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# RELATIONSHIP BETWEEN COMPLEX- AND ENVELOPE-COVARIANCE FOR RICIAN FADING COMMUNICATION CHANNELS

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ABSTRACT

In practice, the measurement of the covariance function (which characterizes a time and frequency selective fading communication channel) involves envelope detecting the received signals before correlating. Envelope covariance is insufficient for an accurate evaluation of system performance and rather a complex covariance is needed. This paper presents a study of these two covariances for Rician statistics. The general properties of the Rician distribution are discussed. The relationship between complex covariance and covariance of the envelopes of a complex Rician fading communication channel is derived. It includes the well known case of Rayleigh fading. For the assumption of symmetric spectrum of the narrow-band random process there exists a monotonic relationship which allows the evaluation of the complex covariance once the envelope covariance is measured. An extension to unsymmetric spectrum is given. The results obtained show that the envelope covariance represents a lower bound of the magnitude of the complex covariance.

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## INTRODUCTION

The transmission characteristics of a time- and frequency-spread channel can be specified by the time-varying transfer function or its Fourier transform (the channel impulse response). Let $s(t)$ denote the received narrow-band waveform resulting from the transmission of a sine wave at a frequency $f + f_0$ ($f_0$ = center frequency) through a fading channel. Then $s(t)$ may be expressed by

$$s(t) = \text{Re}\left\{H(f, t) \exp j 2\pi\left(f + f_0\right)t\right\},\tag{1}$$

where $H(f, t)$ is the complex envelope of the received signal. Actually $H(f, t)$ is the time-variant transfer function of the medium, measured with respect to the center frequency $f_0$. Even if this function could be measured, using it to characterize a given channel would be somewhat akin to using a sample function of a random process to characterize the process. As Gallager (Reference 1) has pointed out, these functions contain too much data to be useful without processing. We are interested only in finding various averages of these quantities. The averages of most interest are the mean value, the variance, and the correlation function (or covariance function).

Let us assume that $H(f, t)$ is a sample function of an ergodic channel which is also stationary in the wide sense—the statistical averages of $H(f, t)$ do not change with changes in $f$ and $t$ (Reference 2). Then the complex time-frequency correlation function is independent of the location of the time interval and frequency band within which the measurement is made:

$$R(\tau, \Omega) = E\left\{H\left(f - \frac{\Omega}{2}, t\right) H^*\left(f + \frac{\Omega}{2}, t + \tau\right)\right\},\tag{2}$$

where $E$ indicates an ensemble average and * denotes the complex conjugate.

## MODIFIED CORRELATION FUNCTION

Bello (Reference 2) presents an experimental technique for the measurement of the complex time-frequency correlation function, using independent frequency standards at transmitter and receiver site. For this method it is necessary to have frequency standards with relatively good long-term stabilities. Unfortunately, such frequency standards in the millimeter-wave region of the electro-magnetic spectrum are difficult to use in earth-to-space experiments, because of the weight and prime-power constraints on present-day spacecraft.

An example of such restriction occurs on the ATS-E Millimeter Wave Propagation Experiment planned for launch in mid 1969 (Reference 3). In this case the long-term stability is not expected to be better than 1 part in $10^6$ per year. Therefore a correlation function will be measured using only the envelopes (magnitudes of the complex envelopes) of the received signal. With

$$H(f, t) = H_r(f, t) + jH_i(f, t) = V(f, t) \exp\left[j\theta(f, t)\right] \tag{3}$$

and Equation 1, we find

$$s(t) = H_r(f, t) \cos\left[2\pi\left(f + f_0\right)t\right] - H_i(f, t) \sin\left[2\pi\left(f + f_0\right)t\right] \tag{4}$$

or

$$s(t) = V(f, t) \cos\left[2\pi\left(f + f_0\right)t + \theta(f, t)\right] . \tag{5}$$

The output of an envelope detector is now proportional to

$$V(f, t) = |H(f, t)| = \left[H_r^2(f, t) + H_i^2(f, t)\right]^{1/2} , \tag{6}$$

which is exactly the magnitude of the complex channel transfer function.

The modified correlation function, denoted by $\hat{R}$, is defined as

$$\hat{R}(\tau, \Omega) = E\left\{\left|H\left(f - \frac{\Omega}{2}, t\right)\right|\left|H\left(f + \frac{\Omega}{2}, t + \tau\right)\right|\right\} . \tag{7}$$

While the modified correlation function is considerably easier to measure, it does not contain information about decorrelation of the phase of the transfer function.

The following estimation shows that for an ergodic process the modified correlation function is always an upper bound of the complex correlation function

$$|R(\tau, \Omega)| \leq \hat{R}(\tau, \Omega) \ . \tag{8}$$

On the assumption that the randomly varying transfer function $H(f, t)$ is a sample of an ergodic process, the correlation function for a certain frequency difference $\Omega$ can be calculated by using the time average

$$R(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} H(t) H^*(t + \tau) \, dt \ . \tag{9}$$

The validity of Equation 8 can be seen from

$$|R(\tau)| = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} H(t) H^*(t + \tau) \, dt \right| \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |H(t)| \, |H(t + \tau)| \, dt = \hat{R}(\tau) \ . \tag{10}$$

Since the received process may have a nonzero mean (this is always the case if the signal envelope is measured), the function really desired is the covariance rather than the correlation function. For a complex random varying process, the two-dimensional covariance is defined (Reference 4) by

$$C(\tau, \Omega) = E\left\{ \left[ H\left(f - \frac{\Omega}{2}, t\right) - E\left\{ H\left(f - \frac{\Omega}{2}, t\right) \right\} \right] \left[ H^*\left(f + \frac{\Omega}{2}, t + \tau\right) - E\left\{ H^*\left(f + \frac{\Omega}{2}, t + \tau\right) \right\} \right] \right\}$$

$$= R(\tau, \Omega) - m_1 m_2^* \ , \tag{11}$$

where $m_1$ and $m_2$ are the complex mean values

$$m_1 = E\left\{ H\left(f - \frac{\Omega}{2}, t\right) \right\}$$

$$m_2 = E\left\{ H\left(f + \frac{\Omega}{2}, t\right) \right\} \ . \tag{12}$$

According to Equation 7 the modified covariance becomes

$$\hat{C}(\tau, \Omega) = \hat{R}(\tau, \Omega) - E\left\{ \left| H\left(f - \frac{\Omega}{2}, t\right) \right| \right\} E\left\{ \left| H\left(f + \frac{\Omega}{2}, t\right) \right| \right\} \ . \tag{13}$$

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From both covariance functions $C$ and $\hat{C}$, the coherence time of the channel, $T_c$, and the coherence bandwidth, $F_c$, can be calculated. $T_c$ is an interval in $\tau$ over which $|C(\tau, 0)|$ is essentially nonzero; $F_c$ is an interval in $\Omega$ over which $|C(0, \Omega)|$ is essentially nonzero (Reference 1). In a parallel manner $\hat{T}_c$ and $\hat{F}_c$ are defined as associated with $\hat{C}(\tau, \Omega)$. But we cannot say that the modified function lead to upper- (Reference 5) or lower-bound estimates on the coherence parameters. This follows from the fact that a general expression similar to Equation 8 cannot be found for the covariance functions. It is necessary to make certain assumptions concerning the statistic of the channel transfer function in order to compare these two covariance functions. For a narrow-band Gaussian random process with zero mean (i.e., Rayleigh envelope distribution) the relationship between the envelope correlation function and the complex correlation function is well known (References 2 and 6).

For a channel with known statistic properties, the different average values can be calculated theoretically from the first and second probability distributions. Time averages, on the other hand, can only be used for experimental measurement techniques.

## CHANNEL WITH RICIAN STATISTICS

The communication channel between spacecraft and ground station can be considered as a multipath channel model with a major strong stable path and additional weak paths. The stable path is a result of line-of-sight propagation. The signal then seems to be the sum of a steady signal and a fading component, the latter having the usual Gaussian quadrature components. The signal envelope resulting from a transmitted continuous sine wave fluctuates according to statistics identical with those for the envelope of a sine wave plus additive bandpass Gaussian noise. This envelope statistic is usually called Rician fading (Reference 7).

If we are looking at the channel transfer function for a certain frequency $f_0$, the received narrow-band waveform is

$$s(t) = \operatorname{Re}\left\{H(t) \exp\left(j 2\pi f_0 t\right)\right\} . \tag{14}$$

During an interval where the channel can be assumed stationary, the transfer function $H(t)$ is the sum of a steady (time-independent) component $\alpha + j\beta$ and a random fluctuation $x(t) + jy(t)$. The sample functions $x(t)$ and $y(t)$ have frequency components only in a narrow band centered on zero frequency (References 6 and 8). The random variables $x$ and $y$, that refer to the possible values of $x(t)$ and $y(t)$, respectively, are Gaussian random variables with zero mean and variance $\psi^2$.

The transfer function may now be written as

$$H(t) = V(t) \exp\left[j\theta(t)\right] , \tag{15}$$

where the envelope $V(t)$ and the phase $\theta(t)$ are slowly varying functions of time. Formally, such a representation need not be restricted to narrow-band signals; the concept of an envelope and

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phase has significance, however, only when the variations of $V(t)$ and $\theta(t)$ are slow compared with those of $\cos 2\pi f_0 t$ ($f_0$ is the carrier frequency). The first-order probability density of the envelope (magnitude) and phase of the complex transfer function $H(t)$ can be found (References 8 and 10) as

$$P_1(V) = \begin{cases} \dfrac{V}{\psi^2} I_0\left(\dfrac{\sqrt{2z}}{\psi} V\right) \exp\left(-\dfrac{V^2}{2\psi^2} - z\right), & \text{for } V \geq 0, \\[2mm] 0, & \text{for } V < 0, \end{cases} \tag{16}$$

and

$$P_1(\theta) = \begin{cases} \dfrac{1}{2\pi}\left[\sqrt{\pi z}\,\cos\theta_0 + {}_1F_1\left(-\dfrac{1}{2};\ \dfrac{1}{2};\ -z\cos^2\theta_0\right)\right]\exp\left(-z\sin^2\theta_0\right), & \text{for } 0 \leq \theta \leq 2\pi, \\[2mm] 0, & \text{otherwise}, \end{cases} \tag{17}$$

where

$$z = \frac{\alpha^2 + \beta^2}{2\psi^2}$$

$$\theta_0 = \theta - \arctan\frac{\beta}{\alpha}. \tag{18}$$

$I_0$ is the modified Bessel function of zero order and ${}_1F_1$ the confluent hypergeometric function.

Figures 1 and 2 are graphs of $P_1(V)$ —or, more strictly speaking, $P_1(V/\sqrt{2}\,\psi)$—and $P_1(\theta)$ for different values of $z$. As expected, the presence of the steady component shifts the average
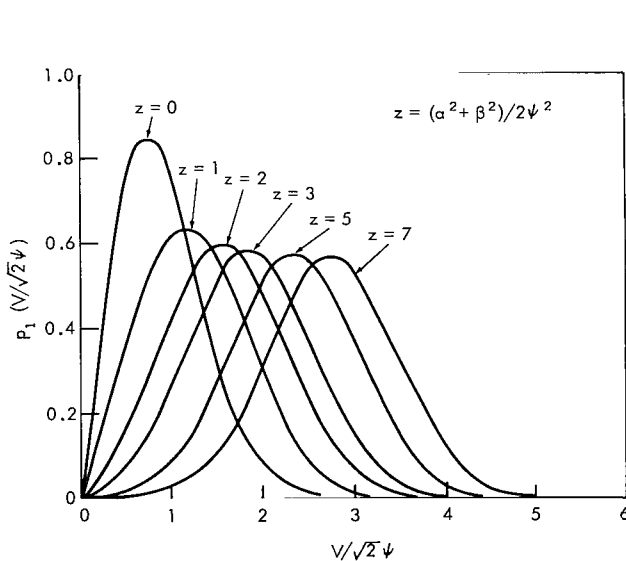


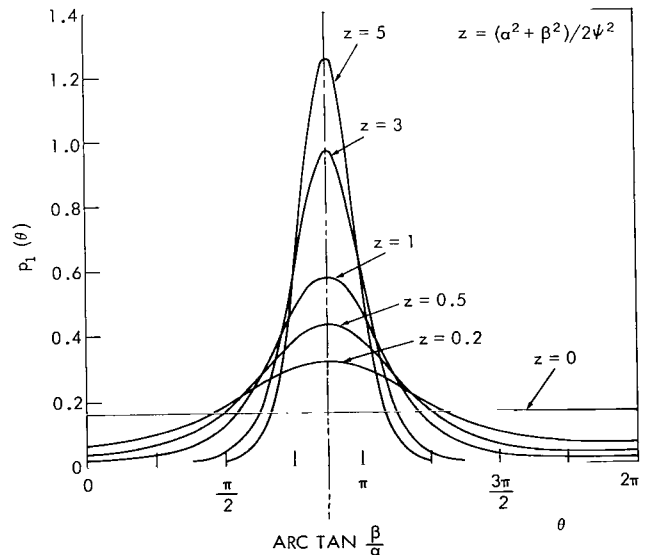Figure 1—Probability density distribution of envelope for Rician fading signals.

Figure 2—Probability density distribution of phase for Rician fading signals.

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and the most probable values of the distribution to larger values of the magnitude. For sufficiently strong coherent signals, the most probable value of V coincides with the amplitude $\sqrt{\alpha^2 + \beta^2}$ of the steady component. No signal, $z = 0$, gives the Rayleigh distribution for the envelope
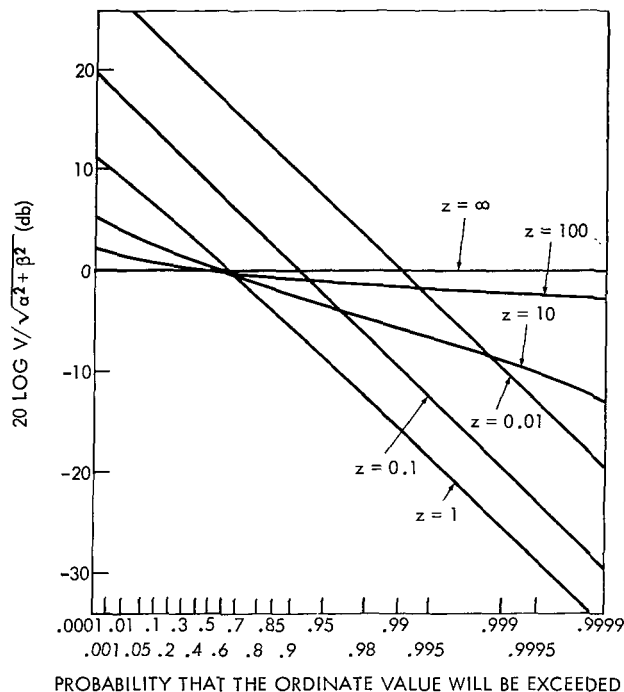
$$p_1(V) = \begin{cases} \dfrac{V}{\psi^2} \exp\left(-\dfrac{V^2}{2\psi^2}\right), & \text{for } V \geq 0, \\ 0, & \text{for } V < 0, \end{cases} \quad (19)$$

and the phase distribution becomes uniform

$$p_1(\theta) = \begin{cases} \dfrac{1}{2\pi}, & \text{for } 0 \leq \theta \leq 2\pi, \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

As the signal strength is increased relative to the rms fluctuation, the phases are grouped progressively closer about the zero-degree phase of the carrier, until for no random signal the density becomes a delta function at $\theta = \arctan \beta/\alpha$ $\left(\theta_0 = 0\right)$.

Figure 3, taken from Reference 9, gives the distribution of $V/\sqrt{\alpha^2 + \beta^2}$ for several values of $z$ on graph paper so designed that the distribution of the amplitude of a Rayleigh-distributed signal ($z = 0$) expressed in decibels is represented by a straight line with a -45-degree slope. When the steady component becomes large, the distribution of the signal level becomes concentrated about the value of the steady component (the 0-db line).



Figure 3—Distribution of the resultant amplitude of a constant vector plus a Rayleigh-distributed vector.

## SECOND-ORDER PROBABILITY DENSITIES

The following sections deal with the determination of auto-correlation function and auto-covariance for a Rician fading channel. An extension to the cross-correlation is mentioned under "Covariance for Envelope Detection—Cross-covariance."

To derive an expression for the correlation function it is convenient to calculate the second-order probability densities. The joint second-order probability density function of real and imaginary parts of $H(t)$ is, as shown by Middleton (Reference 10):

$$p_2\left(H_{r_1}, H_{i_1}, H_{r_2}, H_{i_2}; \tau\right) = \frac{1}{(2\pi)^2 |\Lambda|^{1/2}} \exp\left[-\frac{1}{2} X' \Lambda^{-1} X\right], \quad (21)$$

where X is the column vector

$$
X = \begin{Vmatrix} H_{r_1} - \alpha_1 \\ H_{i_1} - \beta_1 \\ H_{r_2} - \alpha_2 \\ H_{i_2} - \beta_2 \end{Vmatrix} = \begin{Vmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{Vmatrix} ,
$$

whose transpose is the row vector $X'$. The subscripts 1 and 2 refer to two time instants $t_1$ and $t_2$. The time difference $t_2 - t_1$ is denoted by $\tau$. The covariance matrix $\Lambda$ of the random variables $x_1$, $y_1$, $x_2$, and $y_2$ is defined by

$$
\Lambda = \psi^2 \begin{Vmatrix} 1 & 0 & r & \lambda \\ 0 & 1 & -\lambda & r \\ r & -\lambda & 1 & 0 \\ \lambda & r & 0 & 1 \end{Vmatrix} . \tag{22}
$$

The determinant of the covariance matrix is denoted by $|\Lambda|$. The mean square of the random variables is given by

$$
\psi^2 = E\{x_i^2\} = E\{y_i^2\} = \int_{-\infty}^{\infty} w(f)\,df , \qquad i = 1, 2 . \tag{23}
$$

$w(f)$ is the spectral power density, and $f$ is always measured from the carrier frequency $f_0$. The following statistical averages are functions of $\tau = t_2 - t_1$ and denoted by

$$
E\{x_1 x_2\} = E\{y_1 y_2\} = r(\tau) \cdot \psi^2 = \int_{-\infty}^{\infty} w(f) \cos(2\pi f\tau)\,df ,
$$

$$
E\{x_1 y_2\} = -E\{x_2 y_1\} = \lambda(\tau) \cdot \psi^2 = \int_{-\infty}^{\infty} w(f) \sin(2\pi f\tau)\,df ,
$$

$$
\rho^2(\tau) = r^2(\tau) + \lambda^2(\tau) \leq 1 . \tag{24}
$$

The corresponding joint-distribution densities of the envelope and phase may be found from Equation 21 by means of the transformations

$$
V_1 = \left(H_{r_1}^2 + H_{i_1}^2\right)^{1/2} \qquad \theta_1 = \arctan \frac{H_{i_1}}{H_{r_1}}
$$

$$
V_2 = \left(H_{r_2}^2 + H_{i_2}^2\right)^{1/2} \qquad \theta_2 = \arctan \frac{H_{i_2}}{H_{r_2}} . \tag{25}
$$

Specifically, we have

$$p_2\left(V_1, V_2, \theta_1, \theta_2; \tau\right) = p_2\left(H_{r_1}, H_{i_1}, H_{r_2}, H_{i_2}; \tau\right) \cdot J$$

where the jacobian J of this transformation is

$$J = \left|\frac{\partial\left(H_{r_1}, H_{i_1}, H_{r_2}, H_{i_2}\right)}{\partial\left(V_1, V_2, \theta_1, \theta_2\right)}\right| = V_1 V_2 . \tag{26}$$

The regions $\left(-\infty < H_{r_1}, H_{i_1}, H_{r_2}, H_{i_2} < \infty\right)$ for the variables in a rectangular coordinate system map into $\left(0 \le V_1, V_2 < \infty\right)$ and $\left(0 \le \theta_1, \theta_2 \le 2\pi\right)$ for $V_i$ and $\theta_i$. Integrating over all phases $\theta_1$ and $\theta_2$ from 0 to $2\pi$ gives $p_2\left(V_1, V_2; \tau\right)$. These integrations can be performed with the aid of Equation B11.

To simplify considerably the calculation we may assume that the power spectrum $w(f)$ is symmetrical about the carrier frequency $f_0$. In the Millimeter Wave Experiment there is no reason that $w(f)$ should be an odd function of $f$. Equation 24 shows that for an even function $w(f)$ the term $\lambda(\tau)$ becomes zero. For the second-order probability density function of the envelopes, (Reference 10) gives:

$$p_2\left(V_1, V_2; \tau\right) = \frac{V_1 V_2}{\psi^4\left(1 - \rho^2\right)} \exp\left[-\frac{2z}{1 + \rho} - \left(V_1^2 + V_2^2\right)\Big/2\psi^2\left(1 - \rho^2\right)\right]$$

$$\cdot \sum_{m=0}^{\infty} \epsilon_m I_m\left(\frac{\rho V_1 V_2}{\psi^2\left(1 - \rho^2\right)}\right) I_m\left(\frac{\sqrt{2z}}{\psi(1 + \rho)} V_1\right) I_m\left(\frac{\sqrt{2z}}{\psi(1 + \rho)} V_2\right) \quad \text{for} \quad V_1, V_2 \ge 0 , \tag{27}$$

where, from Equation 24, $\rho(\tau) = r(\tau)$, since $\lambda(\tau) \equiv 0$. $I_m(z)$ is the modified Bessel function of order $m$, and

$$\epsilon_0 = 1$$

$$\epsilon_m = 2 \quad \text{for} \quad m \ge 1 .$$

The power S/N ratio $z$ is already defined in Equation 18. When there is no steady signal component ($z = 0$), Equation 27 reduces to

$$p_2\left(V_1, V_2; \tau\right) = \frac{V_1 V_2}{\psi^4\left(1 - \rho^2\right)} \exp\left[-\left(V_1^2 + V_2^2\right)\Big/2\psi^2\left(1 - \rho^2\right)\right] I_0\left[\frac{\rho V_1 V_2}{\psi^2\left(1 - \rho^2\right)}\right]$$

$$\text{for} \quad V_1, V_2 \ge 0 . \tag{28}$$

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From these second-order probability density functions, the first-order probability density can be found by using the general relationship

$$\lim_{\tau \to \infty} P_2 = P_1^2 \ .$$

(29)

From $\rho(\infty) = 0$ and Equations 27 and 29, we obtain the result already found in Equation 16.

## COVARIANCE OF THE COMPLEX TRANSFER FUNCTION

The correlation function of the complex channel transfer function may be determined by solving the fourfold integral (Reference 6)

$$R(\tau) = \int\!\!\!\int\!\!\!\int\!\!\!\int_{-\infty}^{+\infty} dH_{r_1} \, dH_{i_1} \, dH_{r_2} \, dH_{i_2} \, H_1 \, H_2^* P_2 \left( H_{r_1}, H_{i_1}, H_{r_2}, H_{i_2}; \tau \right) \ ,$$

(30)

where $P_2$ is given in Equation 21.

A more convenient way to obtain $R(\tau)$ is the following:  From the definition

$$R(\tau) = E\left\{ H\left(t_1\right) H^*\left(t_2\right) \right\} = E\left\{ H_1 H_2^* \right\} \ ,$$

(31)

with

$$H_i = \alpha_i + x_i + j\left(\beta_i + y_i\right) \ , \qquad i = 1, 2 \ ,$$

$$E\left\{ x_i \right\} = E\left\{ y_i \right\} = 0 \ ,$$

the correlation function becomes

$$R(\tau) = E\left\{ x_1 x_2 \right\} + E\left\{ y_1 y_2 \right\} + j\left[ E\left\{ x_2 y_1 \right\} - E\left\{ x_1 y_2 \right\} \right] + \left( \alpha_1 + j\beta_1 \right)\left( \alpha_2 - j\beta_2 \right) \ .$$

(32)

(Note that Equation 32 is valid for auto-correlation as well as for cross-correlation.)  Combining Equations 24 and 32 gives

$$R(\tau) = 2\psi^2 \left[ r(\tau) - j\lambda(\tau) \right] + \left( \alpha_1 + j\beta_1 \right)\left( \alpha_2 - j\beta_2 \right) \ .$$

(33)

With the mean values of the transfer function

$$m_1 = E\{H_1\} = \alpha_1 + j\beta_1 \, ,$$

$$m_2 = E\{H_2\} = \alpha_2 + j\beta_2 \, , \tag{34}$$

the covariance, defined as

$$C(\tau) = R(\tau) - m_1 m_2^* \, , \tag{35}$$

becomes

$$C(\tau) = 2\psi^2 \left[ r(\tau) - j\lambda(\tau) \right] = 2\psi^2 \rho(\tau) e^{-j\phi(\tau)} \, . \tag{36}$$

This result shows that the covariance does not depend on the coherent part of the transfer function.

The value of $C(\tau)$ for $\tau = 0$ is the variance of the random process. Equation 36 gives, since $r(0) = 1$ and $\lambda(0) = 0$ (compare Equation 24),

$$C(0) = 2\psi^2 \, . \tag{37}$$

The normalized covariance or correlation coefficient becomes

$$\frac{C(\tau)}{C(0)} = r(\tau) - j\lambda(\tau) = \rho(\tau) e^{-j\phi(\tau)} \, , \tag{38}$$

where $\rho(\tau)$ is the magnitude and $\phi(\tau)$ the phase:

$$\phi(\tau) = \arctan\left(\frac{\lambda}{r}\right) \cdot$$

The absolute value of the normalized covariance from which the channel parameters can be determined is

$$\rho(\tau) = \left[ r^2(\tau) + \lambda^2(\tau) \right]^{1/2} \, , \qquad 0 \leq \rho(\tau) \leq 1 \, . \tag{39}$$

The complex normalized covariance depends only on the random part of the complex channel function and therefore yields equal results for Rayleigh and Rician fading channels. On the other hand

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(as will appear later) for envelope detection, the correlation function and mean value both depend on the ratio of the coherent and random parts of the process.

# COVARIANCE FOR ENVELOPE DETECTION

## Auto-covariance

The correlation function of the envelope $\hat{R}(\tau)$ can be calculated by taking the average value of the second-order probability density function, Equation 27. Since this equation is only valid for $\lambda(\tau) \equiv 0$, this restriction is held throughout this discussion of covariance. The integral

$$\hat{R}(\tau) = \iint\limits_0^\infty V_1 V_2 \, p_2 \left( V_1, V_2; \tau \right) dV_1 \, dV_2 \tag{40}$$

can be evaluated in different ways, as shown by Middleton (Reference 11). Because his results are incorrect, Appendix A gives one method in detail.

The simplest way follows after expanding $I_m \left[ \rho V_1 V_2 / \psi^2 \left( 1 - \rho^2 \right) \right]$ according to Appendix B, Equation B9, and using term-wise integration with the help of Equation B10. This yields

$$\hat{R}(\tau) = 2\psi^2 \left( 1 - \rho^2 \right)^2 e^{-2z\rho/1+\rho} \sum_{m=0}^\infty \epsilon_m \frac{\rho^m}{(m!)^2} \left( z \frac{1-\rho}{1+\rho} \right)^m \sum_{n=0}^\infty \frac{\rho^{2n} \Gamma^2 (m+n+1.5)}{n! \, (m+n)!}$$

$$\cdot \, {}_1F_1^{\,2} \left( -n - 0.5; m+1; -z \frac{1-\rho}{1+\rho} \right), \tag{41}$$

where ${}_1F_1 (a; b; z)$ is the confluent hypergeometric function, defined by

$$_1F_1 (a; b; z) = 1 + \frac{az}{b(1!)} + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots \tag{42}$$

The mean value of the received signal envelope may be obtained from the correlation function Equation 41, by setting $\tau = \infty$. With $\rho(\infty) = 0$, we find

$$\hat{R}(\infty) = 2\psi^2 \, \Gamma^2 (1.5) \, {}_1F_1^{\,2} (-0.5; 1; -z) = \hat{m}^2 . \tag{43}$$

The mean and mean-square values can also be calculated in a more direct way from the first-order probability density function, Equation 16. Equation B10 then gives

$$E\{V\} = \hat{m} = \int_0^\infty V p_1(V) \, dV = \psi \sqrt{\frac{\pi}{2}} \, _1F_1(-0.5; 1; -z) \tag{44a}$$

$$E\{V^2\} = \int_0^\infty V^2 p_1(V) \, dV = 2\psi^2(1 + z) . \tag{44b}$$

Equations 43 and 44a yield the same result for $\hat{m}$, because $\Gamma(1.5) = \sqrt{\pi}/2$.

The normalized covariance for the envelopes of the received complex function becomes, using Equation 13,

$$\hat{\rho}(\tau) = \frac{\hat{C}(\tau)}{\hat{C}(0)} = \frac{\hat{R}(\tau) - \hat{m}^2}{\hat{\sigma}^2} , \tag{45}$$

where

$$\hat{C}(0) = \hat{R}(0) - \hat{m}^2 = E\{V^2\} - E^2\{V\} = \hat{\sigma}^2 .$$

The correlation function, the mean value, and variance are defined in Equations 41, 44a, and 44b, respectively. Equation 45 shows a relationship between the normalized covariance $\hat{\rho}$ of the envelopes of the complex process and the magnitude of the normalized complex covariance $\rho$ (defined in Equation 39).

Since this relationship is rather complicated, we develop Equation 41 in a power series of $\rho$ up to $\rho^2$ to obtain an approximate value. This is also to compare the results with those calculated by Lawson and Uhlenbeck (Reference 6), where the correlation function is given also as a power series up to $\rho^2$. Combining

$$\hat{R} = A_0 + A_1 \rho + A_2 \rho^2 + \cdots$$

with

$$A_i = \frac{1}{i!} \left. \frac{\partial^i \hat{R}}{\partial \rho^i} \right|_{\rho=0} \qquad i = 0, 1, 2, \cdots$$

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and Equation B3, gives

$$A_0 = \hat{m}^2 \quad \text{(compare Equation 43)}$$

$$A_1 = \pi\psi^2 \, z\left[\frac{3}{2} \, {}_1F_1(-0.5; 2; -z) - {}_1F_1(-0.5; 1; -z)\right]^2 .$$

$$A_2 = \pi\psi^2 \left\{\left[\frac{3}{4} \, {}_1F_1(-0.5; 2; -z)\right]^2 - \frac{3}{4}(1+z) \, {}_1F_1(-0.5; 2; -z) \, {}_1F_1(0.5; 2; -z)\right.$$

$$\left. + \frac{1}{4}\left(z^2 + 2z + \frac{5}{4}\right) {}_1F_1{}^2(0.5; 2; -z)\right\} .$$

The result is still too complicated; it is convenient to consider separately the cases where $z < 1$ and $z > 1$. With the developments for the confluent hypergeometric functions, Equations B1 and B2, for small coherent parts of the transfer function ($z < 1$) Equation 45 becomes:

$$\hat{\rho} = \frac{\pi\left[\rho z\left(\frac{1}{4} - \frac{1}{8}z\right) + \rho^2\left(\frac{1}{8} - \frac{1}{8}z + \frac{11}{128}z^2\right)\right]}{2\left[1 + z - \frac{\pi}{4}\left(1 + z + \frac{1}{8}z^2\right)\right]}, \qquad z < 1, \ \rho < 1 . \tag{46}$$

For large coherent parts ($z > 1$) Equation 45 becomes

$$\hat{\rho} = \frac{\rho\left(1 - \frac{1}{2z} - \frac{1}{8z^2}\right) + \rho^2\left(\frac{1}{4z} - \frac{1}{8z^2}\right)}{1 - \frac{1}{4z} - \frac{1}{8z^2}}, \qquad z > 1, \ \rho < 1 . \tag{47}$$

In Reference 6, the approximations for the correlation function show only the first term in each expression proportional to $\rho$ and $\rho^2$, respectively. Figure 4 plots $\hat{\rho}$ versus $\rho$, as in Equations 46 and 47, for different values of $z$. The curves are valid only for $\rho < 1$, because only terms up to $\rho^2$ are taken into account. Due to the normalization, $\hat{\rho}$ must become 1 for $\rho = 1$ (which corresponds to $\tau = 0$) and for all values of $z$. For the covariance, it is always true that

$$\hat{\rho} \leq \rho , \tag{48}$$

whereas, for the correlation functions, $\hat{R} \geq |R|$ (see Equation 8).

For an exact calculation of $\hat{R}(\tau)$ Equation 41 is not very useful because it converges less as $\rho$ increases. Since the term $(1 - \rho^2)$ becomes zero for $\rho = 1$ ($\tau = 0$), the sum must become infinite
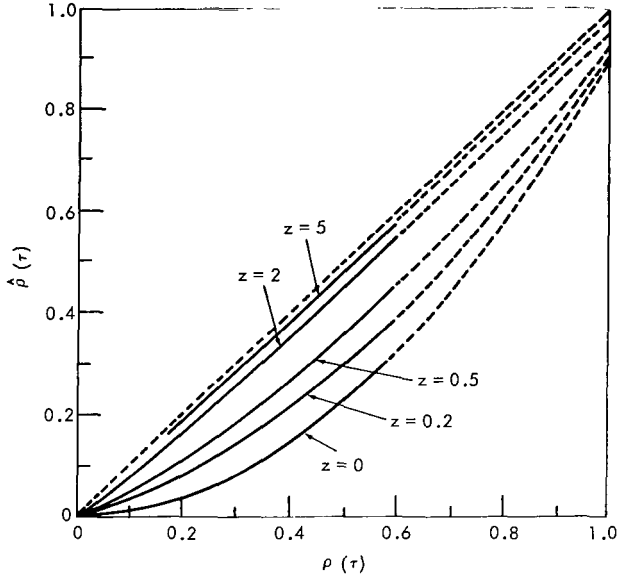
Figure 4—Normalized covariance of envelope versus normalized complex covariance for Rician fading. Approximation valid for $\rho < 1$.

to provide a finite value for $\hat{R}(0)$. For this reason, another way of evaluating the integral in Equation 40 will be used. This is achieved with the aid of the transformation (Reference 11)

$$V_1 = \psi \sqrt{1 - \rho^2} \sqrt{w}\, e^{\mu/2}$$

$$V_2 = \psi \sqrt{1 - \rho^2} \sqrt{w}\, e^{-\mu/2} \; , \qquad (49)$$

followed by the successive application of the expression for the product of two Bessel functions and using the integral form of the modified Bessel function of the second kind. Appendix A evaluates this integral. The correlation function in this case becomes

$$\hat{R}(\tau) = 2\psi^2 e^{-2z/1+\rho} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \frac{\rho^m z^{m+n}}{m!\,(1+\rho)^{2m+2n}}$$

$$\cdot \sum_{k=0}^{n} \frac{\Gamma(m+n-k+1.5)\,\Gamma(m+k+1.5)}{k!\,(n-k)!\,(m+k)!\,(m+n-k)!}\; {}_2F_1\left(-k-0.5,\,-n+k-0.5;\,m+1;\,\rho^2\right) \; , \qquad (50)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function, defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!}\, z^k \; , \qquad (51)$$

with

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) \; , \quad \text{and} \quad (\alpha)_0 = 1 \; .$$

For the limiting case of zero steady component ($z = 0$), Equation 50 reduces to

$$\hat{R}_0(\tau) = 2\psi^2\,\Gamma^2(1.5)\,{}_2F_1\left(-0.5,\,-0.5;\,1;\,\rho^2\right) \; . \qquad (52)$$

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The hypergeometric function $_2F_1$ may now be expressed in terms of the complete elliptic integrals of the first (K) and second (E) kind:

$$_2F_1 \left( -0.5, -0.5; 1; \rho^2 \right) = \frac{4}{\pi} E(\rho) - \frac{2}{\pi} \left( 1 - \rho^2 \right) K(\rho) . \tag{53}$$

We find

$$\hat{R}_0 (\tau) = \psi^2 \left[ 2E(\rho) - \left( 1 - \rho^2 \right) K(\rho) \right] , \tag{54}$$

a well known relationship for Rayleigh-envelope fading characteristics (References 2, 6, and 12). In this case the mean value, by Equation 44a, becomes

$$\hat{m}_0 = \psi \sqrt{\pi/2} , \tag{55}$$

and the variance, by Equation 44b, becomes

$$\hat{\sigma}_0^2 = E\{V^2\} - \hat{m}_0^2 = \psi^2 \left( 2 - \frac{\pi}{2} \right) . \tag{56}$$

These values can, of course, also be achieved from the general formulas:

$$R(\infty) = m^2$$

$$R(0) = E\{V^2\} . \tag{57}$$

From $\rho(0) = 1$, $\rho(\infty) = 0$, and Equations B7 and B8, Equation 52 yields

$$\hat{R}_0 (\infty) = \psi^2 \pi/2$$

$$\hat{R}_0 (0) = 2\psi^2 . \tag{58}$$

Figure 5 plots $\hat{R}$ (that is, $\hat{R}/2\psi^2$) versus $\rho$ for different values of $z$. The normalized covariance $\hat{\rho}$ for the Rician envelope statistic as a function of the magnitude of the complex normalized covariance $\rho$, as defined in Equation 45, is calculated from Equations 50 and 44a and b for different values of $z$ and shown in Figure 6. For Rayleigh distribution ($z = 0$), compare Schwartz (Reference 7, p. 479).

As expected, $\hat{\rho}$ and $\rho$ differ most for the Rayleigh statistic ($z = 0$). This difference decreases with increasing coherent part of the received wave form (increasing $z$). For $z \to \infty$, we find $\hat{\rho} \equiv \rho$.
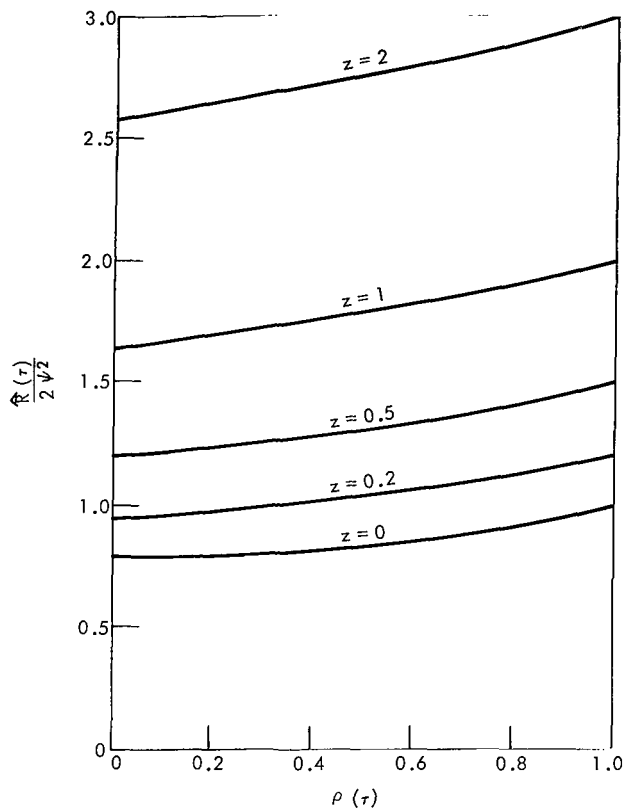
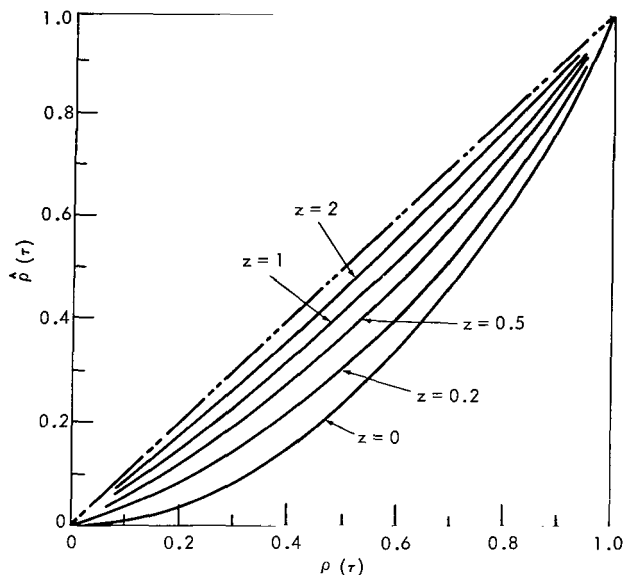Figure 5—Correlation function of envelope versus $\rho(\tau)$.



Figure 6—Normalized covariance of envelope versus normalized complex covariance for Rician envelope statistics.

According to the definition of $\hat{\rho}$ in Equation 45, this value must become 1 for $\rho = 1$, which obtains when delay $\tau = 0$, since $\hat{\sigma}^2 = \hat{R}(0) - \hat{m}^2$. Because of the monotonic relationship between $\hat{\rho}$ and $\rho$ it is possible to determine $\rho$, once $\hat{\rho}$ and the parameter $z$ are measured.

The parameter $z$ can be calculated by measuring the first moment (i.e., the mean value or dc component) and the second central moment (i.e., the mean square of the varying component or ac component) of the recieved signal envelope. Equations 44a and b show that both are functions of $\psi$ and $z$:

$$\hat{m} = E\{V\} = \psi\sqrt{\pi/2} \;\, {}_1F_1(-0.5; 1; -z)$$

$$\hat{\sigma}^2 = E\{V^2\} - \hat{m}^2 = \psi^2\left\{2(1+z) - \frac{\pi}{2}\,{}_1F_1^2(-0.5; 1; -z)\right\} \cdot \tag{59}$$

Once the ratio $\hat{m}^2/2\hat{\sigma}^2$ (which is only a function of $z$) is calculated, the parameter $z$ can be determined with the aid of the following equation:

$$\frac{\hat{m}^2}{2\hat{\sigma}^2} = \frac{1}{2}\left[\frac{4(1+z)}{\pi\,{}_1F_1^2(-0.5; 1; -z)} - 1\right]^{-1} \cdot \tag{60}$$

Figure 7 is a graph of Equation 60.

In case of Rayleigh fading ($z = 0$),

$$\frac{\hat{m}^2}{2\hat{\sigma}^2} = \frac{0.5}{\frac{4}{\pi} - 1} = 1.8299$$

and, for $z \to \infty$, Equation 60 becomes

$$\lim_{z \to \infty} \frac{\hat{m}^2}{2\hat{\sigma}^2} = z + 0.75 \; .$$

Since $\hat{\sigma}^2$ is the mean noise power at the output of the detector and $\hat{m}^2/2$ the mean signal power (1-ohm resistor), $\hat{m}^2/2\hat{\sigma}^2$ is simply the power S/N ratio at the output of the envelope detector. On the other hand, the parameter $z$ is a characteristic of the channel transfer function. It is defined (compare Equation 18) as the ratio of the mean-square values of the steady and fading components.



Figure 7—Output-power S/N ratio versus input-power S/N ratio for Rician fading and envelope detection.

## Cross-covariance

Here the results of the previous sections are extended to the correlation between two signals at different frequencies. Now we must calculate the cross-correlation function of two signals that do not necessarily have equal coherent signal components. Since the normalized complex covariance does not depend on the coherent components, Equation 38 is still valid for the correlation between two different signals. The modified correlation function (correlation function of the envelopes), on the other hand, depends on the coherent parts. This is because the second-order probability density for the envelopes is a function of the two parameters $z_1$ and $z_2$—the S/N ratios of the two different received signals (Reference 10), and is given by

$$P_2(V_1, V_2; \tau) = \frac{V_1 V_2}{\psi^4 (1 - \rho^2)} \exp\left[ -\frac{V_1^2 + V_2^2}{2\psi^2 (1 - \rho^2)} + \frac{z_1 + z_2}{1 - \rho^2} - 2\sqrt{z_1 z_2}\, \frac{\rho}{1 - \rho^2} \right]$$

$$\cdot \sum_{m=0}^{\infty} \epsilon_m I_m\left( \frac{\rho V_1 V_2}{\psi^2 (1 - \rho^2)} \right) I_m\left( \frac{\sqrt{2}}{\psi}\, \frac{\sqrt{z_1} - \rho\sqrt{z_2}}{1 - \rho^2}\, V_1 \right) I_m\left( \frac{\sqrt{2}}{\psi}\, \frac{\sqrt{z_2} - \rho\sqrt{z_1}}{1 - \rho^2}\, V_2 \right) . \quad (61)$$

To obtain the modified cross-correlation function, we can use the method outlined above for the auto-correlation function. The integral in Equation 40 in conjunction with Equation 61 can be solved in the way that is shown in Appendix A. It is convenient to use the two parameters
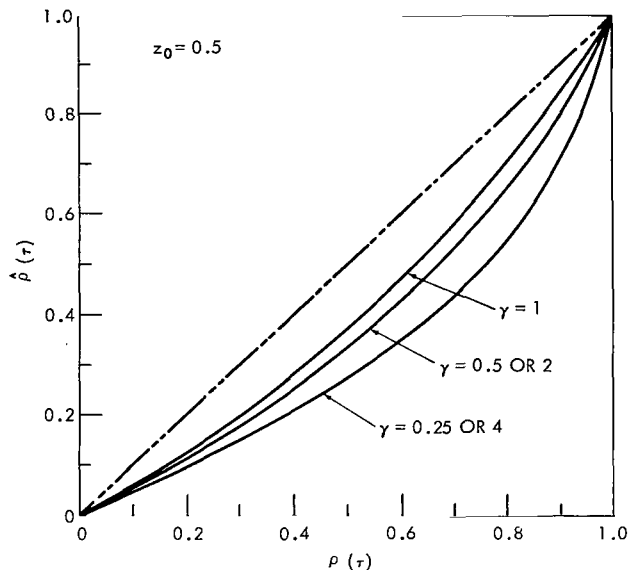
$$z_0 = \sqrt{z_1 z_2}$$

$$\gamma = \sqrt{z_1/z_2} \qquad (62)$$

instead of $z_1$ and $z_2$. The cross-correlation for the envelopes becomes

$$\hat{R}_c(\tau) = 2\psi^2 \exp\left[-z_0 \frac{\gamma + 1/\gamma - 2\rho\gamma}{1-\rho^2}\right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \frac{\rho^m}{m!} z_0^{m+n}$$

$$\cdot \frac{\left[(1-\rho/\gamma)(1-\rho\gamma)\right]^m \left[(\gamma-\rho)^2/\gamma\right]^n}{\left(1-\rho^2\right)^{2m+2n}} \cdot \sum_{k=0}^{n} \left(\frac{1-\rho\gamma}{\gamma-\rho}\right)^{2k} \frac{\Gamma(m+n-k+1.5)\,\Gamma(m+k+1.5)}{k!\,(n-k)!\,(m+k)!\,(m+n-k)!}$$

$$\cdot \; _2F_1\left(-k-0.5,\; -n+k-0.5;\; m+1;\; \rho^2\right) . \qquad (63)$$

We obtain the relationship between the normalized complex cross-covariance $\rho$ and the normalized cross-covariance of the envelopes $\hat{\rho}_c(\tau)$ by

$$\hat{\rho}_c(\tau) = \frac{\hat{R}_c(\tau) - \hat{m}_1 \hat{m}_2}{\hat{\sigma}_1 \hat{\sigma}_2}, \qquad (64)$$

where $\hat{R}_c(\tau)$ is given in Equation 63, and the mean values $\hat{m}_i$ and standard deviations $\hat{\sigma}_i$ are specified in Equation 44. The subscript values $i = 1, 2$ refer to $z_1$ and $z_2$, respectively.
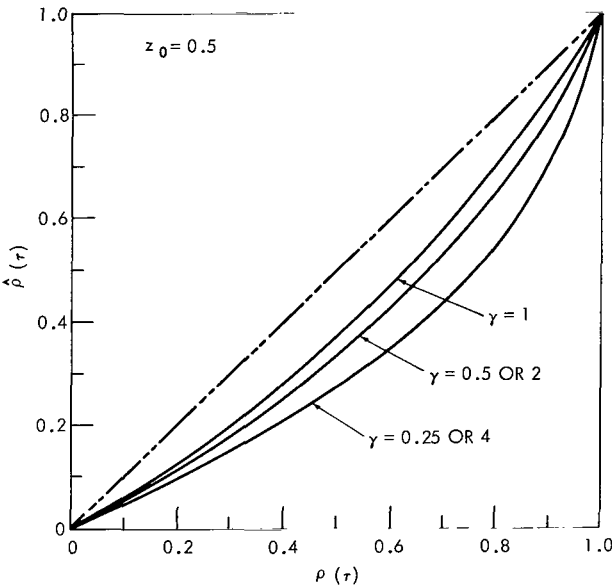
It should be mentioned that $\gamma = 1$ represents the case $z_1 = z_2 = z_0$ (e.g. auto-correlation The parameters $\gamma$ and $z_0$ may be calculated from Equation 62 by measuring the dc and ac components of the two received signals and using Figure 7. Figure 8 is a graph of Equation 64 for $z_0 = 0.5$ and different values of $\gamma$. As expected, the curves for given values of $\gamma$ and $1/\gamma$ are exactly the same. This can also be shown by replacing $\gamma$ by $1/\gamma$ in Equation 63. This leads to

$$\hat{R}_c(\tau, \gamma) = \hat{R}_c(\tau, 1/\gamma) .$$

The modified covariance is highest for $\gamma = 1$ (auto-covariance) and becomes smaller if $\gamma$ decreases or increases (different steady components of the two received signals). Note that



Figure 8— $\hat{\rho}(\tau)$ versus $\rho(\tau)$ for $z_0 = 0.5$ and different values of $\gamma$.

cross-covariance is normalized with respect to the variance of the two received signals and not to some carrier variance.

## Channel Parameters

The preceding consideration shows that for Rician fading channels the following expression is always valid (compare Figures 6 and 8 and see Equation 48):

$$\hat{\rho}(\tau) \leq \rho(\tau) \ . \tag{65}$$

The difference $\rho - \hat{\rho}$ becomes smaller with increasing coherent signal component (increasing $z$).

If the modified covariance functions are used to calculate the coherence parameters of the fading channel (i.e., no correction according to the correspondence between the two covariance functions mentioned above is used), the coherence time $\hat{T}_c$ and the coherence bandwidth $\hat{F}_0$ are always less than the values calculated from the complex covariance:

$$\hat{T}_c \leq T_c$$

$$\hat{F}_c \leq F_c \ . \tag{66}$$

Figure 9 is a 2-dimensional section $(\rho, \tau)$ of a 3-dimensional figure with axes: $\rho$, $\tau$, and $\Omega$ (it shows the case for $\Omega = 0$). $T_c$ and $\hat{T}_c$ are shown, but not $F_c$ and $\hat{F}_c$ (which would require a $\rho$, $\Omega$ graph).
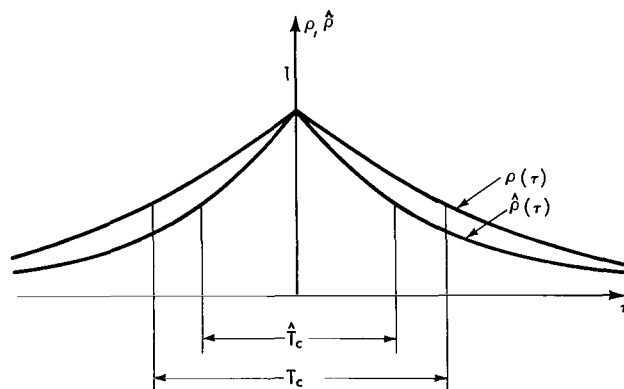


Figure 9—Definition of coherence time and relationship between modified and complex covariance.

On the other hand, the frequency spread $\hat{B}$ and the time spread $\hat{L}$ of the channel are wider than their actual values

$$\hat{B} \geq B \ ,$$

$$\hat{L} \geq L \ . \tag{67}$$

since they are derived from the scattering function, which is a Fourier transform of the two-dimensional normalized covariance.

Few experiments seem to have been made to verify the usefulness of the theoretically derived channel parameters, particularly in the millimeter region of the electromagnetic spectrum. If the results achieved from the Millimeter Wave Propagation Experiment on board the spacecraft ATS-E are

corrected as outlined in the previous chapters, more useful data about the time- and frequency-selective fading properties of the channel between satellite and ground terminal are obtained. Although the stability of the instrumented oscillator allows only the measurement of the modified covariance, the magnitude of the complex covariance for the channel-transfer function can be calculated if the received random signal has the expected Rician probability distribution. From the magnitude of the two-dimensional complex covariance function, the coherence parameters of the channel are characterized and the communication-signal propagation effects of the channel can be determined. The corrected data represent a better description of the fading channel than the data achieved from the measured modified covariance, and allow better and more accurate planning and design of future communication links through the atmosphere for millimeter waves.

## RANDOM SIGNAL WITH UNSYMMETRIC SPECTRUM

The modified correlation function in the preceding sections was calculated for a symmetric power spectrum. In the more general situation where $\lambda(\tau) \neq 0$ (spectrum not symmetrical in the narrow band about center frequency) the second-order probability density function is much more complicated. By introducing polar coordinates in Equation 21 and integrating over all phases (with the aid of Appendix B, Equation B11), we finally obtain

$$P_2\left(V_1, V_2; \tau\right) = \frac{V_1 V_2}{\psi^4\left(1-\rho^2\right)} \exp\left[-\frac{2z(1-\rho\cos\phi)}{1-\rho^2} - \frac{V_1^2 + V_2^2}{2\psi^2\left(1-\rho^2\right)}\right]$$

$$\cdot \sum_{m=0}^{\infty} \epsilon_m I_m\left(\frac{\rho V_1 V_2}{\psi^2\left(1-\rho^2\right)}\right) I_m\left(bV_1\right) I_m\left(bV_2\right) \cos\left(2\xi - \phi\right) , \quad \text{for} \quad V_1, V_2 \geq 0 , \qquad (68)$$

where

$$z = \left(\alpha^2 + \beta^2\right)\big/ 2\psi^2 ,$$

$$b = \frac{\sqrt{2z}}{\psi} \sqrt{\frac{1+\rho^2 - 2\rho\cos\phi}{1-\rho^2}} ,$$

$$\xi = \arctan\frac{\rho\sin\phi}{\rho\cos\phi - 1} ,$$

and, from Equation 24,

$$\rho(\tau) = \left[r^2(\tau) + \lambda^2(\tau)\right]^{1/2}$$

$$\phi(\tau) = \arctan\frac{\lambda(\tau)}{r(\tau)} . \qquad (69)$$

Equation 68 can be found in Middleton (Reference 10) but there is a sign error that leads to a $\cos \phi$ term instead of $\cos (2\xi - \phi)$. For $\lambda = 0$, Equation 68 reduces, of course, to Equation 27.

To obtain the auto-correlation function of the envelopes, we use the same procedure as shown in Appendix A for the case $\lambda = 0$. This leads to

$$\hat{R}_\phi (\tau) \;\mathrel{\widehat{=}}\; 2\psi^2 \exp \left[ -2z \; \frac{1 - \rho \cos \phi}{1 - \rho^2} \right] \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \; \frac{\rho^m \, z^{m+n} \left( 1 + \rho^2 - 2\rho \cos \phi \right)^{m+n}}{m! \left( 1 - \rho^2 \right)^{2m+2n}}$$

$$\cdot \cos \left[ m(2\xi - \phi) \right] \sum_{k=0}^{n} \frac{\Gamma(m + n - k + 1.5) \, \Gamma(m + k + 1.5)}{k! \, (n-k)! \, (m+n-k)! \, (m+k)!} \; {}_2F_1 \left( -k - 0.5, \; -n + k - 0.5; \; m + 1; \; \rho^2 \right) . \quad (70)$$

The correlation function for a symmetric spectrum is a special case of Equation 70, and we obtain Equation 50 by setting $\phi = 0$. Since the first and second moments do not depend on $\phi$, they are the same as shown in Equations 44a and b.

The graphs showing the normalized modified covariance versus complex covariance are similar to those in Figure 8. The modified correlation function and therefore the covariance of the envelopes, becomes smaller with increasing $\phi$ (or $\lambda$); this is due to the decorrelation effect of the asymmetric spectrum. Since $\hat{R}_\phi$ is an even function of $\phi$, the same results are obtained for $\pm \phi$ or $\pm \lambda$. A numerical calculation shows that the difference between the curves for $\phi = 0$ and $|\phi| = \pi/8$ is less than 7.5 percent for all values of $\rho$.

If the spectrum is asymmetric about the center frequency, the complex covariance cannot be calculated by measuring of $\hat{\rho}(\tau)$ alone, since the angle as well as the absolute value of the complex covariance are unknown. The envelope detection yields insufficient information about the complex correlation function. Only for the Rayleigh fading channel ($z = 0$), the above equations become independent of $\phi$. Therefore, in this case there is only one relationship between $\hat{\rho}$ and $\rho$ valid for all values of $\phi$. In Rician fading channels ($z \neq 0$) we must assume that $\lambda = 0$ in order to have a definite relationship between $\hat{\rho}$ and $\rho$. As already mentioned, this assumption is quite reasonable for the received narrow-band signal.

## CONCLUSIONS

This paper describes the properties of the modified correlation functions and presents a comparison of the modified and complex correlation function for the Rician fading channel. The modified correlation function, i.e., the correlation function of the envelope of the received signal, is considerably easier to measure than the complex function, and, in cases where the long-term stability of the measurement system oscillators is not good enough to determine the complex correlation function, is the only function that can be measured.

It has been shown that for Rician fading statistics the modified covariance is a lower bound of the magnitude of the complex covariance. This leads to a useful estimate of the coherence parameters for the fading channel, as determined by the modified functions (under "Channel Parameter").

Because the relationship between $\hat{\rho}(\tau)$ and $\rho(\tau)$ is monotonic and single-valued if the power spectrum of the received narrow-band process is symmetric about the center frequency, we can determine the magnitude of the complex covariance $\rho(\tau)$ by measuring the modified covariance $\hat{\rho}(\tau)$. To test the equivalence of the probability density function for the sampled data to the theoretical Rician probability density function the so-called chi-square goodness-of-fit test described by Bendat and Piersol (Reference 13) can be used. As pointed out under "Auto-covariance," the S/N ratio $z$ can be calculated, once the mean value and the mean-square value of the received signal are measured. Since, in the relationship between $\rho(\tau)$ and $\hat{\rho}(\tau)$ (Equations 45 and 50), $\rho(\tau)$ is the independent variable, a numerical approximation method or the graphs in Figure 6 can be used to calculate $\rho(\tau)$ from the measured value $\hat{\rho}(\tau)$. As expected, the difference between $\rho$ and $\hat{\rho}$ becomes smaller with increasing coherent part of the signal. The greatest difference can be found for zero coherent part, i.e., Rayleigh fading. The coherence parameters can then be found from the magnitude of the complex covariance $\rho(\tau)$, and provide a better knowledge about the properties of the fading channel. For design of future communication links, these corrected data are more useful than those calculated directly from the modified covariance.

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## Appendix A

## Evaluation of Integral of Equation 40

To obtain the auto-correlation function for the envelopes of the received random signal, we must solve the integral

$$\hat{R}(\tau) = \int\!\!\!\int_0^\infty V_1 V_2 p_2 \left(V_1, V_2; \tau\right) dV_1 dV_2 \ , \tag{A1}$$

where $p_2$ is given in Equation 27. To evaluate this integral we use the transformation

$$V_1 = \psi\sqrt{1-\rho^2}\ w^{1/2}\ e^{\mu/2} \ ,$$

$$V_2 = \psi\sqrt{1-\rho^2}\ w^{1/2}\ e^{-\mu/2} \ , \tag{A2}$$

where $0 \leq w < \infty$ and $-\infty < \mu < \infty$. The jacobian of this transformation from $V_1$ and $V_2$ to $w$ and $\mu$ is

$$J = \left|\frac{\partial\left(V_1, V_2\right)}{\partial\left(w, \mu\right)}\right| = \frac{1}{2}\ \psi^2\left(1-\rho^2\right) \ . \tag{A3}$$

In conjunction with Equation A2, the arguments of the Bessel functions in Equation 27 become

$$\frac{\rho V_1 V_2}{\psi^2\left(1-\rho^2\right)} = \rho w \ ,$$

$$\frac{\sqrt{2z}\ V_1}{\psi\ (1+\rho)} = s w^{1/2}\ e^{\mu/2} \ ,$$

$$\frac{\sqrt{2z}\ V_2}{\psi(1+\rho)} = s w^{1/2}\ e^{-\mu/2} \ ,$$

with

$$s = \sqrt{2z}\ \frac{\sqrt{1-\rho^2}}{1+\rho} \ . \tag{A4}$$

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In addition, we need

$$V_1 V_2 = \psi^2 \left(1 - \rho^2\right) w \, ,$$

$$\frac{V_1^2 + V_2^2}{2\psi^2 \left(1 - \rho^2\right)} = w \cosh \mu \, . \tag{A5}$$

The integral in Equation A1 therefore becomes

$$\hat{R} = \frac{1}{2} \psi^2 \left(1 - \rho^2\right)^2 e^{-2z/1+\rho} \sum_{m=0}^{\infty} \epsilon_m \int_0^{\infty} dw \int_{-\infty}^{\infty} d\mu \, w^2 \, e^{-w \cosh \mu} \cdot I_m \left(\rho w\right) I_m \left(sw^{1/2} e^{\mu/2}\right) I_m \left(sw^{1/2} e^{-\mu/2}\right) \cdot \tag{A6}$$

The product of two modified Bessel functions can be written with the aid of Watson (Reference 14, p. 148, Equation 2) as

$$I_m \left(sw^{1/2} e^{\mu/2}\right) I_m \left(sw^{1/2} e^{-\mu/2}\right) = \frac{\left(\frac{s^2}{4} w\right)^m}{m!} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} sw^{1/2} e^{\mu/2}\right)^{2n}}{n! \, (m+n)!} \cdot {}_2F_1 \left(-n, \, -m-n; \, m+1; \, e^{-2\mu}\right) \, . \tag{A7}$$

For the hypergeometric function we use the power series representation, Equation 51:

$$_2F_1 \left(-n, \, -m-n; \, m+1; \, e^{-2\mu}\right) = \sum_{k=0}^{\infty} \frac{(-n)_k \, (-m-n)_k}{k! \, (m+1)_k} \, e^{-2k\mu} \, . \tag{A8}$$

Since $n$ is an integer, $(-n)_k$ becomes zero for $k \geq n + 1$. Therefore the series in Equation A8 consists of a finite number of terms. The upper limit in the sum can be replaced by $n$.

The integration over the variable $\mu$, which may be written in the following form, if we use Watson (Reference 14, p. 182, Equation 7) yields

$$\int_{-\infty}^{\infty} e^{-(2k-n)\mu - w \cosh \mu} \, d\mu = 2K_{2k-n} (w) \, , \tag{A9}$$

where $K_{2k-n}$ is the modified Bessel function of the second kind and order $(2k - n)$. Now we must evaluate the integral over $w$. Using Watson (Reference 14, p. 410, Equation 1) we find

$$\int_0^{\infty} dw \, w^{2+m+n} I_m (\rho w) K_{2k-n} (w) = \rho^m \frac{\Gamma(m+n-k+1.5) \, \Gamma(m+k+1.5)}{2^{-1-m-n} \, \Gamma(m+1)} \cdot {}_2F_1 \left(m+n-k+1.5, \, m+k+1.5; \, m+1; \, \rho^2\right) \cdot \tag{A10}$$

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With the relation

$$_2F_1 (\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} \, _2F_1 (\gamma - \beta, \gamma - \alpha; \gamma; x) ,$$

(A11)

and, by suitable rearrangement of the above formulas, the auto-correlation function becomes finally

$$\hat{R}(\tau) = 2\psi^2 \, e^{-2z/1+\rho} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \frac{\rho^m \, z^{m+n}}{m! \, (1 + \rho)^{2m+2n}}$$

$$\cdot \sum_{k=0}^{n} \frac{\Gamma(m + n - k + 1.5) \, \Gamma(m + k + 1.5)}{k! \, (n - k)! \, (m + k)! \, (m + n - k)!} \, _2F_1 \left(-k - 0.5, \, -n + k - 0.5; \, m + 1; \, \rho^2 \right) .$$

(A12)

Note that in Reference 11 the factor $m!$ as denominator is missing.

## Appendix B

## Special Functions and Integrals

This appendix is an assembly of some mathematical results used throughout this report. Other functions and integrals, already mentioned, will be omitted.

Hypergeometric Functions:

The confluent hypergeometric function is defined by

$$_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!},$$ 
(B1)

where $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$, and $(a)_0 = 1$. The asymptotic series for $_1F_1(a; b; z)$ for large values of $z$ is

$$_1F_1(a; b; -z) \sim \frac{\Gamma(b) z^{-a}}{\Gamma(b-a)} \sum_{m=0}^{\infty} \frac{(a)_m (a-b+1)_m}{m! \, z^m}$$

$$\sim \frac{\Gamma(b) z^{-a}}{\Gamma(b-a)} \left[ 1 + \frac{a(a-b+1)}{z} + \cdots \right].$$ 
(B2)

There exists a set of recurrence relations, which have been used in the major section on covariance:

$$-z F_{11} = b F_{00} - b F_{10},$$

$$a F_{11} = (a-b) F_{01} + b F_{00},$$

$$ab F_{10} = b(a+z) F_{00} - z(b-a) F_{01},$$

$$(b-a) z F_{01} = b(z+b-1) F_{00} + b(1-b) F_{0-1},$$ 
(B3)

where the symbol $F_{k\ell}$ is an abbreviation defined by

$$F_{k\ell} = {}_1F_1(a+k; b+\ell; z).$$

The relation

$$e^{-z} \, {}_1F_1(a; b; z) = {}_1F_1(b-a; b; -z)$$ (B4)

is also needed in the discussion of covariance.

The Gaussian hypergeometric function is represented by

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n , \qquad |z| \leq 1 .$$ (B5)

Useful relations are

$$_2F_1(a, b; c; z) = (1-z)^{c-a-b} \, {}_2F_1(c-b, c-a; c; z)$$ (B6)

$$_2F_1(a, b; c; 1) = \frac{\Gamma(c) \, \Gamma(c-a-b)}{\Gamma(c-a) \, \Gamma(c-b)} , \qquad \mathrm{Re}\{c\} \neq 0, -1, -2, \cdots$$

$$\mathrm{Re}\{c-a-b\} \neq 0, -1, -2, \cdots$$ (B7)

and

$$_2F_1(a, b; c; 0) = 1 .$$ (B8)

Modified Bessel Function:

The modified Bessel function (Bessel function of purely imaginary argument) of the first kind is defined by the power series

$$I_m(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} z\right)^{m+2n}}{n! \, (m+n)!} .$$ (B9)

The following integral, used in the discussion of covariance, can be derived from Hankel's exponential integral; compare Watson (Reference 14, p. 394).

$$\int_0^{\infty} z^{\kappa-1} I_\nu(az) e^{-q^2 a^2} dz = \frac{\Gamma\left(\frac{\nu+\kappa}{2}\right) \left(\frac{1}{2} \frac{a}{q}\right)^\nu}{2q^\kappa \, \Gamma(\nu+1)} \left[\exp\left(\frac{a^2}{4q^2}\right)\right] {}_1F_1\left(\frac{\nu-\kappa}{2}+1; \nu+1; -\frac{a^2}{4q^2}\right) .$$ (B10)

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To solve an integral of the form

$$\int \exp(a \cos \theta) \, d\theta \ ,$$

we can use the following series development:

$$\exp(a \cos \theta) = \sum_{m=0}^{\infty} \epsilon_m I_m(a) \cos(m\theta) \ . \tag{B11}$$

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