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TRANSIENT VIBRATION ANALYSIS OF LINEAR SYSTEMS USING TRANSITION MATRICES

by Anthony Craggs

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ABSTRACT

A step by step Transition matrix method is presented for solving a set of equations of motion. The stability criteria which govern the choice of step size are given in the first part of the report: these are related to the highest eigenvalue of the system under consideration. The use of the technique for evaluating the response to a set of forces and prescribed motions is given in a theoretical discussion. Using the finite element displacement procedure to idealise the structure the method has been applied to several rectangular plate problems and good agreement is obtained between theoretical and experimental results. The results for the transient motion of a simply supported plate are compared with similar results obtained by using a standard 4th order Runge Kutta procedure and it is shown that the transition matrix solutions are more accurate, and there is also a substantial decrease in the computation time per step.

INTRODUCTION

At the present time much work is being carried out on the approximate formulation of the equations of motion in problems of structural dynamics. If the problem is formulated in terms of the system displacements and the system is linear then the resulting equations may be expressed in the matrix form:

$$\underset{\sim}{M}\ddot{\underset{\sim}{W}} + \underset{\sim}{D}\dot{\underset{\sim}{W}} + \underset{\sim}{K}\underset{\sim}{W} = \underset{\sim}{F}(t) \qquad (1)$$

where M, D and K are square symmetric matrices, $\underset{\sim}{W}$ and $\underset{\sim}{F}$ are column vectors representing the unknown displacements and the known forcing function respectively. When obtaining a solution to this equation, most of the work has been in the frequency domain; either for the case where the structure is in a state of free vibration, or, for the response of the structure to a harmonic forcing function. The solution in the time domain may be achieved by the now standard numerical techniques - namely, Runge Kutta and Adams (1) or by the less sophisticated engineering methods suggested by Timoshenko (2) and Newmark (3). While these methods are general in that they may also be applied to non-linear systems, when they are applied to a set of equations of the type shown above they do not utilize the fact that the matrices M, D and K are constant, and consequently lose a significant advantage. This report deals with the step by step method which does utilize the fact and makes an indirect use of the Taylor series.

The Taylor series is first used to deduce a recurrence relationship which, for free vibration, gives the response at one step in terms of the response at a previous step by means of a Transition matrix. When the system is forced it is necessary to add further terms in the recurrence relationship, the actual number depending upon the nature of the forcing function. However, if the step size is small compared with the largest period of the system it is sufficient to include only one additional term. The advantages of the method over the standard techniques are a substantial decrease in computation time and an increase in the accuracy of the solution if a high order Taylor Series is used in evaluating the Transition matrix.

As with the other step-by-step procedures the step size cannot be chosen arbitrarily: it is usually some fraction of the smallest period of the system; otherwise the numerical solution becomes unstable. The stability condition for a different number of terms of the Taylor Series may be found by considering a damped single degree of freedom system; this aspect is covered in the first section of the report.

A general procedure for determining the response to an arbitrary forcing pulse and prescribed boundary motion is given and this is expressed concisely in terms of matrix operations. The theory is applied to several examples using the finite element displacement technique to idealise the structure, and good agreement is obtained where a comparison has been made with some experimental work.

SYMBOLS

w	displacement response
$\underset{\sim}{W}$	displacement response column vector
$\{w\}$	column vector containing displacement and velocity response vector
$\underset{\sim}{q}$	displacement vector for coupled system
$\underset{\sim}{F}$	force column vector
$\underset{\sim}{\xi}$	force column vector for coupled system
$\{F\}$	column vector containing force and force time derivative vectors
$[M]$	square mass matrix
$[D]$	square damping matrix
$[K]$	square stiffness matrix
$[T]$	square Transition matrix for response terms

$[A]$ square Transition matrix for forcing terms

$[B]$ rectangular Boolean transformation matrix relating element coordinates to the overall system coordinates

$[I]$ unit matrix

h step size

Z complex transfer term

λ angular natural frequency, undamped system

β critical damping factor

E Young's modulus

ρ plate density

μ Poisson's ratio

th plate thickness

f_i Hermitian interpolation polynomials

TE kinetic energy

VE strain energy

BE term whose variation gives the work done by the externally applied forces in a virtual displacement

a,b overall dimensions of rectangular plate element

j, n, r integers

α damping factor where damping matrix is proportional to the stiffness matrix ($\alpha < 1$)

STABILITY CRITERIA

In this discussion there is nothing to be gained by dealing with a set of differential equations or forced vibration. All of the relevant points are brought out by considering the equation of motion of a single degree of freedom system:

$$\ddot{w} + 2\beta\lambda\dot{w} + \lambda^2 w = 0 \qquad (2)$$

The exact solution to this equation has the form $w = We^{\Omega t}$ where both W and Ω are complex quantities.

$$\Omega = -\lambda(\beta \pm j\sqrt{1 - \beta^2}) \qquad (3)$$

In the numerical procedure for solving (2) the solution moves forward in a number of small steps and, within each step, the solution is obtained by using only a finite number of terms of the Taylor Series. It will be shown that if the step size does not satisfy certain criteria the solution will become unstable. The Taylor Series gives:

$$w(t + h) = w(t) + h\dot{w}(t) + h^2\ddot{w}(t)/2! + h^3\dddot{w}(t)/3! + h^4 \ddot{\ddot{w}}(t)/4!$$

where h is the step size.

Substituting for w and $\dot{w}$, etc. from the exact solutions then

$$w(t + h)/w(t) = \{1 + \Omega h + \Omega^2 h^2/2! + \Omega^3 h^3/3! + \Omega^4 h^4/4! + \ldots\}$$

$$w(t + nh)/w(t) = \{1 + h + \Omega^2 h^2/2! + \Omega^3 h^3/3! + \Omega^4 h^4/4! + \}^n$$

$$= Z^n.$$

The term Z is complex and for stability the modulus of this quantity should be less than or equal to 1; otherwise the estimate will always become greater with the number of multiplications. To determine the effect of step size values of $|Z|$ have been computed using a different number of terms of the Taylor Series, and various values of the damping factor between $\beta = 0.0$ and $\beta = 0.2$. Some of the results of these calculations are plotted in Fig. 1 and Fig. 2. These show that above a certain value of λh, $|Z|$ becomes greater than unity. This value is λh_{crit} and differs with the order of the Taylor Series used. A summary of the critical values of λh is given in Table 1. In general, it shows that the increase in step size is allowed with an increase in the order of the series. However, there is a fall off in the advantage gained where 6th and 7th order series are used. There is a further gain with an 8th order series.

For systems with further degrees of freedom the step size will need to be related to the highest eigenvalue λ_v, i.e. a fraction of the <u>smallest</u> period, and provided $\lambda_v h < \lambda h_{crit}$ then the method will be stable. This is one of the unfortunate aspects of step-by-step methods as often the response is dominated by the lower modes of vibration, and for good resolution the step size would only need to be a fraction of the <u>largest</u> period.

The conditions which govern the value of λh_{crit} only guarantee a solution which does not diverge, they do not necessarily guarantee a good approximation to the true solution, therefore a criterion for accuracy is needed. This was achieved in a heuristic manner by comparing the computed values of $|Z|$ with the exact $|Z_e|$ value given by

$$|Z| = e^{-\beta(\lambda h)}$$

In Fig. 1, ($\beta = 0.0$), the exact value is the horizontal line at $Z = 1.0$. In Fig. 2 ($\beta = 0.2$), the exact solution is shown as a dotted line. It may be seen that when λh is small the computed curves lie close to the actual

solution, but after a certain value λh_{acc} the curves tend to diverge from the true solution. The approximate solutions will then behave as though the system is more or less damped than it actually is depending upon whether $|Z|$ is above or below $|Z|_e$. A set of curves similar to Fig. 1 and Fig. 2 has been made for $\beta = 0.0002$, 0.002 and 0.02; these are not presented in the paper. The average values for λh_{acc} are shown in Table 3. It may be noted that for this criterion there is always an improvement with an increase in the order of the Taylor Series.

THE APPLICATION OF THE TAYLOR SERIES TO SYSTEMS WITH MANY DEGREES OF FREEDOM

The usual use of Taylor Series involves the calculation of a number of higher derivatives at each step, the number depending upon the order of the series. For some equations this may be a serious drawback as the derivatives may turn out to be complicated functions. However with a linear system the variables have constant coefficients and this leads to a substantial simplification as all of the higher derivatives may be more easily expressed in terms of the displacement and velocity by repeated substitution into the equation of motion.

The equation of the system may be written in the form

$$\ddot{\underset{\sim}{w}} = -M^{-1}K\underset{\sim}{w} - M^{-1}D\dot{\underset{\sim}{w}} + M^{-1}\underset{\sim}{F} \qquad (4)$$

By successive differentiation of this equation with respect to time, the n th derivative of w may be written for $n \geq 2$.

$$\underset{\sim}{w}^{n} = -M^{-1}K\underset{\sim}{w}^{n-2} - M^{-1}D\underset{\sim}{w}^{n-1} + M^{-1}\underset{\sim}{F}^{n-2} \qquad (5)$$

This equation will now be used to express every term in the series in terms of displacement and velocity. In what follows the case of free vibration is considered first and it is shown that the response at one step is related to that of the previous step by a single Transition matrix. The modification to this recurrence relationship due to the forcing term is then considered in the next stage.

Free Vibration

For free vibration $\underset{\sim}{F}$ is zero. The Taylor Series expansion for $\underset{\sim}{w}$ and $\dot{\underset{\sim}{w}}$ is then

$$\underset{\sim}{w}_{r+1} = \underset{\sim}{w}_r + h\dot{\underset{\sim}{w}}_r + h^2\ddot{\underset{\sim}{w}}_r/2! + h^3\dddot{\underset{\sim}{w}}_r/3! + h^4\underset{\sim}{w}^{iv}_r/4! + \ldots$$

$$\dot{\underset{\sim}{w}}_{r+1} = \dot{\underset{\sim}{w}}_r + h\ddot{\underset{\sim}{w}}_r + h^2\dddot{\underset{\sim}{w}}_r/2! + h^3\underset{\sim}{w}^{iv}_r/3! + \ldots$$

Now, by repeated application of equation (5) the second and higher derivatives, all of the terms on the right hand side of these equations may be written in terms of the displacements and velocities $\underset{\sim}{w}_r$ and $\dot{\underset{\sim}{w}}_r$:-

$$\underset{\sim}{w}_{r+1} = T_{11}\underset{\sim}{w}_r + T_{12}\dot{\underset{\sim}{w}}_r, \quad \text{i.e.} \quad \begin{bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{bmatrix}_{r+1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{bmatrix}_r \qquad (6)$$

$$\dot{\underset{\sim}{w}}_{r+1} = T_{21}\underset{\sim}{w}_r + T_{22}\dot{\underset{\sim}{w}}_r$$

The values of T_{11} will depend upon the number of terms used and the value of the step size h. The actual expressions have been avoided as these are complicated. In practice they will be systematically evaluated inside the computer.

The matrix T is the Transition matrix which may be computed to any order within the capacity of a computer. It is important to note that once it has been found a higher order Transition matrix will not require any longer computing time than one of a lower order when generating the response. The response to a given set of initial conditions $\{w_o\}$ is given by

$$\{w\}_r = [T]^r \{w_o\} \qquad (7)$$

where $\{w\} = \begin{Bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{Bmatrix}$

To illustrate the method the solution for an undamped oscillator was found using a 4th order T matrix. These were compared with the exact solution and solutions obtained by other methods obtained from ref. (4). The equation considered was $\ddot{w} + w = 0$, a step size of $h = 0.25$ was used; this satisfied the stability criterion $\lambda h < 2.8$ and the criterion for accuracy $\lambda h < 1.0$, as $\lambda = 1.0$. The appropriate Transition matrix and the response generated to a unit initial displacement is shown in Table 3. This computed response is almost identical to the exact solution to four decimal places.

Forced Vibration

For conciseness only a 4th order description of the forcing is considered. The extension to a higher order will be clear.

The equation of motion is

$$\underset{\sim}{w}^n = - [c1]\underset{\sim}{w}^{n-2} - [c2]\underset{\sim}{w}^{n-1} + [P]\underset{\sim}{F}^{n-2} \qquad (8)$$

where $[c1] = [M^{-1}][K]$; $[c2] = [M^{-1}][D]$.

It may be shown after substitution of equation (8) in the Taylor Series that terms in displacements and velocities are separate from the force terms and the resulting equations for the responses are:-

$$\underset{\sim}{w}_{n+1} = |T_{11}|\underset{\sim}{w}_n + [T_{12}]\dot{\underset{\sim}{w}}_n + \left(\frac{h^2}{2!}[I] - \frac{h^3}{3!}|c2] - \frac{h^4}{4!}[c1] + \frac{h^4}{4!}|c2||c2]\right)|M^{-1}]\underset{\sim}{F} + \ldots$$

$$+ \frac{h^3}{3!}[I] - \frac{h^4}{4!}[c2|) \quad |M^{-1}]\dot{\underset{\sim}{F}} + \frac{h^4}{4!}[M^{-1}|\underset{\sim}{F} + \ldots$$

$$\dot{\underset{\sim}{w}}_{n+1} = [T_{12}]\underset{\sim}{w}_n + [T_{22}]\dot{\underset{\sim}{w}}_n + (h[I] - \frac{h^2}{2!}[c2] - \frac{h^3}{3!}[c1] + \frac{h^2}{3!}[c2][c2][M^{-1}]\underset{\sim}{F} + \ldots$$
$$(\frac{h^2}{2!}[I] - \frac{h^3}{3!}[c2])[M^{-1}]\dot{\underset{\sim}{F}} + \frac{h^3}{3!}[M^{-1}]\ddot{\underset{\sim}{F}} + \ldots$$

that is:

$$\begin{Bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{Bmatrix}_{n+1} = [T]\begin{Bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{Bmatrix}_n + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{Bmatrix} \underset{\sim}{F} \\ \dot{\underset{\sim}{F}} \end{Bmatrix} + \begin{bmatrix} \frac{h^4}{4!} & M^{-1} \\ \frac{h^3}{3!} & M^{-1} \end{bmatrix}\ddot{\underset{\sim}{F}}_n$$

where

$$[a_{11}] = (\frac{h^2}{2!}[I] - \frac{h^3}{3!}[c2] - \frac{h^4}{4!}[c1] + \frac{h^4}{4!}[c2][c2])[M^{-1}]$$
$$[a_{12}] = (\frac{h^3}{3!}[I] - \frac{h^4}{4!}[c2])[M^{-1}]$$
$$[a_{21}] = (h[I] - \frac{h^2}{2!}[c2] - \frac{h^3}{3!}[c1] + \frac{h^3}{3!}[c2][c2])[M^{-1}]$$
$$[a_{22}] = (\frac{h^2}{2!}[I] - \frac{h^3}{3!}[c2])[M^{-1}] \qquad (9)$$

In general, it is difficult to specify the derivatives of the forcing function. However, if the step size is small, which it usually is to satisfy the stability condition, then the effects of these are negligible and it is then sufficient to write the relationship for $\underset{\sim}{w}$:

$$\begin{Bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{Bmatrix}_{n+1} = [T]\begin{Bmatrix} \underset{\sim}{w} \\ \dot{\underset{\sim}{w}} \end{Bmatrix}_n + [A]\begin{Bmatrix} \underset{\sim}{F} \\ \dot{\underset{\sim}{F}} \end{Bmatrix} \qquad (10)$$

Exact solutions may be built up if F has the form e^{kt}, $\sin kt$, $\cos kt$ because then $[A]$ may be formed to the same accuracy as $[T]$.

The response is generated by repeated application of equation (10) it may be shown by substitution that:

$$\underset{\sim}{w}_n = [T]^n\underset{\sim}{w}_o + [T]^{n-1}[A]\{\underset{\sim}{F}_o\} + [T]^{n-2}[A]\{\underset{\sim}{F}_1\} + [T]^{n-3}[A]\{\underset{\sim}{F}_2\} + \ldots [A]\{\underset{\sim}{F}_n\}$$

Response to Prescribed Boundary Motions

Often the system that is under consideration, forms part of a much larger system, e.g. a window in a building with thick walls. When this is the case the dynamic flexibility of this "minor" system is such that when distorted the reacting forces are insufficient to produce any significant effect on the 'major system'. When the major system is disturbed it will force the boundaries of the minor system to move with it. In studying the

minor system it is necessary to solve for these prescribed boundary motions.

The equation is deduced by considering first the response of the system (in the unconstrained state) to a set of forces located at the prescribed coordinates. It is then convenient to arrange the equations in the form:

$$[M]\begin{Bmatrix}\ddot{w}_p\\ \ddot{w}_f\end{Bmatrix} + [K]\begin{Bmatrix}w_p\\ w_f\end{Bmatrix} = \begin{Bmatrix}F_p\\ 0\end{Bmatrix}$$

where the suffices p and f refer to prescribed and free displacements. partitioning the mass and stiffness matrices gives

$$\left[\begin{array}{c|c} M_{pp} & M_{pf} \\ \hline M_{fp} & M_{ff} \end{array}\right]\begin{Bmatrix}\ddot{w}_p\\ \ddot{w}_f\end{Bmatrix} + \left[\begin{array}{c|c} K_{pp} & K_{pf} \\ \hline K_{fp} & K_{ff} \end{array}\right]\begin{Bmatrix}w_p\\ w_f\end{Bmatrix} = \begin{Bmatrix}F_p\\ 0\end{Bmatrix}$$

Multiplying this out gives the two equations

$$[M_{pp}]\ddot{\underset{\sim}{w}}_p + [M_{pf}]\ddot{\underset{\sim}{w}}_f + [K_{pp}]\underset{\sim}{w}_p + [K_{pf}]\underset{\sim}{w}_f = F_p \qquad (11)$$

$$[M_{fp}]\ddot{\underset{\sim}{w}}_p + [M_{ff}]\ddot{\underset{\sim}{w}}_f + [K_{fp}]\underset{\sim}{w}_p + [K_{ff}]\underset{\sim}{w}_f = 0 \qquad (12)$$

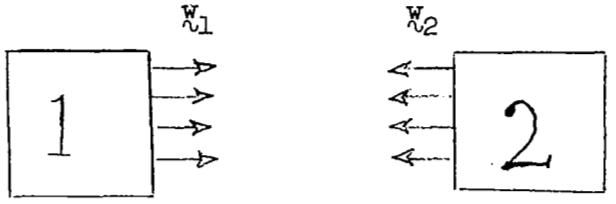
Equation (12) gives the motion of the free coordinates in terms of the known displacement $\underset{\sim}{w}_p$ and the solution for a transient disturbance may be written in terms of Transition matrices

$$\begin{Bmatrix}w_f\\ \dot{w}_f\end{Bmatrix}_{r+1} = [T_{ff}]\begin{Bmatrix}w_f\\ \dot{w}_f\end{Bmatrix}_r + [A_{ff}] - \left[\begin{array}{c|c} M_{fp} & 0 \\ \hline 0 & M_{fp} \end{array}\right]\begin{Bmatrix}\ddot{w}_p\\ \ddot{\dot{w}}_p\end{Bmatrix} - \left[\begin{array}{c|c} K_{fp} & 0 \\ \hline 0 & K_{fp} \end{array}\right]\begin{Bmatrix}w_p\\ \dot{w}_p\end{Bmatrix}_r \qquad (13)$$

The equation shows that the free coordinates are excited through inertia and stiffness coupling terms. If a lumped mass idealisation of the system is made, resulting in a diagonal mass matrix, no inertial coupling will take place because $[M_{fp}]$ is then a null matrix.

Coupling of Transition Matrices

It may be useful to determine the Transition matrix for a coupled system from a knowledge of the transition matrices of the subsidiary systems. The procedure for this is an approximation only, giving good results only if the step size is sufficiently small.



For system 1 $\quad \{\underset{\sim}{w}_1\}_{n+1} = [T_1]\{\underset{\sim}{w}_1\}_n + [A_1]\{F_n\}$

For system 2 $\quad \{\underset{\sim}{w}_2\}_{n+1} = [T_2]\{\underset{\sim}{w}_2\}_n + [A_2]\{F_n\}$

These two equations may now be written as a single matrix equation

$$\begin{Bmatrix} \underset{\sim}{w}_1 \\ \underset{\sim}{w}_2 \end{Bmatrix}_{n+1} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

i.e. $\underset{\sim}{\omega}_{n+1} = [\tau]\,\underset{\sim}{\omega}_n + [\phi]\underset{\sim}{\xi}_n$

If the systems are now linked together, so that certain coordinates are common, the applied forces relative to the coupled system are related to those of the independent systems by the Boolian matrix B, i.e.

$$\underset{\sim}{Q} = [B]\underset{\sim}{\xi}$$

and for compatability,

$$\underset{\sim}{\omega} = [B]^T\underset{\sim}{q}$$

Then substituting for $\underset{\sim}{\omega}$

$$[B]^T\underset{\sim}{q} = [\tau][B]^T\underset{\sim}{q}_n + [\phi]\xi_n$$

pre-multiplying by $[\phi]^{-1}$

$$[\phi]^{-1}[B]^T\underset{\sim}{q} = [\phi]^{-1}[\tau][B]^T\underset{\sim}{q}_n + \underset{\sim}{\xi}_n$$

pre-multiplying by $[B]$

$$[B][\phi]^{-1}[B]^T\underset{\sim}{q}_{n+1} = [B][\phi]^{-1}[\tau][B]^T\underset{\sim}{q}_n + \underset{\sim}{Q}_n$$

therefore,

$$\underset{\sim}{q}_{n+1} = ([B][\phi]^{-1}[B]^T)^{-1}([B][\phi]^{-1}[\tau][B]^T)\underset{\sim}{q}_n + ([B][\phi]^{-1}[B]^T)^{-1}\underset{\sim}{Q}_n$$

The Transition matrix for the complete system then is

$$([B][\phi]^{-1}[B]^T)^{-1}([B][\phi]^{-1}[\tau][B]^T) \qquad (14)$$

The merits of this formulation have to be evaluated and it may only be useful when small additions are made to a larger system.

USE OF FINITE ELEMENTS IN FORMING TRANSITION MATRICES FOR PLATE SYSTEMS

The equation of motion of a structure may be formulated from Hamilton's variational principle: this states that between two instants of time t_1 and t_2 the motion proceeds in such a way that

$$\delta \int_{t}^{t_2} (TE - VE + BE) = 0$$

where TE is the kinetic energy, VE the strain energy and BE a function whose variation gives the work done by the external forces in a virtual displacement (5). Mason (6) uses the finite element displacement technique and invokes this principle to form the mass and stiffness matrices for plates.

An outline of the finite elements adapted to this present report is given below and is fully described in (6). The method consists of dividing the plate into a number of rectangular elements, the displacement in each element is then given by a number of assumed modes and the coefficients of each mode are related to the nodal displacements. Here, 4 unknowns per element corner were used; these were w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x \partial y$. The assumed modes were built up from the following polynomials $f_i(t)$ having the properties

		f(0)	f'(0)	f(1)	f'(1)
$f_1(\varepsilon)$	$= 1 - 3\varepsilon^2 + 2\varepsilon^3$	1	0	0	0
$f_2(\varepsilon)$	$= \varepsilon - 2\varepsilon^2 + \varepsilon^3$	0	1	0	0
$f_3(\varepsilon)$	$= 3\varepsilon^2 - 2\varepsilon^3$	0	0	1	0
$f_4(\varepsilon)$	$= \varepsilon^3 - \varepsilon^2$	0	0	0	1

The element displacement is given by

$$w(x, y) = \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_{ij} f_i(x) f_j(y)$$

There are therefore 16 unknowns α_{ij}, and because of the above properties of $f_i(x)$, each unknown may be expressed in terms of one of the nodal displacements and the element dimensions a, b. The element displacements may then be conveniently written

$$w(x, y) = \{w_e\}^T \{f_x f_y\}$$

where $\{w_e\}$ is a column vector (16 x 1) containing the 4 nodal displacements: w, $a \frac{\partial w}{\partial x}$, $b \frac{\partial w}{\partial y}$, $ab \frac{\partial^2 w}{\partial x \partial y}$, $\{f_x f_y\}$ is a vector containing the 16 polynomial expressions.

Mass and Stiffness Matrices

The derivation of the element mass and stiffness matrices is found from the variation of TE and VE respectively. The procedure is given in (6).

The mass matrix for a rectangular element of dimensions a, b is:-

$$\rho th \int_0^b \int_0^a \{f_x f_y\} \{f_x f_y\}^T \, dxdy$$

The stiffness matrix is

$$\frac{E\,th^3}{12(1-\mu^2)} \int_0^b \int_0^a \{f''_x f_y\}\{f''_x f_y\}^T + \{f_x f''_y\}\{f_x f''_y\}^T + V\{f''_x f_y\}\{f_x f''_y\}^T + V\{f_x f''_y\}\{f''_x f_y\}^T + 2(1-\mu)\{f'_x f'_y\}\{f'_x f'_y\}^T)dxdy$$

If there is a pressure distribution $F(x, y, t)$ acting over the element the function BE is given by

$$BE = \int_0^b \int_0^a w(x, y)\, F(x, y, t)dxdy$$

but

$$w(x, y) = \{w_e\}^T\{f_x f_y\}$$

therefore

$$BE = \{w_e\}^T \int_0^b \int_0^a \{f_x f_y\}\, F(x, y, t)dxdy$$

The forcing vector is found by the variation of this quantity with respect to the nodal displacements w, $\frac{\partial w}{\partial x}$, etc.

$$F_i = \{\frac{\partial BE}{\partial w_i}\} = \Gamma_{ij} \int_0^b \int_0^a \{f_x f_y\}\, F(x, y, t)dxdy$$

where Γ_{ij} is a diagonal matrix with the following non-zero terms in the order

$[1, a, b, ab, 1, a, b\ ab, 1, a, b, ab, 1, a, b, ab.]$

In general these integrals may be quite complicated especially for travelling waveforms. For the purpose of this report only simple functions of F were used:

(a) If F is a uniform pressure normally incident on the plate the forcing vector for an element is

$$\underset{\sim}{F}_i = F(t) \ulcorner\gamma_j \int_0^b \int_0^a \{f_x f_y\} \, dxdy$$

(b) If F is a point force at the position x, y,

$$\underset{\sim}{F}_i = F(t) \ulcorner\gamma_j \{f_x , f_y\}$$

The equations of motion for the complete plate are obtained by assembling the element matrices by applying the conditions of compatability and dynamic equilibrium along the element boundaries. Where the coordinates are common the coefficients of the element mass and stiffness matrices and forcing vectors are additive. The procedure for assembling element matrices is described in (6). The damping matrix was assumed to be viscous and given by a constant times the stiffness matrix.

The equation of motion of the damped system is then of the form

$$M\ddot{\underset{\sim}{W}} + \alpha K\dot{\underset{\sim}{W}} + K\underset{\sim}{W} = \underset{\sim}{F}$$

usually $\alpha << 1$. In this way, the damping does not couple the normal modes, and the critical damping factor, β, is larger for the modes with highest natural frequencies.

Construction of the Transition Matrices [T] and [A]

Before a particular transient vibration problem could be solved it was necessary first to determine the T and A matrices for the given configuration and step-size, h. The A matrix was calculated by carrying out the matrix operations indicated in equation (9). When computing the T matrix use was made of the equation:

$$\{T_j\} = [T] \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ w_j=1 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. the j th column of T is found by post-multiplying T by a column vector in which the j term is unity, the remaining terms being zero. $\{T_j\}$ may be found therefore by using the Taylor Series and equation (5) for the second and higher derivatives to determine the response to a unit value at x_j. Note that for 'n' degrees of freedom system the T matrix is of order 2n x 2n so that there will be 2n of these operations. If they are carried out in order, the first n columns represent the responses to unit displacements at the station j ($1 < j \leq n$), the second n columns represent the responses to unit velocities at station j ($1 < j \leq n$).

RESULTS AND DISCUSSION

Finite Element Applications

The theory on the application of Transition matrices to a system with many degrees of freedom has been applied to several structural idealisations. The idealisations being made by applying the finite element displacement formulation discussed in the previous section. By this means the whole problem may be dealt with on a numerical basis and the formulation and solution of the equations are achieved without any resort being made to normal modes. To further illustrate the stability criterion mentioned in the first section the effect of step size on the maximum values for the forced response of a simply supported plate are shown. The advantages of an 8th order Transition matrix over a standard 4th order Runge-Kutta procedure are also illustrated.

Fig. 3 shows a comparison of Transition matrix solutions with some experimental results which were obtained from reference (7). These were for a rectangular plate of dimensions 18" x 7.5" x 0.282", with one of the edges of dimension 7.5" clamped; the other edges were free. The plate was idealised by using four equal rectangular elements, each element having four unknowns w, $\partial w/\partial x$, $\partial w/\partial y$, $\partial^2 w/\partial x \partial y$ per corner. When the appropriate constraints had been applied the resulting system had 19 degrees of freedom. The agreement between the experimental and theoretical results is good, though the relative frequencies of the upper modes to the lowest mode for the idealised system seem to be high. This is possibly due to the fact that only a small number of elements were used and therefore over constraining the system; it could also be due to the neglect of rotary inertia when forming the mass matrix.

There is an important aspect of this apparently simple system which should be mentioned. Nearly 2000 multiplications by the Transition matrix were necessary to obtain the response diagram shown in Fig. 3. This was because the eigenvalues for the system were well separated, and, since the step size was governed by the highest eigenvalue, it had to be small compared to the largest period to satisfy the stability criterion. A more complex system may not have this property for the same number of degrees of freedom, as the eigenvalues tend to lie closer together; consequently a complete period of the fundamental mode may be obtained with a much smaller number of steps.

Figs. 4 and 5 compare the response of a plate with all edges simply supported and a plate with all edges clamped, to a normally incident 'N' wave. Both plates had the same dimensions: 36.0" x 18.0" x 0.25". The simply supported plate was idealised with 4 elements and the clamped plate with 9 elements. When the appropriate constraints had been applied both configurations then had the same number of degrees of freedom. Due to the additional stiffness the clamped plate has a smaller response and the frequency of the fundamental mode is almost double that for the simply supported plate. (Note: the responses shown in Figs. 4 and 5 do not refer to the same point. For the

simply supported plate the response was at the centre, x = 9", y = 18", for the clamped plate the response was at x = 6", y = 12". The clamped plate response at the plate centre would then be about 50% more than those shown.) In contrast to the cantilever plate system the response histories shown in Fig. 4 were obtained with only 200 steps; this is due to the natural frequencies lying closer together, and the smaller number of degrees of freedom (16 cf. 19). The curves shown in Fig. 5(a) are shock spectra which were obtained by extracting the maximum values for the displacement and velocity responses to 'N' waves of different duration. On the shock spectra the maximum value is seen to occur when the duration of the 'N' wave is equal to the period of the fundamental mode. In Fig. 5(b) a factor N_p gives the number of positive peaks which occur in one period of the fundamental mode; this gives some indication whether higher modes are participating in the response. The results show that the displacement responses are essentially unimodal and most of the velocity responses but for a few cases where the pulse duration is short.

Fig. 6 shows the effect of increasing the step size on the maximum value of the response of a simply supported plate to an 'N' wave. A comparison is made between the 8th order Transition matrix results and the results obtained using a standard 4th order Runge-Kutta process for the forward integration (Appendix 1). All of the results have been normalized by dividing by the Transition matrix results at the smallest step size used (h = 1.0). Two points are illustrated: first, the Transition matrix gives a gain in step size before the onset of instability; second, within the stable range the maximum value for the Transition matrix solution does not diverge so much for increasing step size. This suggests greater accuracy. The points plotted beyond the stable range were the values that occurred at a time that the stable solution gave a maximum. These in themselves seem reasonable, and it is the subsequent motion which is divergent. For the system considered there was a substantial reduction in the computation time per step $\sim$ 0.1295 seconds cf. 0.1915 seconds for the Runge-Kutta. However, this was for displacement and velocity solutions only; had acceleration been computed the advantage would not have been so great.

Discussion on the Method

Throughout this report emphasis has been placed on the direct solution of the equations of motion without first evaluating the normal modes, natural frequencies and the appropriate generalised forces. This is an advantage provided the natural frequencies lie close together and the number of modes affecting the response is of the same order as the number of degrees of freedom of the idealised system. However, if the number of degrees of freedom is large and the natural frequencies well separated, the responses are mainly affected by the lower modes of vibration. This being the case, it would then be an advantage to adopt a normal mode approach and compute the Transition matrices for each of these as for a single degree of freedom. There would then be a reduction in the size of the problem from reducing the matrices involved and by increasing the step size which would previously have been

governed by the highest mode of the idealised system. In practice, then, one of the problems is knowing the spacing of the natural frequencies; this may only be obtained by solving an eigenvalue type problem first.

The method outlined in this report is similar in many respects to the Transfer matrix methods (8) used in structural analysis; the transfer in these cases being made in space and not in time. A similar technique - the Method of Mean Coefficients - was used on a non linear problem for a system with a single degree of freedom (9), the Transition matrices having to be re-evaluated at each step. A Transition matrix technique has also been used in reference (10) when studying the stability of a system with periodic coefficients, but the context was different to that discussed in this paper and no time domain solutions were evaluated.

CONCLUSIONS

A step by step Transition matrix method has been presented for the solution of a transient vibration problem for a system with many degrees of freedom. For the free vibration case, the response at one instant in time is related to that at a previous step by a single transition matrix. This matrix is a function of the system mass, damping and stiffness and the step size. The application to a single degree of freedom system gave an almost exact solution, and the method was much simpler to apply than other numerical methods. For forced vibration additional terms are necessary in the recurrence relationship to account for the forcing and the higher force time derivatives. If the step size is small it is sufficient to include only the force and the first derivative by an additional transition matrix and in general it is not possible to obtain the same order of accuracy as for the free vibration case. However, for forcing functions with certain analytic forms, i.e. e^{kt}, sin kt, cos kt, etc., an accurate solution may be obtained.

A finite element displacement technique has been used to construct the Transition matrices for particular plate systems and several examples on the applications of these has been given. When applied to a cantilever plate good agreement has been obtained with some experimental results where the plate had been excited by simple impulses.

For numerical stability the choice of step size is governed by the highest eigenvalue of the system and the order of the Taylor Series used when computing the Transition matrix. From this point of view there is no significant advantage in going immediately beyond the 5th derivative, though there is an advantage in going to the 8th. However, the accuracy of the method always increases with the order of the Taylor Series used.

An 8th order Transition matrix procedure gave more accurate results for a large step size than a 4th order Runge-Kutta procedure and there was also a substantial reduction in the computation time per step.

APPENDIX

The Runge-Kutta Procedure

The Runge-Kutta procedure which was used for comparison purposes is given below. This is a special application of the general technique given in ref. (1).

The equation of motion,

$$\ddot{\underset{\sim}{w}} = - [c1]\underset{\sim}{w} - [c2]\dot{\underset{\sim}{w}} + [M^{-1}]\underset{\sim}{F},$$

is written for convenience:

$$\ddot{\underset{\sim}{w}} = G(\underset{\sim}{w}, \dot{\underset{\sim}{w}}, \underset{\sim}{F})$$

The method avoids calculating the higher derivatives by making several evaluations of G, within the step, h. These are:

$$\begin{aligned}
\underset{\sim}{v}_o &= h.G(\underset{\sim}{w}(t), \dot{\underset{\sim}{w}}(t), \underset{\sim}{F}(t)) \\
\underset{\sim}{v}_1 &= h.g(\underset{\sim}{w}(t) + h\dot{\underset{\sim}{w}}(t)/2, \dot{\underset{\sim}{w}}(t) + \underset{\sim}{v}_o/2, \underset{\sim}{F}(t + h/2) \\
\underset{\sim}{v}_2 &= h.G(\underset{\sim}{w}(t) + h\dot{\underset{\sim}{w}}(t)/2 + h\underset{\sim}{v}_o/4, \dot{\underset{\sim}{w}}(t) + \underset{\sim}{v}_1/2, \underset{\sim}{F}(t + h/2)) \\
\underset{\sim}{v}_3 &= h.G(\underset{\sim}{w}(t) + h\dot{\underset{\sim}{w}}(t) + h\underset{\sim}{v}_1/2, \dot{\underset{\sim}{w}}(t) + \underset{\sim}{v}_2, \underset{\sim}{F}(t + h))
\end{aligned} \tag{A.1}$$

$\underset{\sim}{w}(t + h)$ and $\dot{\underset{\sim}{w}}(t + h)$ are then given by:

$$\underset{\sim}{w}(t + h) = \underset{\sim}{w}(t) + h\dot{\underset{\sim}{w}}(t) + (\underset{\sim}{v}_o + \underset{\sim}{v}_1 + \underset{\sim}{v}_2)h/6 \tag{A.2}$$

$$\dot{\underset{\sim}{w}}(t + h) = \dot{\underset{\sim}{w}}(t) + (\underset{\sim}{v}_o + 2\underset{\sim}{v}_1 + 2\underset{\sim}{v}_2 + \underset{\sim}{v}_3)/6 \tag{A.3}$$

In equations A.1 ... A.3 there are 12 multiplications of (n x n) matrices by a column matrix plus subsidiary additions at each step. With the Transition matrices, equation (10), there are two matrix multiplications of order (2n x 2n), which is equivalent to 8 multiplications of order (n x n). This implies that the Runge-Kutta would require about 50% more computation time in solving a problem of this type. However, because the matrices are smaller not so much computer storage space is required.

ACKNOWLEDGEMENT

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Damping ratio	λh_{crit} for n^{th} order Taylor Series					
$p=c/c_{crit}$	3	4	5	6	7	8
0.000	1.7	2.8	3.3	2.0	2.2	3.4
0.002	1.7	2.8	3.3	1.8	2.1	3.4
0.020	1.8	2.8	3.3	2.1	2.7	3.7
0.200	2.3	2.9	2.8	3.0	3.7	4.3

Table 1. Limiting values of λh for stability.

	Order of Taylor Series					
	3	4	5	6	7	8
λh	1.0	1.0	2.0	2.0	2.2	3.0

Table 2. Values of λh for accuracy.

Step	Exact Soln.* Disp.	4th Order T matrix Disp.	vel.	Choudhury* Method Ref. 4	Newmark* method
0	1.0000	1.0000	0.0000	1.0000	1.0000
1	.9689	.9689	-.2474	.9688	.9681
2	.8776	.8776	-.4794	.8773	.8782
3	.7317	.7317	-.6816	.7310	.7330
4	.5403	.5403	-.8414	.5392	.5425
5	.3153	.3154	-.9490	.3138	.3184
6	.0707	.0708	-.9975	.0688	.0746
7	-.1782	-.1782	-.9840	-.1805	-.1738
8	-.4161	-.4161	-.9093	-.4185	-.4114
9	-.6282	-.6281	-.7781	-.6305	-.6236
10	-.8011	-.8011	-.5985	-.8031	-.7973
11	-.9243	-.9243	-.3817	-.9257	-.9216
12	-.9900	-.9900	-.1417	-.9905	-.9889

*These results were taken from reference (4).

Table 3. Comparison of 4th order Transition matrix solution with exact solution and solutions by other methods for an undamped oscillator.

Equation of motion: $\ddot{w} + w = 0, \quad h = 0.25$

T matrix: $\begin{bmatrix} .968913 & .247396 \\ -.246396 & .968913 \end{bmatrix}$

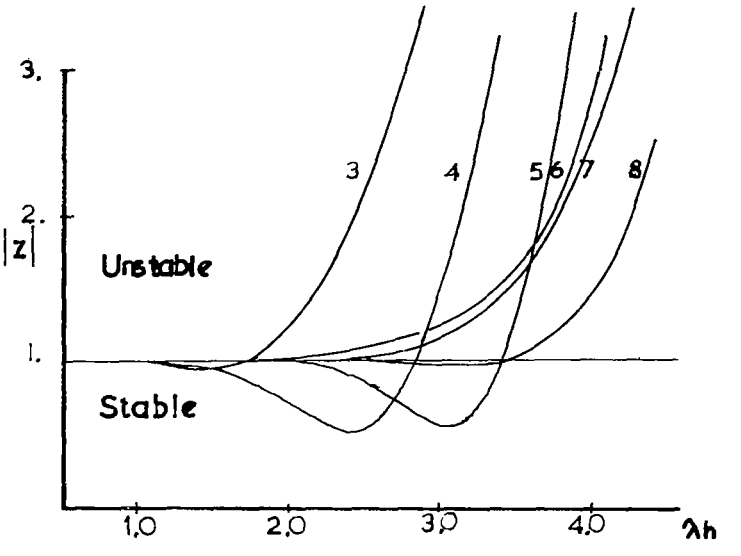


Fig 1 Variation of Z , undamped oscillator($\beta = 0.0$)

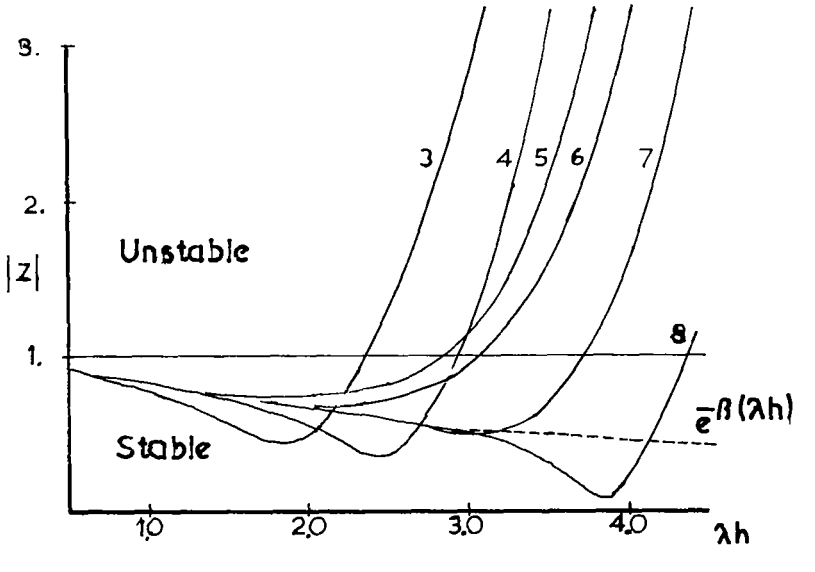
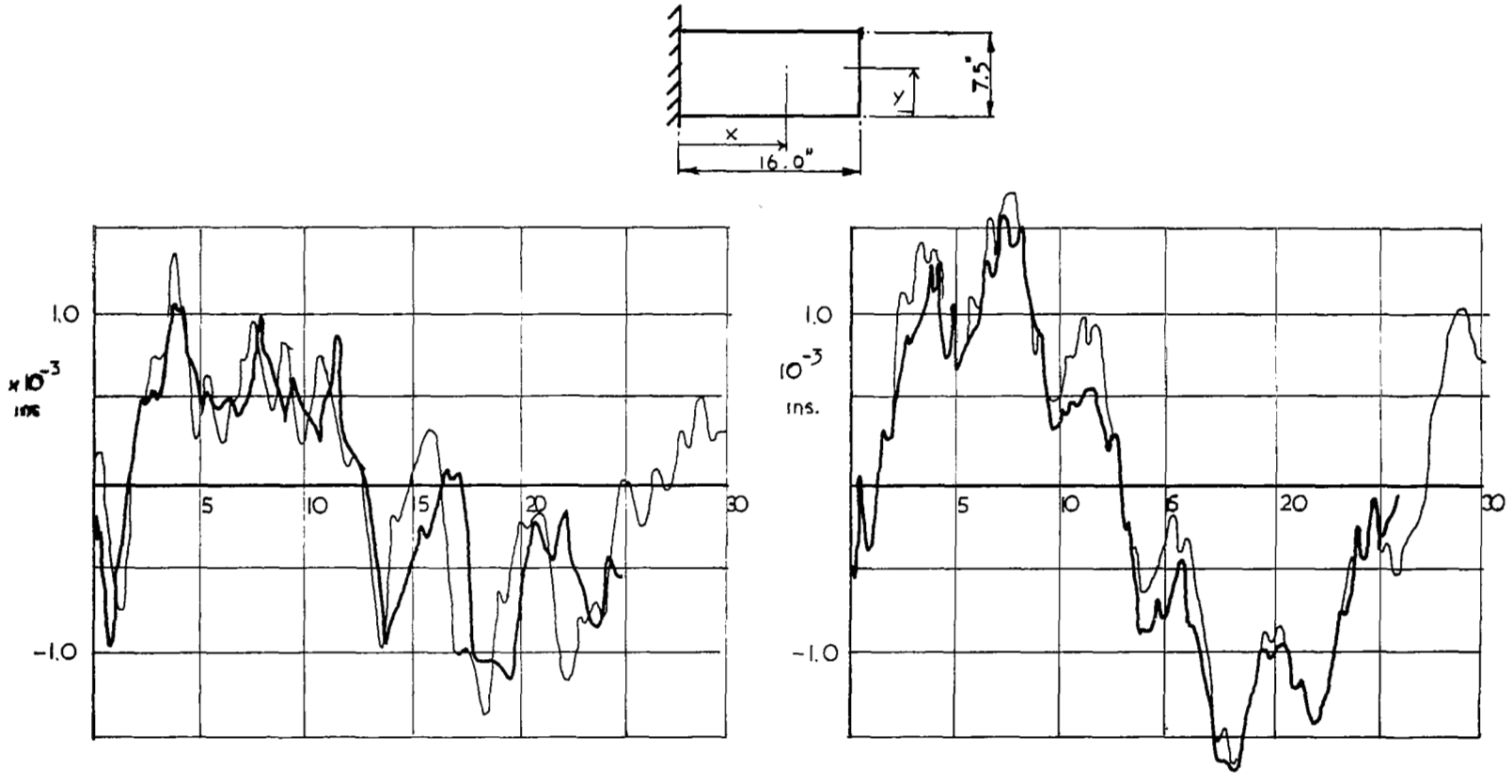


Fig 2 Variation of Z ($\beta = 0.2$)



Displacement at x = 16.0 ins y = 0.0 ins
impact at x = 6.0 ins y = 5.5 ins

Displacement at x = 16.0 ins y = 7.5 ins
impact at x = 8.0 ins y = 3.75 ins

Fig 3 Comparison between Experimental & Theoretical results, —— theory , — exp.
for plate with one edge clamped.

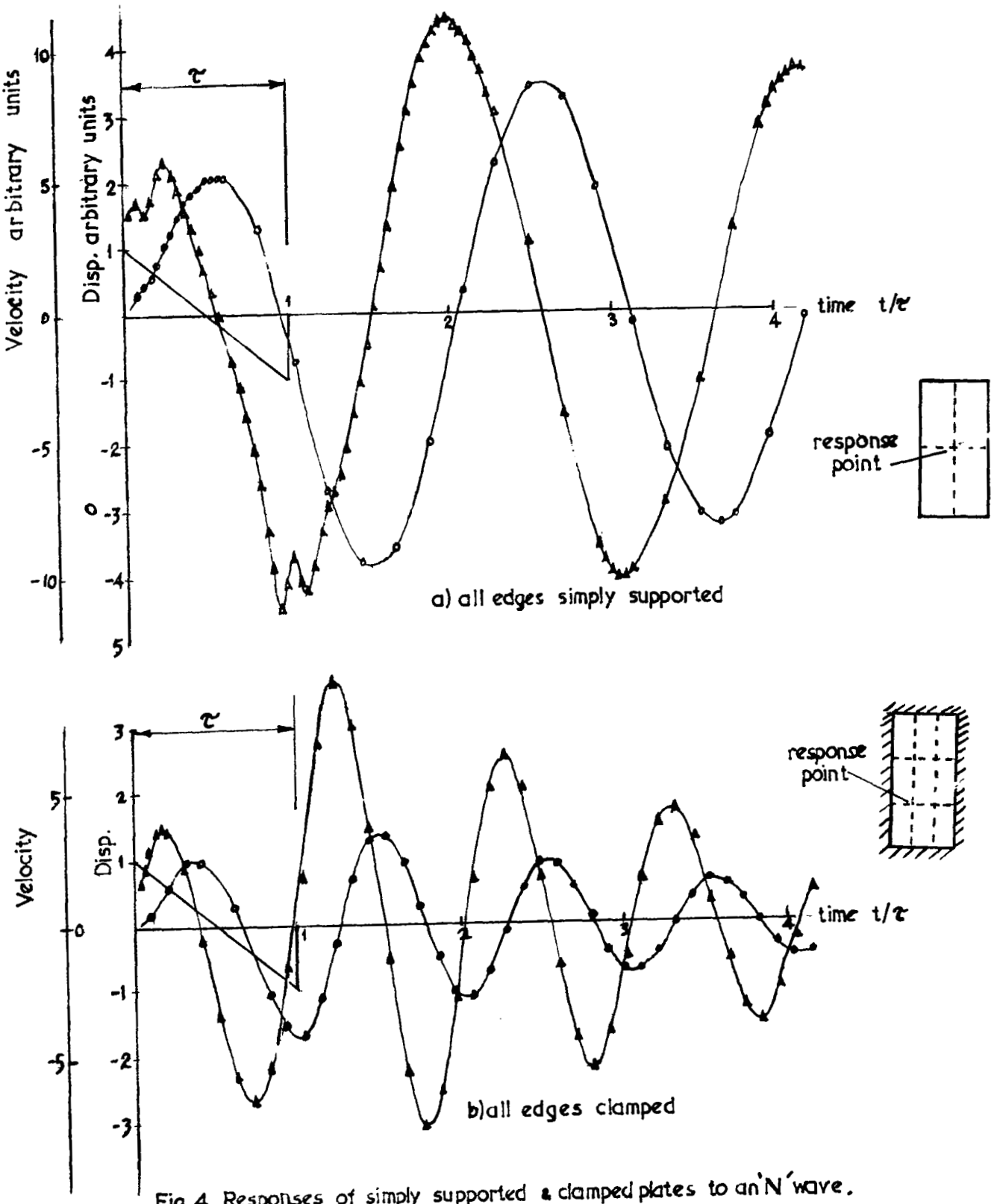


Fig 4. Responses of simply supported & clamped plates to an 'N' wave.
Δ velocity. o displacement.

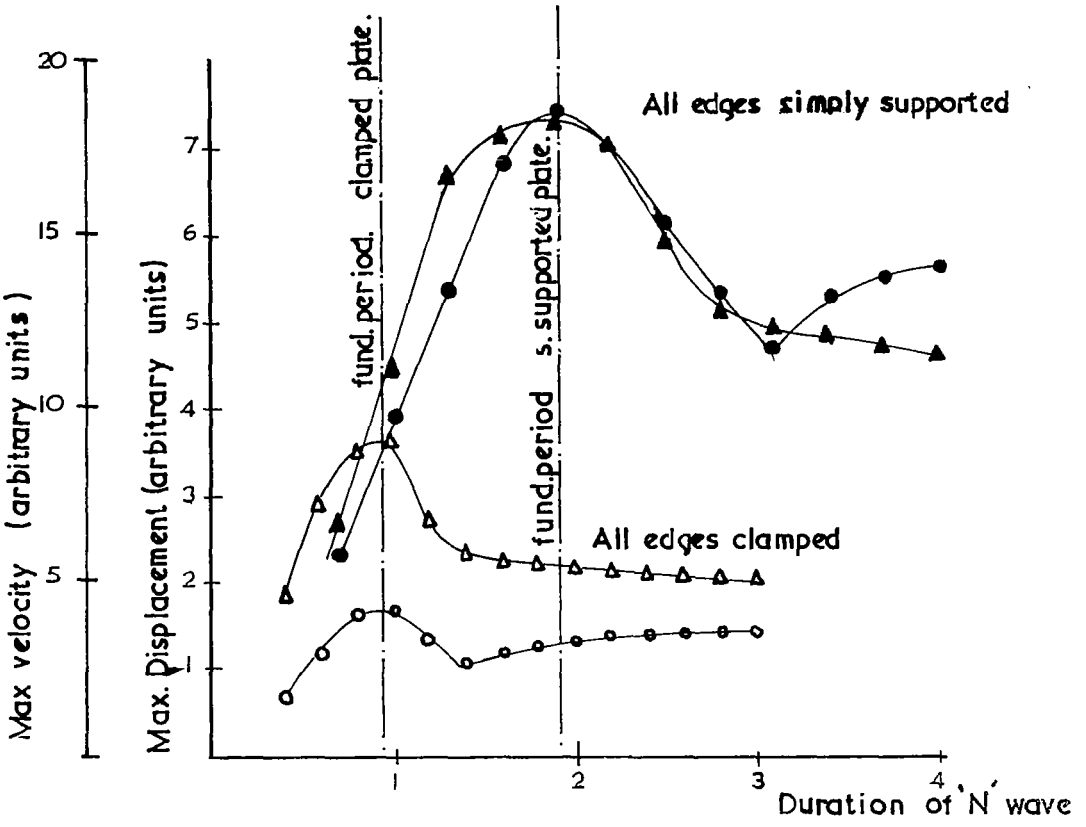


Fig 5 a) Shock spectra of Clamped & Simply Supported plates to N wave

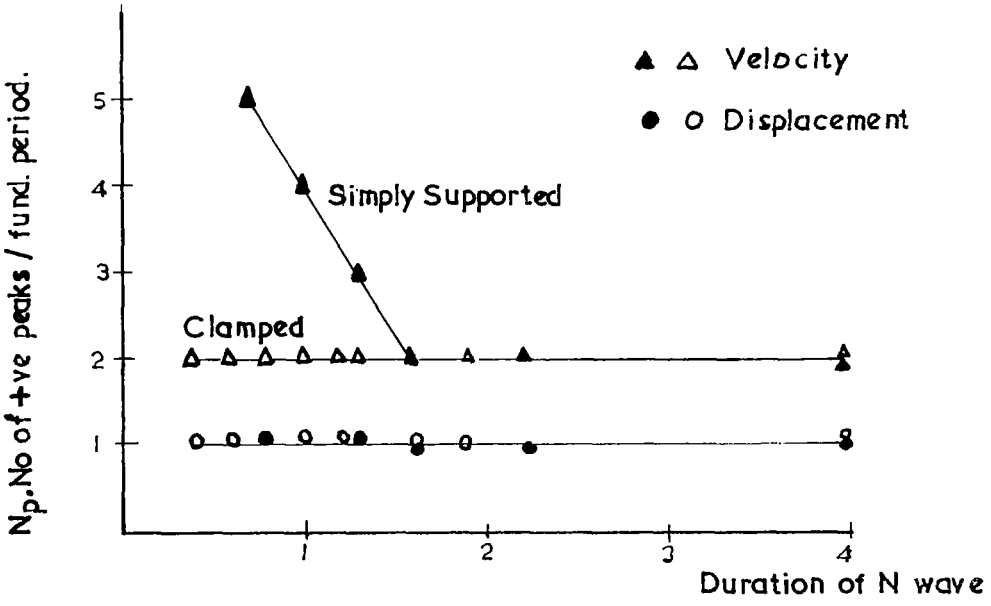


Fig 5 b) Variation of N_p with pulse duration

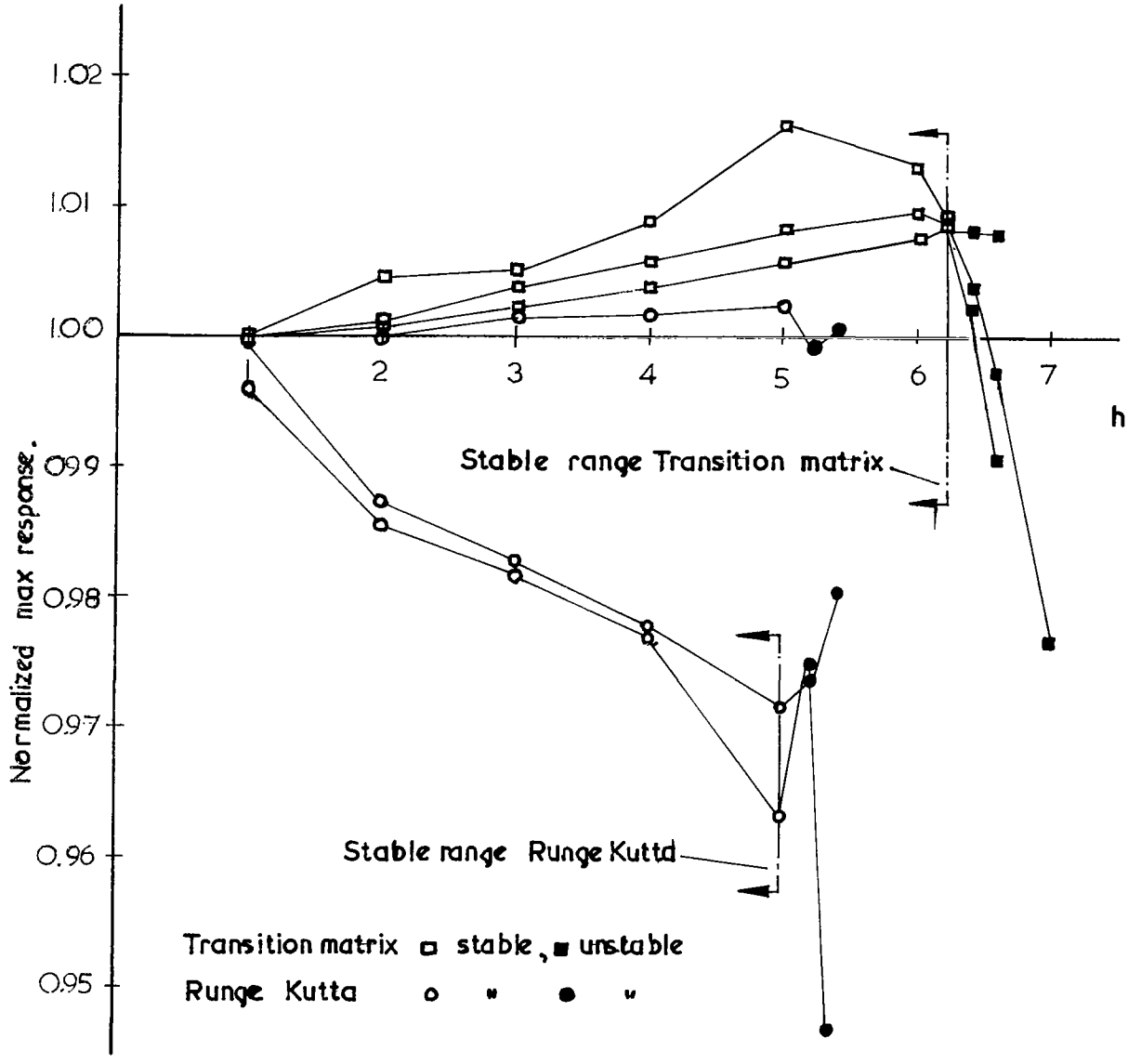


Fig 6 Comparison between normalized maximum responses obtained by 8th. order Transition matrix & 4th. order Runge Kutta

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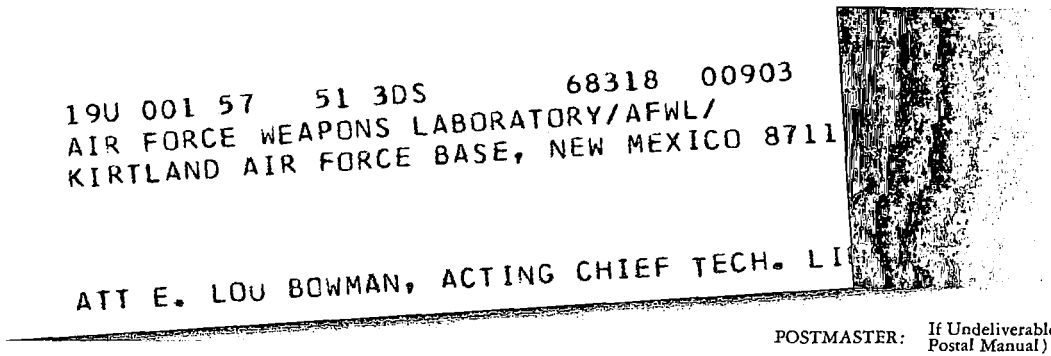

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