Minimum-Weight Springs

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Springs are considered as energy accumulators. Efficient load-deflection curves and ways of obtaining them are shown. The resilience of different materials in different modes of stressing and methods of increasing the apparent resilience are discussed. Formulas are given for the design of torsion bars with rectangular cross section and of coil springs with egg-shaped cross section, which are very efficient springs.

I. Introduction

The weight of a spring is determined by the amount of energy that must be absorbed in the spring, by the qualities of the spring material, by the stress distribution in the cross sections of the spring, and by the extent to which all the cross sections are active in performing spring duties. We will denote these factors as

R material resilience

f_1 efficiency of the cross section

f_2 efficiency of the configuration

ρ density

U maximum energy stored in the spring

so that the total weight required for a spring will be

$$\frac{\rho U}{R f_1 f_2}$$

This formula is trivial by itself. Our interest will be in minimizing the energy U, maximizing the efficiency factors f, choosing a material of high resilience R, looking for tricks to coax more resilience R out of a given material, and keeping in mind that the performance of associated functions such as the provision of leverage has a large effect on the overall efficiency of the device.

To dispose of this last point first, we remember that a torsion bar may be more efficient by itself than a coil spring, but that the levers and anchors which are required to transmit forces into the torsion bar may reverse the situation. We observe that springs are most manageable when they incorporate leverage, such as the distance from coil center to wire center in a coil spring.

II. Minimizing the Absorbed Energy

The maximum energy stored in a spring depends primarily on the maximum load and the travel from zero

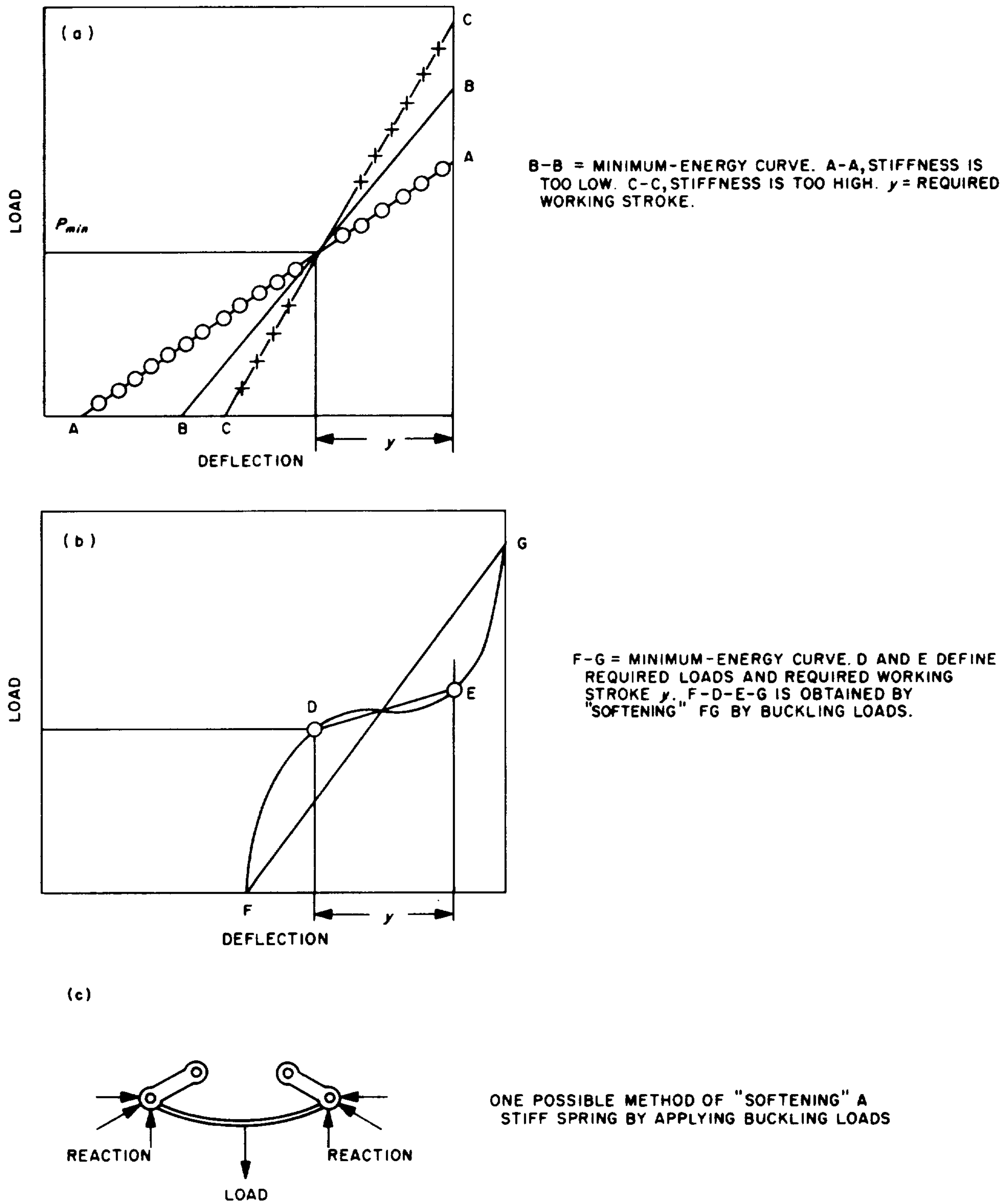


Fig. 1. Minimizing the energy stored in a spring

load to maximum load. It also depends on the shape of the load-deflection curve.

With the usual constraints, namely, a given minimum load P_{min}, a given working stroke y, and a linear load-deflection curve, the minimum energy condition is defined by the equation

$$U_{min} = 2yP_{min}$$

which implies

$$P_{max} = 2P_{min}$$

This follows from symmetry considerations (Fig. 1a).

Fortunately this minimum condition is not very sensitive, as shown by Table 1.

Table 1. Variation of maximum energy *U* stored in linear springs of equal minimum load P_{min} and different maximum loads P_{max}

P_{max}/P_{min}	1.1	1.3	1.5	2	3	5	10
U/U_{min}	3	1.4	1.12	1	1.12	1.56	2.77

The ratios of P_{max}/P_{min} between 1.5 and 3 are quite innocuous. For the low ratios that are sometimes required, the situation is more serious, and we are well advised to deviate from the usual linear load-deflection relation. This can be done in two ways: to use the spring piece-wise, as in the Negator, or to approach buckling. To do the latter, we use a basically stiff spring, of somewhat longer stroke, with the ratio $P_{max} = 2P_{min}$, and we soften this spring by applying a buckling load (Fig. 1b).

In leaf springs, the buckling load can be provided by a suitable shackle arrangement (Ref. 1); in Belleville springs, the action is built in. The low lateral stiffness of coil springs near their buckling load has also been used. All these can be regarded as modifications of snap actions like those in oil cans, light switches, or microswitches. They can be very useful where a low ratio of P_{max}/P_{min} is required.

For cushioning springs with nonlinear load-deflection curves, it is well to remember that increasing the stiffness by bottoming out some of the spring before it has reached the maximum permissible stress is bound to be wasteful. Clever ways can be devised to overcome this, but in general it is more effective and more efficient to simply add bumper springs which come into action later in the working stroke.

III. Maximizing Resilience

The energy that must be stored in a spring is the integral under the entire load-deflection curve, from zero load to maximum deflection. We have seen how this integral can be minimized. Now we want to find the material which can store the energy in the smallest mass. If all the spring material could be stressed up to the permissible limit, then the specific resilience of the springs (energy stored per unit mass) would equal the specific resilience of the material.

We define the specific resilience as R/ρ and the resilience as

$$R_1 = \frac{\sigma^2}{2E}$$

or

$$R_2 = \frac{\tau^2}{2G}$$

depending on the type of stressing, normal or shear. The permissible stress in tension or shear is σ or τ respectively; the corresponding modulus of elasticity is E or G.

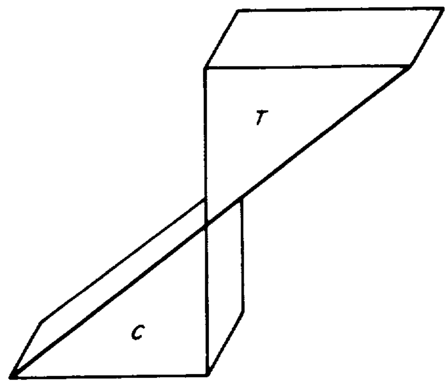
In springs, the stresses are fortunately either predominantly normal, as in bending, or predominantly shear, as in torsion; we need not here be concerned about intermediate cases and triaxial states of stress. We define R as energy stored per unit volume mainly to postpone the nuisance of pounds force and pounds mass. The units of R are inch pounds per cubic inch or lb/in.^{2}. The units of R/ρ, with ρ in pounds mass per cubic inch, are inches × standard acceleration.

If we compare the two resiliences, we find that they would be equal if

$$\frac{\sigma}{\tau} = \left(\frac{E}{G}\right)^{1/2}$$

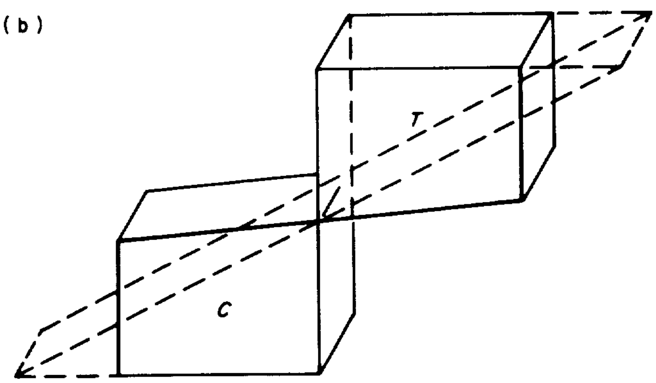
For steel, aluminum, titanium, magnesium, etc., E/G is about 2.8. We would be indifferent about using bending or shear if the ratio of permissible tensile stress to permissible shear stress were $2.8^{1/2}$ or about 1.7. This is not far from the ratio of yield stresses according to the Mises–Hencky or octahedral shear stress theory, but it may be very far from the ratio of permissible bending

(a)



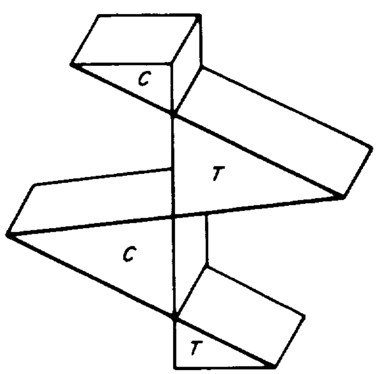
THEORETICAL ELASTIC STRESS DISTRIBUTION ON A RECTANGULAR SECTION IN BENDING
T = TENSION
C = COMPRESSION

(b)



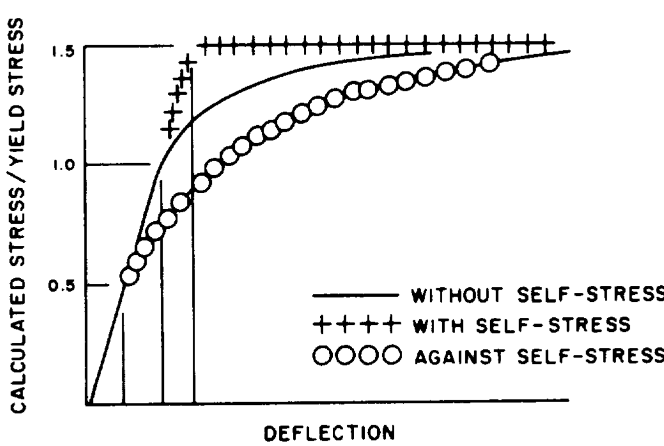
THEORETICAL LIMIT-STRESS DISTRIBUTION WITH ENTIRE SECTION AT YIELD STRESS Y. DASHED LINE SHOWS FICTITIOUS ELASTIC STRESS DISTRIBUTION WHICH WOULD PRODUCE SAME BENDING MOMENT. MAXIMUM FICTITIOUS STRESS = 1.5 Y.

(c)



STRESS DISTRIBUTION AFTER RELEASE FROM THE CONDITION OF 2(b). SELF-STRESS AT EXTREME FIBER = 0.5 Y.

(d)



THEORETICAL ELASTIC-PLASTIC LOAD-DEFLECTION CURVES

Fig. 2. Self-stress

stress to permissible torsional stress in spring design. The permissible torsional stress in compression springs is much higher than would be expected on this basis. Comparing different materials, we find the following conservative values of permissible stresses recommended in the SAE Manual on Helical Springs (Ref. 2) and calculate the apparent values of resilience with the moduli given there, as shown in Table 2.

The apparent resiliences in the different modes of loading are far from equal. The high values for compression springs are explained by the existence of beneficial self-stresses. In those helical torsion springs (stressed in bending) which are cold-wound from small wire, beneficial self-stresses also exist, but are less effective. In the hot-wound 0.50-in. alloy steel spring, the self-stresses induced by coiling are removed by heat-treating. Figure 2 illustrates self-stresses.

The much higher apparent resilience that can be obtained from the material in compression springs explains why weight can be saved by replacing an extension spring by a pair of long "hooks" which compress a spring between their inner ends when the outer ends are pulled apart.

Table 2 also illustrates the fact that the level of permissible design stresses is much more important in springs than in structural members, because the weight of a spring will be inversely proportional to the *square of the stress.*

In music wire and in hard-drawn stainless, the decrease in diameter from 0.10 to 0.05 in. corresponds to an increase in permissible stress of about 13%, but to an increase in resilience of about 28%. The dependence on the square of the stress explains also why springs were among the first products that utilized the stress increase made possible by shotpeening.

Steel is hard to beat as a spring material. Any competing material will have to be evaluated on the basis of specific resilience. Aluminum alloys, whose modulus of elasticity and density each are about ⅓ that of steel, will save weight only if their permissible stresses exceed ⅓ of the corresponding stresses for steel. Glass fiber, which has even lower values of modulus and of density, seems to be worthy of serious consideration for certain applications.

IV. Maximizing the Cross-Section Efficiency

We define the efficiency f_1 of a cross section as the ratio of the elastic energy stored per unit volume of the cross section in bending or in torsion to the resilience of the material. The factor f_1 is a measure of the uniformity of stress distribution in the cross section. Anything we can do to make the stress distribution more uniform will increase f_1 and decrease the weight of the spring.

The efficiencies of a few cross sections are shown in Table 3. Note that the data are restricted to sections that are free of stresses at zero load. The very important effect of self-stresses will be discussed later.

Table 2. Apparent resilience of spring material

Material	Modulus, Msi		Diameter, in.	Torsion springs		Compression springs		Tension springs	
	E	G		σ, ksi	R_0, psi	τ_1, ksi	R_1, psi	τ_2, ksi	R_2, psi
Alloy steel (hot wound)	29	11	0.50	155	410	146	985	106	515
Music wire (cold wound)	30	11.5	0.10	212	750	154	1030	114	565
			0.05	240	960	174	1320	128	710
302 stainless hard drawn (cold wound)	25.5	10	0.10	148	430	106	560	91	415
			0.05	170	565	123	755	105	550
Phosphor bronze (cold wound)	15	6.2	0.10	90	270	70	395	55	245
			0.05	98	320	76	460	60	290

Msi = 10^6 lb/in.^{2}

Table 3. Efficiencies of cross sections of straight bars, free of stresses at zero load

Loading	Round	Square	Rectangle 2:1	Rectangle 10:1
Bending	0.25	0.33	0.33	0.33
Torsion	0.50	0.31	0.26	0.31

First, we note the good showing of round sections in torsion; they are more efficient than any of the others by a ratio of 3:2 or more. By extending uniformly all around the circumference, the high stresses cover a larger fraction of the total area than in other sections or other modes of loading. We know that we can redistribute the stresses at maximum load by introducing self-stresses. Their exact distribution depends on strain-hardening. For purposes of comparison we assume a perfect plastic–elastic material, prestressed to the extreme limit, and then find the efficiencies listed in Table 4.

Table 4. Efficiency of cross sections of straight bars, prestressed to a theoretical limit

Loading	Round	Square	Rectangle 2:1	Rectangle 10:1
Bending	0.72	0.75	0.75	0.75
Torsion	0.89	0.77	0.75	0.80

Comparing Table 4 with Table 3, we note that all cross-section efficiencies have increased dramatically, and that the differences between various cross sections have become much less. The most remarkable increase is for round sections in bending. The makers of roller shades and mouse traps were not as inefficient as one might think.

In passing, we note that these efficiencies apply only if the maximum stress is the failure criterion, not if the stress range is the failure criterion. For infrequent loading or for shotpeened springs in fatigue, the maximum stress is the better criterion. The stress values used in spring design calculations are the ranges from zero load (where we usually have a negative self-stress) to full load. The increase in section efficiency produced by prestressing is reflected by an increase in calculated permissible stress as in Table 2. The greater improvement by prestressing of the originally less efficient sections is reflected in the higher calculated stresses permissible for these sections. The permissible design stress for round wire stressed in bending in a torsion spring will thus be 13% higher than the permissible stress for square wire of the same size and material (the increase in section efficiency is 50% greater). The stress values quoted in the literature include the effect of presetting. In choosing springs, one must take full advantage of presetting, but not make a double allowance for it, once by the efficiency factor and then again by the higher permissible stress. The efficiency factor is used to choose a cross-section shape in preliminary design, the stress value in final dimensioning of the chosen shape.

Returning now to the real implications of Table 4, we observe that in limit design all quoted cross sections are better in torsion than in bending. The table actually understates the case for torsion, because prestressing is easier in torsion than in bending.

We also observe that with limit design the round section is only about 11% more efficient in torsion than a narrow rectangle. If we apply this to torsion bars, the narrow rectangle turns out to be a far better spring for the following reasons:

(1) Flat torsion bars do not require inactive ends (Ref. 3).

(2) The ends are easier to hold, with better leverage.

(3) Flat torsion bars are far easier to manufacture.

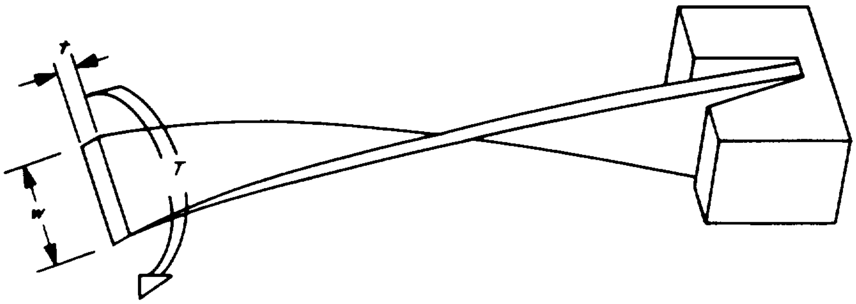
(4) For equal spring rates, flat torsion bars are shorter.

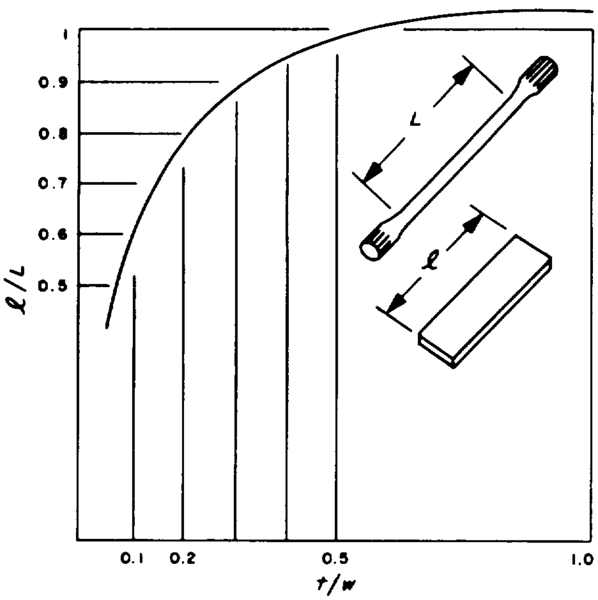
(5) Several flat torsion bars can be used as a bundle in parallel.

To facilitate the use of flat torsion bars, a few relevant formulas are quoted in Fig. 3.

The efficiency factors of Tables 3 and 4 are calculated for straight bars. In coiled springs, we must modify these factors to allow for the higher strain on the shorter fibers near the coil center and for the direct shear in tension springs and in compression springs.

For the usual round wire of diameter d, coiled into a compression spring of mean diameter D, the combined correction factor for curvature and direct shear can be approximated as $K = 1 + 1.6d/D$. (A more accurate approximation is due to Wahl, Ref. 4.) Taking $D = 6d$, which is a very reasonable proportion, the efficiency of the cross section decreases in the ratio $(1/K)^2 = 0.63$. As





RATIO ℓ/L OF LENGTH OF RECTANGULAR BAR OVER ACTIVE LENGTH OF ROUND BAR OF SAME STIFFNESS, SAME LOAD CAPACITY, AND SAME MAXIMUM STRESS (INCLUDING SELF-STRESS). BOTH BARS PRESET, CALCULATED BY LIMIT DESIGN.

FORMULAS FOR RECTANGULAR TORSION BAR SPRINGS

$$S/A = Gt/\ell$$
$$T/A = (0.333 - 0.2\,t/w)Gt^3w/\ell$$
$$T/S = (0.333 - 0.2\,t/w)t^2w$$

A = ANGLE OF TWIST
G = SHEAR MODULUS
S = STRESS (EXCLUDING SELF-STRESS)
T = TORQUE
ℓ = LENGTH
t = THICKNESS
w = WIDTH

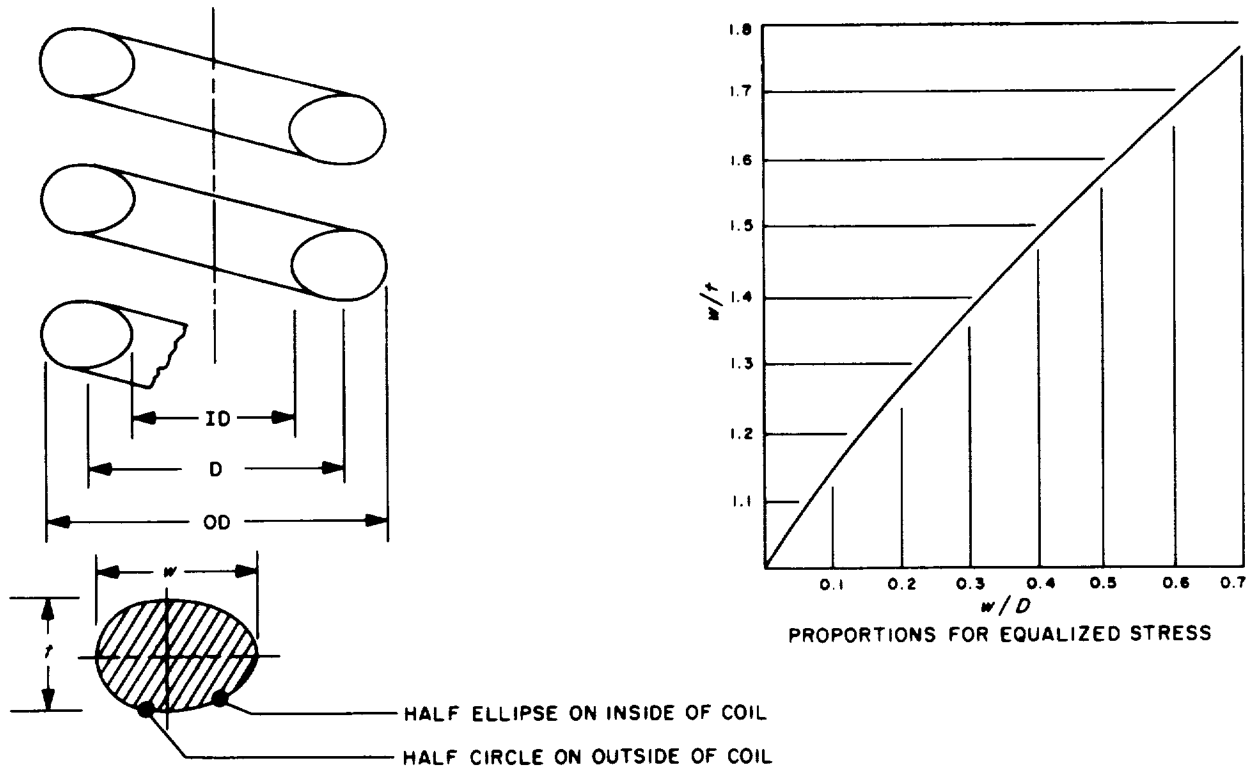
Fig. 3. Rectangular torsion bars with active ends

a result, the efficiency of the round section without presetting is decreased from 0.5 to 0.31, a very substantial decrease. The efficiency of the heavily preset round section will be decreased by a lesser but still substantial amount.

To overcome this unfavorable stress distribution, it has been proposed to make coil springs not from round wire but from egg-shaped wire (Ref. 5). By this method, weight savings of 30% are readily obtainable without presetting, on the basis of calculations. With presetting, the difference becomes less. A series of tests was made, in which eight preset springs experimentally hot coiled from egg-shaped bars were fatigue tested and compared with preset production springs which had a better surface because of the more expeditious hot processing of the production springs. The springs were not shot-peened. The tests showed the following values of energy stored per pound of spring at a median fatigue life of 200,000 cycles:

Round bars	325 in.-lb per lb of spring	100%
Egg-shaped bars	380 in.-lb per lb of spring	117%

Note that the greater resilience was obtained in spite of the poorer surface. The saving in steel weight was considered insufficient to justify the extra effort for railway use, but might be very attractive for aerospace. The saving increases rapidly when springs are coiled to smaller diameters. A collection of formulas relevant to the design of such coil springs is shown in Fig. 4.



$$S/P = 2.55D/wt^2$$
$$P/f = Gt^4(2.1w/t - 1.1)/8ND^3$$
$$A = \pi wt/4$$
$$D = 0.5(OD + ID) + 0.152(w - t)$$

A = AREA OF CROSS-SECTION
D = COIL DIAMETER (OF CENTROID OF SECTION)
f = DEFLECTION
G = SHEAR MODULUS
N = NUMBER OF ACTIVE COILS
P = LOAD
S = MAXIMUM SHEAR STRESS (EXCLUDING SELF-STRESS)
t = THICKNESS
w = WIDTH

Fig. 4. Coil springs with cross section compensated for curvature correction

Knowing that hollow sections are more efficient than solid sections, one might be tempted to make springs of tubes instead of bars or wires. This approach is reasonable for springs which must only maintain a static load, but it will not work for springs in fatigue service because it is too difficult to shotpeen the inside of small straight hollow sections and impossible to shotpeen the inside of a coiled tube. Unless the surfaces are shotpeened, the permissible stress is so much less that a weight increase results instead of a weight saving.

Shotpeening works, of course, because surfaces need fatigue protection and because fatigue damage is propagated by tensile stresses. This suggests a potential improvement for sections stressed in bending: by making the section unsymmetrical, the neutral axis can be located closer to the tensile surface. Automotive leaf springs of this type have been used successfully (Ref. 1), archery bows have long been made of cross sections unsymmetric about the neutral axis, and skis used to be made with such sections, no doubt for the same reasons.

V. Configuration Efficiencies

We have already mentioned the superior configuration efficiency of flat torsion bars as compared with round torsion bars. Greater ease of force transmission and the absence of enlarged inactive ends account for it. It is difficult to give a general numerical ratio, but in a 1-in.-diameter torsion bar, 20 in. long, with 1.3-in.-diameter ends each 1.5 in. long, the inactive ends account for 25% of the total spring weight. With a value of $R = 985$ from Table 2, $f_1 = 0.50$ from Table 3, $f_2 = 0.75$ from the consideration of the inactive ends, and 0.28 lb/in.^{3} for the density of steel, we obtain an overall energy/weight ratio of $985 \times 0.5 \times 0.75/0.28 = 1300$ in.-lb/lb for such a torsion bar. The SAE Manual (Ref. 1) gives 1000 to 1500 in.-lb energy per lb of spring for torsion bars.

A compression coil spring requires one or two inactive end coils, which will account for 10 to 20% of the weight of a spring with 10 coils. Heavy-duty die springs show energy/weight ratios quite close to those of torsion bars.

If minimum weight is a serious consideration, cantilever springs should not be clamped between plates at their fixed ends, because the clamped part is obviously inactive, the end of the clamp introduces a stress concentration, and the clamp itself must be very rigid to perform its intended function. A three-point support is far superior because it leaves the material between support points active.

By themselves, flat springs in bending are not as efficient as springs stressed in torsion. They can be more efficient if the spring can also serve as a guide or link. Leaf springs in automobile suspensions are good examples of this: they compete successfully against a combination of coil spring plus guiding link.

We have already mentioned the importance of using high stresses in order to reduce spring weight in inverse proportion to the square of the stresses. As the spring weight is reduced, the spring becomes smaller and all attachments and enclosures also decrease in weight.

VI. Conclusion

The choice of material of high resilience is the paramount consideration in reducing spring weight. Anything that can be done to increase the permissible working stress will pay off doubly in decreased spring weight. Shotpeening and presetting are the chief tools in raising permissible levels of working stress. Permissible stresses are generally higher in smaller sections of material than in larger sections.

The fact that extension springs can not be preset and are difficult to peen decreases their utility. Compression springs and torsion bar springs are most amenable to presetting.

Flat torsion bar springs and compression springs made of egg-shaped wire deserve consideration where weight must be minimized. These two types of springs have the best overall efficiencies.

A value of 1000 in.-lb energy stored per pound of spring is a good round target figure. It can be exceeded with careful design, testing, and process control. Average design and manufacture may give as low as 300 in.-lb energy stored per pound of spring.

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