6) may be put into a much more convenient form by expressing it as an integral over  $z$ . We then obtain

$$I(E_\gamma) = \int_0^{z_{\max}} dz n(z) \frac{I(z)}{I_g} \frac{f(E_\gamma, z)}{(1+z)^4} e^{-\tau(E_\gamma, z)} \frac{d\ell}{dz} \quad (7)$$

Since the energy of a gamma-ray is directly proportional to its frequency, it follows that

$$G_g(E_\gamma, z) = (1+z) G_g[(1+z) E_\gamma] \quad (8)$$

It also follows from (3) that

$$n(z) = n_0 (1+z)^3 \quad (9)$$

The quantity

$$\frac{d\ell}{dz} = R(z) \frac{du}{dz} \quad (10)$$

depends, in general, both upon the cosmological model involved and the epoch of world-time which defines the acceleration (or deceleration) of the expansion. In Friedmann-type solutions to the Einstein equations, it is found that the expansion of the universe is decelerating. The magnitude of this deceleration is usually denoted by the deceleration parameter  $q$ . In the usual notation, the Hubble expansion parameter,  $H$ , and the quantity  $q$  are defined by the relations

$$H \equiv \frac{\dot{R}(t)}{R(t)} \quad (11)$$

and

$$q \equiv - \frac{\ddot{R}(t)}{R(t) H^2} \quad (12)$$

In a decelerating universe, therefore,  $q > 0$ .

For the Einstein-de Sitter model,  $R(t)$  can be expressed explicitly in terms of  $t$  by the relation

$$R(t) = (6\pi G \rho R^3)^{1/3} t^{2/3} \quad (13)$$

From (11) and (13) it then follows that

$$q = \frac{1}{2} \quad \text{for all } t. \quad (14)$$

The Einstein-de Sitter model is a good approximation to the universe if it has not yet reached a highly evolved state. It is also compatible with the most probable values of  $q$  as discussed by Sandage (1961a, 1962), based on the observed magnitude-red shift relation, and is consistent with the X-ray observation by Henry, Fritz, Meekins, Friedman and Byram (1968) interpreted to be bremsstrahlung from a metagalactic gas having a density of the order of  $10^{-5} \text{ cm}^{-3}$ .

Under the assumption of a Euclidean model, we will now determine the cosmological effects on the metagalactic gamma-ray spectrum. It can be shown (Sandage, 1961b) that

$$\frac{d\ell}{dz} = \frac{cH_0^{-1}}{(1+z)^2 (1+2q_0 z)^{1/2}} \quad (15)$$

where  $cH_0^{-1} = 10^{28} \text{ cm}$ . In the Einstein-de Sitter case,  $q_0 = 1/2$  and we may take in equation (7)

$$\frac{d\ell}{dz} = \frac{10^{28}}{(1+z)^{5/2}} \quad (16)$$

#### ABSORPTION OF METAGALACTIC GAMMA-RAYS

An excellent discussion of the absorption processes affecting cosmic gamma-rays has been given by Fazio (1967). The principal absorption process to be

considered is that of electron-positron pair production through interaction with the universal black-body radiation field, i.e., the reaction

$$\gamma + \gamma \rightarrow e^+ + e^- \quad (17)$$

Detailed calculations of the energy-dependent absorption probability for this process have been performed by Gould and Schréder (1967). They have shown that for a gamma-ray of energy  $E_\gamma$  interacting with a black-body radiation field of temperature  $T_\gamma$

$$\frac{d\tau}{d\ell} \sim \frac{\alpha^2}{2\pi\Lambda} \left( \frac{kT_\gamma}{mc^2} \right)^3 \sqrt{\xi} e^{-\xi} \quad (18)$$

where

$$\xi \equiv \frac{(mc^2)^2}{kT_\gamma E_\gamma} \gg 1$$

$\alpha \simeq 1/137$  is the fine-structure constant,  $\Lambda = \hbar/mc = 3.86 \times 10^{-11}$  cm, and  $k$  here is Boltzmann's constant. The local black-body temperature has been found by Stokes, Partridge and Wilkinson (1967) to be

$$T_0 = 2.7 \text{ }^\circ\text{K}. \quad (19)$$

so that the condition  $\xi \gg 1$  corresponds to the condition

$$E_\gamma \ll \frac{1.12 \times 10^6 \text{ GeV}}{(1+z)^2} \quad (20)$$

(see equation (4) and (8)).

We will restrict ourselves here to a determination of the gamma-ray spectrum below 1 GeV and  $z \leq 10^2$  (as will be discussed later) so that the approximation given by equation (18) will be generally valid. Therefore, from (16) and (18), we find\*

$$\tau(E_\gamma, z) = 3.9 \times 10^8 E_\gamma^{-1/2} \int_0^z dy \frac{\exp \left[ \frac{1.12 \times 10^6}{(1+y)^2 E_\gamma} \right]}{(1+y)^{1/2}} \quad (21)$$

## THE METAGALACTIC COSMIC-RAY SPECTRUM

It now remains only to specify a suitable model for the metagalactic cosmic ray flux. We will assume that at some early epoch, corresponding to  $z \geq z_{\max}$  conditions were unsuitable for the acceleration of cosmic-rays. We will consider  $z_{\max}$  to correspond to the epoch of galaxy formation and consider two possible models for the origin of a metagalactic cosmic-ray flux. For model I,

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\*As an intermediate solution to the problem considered in the text, numerical solutions were obtained for the implicit relation,  $\tau(E_\gamma, z_\gamma) \equiv 1$ , which defines the red-shift,  $z_\gamma$ , beyond which the universe becomes opaque to gamma-rays of local energy  $E_\gamma$ . It was found that the numerical solution to equation (21) may be quite well approximated by the relation  $1 + z_\gamma \simeq 2.30 \times 10^2 E_\gamma^{-0.484}$  with  $E_\gamma$  expressed in GeV. Thus, although absorption was taken into account in this calculation, the effect is negligible as the universe may be considered to be transparent out to  $z_\gamma$ .

we will assume that the metagalactic flux arises through leakage from the halos of radio galaxies from  $z = z_{\max}$  to  $z = 0$ . For model II, we will assume that this flux was created primarily in a burst at the time of galaxy formation. (Other possibilities will be considered in a future paper.) For  $z_{\max}$ , we will also consider two extremes. One extreme is  $z_{\max} = 10^3$ , which corresponds to the earliest epoch when galaxy formation could probably occur. At  $z = 10^3$ , the black-body temperature of the universe was of the order of  $10^3 - 10^4$ °K, cool enough for ionized hydrogen to combine to form a neutral gas. According to Peebles (1965),  $z = 10^3$  also corresponds to the epoch when gas clouds may begin to form gravitationally bound systems.

The other extreme for  $z_{\max}$  which we may consider corresponds to the highest red-shift yet observed for a quasar, viz., 2.2. This is, of course, an extreme which is technique-limited rather than being limited by any physical criteria, and it is included mainly for purposes of discussion. We will also consider various intermediate values for  $z_{\max}$  of 4, 9, and  $10^2$ . (Doroshkevich, et. al. (1967) suggest that galaxy formation took place at  $z = 10 - 20$  whereas Weymann (1967) suggests  $z = 10^2$ .)

It is important to note here that the upper limit,  $z_{\max}$ , may be effectively restricted, not by the epoch of galaxy formation, but by attenuation of the metagalactic cosmic-ray flux due to the collisions themselves. The cross-section for inelastic cosmic-ray p-p collisions is of the order of 30 mb. Therefore, the lifetime of the metagalactic cosmic-rays against collisional losses is given by

$$\tau_c = \frac{1}{n\sigma c} = \frac{1}{n_0 \sigma c (1+z)^3}$$

$$\simeq 10^{20} (1+z)^{-3} \text{ sec} \quad (22)$$

for  $n_0 = 10^{-5} \text{ cm}^{-3}$ .

The lifetime of the universe at a red-shift  $z$  is given by

$$\tau_u \simeq 10^{17} (1+z)^{-3/2} \text{ sec} \quad (23)$$

Cosmic-rays cannot accumulate in the metagalaxy if the ratio,  $\tau_c/\tau_u < 1$ . The condition  $\tau_c/\tau_u = 1$  therefore defines a critical value of  $z_{\text{max}} = 10^2$  beyond which a further buildup of metagalactic cosmic-rays cannot occur. With these limitations on  $z$  in mind, we will now consider the various ideal models for describing the metagalactic cosmic-ray flux.

For model I, we assume a constant leakage rate so that the total number of cosmic-rays in the metagalaxy is proportional to the time elapsed since galaxy formation. It follows from (13) that this time is given by

$$\tau_g \simeq 10^{10} [(1+z)^{-3/2} - (1+z_{\text{max}})^{-3/2}] \text{ yrs.} \quad (24)$$

The cosmic-ray density will then increase with red-shift according to the relation



$$\frac{I'(g)}{I_g} \sim (1+z)^3 [(1+z)^{-3/2} - (1+z_{\max})^{-3/2}] \quad (25)$$

However, the cosmic-rays which produce the neutral pi-mesons necessary for gamma-ray production are only those above a threshold kinetic energy,  $E_{\text{th}} - M_p$ , of about 300 MeV (Stecker, 1966). We must therefore determine

$$I(z) = I'(E > E_{\text{th}}; z) \quad (26)$$

For a power law cosmic-ray spectrum of the form

$$I(> E) \sim E^{-3/2}, \quad (27)$$

as is approximately valid in the energy region where most pi-mesons are produced (Stecker 1967), it follows from the red-shift relation that

$$\begin{aligned} I(E > E_{\text{th}}; z) &= I' \left( E > \frac{E_{\text{th}}}{(1+z)} \right) \\ &= I'(z) \left[ \frac{1+z}{1+z_{\max}} \right]^{3/2} \end{aligned} \quad (28)$$

so that we must use an effective flux of

$$\frac{I(z)}{I_g} \sim (1+z)^3 \left[ \frac{1+z}{1+z_{\max}} \right]^{3/2} [(1+z)^{-3/2} - (1+z_{\max})^{-3/2}]$$

For model II, we assume that the metagalactic cosmic-rays were created in a burst at the time of galaxy formation. We thus find for this model that

$$\frac{I(z)}{I_g} \sim (1+z)^3 \left[ \frac{1+z}{1+z_{\max}} \right]^{3/2} \quad (30)$$

Using the models defined by equations (29) and (30), together with equations (7), (8), (9), (16), and (21), the metagalactic gamma-ray spectra produced by models I and II were calculated. The results were normalized by requiring the integral gamma-ray spectrum above 100 MeV to be equal to  $1.1 \times 10^{-4} \text{ cm}^{-2} \text{ sec}^{-1} \text{ sr}^{-1}$ , according to the results of Clark, Garmire and Kraushaar (1968), measured by the detector aboard the OSO-3 satellite. As has been noted previously, such a normalization makes possible the determination of upper limits on the value  $I_0/I_g$ , i.e., the present metagalactic intensity of cosmic-ray nucleons. The upper limits are given by Stecker (1968).

The gamma-ray fluxes thus calculated are given in Figures 1-4. Figure 5 shows the gamma ray flux expected from the galactic halo in the direction of the pole, taking  $\langle nL \rangle = 3 \times 10^{20} \text{ cm}^{-2}$  and based on previous calculations (Stecker, 1967; Stecker, et. al. 1968).<sup>\*</sup> It can be seen that the local gamma-ray flux from the galactic halo will not explain the data of Clark, et. al. and should be unimportant compared to the metagalactic flux. The metagalactic gamma-ray spectra tend to

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<sup>\*</sup>Thus  $I_{\text{pole}}(E_\gamma) = 3 \times 10^{20} G_g(E_\gamma)$ .

to peak near  $7 \times 10^{-2} / (1 + z_{\max})$ , GeV, being weighted toward higher red-shifts by the effect of greater densities at earlier epochs. Because of the density effect, a cosmic-ray burst at large red-shifts is much more effective in producing gamma-rays than a continuous production of the same number of cosmic-rays.

## CONCLUSIONS

Present evidence about the flux of cosmic-rays between the galaxies is quite meager. The most promising way to study the flux is by a satellite experiment measuring the isotropic gamma-ray flux in the region between 1 and 100 MeV. Such gamma-rays can supply us with direct information on metagalactic cosmic-rays, because they travel to us in straight lines and suffer little absorption. Theoretical metagalactic gamma-ray fluxes from  $\pi^0$  decay are presented here under various assumptions as to the metagalactic cosmic-ray flux. These predictions indicate that an experimental determination of the isotropic gamma-ray spectrum at high galactic latitudes and in the energy range 1 - 100 MeV, could supply valuable information, not only about metagalactic cosmic-rays, but also about such fundamental questions as when the galaxies were formed, since the metagalactic gamma-ray spectrum will peak near  $70 (1 + z_{\max})^{-1}$  MeV.

## ACKNOWLEDGMENTS

The author is indebted to Dr. Richard C. Henry of the Naval Research Laboratory for supplying him with the details of the NRL X-ray observations prior to publication and for several stimulating discussions. The author would also like

to acknowledge and thank Dr. Frank C. Jones and Dr. Joseph Silk for their comments and Mr. Joseph Bredekamp for programming the numerical calculations essential to this paper.

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## FIGURE CAPTIONS

- Figure 1. Metagalactic differential gamma-ray spectra from cosmic-ray p-p interactions based on a cosmic-ray flux produced by constant leakage from radio galaxies (Model I) and shown for various maximum red-shifts as discussed in the text.
- Figure 2. Metagalactic integral gamma-ray spectra from cosmic-ray p-p interactions based on a cosmic-ray flux produced by constant leakage from radio galaxies (Model I) and shown for various maximum red-shifts as discussed in the text.
- Figure 3. Metagalactic differential gamma-ray spectra from cosmic-ray p-p interactions based on a cosmic-ray flux produced by a burst of cosmic rays at  $z_{\max}$  (Model II) as discussed in the text.
- Figure 4. Metagalactic integral gamma-ray spectra from cosmic-ray p-p interactions based on a cosmic-ray flux produced by a burst of cosmic rays at  $z_{\max}$  (Model II) as discussed in the text.
- Figure 5. The expected gamma-ray flux from the galactic halo in the direction of the pole, taking  $\langle nL \rangle = 3 \times 10^{20} \text{ cm}^{-2}$  and based on previous calculations (Stecker, 1967; Stecker, et. al, 1968).

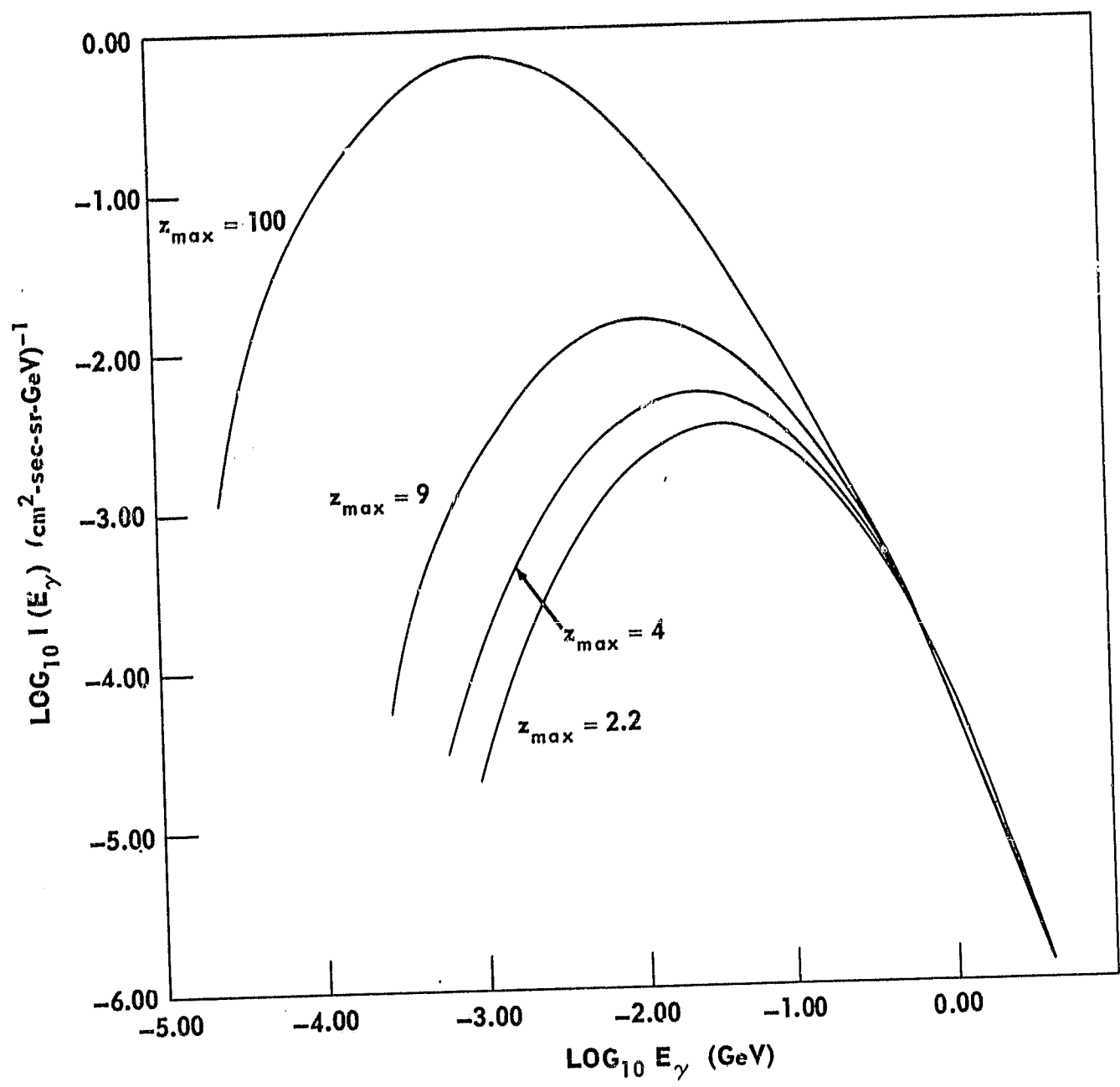


Figure 1—Differential Spectrum for Leakage Model (Model 1)



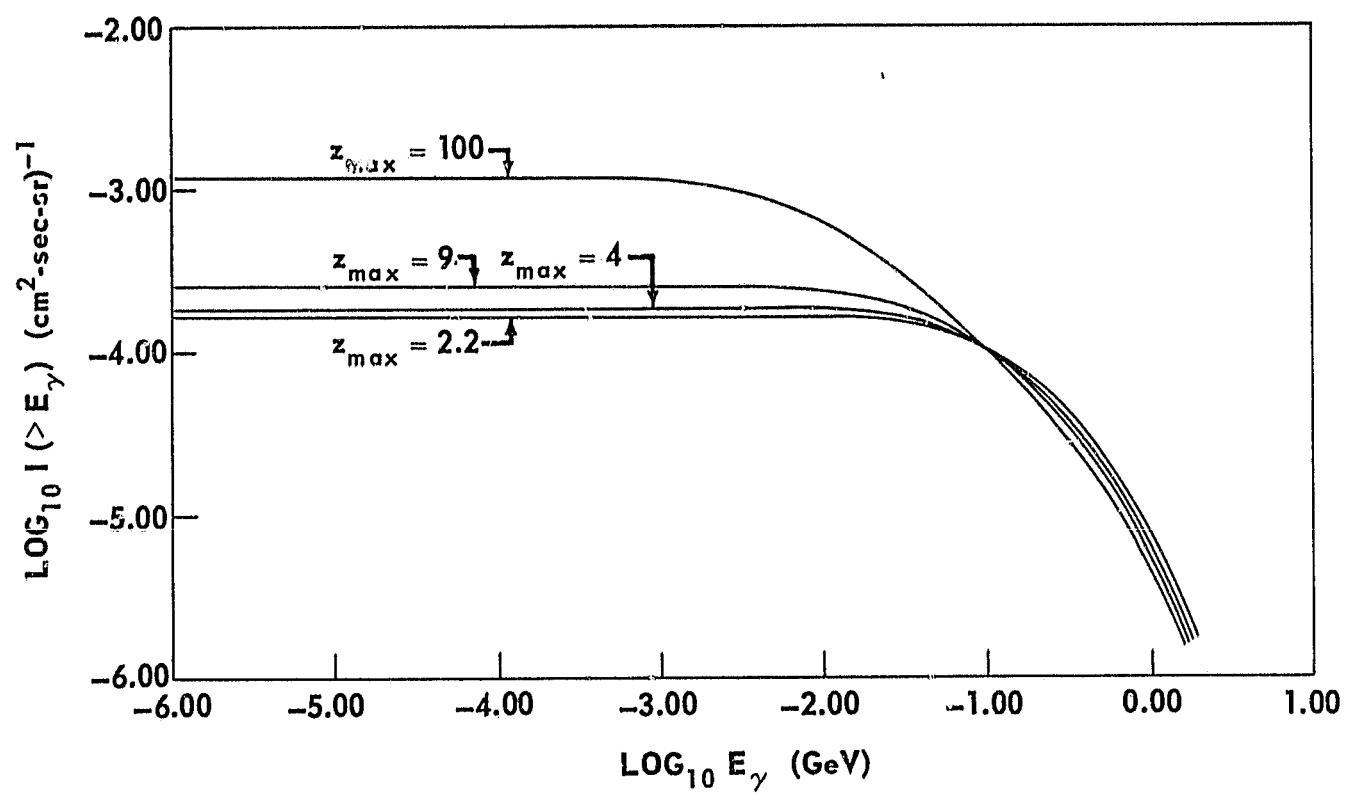


Figure 2—Integral Spectrum for Leakage Model (Model I)

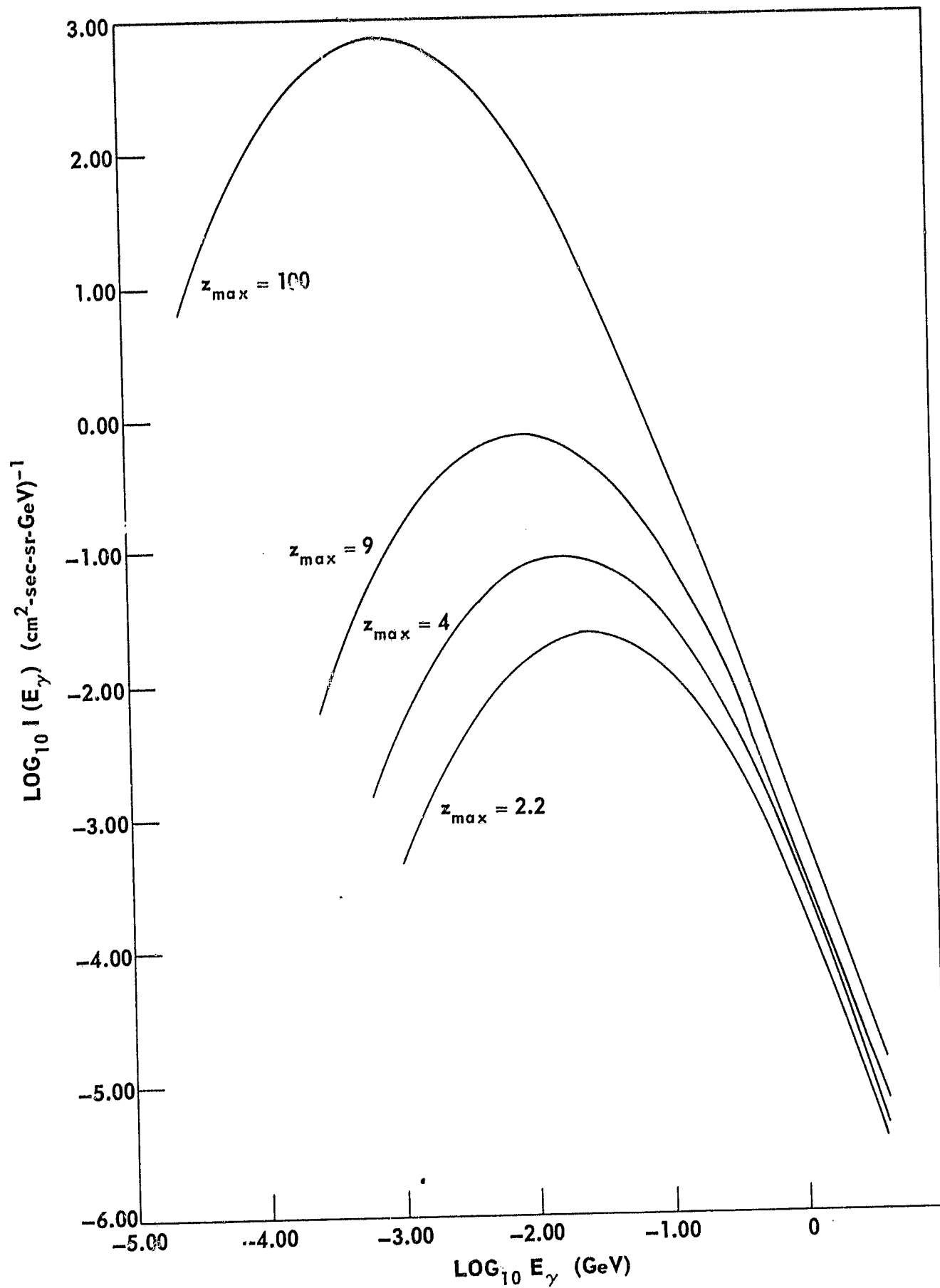


Figure 3-Differential Spectrum for Burst Model (Model II)

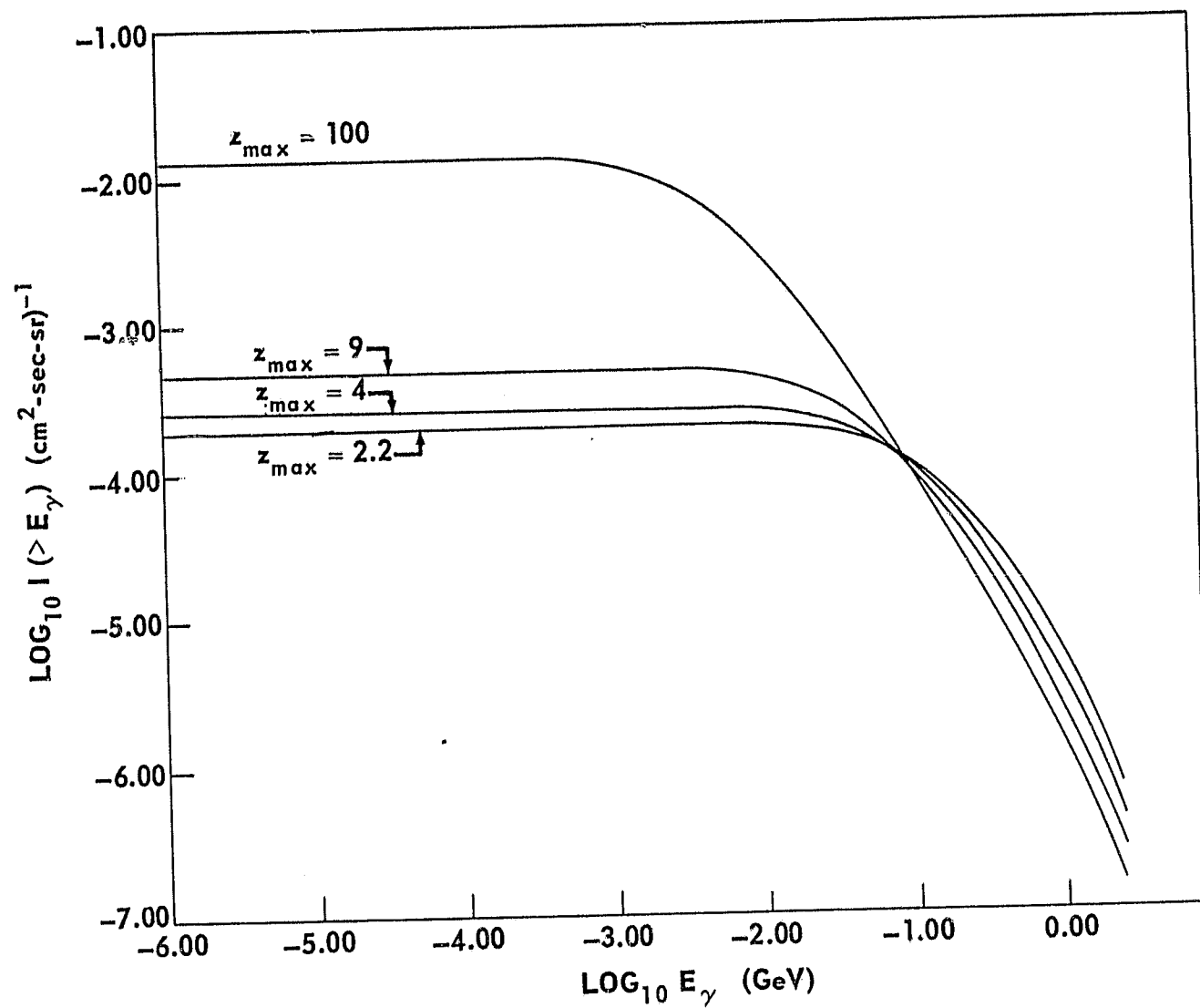


Figure 4—Integral Spectrum for Burst Model (Model II)

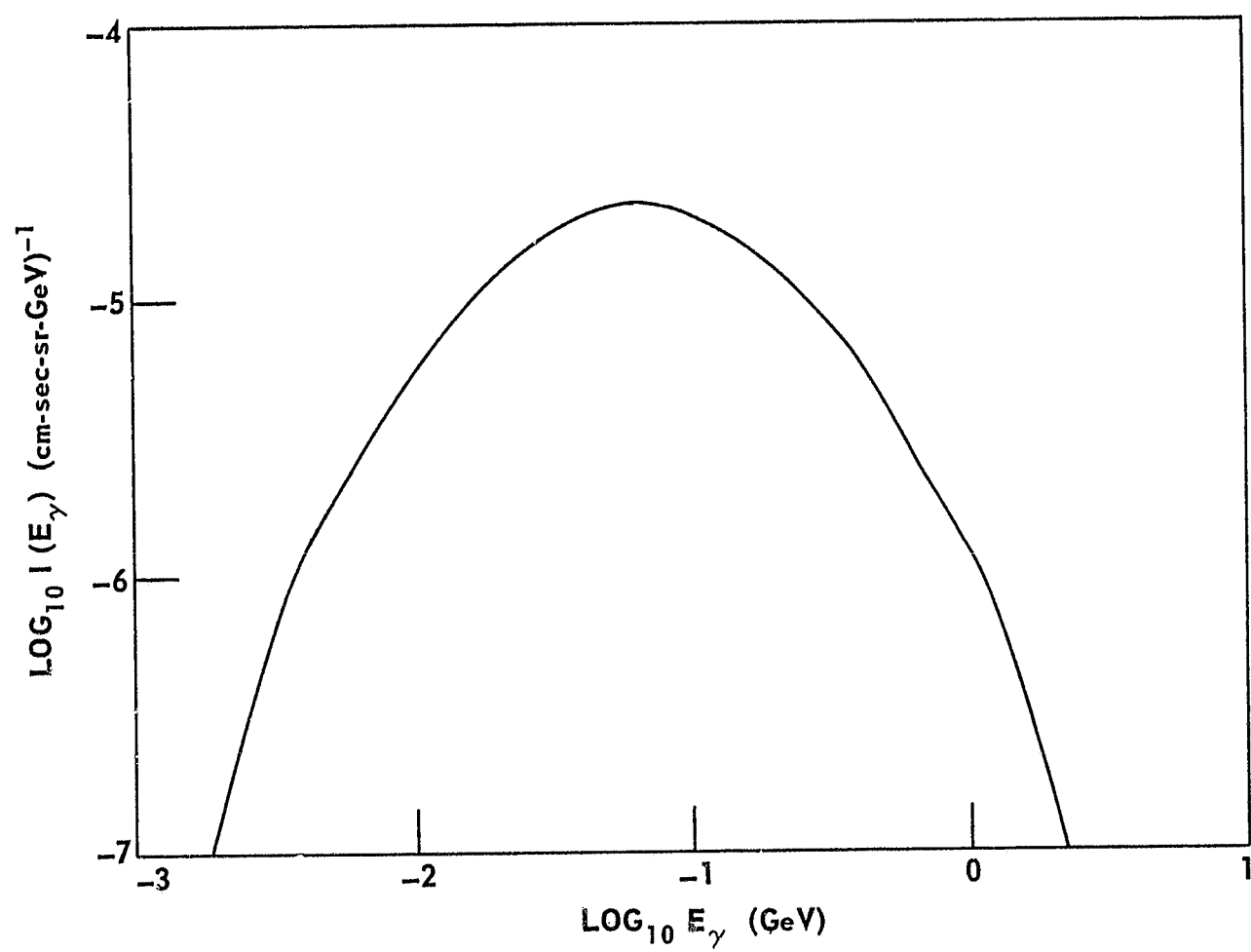


Figure 5—Differential Spectrum for the Galactic Halo ( $\langle nL \rangle = 3 \times 10^{20} \text{ cm}^{-2}$ )