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FLUCTUATIONS IN MONATOMIC GASES*

by

David Montgomery

Department of Physics and Astronomy
THE UNIVERSITY OF IOWA
Iowa City, Iowa

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Errata for "Fluctuations in Monatomic Gases"

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1. page 4, last equation before bottom of page; replace $\delta(\vec{v}-\vec{v}_i(t_1))$ by $\delta(\vec{v}_1-\vec{v}_i(t_1))$.

2. page 19, beginning with last new paragraph and continuing on through page 20; replace t by τ everywhere it appears (a total of nine replacements).

3. page 21, Equations (22) and (24); replace $\frac{\partial}{\partial\tau}$ by $\frac{\partial}{\partial t}$.

4. page 21, middle equation should be numbered (23).

5. page 24, Equation (26); replace t by τ everywhere (a total of nine replacements).

ABSTRACT

We treat the problem of calculating distribution-function autocorrelations of the form $\langle f(\vec{x}_1,\vec{v}_1,\vec{t}_1)f(\vec{x}_2,\vec{v}_2,\vec{t}_2)\rangle$ for a dilute monatomic gas. Two-time probability distributions of the type introduced by Rostoker for the plasma case are used. A perturbation expansion in the density is performed on the generalized BBGKY hierarchy which results. It is shown that the problem of determining the fluctuation spectra can be reduced to solving for a function which obeys the linearized kinetic equation for a dilute neutral gas with a particular choice of initial conditions, a result previously obtained by van Leeuwen and Yip, using diagrammatic perturbation theory. In the limit of infinite wavelengths and hard-sphere interactions, this equation <u>reduces</u> to the linearized Boltzmann equation.

I. INTRODUCTION

Scattering from many-particle systems is often governed by correlation functions referred to two points which are separated in time as well as in space.[1] For example, in a wide class of scattering experiments one measures the <u>square</u> of some scattered "amplitude" $S(\vec{x},t)$, where

$$S(\vec{x},t) = \int d\vec{x}_1 dt_1 \; I(\vec{x}_1,t_1) G(\vec{x}-\vec{x}_1, t-t_1) n(\vec{x}_1,t_1).$$

$I(\vec{x}_1,t_1)$ is the incident "amplitude", $G(\vec{x}-\vec{x}_1, t-t_1)$ is some known Green's function, and $n(\vec{x}_1,t_1)$ is the number density of a volume distribution of scatterers. The integration $\int d\vec{x}_1$ runs over the scattering volume, and the $\int dt_1$ over the duration of the incident beam of particles or waves.

Exact calculations of S^2 are usually far too hard to carry out, and so one ends up averaging it over an appropriate <u>ensemble</u> of distributions of scatterers. We shall denote such ensemble averages by a bracket $\langle \;\; \rangle$. Since I and G are assumed to be the same for all members of the ensemble,

$$\langle S^2(\vec{x},t) \rangle = \int d\vec{x}_1 dt_1 \int d\vec{x}_2 dt_2 \; I(\vec{x}_1,t_1) \; I(\vec{x}_2,t_2) G(\vec{x}-\vec{x}_1,t-t_1) \cdot$$

$$G(\vec{x}-\vec{x}_2,t-t_2) \; \langle n(\vec{x}_1,t_1)n(\vec{x}_2,t_2) \rangle .$$

It is clear from this expression that the quantity of central theoretical interest is the number density auto-correlation $\langle n(\vec{x}_1,t_1)n(\vec{x}_2,t_2) \rangle$.

The calculation of $\langle n(\vec{x}_1,t_1)n(\vec{x}_2,t_2)\rangle$ for classical many-particle systems of point particles, in turn, depends upon being able to calculate the auto-correlation of the particle distribution function $f(\vec{x},\vec{v},t)$, because

$$\langle n(\vec{x}_1,t_1)n(\vec{x}_2,t_2) \rangle = n_o^2 \int d\vec{v}_1 d\vec{v}_2 \; \langle f(\vec{x}_1,\vec{v}_1,t_1) f(\vec{x}_2,\vec{v}_2,t_2) \rangle ,$$

where the exact distribution for N particles is

$$f(1) \equiv f(\vec{x}_1,\vec{v}_1,t_1) = n_o^{-1} \sum_{i=1}^{N} \delta(\vec{x}_1-\vec{x}_i(t_1))\,\delta(\vec{v}-\vec{v}_i(t_1))$$

The average number density is $n_o=N/V$, V is the (arbitrarily large) volume occupied by the scatterers, and $\vec{x}_i(t),\vec{v}_i(t)$ is the instantaneous phase space location of the $i^{\underline{th}}$ particle at time t.

For thermal equilibrium systems, we must compute $\langle f(1)f(2)\rangle$

over an ensemble which is the Gibbs distribution. The evaluation of this average for dilute monatomic gases is the subject of this paper.

The same problem was considered some time ago for plasmas by Rostoker,[2] Dougherty and Farley,[3] Salpeter,[4] and others. Recently, it has been considered for neutral gases by van Leeuwen and Yip,[5] Yip and Nelkin,[6] and Gross[7] (for a comprehensive bibliography, see also Gross and Wisnivesky[8]). In both situations, the calculation of $\langle f(1)f(2) \rangle$ resolves itself into the solution of the linearized version of an appropriate kinetic equation.

The proofs of Rostoker[2] and van Leeuwen and Yip[5] look as dissimilar as BBGKY theory and diagrammatic perturbation theory often do. The purpose here is, first, to re-cast Rostoker's formalism in a sufficiently general way to include other physical limits to which the BBGKY approach has been successfully applied to compute single-time ensemble averages (for example, the low-density limit, or weak-coupling limit). Secondly, we then specialize to the low-density limit, and give what (to the author, at least) appears to be a more intuitively accessible treatment of the problem of van Leeuwen and Yip.[5]

The problem of calculating $\langle f(1)f(2) \rangle$ is shown to be equivalent to the solution of the linearized version of the kinetic equation for dilute gases. As the authors remark,[5] this equation is not quite the Boltzmann equation, but reduces to it for hard-sphere interactions and long wavelengths. This observation was also made by Bogolyubov[9] in 1946 when the equation first appeared in a different context.

II. THE TWO-TIME BBGKY HIERARCHY

For shorthand, we introduce the notation

$$X \equiv X_1 X_2 \ldots X_N \equiv \vec{x}_1 \vec{v}_1 \vec{x}_2 \vec{v}_2 \ldots \vec{x}_N \vec{v}_N$$

as the 6N-dimensional vector which completely specifies the phase space location of the N-particle system. The probability of finding a member of the ensemble at X is

$$D_N(X) \equiv \frac{\exp(-E_N/\theta)}{\int dX \exp(-E_N/\theta)} , \tag{1}$$

where

$$E_N \equiv \sum_{i=1}^{N} \tfrac{1}{2} m \vec{v}_i^{\,2} + \sum_{i<j=1}^{N} \varphi_{ij} .$$

The particle mass is m, the two-particle energy of interaction is $\varphi_{ij}=\varphi(|\vec{x}_i-\vec{x}_j|)$ and θ is the temperature in energy units. The $\int dX$ is a 6N-fold integration which runs over all the phase space accessible to the system. Clearly $\int dX D_N=1$, and D_N is time-independent.

The joint probability Δ_N introduced by Rostoker is the probability of finding the system at X at time t <u>and</u> at X′ at time t+τ:

$$\Delta_N(X,t;X',t+\tau)=D_N(X)\delta(X'-X(\tau)). \tag{2}$$

$X(\tau) \equiv X_1(\tau)\ldots X_N(\tau)$ is that solution of the N-particle equations of motion which passes through X at time $\tau = 0$. The delta-function is an abbreviation:

$$\delta(X'-X(\tau)) = \prod_{i=1}^{N} \delta(X_i'-X_i(\tau)) = \prod_{i=1}^{N} \delta(\vec{x}_i'-\vec{x}_i(\tau))\delta(\vec{v}_i'-\vec{v}_i(\tau)).$$

Δ_N will depend on τ only, not on the absolute location of t, and is normalized so that

$$\int dX \int dX' \Delta_N(X,t;X',t+\tau) = 1$$

for all τ.

In terms of Δ_N we may express the probability of finding particles $1,2,\ldots,r$ at $X_1X_2\ldots X_r$ and particles $1,2,\ldots,s$ at $X_1'X_2'\ldots X_s'$, τ seconds later, <u>regardless</u> of the coordinates of the other particles:

$$\underset{12\ldots r,12\ldots s}{W}(X_1X_2\ldots X_r;X_1'X_2'\ldots X_s';\tau) \equiv V^{r+s}\int\Delta_N dX_{r+1}\ldots dX_N dX_{s+1}'\ldots dX_N'. \tag{3}$$

These **W**-functions play a role analogous to the reduced probability distributions of the usual BBGKY theory,[9, 10] which are defined (for equilibrium <u>or</u> non-equilibrium) by

$$f_s(X_1\ldots X_s) = V^s\int D_N dX_{s+1}\ldots dX_N.$$

The **W**-functions are symmetric under the simultaneous interchanges

$$X_k \leftrightarrow X_j$$

$$X_k' \leftrightarrow X_j'$$

but not under either interchange separately. Such complexities (beyond those in the usual BBGKY theory) as exist largely stem from this lack of symmetry.

Δ_N obeys the Liouville equation in the primed variables, which we write as

$$\left(\frac{\partial}{\partial \tau} + H_N'\right) \Delta_N = 0, \tag{4}$$

where for any value of n,

$$H_n' \equiv \sum_{i=1}^{n} \vec{v}_i' \cdot \frac{\partial}{\partial \vec{x}_i'} - \frac{1}{m} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{\partial \varphi'_{ij}}{\partial \vec{x}_i'} \cdot \frac{\partial}{\partial \vec{v}_i'} \, . \tag{5}$$

Primes on functions or operators will in general indicate that they are functions of the X_i' rather than the X_i.

By integrating the Liouville equation over $X_{r+1} \ldots X_N$ and $X_{s+1}' \ldots X_N'$, a chain of equations with a structure similar to that of the BBGKY hierarchy results. In practice we do not need all of these, for it is readily shown that

$$\langle\, f(\vec{x}_a, \vec{v}_a, t)\; f\; (\vec{x}_b, \vec{v}_b, t+\tau)\, \rangle \equiv \langle\, f(a) f(b)\, \rangle$$

$$= \int dX D_N\; f(\vec{x}_a, \vec{v}_a, t)\; f\; (\vec{x}_b, \vec{v}_b, t+\tau)$$

$$= \int dX dX' \Delta_N \sum_{i=1}^{N} \frac{\delta(X_a - X_i)}{n_o} \sum_{j=1}^{N} \frac{\delta(X_b - X_j')}{n_o}$$

$$= W_{1,2}(X_a;X_b;\tau) + \frac{1}{N} W_{1,1}(X_a;X_b;\tau) \tag{6}$$

for any two points $X_a = \vec{x}_a, \vec{v}_a$ and $X_b = \vec{x}_b, \vec{v}_b$. Here $W_{1,2} \equiv \int \frac{dX_1'}{V} W_{1,12}$.

The problem is thus reduced to getting satisfactory approximations to $W_{1,1}$ and $W_{1,12}$.

Integrating Eq. (4) over $X_2 \ldots X_N$ and $X'_{s+1} \ldots X'_N$ gives, upon multiplying by V^{1+s},

$$\left(\frac{\partial}{\partial\tau} + H_s'\right) \underset{1,12\ldots s}{W}(X_1;X_1'\ldots X_s';\tau) = L_s' \underset{1,12\ldots s\ s+1}{W}(X_1;X_1'\ldots X_s'X_{s+1}';\tau). \tag{7}$$

The operator H_s' is defined by Eq. (5) and L_s' by $L_s' = \sum_{i=1}^{s} L_1'(i,s+1)$, where

$$L_1'(i,s+1) \equiv \frac{n_o}{m} \int dX_{s+1}' \, \frac{\partial\varphi_{i,s+1}'}{\partial\vec{x}_i'} \cdot \frac{\partial}{\partial\vec{v}_i'} \, .$$

(We have ignored s/N compared to unity -- i.e., we assume the "thermodynamic limit" of $N \to \infty$, $V \to \infty$, $N/V = n_o$). Eqs. (7) are quite similar to the usual BBGKY equations.

The first two members are:

$$\left\{\frac{\partial}{\partial\tau} + H_1'(1)\right\} W_{1,1}$$

$$= \left\{\frac{\partial}{\partial\tau} + \vec{v}_1'\cdot\frac{\partial}{\partial\vec{x}_1'}\right\} \quad W_{1,1}(X_1;X_1';\tau)$$

$$= \frac{n_o}{m}\frac{\partial}{\partial\vec{v}_1'} \cdot \int dX_2' \quad \frac{\partial\varphi_{12}'}{\partial\vec{x}_1'} \quad W_{1,12}(X_1;X_1'X_2';\tau) \tag{8}$$

and

$$\left\{\frac{\partial}{\partial\tau} + H_2'\ (1,2)\right\} W_{1,12}$$

$$= \left\{\frac{\partial}{\partial\tau} + \vec{v}_1'\cdot\frac{\partial}{\partial\vec{x}_1'} + \vec{v}_2'.\ \frac{\partial}{\partial\vec{x}_2'} - \frac{1}{m}\ \frac{\partial\varphi_{12}'}{\partial\vec{x}_1'}\cdot\left(\frac{\partial}{\partial\vec{v}_1'} - \frac{\partial}{\partial\vec{v}_2'}\right)\right\} W_{1,12}$$

$$= \frac{n_o}{m}\ \int dX_3'\ \left(\frac{\partial\varphi_{13}'}{\partial\vec{x}_1'}.\ \frac{\partial}{\partial\vec{v}_1'} + \frac{\partial\varphi'_{23}}{\partial\vec{x}_2'}\cdot\frac{\partial}{\partial\vec{v}_2'}\right)\ W_{1,123}. \tag{9}$$

The dilute gas problem now amounts to our finding a well-behaved perturbation expansion to these equations in powers of the density. Hereafter, we make the formal replacement

$$n_o \rightarrow \epsilon n_o$$

in Eqs. (8) and (9), letting $\epsilon \rightarrow 1$ at the end, after carrying out the expansion in ϵ.

Any such procedure necessarily leaves open some questions of convergence for the larger values of s. Similar unsatisfactory features exist in the ordinary BBGKY theory,[10] ultimately because we know very few properties of the solution to the s-body problem. These questions are swept under the rug in the usual theory, and will be here, also. But it is not to be expected that the procedure given will generalize readily to arbitrarily high powers of the density.

III. DERIVATION OF THE KINETIC EQUATION FOR $W_{1,1}$

In the most abbreviated notation, the problem is to find a well-behaved perturbation expansion for

$$\left\{\frac{\partial}{\partial\tau} + H_1'\right\} W_{1,1} = \epsilon L_1' W_{1,12} \qquad (\tilde{8})$$

and

$$\left\{\frac{\partial}{\partial\tau} + H_2'\right\} W_{1,12} = \epsilon L_2' W_{1,123}. \qquad (\tilde{9})$$

We shall write

$$W_{1,1} = W_{1,1}^{(0)} + \epsilon W_{1,1}^{(1)} + 0\,(\epsilon^2) + \ldots$$

$$W_{1,12} = W_{1,12}^{(0)} + \epsilon\, W_{1,12}^{(1)} + 0\,(\epsilon^2) + \ldots\,. \qquad (10)$$

We see that the $0\,(\epsilon)$ relation for $W_{1,1}$ will involve only $W_{1,12}^{(0)}$.

We anticipate the (easily verified) fact that a straightforward expansion in ϵ leads to "secular", or τ-proportional, terms in $W_{1,1}^{(1)}$, and thus becomes useless for $\tau \geqslant 0\,(1/\epsilon)$.

[Rostoker would have found secular terms if he had gone to the next order in the expansion of $W_{1,1}$ in powers of the plasma parameter, but he did not need to.] Thus anticipating, we introduce the multiple time scale procedure of Frieman[11] and Sandri[12] in the form presented by the author.[10] We refer to Ref. 10 for a discussion of the method and the terminology.

We replace Eqs. $(\tilde{8})$, $(\tilde{9})$ in the "extended domain" by

$$\left(\frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1} + \ldots + H_1'\right) W_{1,1} = \epsilon L_1' W_{1,12}, \tag{11}$$

$$\left(\frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1} + \ldots + H_2'\right) W_{1,12} = \epsilon L_2' W_{1,123}. \tag{12}$$

The W-functions are understood to be functions of the "fast" time variable τ_0 and the "slow" time variable τ_1; we need consider no others if we are content with an $O(\epsilon)$ theory. These correspond to the usual terminology of "initial stage" and "kinetic stage"[13] in BBGKY jargon.

The initial conditions to be obeyed by the W-functions at $\tau = 0$ are:

$$W_{12\ldots r,12\ldots s}(X_1\ldots X_r;X_1'\ldots X_s';\ \tau = 0)$$

$$= \begin{cases} V^r f_s(X_1'\ldots X_s')\,\delta(X_1-X_1')\ldots\delta(X_r-X_r'), & s \geq r, \\ V^s f_r(X_1\ldots X_r)\,\delta(X_1-X_1')\ldots\delta(X_s-X_s'), & r > s. \end{cases} \tag{13}$$

The equilibrium theory[9,10] provides a well-behaved expansion in ϵ for the f_s. For example, $f_s^{(0)} = (m/2\Pi\theta)^{3s/2} \exp\,(-E_s/\theta)$. The problem is to find what these initial values evolve into as τ increases from zero.

Substituting Eqs. (10) into (11) and (12) and equating the O (1) terms,

$$\left(\frac{\partial}{\partial\tau_o} + H_1'\right) W_{1,1}^{(0)} = 0 \tag{14}$$

$$\left(\frac{\partial}{\partial\tau_o} + H_2'\right) W_{1,12}^{(0)} = 0. \tag{15}$$

The solution of Eq. (14) is

$$W_{1,1}^{(0)}\,(X_1;X_1';\tau_o,\tau_1) = e^{-\tau_o H_1'}\, W_{1,1}^{(0)}(X_1;X_1';0,\tau_1). \tag{16}$$

The properties of the "streaming operators" $e^{-\tau_o H_s'}$ are discussed many places.[9,10,13] Essentially they trace back along the s-body trajectories τ_o units in time, in the s-particle phase space.

The τ_1 dependence is not determined at this stage; $W_{1,1}^{(0)}(X_1;X_1';0,\tau)$ can at this point be <u>any</u> function of τ_1 which reduces to $Vf(\vec{v}_1')\delta(X_1-X_1')$ at $\tau_1 = 0$. [$f_1(\vec{v}_1')$ is just a Maxwellian.] In the manner characteristic of the multiple time scale method, we shall <u>choose</u> the τ_1 dependence to avoid unbounded growth of $W_{1,1}^{(1)}$ for large τ_o. The "slow" ($\sim n_o\tau$) time evolution of $W_{1,1}^{(0)}$ is provided only by consideration of the first order terms.

Similarly, $W_{1,12}^{(0)}(X_1;X_1'X_2';\tau_o,\tau_1)$ is the streaming operator $e^{-\tau_o H_2'}$ applied to <u>any</u> function of τ_1 which reduces to

$$V\delta(X_1-X_1')\ (m/2\Pi\theta)^3 \exp\left\{-\frac{m(\vec{v}_1'^2+\vec{v}_2'^2)}{2\theta} - \frac{\varphi_{12}'}{\theta}\right\},\ \text{at}\ \tau_1 = 0.$$

Just as in the usual BBGKY theory it has been important to approximate $f_2^{(0)}(X_1X_2;0,\tau_1)$ correctly in terms of $f_1^{(0)}(X_1;0,\tau_1)$ in order to get the kinetic equation, so here we must approximate $W_{1,12}^{(0)}$ correctly in terms of $W_{1,1}^{(0)}$. The obvious choice is

$$W_{1,12}^{(0)}(X_1;X_1'X_2';\tau_o\tau_1) =$$

$$e^{-\tau_o H_2'} W_{1,12}^{(0)}(X_1;X_1'X_2';0,\tau_1)$$

$$= e^{-\tau_o H_2'} \left\{ W_{1,1}^{(0)}(X_1;X_1';0,\tau_1) f_1(\vec{v}_2') \left[1 + \left(1 - e^{-\varphi_{12}'/\theta} \right) \right] \right\}$$

$$= e^{-\tau_o H_2'} e^{\tau_o H_1'(1)} e^{\tau_o H_1'(2)} W_{1,1}^{(0)}(X_1;X_1';\tau_o,\tau_1) f_1(\vec{v}_2')$$

$$+ \left\{ \begin{array}{l} \text{terms which} \Rightarrow 0 \text{ as } \tau_o \Rightarrow \infty \\ \text{for finite } |\vec{x}_1' - \vec{x}_2'| \end{array} \right\}. \tag{17}$$

Now consider the $O(\epsilon)$ terms of Eq. (11):

$$\left(\frac{\partial}{\partial \tau_o} + H_1'(1) \right) W_{1,1}^{(1)}(X_1;X_1';\tau_o,\tau_1) + \frac{\partial}{\partial \tau_1} W_{1,1}^{(0)}(X_1;X_1';\tau_o\tau_1)$$

$$= L_1' \; W_{1,12}^{(0)}(X_1;X_1'X_2';\tau_o,\tau_1). \tag{18}$$

Using Eq. (17), multiplying through by $e^{\tau_o H_1'(1)}$, and noting that only $O(1)$ values of $|\vec{x}_1' - \vec{x}_2'|$ contribute to L_1', Eq. (18) becomes

$$\frac{\partial}{\partial \tau_o} \left[e^{\tau_o H_1'(1)} W_{1,1}^{(1)}(X_1;X_1';\tau_o\tau_1) \right] + \frac{\partial}{\partial \tau_1} W_{1,1}^{(0)}(X_1;X_1';0,\tau_1) =$$

$$e^{\tau_o H_1'(1)} L_1' e^{-\tau_o H_2'} e^{\tau_o H_1'(1)} e^{\tau_o H_1'(2)} W_{1,1}^{(0)}(X_1;X_1';\tau_o,\tau_1) f_1(\vec{v}_2')$$

$$+ \left\{ \begin{array}{l} \text{terms which} \to 0 \text{ after a few} \\ \text{units of } \tau_o \end{array} \right\}. \tag{19}$$

To avoid τ_o- proportional terms in $W_{1,1}^{(1)}$ at large τ_o, we see that we must choose

$$\frac{\partial}{\partial \tau_1} W_{1,1}^{(0)}(X_1;X_1';0,\tau_1) =$$

$$\lim_{\tau_o \to \infty} e^{\tau_o H_1'(1)} L_1' \, e^{-\tau_o H_2'} e^{\tau_o H_1'(1)} e^{\tau_o H_1'(2)} W_{1,1}(X_1;X_1';\tau_o \tau_1) f_1(\vec{v}_2'). \tag{20}$$

The content of Eqs. (16) and (20) on the "physical line" $\tau_o = t$, $\tau_1 = \epsilon t$, where $\frac{\partial}{\partial t} = \frac{\partial}{\partial t_o} + \epsilon \frac{\partial}{\partial t_1}$ is:

$$\left\{ \frac{\partial}{\partial \tau} + H_1' \right\} W_{1,1}^{(0)}(X_1;X_1';t)$$

$$= \lim_{\xi \to \infty} L_1' \, e^{-\xi H_2'} e^{\xi H_1'(1)} e^{\xi H_1'(2)} W_{1,1}^{(0)}(X_1;X_1';t) f_1(\vec{v}_2') \tag{21}$$

for $t \geqslant$ a few durations of a two-body collision. The initial condition to be satisfied by the solution of Eq. (21) is clearly $W_{1,1}^{(0)}(t = 0) = V\delta(X_1 - X_1')f_1(\vec{v}_1')$. This is the desired kinetic equation. Its relation to the linearized Boltzmann equation is discussed in the next section.

IV. RELATION TO THE LINEARIZED BOLTZMANN EQUATION

It is not yet apparent that Eq. (21) is the kinetic equation for a dilute gas. The equation derived by Bogolyubov [9,13] is, in the present notation,

$$\left\{\frac{\partial}{\partial \tau} + H_1(1)\right\} f(X_1,t) = \lim_{\xi \to \infty} L_1 e^{-\xi H_2} e^{\xi H_1(1)} e^{\xi H_1(2)} f(X_1,t)f(X_2,t). \tag{22}$$

Bogolyubov showed[9] that for spatially uniform f_1 and hard-sphere interactions, Eq.(22) reduces to Boltzmann's equation. (For a clear proof, see Uhlenbeck and Ford.[13]) If we linearize Eq. (22) about a Maxwellian, writing

$$f = \left(\frac{m}{2\Pi\theta}\right)^{3/2} e^{-m\vec{v}_1^{\,2}/2\theta} + g(X_1,t) \equiv f_1(\vec{v}_1)+g,$$

and discarding second-order terms in g, we get

$$\left\{\frac{\partial}{\partial \tau} + H_1(1)\right\} g(X_1,t) = \lim_{\xi \to \infty} L_1 \, e^{-\xi H_2} e^{\xi H_1(1)} e^{\xi H_1(2)} \Big[f_1(\vec{v}_1)g(X_2,t)$$
$$+ f_1(\vec{v}_2)g(X_1,t)\Big]. \tag{24}$$

Comparing Eqs. (24) and (21), we see that to prove that $W_{1,1}^{(0)}$ obeys the linearized dilute-gas kinetic equation, we must show that

$$\lim_{\xi \Rightarrow \infty} L_1' \, e^{-\xi H_2'} e^{\xi H_1'(1)} e^{\xi H_1'(2)} W_{1,1}^{(0)}(X_1;X_2';t) f_1(\vec{v}_1') = 0. \quad (25)$$

The demonstration depends upon a property of $e^{\xi H_1'(2)}$. $W_{1,1}^{(0)}(X_1;X_2';t)$ which, while it seems physically obvious, it has not been possible to prove rigorously directly from Eq. (21). We must assume that $e^{\xi H_1'(2)} W_{1,1}^{(0)}(X_1;X_2';t)$ has a finite range of values of $|\vec{x}_1 - \vec{x}_2'|$ over which it can be non-zero. Surely this property is obeyed at t = 0, for

$$e^{-\xi H_1'(2)} W_{1,1}^{(0)}(X_1;X_2',0) =$$

$$f_1(\vec{v}_1) f_1(\vec{v}_2') \exp\left\{- \varphi(\vec{x}_1 - \vec{x}_2' + \vec{v}_2'\xi)\right\} \delta(\vec{v}_1 - \vec{v}_2')\delta(\vec{x}_1 - \vec{x}_2' + \vec{v}_2'\xi)$$

and is non-zero only at one value of $\vec{x}_1 - \vec{x}_2'$. Assume now that the property holds for all finite t, and consider the integral operator L_1'. Only points $\vec{x}_1', \vec{x}_2'$ which are separated by less than the range of the interaction φ contribute to it. For purposes of Eq. (25), $e^{\xi H_1'(1)}$ just acts like the identity operator. The operator $e^{-\xi H_2'}$ traces back a long distance along the two-particle trajectories to two points $\vec{x}_1'(-\xi)$, $\vec{x}_2'(-\xi)$. By virtue of our assumption, however, $\vec{x}_1 - \vec{x}_2'(-\xi)$ will lie outside the domain where $W_{1,1}^{(0)}$ is non-vanishing unless the tracing operation leaves us with $x_2'(-\xi)$ near $\vec{x}_1$. However, for larger and larger ξ, this leaves us with a narrower and narrower range of solid angle into which a given pair of velocities $|\vec{v}_1'|$, $|\vec{v}_2'|$ have to be aimed in order to lead to such a configuration. In the limit of $\xi \to \infty$, the set which can contribute to $\int d\vec{v}_2'$ is measure zero, and Eq. (25) follows.

Note that no similar argument can be constructed for the non-zero term of Eq. (21). For each value of X_1', there always exists a set of X_1 of finite measure over which the whole integrand contributes to Eq. (21).

The question of the compact non-zero domain of the joint probability $W_{1,1}^{(0)}$ is a sticky one. Initially it is true; but such soluble examples of initially singular conditional joint probability functions as exist (those from the theory of Brownian motion[14] are the only ones known to the author) indicate that what may happen is that the delta-function which $W_{1,1}^{(0)}$ is initially, is converted instantaneously into a function which falls off

<u>exponentially</u> at large separations rather than going strictly to zero. Such a result would also be adequate for a proof of Eq. (25). But until much more is known about solutions to the linearized kinetic equations than is now known, the property remains, strictly speaking, only a plausible physical conjecture.

Finally, given the solution to Eq. ~~(21) and the definition~~ of $W_{1,2}$, we have

$$W_{1,2}^{(0)} = \lim_{V\to\infty} \int \frac{dX_1'}{V} e^{-\tau_0 H_2'} \left\{ W_{1,1}^{(0)}(X_1, X_1'; 0, \tau_1) \right.$$

$$\left. \cdot f_1(\vec{v}_2') \left[1 + G(\vec{x}_1' - \vec{x}_2') \right] \right\},$$

where $G \equiv e^{-\varphi_{12}'/\theta} - 1$ is the equilibrium pair correlation. After a few times the duration of a collision, and on the physical line, this becomes

$$W_{1,2}^{(0)}(X_1; X_2'; t) =$$

$$\lim_{V\to\infty} \int \frac{dX_1'}{V} e^{-tH_2'} e^{tH_1'(1)} e^{tH_1'(2)} W_{1,1}^{(0)}(X_1; X_1'; t) f_1(\vec{v}_2')$$

$$+ \lim_{V\to\infty} \int \frac{dX_1'}{V} e^{-tH_2'} \left\{ e^{tH_1'(1)} W_{1,1}^{(0)}(X_1; X_1'; t) \right.$$

$$\left. \cdot \left[e^{tH_1'(2)} f_1(\vec{v}_2') \right] G(\vec{x}_1' - \vec{x}_2') \right\} \qquad (26)$$

This relation is similar to van Leeuwen and Yip's Eq.(2.33). They agree at $\tau = 0$. We have not been able to establish agreement or disagreement, however, for $\tau > 0$.

V. SUMMARY

We have shown how to express the density-density correlation function of a dilute classical gas in terms of the joint probability functions of Rostoker. We have shown how to obtain a well-behaved density expansion of the hierarchy which these joint probabilities obey. The crucial function, called $W_{1,1}^{(0)}$, turns out to obey the linearized dilute gas kinetic equation (Eq.(21)) with delta-function initial conditions. For hard-sphere interactions and infinite wave-lengths, this equation reduces to the linearized Boltzmann equation. Our conclusions, arrived at by quite different methods, confirm most (but not quite all) the conclusions of Van Leeuwen and Yip.

Other expansions of the hierarchy (e.g., in the coupling constant or the plasma parameter) will lead $W_{1,1}^{(0)}$ to obey the appropriate kinetic equation. We have not considered the related problem of solving[6] the linearized kinetic equation (Eq. (21)).

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