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TRAPPED WAVES PROPAGATING ALONG A WARM PLASMA SLAB

by
J. D. Mitchell

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IONOSPHERE RESEARCH LABORATORY

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TABLE OF CONTENTS

ABSTRACT

Trapped modes in a warm plasma slab partially filling a parallel plate waveguide are studied. A two fluid model with a scalar pressure term for electrons is used to obtain the field solutions and dispersion equations for both a symmetric and antisymmetric longitudinal $\overline{E}$ field. The dependence of the dispersion relations on electron temperature, slab thickness, and waveguide width is studied.

The quasistatic dispersion equation and a dispersion relation obtained by a heuristic argument are compared with the more rigorous dispersion relation and the regions of validity are discussed. Good agreement is found between experimental data measured in a cylindrical system and the theoretical results of the slab geometry. Finally, the effects of Landau damping and electron-neutral collisional damping are considered.

1 INTRODUCTION

1.1 STATEMENT OF THE PROBLEM AND RELATED STUDIES

In recent years, much interest has been focused on the propagation of trapped modes in homogeneous bounded plasmas[1-12]. We will define trapped modes to be those waves whose field solutions are damped exponentially in the outward direction from the plasma boundary. The first six references cited above study wave propagation in cold plasmas and the remaining six consider propagation in warm plasmas. Including the electron temperature in the development effects the trapped wave modes in two significant ways. First, it serves as a mechanism for ion body waves or what are often called Tonks-Dattner resonances[11,13]. Second, the electron body and surface waves have a finite group velocity for large wave numbers which asymptotically approaches the electron thermal velocity.

In one of the first significant papers on wave propagation in a confined plasma, Trivelpiece and Gould[2] used a quasistatic treatment to study electron surface waves in a cold plasma column. This work was extended by Diament, Granatstein, and Schlesinger[11] to include finite electron temperatures. Their analysis employed a fluid equation for electrons and the full set of Maxwell's equations. Ions were neglected. Andersson and Weissglas[10] included ions via a two fluid model and considered the propagation of surface waves and body waves in a partially filled circular plasma waveguide for frequencies below ω_{pi}.

In the present work, the field equations and dispersion relations are developed for a warm, homogeneous plasma slab partially filling a parallel plate waveguide. A plane geometry was chosen because of the simplification in the analysis. In the second chapter, the field components and dispersion equations are obtained for both a symmetric and antisymmetric longitudinal $\overline{E}$ field component. Symmetric longitudinal $\overline{E}$ field solutions are similar to the circularly symmetric case in cylindrical geometry while antisymmetric longitudinal $\overline{E}$ field solutions bear similarity to the dipole mode. Ion effects and electron temperature effects are included in this development. Discussion of the dispersion equation in the third chapter is restricted to trapped modes. The effects of changing the electron temperature, the slab width, and the waveguide size are studied. Finally, in Chapter 4 we examine the different approximations used to simplify the study of trapped mode propagation. In particular, the heuristic and quasi-static approximations are discussed and their limitations determined.

1.2 DEFINITION OF SYMBOLS

The following symbols will be used in this paper:

a	one half the width of the plasma slab
b	one half the distance separating the conducting plates
c	free space speed of light

e	absolute value of the charge of an electron
k	Boltzmann constant
m_e	mass of an electron
m_i	mass of an ion
n_e	perturbed electron density about n_o
n_i	perturbed ion density about n_o
n_o	unperturbed electron and ion density
T_e	electron temperature
T_i	ion temperature
$W = \left(\frac{kT_e}{m_e}\right)^{\frac{1}{2}}$	electron thermal velocity
$X = \frac{c\beta}{\omega_{pe}}$	normalized longitudinal propagation constant
$Y = \frac{\omega}{\omega_{pe}}$	normalized wave frequency
$\alpha = \frac{\omega_{pe} a}{c}$	normalized electron plasma frequency
β	longitudinal propagation constant
$\gamma_1 = \left(\beta^2 - \frac{\omega^2 \hat{\epsilon}}{c^2}\right)^{\frac{1}{2}}$	transverse propagation constant in the plasma
$\gamma_2 = \left(\beta^2 - \frac{\omega^2 \epsilon}{W^2 \epsilon_+}\right)^{\frac{1}{2}}$	transverse propagation constant in the plasma
$\delta = \left(\beta^2 - \frac{\omega^2}{c^2} K\right)^{\frac{1}{2}}$	transverse propagation constant in the dielectric

ϵ_o		dielectric constant in a vacuum
ϵ	$= 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$	
ϵ_+	$= 1 - \frac{\omega_{pi}^2}{\omega^2}$	
K		relative dielectric constant
λ		wavelength of the wave
λ_{De}	$= \left(\frac{\epsilon_o kT_e}{n_o e^2}\right)^{\frac{1}{2}}$	electron Debye length
λ_{Di}	$= \left(\frac{\epsilon_o kT_i}{n_o e^2}\right)^{\frac{1}{2}}$	ion Debye length
ω		frequency of the wave
ω_{pe}		electron plasma frequency
ω_{pi}		ion plasma frequency

2 FIELD SOLUTIONS AND DISPERSION EQUATIONS

2.1 DESCRIPTION OF THE MODEL

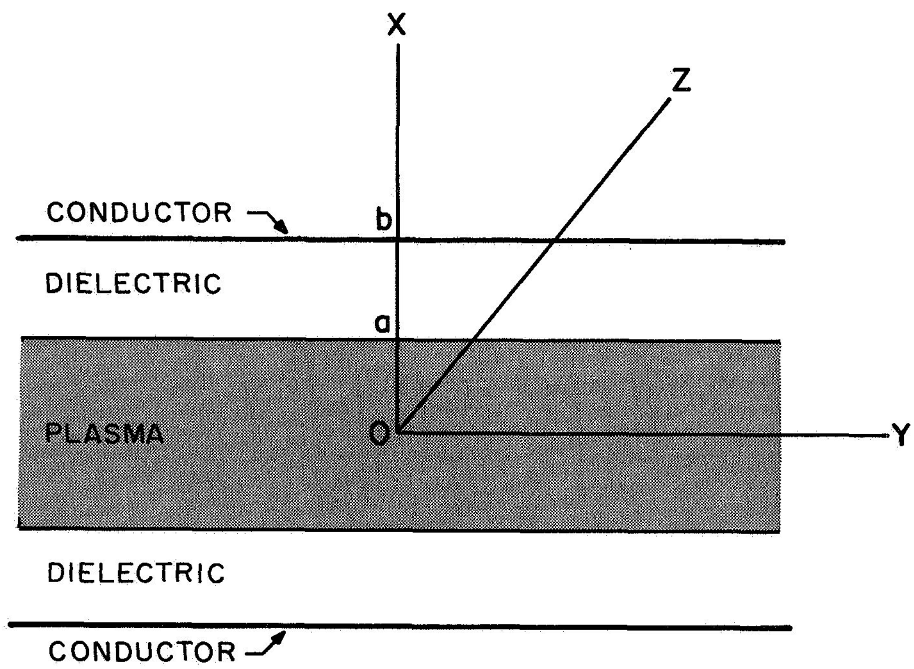
The plasma slab as drawn in Fig. 1 will be assumed infinite in the y and z directions and contained between $x = \pm a$ by a dielectric having a relative dielectric constant K. The dielectric is bounded at $x = \pm b$ by parallel plates having an infinite conductivity. We will assume that the plasma is homogeneous and isotropic. The effects of electron temperature are included by assuming a scalar pressure term proportional to T_e in the fluid momentum equation for electrons. The ion temperature is assumed to be much smaller than the electron temperature and therefore is neglected.

For mathematical convenience, we will consider a monochromatic wave of the form $e^{i(\omega t - \beta y)}$ propagating in the y direction. Because of the geometry, we may assume that none of the field expressions are dependent on z. The linearized fluid equations and Maxwell's equations necessary to describe the field quantities inside the plasma are:

$$n_e = \frac{i}{\omega} n_o \nabla \cdot \bar{v}_e \tag{1}$$

$$n_i = \frac{i}{\omega} n_o \nabla \cdot \bar{v}_i \tag{2}$$

$$\bar{v}_e = \frac{ie}{m_e \omega} \bar{E} + \frac{iW^2}{n_o \omega} \bar{\nabla} n_e \tag{3}$$



A PLASMA SLAB PARTIALLY FILLING A PARALLEL PLATE WAVEGUIDE

FIGURE I

$$\overline{v}_i = -\frac{ie}{m_i\omega}\overline{E} \quad (4)$$

$$\nabla \cdot \overline{E} = \frac{e}{\epsilon_o}(n_i - n_e) \quad (5)$$

$$\nabla \cdot \overline{H} = 0 \quad (6)$$

$$\nabla \times \overline{E} = -i\omega\mu_o \overline{H} \quad (7)$$

$$\nabla \times \overline{H} = n_o e(\overline{v}_i - \overline{v}_e) + i\omega\epsilon_o\overline{E} \quad (8)$$

In the dielectric region, Eqs. 5 and 8 must be modified as follows:

$$\nabla \cdot \overline{E} = 0 \quad (9)$$

$$\nabla \times \overline{H} = i\omega\epsilon_o K \overline{E} \quad (10)$$

In the above equations, quantities describing the electrons are subscripted with an "e" and similarly, ion quantities are subscripted with an "i"; $\overline{v}_e$ and $\overline{v}_i$ are the perturbed values for the electron and ion velocities, respectively, and n_e and n_i are the respective electron and ion densities perturbed about the value n_o, which is the unperturbed number density for both ions and electrons.

2.2 DERIVATION OF THE FIELD SOLUTIONS

Let us assume a transverse magnetic mode propagating in the y direction. For $a < x < b$, taking the curl of both sides of Eq. 10 gives us:

$$\nabla \times \nabla \times \bar{H} = \nabla(\nabla \cdot \bar{H}) - \nabla^2 \bar{H} = i\omega\epsilon_o K \quad \nabla \times \bar{E}$$

or

$$\frac{\partial^2 H_z}{\partial x^2} - (\beta^2 - \frac{\omega^2}{c^2} K) H_z = 0$$

If we consider only a symmetric expression for E_y, i.e. an antisymmetric expression for the transverse $\bar{H}$ field, the solution to the above homogeneous wave equation for H_z is

$$H_z = A \sinh \delta x + B \operatorname{sgn} x \cosh \delta x$$

where δ is defined by the equation

$$\delta = (\beta^2 - \frac{\omega^2}{c^2} K)^{\frac{1}{2}}$$

and sgn x equals plus one when its argument is greater than zero and minus one when its argument is negative.

Using Eq. 10 again, the remaining two field quantities for $a < x < b$ may be written as follows:

$$E_y = \frac{i\delta}{\omega\epsilon_o K} (A \cosh \delta x + B \operatorname{sgn} x \sinh \delta x)$$

$$E_x = \frac{-\beta}{\omega\epsilon_o K} \quad (A \sinh \delta x + B \operatorname{sgn}x \cosh \delta x)$$

To obtain field solutions for the quantities inside the plasma ($0 \leq x < a$), a similar procedure will be used. Taking the curl of Eq. 8

$$\nabla \times \nabla \times \overline{H} = -\nabla^2 \overline{H} = n_o e \; \nabla \times (\overline{v}_i - \overline{v}_e) + i\omega\epsilon_o \; \nabla \times \overline{E}$$

and making the same assumptions as mentioned previously in this section, we obtain the following differential equation for H_z:

$$\frac{\partial^2 H_z}{\partial x^2} - (\beta^2 - \frac{\omega^2\epsilon}{c^2})H_z = 0$$

where ϵ is defined by the equation

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$$

The solution that satisfies this differential equation for H_z is

$$H_z = C \sinh \gamma_1 x \tag{11}$$

where

$$\gamma_1 = (\beta^2 - \frac{\omega^2\epsilon}{c^2})^{\frac{1}{2}}$$

Equation 8 is used to obtain E_y. Using Eqs. 1-5, the quantities $\overline{v}_i$ and $\overline{v}_e$ can be written in terms of the unknown quantity $\overline{E}$. The solutions are:

$$\overline{v}_i = -\frac{ie}{m_i\omega}\overline{E}$$

$$\overline{v}_e = \frac{ie}{m_e\omega}\overline{E} + \frac{iW^2\epsilon_o}{n_o e\omega}\left(\frac{\omega_{pi}^2}{\omega^2} - 1\right)\overline{\nabla}(\nabla\cdot\overline{E})$$

Substituting these expressions into the y component of Eq. 8 gives us

$$i\omega\epsilon_o\,\epsilon\,E_y + \frac{iW^2\epsilon_o\epsilon}{\omega}\frac{\partial}{\partial y}(\nabla\cdot\overline{E}) = -\frac{\partial H_z}{\partial x}$$

where ϵ_+ is defined by

$$\epsilon_+ = 1 - \frac{\omega_{pi}^2}{\omega^2}$$

Using Eq. 7 to obtain $\frac{\partial}{\partial y}(\nabla\cdot\overline{E})$ and substituting the expression for H_z from Eq. 11 we get the following equation for E_y:

$$\frac{iW^2\epsilon_o\epsilon_+}{\omega}\frac{\partial^2 E_y}{\partial_x^2} + i\omega\epsilon_o\left(\epsilon - \frac{\beta^2W^2\epsilon_+}{\omega^2}\right)E_y = \left(\frac{W^2}{c^2}\epsilon_+ - 1\right)\gamma_1 C\cosh\gamma_1 x$$

The solution to this differential equation is

$$E_y = \frac{i\gamma_1}{\omega\epsilon_o\epsilon} C \cosh \gamma_1 x + iD \cosh \gamma_2 x$$

where

$$\gamma_2 = (\beta^2 - \frac{\omega^2\epsilon}{W^2\epsilon_+})^{\frac{1}{2}}$$

By a similar method, we obtain the following solution for E_x:

$$E_x = - \frac{\beta}{\omega\epsilon_o\epsilon} C \sinh \gamma_1 x - \frac{\gamma_2}{\beta} D \sinh \gamma_2 x$$

We have obtained our full set of T M mode field solutions for a traveling wave having a symmetric electric field component in the direction of propagation. By using the same procedure, it is also possible to get a set of solutions inside the plasma and the dielectric for a traveling wave having an antisymmetric longitudinal electric field component. A summary of the solutions are presented below:

2.2.1 Field Solutions for a Symmetric Longitudinal $\bar{E}$ Field

1.) Inside the plasma $(0 \leq x < a)$

$$H_z = (C \sinh \gamma_1 x) e^{i(\omega t - \beta y)} \qquad (12)$$

$$E_y = i(\frac{\gamma_1}{\omega\epsilon_o\epsilon} C \cosh \gamma_1 x + D \cosh \gamma_2 x)e^{i(\omega t-\beta y)} \tag{13}$$

$$E_x = -(\frac{\beta}{\omega\epsilon_o\epsilon} C \sinh \gamma_1 x + \frac{\gamma_2}{\beta} D \sinh \gamma_2 x)e^{i(\omega t-\beta y)} \tag{14}$$

2.) Inside the dielectric $(a < x < b)$

$$H_z = (A \sinh \delta x + B \operatorname{sgn} x \cosh \delta x)e^{i(\omega t-\beta y)} \tag{15}$$

$$E_y = \frac{i\delta}{\omega\epsilon_o K}(A \cosh \delta x + B \operatorname{sgn} x \sinh \delta x)e^{i(\omega t-\beta y)} \tag{16}$$

$$E_x = -\frac{\beta}{\omega\epsilon_o K}(A \sinh \delta x + B \operatorname{sgn} x \cosh \delta x)e^{i(\omega t-\beta y)} \tag{17}$$

2.2.2 Field Solutions for an Antisymmetric Longitudinal $\overline{E}$ Field

1.) Inside the plasma $(0 \leq x < a)$

$$H_z = (C' \sinh \gamma_1 x)e^{i(\omega t-\beta y)} \tag{18}$$

$$E_y = i(\frac{\gamma_1}{\omega\epsilon_o\epsilon} C' \sinh \gamma_1 x + D' \sinh \gamma_2 x)e^{i(\omega t-\beta y)} \tag{19}$$

$$E_x = -(\frac{\beta}{\omega\epsilon_o\epsilon} C' \cosh \gamma_1 x + \frac{\gamma_2}{\beta} D' \cosh \gamma_2 x)e^{i(\omega t-\beta y)} \tag{20}$$

2.) Inside the dielectric $(a < x < b)$

$$H_z = (A' \cosh \delta x + B' \operatorname{sgn} x \sinh \delta x) e^{i(\omega t - \beta)} \tag{21}$$

$$E_y = \frac{i\delta}{\omega \epsilon_o K} (A' \sinh \delta x + B' \operatorname{sgn} x \cosh \delta x) e^{i(\omega t - \beta y)} \tag{22}$$

$$E_x = - \frac{\beta}{\omega \epsilon_o K} (A' \cosh \delta x + B' \operatorname{sgn} x \sinh \delta x) e^{i(\omega t - \beta y)} \tag{23}$$

2.3 DERIVATION OF THE DISPERSION EQUATIONS

In order to determine a dispersion relation for either of the previously mentioned cases, we impose the following boundary conditions:

$$E_y \big|_{x=b} = 0 \tag{24}$$

$$E_y \big|_{x=a_-} = E_y \big|_{x=a_+} \tag{25}$$

$$H_z \big|_{x=a_-} = H_z \big|_{x=a_+} \tag{26}$$

$$v_{ex} \big|_{x=a_-} = 0 \tag{27}$$

The first three of these conditions are obtainable from Maxwell's equations. The fourth condition is a necessary assumption in order to close the set of equations. In writing this equation, we are assuming that the plasma-dielectric interface acts as a reflector to electrons having a component of velocity in the x direction. Applying these boundary conditions gives us the following dispersion equations:

1.) For symmetric E_y

$$\frac{\delta\epsilon}{K} \tanh \delta(a-b) = \gamma_1 \coth \gamma_1 a - \frac{\omega_{pe}^2}{\omega^2} \frac{\beta^2}{\gamma_2 \epsilon_+} \coth \gamma_2 a \tag{28}$$

2.) For antisymmetric E_y

$$\frac{\delta\epsilon}{K} \tanh \delta(a-b) = \gamma_1 \tanh \gamma_1 a - \frac{\omega_{pe}^2}{\omega^2} \frac{\beta^2}{\gamma_2 \epsilon_+} \tanh \gamma_2 a \tag{29}$$

In order to simplify graphing the above dispersion equations, we will use the parameters $Y = \omega/\omega_{pe}$ and $X = c\beta/\omega_{pe}$. Since $\omega_{pi} << \omega_{pe}$, the dispersion equations in terms of X and Y may be written as follows:

1.) For symmetric E_y

$$\frac{(X^2-Y^2)^{\frac{1}{2}}}{K} \tanh \alpha\left(\frac{b}{a} -1\right) (X^2-Y^2)^{\frac{1}{2}} = (X^2+1-Y^2)^{\frac{1}{2}} \coth\alpha (X^2+1-Y^2)^{\frac{1}{2}}$$

$$- \frac{X^2}{Y^2\left(1 - \frac{m_e}{m_i Y^2}\right)\left[X^2 + \frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_i Y^2}\right)}\right]^{\frac{1}{2}}} \coth \alpha \left[X^2 + \frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_i Y^2}\right)}\right]^{\frac{1}{2}} \tag{30}$$

2.) For antisymmetric E_y

$$\frac{(X^2-Y^2)^{\frac{1}{2}}}{K}\tanh\alpha\left(\frac{b}{a}-1\right)(X^2-Y^2)^{\frac{1}{2}} = (X^2+1-Y^2)^{\frac{1}{2}}\tanh\alpha(X^2+1-Y^2)^{\frac{1}{2}}$$

$$-\frac{X^2}{Y^2\left(1-\frac{m_e}{m_i Y^2}\right)\left[X^2+\frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_i Y^2}\right)}\right]^{\frac{1}{2}}}\tanh\alpha\left[X^2+\frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_i Y^2}\right)}\right]^{\frac{1}{2}} \qquad (31)$$

3 DISCUSSION OF THE DISPERSION EQUATIONS

3.1 GENERAL DISCUSSION

As was mentioned in the introduction, only trapped modes will be considered in this paper. Trapped modes may be classified either as body waves or surface waves. The amplitudes of both of these waves decay exponentially as one goes out normally from the plasma-dielectric interface. Their behavior inside the plasma, however, does differ in that a body wave has an oscillatory solution while a surface wave has a damped solution as one travels into the center from the plasma boundary. If we now look at our field solutions given by Eqs. 12 through 23, what we have just said is that δ must be real and γ_1 and γ_2 can be either real or imaginary. If δ is to be real, then δ written in terms of the new parameters X and Y gives us

$$\delta = \left(\beta^2 - \frac{\omega^2}{c^2} K\right)^{\frac{1}{2}} = \frac{\omega_{pe}}{c} (X^2 - KY^2)^{\frac{1}{2}}$$

which shows us that X must be greater than $\sqrt{KY}$. For the remainder of the discussion, we will assume that K = 1 and thus only the region to the right of the light line is of interest to us in our analysis. Since δ must be real for trapped modes to occur, we may conclude from the following inequality that γ_1 must also be real:

$$\gamma_1{}^2 = \beta^2 - \frac{\omega^2}{c^2} \epsilon > \delta^2 > 0$$

Thus, surface waves are those waves having γ_1 and γ_2 real. Solutions for which γ_1 is real and γ_2 is imaginary exhibit both damped and oscillatory behavior. For shorter wavelengths, however, the oscillatory behavior tends to dominate so these solutions will be referred to as body wave solutions. Looking at Fig. 2, the trapped mode region lies to the right of the light line and the oscillatory or waveguide modes occur in the area to the left of the light line. Those sections in which γ_2 is imaginary, i.e. possible body wave regions, have been distinguished from possible areas in which surface wave solutions may occur. At very low frequencies, the slope of the line separating the ion body and surface wave regions is equal to the large wavelength limit of the ion acoustic wave phase velocity. Thus, this line is called the ion acoustic line. The line separating the electron body and surface waves at very high frequencies is called the electron thermal line.

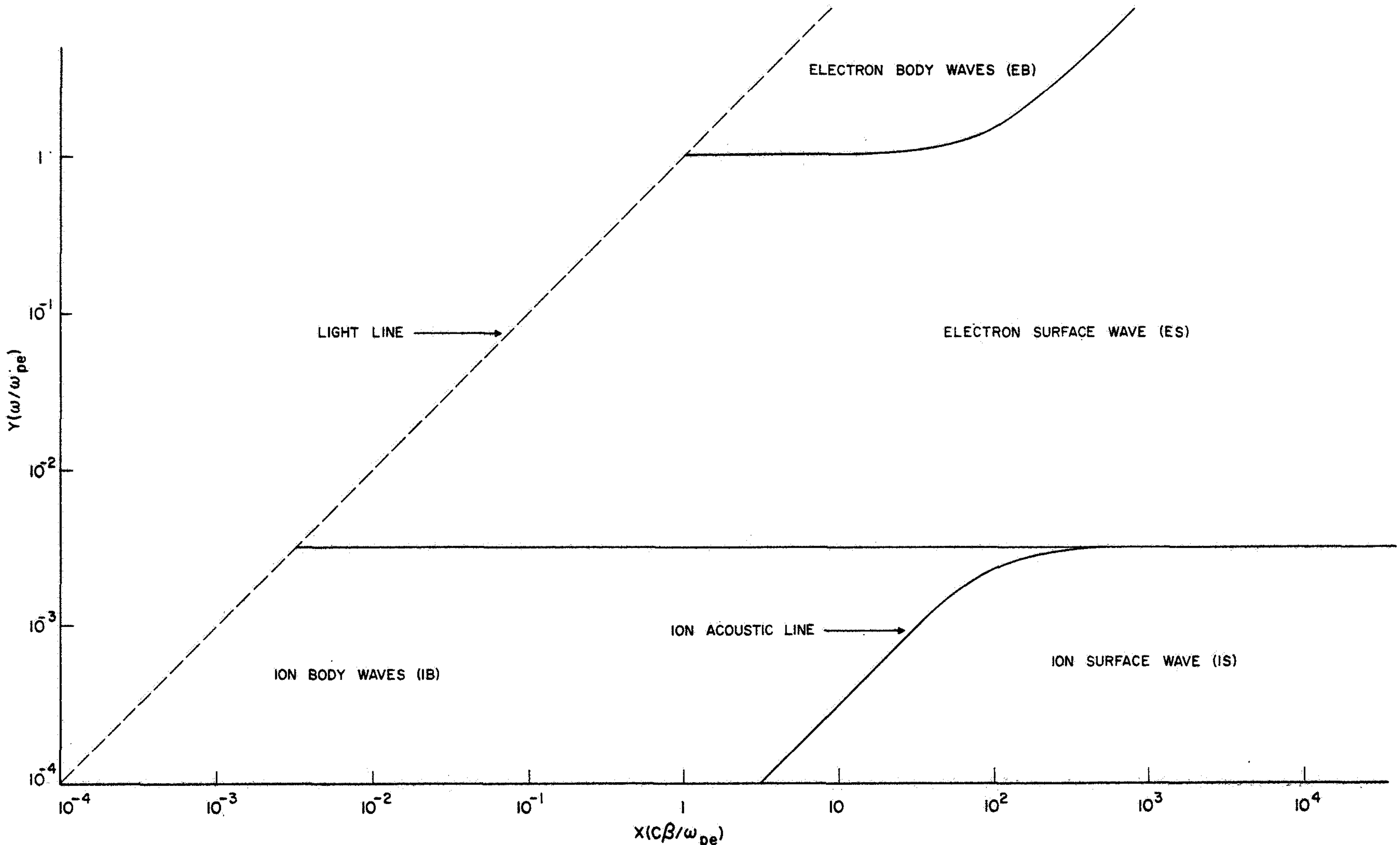
If one examines either of the two dispersion equations (Eqs. 28-29), it is readily noticed that they are dependent on five different parameters:

1.) m_i/m_e--the ion to electron mass ratio

2.) K--the relative dielectric constant

3.) α--the normalized electron plasma frequency

4.) b/a--the ratio of the distance between the conducting plates to the plasma slab width



TRAPPED MODE REGIONS FOR $c/W = 10^2$ AND $m_i/m_e = 10^5$

FIGURE 2

5.) c/W--the ratio of the free space speed of light to the electron thermal velocity.

For purposes of numerical calculation, it is assumed that $m_i/m_e = 10^5$ and K = 1 unless otherwise specified. Both the symmetric and antisymmetric dispersion equations are studied for different values of the remaining three parameters α, c/W, and b/a. In particular, the following cases will be considered in this paper:

1.) For symmetric E_y

Case I -- $\alpha = 0.1$, $c/W = 10^2$, $b/a = 2$

Case II -- $\alpha = 0.1$, $c/W = 10^2$, $b/a = 1$

Case III -- $\alpha = 0.1$, $c/W = 10^2$, $b/a \rightarrow \infty$

Case IV -- $\alpha = 0.1$, $c/W = 10^3$, $b/a = 2$

Case V -- $\alpha = 1.5$, $c/W = 4.2\times10^2$, $b/a \rightarrow \infty$,

$m_i/m_e = 3.66\times10^5$

2.) For antisymmetric E_y

Case VI -- $\alpha = 0.1$, $c/W = 10^2$, $b/a = 2$

Case VII -- $\alpha = 0.1$, $c/W = 10^2$, $b/a = 1$

Case VIII-- $\alpha = 0.1$, $c/W = 10^2$, $b/a \rightarrow \infty$

Case IX -- $\alpha = 0.1$, $c/W = 10^3$, $b/a = 2$

Case X -- $\alpha = 1.0$, $c/W = 10^2$, $b/a \rightarrow \infty$

It should be noted that the values used in Case I are the same as those used by Andersson and Weissglas[10] in which similar work was done in the low frequency range for a plasma cylinder. In analyzing our dispersion equations, we will extend their work in three ways. First, we will study and compare the dispersion relations for both symmetric and antisymmetric longitudinal $\overline{E}$ fields. Also, we will consider the effects of changing the plasma width and the distance separating the conducting plates. And finally, we will study electron body and surface waves as well as trapped ion modes.

By letting b/a go to one as in Case II and to infinity as in Case III, we can see how moving the conducting plates influences the behavior of our dispersion curves. Case IV is included to show the changes resulting from decreasing the electron temperature. Case V allows us to compare our theoretical results with the experimental data of Little and Jones[9]. In Cases VI through IX, we will be looking for differences between the antisymmetric dispersion equations and their respective symmetric counterparts. Case X is of interest since it will enable us to see what effects changing the plasma width has on the trapped modes.

When we apply the above cases to physical situations, we see that changing the distance between the conducting plates gives us:

1.) for $b/a \rightarrow \infty$, a plasma slab confined by a dielectric[9] which may serve as a model for the ionosphere or for a plasma discharge in free space
2.) for $b/a = 1$, a plasma-filled parallel plate waveguide[12]
3.) for $b/a = 2$, a partially-filled, parallel plate waveguide[14].

3.2 THE SYMMETRIC DISPERSION EQUATION

Let us consider Case I in detail. Looking at Fig. 3, we notice that a surface wave exists in the frequency ranges $\omega < \omega_{pi}/\sqrt{2}$ and $\omega > \omega_{pi}$ while body waves are supported by the plasma slab for frequencies below ω_{pi} and above ω_{pe}. The ion and electron body waves are labeled in such a way that the number associated with each of them is equal to the number of half wavelengths between $x = -a$ and $x = +a$.

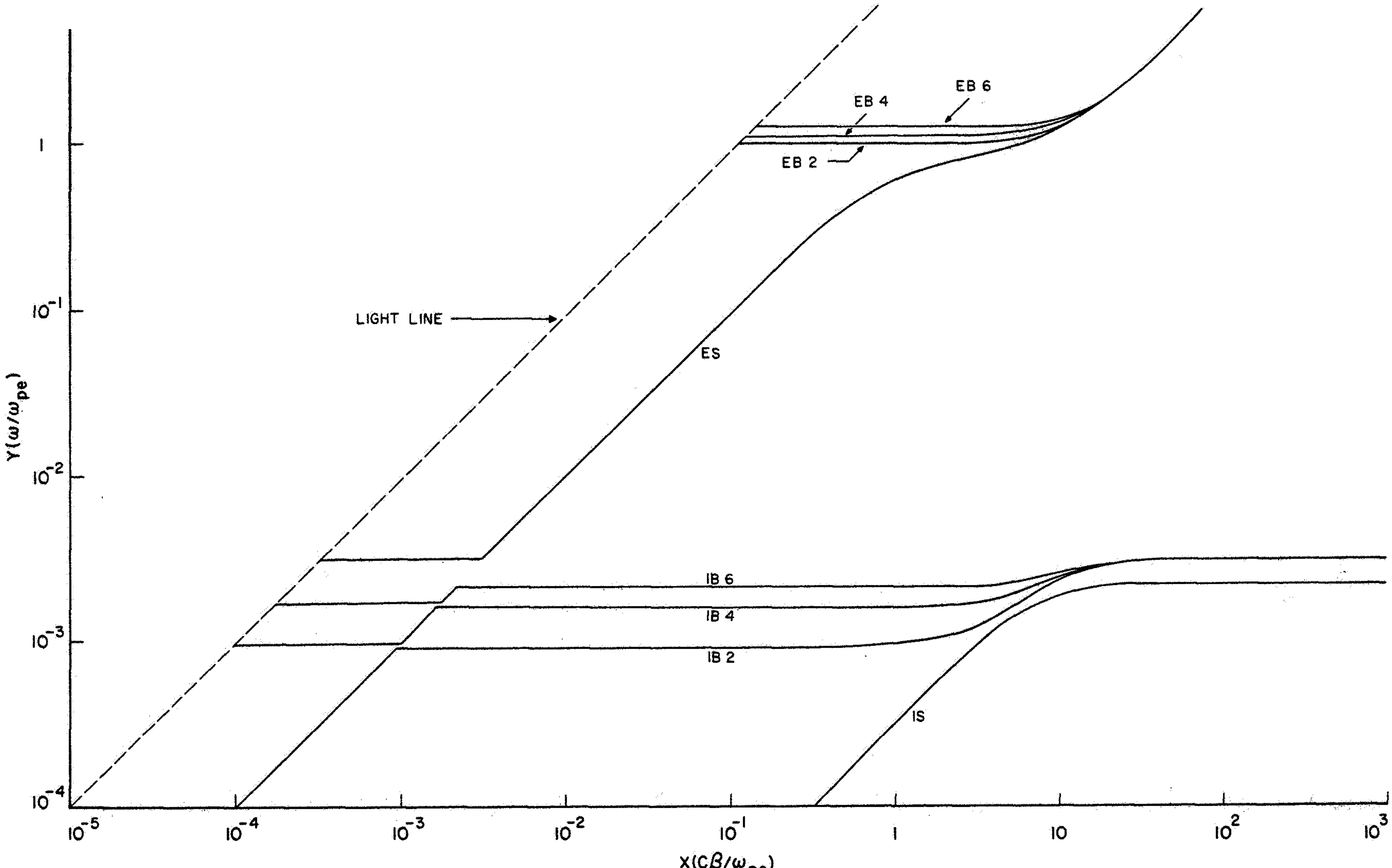
Only the lowest harmonic body wave exists for frequencies much below ω_{pi}. The low frequency linear approximation for this wave is

$$Y \approx \alpha\left(\frac{b}{a} - 1\right) X = 0.1\,X \tag{32}$$

For shorter wavelengths, all of the ion body waves asymptotically approach the ion plasma frequency. The ion surface wave has the low frequency representation (Appendix A):

$$Y \approx \frac{W}{c}\,\frac{\omega_{pi}}{\omega_{pe}}\,X = 10^{-4.5}\,X$$

For large values of β, the frequency of this surface wave approaches $\omega_{pi}/\sqrt{2}$. This is analogous to the electron surface wave in a cold plasma-filled waveguide (Trivelpiece and Gould[2]).

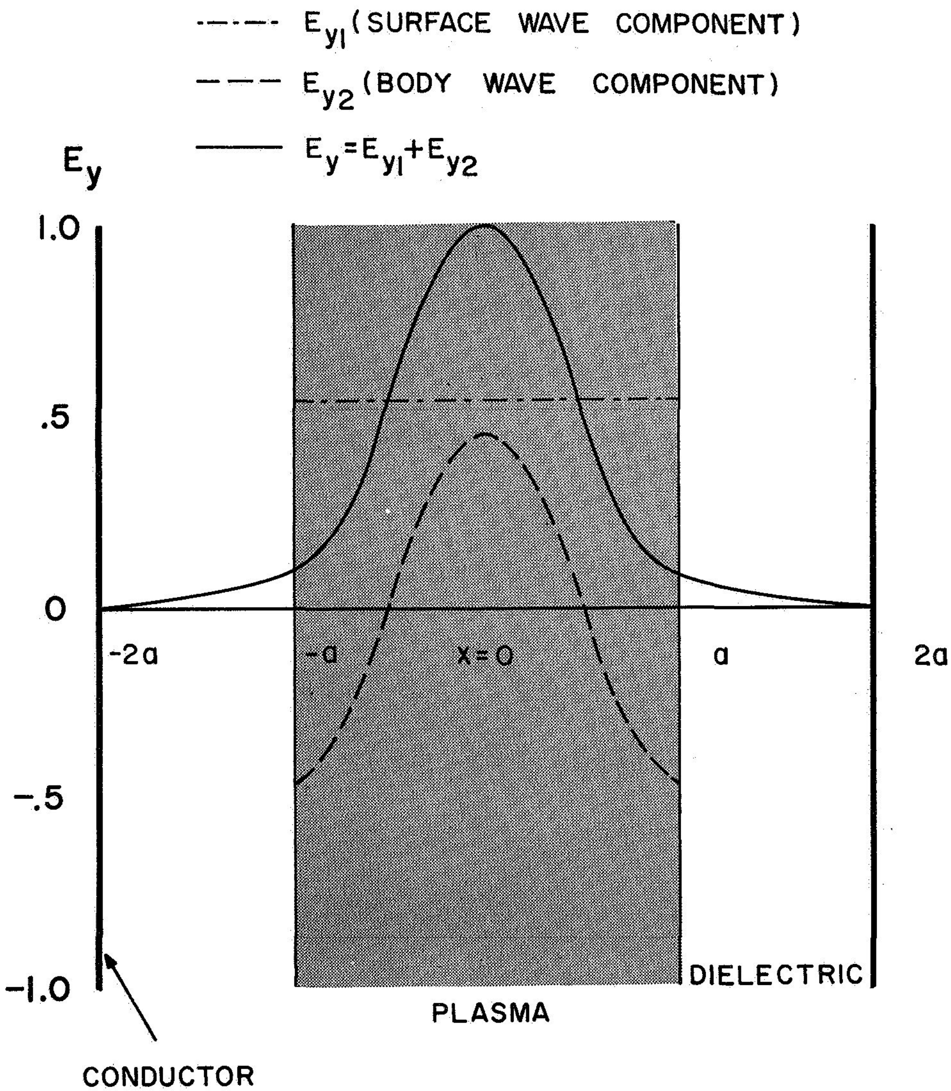


DISPERSION CURVES FOR SYMMETRIC Ey, $\alpha=0.1$, $c/W=10^2$ AND $b/a=2$

FIGURE 3

For wave numbers near the light line, all of the ion body waves except for the lowest order one are shifted vertically. This region will be referred to as the transition region. To the right of the transition region, the oscillatory body wave behavior dominates. The ion body waves are labeled for this region. To the left of the transition region, we have a superposition of body and surface waves and neither wave dominates. This behavior is indicated for E_y in Figs. 4 and 5. As we increase X and go through the transition region, the imaginary propagation constant γ_2 becomes larger. Indeed, the lowest order body wave existing to the left of the transition region (IB3) displays the same oscillatory behavior in this region as does the lowest order ion body wave (IB1) that lies to the right of the transition region. This indicates a rather strong coupling of the body wave solutions in the transition region. It should be noted that the transition region occurs along the line given by Eq. 32. The presence of the parameter b/a indicates that the coupling depends on the location of the conducting boundary.

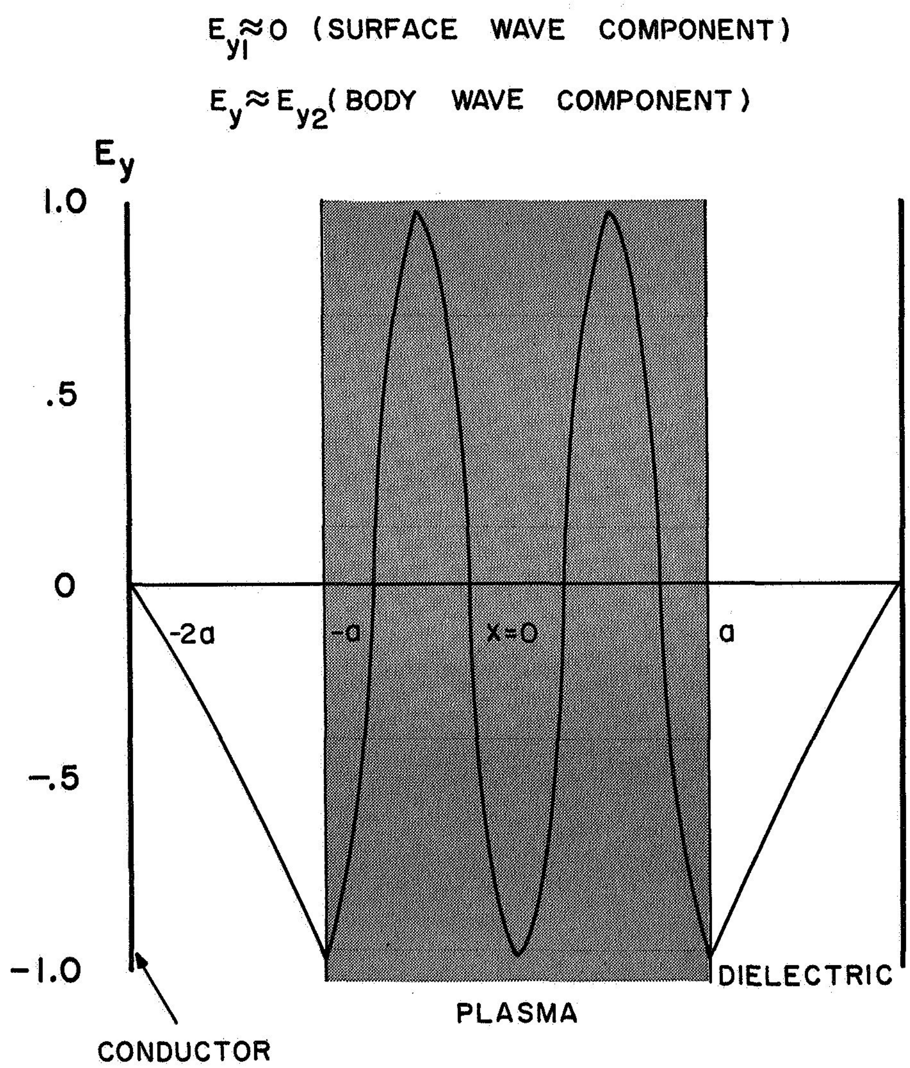
For frequencies between the ion and electron plasma frequencies, only an electron surface wave is found to exist. When $\omega << \omega_{pe}$, this surface wave solution has the same slope as the lowest order ion body wave has when $\omega << \omega_{pi}$.



E_{y1}, E_{y2} AND E_y VS DISTANCE FOR $Y=9.3\times10^{-4}$ AND $X=10^{-3}$ IN FIGURE 3

(NORMALIZED FOR $E_{y\,max.}=1$)

FIGURE 4



$E_y \approx E_{y2}$ VS DISTANCE FOR $Y = 1.6 \times 10^{-3}$ AND $X = 10^{-1}$ IN FIGURE 3

(NORMALIZED FOR $E_{y\ max.} = 1$)

FIGURE 5

The first three body wave modes have been drawn for frequencies above ω_{pe}. For $\omega > \omega_{pe}$ and large values of X, the dispersion relation can be simplified to the linear equation (Appendix C):

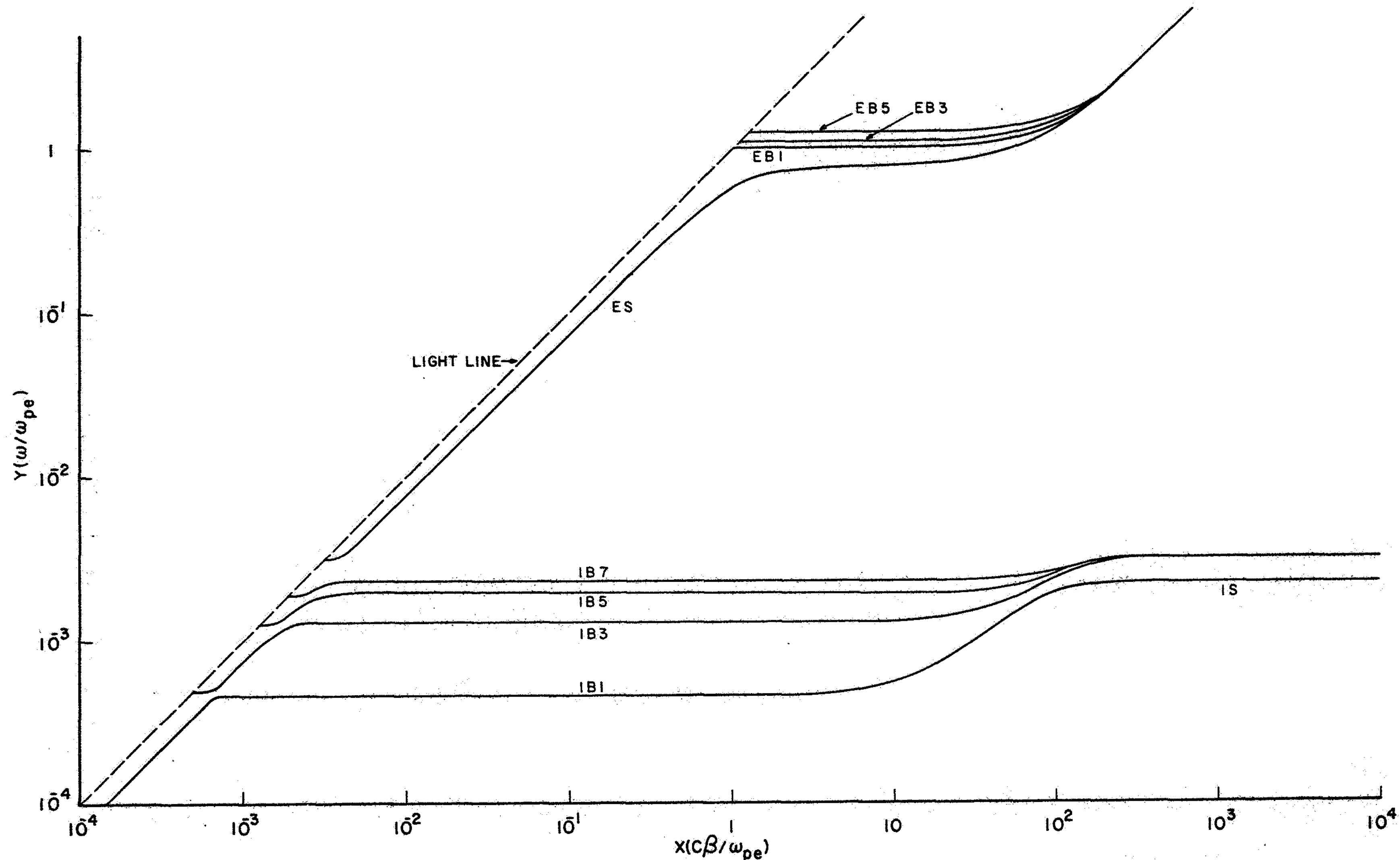
$$Y = \frac{W}{c} X = 10^{-2} X$$

Thus the group and phase velocities in this region approach the electron thermal velocity.

3.3 THE ANTISYMMETRIC DISPERSION EQUATION

For antisymmetric E_y, Case VI (Fig. 6) is comparable to Case I. The most significant difference between the antisymmetric and symmetric dispersion equations is seen to be the absence of a very low frequency ($\omega << \omega_{pi}$) ion surface wave. When β is small, the lowest order solution is to the left of the ion acoustic line and is a body wave. For large β, this curve crosses the ion acoustic line and becomes a surface wave which asymptotically approaches $\omega_{pi}/\sqrt{2}$.

For $\omega_{pi} << \omega << \omega_{pe}$, the electron surface wave for the antisymmetric case is found to be shifted more to the left than for the symmetric case. It should be noted that this is true for all cases except $b/a \rightarrow \infty$ (For that special case, the surface wave solution coincides with the dielectric light line). We therefore may conclude that the group and phase velocities for the surface waves given by



DISPERSION CURVES FOR ANTISYMMETRIC E_y, $\alpha = 0.1$, $c/W = 10^2$ AND $b/a = 2$

FIGURE 6

the antisymmetric cases are greater than or equal to their respective symmetric counterparts.

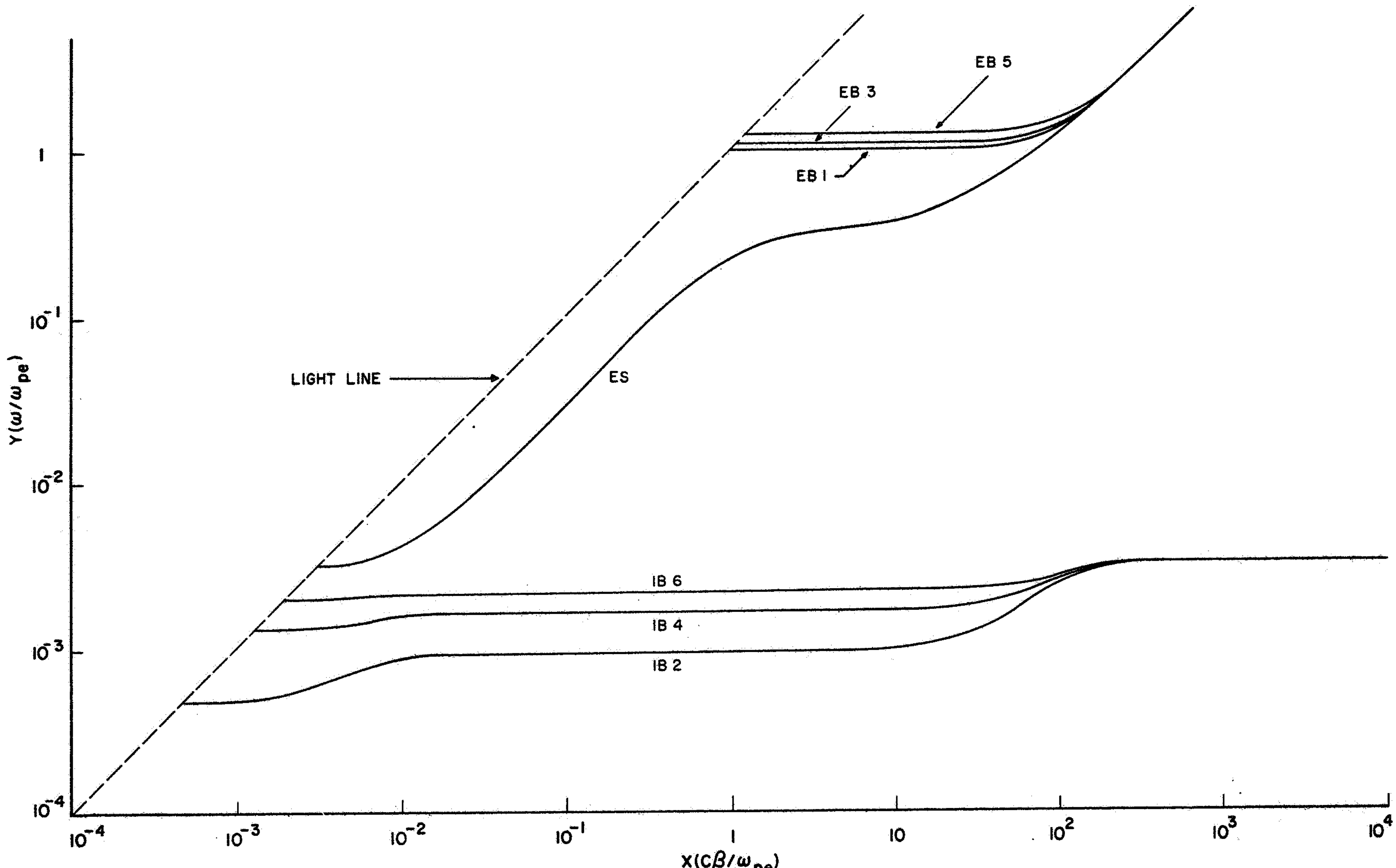
For $\omega > \omega_{pe}$ and large values of β, the surface wave and body wave solutions for the antisymmetric cases also approach the electron thermal line.

3.4 EFFECTS OF THE CONDUCTING PLATES

Setting $b/a = 1$ for antisymmetric E_y (Fig. 7) results in a low frequency cutoff below which no ion body or surface waves can propagate. Similarly, it is found that no ion surface wave exists for symmetric E_y (Fig. 8) when $b/a = 1$. For $\omega << \omega_{pi}$, increasing the distance between the conducting plate and the plasma-dielectric interface causes the lowest order body wave solution to shift to the left until at $b/a \rightarrow \infty$, it coincides with the light line (Figs. 3,6,9,10). The very low frequency ion surface wave is not noticeably affected by changing b/a as long as $b/a > 1$.

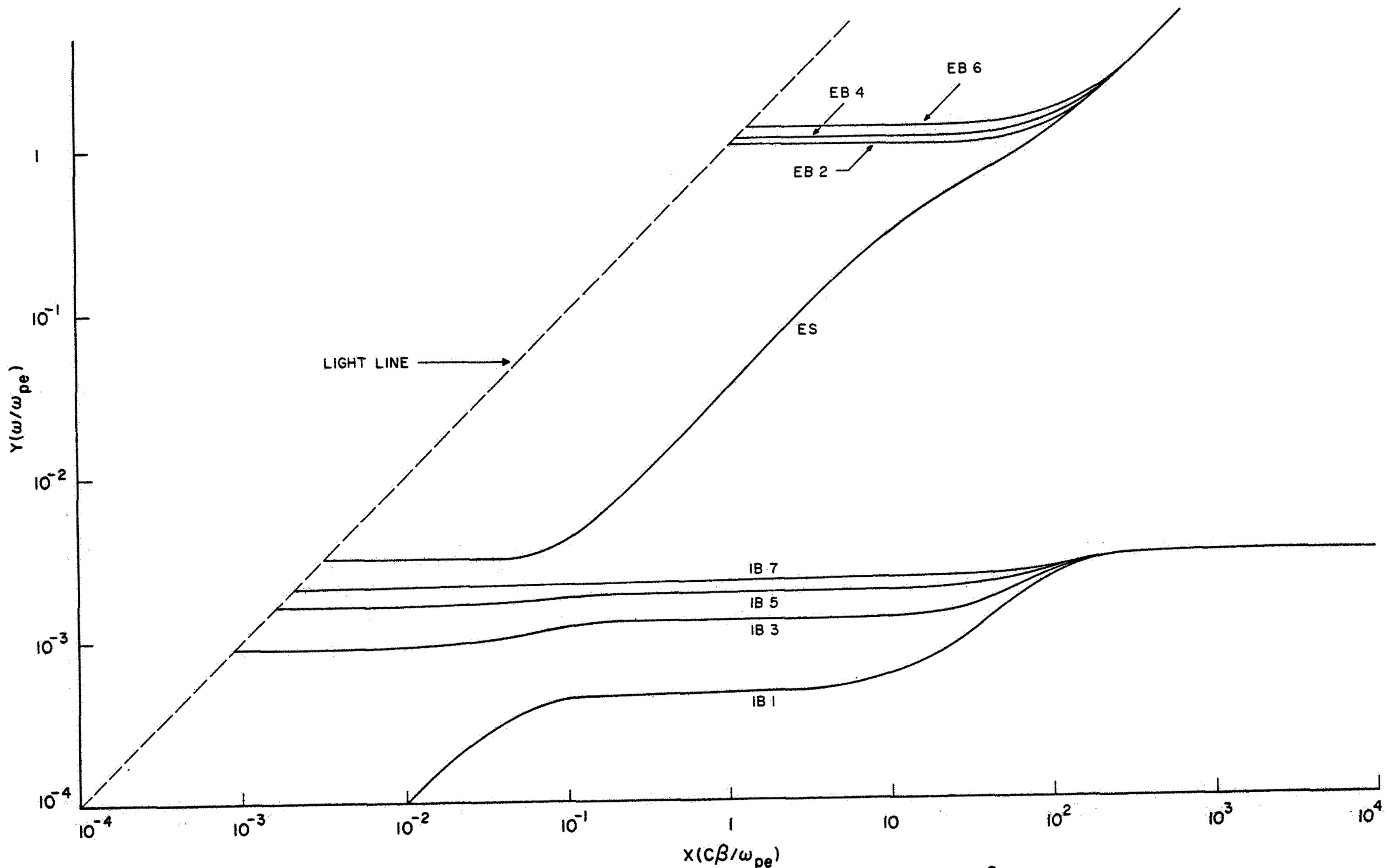
For smaller values of X and ω near ω_{pi}, it has been mentioned that there is a transition region in which there appears a coupling between the different body wave harmonics[10]. If we compare Cases I - III or VI - VIII, we see that decreasing b/a weakens this coupling and shifts the transition region more to the right.

For $\omega << \omega_{pe}$, the electron surface wave, like the lowest order ion body wave for $\omega << \omega_{pi}$, is shifted to the left when b/a is increased. We therefore may



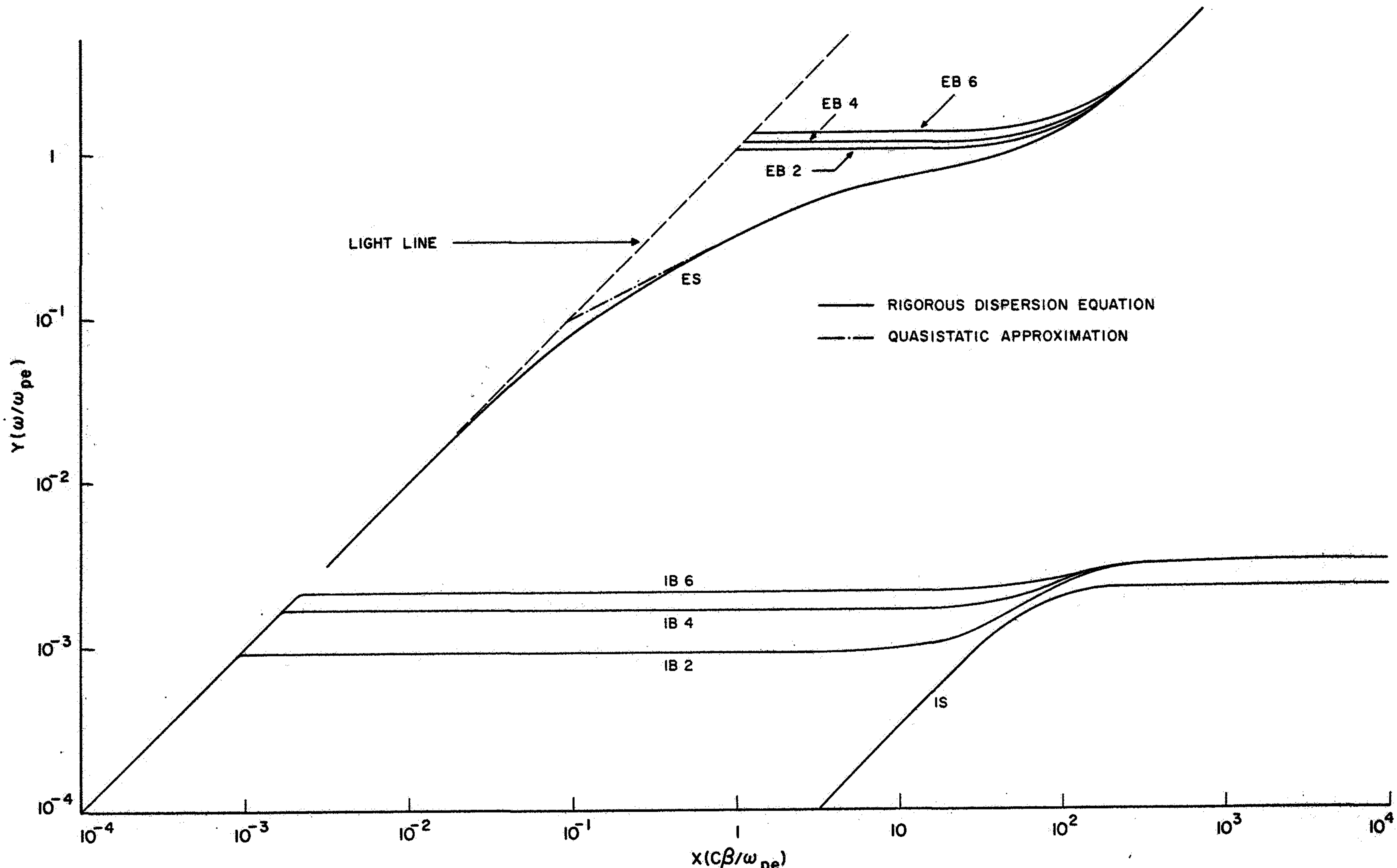
DISPERSION CURVES FOR ANTISYMMETRIC Ey, $\alpha = 0.1$, $c/W = 10^2$ AND $b/a = 1$

FIGURE 7



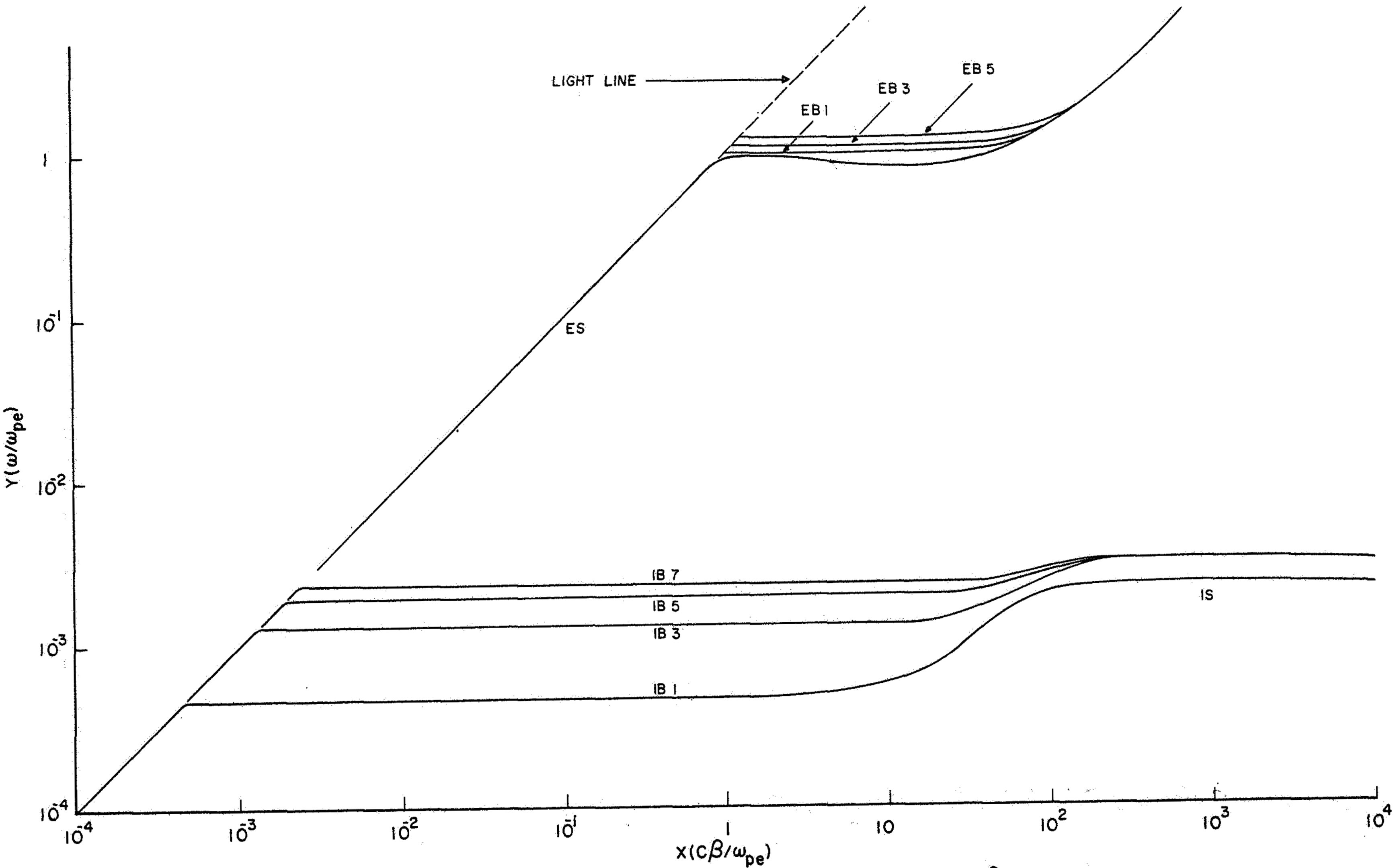
DISPERSION CURVES FOR SYMMETRIC Ey, $\alpha = 0.1$, $c/W = 10^2$ AND $b/a = 1$

FIGURE 8



DISPERSION CURVES FOR SYMMETRIC E_y, $\alpha = 0.1$, $c/W = 10^2$ AND $b/a \rightarrow \infty$

FIGURE 9



DISPERSION CURVES FOR ANTISYMMETRIC Ey, $\alpha=0.1$, $c/W=10^2$, AND $b/a \rightarrow \infty$

FIGURE 10

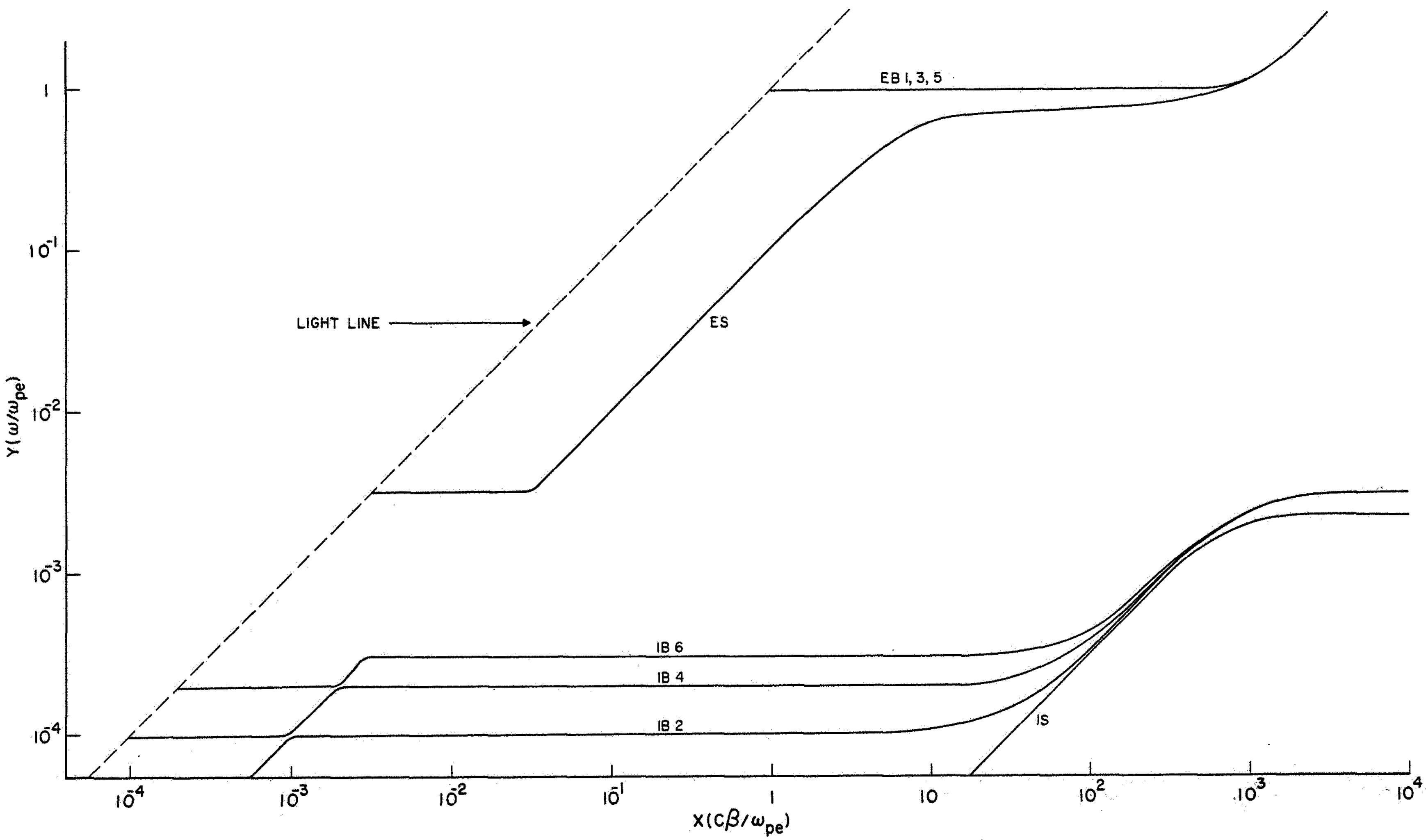
conclude that increasing b/a increases the phase and group velocities of this surface wave. When $\omega >> \omega_{pe}$, changing b/a has negligible effect on the body and surface waves.

3.5 TEMPERATURE EFFECTS

Since the approximate low frequency ion surface wave solution for symmetric E_y is proportional to the electron thermal velocity (Appendix A), decreasing this parameter as in Case IV (Fig. 11) causes a shift to the right in the surface wave curve. The lowest order ion body wave for very small wave numbers is not affected by changing T_e (Figs. 11,12). The higher order body wave harmonics are found to be more spread out and occur at lower frequencies when the electron temperature is decreased. In contrast, it is noticed that the electron body waves are bunched together so as to form what might be considered a continuum. For large wave numbers, the electron body and surface waves asymptotically approach the electron thermal line which is shifted to the right when T_e decreases.

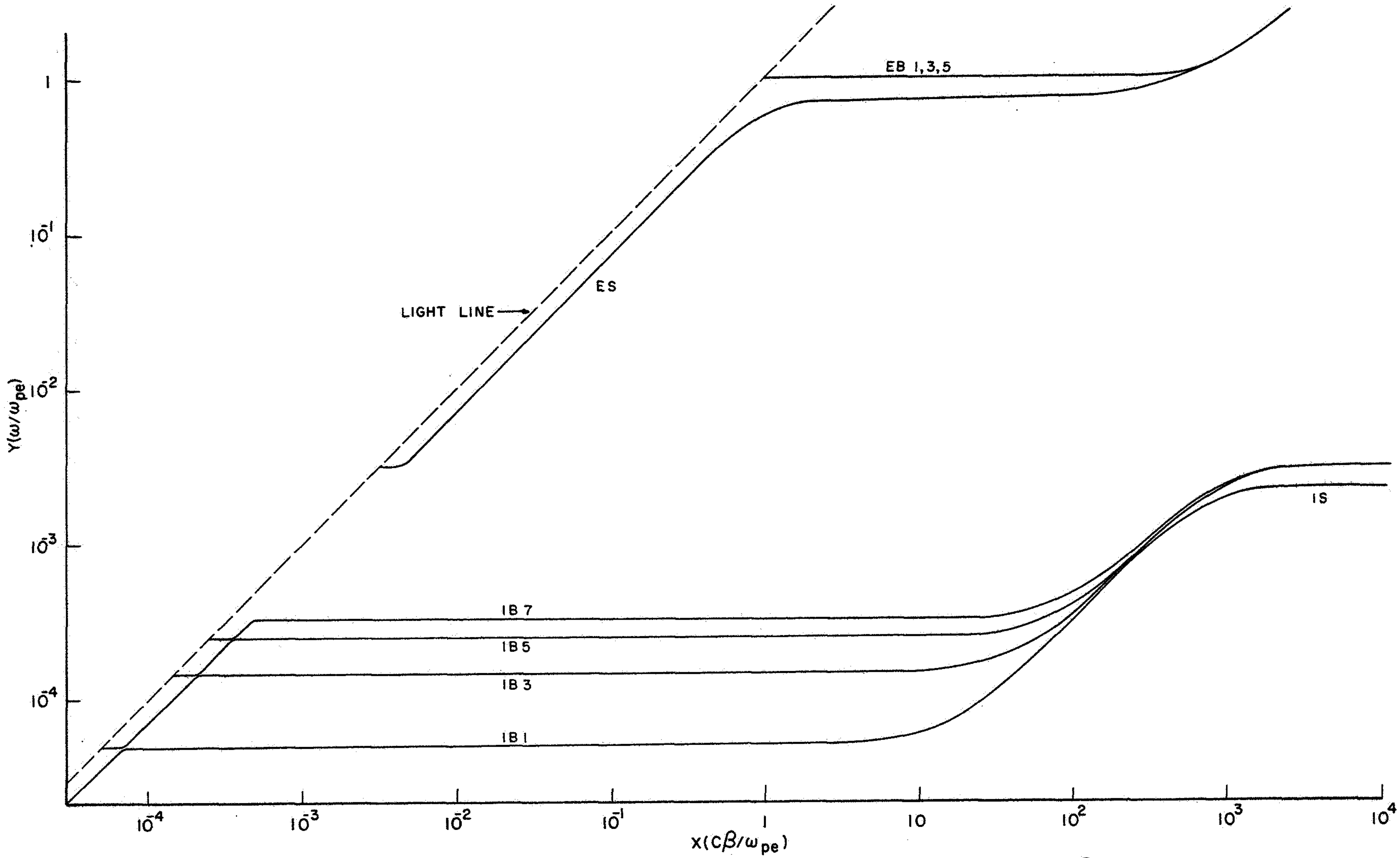
3.6 INFLUENCE OF THE PLASMA WIDTH

Increasing the width of the plasma slab (Case X--Fig. 13) results in phenomena similar to those that were noticed when T_e was made smaller. The higher order ion body waves near the light line are more spread out and occur at lower frequencies. The electron body waves are bunched about ω_{pe}. The most noticeable change is in the electron surface wave for $\omega < \omega_{pe}$. A backward surface



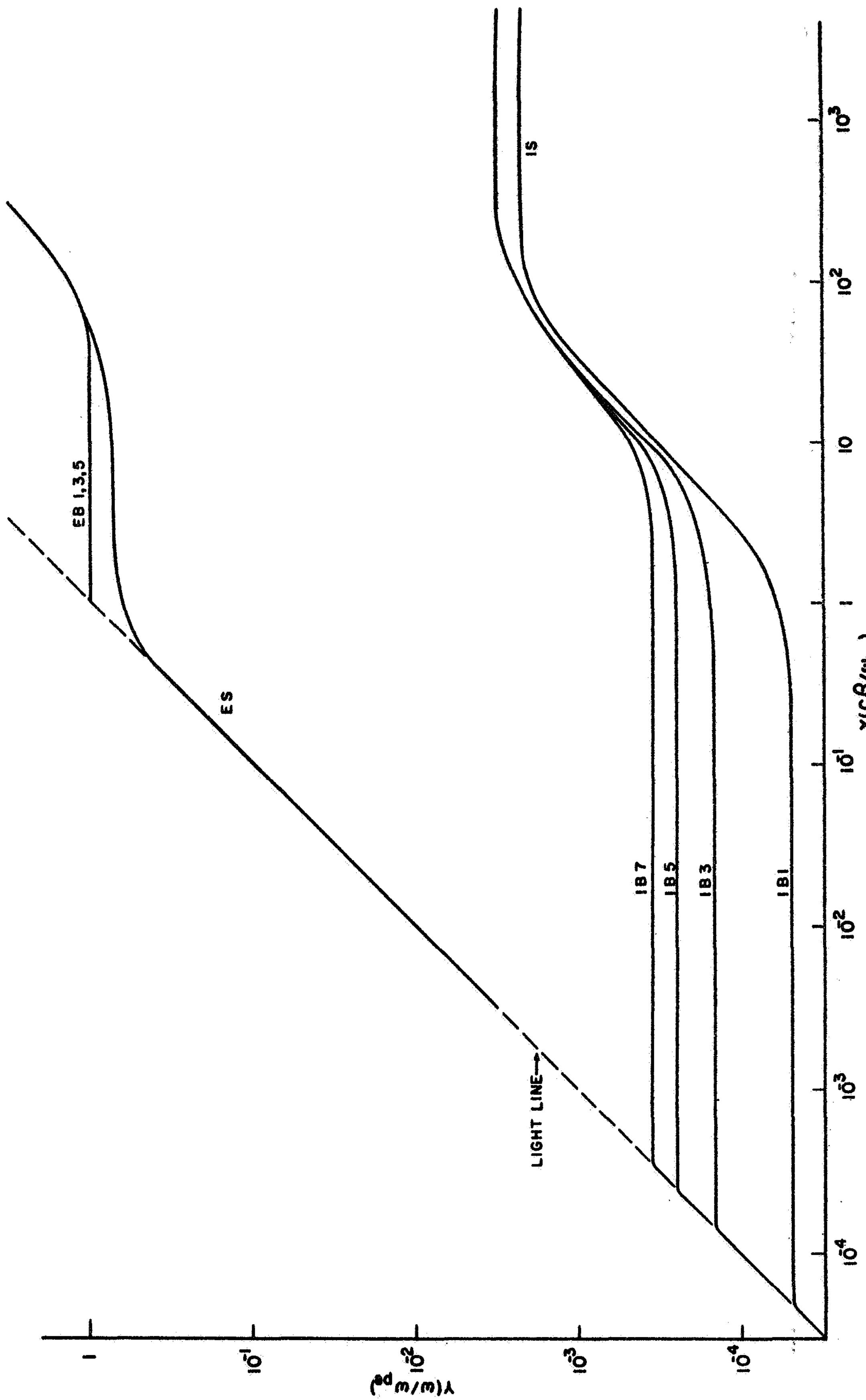
DISPERSION CURVES FOR SYMMETRIC Ey, $\alpha = 0.1$, $c/W = 10^3$ AND $b/a = 2$

FIGURE 11



DISPERSION CURVES FOR ANTISYMMETRIC E_y, $a = 0.1$, $c/W = 10^3$ AND $b/a = 2$

FIGURE 12



DISPERSION CURVES FOR ANTISYMMETRIC E_y, $\alpha=1.0$, $c/W=10^2$ AND $b/a \to \infty$

FIGURE 13

wave was found to exist when $\alpha = 0.1$. By increasing the plasma width, we now have a forward wave. This phenomenon was previously observed by Diament, Granatstein, and Schlesinger[11] for dipole modes. No analogous behavior occurs for the symmetric case.

4 APPROXIMATIONS

There are certain approximations that are sometimes used to simplify the analysis of the dispersion equation. Two of these which are appropriate in studying trapped mode propagation are the quasistatic approximation (Trivelpiece and Gould[2], Crawford and Tataronis[14], O'Brien[15]) and the heuristic approximation (Little and Jones[9], Andersson and Weissglas[10]).

The quasistatic approximation is obtained by letting $c \to \infty$ in δ and γ_1 so that δ and $\gamma_1 \to \beta$. As shown in Appendix D, this is equivalent to taking

$$\nabla \times \bar{E} = 0$$

at the beginning of the analysis. The resulting symmetric dispersion relation is

$$\frac{\epsilon}{K} \tanh \beta(a-b) = \coth \beta a - \frac{\omega_{pe}^2 \beta}{\omega^2 \gamma_2 \epsilon_+} \coth \gamma_2 a \qquad (33)$$

or, written in terms of X and Y($\omega_{pi} \ll \omega_{pe}$),

$$\frac{\frac{1}{Y^2} - 1}{K} \tanh \alpha\left(\frac{b}{a} - 1\right) X = \coth \alpha X -$$

$$\frac{X}{Y^2\left(1-\frac{m_e}{m_iY^2}\right)\left[X^2+\frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_iY^2}\right)}\right]^{\frac{1}{2}}}\coth \alpha\left[X^2+\frac{c^2(1-Y^2)}{W^2\left(1-\frac{m_e}{m_iY^2}\right)}\right]^{\frac{1}{2}} \tag{34}$$

This approximation is valid in the region for which $\delta \approx \beta$ i.e. $X >> Y$. For Cases I or II (Figs. 3 and 8), the significant portion of the body wave and surface wave curves lies to the right of the light line where $X >> Y$ and thus our results are in very good agreement with the rigorous results. There is even good agreement near the light line since the second term on the right hand side of the dispersion equation dominates and this term has been unaltered by the approximation. For Case III in which $b/a \to \infty$ (Fig. 9), the quasistatic approximation breaks down near the light line. It no longer gives us an ion body wave going into the origin along the light line, but instead, it goes into the origin to the left of the light line as

$$Y = \sqrt{\alpha X}$$

This behavior also occurs for the electron surface wave in the midfrequency region ($\omega_{pi} << \omega << \omega_{pe}$). It should be noted that for confined plasmas having smaller widths, the differences between the quasistatic and rigorous results are much smaller than those predicted by Diament et. al.[11] for a plasma column whose width was assumed large. As a matter of fact, for $b/a \leq 2$ and $\alpha = 0.1$, the differences were found to be negligible.

We formulate the heuristic approximation by knowledge of the behavior of the field solutions. It will be shown that this approximation predicts the ion body waves fairly accurately. We begin by using the dispersion relation for ion waves in an infinite medium

$$\frac{\omega^2}{\omega_{pi}^2} = \frac{k^2}{k^2 + 1/\lambda_{De}^2}$$

where k^2 is the square of the wave number and λ_{De} is the electron Debye length. For our particular problem, we separate k into both transverse and longitudinal components which are expressed by the equation

$$k^2 = k_{\parallel}^2 + k_{\perp}^2 = \beta^2 + (\frac{2\pi}{\lambda})^2$$

For $b/a > 1$, examination of the field solutions (Figs. 4,5) gives us

$$\lambda = \frac{4a}{n} \; ; \; n = 1,2,3\ldots$$

where n is the number of half wavelengths that exist in the x direction inside the plasma slab. Even integers of n represent wavelengths for symmetric E_y and odd integers of n give λ for antisymmetric E_y. Replacing k^2 by our heuristic approximation, the ion body wave equation is

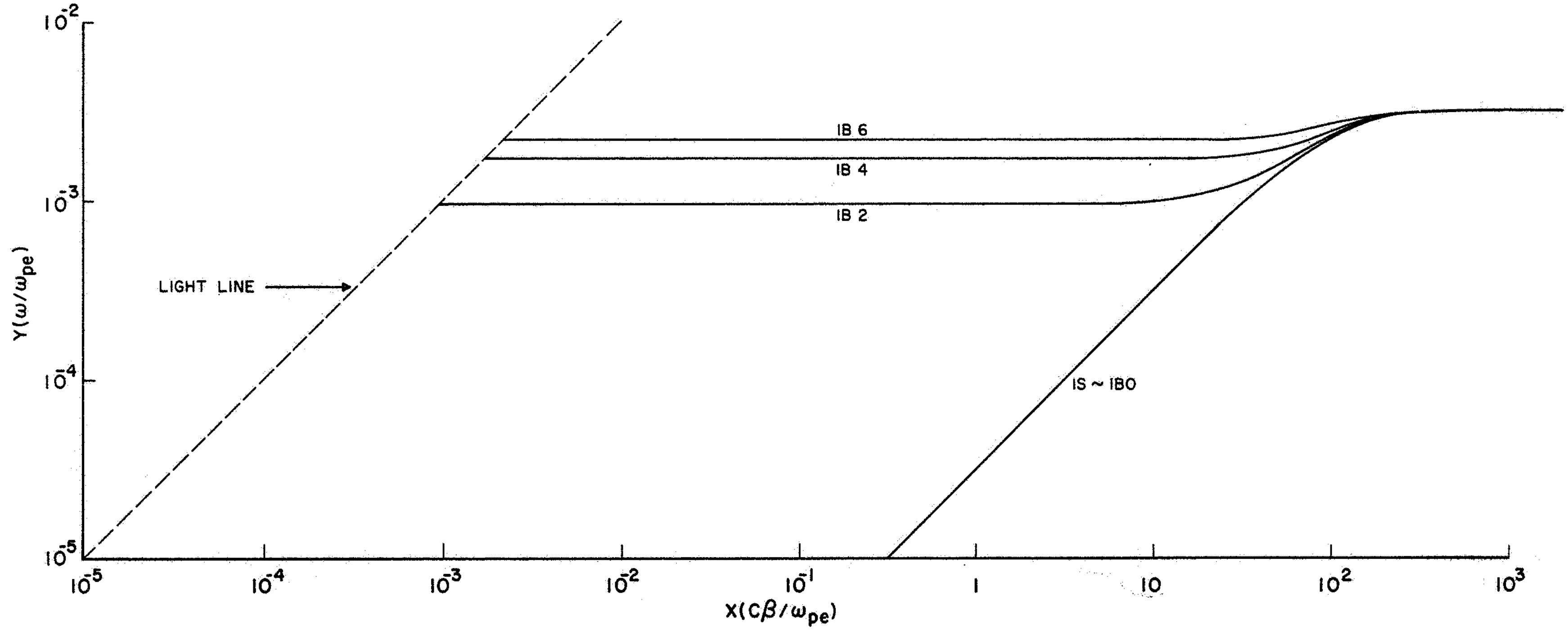
$$\frac{\omega^2}{\omega_{pi}^2} = \frac{\beta^2 + (\frac{n\pi}{2a})^2}{\beta^2 + (\frac{n\pi}{2a})^2 + 1/\lambda_{De}^2}\ ; \quad n = \begin{array}{l} 1,3,\ldots \text{for antisymmetric } E_y \\ 2,4,\ldots \text{for symmetric } E_y \end{array} \tag{35}$$

or in terms of X and Y,

$$Y = \frac{\omega_{pi}\left[X^2 + (\frac{n\pi}{2\alpha})^2\right]^{\frac{1}{2}}}{\omega_{pe}\left[X^2 + (\frac{n\pi}{2\alpha})^2 + \frac{c^2}{W^2}\right]^{\frac{1}{2}}}\ ; \quad n = \begin{array}{l} 1,3,\ldots \text{for antisymmetric } E_y \\ 2,4,\ldots \text{for symmetric } E_y \end{array} \tag{36}$$

Using the values given in Case III, a Brillouin diagram of the above equation for symmetric E_y is drawn in Fig. 14. There are some significant differences between the results shown in Fig. 14 and those obtained from the full set of equations (Fig. 9). First, the coupling phenomena of the different ion body waves is not found using the heuristic model. Also, we do not obtain the lowest order ion body wave going into the origin for very small wave numbers. Finally, if we examine the zeroth order heuristic solution, we find that it correctly describes the low frequency ion surface wave but at large wave numbers, it approaches ω_{pi} rather than $\omega_{pi}/\sqrt{2}$.

Experimental results on low frequency waves have been obtained for a cylindrical plasma column by Little and Jones[9]. Although their geometry was cylindrical, the



HEURISTIC APPROXIMATION OF ION WAVES IN FIGURE 8

FIGURE 14

agreement between their experimental data without a magnetic field and our theoretical values is seen in Fig. 15 to be very good. It would appear that Little and Jones were measuring the ion surface wave as well as the lowest order body wave. If we use the values in Case V and look at the Brillouin diagram for antisymmetric E_y (Fig. 16), it would seem that it should be possible to also measure the dipole mode corresponding to our lowest order antisymmetric mode.

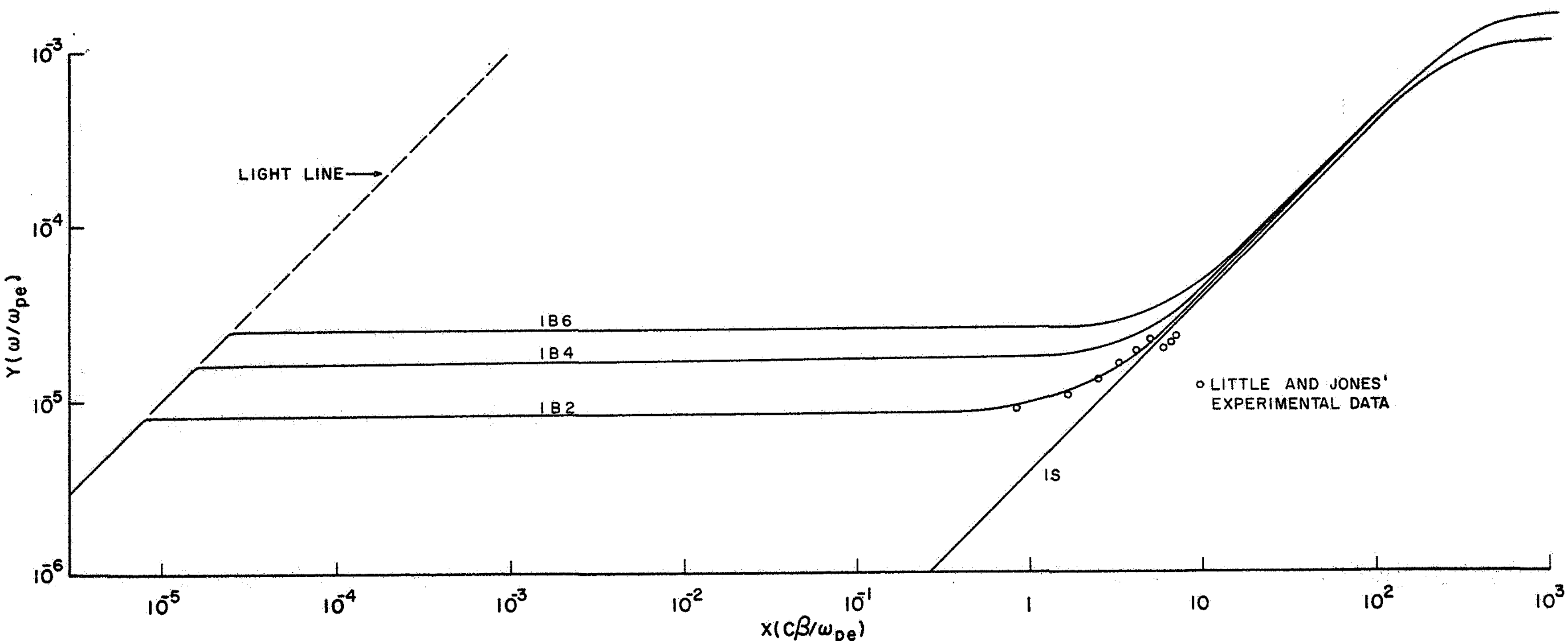
Finally, we wish to examine the influence of collisional and Landau damping which so far have been neglected. In order to include the effects of collisions in our previously obtained dispersion relations, we let

$$\omega^2 = \omega(\omega - i\nu)$$

and

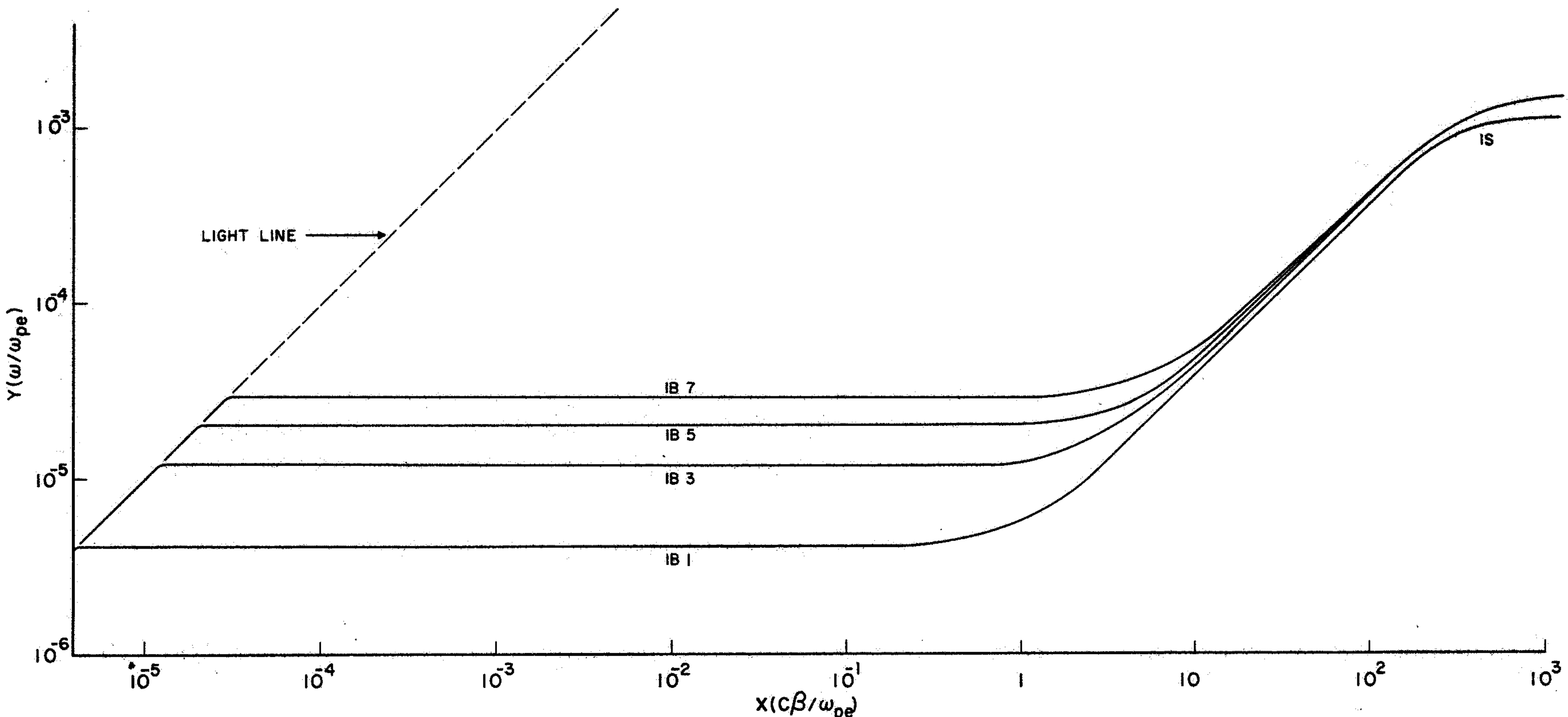
$$\beta = \beta_R - i\beta_I$$

where ν is the electron-neutral collision frequency for momentum transfer and β_R and β_I are the respective real and imaginary parts of the longitudinal propagation constant. For reason of simplification, we will use the heuristic approximation for symmetric E_y to study the collisional damping of ion waves. Assuming



DISPERSION CURVES FOR SYMMETRIC E_y, $\alpha = 1.5$, $c/W = 4.2 \times 10^2$, $b/a \rightarrow \infty$, $m_i/m_e = 3.66 \times 10^5$

FIGURE 15



DISPERSION CURVES FOR ANTISYMMETRIC Ey, $\alpha=1.5$, $c/W=4.2\times10^2$, $b/a\rightarrow\infty$ AND $m_i/m_e=3.66\times10^5$

FIGURE 16

$$\frac{1}{\lambda_{De}^{2}} \gg \beta^2 + \left(\frac{n\pi}{a}\right)^2$$

and substituting the above equations into this approximation, we get the following two equations:

1.) Real part

$$\frac{\omega^2}{\omega_{pi}^{2}} = \left[\beta_R^{2} - B_I^{2} + \left(\frac{n\pi}{a}\right)^2\right]\lambda_{De}^{2} \tag{37}$$

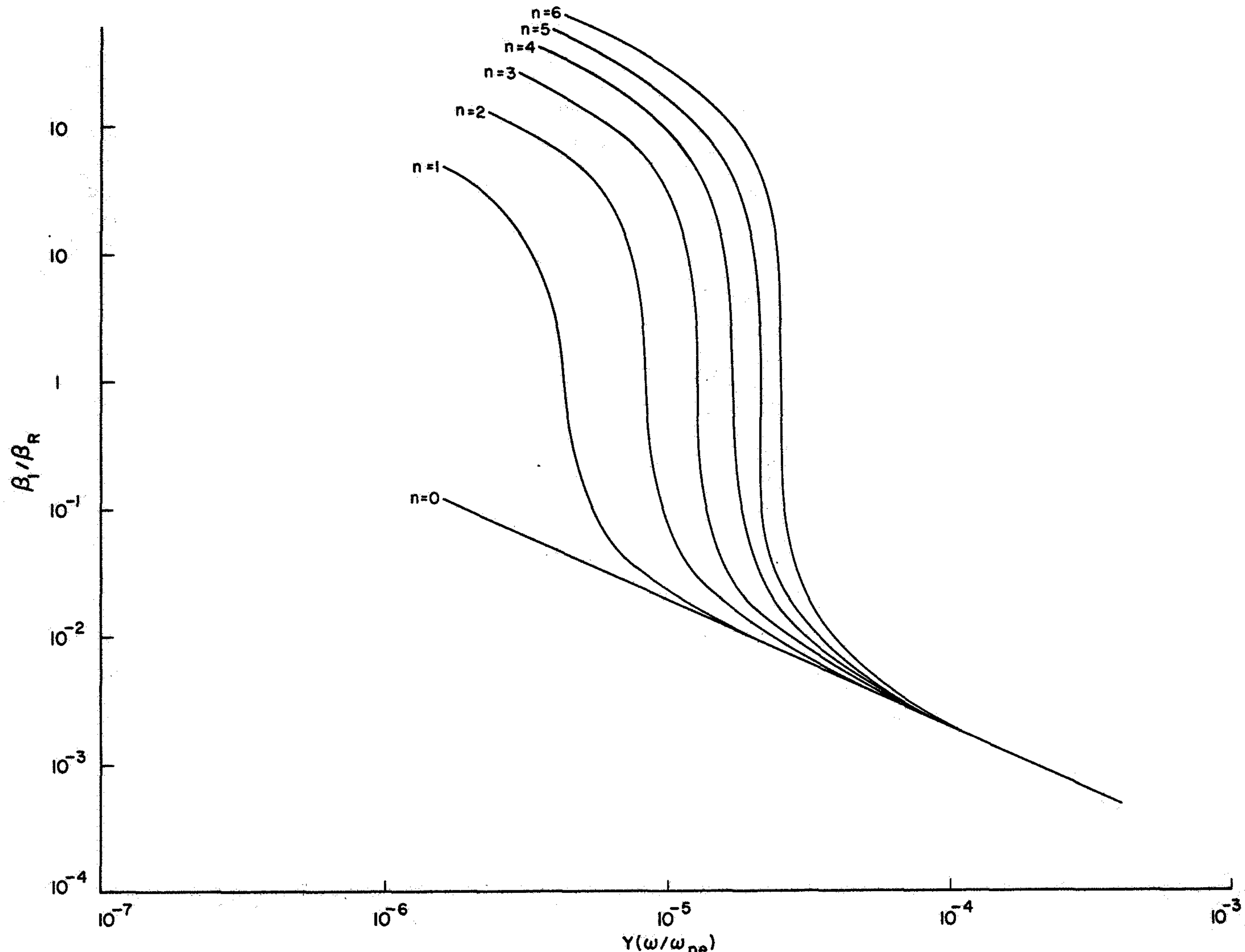
2.) Imaginary part

$$\frac{\nu\omega}{\omega_{pi}^{2}} = 2\ \beta_R\ \beta_I\ \lambda_{De}^{2} \tag{38}$$

From these two equations, we obtain the following:

$$\frac{\beta_I}{\beta_R} = \frac{\nu\omega}{\omega^2 - \left(\frac{n\pi W\omega_{pi}}{a\omega_{pe}}\right)^2 \pm \sqrt{\left[\omega^2 - \left(\frac{n\pi W\omega_{pi}}{a\omega_{pe}}\right)^2\right]^2 + \nu^2\omega^2}} \tag{39}$$

Using the experimental values for a, W, ω_{pi}/ω_{pe}, and ν measured by Little and Jones[9], we see that β_I/β_R gets larger when ω is made smaller (Fig. 17). Thus, there is some low frequency cutoff below which the ion body waves and surface wave ($n = 0$) will be significantly damped by collisions. It also should be noted that for low frequencies, the wave damping becomes more dominant as n



β_I/β_R VS FREQUENCY FOR α=1.5, $c/W=4.2\times10^2$, $m_i/m_e=3.66\times10^5$, a=2.5 cm AND $\nu=7\times10^3$ sec.$^{-1}$

FIGURE 17

is increased. This would explain why Little and Jones[9] were only able to measure the lowest order ion body wave. For ω sufficiently large, β_I/β_R asymptotically approaches the equation

$$\frac{\beta_I}{\beta_R} = \frac{\nu}{2\omega} \tag{40}$$

and the collisional damping becomes insignificant. If we examine Fig. 15 at these frequencies, however, we see that the different body wave harmonics are bunched together and so distinguishing between them would be difficult.

It is well known that Landau damping of electron waves is negligible when $\beta\lambda_{De} << 1$. As shown by Klevans and Primack[16], Landau damping of ion waves is negligible when $\beta\lambda_{Di} << 1$. Writing these inequalities in terms of our parameters Y and X, we have that:

1.) ion wave Landau damping is negligible when

$$X << \frac{T_e}{T_i}\,\frac{c}{W}$$

2.) electron wave Landau damping is negligible when

$$X << \frac{c}{W}$$

The above inequality for ion waves helps to explain why it would be difficult for Little and Jones[9] to measure field quantities near the ion plasma frequency.

5 SUMMARY

The field solutions and dispersion relations, assuming both symmetric and antisymmetric longitudinal electric field components, have been determined for the slab configuration in Fig. 1. A two fluid model with a scalar pressure term was used to take into account the effects of ions and electrons. The dependence of the symmetric and antisymmetric dispersion relations on electron temperature, slab thickness, and waveguide width has been discussed in this report. For $\omega < \omega_{pi}$, an interesting coupling phenomena between the different ion body waves was found to occur for wave numbers near the light line. Unfortunately, the possibility of measuring such behavior is poor since collisional damping dominates at these frequencies.

Approximate dispersion equations were obtained by using a heuristic argument and the quasistatic approximation. The heuristic approximation gives a fairly accurate representation of the ion body waves and the low frequency portion of the ion surface wave. The advantage of this approximation is that we have a simplified expression to work with when studying such phenomena as collisional wave damping. The agreement between the quasistatic and rigorous results was found to be quite good for smaller slab thicknesses and waveguide widths. It thus serves as a useful and valid approximation in many physical problems[14,15].

Finally, our lowest order ion body wave for symmetric E_y was found to agree favorably with experimental data measured by Little and Jones[9] in a cylindrical geometry.

Since experiments are actually performed in inhomogeneous plasmas, a significant extension of this report would be a study of ion waves propagating in an inhomogeneous plasma. This work would represent an extension of Crawford and Tataronis'[14] analysis on electron waves to include ions and ion waves. Another related topic would be the investigation of the dipole ion mode in a plasma cylinder. The circularly symmetric mode has been treated by Andersson and Weissglas[10]. It would be interesting to see what similarities exist between the dipole mode and the antisymmetric mode in our problem.

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APPENDIX A

LOW FREQUENCY ASYMPTOTIC SURFACE WAVE AND BODY WAVE SOLUTIONS ($\omega \ll \omega_{pi}$)

It will be assumed in these appendices that K = 1 and $m_i/m_e = 10^5$.

A.1 SYMMETRIC E_y

1.) Cases I and IV--α = 0.1, c/W = 10^2 or 10^3, b/a = 2

Assuming X > Y, the dispersion relation (Eq. 30) simplifies to

$$(X^2 - Y^2)\ \frac{\alpha}{Y^2}\ (\frac{b}{a} - 1) \approx \frac{1}{\alpha} + \frac{m_i X^2}{\alpha m_e (X^2 - \frac{m_i c^2}{m_e W^2} Y^2)} \qquad \text{(A.1)}$$

This equation may be rearranged to the following one which is quadratic in Y^2:

$$\frac{m_i c^2}{m_e W^2} \left[\alpha^2\ (\frac{b}{a} - 1) + 1\right] Y^4$$

$$- \left[\alpha^2\ (\frac{b}{a} - 1)\ (1 + \frac{m_i c^2}{m_e W^2}) + 1 + \frac{m_i}{m_e}\right] X^2 Y^2 + \alpha^2\ (\frac{b}{a} - 1) X^4 \approx 0 \qquad \text{(A.2)}$$

For the values given above, Eq. A.2 may be simplified to the following:

$$Y^4 - \alpha^2\ (\frac{b}{a} - 1)\ X^2 Y^2 + \frac{m_e W^2}{m_i c^2}\ \alpha^2\ (\frac{b}{a} - 1)\ X^4 \approx 0 \quad \text{(A.3)}$$

Solving for Y, we obtain

$$Y^2 \approx \frac{X^2}{2}\left[\alpha^2 \left(\frac{b}{a}-1\right) \pm \sqrt{\alpha^4 \left(\frac{b}{a}-1\right)^2 - \frac{m_e W^2}{m_i c^2} \alpha^2 \left(\frac{b}{a}-1\right)}\right]$$

or

$$Y^2 \approx \alpha^2 X^2 \left(\frac{b}{a}-1\right)\left[\frac{1}{2} \pm \left(\frac{1}{2} - \frac{m_e W^2}{\alpha^2 m_i c^2 \left(\frac{b}{a}-1\right)}\right]\qquad (A.4)$$

The body wave solution is

$$Y \approx \alpha \left(\frac{b}{a}-1\right)^{\frac{1}{2}} X \qquad (A.5)$$

and the surface wave solution is

$$Y \approx \frac{\omega_{pi} W}{\omega_{pe} c} X \qquad (A.6)$$

2.) Case II--$\alpha = 0.1$, $c/W = 10^2$, $b/a = 1$

Assuming the same conditions as in part 1 except that $b/a = 1$, Eq. A.2 gives us the following:

$$\frac{m_i c^2}{m_e W^2} Y^4 - \left(\frac{m_i}{m_e} + 1\right) X^2 Y^2 \approx 0 \qquad (A.7)$$

Thus, the body wave solution for very low frequencies is

$$Y \approx \frac{W}{c} X \qquad (A.8)$$

3.) Case III--$\alpha = 0.1$, $c/W = 10^2$, $b/a \to \infty$

The simplified dispersion equation for $b/a \to \infty$ is

$$(X^2 - Y^2)^{\frac{1}{2}} \approx \frac{Y^2}{\alpha} + \frac{m_i X^2 Y^2}{\alpha m_e \left(X^2 - \frac{m_i c^2}{m_e W^2} Y^2\right)} \tag{A.9}$$

If X and Y are very small and are of the same order in magnitude, then the left hand side of the equation predominates and we get

$$Y \approx X \tag{A.10}$$

which is the body wave solution. If $X \gg Y$, the above equation simplifies to

$$X \approx \frac{Y^2}{\alpha} + \frac{m_i X^2 Y^2}{\alpha m_e \left(X^2 - \frac{m_i c^2}{m_e W^2} Y^2\right)}$$

or

$$(\alpha X - Y^2)\left(X^2 - \frac{m_i c^2}{m_e W^2} Y^2\right) \approx \frac{m_i}{m_e} X^2 Y^2 \tag{A.11}$$

Since X and Y are assumed small, the lowest ordered terms will predominate and give us

$$Y \approx \frac{\omega_{pi} W}{\omega_{pe} c} X \tag{A.12}$$

which is the surface wave solution.

A.2 ANTISYMMETRIC E_y

4.) Cases VI and IX--α = 0.1, c/W = 10^2 or 10^3, b/a = 2

Assuming X > Y and Y << ω_{pi}/ω_{pe} gives us the following simplified dispersion equation:

$$(X^2 - Y^2) \frac{\alpha}{Y^2} \left(\frac{b}{a} - 1\right) \approx \alpha + \alpha \frac{m_i}{m_e} X^2 \qquad (A.13)$$

Keeping only the significant terms, the approximate body wave solution is

$$Y \approx \left(1 - \frac{a}{b}\right)^{\frac{1}{2}} X \qquad (A.14)$$

5.) Case VII--α = 0.1, c/W = 10^2, b/a = 1

Letting b/a = 1 in Eq. A.14 we see that there is no low frequency body or surface wave solution.

6.) Cases VIII and X--α = 0.1 or 1.0, c/W = 10^2, b/a$\rightarrow\infty$

The approximate low frequency dispersion equation for the above conditions is

$$(X^2 - Y^2)^{\frac{1}{2}} \approx \alpha Y^2 \left(1 + \frac{m_i}{m_e} X^2\right) \qquad (A.15)$$

For very small values of X and Y, the left hand side of this equation predominates and thus the body wave solution is

$$Y \approx X \qquad (A.16)$$

APPENDIX B

MIDFREQUENCY APPROXIMATE SURFACE WAVE SOLUTIONS

$(\omega_{pi} \ll \omega \ll \omega_{pe})$

B.1 SYMMETRIC E_y

1.) Cases I and IV--$\alpha = 0.1$, $c/W = 10^2$ or 10^3, $b/a = 2$

Assuming $X > Y$, the dispersion equation simplifies to

$$(X^2 - Y^2)\frac{\alpha}{Y^2}\left(\frac{b}{a} - 1\right) \approx \frac{1}{\alpha} - \frac{W}{c}\frac{X^2}{Y^2}$$

or

$$\left[\frac{1}{\alpha} + \alpha\left(\frac{b}{a} - 1\right)\right] Y^2 \approx \left[\alpha\left(\frac{b}{a} - 1\right) + \frac{W}{c}\right] X^2 \qquad \text{(B.1)}$$

For the above values of α, c/W, and b/a, the equation is approximately

$$Y \approx \alpha\left(\frac{b}{a} - 1\right)^{\frac{1}{2}} X \qquad \text{(B.2)}$$

2.) Case II--$\alpha = 0.1$, $c/W = 10^2$, $b/a = 1$

When $b/a = 1$, equation B.1 gives us the following surface wave solution:

$$Y \approx \left(\frac{\alpha W}{c}\right)^{\frac{1}{2}} X \qquad \text{(B.3)}$$

3.) Case III--$\alpha = 0.1$, $c/W = 10^2$, $b/a \to \infty$

The approximate dispersion equation for the above conditions is

$$(X^2 - Y^2)^{\frac{1}{2}} \approx \frac{Y^2}{\alpha} - \frac{W}{c} X^2 \qquad (B.4)$$

Again for this case, the left hand side of the equation predominates and our solution for Y is

$$Y \approx X \qquad (B.5)$$

B.2 ANTISYMMETRIC E_y

4.) Cases VI and IX--α = 0.1, c/W = 10^2 or 10^3, b/a = 2

The approximate dispersion equation for the above conditions is

$$(X^2 - Y^2) \frac{\alpha}{Y^2} (\frac{b}{a} -1) \approx \alpha - \frac{WX^2}{cY^2}$$

or

$$Y^2 \approx \frac{a}{\alpha b} \left[\alpha(\frac{b}{a} -1) + \frac{W}{c}\right] X^2 \qquad (B.6)$$

Thus the solution for Y is

$$Y \approx (1 - \frac{a}{b})^{\frac{1}{2}} X \qquad (B.7)$$

5.) Case VII--α = 0.1, c/W = 10^2, b/a = 1

Equation B.6 for b/a = 1 gives us the following surface wave solution:

$$Y \approx (\frac{W}{\alpha c})^{\frac{1}{2}} X \qquad (B.8)$$

6.) Cases VIII and X-- $\alpha = 0.1$ or 1.0, $c/W = 10^2$, $b/a \rightarrow \infty$

The dispersion relation for this case simplifies to

$$(X^2 - Y^2)^{\frac{1}{2}} \approx \alpha Y^2 - \frac{W}{c} X^2 \qquad (B.9)$$

Over the particular range of values we are considering, the left hand side of the equation predominates and thus

$$Y \approx X \qquad (B.10)$$

APPENDIX C

HIGH FREQUENCY ASYMPTOTIC SURFACE WAVE AND BODY WAVE SOLUTION ($\omega \gg \omega_{pe}$)

The following development is valid for all cases studied in this paper. Assume that X >> Y and that X is linearly related to Y. Then either the symmetric or antisymmetric dispersion equation for large Y may be approximated as follows:

$$-X \approx X - \frac{X^2}{Y^2\left[X^2 - \frac{c^2}{W^2}Y^2\right]} \tag{C.1}$$

Rearranging this equation, we get

$$X^2 - \frac{c^2}{W^2}Y^2 \approx \frac{X^2}{4Y^4} \tag{C.2}$$

Using the above assumption that Y is linear in X, we obtain the following high frequency solution for our dispersion equation:

$$Y \approx \frac{W}{c} X \tag{C.3}$$

APPENDIX D

DERIVATION OF THE QUASISTATIC DISPERSION EQUATION

For the quasistatic approximation, we approximate Eq. 7 by the following:

$$\nabla \times \bar{E} = 0 \tag{D.1}$$

which implies that

$$\bar{E} = - \bar{\nabla} \phi \tag{D.2}$$

We shall begin by solving for the field solutions inside the plasma ($0 \leq x < a$). If we make the same assumptions that were used in developing the field solutions from the full set of Maxwell's equations, taking the divergence and d/dt of Eq. 8 gives us

$$\nabla \cdot \left(\frac{W^2 e}{\omega^2 \epsilon_o} \bar{\nabla} n_e - \epsilon \bar{E} \right) = 0 \tag{D.3}$$

Using Eqs. 2,4, and 5 to solve for n_e, we obtain

$$n_e = - \frac{\epsilon_o \epsilon_+}{e} \nabla^2 (\nabla \cdot \bar{E}) + \epsilon \quad \nabla \cdot \bar{E} \tag{D.4}$$

Thus, Eq. D.3 may be written as

$$\frac{W^2}{\omega^2} \epsilon_+ \quad \nabla^2 (\nabla \cdot \bar{E}) + \epsilon \quad \nabla \cdot \bar{E} = 0 \tag{D.5}$$

If we now let $\bar{E} = - \bar{\nabla} \phi$ and assume that E_y is symmetric, the solution for ϕ is

$$\phi = A \cosh \beta x + B \cosh \gamma_2 x \tag{D.6}$$

which gives us

$$E_y = i\beta(A \cosh \beta x + B \cosh \gamma_2 x) \tag{D.7}$$

$$E_x = -\beta A \sinh \beta x - \gamma_2 B \sinh \gamma_2 x \tag{D.8}$$

To write H_z in terms of $\bar{E}$, we use Eqs. 3,4,8, and D.4 to obtain the following:

$$H_z = - \frac{\omega\epsilon_o\epsilon}{\beta} E_x - \frac{W^2\epsilon_o\epsilon_+}{\omega\beta} \frac{\partial}{\partial x} (\nabla \cdot \bar{E}) \tag{D.9}$$

Using Eqs. D.7 and D.8, the solution for H_z is

$$H_z = A\omega\epsilon_o\epsilon \sinh \beta x \tag{D.10}$$

To obtain the field solutions inside the dielectric $(a < x < b)$, solving Eqs. 5 and D.1 simultaneously gives us

$$E_y = C \cosh \beta x + D \operatorname{sgn} x \sinh \beta x \tag{D.11}$$

$$E_x = iC \sinh \beta x + iD \operatorname{sgn} x \cosh \beta x \tag{D.12}$$

From Eqn. 8, the solution for H_z is

$$H_z = - \frac{i\omega\epsilon_o K}{\beta} (C \sinh \beta x + D \operatorname{sgn} x \cosh \beta x) \quad (D.13)$$

Using the boundary conditions given by Eqs. 24 to 27 and the above field solutions, the dispersion equation for a symmetric E_y component is

$$\frac{\epsilon}{K} \tanh \beta(a-b) = \coth \beta a - \frac{\omega_{pe}^2}{\omega^2} \frac{\beta}{\gamma_2 \epsilon_+} \coth \gamma_2 a \quad (D.14)$$

It should be mentioned that the same results could have been obtained from our previously determined dispersion equation (Eq. 28) for symmetric E_y by setting δ and γ_1 equal to β.

A similar procedure as just outlined will give the following quasistatic dispersion equation for antisymmetric E_y:

$$\frac{\epsilon}{K} \tanh \beta(a-b) = \tanh \beta a - \frac{\omega_{pe}^2}{\omega^2} \frac{\beta}{\gamma_2 \epsilon_+} \tanh \gamma_2 a \quad (D.15)$$