Some Free Boundary Value Problems in Elasticity with Electroelastic Applications

by

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1. Introduction

It is the purpose of this paper to consider the following free boundary value problem in elasticity: Can we determine uniquely a plane simply connected region in which the plane linear theory of elasticity holds and on the *a priori* unknown boundary we are not given the forces or displacements but only a non-linear relationship between them? We shall restrict ourselves to regions which, when unstressed, are conformal images by polynomial maps of the unit disc. We shall also restrict ourselves to the case that the boundary stresses and displacements are expressed parametrically in terms of the unknown boundary.

We shall show that the question is equivalent to solving a non-linear singular integral equation of the variety considered in [7]. The solution of the equation will give the unknown boundary. As a by-product, since the boundary stresses and displacements are expressed in terms of the unknown boundary, the first and second fundamental boundary value problems can be solved, once the boundary is determined, and thus all elastic information can be obtained.

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As an application, we shall solve the following free boundary value problem of electroelasticity: Given an elastic, electrically conducting cylinder in a uniform electric field, which is perpendicular to the axis of the cylinder, determine the deformation of the cylinder due to the boundary stresses of the electric field. This problem has been treated in [8] for the case when the cylinder is circular. We shall consider the case when the cylinder is the conformal image of the unit circle by a polynomial map.

To understand the general question more clearly, consider that we are given a bounded simply connected region S in the ζ-plane with boundary L which is the image under the one-to-one conformal map

$$\zeta = m(z)$$

of the unit circle $|z| < 1$, where $m'(z) \neq 0$ on $|z| \leq 1$ and $m'(z)$ is Hölder continuous on $|z| \leq 1$. The first and second fundamental boundary value problem of the plane theory of elasticity can be reduced [6] to finding functions $\varphi(z)$ and $\psi(z)$ analytic for $|z| < 1$, with $\varphi'(z)$ and $\psi(z)$ Hölder continuous on $|z| \leq 1$ and which satisfy respectively:

$$\varphi(t) + \frac{m(t)}{\overline{m'(t)}} \overline{\varphi'(t)} + \overline{\psi(t)} = f(\theta) + \text{const}, \qquad t = e^{i\theta}, \tag{1.1}$$

$$\varkappa\varphi(t) - \frac{m(t)}{\overline{m'(t)}} \overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu^* g(\theta), \qquad t = e^{i\theta}, \tag{1.2}$$

where

$$f(\theta) = F[s(\theta)]$$

with

$$F[s] = i\int_0^s (X_n + iY_n)ds ,$$

s = arc length of L, $X_n + iY_n$ = stress vector on L,

$$g(\theta) = g_1 + ig_2$$

is the displacement vector on L referred to $|t| = 1$, and $\varkappa$ and μ^* are constants, $\varkappa > 1$. We shall assume that we do not know f and g but only the non-linear parametric relationship,

$$\frac{df}{d\theta} = \mathfrak{F}_1[t,\lambda,w'(t)] , \tag{1.3}$$

$$t = e^{i\theta},$$

$$\frac{dg}{d\theta} = \mathcal{G}_1[t,\lambda,w'(t)] , \tag{1.4}$$

where $w'(t)$ is a "parameter function" which is assumed to be Hölder continuous on $|t| = 1$ and $w(t)$ maps $|t| = 1$ onto the *a priori* unknown free boundary. $\mathfrak{F}_1$ and $\mathcal{G}_1$ are either of the form

$$\begin{aligned} \mathfrak{F}_1[t,w'(t),\lambda] &= \lambda\mathfrak{F}[t,w'(t),\lambda] \\ \mathcal{G}_1[\quad \cdots \quad] &= a(t) + b(t)w'(t) + \lambda\mathcal{G}[t,w'(t),\lambda] , \end{aligned} \tag{1.5}$$

or

$$
(1.6)\qquad
\begin{aligned}
\mathcal{F}_1[t,w'(t),\lambda] &= a(t) + b(t)w'(t) + \lambda\,\mathcal{F}[t,w'(t),\lambda] \\
\mathcal{G}_1[\quad \cdots \quad] &= \lambda\,\mathcal{G}[t,w'(t),\lambda] ,
\end{aligned}
$$

where $a(t) \neq 0$ for $|t| = 1$ and λ is a real or complex parameter. By assuming $\mathcal{F}$ and $\mathcal{G}$ are analytic functions of $\operatorname{Re} w'(t)$, $\operatorname{Im} w'(t)$ and λ with Hölder continuous coefficients (see below), $m(z)$ is a certain general class of polynomials; we shall show that for $|\lambda|$ sufficiently small there exists a unique set of functions $\varphi(z,\lambda)$, $\psi(z,\lambda)$, $w'(t,\lambda)$; with $\varphi(z,\lambda)$ and $\psi(z,\lambda)$ analytic in z for $|z| < 1$; $\frac{\partial\varphi(t,\lambda)}{\partial t}$, $\psi(t,\lambda)$, $w'(t,\lambda)$ Hölder continuous for $|t| = 1$ which satisfy (1.1), (1.2), (1.3) and (1.4). Moreover, $w'(t)$ must satisfy a given non-linear singular integral equation, the unique solution of which (provided $|\lambda|$ is sufficiently small) is guaranteed by the results of [7]. Thus the solution of the integral equation will give the <u>a priori</u> unknown free boundary provided $|\lambda|$ is sufficiently small.

In the electroelastic problem, $w(t)$ $(|t| = 1)$ gives the boundary value of the conformal map of the exterior of the unit circle $|t| = 1$ onto the exterior of the deformed and <u>a priori</u> unknown boundary. In setting up the problem, it is assumed that the deformed region is bounded by a Jordan curve with Hölder continuous tangent and that $\operatorname{Re} m'(t) \neq 0$ on $|t| = 1$ or $\operatorname{Im} m'(t) \neq 0$ on $|t| = 1$. Under these assumptions, the unique boundary is given by the unique solution of a non-linear singular integral equation. Thus, within the class of regions bounded by Jordan curves with Hölder continuous tangents, there is a unique such region which solves the free boundary problem.

Elastic free boundary values problems have been treated in: (1) hydroelasticity by the author and Finnila in [3]; (2) electroelasticity by Landau-Lifshitz in [4], and by the author in [8]; (3) plastoelasticity by Cherepanof in [1], [2].

§ 2. Reduction of the Free Boundary Problem to an Integral Equation

Let us assume that the polynomial mapping

$$(2.1) \qquad m(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n, \qquad n \geq 1, \quad c_1 \neq 0, \quad c_n \neq 0,$$

maps the unit circle $|z| \leq 1$ conformally and 1-1 onto the region S and its boundary L. Thus $m'(z) \neq 0$ on $S \cup L$. Using the ideas of Muskhelishvili [6], note that

$$\frac{m(z)}{\overline{m'(z)}} = \frac{c_0 + c_1 z + \cdots + c_n z^n}{\bar{c}_1 + \bar{c}_2 \bar{z} + \cdots + n\bar{c}_n \bar{z}^{n-1}} .$$

But for $|z| = |\sigma| = 1$, $\bar{z} = z^{-1}$, thus $m(\sigma)/\overline{m'(\sigma)}$ is the boundary value of the rational function

$$\frac{m(z)}{\overline{m}'(1/z)} = \frac{c_0 + c_1 z + \cdots c_n z^n}{\bar{c}_1 + 2\bar{c}_2 z^{-1} + \cdots + n\bar{c}_n z^{-n}}$$

which has no poles in $|z| \geq 1$ except at $z = \infty$ since $m'(z) \neq 0$ for $|z| \leq 1$. And at $z = \infty$ this function has a pole of order n. Thus

$$(2.2) \qquad \frac{m(z)}{\overline{m}'(1/z)} = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0 + \sum_{k=1}^{\infty} b_{-k} z^{-k}$$

for $|z| \geq 1$, $b_n \neq 0$.

We shall restrict the regions to be considered further by insisting that the conformal map $m(z)$ satisfy the condition

$$|I - \bar{B}B| \neq 0 \tag{2.3}$$

where

$$B = \begin{bmatrix} b_1 & 2b_2 & \cdots & (n-1)b_{n-1} & nb_n \\ b_2 & 2b_3 & \cdots & (n-1)b_n & 0 \\ \vdots & & & & \\ b_n & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.4}$$

where the b_j's are given in (2.2).

Now we can state

<u>Theorem 1. Given</u>

(i) <u>the conditions:</u>

$$\text{(I)} \quad \begin{aligned} &\varphi(t) + \frac{m(t)}{m'(t)}\overline{\varphi'(t)} + \overline{\psi(t)} = f(\theta) + \underline{\text{const}}, \\ &\varkappa\varphi(t) - \frac{m(t)}{m'(t)}\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu^* g(\theta) , \end{aligned} \qquad t = e^{i\theta},$$

<u>where</u>

$\varkappa = (\lambda^* + 3\mu^*)/(\lambda^* + \mu^*) > 1,$ $\quad \lambda^*, \mu^*$ <u>the Lamé constants;</u>

$$\text{(II)} \quad \begin{aligned} &\frac{df}{d\theta} = \mathcal{F}_1[t, w'(t), \lambda] , \\ &\frac{dg}{d\theta} = \mathcal{G}_1[t, w'(t), \lambda] , \end{aligned} \qquad t = e^{i\theta},$$

where $f(\theta)$ *and* $g(\theta)$ *are periodic of period* 2π, *with*

$$\text{(II.1)}\qquad \begin{aligned} \mathcal{F}_1[t,w'(t),\lambda] &= a_1(t) + \lambda\,\mathcal{F}[t,w'(t),\lambda]\,, \\ \mathcal{G}_1[\quad\cdots\quad] &= a_2(t) + b_2(t)w'(t) + \lambda\,\mathcal{G}[t,w'(t),\lambda], \end{aligned} \qquad t = e^{i\theta},$$

where $b_2(t) \neq 0$ *for* $|t| = 1$, λ *is a real or complex parameter*, $a_i(t), b_2(t) \in H(\mu)$, $0 < \mu < 1$ $(i = 1,2)$ *for* $|t| = 1$; $\mathcal{F}[t,\zeta,\lambda]$ *and* $\mathcal{G}[t,\zeta,\lambda]$ *are complex-valued functions defined for* $|t| = 1$, $\zeta \in \mathcal{R}$, $\lambda \in \mathcal{S} \ni \lambda_0$. *Moreover* $\mathcal{F}[t,w'(t),\lambda]$ *and* $\mathcal{G}[t,w'(t),\lambda]$ *are in* $H(\mu)$ *with respect to* t *provided* $w'(t) \in H(\mu)$ *and* $w'(t) \subset \mathcal{R}$ *for* $|t| = 1$;

(ii) *the polynomial conformal map* $m(z)$ *of* (2.1) *satisfies condition* (2.3).

Then a necessary and sufficient condition that there exist three functions $\varphi(t)$, $\psi(t)$ *and* $w'(t)$ *for which* $\varphi(z)$ *and* $\psi(z)$ *are analytic in* $|z| < 1$, $\varphi'(t)$, $\psi(t)$ *and* $w'(t)$ *are in* $H(\mu)$ *on* $|t| = 1$, $w'(t) \subset \mathcal{R}$ *for* $|t| = 1$, *and that satisfy* (I) *and* (II) *on* $|t| = 1$ *is that* $w'(t)$ *be in* $H(\mu)$ *and be a solution of:*

$$\text{(III)}\qquad w'(t) = R(t) + \lambda \sum_{j=1}^{n} [t^j/b_2(t)]L_j[\mathcal{F}\{\sigma,w'(\sigma),\lambda\}] + \lambda D[t,w'(t),\lambda]$$
$$+ \lambda \frac{(\varkappa+1)t}{4\pi i\mu^* b_2(t)} \int_{|\sigma|=1} \frac{\mathcal{F}[\sigma,w'(\sigma),\lambda]}{\sigma - t}\, d\sigma$$

where for $\sigma = e^{i\alpha}$,

$$R(t) = \left\{R_0(t) + \sum_{j=1}^{n} L_j[a_1(\sigma)]t^j\right\} \Big/ 2\mu^* b_2(t) \ ,$$

(III.1)

$$R_0(t) = -2\mu^* a_2(t) + (\tfrac{1}{2})(\varkappa-1)a_1(t) + \frac{t(\varkappa+1)}{m\, 2\pi} \int_0^{2\pi} \frac{a_1(\sigma)}{\sigma - t}\, d\alpha \ ,$$

$$L_j[g(\sigma)] = -\frac{\varkappa+1}{4\pi\mu^*} \int_0^{2\pi} d\alpha \sum_{k=1}^{n} \frac{1}{k} \{\sigma^{-k} g(\sigma) p_{kj}^2 - \sigma^k \overline{g(\sigma)}\, p_{kj}^1\} \ ,$$

$$D[t, w'(t), \lambda] = \{(\tfrac{1}{2})(\varkappa-1)\,\mathfrak{F}[t, w'(t), \lambda] - 2\mu^* \mathcal{A}[t, w'(t), \lambda]\}/2\mu^* b_2(t) \ ,$$

where

$$p_{kj}^1 = \text{the } kj \text{ entry in the matrix } C^T B^T,$$

$$p_{kj}^2 = \text{the } kj \text{ entry in the matrix } \bar{C}^T \bar{B}^T B^T \ ,$$

in which B is given in (2.4) and $C = [I - \bar{B}B]^{-1}$. Moreover if $w'(t)$ is a solution of (III) then $\varphi(t)$ and $\psi(t)$ on $|t| = 1$ are given uniquely in terms of $w'(t)$ provided $w'(t) \subset \mathcal{R}$ for $|t| = 1$.

Proof: We shall first show that the condition is necessary. Suppose we are given three functions $\varphi(t)$, $\psi(t)$ and $w(t)$, $\varphi(z)$ and $\psi(z)$ analytic on $|z| < 1$, $\varphi'(z)$ and $\psi(z) \in H(\mu)$ on $|z| \le 1$ and $w'(t) \in H(\mu)$ for $|t| = 1$, and which satisfy (I) and (II) on $|t| = 1$ where $\mathfrak{F}_1$ and $\mathcal{A}_1$ are given by (II.1).

In what is to follow, we shall use many of the ideas of Muskhelishvili [6]. Assume

(2.5)

$$\varphi(z) = a_1 z^2 + a_2 z^2 + \cdots + a_n z^n + \cdots \ ,$$

$$\varphi'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1} + \cdots \ ,$$

for $|z| < 1$, $z = re^{i\theta}$, $r < 1$. Then the function

$$\overline{\varphi'}(z) = \bar{a}_1 + 2\bar{a}_2 z + \cdots + n\bar{a}_n z^{n-1} + \cdots$$

converges for $|z| < 1$ and

$$\overline{\varphi'}(1/z) = \bar{a}_1 + 2\bar{a}_2 \frac{1}{z} + \cdots + n\bar{a}_n \frac{1}{z^{n-1}} + \cdots$$

converges for $|z| > 1$, whereas on $|t| = 1$, we have

$$\overline{\varphi'}(1/t) = \overline{\varphi'}(\bar{t}) = \overline{\varphi'(t)} .$$

Thus for $|z| > 1$,

$$\text{(2.6)} \qquad \frac{m(z)}{\overline{m'}(1/z)} \overline{\varphi'}(1/z) = K_0 + K_1 z + K_2 z^2 + \cdots + K_n z^n + O(1/z) ,$$

where with the aid of (2.2),

$$\text{(2.7)} \qquad \begin{aligned} K_0 &= \bar{a}_1 b_0 + 2\bar{a}_2 b_1 + \cdots + (n-1)\bar{a}_{n-1} b_{n-2} + n\bar{a}_n b_{n-1} , \\ K_1 &= \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \cdots + (n-1)\bar{a}_{n-1} b_{n-1} + n\bar{a}_n b_n , \\ K_2 &= \bar{a}_1 b_2 + 2\bar{a}_2 b_3 + \cdots + (n-1)\bar{a}_{n-1} b_n , \\ &\vdots \\ K_n &= \bar{a}_1 b_n . \end{aligned}$$

But $\dfrac{m(z)}{\overline{m'}(1/z)} \overline{\varphi'}(1/z)$ has for its boundary values $\dfrac{m(t)}{\overline{m'(t)}} \overline{\varphi'(t)}$, $z = rt$,

$r < 1$, $t = e^{i\theta}$. Now we apply the integral operator

$$\frac{1}{2\pi i} \int_{|\sigma|=1} \frac{d\sigma}{\sigma - z}, \qquad |z| < 1, \quad \sigma = e^{i\alpha},$$

to the first of equations (I) and get:

$$\varphi(z) + K_0 + K_1 z + K_2 z^2 + \cdots + K_n z^n + \tilde{a} = \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{f(\alpha)}{\sigma - z}\, d\sigma \tag{2.8}$$

where

$$\tilde{a} = \overline{\psi(0)} .$$

Note that

$$\frac{1}{2\pi i} \int_{|\sigma|=1} \frac{f(\alpha)}{\sigma - z}\, d\sigma = A_0 + A_1 z + \cdots + A_n z^n + z^{n+1} \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{\sigma^{-(n+1)} f(\alpha)\, d\sigma}{\sigma - z}, \tag{2.9}$$

where

$$A_k = \frac{1}{2\pi i} \int_{|\sigma|=1} f(\alpha)\, \sigma^{-(k+1)} d\sigma, \qquad k = 0,1,2,\ldots,n. \tag{2.10}$$

Combining (2.5), (2.8) and (2.9), we see that the K's must be determined by solving

$$\begin{aligned}
&\tilde{a} + K_0 = A_0 \\
&a_1 + \bar{a}_1 b_1 + 2\bar{a}_2 b_2 + \cdots + (n-1)\bar{a}_{n-1} b_{n-1} + n\bar{a}_n b_n = A_1 \\
&a_2 + \bar{a}_1 b_2 + 2\bar{a}_2 b_3 + \cdots + (n-1)\bar{a}_{n-1} b_n = A_2 \\
&\cdots \\
&a_n + \bar{a}_1 b_n = A_n
\end{aligned} \tag{2.11}$$

for the a_k's and using the a_k's to determine the K's from (2.7).

In matrix notation, we get the last n equations of (2.11) become

$$Ia^* + B\bar{a}^* = A \tag{2.12}$$

where

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & 2b_2 & \cdots & (n-1)b_{n-1} & nb_n \\ b_2 & 2b_3 & \cdots & (n-1)b_n & 0 \\ \vdots & & & & \vdots \\ b_n & 0 & \cdots & & 0 \end{bmatrix}, \tag{2.13}$$

with

$$a^* = (a_1, a_2, \ldots, a_n)', \qquad \bar{a}^* = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)', \tag{2.13.1}$$

$$A = (A_1, A_2, \ldots, A_n)'. \tag{2.13.2}$$

Taking the conjugate (2.12) gives

$$I\bar{a}^* + \bar{B}a^* = \bar{A}\ . \tag{2.14}$$

Thus (2.12) and (2.14) can be written

$$\begin{bmatrix} I & B \\ \bar{B} & I \end{bmatrix} \begin{bmatrix} a^* \\ \bar{a}^* \end{bmatrix} = \begin{bmatrix} A \\ \bar{A} \end{bmatrix}, \tag{2.15}$$

where $\begin{bmatrix} a^* \\ \bar{a}^* \end{bmatrix}$ and $\begin{bmatrix} A \\ \bar{A} \end{bmatrix}$ are vectors of length $2n$, I, B and $\bar{B}$ are $n \times n$ matrices. Note that

$$\begin{bmatrix} I & B \\ \bar{B} & I \end{bmatrix} \begin{bmatrix} I & -B \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ \bar{B} & I-\bar{B}B \end{bmatrix} .^{\dagger}$$

Thus

$$\begin{vmatrix} I & B \\ \bar{B} & I \end{vmatrix} = \left| I - \bar{B}B \right| . \tag{2.16}$$

In this notation the last n equation of (2.7) becomes

$$K = B\bar{a}^* \tag{2.17}$$

where

$$K = (K_1, K_2, \ldots, K_n)^T.$$

Thus

$$\begin{bmatrix} \bar{K} \\ K \end{bmatrix} = \begin{bmatrix} \bar{B} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} a^* \\ \bar{a}^* \end{bmatrix} . \tag{2.18}$$

Since $|I - \bar{B}B| \neq 0$ by assumption (ii), then by (2.16), $\begin{bmatrix} I & B \\ \bar{B} & I \end{bmatrix}$ has an inverse and thus we can combine (2.15) and (2.18) to give:

$$\begin{bmatrix} \bar{K} \\ K \end{bmatrix} = M \begin{bmatrix} A \\ \bar{A} \end{bmatrix} , \tag{2.19}$$

$\dagger$ This simplification was kindly suggested to me by M. Marcus.

where

$$\text{(2.20)} \qquad M = \begin{bmatrix} \bar{\bar{B}} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & B \\ \bar{\bar{B}} & I \end{bmatrix}^{-1} .$$

Next we let

$$\Theta(z) = (0,0,\ldots,0,z,z^2,\ldots,z^n)' .$$

Then

$$K_1 z + K_2 z^2 + \cdots + K_n z^n = \left(\begin{bmatrix} \bar{\bar{K}} \\ K \end{bmatrix}, \overline{\Theta(z)} \right)$$

$$= \left(M \begin{bmatrix} A \\ \bar{A} \end{bmatrix}, \overline{\Theta(z)} \right) ,$$

where the complex inner product is used. Thus from (2.8) we get

$$\text{(2.21)} \qquad \varphi(z) = - A_0(f) - \left(M \begin{bmatrix} A(f) \\ \bar{A}(f) \end{bmatrix}, \overline{\Theta(z)} \right) + \frac{1}{2\pi i} \int_{|\sigma| = 1} \frac{f(\alpha)}{\sigma - z} \, d\sigma ,$$

where we have written $A_0(f)$ and $A(f)$ to emphasize that A is a functional of f, A and A_0 are given in (2.10) and (2.13.2).

We next note that $\varphi'(z)/m'(z)$ is analytic for $|z| < 1$ since $m'(z) \neq 0$ for $|z| \leq 1$. Thus we can write for $|z| < 1$,

$$\varphi'(z)/m'(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_n z^n + \cdots .$$

Also observe that if

$$\bar{m}(z) = \bar{c}_0 + \bar{c}_1 z + \bar{c}_2 z^2 + \cdots + \bar{c}_n z^n ,$$

then

$$\frac{\overline{m(t)}}{m'(t)}\,\varphi'(t) = \frac{\bar{m}(1/t)}{m'(t)}\,\varphi'(t) , \qquad t = e^{i\theta},$$

is the boundary value of the function

$$\frac{\bar{m}(1/z)}{m'(z)}\,\varphi'(z) . \tag{2.22}$$

But (2.22) is the same as (2.6) if we replace z by $1/z$ and replace the coefficients in the expansions by their conjugates. Thus

$$\frac{\bar{m}(1/z)}{m'(z)}\,\varphi'(z) = \bar{K}_0 + \bar{K}_1 \frac{1}{z} + \bar{K}_2 \frac{1}{z^2} + \cdots + \bar{K}_n \frac{1}{z^n} + H(z) , \tag{2.22.1}$$

where $H(z)$ is holomorphic for $|z| < 1$ and where $\bar{K}_j$ is the complex conjugate of K_j.

Now note that

$$\frac{1}{2\pi i} \int_{|\sigma| = 1} \frac{\bar{\sigma}^j}{\sigma - z}\, d\sigma = 0 ,$$

if $j \geq 1$, $|z| < 1$. Thus if we apply the integral operator

$$\frac{1}{2\pi i} \int_{|\sigma| = 1} \frac{d\sigma}{\sigma - z} , \qquad |z| < 1,$$

to the complex conjugate of the first equation of (I) and rearrange terms, we get:

(2.23)
$$\psi(z) = \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{\overline{f(\alpha)}}{\sigma - z}\, d\sigma - H(z) + \bar{\bar{K}}_0 + \text{const}$$
$$= \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{\overline{f(\alpha)}}{\sigma - z}\, d\sigma - \frac{\bar{\bar{m}}(1/z)}{m'(z)}\, \varphi'(z) + \bar{\bar{K}}_1 \frac{1}{z} +$$
$$+ \bar{\bar{K}}_2 \frac{1}{z^2} + \cdots + \bar{\bar{K}}_n \frac{1}{z^n} + \text{const} ,$$

where we have also used

$$\frac{1}{2\pi i} \int_{|\sigma| = 1} \frac{\overline{\varphi(\sigma)}}{\sigma - z}\, d\sigma = \varphi(0) = 0 .$$

Since by assumption, the two equations of (I) hold simultaneously for φ and ψ, thus we get upon adding that:

(2.24)
$$(1+\kappa)\varphi(t) = f(\theta) + 2\mu^* g(\theta) + \text{const} , \qquad t = e^{i\theta}.$$

Upon differentiating with respect to θ, we get

(2.25)
$$it(1+\kappa)\, \varphi'(t) = f'(\theta) + 2\mu^* g'(\theta) .$$

But upon differentiating (2.21) with respect to z, we get

(2.26)
$$\varphi'(z) = -\left(\begin{bmatrix} A(f) \\ \bar{A}(f) \end{bmatrix} , \overline{\Theta'(z)} \right) + \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{f'(\alpha)}{i\sigma(\sigma - z)}\, d\sigma ,$$

where we have used the periodicity of f and integrated the second term by parts. Letting $z \to t = e^{i\theta}$ and using the Plemelj formula, (2.26) becomes

$$\varphi'(t) = -\left(M\left[\frac{A(f)}{\bar{A}(f)}\right], \overline{\Theta'(t)}\right) + \frac{1}{2it} f'(\theta) + \frac{1}{2\pi i} \int_{|\sigma|=1} \frac{f'(\alpha)}{i\sigma(\sigma - t)} d\sigma , \tag{2.27}$$

where the improper integral is taken as the Cauchy principal value.

Combining (2.25) and (2.27) yields:

$$2\mu^* g'(\theta) = \tfrac{1}{2}(\varkappa - 1) f'(\theta) - it(\varkappa + 1)\left(M\left[\frac{A(f)}{\bar{A}(f)}\right], \overline{\Theta'(t)}\right) + t\,\frac{\varkappa+1}{2\pi i} \int_{|\sigma|=1} \frac{f'(\alpha)}{\sigma(\sigma - t)} d\sigma , \qquad \sigma = e^{i\alpha}.$$

Combining this with (II) and (II.1) yields:

$$w'(t) = B_0(t) + \lambda D_0[t, w'(t), \lambda] - \frac{it(\varkappa+1)\left(M\left[\frac{A}{\bar{A}}\right], \overline{\Theta'(t)}\right)}{2\mu^* b_2(t)} + \lambda\,\frac{t(\varkappa+1)}{4\pi i \mu^* b_2(t)} \int_{|\sigma|=1} \frac{\mathfrak{F}[\sigma, w'(\sigma), \lambda]}{\sigma - t} d\sigma , \tag{2.28}$$

where

(2.28.1)

$$B_0(t) = \left[-2\mu^* a_2(t) + \tfrac{1}{2}(\varkappa - 1) a_1(t) + \frac{t(\varkappa+1)}{2\pi i} \int_{|\sigma|=1} \frac{a_1(\sigma)}{\sigma(\sigma - t)} d\sigma\right] \Big/ 2\mu^* b_2(t) ,$$

(2.28.2)

$$D_0[t,\lambda,w'(t)] = \{(\tfrac{1}{2})(\kappa-1)\mathfrak{J}[t,w'(t),\lambda] - 2\mu^*\mathscr{S}[t,w'(t),\lambda]\}/2\mu^* b_2(t).$$

Next we note that for A_k of (2.10) we have, upon integrating by parts and using the periodicity of $f(\alpha)$, that

(2.29)
$$A_k = -\frac{1}{2\pi k}\int_{|\sigma|=1} f'(\alpha)\,\sigma^{-(k+1)}d\sigma\,, \qquad k = 0,1,\ldots,n.$$

Now we note that

$$\begin{bmatrix} I & B \\ \bar{B} & I \end{bmatrix}^{-1} = \begin{bmatrix} \bar{C} & -BC \\ -\bar{B}\bar{C} & C \end{bmatrix}$$

where

$$C = [I - \bar{B}B]^{-1}.$$

Thus by (2.20):

$$M = \begin{bmatrix} \bar{B} & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I & B \\ \bar{B} & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \bar{B}\bar{C} & -\bar{B}BC \\ -B\bar{B}\bar{C} & BC \end{bmatrix}.$$

This yields

$$M\begin{bmatrix} A \\ \bar{A} \end{bmatrix} = \begin{bmatrix} \beta \\ \bar{\beta} \end{bmatrix}.$$

where

$$\beta = \bar{B}\bar{C}A - \bar{B}BC\bar{A}\,.$$

Combining this with the definition of $\Theta(z)$ yields

(2.30) $$\left(M\begin{bmatrix} A \\ \bar{A} \end{bmatrix}, \overline{\Theta'(z)}\right) = \left(\bar{\beta}, \overline{\Theta_o'(z)}\right)$$

where

$$\Theta_o(z) = (z, z^2, \ldots, z^n) \; .$$

Thus (2.30) becomes

$$\left(M\begin{bmatrix} A \\ \bar{A} \end{bmatrix}, \overline{\Theta'(z)}\right) = \left(BC\bar{A}, \overline{\Theta_o'(z)}\right) - \left(B\bar{B}\bar{C}A, \overline{\Theta_o'(z)}\right)$$

$$= (\bar{A}, \overline{Q_1(z)}) + (A, \overline{Q_2(z)})$$

where

$$Q_j(z) = (p_1^j(z), p_2^j(z), \ldots, p_n^j(z)) \; , \qquad j = 1,2,$$

$$p_k^j(z) = \sum_{\ell=1}^{n} p_{k\ell}^j z^{\ell-1} \; ,$$

$p_k^1(z)$ is the k^{th} component of the vector

$$C^T B^T \Theta_o'(z) \; ,$$

and $p_k^2(z)$ is the k^{th} component of the vector

$$\bar{C}^T \bar{B}^T B^T \Theta_o'(z) \; ,$$

i.e.,

(2.30.1) $$p_{k\ell}^1 = (C^T B^T)_{k\ell} \; , \qquad p_{k\ell}^2 (\bar{C}^T \bar{B}^T B^T)_{k\ell} \; .$$

Thus

$$(2.31)\qquad \left(M\left[\frac{A}{\bar{A}}\right], \overline{\Theta'(z)}\right) = \sum_{k=1}^{n} \bar{A}_k p_k^1(z) + \sum_{k=1}^{n} A_k p_k^2(z)$$

$$= \sum_{k=1}^{n} \left\{\bar{A}_k \sum_{\ell=1}^{n} p_{k\ell}^1 z^{\ell-1} + A_k \sum_{\ell=1}^{n} p_{k\ell}^2 z^{\ell-1}\right\}$$

$$= \sum_{\ell=1}^{n} \mathcal{L}_\ell[w',\lambda] z^{\ell-1} ,$$

where

$$\mathcal{L}_\ell[w',\lambda] = \sum_{k=1}^{n} \{\bar{A}_k p_{k\ell}^1 + A_k p_{k\ell}^2\}$$

$$= -\frac{1}{2\pi} \int_{|\sigma|=1} \left\{\overline{f'(\alpha)} \sum_{k=1}^{n} \frac{1}{k} \sigma^{+(k+1)} p_{k\ell}^1\right\} d\bar{\sigma}$$

$$+ \left\{f'(\alpha) \sum_{k=1}^{n} \frac{1}{k} \sigma^{-(k+1)} p_{k\ell}^2\right\} d\sigma .$$

Thus with the aid of (II) and (II.1),

$$(2.32)\qquad \mathcal{L}_\ell[w',\lambda] = L_\ell^o[a_1(\sigma)] + \lambda L_\lambda^o[\mathcal{F}\{\sigma, w'(\sigma), \lambda\}]$$

where

$$(2.33)\qquad L_\ell^o[g(\sigma)] = \frac{1}{2\pi i} \int_0^{2\pi} d\alpha \sum_{k=1}^{n} \frac{1}{k} \{\sigma^{-k} g(\sigma) p_{k\ell}^2 - \sigma^k \overline{g(\sigma)}\, p_{k\ell}^1\} .$$

Upon combining (2.28), (2.31), (2.32) and (2.33) we get that a necessary condition that there exist three functions $\varphi(t)$, $\psi(t)$ and $w(t)$ that satisfy (I), (II) and (II.1) on $|t| = 1$ where $\varphi(z)$ and $\psi(z)$ are analytic in $|z| < 1$, $\varphi'(t)$, $\psi(t)$ and $w'(t)$ are in $H(\mu)$ on $|t| = 1$ is that $w'(t)$ be a solution of (III) and $w'(t)$ is in $H(\mu)$.

Next we must show that if $w'(t) \in H(\mu)$, $w'(t) \subset R$ for $|t| = 1$, and $w'(t)$ is a solution of (III), then there will exist two functions $\varphi(z)$ and $\psi(z)$ on $|z| \leq 1$ for which $\varphi'(t)$ and $\psi(t)$ are in $H(\mu)$ on $|t| = 1$, $\varphi(z)$ and $\psi(z)$ are analytic in $|z| < 1$ and combined with $w'(t)$ satisfy (I) and (II). To see this we define

$$(2.34) \qquad f(\theta) = \int^{\theta} \{a_1(t) + \lambda\mathcal{F}[t,\lambda,w'(t)]\}\, d\theta + \text{const}, \qquad t = e^{i\theta},$$

$$(2.35) \qquad g(\theta) = \int^{\theta} \{a_2(t) + b_2(t)w'(t) + \lambda\mathcal{G}[t,\lambda,w'(t)]\}\, d\theta + \text{const}.$$

With the aid of (2.34) we define A_k by means of (2.10). Note that the A_k are uniquely defined independent of what the constant is in (2.34). From the A_k's, we define $\varphi(z)$ from (2.21), the b's are known from $m(z)$. It is clear that $\varphi(z)$, so defined, is analytic for $|z| < 1$. Moreover, by differentiation of (2.21) with respect to z, noting that

$$\int_{|\sigma|=1} \frac{f(\alpha)}{(\sigma-z)^2}\, d\sigma = \int_{|\sigma|=1} \frac{f'(\alpha)}{\sigma-z}\, d\sigma$$

since $f(\alpha)$ is periodic, remembering that $w'(t)$ in $H(\mu)$ implies $\mathcal{F}$ is in $H(\mu)$ as a function of t and that $a_1(t) \in H(\mu)$ by (II.1), and letting $z \to t$ on $|\sigma| = 1$, we get $\varphi'(t) \in H(\mu)$ for $|z| \le 1$. From the A_k's, we define the K_k's by (2.19), $k = 1,2,\dots,n$. Knowing the K's, we define $\psi(z)$ by (2.23). It is clear from this definition (using 2.22.1) that $\psi(z)$ is analytic on $|z| < 1$ and in $H(\mu)$ on $|z| \le 1$. Moreover $\psi(z)$ is uniquely defined up to a constant.

By Harnack's Theorem, a necessary and sufficient condition for the first equation of (I) to hold, i.e.,

$$\mathcal{R}(t) = \varphi(t) + \frac{m(t)}{m'(t)}\,\overline{\varphi'(t)} + \psi(t) - f(\theta) - \text{const} = 0, \qquad t = e^{i\theta},$$

is that for all $|z| < 1$,

$$\frac{1}{2\pi i} \int_{|\sigma|=1} \frac{a(\sigma)}{\sigma - z}\, d\sigma = 0 \tag{2.36}$$

and

$$\frac{1}{2\pi i} \int_{|\sigma|=1} \frac{\overline{a(\sigma)}}{\sigma - z}\, d\sigma = 0 \ . \tag{2.37}$$

But (2.36) is equivalent to the definition of $\varphi(z)$ given above and (2.37) is equivalent to the definition of $\psi(z)$ given above. Thus φ and ψ satisfy the first of the boundary conditions (I).

Next we must show that φ and ψ defined in terms of w' also satisfy the second equation of (I). To this end note that since $w'(\sigma)$ is a solution of (III), or equivalently a solution of (2.28) where $f(\theta)$ and $g(\theta)$ are given in (2.34) and (2.35), we have, upon rearranging the terms of (2.28) and using (2.34), (2.35), and the definition of $\varphi(z)$ that

$$f'(\theta) + 2\mu^* g'(\theta) = it(1+\varkappa)\, \varphi'(t) \ . \tag{2.38}$$

Upon integrating with respect to θ, we get

$$f(\theta) + 2\mu^* g(\theta) = (1+\varkappa)\, \varphi(t) + \text{const.}$$

Combining this with the fact that φ and ψ satisfy the first equation of (I), we get

$$\varkappa\, \varphi(t) - \frac{m(t)}{\overline{m'(t)}}\ \overline{\varphi'(t)} - \overline{\psi(t)} = g(\theta) \ ,$$

which is the second equation of (I). (We can adjust the constant of the first equation so that the constant of the second equation is zero.) Thus φ and ψ satisfy (I), (II), and (II.1) simultaneously if w' is a solution of (III), i.e., the condition is not only necessary but sufficient and the theorem is proved.

§ 3. Determination of the Parameter Function $w'(t,\lambda)$

We shall next consider the free boundary problem expressed in Theorem 1 and show that by restricting $\mathcal{F}$ and $\mathcal{B}$ to be "analytic in $\operatorname{Re} w'$ and λ" (see below for definition), we shall be able to use the results of [7] to show that there exists one and only one set of functions $\varphi(t)$, $\psi(t)$, $w'(t,\lambda)$ for which: $\varphi(z)$ and $\psi(z)$ are defined on $|z| \leq 1$; $w'(t,\lambda)$ is defined on $|t| = 1$, $|\lambda|$ sufficiently small; $\varphi(z)$ and $\psi(z)$ are analytic on $|z| < 1$; $w'(t,\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, $|t| = 1$; $\varphi'(t)$, $\psi(t)$ and $w'(t,\lambda)$ are in $H(\mu)$ for $|t| = 1$.

Before we proceed, we shall recall some definitions from [7]. Let Γ be a simple closed contour. Let $H(\mu)$, $0 < \mu < 1$, be the set of all complex-valued Hölder continuous functions defined on Γ with exponent μ. For $\varphi(t)$ in $H(\mu)$ let

$$\|\varphi(t)\|_\mu = M_\varphi + k_\varphi(\mu)$$

where

$$M_\varphi = \max_{t \in \Gamma} |\varphi(t)| ,$$

$$k_\varphi(\mu) = \sup_{t,z \in \Gamma} \frac{|\varphi(t) - \varphi(z)|}{|t-z|^\mu} .$$

Let $\varphi(t_1,t_2)$ be in $H(\mu_1,\mu_2)$, i.e., in $H(\mu_1)$ with respect to t_1 and in $H(\mu_2)$ with respect to μ_2, and let

$$\|\varphi(t_1,t_2)\|_{(\mu_1,\mu_2)} = M_\varphi + k_\varphi(\mu_1,\mu_2) ,$$

where

$$M_\varphi = \max_{t_1, t_2 \in \Gamma} |\varphi(t_1, t_2)| ,$$

$$k_\varphi(\mu_1, \mu_2) = \sup_{t_1, t_2, z_1, z_2 \in \Gamma} \frac{|\varphi(z_1, z_2) - \varphi(t_1, t_2)|}{|z_1 - t_1|^{\mu_1} + |z_2 - t_2|^{\mu_2}} .$$

Note that $\| \ \|_\mu$ is a norm for $H(\mu)$ and $\| \ \|_{(\mu_1, \mu_2)}$ is a norm for $H(\mu_1, \mu_2)$ and that $H(\mu)$ and $H(\mu_1, \mu_2)$ equipped with these respective norms are Banach spaces.

Let $\mathcal{D}_w$ and $\mathcal{D}_\lambda$ be open connected subsets of the complex $w = u + iv$ and λ-planes respectively and let

$$\mathcal{D}_w(\mu) = \{(u,v) : u(x,y) + iv(x,y) \in H(\mu) \text{ and } w(z) \subset \mathcal{D}_w \text{ provided } z \in \Gamma\} .$$

<u>Definition</u> 1. Let $\lambda_o \in \mathcal{D}_\lambda$ and $\gamma(x,y,\lambda)$ be a complex-valued function

$$\gamma(x,y,\lambda) : \quad \Gamma \times \mathcal{D}_\lambda \to C = \text{complex plane},$$

we shall say "$\gamma(x,y,\lambda)$ is an analytic function at λ_o with $H[\mu, \lambda_o]$ coefficients" and mean that there exist a $\delta(\lambda_o) > 0$ and an $A^*(\lambda_o) > 0$ independent of (x,y) such that γ can be represented uniquely as

$$\gamma(x,y,\lambda) = \sum_{j=0}^{\infty} \frac{1}{j!} c_j(x,y,\lambda_o)(\lambda - \lambda_o)^j ,$$

provided $|\lambda - \lambda_o| < \delta(\lambda_o)$ where c_j are complex-valued functions in $H(\mu)$ for which

$$\|c_j(x,y,\lambda_0)\|_\mu \le A^*(\lambda_0) .$$

Definition 2. Let $\lambda_0 \in \mathcal{D}_\lambda$ and

$$A[z,u,v,\lambda]: \quad (z,u,v,\lambda) \in \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda \to C ,$$

and let $(q_1,q_2,\lambda_0) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$. We shall say "$A[z,u,v,\lambda]$ is an analytic function at (q_1,q_2,λ_0) with $H[\mu,A^*,a,b,c]$ coefficients" and mean that there exist positive constants $\delta_1(\lambda_0)$, $\delta_2(\lambda_0)$, $A^*(q_1,q_2,\lambda_0)$, $a(q_1,q_2,\lambda_0)$, $b(q_1,q_2,\lambda_0)$, $c(q_1,q_2,\lambda_0)$ independent of z such that A can be represented by:

(3.1) $$A[z,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} a_{jk\ell}(z)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell$$

with

$$a_{jk\ell}(z) \in H(\mu)$$

and

$$\|a_{jk\ell}(z)\|_\mu \le A^*(q_1,q_2,\lambda_0)a^j(\cdots)b^k(\cdots)c^\ell(\cdots)$$

provided

$$|u-q_1| < \delta_1(\lambda_0), \quad |v-q_2| < \delta_2(\lambda_0) \text{ and } |\lambda-\lambda_0| < \delta_3(\lambda_0) .$$

Definition 3. Let $\mathcal{D}_w$ and $\mathcal{D}_\lambda$ be open arc-wise connected sets in the w-plane and λ-plane respectively, then for

$$A[z,u,v,\lambda]: \quad \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda \to C = \text{complex plane},$$

we shall say "$A[z,u,v,\lambda]$ is an analytic function on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\mu,A^*,a,b,c,q_1,q_2,\lambda_0]$ coefficients" and mean that A is analytic at each point $(q_1,q_2,\lambda_0) \in \mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\mu,A^*,a,b,c]$ coefficients.

Definition 3A. It is clear what is meant by "$\gamma(x,y,\lambda)$ is an analytic function on $\mathcal{D}_\lambda$ with $H[\mu,\lambda]$ coefficients".

Definition 4: We shall say "$A[z,u,v,\lambda]$ is a uniformly analytic function on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with $H[\mu,A^*,a,b,c]$ coefficients" and mean there exist positive constants δ_1, δ_2, δ_3, A^*, a, b, c independent of λ_0, q_1 and q_2 such that for every point of $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$, A can be represented by a series of the form (3.1) in which

$$a_{jk\ell}(z) \in H(\mu)$$

and

$$\|a_{jk\ell}\|_\mu \leq A^* a^j b^k c^\ell ,$$

provided

$$|u-q_1| < \delta_1, \qquad |v-q_2| < \delta_2, \qquad |\lambda-\lambda_0| < \delta_3.$$

Definition 4A. It is clear what is meant by "$\gamma(x,y,\lambda)$ is a uniformly analytic function on $\mathcal{D}_\lambda$ with $H[\mu]$ coefficients".

We are now in a position to state the results of [7].

Theorem 2. Given the non-linear singular integral equation with real or complex parameter λ:

$$\text{(E)} \qquad w(z) = B(z) + \sum_{j=1}^{n^*} B_j^*(z) L\{P^j[\sigma, u(\xi,\eta), v(\xi,\eta), \lambda]$$

$$+ F[z, u(x,y), v(x,y), \lambda] + \frac{1}{2\pi i} \int_\Gamma \frac{N[\sigma, z, u(\xi,\eta), v(\xi,\eta), \lambda]}{\sigma - z} \, d\sigma ,$$

where $z = x + iy \in \Gamma$, $\sigma = \xi + i\eta \in \Gamma$, $w = u + iv$, and by the integral, we mean the Cauchy principal value.

Assume:

(H.1) $B(z)$ and $B_j^*(z)$ are single-valued functions of z which are in $H(\mu)$, $0 < \mu < 1$, for z on Γ, $\|B_j^*(z)\|_\mu \leq b^0$;

(H.2) L is a complex-valued linear bounded functional (with bound L^*) defined on the set of complex-valued functions continuous on Γ;

(H.3) $w_0^*(z) = u_0^*(x,y) + iv_0^*(x,y) \in H(\mu)$;

(H.4) $F[z,u,v,\lambda]$ and $P^j[\cdots]$ are defined and single-valued on $\Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$ and $N[\cdots]$ is defined on $\Gamma \times \Gamma \times \mathcal{D}_w \times \mathcal{D}_\lambda$ where $\mathcal{D}_w$ is a neighborhood of $\{w_0^*(z) : z \in \Gamma\}$ and $\mathcal{D}_\lambda$ is a neighborhood of λ_0 and they are analytic functions on $\mathcal{D}_w(\mu) \times \mathcal{D}_\lambda$ with

$H[\mu, p_0, a_p, b_p, c_p, q_1, q_2, \lambda_0]$ coefficients for P^j, $H[\mu, f_0, a_f, b_f, c_f, q_1, q_2, \lambda_0]$ coefficients for F, and $H[\mu, \mu_1, a_n, b_n, c_n, q_1, q_2, \lambda_0]$ for N, $0 < \mu < \mu_1 < 1$;

(H.5) $w_0^*(z)$ is a solution of (E) for $\lambda = \lambda_0$.

Conclusion:

(C.1) There exists a unique solution $w(z, \lambda)$ of (E) with $w(z, \lambda_0) = w_0^*(z)$ which is an analytic function of λ on $\mathcal{D}_{\lambda_0}$ with $H[\mu, \lambda]$ coefficients where $\mathcal{D}_{\lambda_0}$ is the circle with center at $\lambda_0 - ec$ and radius $\sqrt{e^2c^2 + e}$ where

$$e = [(2d^{-1}U^*)^2 - c^2]^{-1}$$

with

$$d = [2(a+b)]^{-1} ,$$

$$a = \max\{a_p(u_0, v_0, \lambda_0),\ a_f(\cdots),\ a_n(\cdots)\} ,$$

$$b = \max\{b_p(\quad \cdots \quad),\ b_f(\cdots),\ b_n(\cdots)\} ,$$

$$c = \max\{c_p(\quad \cdots \quad),\ c_f(\cdots),\ c_n(\cdots)\} ,$$

and

$$U^* = c[L^* n^* b^0 p_0 + f_0 + (2\pi)^{-1} K^*(\mu) n_0] ,$$

where $K^*(\mu)$ is a positive constant $\geq \pi$ depending only on Γ and μ, L^* is such that

$$|L[f(\sigma)]| \le L^* \max_{\sigma\in\Gamma} |f(\sigma)|$$

for all f continuous on Γ.

(C.2) If λ is real, then the solution is valid for

$$\lambda_o - \frac{d}{2U^*-dc} < \lambda < \lambda_o + \frac{d}{2U^*+dc} .$$

(C.3) $$\|w(z,\lambda) - w_o(z)\|_\mu \le d - \sqrt{d^2 - t(|\lambda-\lambda_o|)}$$

where

$$t(\tilde{\lambda}) = U^*[a+b]^{-1} \frac{\tilde{\lambda}}{1-c\tilde{\lambda}} = 2dU^* \frac{\tilde{\lambda}}{1-c\tilde{\lambda}} .$$

We shall also need the following, which was proved in [7].

Lemma. If $\varphi(t_1,t_2) \in H(\nu_1,\nu_2)$ and $\psi_j(t_j) \in H(\nu_j)$, $j = 1,2,$ then

1. $\varphi(t_1,t_2)\, \psi_j(t_j) \in H(\nu_1,\nu_2)$, $j = 1,2$,

and

2. $\|\varphi(t_1,t_2)\, \psi_j(t_j)\|_{(\nu_1,\nu_2)} \le \|\varphi(t_1,t_2\|_{(\nu_1,\nu_2)} \|\psi_j(t_j)\|_{\nu_j}$.

By combining the results of Theorems 1 and 2 and the Lemma, we can prove

Theorem 3. Given:

(H.1) the polynomial map $m(z)$ of (2.1) with n replaced by n^* for which (2.3) holds;

(H.2) a function $w_o^{*\prime}(t) = u_o^*(x,y) + iv_o^*(x,y)$, $t = x + iy$, defined, single-valued and in $H(\mu)$ for $|t| = 1$, $0 < \mu < 1$;

(H.3)

$$\mathfrak{F}_1[t, w'(t), \lambda] = a_1(t) + \lambda \mathfrak{F}[t,u,v,\lambda], \qquad t = e^{i\theta},$$

$$\mathcal{G}_1[\quad \cdots \quad] = a_2(t) + b_2(t)w'(t) + \lambda \mathcal{G}[t,u,v,\lambda] ,$$

where $w'(t) = u(x,y) + iv(x,y)$; $t = x + vy$; u, v real; $b_2(t) \neq 0$ for $|t| = 1$; λ is a real parameter; $a_j(t) \in H(\mu)$, $b_2(t) \in H(\mu_1)$, $0 < \mu < \mu_1 < 1$, $j = 1,2$, for $|t| = 1$; $\lambda\mathfrak{F}$ and $\lambda\mathcal{G}$ are single-valued complex functions defined on $\{t : |t| = 1\} \times \mathcal{B}_w \times \mathcal{B}_\lambda$ where $\mathcal{B}_w$ is a neighborhood of $\{w_o^{*\prime}(t) : |t| = 1\}$ and $\mathcal{B}_\lambda$ is a neighborhood of λ_o; $\mathfrak{F}$ and $\mathcal{G}$ are analytic functions on $\mathcal{B}_w(u) \times \mathcal{B}_\lambda$ with $H[\mu, f_0^*, a_f, b_f, c_f, q_1, q_2, \lambda_0]$ coefficients for $\mathfrak{F}$ and $H[\mu, g_0^*, a_g, b_g, c_g, q_1, q_2, \lambda_0]$ coefficients for $\mathcal{G}$;

(H.4) $\varphi_o(z)$, $\psi_o(z)$ are defined on $|z| \leq 1$, analytic in $|z| < 1$, $\varphi_o'(t)$ and $\psi_o(t)$ are in $H(\mu)$ on $|t| = 1$ and for $\lambda = \lambda_o$, $\varphi = \varphi_o$, $\psi = \psi_o$, and $w' = w_o^{*\prime}$ satisfy:

$$\text{(H.4.1)} \qquad \varphi(t) + \frac{m(t)}{\overline{m'(t)}}\, \overline{\varphi'(t)} + \overline{\psi(t)} = f(\theta) + \text{const}, \qquad t = e^{i\theta},$$

and

(H.4.2) $$\varkappa\varphi(t) - \frac{m(t)}{m'(t)}\,\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu^* g(t) ,$$

where

$$\varkappa = (\lambda^* + 3\mu^*)/(\lambda^* + \mu^*) > 1, \qquad \lambda^*,\ \mu^* \text{ Lamé constants,}$$

(H.4.3) $$\frac{df}{d\theta} = \mathcal{F}_1[t, w'(t), \lambda] ,$$

(H.4.4) $$\frac{dg}{d\theta} = \mathcal{G}_1[t, w'(t), \lambda] ,$$

$f(\theta)$ and $g(\theta)$ are periodic functions of θ of period 2π.

Conclusion:

(C.1) There exists one and only one set of functions $\varphi(t,\lambda)$, $\psi(t,\lambda)$, $w'(t,\lambda)$ for which $\varphi(z,\lambda)$, $\psi(z,\lambda)$ are defined for $|z| \le 1$, $\lambda \in I$ = open interval containing λ_o; $w'(t,\lambda)$ is defined for $|t| = 1$, $\lambda \in I$; $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are analytic for $|t| < 1$, $\lambda \in I$; $w'(t,\lambda)$ is an analytic function of λ on I with $H[\mu,\lambda]$ coefficients; $\frac{\partial\varphi}{\partial t}(t,\lambda)$ and $\psi(t,\lambda) \in H(\mu)$ for each $\lambda \in I$; $\varphi(t,\lambda)$, $\psi(t,\lambda)$, $w'(t,\lambda)$ satisfy (H.4.1), (H.4.2), (H.4.3) and (H.4.4).

(C.2) For $|t| = 1$,

$$w'(t,\lambda_o) = w_o^{*\prime}(t) ,$$

$$\varphi(t,\lambda_o) = \varphi_o(t) ,$$

$$\psi(t,\lambda_o) = \psi_o(t) .$$

(C.3)

$$I = J_1 \cap J_2$$

where

$$J_1 = \{\lambda : \lambda_o - \frac{d}{2U^* - dc} < \lambda < \lambda_o + \frac{d}{2U^* + dc}\} ,$$

$$J_2 = \{\lambda : w'(t,\lambda) \subset \mathcal{B}_W \text{ for } |t| = 1\} ,$$

$$d = [2(a+b)]^{-1}$$

$$a = a_f(u_o, v_o, \lambda_o) + a_g(u_o, v_o, \lambda_o)$$

$$b = b_f(\cdots) + b_g(\cdots)$$

$$c = c_f(\cdots) + c_g(\cdots)$$

$$U^* = c[L^* n^* b^o p_o + f_o + (2\pi)^{-1} K^*(\mu) n_o]$$

where $L^* = \frac{\kappa+1}{2\mu^*}$, n^* = degree of m,

$$b^o = [1 + n^* 2^{1-\mu}] \|b_2^{-1}(t)\|_\mu ,$$

$$p_o = \max_{1 \le r, m \le n^*} \{|p^1_{mr}|, |p^2_{mr}|\} ,$$

p^1_{mr} and p^2_{mr} are the mr entries in the matrices $C'B'$ and $\bar{C}'\bar{B}'B'$ respectively where B is given in (2.4) and $C = [I - \bar{B}B]^{-1}$,

$$f_o = \|b_2^{-1}(t)\|_\mu \max\{ \frac{\kappa-1}{4\mu^*} f_o^*, g_o^* \} ,$$

$$n_o = \frac{\kappa+1}{2\mu^*} (1 + 2^{1-\mu_1}) \|b_2^{-1}(t)\|_{\mu_1} ,$$

$K^*(\mu)$ = positive constant, depending on μ and $\ge \pi$.

(C.4)

$$\|w'(z,\lambda) - w_o^{*'}(z)\|_\mu \leq d - \sqrt{d^2 - t(|\lambda-\lambda_o|)}$$

for $\lambda \in I$, *where*

$$t(\tilde{\lambda}) = U^*[a+b]^{-1} \frac{\tilde{\lambda}}{1 - \tilde{\lambda}} .$$

(C.5) $w'(t,\lambda)$ *is the solution of the integral equation* (III) *with* n *replaced by* n^*.

Proof: According to Theorem 1, for fixed λ, a necessary and sufficient condition that there exist a set of three functions $\varphi(t,\lambda)$, $\psi(t,\lambda)$, $w'(t,\lambda)$ of the type described in (C.1) is that $w'(t,\lambda)$ be a solution of an equation of the form (III). In order to apply Theorem 2, we check the hypotheses in Theorem 2:

(H.1) $R(t)$ and $t^j/b_2(t)$ n are single-valued complex functions given in (III.1),

$$\left\| \frac{t^j}{b_2(t)} \right\|_\mu \le \|t^j\|_\mu \; \|b_2^{-1}(t)\|_\mu \le [1 + j2^{1-\mu}] \; \|b_2^{-1}(t)\|_\mu .$$

Thus we can take

$$b^0 = [1 + n^* 2^{1-\mu}] \; \|b_2^{-1}(t)\|_\mu . \tag{3.2}$$

(H.2) In order to find L^* note that L_j of (III.1) can be expressed

$$\lambda L_j[\mathfrak{F}\{\sigma, w'(\sigma),\lambda\} = L\{P^j[\sigma, u(\xi,\eta), v(\xi,\eta), \lambda]\} , \qquad 1 \le j \le n^*,$$

where

$$L\{P^j[\sigma, \cdots]\} = \frac{1}{4\mu^*\pi} (\varkappa + 1) \int_0^{2\pi} d\alpha P^j[\sigma, \cdots] ,$$

for $\sigma = e^{i\alpha}$, $1 \le j \le n^*$; and

$$P^j[\sigma, \cdots] = -\lambda \sum_{k=1}^{n^*} \frac{1}{k} \{\sigma^{-k} \mathfrak{F}[\sigma, w'(\sigma),\lambda] p_{kj}^2 - \sigma^k \, \overline{\mathfrak{F}[\sigma, w'(\sigma),\lambda]} \, p_{kj}^1\} \tag{3.3}$$

$1 \le j \le n^*$, Thus we can take

$$(3.4) \qquad L^* = \frac{\varkappa+1}{2\mu^*} .$$

(H.3) and (H.5) Since by assumption $w_o'(t) = u_o + iv_o \in H(\mu)$, and by assumption (H.4), φ_o, ψ_o, w_o' satisfy (H.4.1) and (H.4.2) for $\lambda = \lambda_o$, then by Theorem 1,

$$w_o'(t) = R(t) + \sum_{j=1}^{n^*} \left[\frac{t^j}{b_2(t)}\right] L\{P^j[\sigma,u_o(\xi,\eta),v_o(\xi,\eta),\lambda_o]\}$$

$$+ \lambda_o D[t,w_o'(t),\lambda_o] + \lambda_o \frac{(\varkappa+1)t}{4\pi i\mu^* b_2(t)} \int_{|\sigma|=1} \frac{\mathcal{F}[\sigma,w_o'(t),\lambda_o]}{\sigma - t} d\sigma .$$

(H.4)

$$F[t,u,v,\lambda] = \lambda D[t,w'(t),\lambda]$$

$$= \frac{\varkappa-1}{4\mu^*} \frac{\lambda\mathcal{F}[t,w'(t),\lambda]}{b_2(t)} - \frac{\lambda\mathcal{E}[t,w'(t),\lambda]}{b_2(t)} ,$$

and for $\sigma = \xi + i\eta$,

$$N[\sigma,t,u(\xi,\eta),v(\xi,\eta),\lambda] = \frac{\varkappa+1}{2\mu^*} \frac{\lambda\mathcal{F}[\sigma,w'(\sigma),\lambda]}{b_2(t)} t .$$

According to the hypotheses, for all $q_1 + iq_2 \in \mathcal{D}_w(\mu)$ we have

$$\lambda\mathcal{F}[t,w'(t),\lambda] = \sum_{j,k,\ell=0}^{\infty} f_{jk\ell}^*(t)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell ,$$

with

$$f_{jk\ell}^*(t) \in H(\mu)$$

and

$$\|f^*_{jk\ell}(t)\|_\mu \le f^*_0 a^j_f b^k_f c^\ell_f$$

provided $|u-q_1|$, $|v-q_2|$ and $|\lambda-\lambda_0|$ are sufficiently small, where f^*_0, a_f, b_f, c_f depend on q_1, q_2 and λ_0 but not on t. Similarly

$$\lambda\mathcal{G}[t,w'(t),\lambda] = \sum_{j,k,\ell=0}^{\infty} g^*_{jk\ell}(t)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell ,$$

with . . . .

Thus we have for $q_1+iq_2 \in \mathcal{D}_w(\mu)$ and $\lambda_0 \in \mathcal{D}_\lambda$,

$$F[t,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} f_{jk\ell}(t)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell ,$$

where

$$f_{jk\ell}(t) = b_2^{-1}(t)\left[\frac{\varkappa-1}{4\mu^*} f^*_{jk\ell}(t) - g^*_{jk\ell}(t)\right] ;$$

and thus

$$\|f_{jk\ell}(t)\|_\mu \le \|b_2^{-1}(t)\|_\mu \left[\frac{\varkappa-1}{4\mu^*} f^*_0 a^j_f b^k_f c^\ell_f + g^*_0 a^j_g b^k_g c^\ell_g\right] \tag{3.5}$$

$$\le f_0(a_f+a_g)^j\,(b_f+b_g)^k\,(c_f+c_g)^\ell$$

where

$$f_0 = \|b_2^{-1}(t)\|_\mu \max\left\{\frac{\varkappa-1}{4\mu^*} f^*_0,\ g^*_0\right\} . \tag{3.6}$$

For N we get

$$(3.7) \qquad N[\sigma,t,u(\xi,\eta),v(\xi,\eta),\lambda] = \sum_{j,k,\ell=0}^{\infty} n_{jk\ell}(\sigma,t)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell$$

where

$$(3.8) \qquad n_{jk\ell}(\sigma,t) = \frac{\varkappa+1}{2\mu^*} \frac{f^*_{jk\ell}(\sigma)}{b_2(t)} t ,$$

and thus by the lemma

$$\|n_{jk\ell}(\sigma,t)\|_{(\mu,\mu_1)} \le \frac{\varkappa+1}{2\mu^*} \|f^*_{jk\ell}(\sigma)\|_\mu \|b_2^{-1}(t)\|_{\mu_1} \|t\|_{\mu_1} ,$$

i.e.

$$(3.9) \qquad \|n_{jk\ell}(\sigma,t)\|_{(\mu,\mu_1)} \le n_0 a_f^j b_f^k c_f^\ell$$

where

$$(3.10) \qquad n_0 = \frac{\varkappa+1}{2\mu^*} (1 + 2^{1-\mu_1})\|b_2^{-1}(t)\|_{\mu_1} .$$

For P^r we have, since λ is real,

$$(3.11) \qquad P^r[\sigma,u,v,\lambda] = \sum_{j,k,\ell=0}^{\infty} p^r_{jk\ell}(\sigma)[u-q_1]^j[v-q_2]^k[\lambda-\lambda_0]^\ell, \qquad 1 \le r \le n^*,$$

where

$$p^r_{jk\ell}(\sigma) = -\sum_{m=1}^{n^*} \frac{1}{m} \{\sigma^{-m} f^*_{jk\ell}(\sigma) p^2_{mr} - \sigma^m \overline{f^*_{jk\ell}(\sigma)}\, p^1_{mr}\} , \qquad \sigma = e^{i\alpha};$$

and thus

$$\|p^r_{jk\ell}(\sigma)\|_\mu \le p^r_0 a^j_f b^k_f c^\ell_f \tag{3.12}$$

where

$$\begin{aligned} p^r_0 &= \sum_{m=1}^{n^*} \frac{1}{m} \{\|\sigma^{-m}\|_\mu |p^2_{mr}| + \|\sigma^m\|_\mu |p^1_{mr}|\} \\ &\le \sum_{m=1}^{n^*} \frac{1}{m} \{[1 + m2^{1-\mu}]\ |p^2_{mr}| + [1 + m2^{1-\mu}]\ |p^1_{mr}|\} \\ &\le [1 + n^*2^{1-\mu}] \sum_{m=1}^{n^*} \{|p^2_{mr}| + |p^1_{mr}|\} \\ &\le p_0 \end{aligned} \tag{3.13}$$

where

$$p_0 = \max_{1\le r, m\le n^*} \{|p^2_{mr}|, |p^1_{mr}|\} . \tag{3.14}$$

Since the hypotheses of Theorem 2 are satisfied we can conclude there exists a unique solution $w'(z,\lambda)$ of the integral equation of the form (III) provided $\lambda \in I_1$ with $w'(z,\lambda_o) = w'_o(z)$ which is an analytic function on I_1 with $H[\mu,\lambda]$ coefficients. The $\varphi(z,\lambda)$ and $\psi(z,\lambda)$ are determined uniquely in terms of $w'(t,\lambda)$, according to Theorem 1, provided $w'(z,\lambda) \subset \mathcal{D}_w$, i.e. for $\lambda \in I_2$. Thus the conclusions hold by Theorems 1 and 2.

§4. Reduction of the Physical Free Boundary Value Problem to a Mathematical Problem

1. Elastic problem:

Let us assume that we are given a bounded simply connected region S with smooth boundary L_o in the ζ-plane which contains the origin. Within S, we assume the plane linear theory of elasticity to hold, i.e., we assume the stresses and strains within S are governed by the biharmonic equation $\Delta^2 u = 0$. According to Muskhelishvili [6], the general solution is of the form

$$u = \mathrm{Re}\{\overline{\zeta}\,\Phi(\zeta) + \chi(\zeta)\},$$

where $\Phi(\zeta)$ and $\chi(\zeta)$ are functions regular in S. If we assume $\Phi'(\zeta)$ and $\chi'(\zeta)$ are continuous on the closed region $S \cup L_o$, then the stress and displacement boundary conditions are:

$$\Phi(t) + t\overline{\Phi'(t)} + \overline{\Psi(t)} = F(t) + \text{const}, \qquad t \in L_o, \tag{4.1}$$

$$\varkappa\Phi(t) - t\overline{\Phi'(t)} - \overline{\Psi(t)} = 2\mu^* G(t), \qquad t \in L_o, \tag{4.2}$$

where

$$F(s) = F_1 + iF_2 = i\int_0^s (X_n + iY_n)\,ds, \tag{4.3}$$

$$s = \text{arc length of } L_o = 0,$$

with $X_n + iY_n$ the external stress acting on L_o, $\varkappa = (\lambda^* + 3\mu^*)/(\lambda^* + \mu^*) > 1$, where λ^* and μ^* are the Lamé constants and $g(t)$ is the complex displacement of the boundary and $\chi'(\zeta) = \Psi(\zeta)$.

Let L_1 be the unknown boundary of the deformed region and assume L_1 contains the origin.

<u>We assume</u> L_1 <u>has a Hölder continuous tangent.</u>

Let $w_i(z)$ map the exterior of the unit circle $|z| > 1$ one to one and conformally onto the exterior of L_i, $i = 0, 1$, so that points at infinity correspond and the direction of the positive real axis at z = infinity corresponds to the direction of the positive real axis at ζ = infinity. Since L_i $(i=0,1)$ have Hölder continuous tangents, then $w_i'(z)$ $(i=0,1)$ will be Hölder continuous on $|z| \geq 1$ by Warshawski's theorem [5].

<u>We shall assume the displacement at</u> $\zeta_0 = w_0(z_0)$ <u>to be</u>

$$(4.4) \qquad D(t_0) = w_1(t_0) - w_0(t_0), \qquad t_0 = e^{i\theta_0} .$$

Thus the displacement, referred to ζ_0 on L_0 is

$$(4.5) \qquad G(\zeta_0) = w_1[w_0^{-1}(\zeta_0)] - \zeta_0 .$$

Let us assume next that $|z| \leq 1$ is mapped one to one and conformally onto the interior of L_0 by the unique mapping $m(z)$ given in (2.1). As above, assume $m'(t)$ is Hölder continuous on $|t| \leq 1$. Then (4.1) and (4.2) transform into:

$$(4.6) \qquad \varphi(t) + \frac{m(t)}{\overline{m'(t)}}\, \overline{\varphi'(t)} + \overline{\psi(t)} = f(\theta) + \text{const}, \qquad t = e^{i\theta},$$

$$\varkappa\varphi(t) - \frac{m(t)}{m'(t)}\,\overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu^* g(\theta), \tag{4.7}$$

where we have set

$$f(\theta) = F[s(\theta)], \quad \varphi(t) = \Phi[m(t)], \quad \psi(t) = \Psi[m(t)],$$

and

$$g(\theta) = G[m(t)] = w_1\{w_0^{-1}[m(t)]\} - m(t), \qquad t = e^{i\theta}. \tag{4.8}$$

2. Electrostatic problem:

Exterior to L_1, the electric field is governed by the harmonic equation. More specifically, for the electric field E we have:

$$E = E_\xi + iE_\eta = -\nabla\alpha = -(\partial\alpha/\partial\xi, \partial\alpha/\partial\eta)$$

where α is the field potential and

$$\Delta\alpha = 0.$$

If $\beta(\xi,\eta)$ is the complex conjugate of $-\alpha$; i.e.,

$$\partial\beta/\partial\eta = -\partial\alpha/\partial\xi$$
$$\partial\beta/\partial\xi = \partial\alpha/\partial\eta\ ,$$

then the complex potential,

$$A(\zeta) = \alpha - i\beta, \tag{4.9}$$

is an analytic function of $\zeta = \xi + i\eta$ outside of L_1 and thus

(4.10) $$-\overline{E} = -E_\xi + iE_\eta = \partial\alpha/\partial\xi - i\,\partial\alpha/\partial\eta = \partial\alpha/\partial\xi - i\,\partial\beta/\partial\xi = dA/d\zeta .$$

In order to find the field E, uniform in the direction of the ξ-axis at ∞, exterior to the conductor with boundary L_1, it is sufficient to find:

(i) an analytic function $A(\zeta)$, analytic outside L_1,

(ii) with $\operatorname{Re} A(\zeta)$ = real constant for ζ on L_1,

(iii) $-\overline{\dfrac{dA}{d\zeta}} \to E_\infty$ as $\zeta = \xi + i\eta \to \infty$.

Such a function is given by:

$$A(\zeta) = -E_\infty\, w_1'(\infty)[Q(\zeta) - 1/Q(\zeta)] + i\,b, \qquad b \text{ real}$$

where

$$z = Q(\zeta) = w_1^{-1}(\zeta).$$

This follows since

(i) clearly $A(\zeta)$ is analytic outside L_1,

(4.11) (ii) $\operatorname{Re} A(\zeta) = 0$ on L_1 since $|Q(\zeta)|^2 = 1$ for ζ on L_1,

(iii) $$E(\zeta) = -\overline{\frac{dA}{d\zeta}} = E_\infty\, \overline{w_1'(\infty)}\ \ \overline{Q'(\zeta)}\ [1 + \overline{1/Q^2(\zeta)}]$$

$$= E_\infty \frac{\overline{w_1'(\infty)}}{\overline{w_1'(z)}}\ [1 + 1/\bar{z}^2] \to E_\infty ,$$

as $\zeta = \xi + i\eta \to \infty$, since then $z = x + iy \to \infty$.

Thus the field is given by (iii) above.

Next we note that the tension on the conductor at ζ_1 on L_1 due to the electric field is [4]

$$(8\pi)^{-1} |E(\zeta_1)|^2 \qquad \text{in c.g.s. units.}$$

Consistent with the linear theory of elasticity we assume the tension is applied to the undeformed conductor, i.e., applied to the point ζ_0 on L_0 where

$$\zeta_1 = w_1(t_0), \quad \zeta_0 = w_0(t_0), \quad \zeta_0 = m(t) .$$

See Figure 1.

Then the force/arc length at ζ_0 is given by:

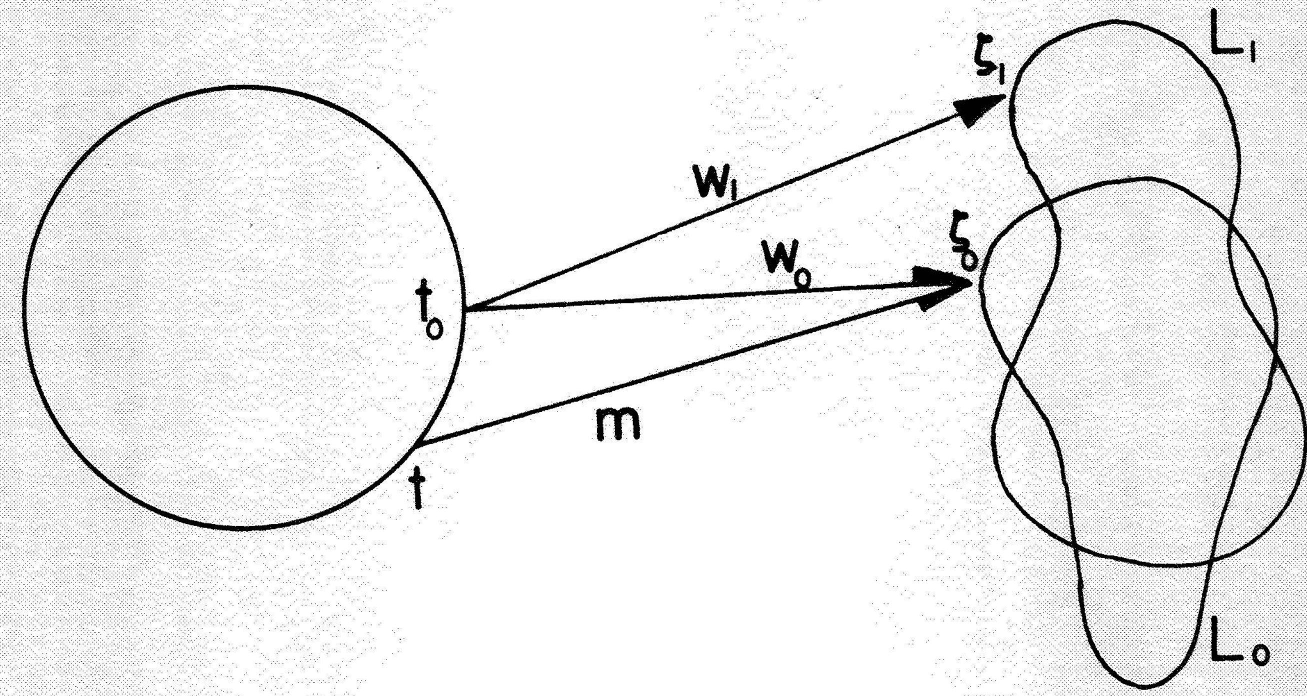


FIG. 1

$$X_n + i\,Y_n = (8\pi)^{-1}|E[w_1[w_0^{-1}(\zeta_0)]]|^2[\cos(n,\xi_0) + i\cos(n,\eta_0)]$$

$$= -i(8\pi)^{-1}|E[w_1[w_0^{-1}(\zeta_0)]]|^2 \frac{d\zeta_o}{ds}$$

since

$$(n,\xi_0) = (t,\eta_0), \qquad (n,\eta_0) = (t,\xi_0) \pm \pi ,$$

and thus

$$\cos(n,\xi_0) + i\cos(n,\eta_0) = \cos(t,\eta_0) - i\cos(t,\xi_0)$$

$$= \frac{d\eta_o}{ds} - i\frac{d\xi_o}{ds} = -i\frac{d\zeta_o}{ds} .$$

Thus from (4.3), we get

$$F[s] = (8\pi)^{-1}\int_o^s |E[w_1[w_0^{-1}(\zeta_0)]]|^2 \frac{d\zeta_o}{ds}\,ds .$$

Thus for

$$f(\theta) = F[s(\theta)] ,$$

we have, since $t = e^{i\theta}$, $\zeta_0 = m(t)$,

$$\frac{\partial}{\partial\theta} f(\theta) = f'(\theta) = (8\pi)^{-1}|E[w_1[w_0^{-1}[m(t)]]]|^2 \frac{d\zeta_o}{ds}\,\frac{ds}{d\theta} .$$

But

$$\frac{d\zeta_o}{ds}\frac{ds}{d\theta} = \frac{d\zeta_o}{d\theta} = itm'(t).$$

Thus

$$f'(\theta) = it(8\pi)^{-1} \left| E[w_1[w_0^{-1}[m(t)]]] \right|^2 m'(t) \tag{4.12}$$

$$= A_\infty t \left| \frac{1 + [w_0^{-1}[m(t)]]^2}{w_1'[w_0^{-1}[m(t)]]} \right|^2 m'(t)$$

with

$$A_\infty = i(8\pi)^{-1} E_\infty{}^2 |w_1'(\infty)|^2 \tag{4.13}$$

where we have made use of (4.11) and the fact that

$$\bar{t}_0^{-1} = t_0$$

since $|t_0| = 1$.

The physical problem is thus reduced to the following mathematical problem: find three functions $\varphi(z)$, $\psi(z)$ and $w_1(z)$; $\varphi(z)$, $\psi(z)$ defined on $|z| \le 1$, analytic in $|z| < 1$, with $\varphi'(z)$ and $\psi(z)$ Hölder continuous (exponent μ) on $|z| \le 1$; $w_1(z)$ defined on $|z| \ge 1$, analytic in $|z| > 1$ with $w_1(\infty) = \infty$, $w_1'(\infty) > 0$, $w_1'(z)$ Hölder continuous (exponent μ) on $|z| \ge 1$ and satisfying

$$\varphi(t) + \frac{m(t)}{m'(t)} \overline{\varphi'(t)} + \overline{\psi(t)} = f(\theta) + \text{const} , \qquad t = e^{i\theta}, \tag{4.14}$$

$$\varkappa \varphi(t) - \frac{m(t)}{m'(t)} \overline{\varphi'(t)} - \overline{\psi(t)} = 2\mu^* g(\theta), \tag{4.15}$$

where $m(t)$ is given by (2.1) and

(4.16) $$\frac{df}{d\theta} = +A_\infty \, t m'(t) \left| \frac{1 + [w_0^{-1}[m(t)]]^2}{w_1'[w_0^{-1}[m(t)]]} \right|^2 , \qquad t = e^{i\theta},$$

(4.17) $$\frac{dg}{d\theta} = i t \left[\frac{w_1'[w_0^{-1}[m(t)]]}{w_0'[m(t)]} - 1 \right] m'(t) , \qquad t = e^{i\theta},$$

where we have used (4.12) and (4.8) with A_∞ given by (4.13) and where $w_0(z)$ is the conformal mapping of $|z| > 1$ onto the exterior of $m(e^{i\theta})$, $0 \le \theta < 2\pi$ with the points at infinity corresponding and $w_0'(\infty) > 0$ and therefore $w_0(z)$ is theoretically known.

§ 5. Solution of the Mathematical Problem

Let

$$w(t) = w_1[w_0^{-1}[m(t)]] , \tag{5.1}$$

then

$$w'(t) = \frac{w_1'[w_0^{-1}[m(t)]]}{w_0'[m(t)]} \; m'(t) = u(x,y) + i\,v(x,y),$$

$t = x + iy$, u, v real.

In the notation of Theorem 3 we have from (4.16) and (4.17),

$$\lambda = A_\infty$$

$$\mathcal{F}\,[t,w'(t)] = t\,m'(t)\left|\frac{1+\{w_0^{-1}[m(t)]\}^2}{w'(t)}\;\frac{m'(t)}{w_0'[m(t)]}\right|^2 \tag{5.2}$$

$$\mathcal{F}_1[t,w'(t),\lambda] = \lambda\,\mathcal{F}[t,w'(t)]$$

$$\mathcal{G}_1[t,w'(t)] = -\,i\,t\,m'(t) + i\,t\,w'(t)$$

$$\mathcal{G} \equiv 0$$

$$a_1 \equiv 0, \quad a_2(t) = -\,i\,t\,m'(t), \quad b_2(t) = i\,t$$

$$\lambda_0 = 0$$

$$w_0^{*\prime} = R(t) \quad \text{(of (III.1) Theorem 1)}$$
$$= m'(t) \; .$$

$$\mathfrak{F}[t,w'(t)] = t\,m'(t)\left|\frac{m'(t)[1+\{w_0^{-1}[m(t)]\}^2]}{w_0'[m(t)]}\right|^2 \sum_{j,k=0}^{\infty} f_{jk}(t)[u-m_1]^j[v-m_2]^k$$

where

$$m' = m_1 + i\,m_2$$

and where f_{jk} are given as the algebraic solution of the system of $1+2+3+\cdots+(j+k-1)+k+1 = \frac{(j+k)(j+k-1)}{2}+k+1$ equations.

$$(5.3)\quad \begin{aligned}
&a_{00}f_{00} &&= 1\\
&a_{10}f_{00} + a_{00}f_{10} &&= 0\\
&a_{01}f_{00} + \qquad\qquad + a_{00}f_{01} &&= 0\\
&\qquad\cdots\\
&a_{jk}f_{00} + a_{j-1,k}f_{10} + a_{j,k-1}f_{01} + a_{j-2,k}f_{20} + a_{j-1,k-1}f_{11} + a_{j,k-2}f_{02} + \cdots &&= 0
\end{aligned}$$

in which

$$a_{00} = 2|m'(t)|^2,\quad a_{10} = 2m_1,\quad a_{01} = 2m_2,\quad a_{20} = 1,\quad a_{11} = 0,$$

$$a_{02} = 1,\quad a_{jk} = 0 \quad \text{for } j+k \geq 3.$$

The system (5.3) can be solved easily and explicitly for the f_{jk} in terms of the a's since the coefficient matrix is triangular.

The integral equation in this case is

$$(5.4)\qquad w'(t) = m'(t) + \lambda\sum_{j=1}^{n^*} t^{j-1} L\{P^j[\sigma,w'(\sigma)]\}$$

$$+\lambda\frac{\kappa-1}{4\mu^* it}\mathfrak{F}[t,w'(t)] - \lambda\frac{(\kappa+1)}{4\pi\mu^*}\int_{|\sigma|=1}\frac{\mathfrak{F}[\sigma,w'(\sigma)]}{\sigma-t}\,d\sigma\,,$$

where

$$L = \frac{(\varkappa+1)i}{4\pi\mu^*} \int_0^{2\pi} d\alpha ,$$

$$P^j[\sigma,w'(\sigma)] = \sum_{k=1}^{n^*} \frac{1}{k} \{\sigma^{-k}\mathfrak{F}[\sigma,w'(\sigma)]p_{kj}^2 - \sigma^k \overline{\mathfrak{F}[\sigma,w'(\sigma)]}\, p_{kj}^2\} , \qquad \sigma = e^{i\alpha},$$

where

$$p_{kj}^1 = kj \text{ element of the matrix } C^T B^T,$$

$$p_{kj}^2 = kj \text{ element of the matrix } \bar{C}^T \bar{B}^T B^T,$$

where B is given in (2.4) and

$$C = [I - \bar{B}B]^{-1} ,$$

and $\mathfrak{F}$ is given by (5.2).

In an effort to apply Theorem 3, our next goal is to obtain estimates for $\|f_{jk}(t)\|\mu$. We shall restrict ourselves to the case

$$m_1(t) = \operatorname{Re} m'(t) \neq 0 \quad \text{for} \quad |t| = 1$$

(5.4.1) or

$$m_2(t) = \operatorname{Im} m'(t) \neq 0 \quad \text{for} \quad |t| = 1 .$$

Without loss of generality assume $m_1(t) \neq 0$ for $|t| = 1$ and further, without loss of generality assume $m_1(t) > 0$ for $|t| = 1$. Let

$$0 < \alpha_1 \le m_1(t) \le \beta_1 \quad \text{on} \quad |t| = 1$$
$$\alpha_2 \le m_2(t) \le \beta_2 \quad \text{on} \quad |t| = 1$$

and

(5.5) $$\mathcal{D}_w = \{w = u + iv : (1-\theta)\alpha_1 < u,\ 0 < \theta < 1\}.$$

Then $P^j[t,w']$ and $\mathfrak{Z}[t,w']$ are defined, single-valued and analytic on $\Gamma \times \mathcal{D}_w$ with $H(\mu)$ coefficients. Let

$$I_1 = \{\zeta_1 = \xi_1 + i\eta_1 : \alpha_1 \le \xi_1 \le \beta_1,\ \eta_1 = 0\}$$

$$I_2 = \{\zeta_2 = \xi_2 + i\eta_2 : \alpha_2 \le \xi_2 \le \beta_2,\ \eta_2 = 0\}$$

and let ρ be so small that the circle of radius ρ centered at $(\alpha_1, 0)$ does not intersect the lines $\xi_1 = \pm \eta_1$. Then let

$$R_1(\rho) = I_1 + \mathcal{D}(0,\rho) ,$$

$$R_2(r_2) = I_2 + \mathcal{D}(0,r_2) , \qquad r_2 > 0,$$

where by the sum of two sets A, B we mean:

$$A + B = \{c : c = a+b, \ a \in A, \ b \in B, \ a,b \text{ vectors from origin to points of sets } A \text{ and } B \text{ respectively}\} ,$$

and $\mathcal{D}(0,r)$ is the closed disc with center at the origin and of radius r, (see Fig. 2).

For $\xi_1 + i\eta_1$ in R_1 we have

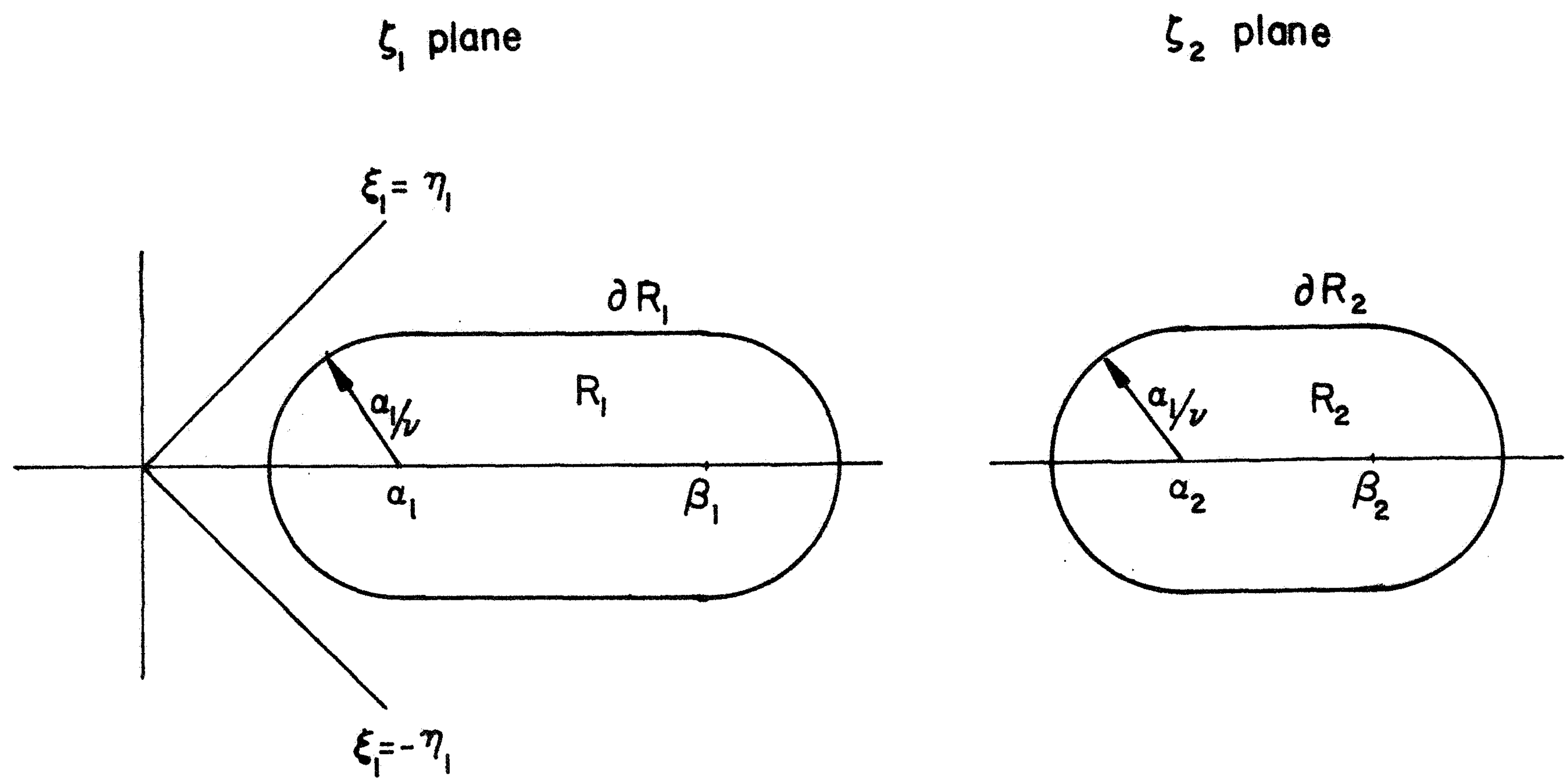
$$\begin{aligned} \xi_1^{\,2} - \eta_1^{\,2} &= (\xi_1 - \eta_1)(\xi_1 + \eta_1) \\ &\geq (\alpha_1 - \rho - \rho)(\alpha_1 - \rho) \\ &> 0 \end{aligned}$$

provided

$$2\rho < \alpha_1 .$$

Let ν = real number ≥ 3, and let

(5.6) $$A[\zeta_1,\zeta_2] = \frac{1}{\zeta_1^{\,2} + \zeta_2^{\,2}} .$$



ζ_1 plane
ζ_2 plane
$\xi_1 = \eta_1$
$\xi_1 = -\eta_1$
∂R_1
R_1
a_1/ν
a_1
β_1
∂R_2
R_2
a_1/ν
a_2
β_2

FIG. 2

Then for

$$(\zeta_1,\zeta_2) \in \text{polycylinder } R_1\left(\frac{\alpha_1}{\nu}\right) \times R_2\left(\frac{\alpha_1}{\nu}\right) ,$$

$A[\zeta_1,\zeta_2]$ is an analytic function of the two complex variables $\zeta_1 = \xi_1 + i\eta_1$ and $\zeta_2 = \xi_2 + i\eta_2$ since on this polycylinder

$$(5.6.1) \qquad \begin{aligned} \operatorname{Re}\{\zeta_1^{\,2} + \zeta_2^{\,2}\} &= \xi_1^{\,2} - \eta_1^{\,2} + \xi_2^{\,2} - \eta_2^{\,2} \\ &\geq (\xi_1 - \eta_1)(\xi_1 + \eta_1) - \eta_2^{\,2} \\ &> \left(\alpha_1 - \frac{\alpha_1 m}{\nu} - \frac{\alpha_1}{\nu}\right)\left(\alpha_1 - \frac{\alpha_1}{\nu}\right) - \left(\frac{\alpha_1}{\nu}\right)^2 \\ &= \alpha_1^{\,2}\left\{\left(1 - \frac{2}{\nu}\right)\left(1 - \frac{1}{\nu}\right) - \frac{1}{\nu^2}\right\} \\ &\geq \left(\frac{\alpha_1}{\nu}\right)^2 > 0 , \end{aligned}$$

since $\nu \geq 3$. $A[\zeta_1,\zeta_2]$ is the analytic continuation of $A[u,v]$ on $I_1 \times I_2$ into $R_1(\alpha_1/\nu) \times R_2(\alpha_1/\nu)$, and by (5.2):

$$(5.7) \qquad \mathfrak{F}[t,w'] = tm'(t)\left|\frac{m'(t)[1 + \{w_0^{-1}[m(t)]\}^2]}{w_0'[m(t)]}\right|^2 A[u,v] ,$$

$w' = u + iv$. By Cauchy's formula, we see for $u \in I_1$, $v \in I_2$ that:

$$A[u,v] = (2\pi i)^{-2} \int_{\partial R_1} \int_{\partial R_2} \frac{d\zeta_1 d\zeta_2 A[\zeta_1,\zeta_2]}{(\zeta_1-u)(\zeta_2-v)} \tag{5.8}$$

$$= \sum_{j,k=0}^{\infty} a_{jk}(m_1,m_2)(u-m_1)^j(v-m_2)^k$$

where $R_1 = R_1[\alpha_1/\nu]$, $R_2 = R_2[\alpha_1/\nu]$,

$$a_{jk}(m_1,m_2) = (2\pi i)^{-2} \int_{\partial R_1} \int_{\partial R_2} \frac{A[\zeta_1,\zeta_2]\, d\zeta_1\, d\zeta_2}{(\zeta_1-m_1)^{j+1}(\zeta_2-m_2)^{k+1}}$$

since

$$m_1 = \operatorname{Re} m'(t) \subset I_1 \quad \text{for} \quad |t| = 1$$

$$m_2 = \operatorname{Im} m'(t) \subset I_2 \quad \text{for} \quad |t| = 1 .$$

Thus

$$|a_{jk}| \le \gamma(2\pi)^{-2}\, \ell_1 \ell_2 A^* \left(\frac{\nu}{\alpha_1}\right)^{j+1}\left(\frac{\nu}{\alpha_1}\right)^{k+1} \tag{5.9}$$

where

$$\gamma = \text{arbitrary number} \ge 1 ,$$

$$\begin{aligned} \ell_1 &= \text{length of } \partial R_1 = 2\{\beta_1 - \alpha_1 + \pi(\alpha_1/\nu)\} , \\ \ell_2 &= \text{length of } \partial R_2 = 2\{\beta_2 - \alpha_2 + \pi(\alpha_1/\nu)\} , \\ A^* &= \max_{\zeta_i \in \partial R_i} |A(\zeta_1,\zeta_2)| \le \frac{\nu^2}{\alpha_1^{\,2}} . \end{aligned} \tag{5.10}$$

In order to compute

(5.11)
$$\sup_{|t_1|=|t_2|=1} \frac{|a_{jk}[m_1(t_2),m_2(t_2)] - a_{jk}[m_1(t_1),m_2(t_1)]|}{|t_2 - t_1|^{\mu}},$$

we note that

$$\frac{1}{(\zeta_1-q_2)^{j+1}} - \frac{1}{(\zeta_1-q_1)^{j+1}} = (q_1-q_2) \sum_{\ell=0}^{j} \frac{1}{(\zeta_1-q_1)^{\ell+1}(\zeta_1-q_2)^{j+1-\ell}} ,$$

thus

$$\Delta_2 a_{jk} \equiv a_{jk}[m_1(t_2),m_2(t_2)] - a_{jk}[m_1(t_1),m_2(t_2)] =$$

$$= [m_1(t_2)-m_1(t_1)](2\pi i)^{-2} \int_{\partial R_1} \int_{\partial R_2} \frac{d\zeta_1 d\zeta_2 A[\zeta_1,\zeta_2]}{[\zeta_2-m_2(t_2)]^{k+1}} \sum_{\ell=0}^{j} \frac{1}{[\zeta_1-m_1(t_1)]^{\ell+1}[\zeta_1-m_1(t_2)]^{j+1-\ell}} .$$

Thus

(5.12)
$$|\Delta_2 a_{jk}| \le \gamma|m_1(t_2) - m_1(t_1)|(2\pi)^{-2} \ell_1 \ell_2 A^* \left(\frac{\nu}{\alpha_1}\right)^{k+1} (j+1) \left(\frac{\nu}{\alpha_1}\right)^{j+2} .$$

and similarly for

$$\Delta_1 a_{jk} \equiv a_{jk}[m_1(t_1),m_2(t_2)] - a_{jk}[m_1(t_1),m_2(t_1)] ,$$

we have

$$|\Delta_1 a_{jk}| \le \gamma|m_2(t_2) - m_2(t_1)|(2\pi)^{-2} \ell_1 \ell_2 A^*(k+1) \left(\frac{\nu}{\alpha_1}\right)^{k+2} \left(\frac{\nu}{\alpha_1}\right)^{j+1}.$$

Thus

$$|a_{jk}[m_1(t_2),m_2(t_2)] - a_{jk}[m_1(t_1),m_2(t_1)]| \equiv |\Delta_2 a_{jk} + \Delta_1 a_{jk}|$$

$$\le |t_2-t_1|^{\mu} \gamma k_{m'}(\mu)(2\pi)^{-2} \ell_1 \ell_2 A^* \left(\frac{\nu}{\alpha_1}\right)^{j+k+3} (j+k+2) ,$$

where

$$\sup_{|t_1|=|t_2|=1} \frac{m'(t_2)-m'(t_1)}{|t_2-t_1|^{\mu}} = k_{m'}(\mu) .$$

Thus we get

(5.13) $$\|a_{jk}\|_{\mu} \le a^* r_3^{\,j+k+2}\left(\frac{\nu}{\alpha_1}\right)^{j+k+2},$$

where

(5.14)
$$a^* = \gamma(2\pi)^{-2}\,\ell_1\ell_2 A^*$$
$$r_3 = 1 + k_{m'}(\mu)\left(\frac{\nu}{\alpha_1}\right)(j+k+2) .$$

From (5.7) and (5.8) we see that if

(5.15) $$\mathfrak{F}[t,w'] = \sum_{j,k=0}^{\infty} F_{jk}(t)[u-m_1]^j[v-m_2]^k ,$$

then

(5.16) $$F_{jk}(t) = t\,m'(t)\left|\frac{1+\{w_0[m(t)]\}^2 m'(t)}{w_0'[m(t)]}\right|^2 a_{jk}[m_1,m_2].$$

Thus

(5.17) $$\|F_{jk}(t)\|_{\mu} \le F_0\|a_{jk}[m_1,m_2]\|_{\mu} ,$$

where

(5.18) $$F_0 = [1+2^{1-\mu}]\,\|m'(t)\|_{\mu}\,\|[1+\{w_0[m(t)]\}^2]m'(t)\|_{\mu}^2\,\|\{w_0'[m(t)]\}^{-1}\|_{\mu}^2 .$$

By combining (5.13) with this, we get

$$\|F_{jk}(t)\|_\mu \le f_0^*(\nu r_3/\alpha_1)^j(\nu r_3/\alpha_1)^k , \tag{5.19}$$

where

$$f_0^* = F_0 a^*(\nu r_3/\alpha_1)^2. \tag{5.20}$$

We are now in a position to verify the hypotheses of Theorem 3.

(H.1) By hypotheses.

(H.2) $w_0^{*\prime}(t) = m'(t)$.

(H.3) $\mathfrak{F}_1$, a_1, $\mathfrak{F}$, $\mathscr{E}_1$, a_2, $b_2 = it$, $\mathscr{E} \equiv 0$ given in (5.2), $|b_2(t)| = 1 \ne 0$ for $|t| = 1$, $a_1 \equiv 0$ and $a_2(t) = -i\,t\,m'(t) \in H(\mu_1)$, $0 < \mu < \mu_1 < 1$ for $|t| = 1$. $\mathfrak{F}$ is single-valued complex function defined on $\{t : |t| = 1\} \times \mathscr{D}_w$, where by (5.5) $\mathscr{D}_w = \{w = u + iv : 0 < \alpha_1(1-\theta) \le u,\ 0 < \theta < 1\}$; $\lambda_0 = 0$; $\mathfrak{F}$ is an analytic function on $\mathscr{D}_w(\mu)$ with $H[\mu, f_0^*, a_f, b_f, c_f, q_1, q_2]$ coefficients given in (5.15), (5.16); f_0^* is given by (5.20), (5.18), (5.14), (5.10), $a_f = \nu r_3/\alpha_1$, $b_f = \nu r_3/\alpha_1$, $c_f = 0$, $q_1 = m_1 = \operatorname{Re} m'(t)$, $q_2 \equiv m_2 = \operatorname{Im} m'(t)$, $\lambda_0 = 0$; $g_0^* = a_g = b_g = c_g \equiv 0$.

(H.4) In order to find φ_0 and ψ_0, we note that equation III of Theorem 1 becomes by (5.4) for $\lambda = \lambda_0 = 0$

$$w'(t) = m'(t).$$

But the fact that $w'(t)$ satisfies III for $\lambda = 0$, i.e., $w'(t) = m'(t)$, is sufficient to insure the existence of two functions φ and ψ of the proper type, and these can be given explicitly by consulting the proof of Theorem 1.

Since all the hypotheses of Theorem 3 are satisfied, we have that there exist a unique triplet of functions $\varphi(z,\lambda)$, $\psi(z,\lambda,w'(t,\lambda)$; defined for $|z| \le 1$, $|t| = 1$, $\lambda \in J$;

$$J = J_1 \cap J_2,$$

$$J_1 = \{\lambda : -d/(2U^* - dc) < \lambda < d/(2U^* + dc)\},$$

$$J_2 = \{\lambda : \text{for the solution } w'(t,\lambda),\ \operatorname{Re} w'(t,\lambda) > (1-\theta)\alpha_1,$$
$$0 < \theta < 1,\ |t| = 1\},$$

with α_1 given in (5.5), d, c and U^* given in Theorem 3; (note that J is an open interval since $\operatorname{Re} w'(t,0) = \operatorname{Re} m'(t) \ge \alpha_1$) where $w'(t,\lambda)$ is an analytic function of λ on J with $H[\mu,\lambda]$ coefficients, $\frac{\partial\varphi}{\partial t}(t,\lambda)$ and $\psi(t,\lambda) \in H(\mu)$ for each $\lambda \in J$; and which satisfy (4.14), (4.15), (4.16) and (4.17) ($w(t)$ given by (5.1)). Moreover $w'(t,\lambda)$ is the unique solution of the integral equation (5.4).

Thus, within the class of regions bounded by Jordan curves with $H(\mu)$ tengents, there is a unique such region whose boundary $w(t,\lambda)$ for $|t| = 1$ has $w'(t,\lambda)$ analytic in λ with $H(\mu)$ coefficients and which solves the electroelastic free boundary problem where it is assumed that $m(z)$ is of the form (2.1), satisfies (2.3) and $\operatorname{Re} m'(t) \neq 0$ for $|t| = 1$ or $\operatorname{Im} m'(t) \neq 0$ for $|t| = 1$. Moreover $w'(t,\lambda)$ satisfies the non-linear singular integral equation (5.4).

§6. Example

In this section we shall illustrate that the conditions (2.3) and (5.4.1) are satisfied for the epitrochoids (see Fig. 3):

$$m(z) = c_0(z + c_n z^n), \qquad c_0 > 0, \quad 0 < nc_n < 1, \quad n > 1.$$

In this case

$$m'(z) = c_0(1 + nc_n z^{n-1})$$

and (2.2) becomes

$$\frac{m(z)}{\overline{m'}(1/z)} = \frac{(z + c_n z^n)z^{n-1}}{z^{n-1} + nc_n} = c_n z^n + (1 - nc_n^2)z - nc_n(1 - nc_n^2)\frac{z}{z^{n-1} + nc_n} .$$

Thus

$$b_n = c_n, \qquad b_{n-1} = b_{n-2} = \cdots = b_2 = 0,$$

$$b_1 = (1 - nc_n^2), \qquad b_0 = 0 ,$$

and

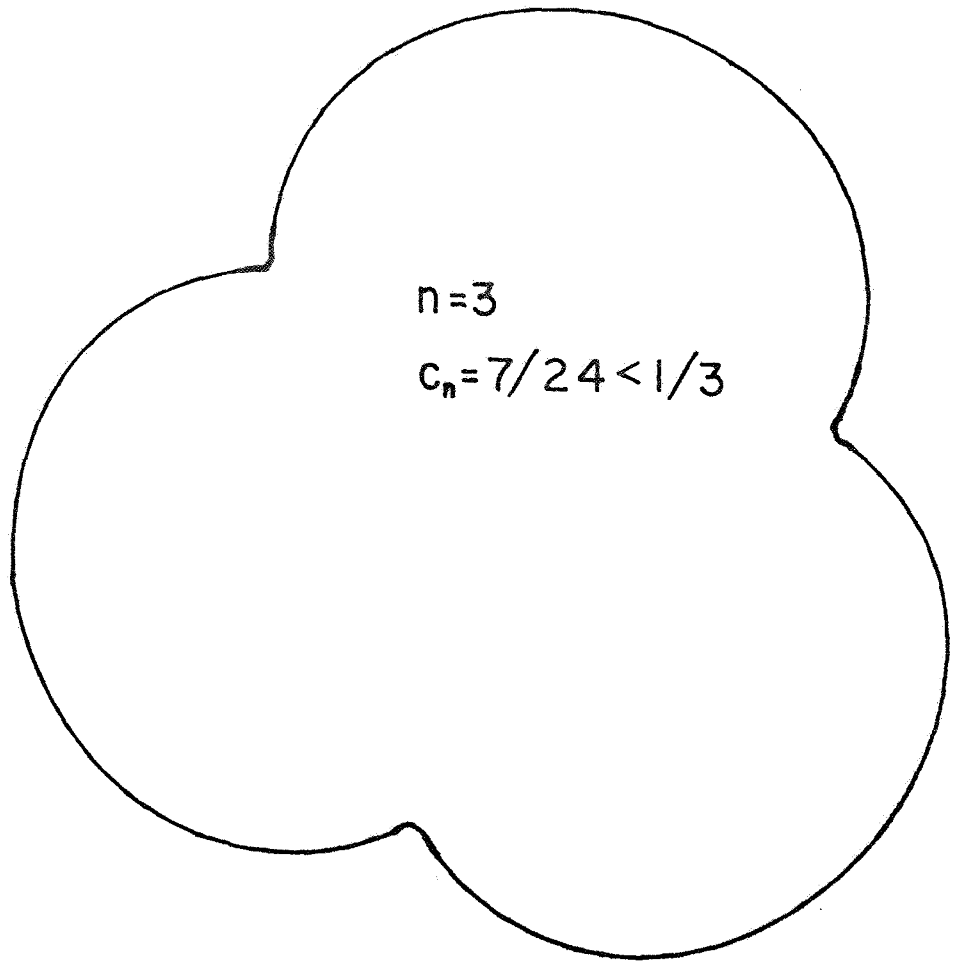


FIG. 3

$$B = \begin{bmatrix} 1-nc_n^2 & 0 & 0 & \cdots & 0 & 0 & nc_n \\ 0 & 0 & 0 & \cdots & 0 & (n-1)c_n & 0 \\ \vdots & \vdots & & & \cdot^{\cdot^{\cdot}} & & \vdots \\ 0 & 0 & & & & & \\ 0 & 2c_n & 0 & & & & \\ c_n & 0 & 0 & & \cdots & & 0 \end{bmatrix}$$

$$\bar{\bar{B}}B = \begin{bmatrix} (1-nc_n^2)^2+nc_n^2 & 0 & 0 & \cdots & 0 & nc_n(1-nc_n^2) \\ 0 & 2(n-1)c_n^2 & 0 & & & 0 \\ 0 & 0 & 3(n-2)c_n^2 & & & \vdots \\ \vdots & & & \ddots & & \\ 0 & & & & & 0 \\ c_n(1-nc_n) & 0 & 0 & \cdots & 0 & nc_n^2 \end{bmatrix} .$$

Thus

$$|I-\bar{\bar{B}}B| = \begin{vmatrix} (1-nc_n^2)nc_n^2 & 0 & 0 & \cdots & 0 & nc_n(1-nc_n^2) \\ 0 & 1-2(n-1)c_n^2 & 0 & & & 0 \\ 0 & 0 & 1-3(n-2)c_n^2 & & & \vdots \\ \vdots & & & \ddots & & \\ 0 & & & & & 0 \\ c_n(1-nc_n) & 0 & 0 & \cdots & 0 & 1-nc_n^2 \end{vmatrix}$$

$$= n^2c_n^3[1-c_n][1-nc_n^2][1-2(n-1)c_n^2][1-3(n-2)c_n^2] \cdots [1-(n-1)2c_n^2] .$$

But $0 < nc_n < 1$, thus for $m = 1,2,3,\ldots,n-1$,

$$1 - m[n-(m-1)]c_n^2 > 0 .$$

Thus

$$|I - \bar{\bar{B}}B| > 0$$

and condition (2.2) is satisfied for all epitrochoids of the form considered.

To check (5.4.1) we simply note that

$$\operatorname{Re} m'(z) = c_0\{1 + nc_n \cos[(n-1)\theta]\} > 0$$

since $c_0 > 0$ and $0 < nc_n < 1$.

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