Stability Criteria for a Cantilever Subjected to a Time-Dependent Follower Force

by

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Introduction

In this Note we investigate the stability of a nonconservative nonautonomous system, a cantilevered column subjected at its free end to a compressive follower force which varies with time. Viscous damping is assumed to be present. With the use of a Liapunov type of approach, conditions are obtained which guarantee asymptotic stability or almost sure asymptotic stability of the column.

Consider a linear elastic column, built-in at one end and free at the other. In nondimensional terms, the equation of motion for the lateral displacement $w(x,t)$ is assumed to be

$$w_{,xxxx} + p(t)w_{,xx} + 2\xi w_{,t} + w_{,tt} = 0 \tag{1}$$

where $0 < x < 1$, $t \geqq 0$, and a comma denotes partial differentiation. The x axis lies along the undisturbed straight column, with $x = 0$ at the built-in end and $x = 1$ at the free end. ξ (> 0) is the damping coefficient. The compressive force $p(t)$ is applied at the free end and remains tangent to the column during motion. We assume $p(t)$ is continuous for $t > 0$. The boundary conditions are

$$w(0,t) = w_{,x}(0,t) = w_{,xx}(1,t) = w_{,xxx}(1,t) = 0 \quad (t \geqq 0) \tag{2}$$

and the initial conditions are given by

$$w(x,0) = w_o(x), \quad w_{,t}(x,0) = v_o(x). \tag{3}$$

Stability Analysis I

It is assumed that $p(t)$ is strictly stationary and satisfies an ergodic property which insures the equality of time averages and ensemble averages. Then

$$E\{|p(t)|\} = \lim_{t\to\infty} \frac{1}{t}\int_0^t |p(\tau)|\,d\tau \tag{4}$$

exists with probability one [1].[1]

Consider the functional

$$V(w,v) = \int_0^1 [w_{,xx}^2 + v^2 + 2\xi vw + c\xi^2 w^2]dx \tag{5}$$

where $w_{,xx}^2 = (w_{,xx})^2$, $v = w_{,t}$ and

$$c = \begin{cases} 1 & \text{if } 0 < \xi \leq \pi^2/2 \\ (8\xi^2-\pi^4)/4\xi^2 & \text{if } \xi > \pi^2/2 \end{cases} \tag{6}$$

Note that $V = 0$ if $w = v = 0$, with $V > 0$ otherwise. The total time derivative of V along solutions of (1) and (2) can be written in the form

$$\dot{V}(w,v,t) = -2\int_0^1 [\xi w_{,xx}^2 + \xi v^2 + (2-c)\xi^2 vw + p(t)w_{,xx}(v+\xi w)]dx \tag{7}$$

with the use of integration by parts.

For w and v not both zero consider the ratio $\dot{V}/V$, which

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[1] Numbers in brackets designate References at end of Note.

we write as

$$\dot{V}/V = -\lambda - \gamma p(t) \tag{8}$$

where

$$\lambda = \frac{2\xi}{V}\int_0^1 [w_{,xx}^2 + v^2 + (2-c)\xi vw]dx, \quad \gamma = \frac{2}{V}\int_0^1 (v + \xi w)w_{,xx}dx. \tag{9}$$

Then

$$\dot{V}/V \leqq -\lambda + |\gamma||p(t)|. \tag{10}$$

Using the calculus of variations we can show that $\lambda \geqq \lambda_m$ where

$$\lambda_m = \begin{cases} 2\xi(\pi^2 - \xi\sqrt{2})/\pi^2 & \text{if} \quad 0 < \xi \leqq \pi^2/2 \\ 2\xi - [(8\xi^2 - \pi^4)/2]^{1/2} & \text{if} \quad \xi > \pi^2/2 \end{cases} \tag{11}$$

and from the inequality

$$\left|\int_0^1 2(v+\xi w)w_{,xx}dx\right| \leqq \int_0^1 [w_{,xx}^2 + (v+\xi w)^2]dx \leqq V(w,v) \tag{12}$$

it follows that $|\gamma| \leqq 1$. Therefore

$$\dot{V}/V \leqq -\lambda_m + |p(t)|. \tag{13}$$

Integrating this expression between 0 and t yields

$$V(t) \leqq V(0)\exp\{t[-\lambda_m + \frac{1}{t}\int_0^t |p(\tau)|d\tau]\} \tag{14}$$

where $V(t) = V[w(x,t), v(x,t)]$ and $V(0) = V[w_o(x), v_o(x)]$.

Now assume that

$$E\{|p(t)|\} \leqq \lambda_m - \varepsilon \tag{15}$$

for some constant $\varepsilon > 0$, as small as desired. Since $V(t) \geqq 0$ it then follows from (4) and (14) that, with probability one, $V(t) \to 0$ as $t \to \infty$, and therefore also $\int_0^1 w_{,xx}^2 dx \to 0$, $\int_0^1 w^2 dx \to 0$, and $\int_0^1 v^2 dx \to 0$ as $t \to \infty$. We conclude that (15) is a sufficient condition for almost sure asymptotic stability in the large [2][2].

The boundary of the stability region defined by (15) is depicted by the solid line in Fig. 1.

[2]To be explicit we should define the stability in terms of a norm; for example, we could use the norm $\rho(w,v) = [\int_0^1(w_{,xx}^2 + v^2)dx]^{1/2}$.

Stability Analysis II

In this section we obtain stability criteria in terms of $\max\limits_{t \geq 0} |p(t)|$.

From (13) one can show that

$$V(t) \leq V(0)e^{-\varepsilon t} \tag{16}$$

if

$$\max_{t \geq 0} |p(t)| \leq \lambda_m - \varepsilon \tag{17}$$

for some $\varepsilon > 0$. Therefore (17) is a sufficient condition for asymptotic stability of the column (see Fig. 1).

For $\xi > 4.66$ a stronger condition than (17) can be obtained by the use of a different technique. Consider the functional V as in (5), except let $c = 2(1-\delta)$ where $0 < \delta < 1/2$. We can write $\mathring{V}$ in the form

$$\mathring{V}(w,v,t) = -2\xi\delta V(w,v) - 2W(w,v,t) \tag{18}$$

where

$$W(w,v,t) = \int_0^1 [\xi(1-\delta)(w_{,xx}^2 + v^2 - 2\xi^2\delta w^2) + \xi p(t)ww_{,xx} + p(t)vw_{,xx}]dx. \tag{19}$$

We now seek conditions on $p(t)$ under which $W \geq 0$.

The calculus of variations can be used to show that

$$\int_0^1 w_{,xx}^2 dx \geq \frac{\pi^4}{8} \int_0^1 w^2 dx \tag{20}$$

where w satisfies (2). We then can write

$$W \geq \int_0^1 \{\xi(1-\delta)[(1-\alpha)w_{,xx}^2 + v^2 + (\frac{\pi^4}{8}\alpha - 2\xi^2\delta)w^2] + \xi p(t) w w_{,xx} + p(t) v w_{,xx}\} dx \tag{21}$$

where $0 < \alpha < 1$. The integrand in (21) has the form

$$(a w_{,xx}^2 + b w_{,xx} v + c v^2) + (A w_{,xx}^2 + B w_{,xx} w + C w^2),$$

which will be non-negative if $a \geq 0$, $A \geq 0$, $b^2 \leq 4ac$, and $B^2 \leq 4AC$. If the optimum values of α, A/a, and δ are chosen, it can be shown that these conditions are satisfied if

$$\max_{t \geq 0} |p(t)| \leq 2\xi(\sqrt{8\xi^2 + \pi^4} - 2\sqrt{2}\xi)/\pi^2 \tag{22}$$

for some $\varepsilon > 0$. It then follows that $W \geq 0$, $\dot{V} \leq -2\xi\delta V$, and

$$0 \leq V(t) \leq V(0)e^{-2\xi\delta t} \tag{23}$$

Therefore $V(t) \to 0$ as $t \to \infty$, and we conclude that (22) is a sufficient condition for asymptotic stability in the large.

The boundary of the stability region defined by (22) is depicted by the dashed line in Fig. 1. For $\xi > 4.66$ condition (22) is stronger than (17).

Concluding Remarks

It is of interest to compare conditions (15), (17), and (22) to the stability condition for the case of a constant follower force. If $p(t) = p_o$ = constant and $\xi = 0$, Beck [3] showed that the column is stable if and only if

$$p_o < 2.008\pi^2. \tag{24}$$

For $\xi > 0$ (24) is a sufficient condition for asymptotic stability [4]; it is also a necessary condition if ξ is small enough.

In conclusion, we point out that the stability regions derived here and shown in Fig. 1 surely comprise only a part of the total stability region. However it is believed that these results represent the first stability criteria obtained for a cantilevered column subjected to a time-dependent follower force.

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References

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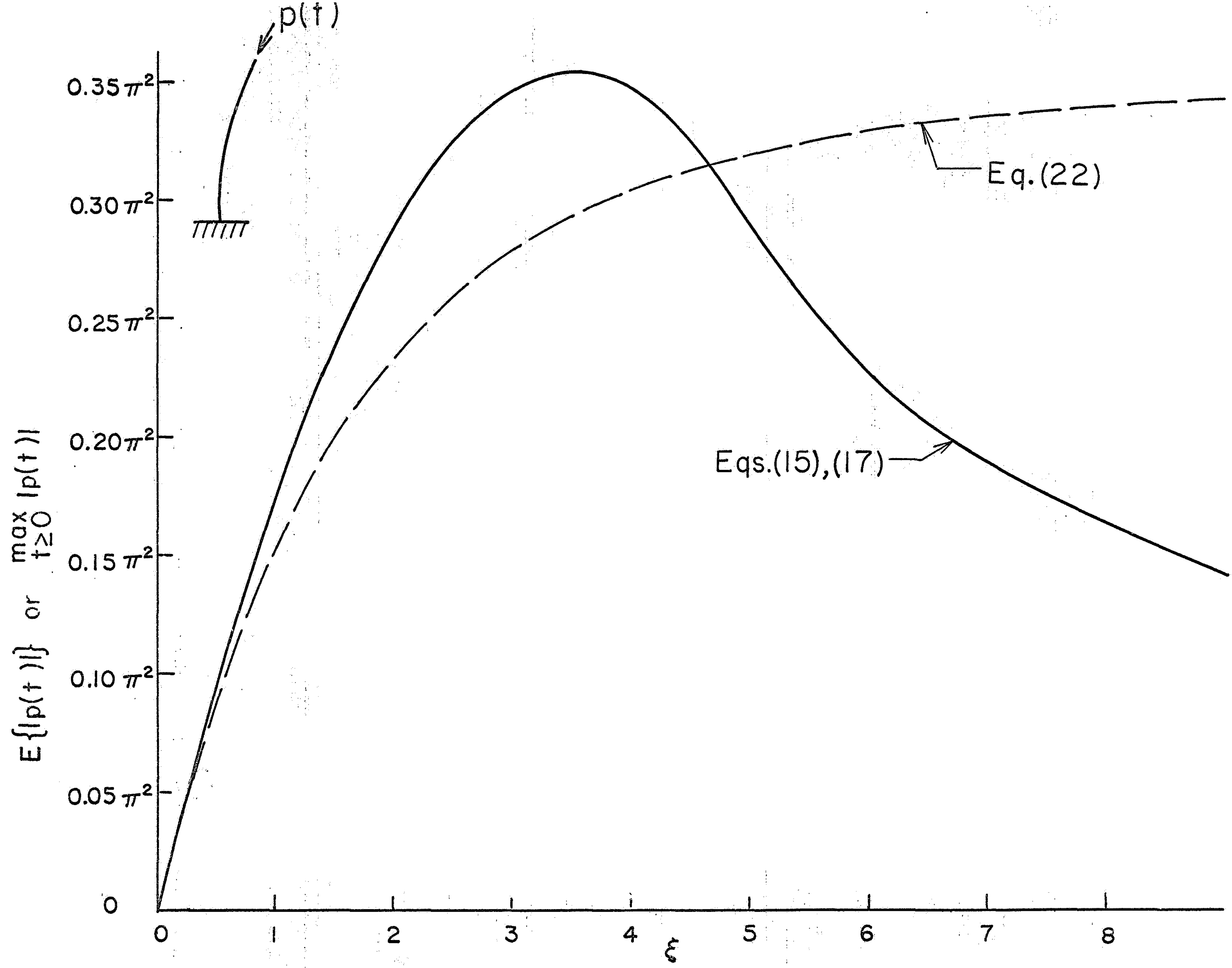
Fig. 1 Stability Conditions



p(t)
0.35 π²
0.30 π²
0.25 π²
0.20 π²
0.15 π²
0.10 π²
0.05 π²
0
E{|p(t)|} or max t≥0 |p(t)|
Eq.(22)
Eqs.(15),(17)
0
1
2
3
4
5
6
7
8
ξ