WICHITA STATE UNIVERSITY

A LOCAL EXISTENCE THEOREM FOR QUASI-LINEAR

PARTIAL DIFFERENTIAL EQUATIONS*

by

Alan R. Elcrat and Lucio Artiaga

MATHEMATICS DEPARTMENT

WICHITA, KANSAS
67208

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Alan R. Elcrat and Lucio Artiaga

Mathematics Department
Wichita State University

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I. Introduction.

We investigate here conditions on a quasi-linear partial differential equation of second order sufficient to guarantee existence of a smooth solution. More precisely we consider an equation

$$(1) \qquad A_{ij}u_{x_i x_j} + D = 0$$

in a bounded region R in E^n subject to the condition

$$u = 0 \quad \text{on} \quad \partial R.$$

We use the subscript x_i to denote partial differentiation with respect to x_i and make free use of the summation convention on repeated indices (two terms with the same index are summed over the common index). The coefficients A_{ij}, D are functions of $2n$ variables, the first n being $x = (x_1, \ldots, x_n)$, and the second n being $u_x = (u_{x_1}, \ldots, u_{x_n})$.

We will seek solutions which together with their derivatives through second order, are uniformly Hölder continuous, that is which belong to the Banach Space $C_{2+\alpha}(R)$, where α is a positive number less than one. Therefore we assume immediately that ∂R is sufficiently smooth, that is of class $C_{2+\alpha}$ (see [1]), and that the functions A_{ij} are "uniformly elliptic" with respect to elements of $C_{2+\alpha}(R)$, that is

$$A_{ij}(x,u_x(x))\lambda_i\lambda_j \geq \text{const.}|\lambda|^2, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \in E^n,$$

where the constant may depend on the $u \in C_{2+\alpha}(R)$ appearing but does not depend on x.

The approach taken will be to give sufficient conditions for the differential operator involved to map one Banach Space of Hölder continuous functions into another in a way sufficiently nice for the theorem of Kantorovich on convergance of Newtons method in Banach Spaces to be applied.

Before proceeding to the proof of our main theorem we must make a survey of some prerequisite material.

II. Preliminaries.

The work done below will be carried out in two Banach Spaces of Hölder continuous functions, $C_{2+\alpha}(R)$ and $C_\alpha(R)$. The latter consists of functions uniformly Hölder continuous in R, and is normed by

$$|\phi|_\alpha = |\phi|_0 + H(\phi)$$

where $|\ |_0$ denotes the maximum norm over R and

$$H(\phi) = \sup_{x,y\in R} \frac{|\phi(x) - \phi(y)|}{|x - y|^\alpha} .$$

The former consists of functions which, together with their derivatives through second order, belong to $C_\alpha(R)$ and is normed by

$|\phi|_{2+\alpha} = |\phi|_\alpha$ + sum of C_α norms of derivatives through 2nd order.

The reader is referred to [1] for relevant facts about these spaces.

We need two results on linear elliptic equations, which follow from theorems given in [1], the first an a priori estimate, the second an existence theorem. We consider an equation

$$a_{ij}u_{x_i x_j} + b_i u_{x_i} = f$$

in R together with prescription of zero boundary values. The functions

a_{ij}, b_i, f are assumed to be elements of $C_\alpha(R)$, and it is assumed that

$$a_{ij}\,\lambda_i\lambda_j \geq m|\lambda|^2 \qquad \lambda = (\lambda_1, \ldots, \lambda_n) \in E^n$$

where m is a positive number independent of x. The needed results are

(1) There is a constant C depending only on α, R, m and an upper bound on the C_α norms of a_{ij}, b_i such that for any solution u of the above linear problem that is an element of $C_{2+\alpha}(R)$ the inequality

$$|u|_{2+\alpha} \leq C|f|_\alpha$$

holds.

and

(2) Under the above assumptions for any f in $C_\alpha(R)$ there is a solution u in $C_{2+\alpha}(R)$ of the above linear problem.

It is worth noting here that a solution of the above problem is unique by the maximum principle. (See [2].)

We assume that the reader is familiar with elementary facts about Frechét derivatives, which we will call F-derivatives. (These are available in [3] or [4] for example.) If X and Y are Banach Spaces, we denote by $B(X,Y)$ the space of continuous linear operators from X into Y, and by $B(X,X;Y)$ the space of continuous bilinear operators from $X \times X$ into Y.

The theorem of Kantorovich on convergance of Newtons method is given below. A proof can be found in [4] .

Theorem. (Kantorovich) Suppose P is a continuous mapping of the sphere $A = \{x \mid \|x - x_0\| < R\}$ in the Banach space X into the Banach space Y, and that P is twice continuously differentiable on the closed sphere $A_0 = \{x \mid \|x - x_0\| \leq r\}$, where $r < R$. Further, suppose that

(1) $[P'(x_0)]^{-1}$ exists,

(2) $\| [P'(x_0)]^{-1} \| \leq B$,

(3) $\| P(x_0) \| \leq \eta$,

(4) $\| P''(x) \| \leq K$ for x in A.

Then, if $h = KB^2\eta < 1/2$ and $r_0 = (1 - \sqrt{1 - 2h})B\eta/h \leq r$, the equation

$$P(x) = 0$$

has a solution x^* in the sphere $\{x \mid \|x_0 - x\| \leq r_0\}$ to which the sequence

$$x'_{n+1} = x'_n - [P'(x_0)]^{-1}P(x'_n)$$

converges. In addition, the sequence

$$x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n)$$

is well defined and converges to x^*. The following error bounds hold:

$$\|x^* - x_n\| \leq (2h)^{2^n}\eta/h2^n, \text{ and}$$
$$\|x^* - x'_n\| \leq (1 - \sqrt{1 - 2h})^{n+1}\eta/h.$$

If

$$r < r_1 = (1 + \sqrt{1 - 2h})B\eta/h.$$

the solution x^* is unique in A_0.

The space Y which we will utilize is just $C_\alpha(R)$. A moments reflection shows that the functions in $C_{2+\alpha}(R)$ which vanish on ∂R, form a Banach Space and this is the space X which we use. The operator P is defined by letting $P(u)$ be the left hand side of (1) with u substituted. Therefore we have translated our problem into that of existence of a zero in X for the operator P.

III. The Existence Theorem.

The main task in what follows is to show that under appropriate conditions the operator P, and the spaces X and Y defined above, satisfy the conditions needed to apply the Kantorovich theorem. The first step is to see that P does in fact map X into Y. We must have

$$(2) \qquad P(u) = A_{ij}u_{x_ix_j} + D \in Y$$

for u in X. Since Y is closed under (pointwise) multiplication, it suffices to show that $A_{ij}(x,u_x(x))$, $D(x,u_x(x))$ are uniformly Hölder continuous for u in X. Letting F denote anyone of A_{ij}, or D, we propose the following sufficient condition on F in order that this be true:

$$\text{E)} \qquad |F(x,\xi) - F(y,\eta)| \leq K\left[|x - y|^\alpha + |\xi - \eta|\right]$$

for $x,y \in \bar{R}$ $|\xi|,|\eta| \leq c$ where K depends only on c.

Lemma 1. *If each of* A_{ij}, D *satisfy* E, P *maps* X *into* Y.

PROOF: Using F as above, and applying E, we have

$$|F(x,u_x(x)) - F(y,u_x(y))| \leq K\left[|x - y|^\alpha + |u_x(x) - u_x(y)|\right]$$

where K is determined by u. Then, since $u \varepsilon X$,

$$|F(x,u_x(x)) - F(y,u_x(y)| \leq K[|x - y|^\alpha + K_1|x - y|^\alpha] = K(1 + K_1)|x - y|^\alpha ,$$

and the lemma is proven.

In order to investigate continuity of P, suppose that we have chosen u^o in X and consider

$$P(u) - P(u^o) = A_{ij}u_{x_ix_j} - A^o_{ij}u^o_{x_ix_j} + D - D^o , \tag{3}$$

where we have denoted the coefficient with u^o substituted by the same superscript. We let

$$\begin{aligned} A_{ij} &= A^o_{ij} + \Delta A_{ij}, \\ D &= D^o + \Delta D , \\ u &= u^o + h \end{aligned}$$

and (3) becomes

$$P(u) - P(u^o) = \Delta A_{ij}u^o_{x_ix_j} + A^o_{ij}h_{x_ix_j} + \Delta A_{ij}h_{x_ix_j} ,$$

so that

$$|P(u) - P(u^o)|_\alpha \leq |\Delta A_{ij}|_\alpha |u^o_{x_ix_j}|_\alpha + |A^o_{ij}|_\alpha |h_{x_ix_j}|_\alpha + |\Delta A_{ij}|_\alpha |h_{x_ix_j}|_\alpha .$$

Clearly then, a condition which guarantees that

$$|\Delta A_{ij}|_\alpha \text{ and } |\Delta D|_\alpha \text{ go to zero as } |h|_{2+\alpha}$$

does will imply continuity of P at u^o. We assert that it suffices to assume that the first and second derivatives of A_{ij} and D with respect to their "gradient" variables satisfy the condition E given above. For brevity we introduce the class of functions G_m: a real-valued function with domain $\bar{R} \times E^n$ which has continuous derivatives through the mth order with respect to its last n variables and which is such that it and all of these derivatives satisfy condition E is in G_m.

Lemma 2. If A_{ij} and D are in G_2, P is continuous on X.

PROOF: Retaining the above notation, and letting F represent any of A_{ij}, D as before, it suffices to show that

$$|\Delta F|_\alpha \to 0 \quad \text{as} \quad |h|_{2+\alpha} \to 0.$$

The assumptions on F clearly allow application of Taylors' theorem and imply the validity of

$$\text{(4)} \qquad \Delta F = F_{u_{x_i}}(x, u_x^0)h_{x_i} + F_{u_{x_i}u_{x_j}}(x, u_x^0 + \theta h_x)h_{x_i}h_{x_j} ,$$

where θ is a positive number less than one.

Since functions which satisfy E yield elements of Y when elements of X are substituted in them, we have

$$|\Delta F|_\alpha \le |F^0_{u_{x_i}}|_\alpha |h_{x_i}|_\alpha + |F^\theta_{u_{x_i}u_{x_j}}|_\alpha |h_{x_i}|_\alpha |h_{x_i}|_\alpha$$

where $F^\theta_{u_{x_i}u_{x_j}}$ denotes the appropriate function from (4). It remains to show that there is an $r > 0$ such that $|F^\theta_{u_{x_i}u_{x_j}}|_\alpha$ is bounded for $|h|_{2+\alpha} < r$. Let G denote the function $F_{u_{x_i}u_{x_j}}(\cdot,\cdot)$. We have, since $F \in G_2$,

$$|G(x, u_x^0(x) + \theta h_x(x)) - G(y, u_x^0(y) + \theta h_x(y))| \le$$
$$|G(x, u_x^0(x) + \theta h_x(x)) - G(x, u_x^0(x) + \theta h_x(y))| + |G(x, u_x^0(x) + \theta h_x(y)) - G(y, u_x^0(y) + \theta h_x(y))| ,$$

which is not greater than

$$\theta|h_x(x) - h_x(y)| + K|x - y|^\alpha + K|u_x^0(x) - u_x^0(y)|$$

for x,y in $\bar{R}$. Dividing by $|x - y|^\alpha$ and taking the supremum for x,y in R we get the desired result.

We are now ready to investigate differentiability properties of the operator P. To this end we note that assuming each of A_{ij}, D belong to G_3 guarantees the validity of

$$F(x,u_x + h_x) = F(x,u_x) + F_{u_{x_i}}(x,u_x)h_{x_i} + F_{u_{x_i}u_{x_j}}(x,u_x)h_{x_i}h_{x_j}$$

$$+ F_{u_{x_i}u_{x_j}u_{x_k}}(x,u_x + \theta h_x)h_{x_i}h_{x_j}h_{x_k}, \quad 0 < \theta < 1,$$

where F represents any of A_{ij}, D, and $u,h \in X$. Then we have

$$\begin{aligned} P(u + h) &= A_{ij}(x,u_x + h_x)u_{x_ix_j} + A_{ij}(x,u_x + h_x)h_{x_ix_j} + D(x,u_x + h_x) \\ &= A_{ij}(x,u_x)u_{x_ix_j} + (A_{ij})_{u_{x_k}}u_{x_ix_j}h_{x_k} + (A_{ij})_{u_{x_k}u_{x_l}}u_{x_ix_j}h_{x_k}h_{x_l} \\ &\qquad + (A_{ij}^\theta)_{u_{x_k}u_{x_l}u_{x_m}}u_{x_ix_j}h_{x_k}h_{x_l}h_{x_m} \\ &+ A_{ij}(x,u_x)h_{x_ix_j} + (A_{ij})_{u_{x_k}}h_{x_ix_j}h_{x_k} + (A_{ij})_{u_{x_k}u_{x_l}}h_{x_ix_j}h_{x_k}h_{x_l} \\ &\qquad + (A_{ij}^\theta)_{u_{x_k}u_{x_l}u_{x_m}}h_{x_ix_j}h_{x_k}h_{x_l}h_{x_m} \\ &+ D(x,u_x) + D_{u_{x_i}}h_{x_i} + D_{u_{x_ix_j}}h_{x_i}h_{x_j} + D^\theta_{u_{x_i}u_{x_j}u_{x_k}}h_{x_i}h_{x_j}h_{x_k} \end{aligned}$$

so that

$$P(u + h) = P(u) + L_u(h) + B_u(h,h) + R_u(h)$$

where

(5) $$L_u(h) = A_{ij}h_{x_ix_j} + (A_{ij})_{u_{x_k}} u_{x_ix_j} h_{x_k} + D_{u_{x_i}} h_{x_i},$$

(6) $$B_u(h,h) = (A_{ij})_{u_{x_k}u_{x_l}} u_{x_ix_j} h_{x_k} h_{x_l} + (A_{ij})_{u_{x_k}} h_{x_ix_j} h_{x_k} + D_{u_{x_i}u_{x_j}} h_{x_i} h_{x_j} ,$$

and

(7) $$R_u(h) = (A^{\theta}_{ij})_{u_{x_k}u_{x_l}u_{x_m}} \left[u_{x_ix_j} + h_{x_ix_j}\right] h_{x_k} h_{x_l} h_{x_m}$$
$$+ D^{\theta}_{u_{x_i}u_{x_j}u_{x_k}} h_{x_i} h_{x_j} h_{x_k} + (A_{ij})_{u_{x_k}u_{x_l}} h_{x_ix_j} h_{x_k} h_{x_l}$$

A quick check shows that, for fixed u in X, L_u and B_u are elements of $B(X,Y)$, $B(X,X;Y)$, respectively. Therefore, application of the reasoning used at the end of the proof of lemma 2 to the terms of $R_u(h)$ with the superscript θ proves

Lemma 3. If A_{ij}, D are in G_3, P is F-differentiable at each u in X, and $P'(u)$ is given by (6).

To proceed we must guarantee continuity of the mapping

$$u \to P'(u).$$

Consider

$$\left[P'(u) - P'(v)\right](h) = (A^u_{ij} - A^v_{ij})h_{x_ix_j} + \left[(A^u_{ij})_{u_{x_k}} u_{x_ix_j} - (A^v_{ij})_{u_{x_k}} v_{x_ix_j}\right]h_{x_k}$$
$$+ (D^u_{u_{x_k}} - D^v_{u_{x_k}})h_{x_k} ,$$

where a function F with say u substituted is denoted F^u. It follows from the above that

$$\|P'(u) - P'(v)\| \leq C \max\Big\{|A^u_{ij} - A^v_{ij}|_\alpha, \ |(A^u_{ij})_{u_{x_k}} u_{x_i x_j} - (A^v_{ij})_{u_{x_k}} v_{x_i x_j}|_\alpha,$$
$$|D^u_{u_{x_k}} - D^v_{u_{x_k}}|_\alpha\Big\},$$

where C is independent of u,v. We must insure that this quantity go to zero as v approaches u in the norm of X. That this is true for the first and third terms in brackets follows from the assumption that A_{ij}, D belong to G_3 and the content of lemma 2. In order to deal with the other term we denote $(A_{ij})_{u_{x_k}}$ by F, observe that

$$|F^u u_{x_i x_j} - F^v v_{x_i x_j}|_\alpha \leq |u_{x_i x_j}|_\alpha |F^u - F^v| + |F^v|_\alpha |v - u|_{2+\alpha} \quad ,$$

and again use the reasoning applied at the end of the proof of lemma 2. Thus the hypothesis of lemma 3 yields <u>continuous</u> differentiability as well. Since $|R_u(h)|_\alpha$ is clearly of order 3 with respect to the norm of X, it now follows that P has a second Frechét derivative at each u, which is identified in the usual way (see either [3] or [4]) with the bilinear operator B_u. In fact, an unexciting return to the ground just covered shows that with no further assumptions P is twice continuously differentiable.

<u>Corollary</u> (<u>to</u> <u>lemma</u> 3). <u>If</u> A_{ij}, $D \in G_3$, P <u>is</u> <u>twice</u> <u>continuously</u> F-<u>differentiable</u>.

We now observe that the assumptions we made in the Introduction guarantee that the operator $P'(u)$ is invertible for each u in X. In fact, the regularity assumptions, and the "uniform ellipticity" assumption made on the A_{ij} in that section allow application of the existence theorem for linear equations given in section 2, so that we know $P'(u)$ maps X onto Y. Then the maximum principle implies that $P'(u)$ is one-to-one, and the closed graph theorem guarantees continuity of the inverse operator. We have

Lemma 4. *The uniform ellipticity assumption implies that* $P'(u)$ *is invertible for each* u *in* X.

To complete the preparation for use of the Kantorovich theorem, we give the following lemma which says, roughly, that the norm of $P''(u)$ is bounded on spheres.

Lemma 5. *For each* u *in* X, $r > 0$ *there is a number* $K = K(u,r)$ *such that*

$$\|P''(u)\| \leq K$$

for u *in* $S(u,r)$.

PROOF: Looking back at the expression giving $P''(u)$ in (6), we see that one further application of the reasoning given at the end of the proof of lemma 2 suffices.

By simply looking applying the Kantorovich Theorem to the operator P we can now deduce the following theorem.

Theorem. *If* $u \varepsilon X$ *and* $r > 0$ *are such that*

$$K(u,r)\ \|[P'(u)]^{-1}\|^{2}\,|P(u)|_{\alpha} < 1/2\ ,$$

and

$$1/\ K(u,r)\|[P'(u)]^{-1}\| \leq r,$$

then there is an element u^* *in* $S(u,r)$ such that

(8) $$P(u^*) = 0$$

to which both the Newton sequence and the modified Newton sequence (begun at u*) converge.*

Of course, the uniformly ellipticity condition, and the assumption that the coefficients in P are of class G_3 have been implicitly assumed in order to obtain this theorem.

We now deduce the following

Corollary. For each u *in* X *there is a number* H *such that*

$$|P(u)|_\alpha < H$$

implies that there is a solution of the equation (8) *in a sphere around* u *in* X.

PROOF: We first need to observe that, for fixed u, the second of the inequalities needed in the above theorem can always be fulfilled by taking, if necessary, a larger upper bound for $\|P''(u)\|$. More precisely, denoting $\|[P'(u)]^{-1}\|$ by B, for any given r we need only guarantee that

$$K \geq 1/Br$$

which is certainly possible since K is an upper bound. Then observing the first inequality needed in the theorem we deduce that

$$|P(u)|_\alpha < 1/2KB^2$$

implies that there is a solution of (8) in $S(u,r)$, and the corollary is proven.

Remark 1. The conclusions of the Kantorovich theorem could also be used to deduce a "local uniqueness" result. However, for the partial differential equation under consideration a solution can be shown to be unique using the maximum principle (see for example [2]), so this is of limited interest here.

Remark 2. The theorem and corollary proven above say something essentially about retention of existence under a perturbation. To see this more clearly, suppose we assume for the moment that the function u in the above is the function identically zero. Also, note that when

$$D(x,0) \equiv 0$$

the function identically zero is a solution of the equation being studied. Therefore, the corollary says that under a sufficiently small perturbation of the function

$$D(x,0) = P(0)$$

from zero (in the norm of X) there still exists a solution (in X) of the equation.

Remark 3. We give here a more concrete class of coefficients which is contained in the class G_m utilized above. This class consists of functions with domain $\bar{R} \times E^n$ which are polynomials in the last n variables with coefficients that are uniformly Hölder continuous functions of the first n. More precisely we consider functions

$$F(x,\xi) = a_{i\ldots j}(x)\xi_1^{v_i}\ldots\xi_n^{v_j}$$

(recall the summation convention) where $v_i, \ldots, v_j$ vary, over the lowest through the highest powers appearing on $\xi_1, \ldots, \xi_n$, respectively, and each function $a_{i\ldots j}$ is an element of Y.

To see that such a function is in G_m we note that this new class is closed under partial differentiations with respect to any of the last n variables, and that

$$F(x,\xi) - F(y,\eta) = \left[a_{i\ldots j}(x) - a_{i\ldots j}(y)\right]\xi_1^{\nu_i} \ldots \xi_n^{\nu_j}$$
$$+ a_{i\ldots j}(y)\left[\xi_1^{\nu_i} \ldots \xi_n^{\nu_j} - \eta_1^{\nu_i} \ldots \eta_n^{\nu_j}\right]$$

The result follows from Hölder continuity of $a_{i\ldots j}$ and the fact that a continuously differentiable function of n variables satisfies a uniform Lipschitz condition on any compact set (the Lipschitz constant depending, of course, on the compact set.).

We will deal with an equation whose coefficients are of this type in the appendix.

<u>Appendix</u>. <u>An example: the Minimal Surface Equation</u>.

We apply the results obtained above to the n-dimensional "minimal surface" equation. We seek a function u satisfying

$$\text{(9)} \qquad M(u) \equiv (1 + u_i u_i)\Delta u - u_i u_j u_{ij} = 0$$

in R, and

$$u = \phi$$

on ∂R. We have used here and will use henceforth the notation

$$u_i = \partial u/\partial x_i \ , \ u_{ij} = \partial^2 u/\partial x_i \partial x_j \ ,$$

that is partial derivatives are indicated simply by subscripts.

In order to pose this problem in the setting of the previous paragraphs we assume immediately that the given boundary function is the restriction to ∂R of some function in $C_{2+\alpha}(R)$ which we will also call ϕ. Then, setting

$$v = u - \phi,$$

we obtain from (9) the equation

$$P(v) \equiv M(v + \phi) = 0$$

for v subject to a homogeneous Dirichlet boundary condition. Clearly, we may think of P as an operator mapping X into Y, and a solution of the operator equation

$$P(v) = 0$$

in X is a solution of our problem. In fact, the last remark of paragraph 3 together with the lemmas of that paragraph show that P is twice continuously F-differentiable on X, and that the norm of its second derivative is bounded on spheres. In order to make use of the theorem of paragraph 3 we need only show that the F-derivative of P is a (uniformly) elliptic operator for each v in X. We have

$$P(v) = N(v) + L(v) + M(\phi)$$

where the operator M is defined in (9), and

$$N(v) = (v_i v_i + 2\phi_i v_i)\Delta v - v_i v_j v_{ij} - 2\phi_i v_j v_{ij} + \Delta\phi v_i v_i - \phi_{ij} v_i v_j \,,$$

$$L(v) = (1 + \phi_i \phi_i)\Delta v - \phi_i \phi_j v_{ij} - 2\phi_i \phi_{ij} v_j + 2\Delta\phi \; \phi_i v_i .$$

It is clear that $M(\phi)$ is an element of Y and that the linear operator L defined above is an element of $B(X,Y)$. Thus, the F-derivative of P is that of N added to L. (The derivative of a linear operator is the operator itself and the derivative of a sum is the sum of the derivatives.). Routine manipulations show that

$$N(v + h) = N(v) + Q_v(h) + R_v(h)$$

where

$$Q_v(h) = (v_i v_i + 2\phi_i v_i)\Delta h - (v_i v_j + 2\phi_i v_j)h_{ij} + 2\Delta v(v_i h_i + \phi_i h_i)$$

$$+ 2v_{ij}(v_i h_j + \phi_i h_j) + 2\Delta\phi \; v_i h_i - 2\phi_{ij} v_i h_j ,$$

and $R_v(h)$, for fixed v, vanishes quadratically in $|h|_{2+\alpha}$ as $|h|_{2+\alpha}$ goes to zero. Since the operator Q_v so defined is an element of $B(X,Y)$ we have

$$Q_v = N'(v).$$

Therefore the principle part (the part involving only second partials of h) of $N'(v)$ is given by

$$\text{pr } N'(v)(h) = (v_i v_i + 2\phi_i v_i)\Delta h - (v_i v_j + 2\phi_i v_j)h_{ij}$$

and that of L by

$$\text{pr } L(h) = (1 + \phi_i \phi_i)\Delta h - \phi_i \phi_j h_{ij}$$

so that

$$\text{pr } P'(v)(h) = \big(1 + (\phi_i + v_i)(\phi_i + v_i)\big)\Delta h - (\phi_i + v_i)(\phi_j + v_j)h_{ij}.$$

Setting $u = \phi + v$, the relevant quadratic form is given by

$$q(\lambda) = (1 + u_i u_i)|\lambda|^2 - u_i u_j \lambda_i \lambda_j.$$

Since, by the Schwartz inequality,

$$u_i u_j \lambda_i \lambda_j = (u_i \lambda_i)^2 \leq (u_i u_i)|\lambda|^2 ,$$

we have

$$q(\lambda) \geq |\lambda|^2 ,$$

and the required uniform ellipticity of the F-derivative is demonstrated.

We are now able to apply the theorem of the previous paragraph to the operator P. We restrict ourselves to asking whether a solution exists in a neighborhood in X of 0 for the operator equation

$$P(v) = 0. \tag{10}$$

Let $K = K(r)$ be such that

$$\|P''(v)\| \leq K, \quad v \in S(0,r).$$

Then the above mentioned theorem guarantees that if

$$KB^2|P(0)|_\alpha < 1/2 ,$$

and

$$1/KB \leq r ,$$

where

$$B = \|[P'(0)]^{-1}\|,$$

there exists a solution of (10) in $S(0,r)$. We force the second of the above inequalities to hold simply by choosing

$$K \geq 1/Br ,$$

that is by choosing K to be the maximum of $1/Br$ and some upper bound for $\|P''(v)\|$ on $S(0,r)$. Then, noting that

$$P(0) = M(\phi),$$

we know there is a solution of (10) in $S(0,r)$ whenever

$$|M(\phi)|_\alpha < 1/2\ KB^2.$$

We can now state

<u>Theorem</u>. <u>For each</u> $r > 0$ <u>there is a number</u> H <u>depending only on</u> r <u>and a constant bounding the</u> C_α <u>norms of the first and second de-</u>

rivatives of ϕ such that

$$|M(\phi)|_{\alpha} < H$$

implies existence of a solution u of the equation (9) such that u is in $C'_{2+\alpha}(R)$, and

$$|u - \phi|_{2+\alpha} < r.$$

PROOF: We will, in fact, give an H explicitly in terms of r, B, and C_α norms of the first and second derivatives of ϕ. The result then follows from the fact that B is the smallest constant possible in the a priori estimate

$$|h|_{2+\alpha} \leq \text{const. } |P'(0)(h)|_{\alpha}$$

which holds for the linear elliptic boundary value problem

$$P'(0)(h) = f \quad (f \text{ in } Y),$$

since these constants depend only on an upper bound for the C_α norms of the coefficients of the differential operator $P'(0)$ (which are quadratic expressions in the first and 2nd derivative of ϕ) and the ellipticity constant for $P'(0)$ which is one. It must be noted here that we have suppressed the fact that such a constant also depends on α, n, and R so that H does as well.

We must now give an upper bound for $\|P''(u)\|$ on $S(0,r)$. We have

$$P''(v)(h,h) = (\Delta v + \Delta\phi)h_i h_i - (v_{ij} + \phi_{ij})h_i h_j$$
$$+ 2(v_i + \phi_i)h_i \Delta h - 2(v_i + \phi_i)h_j h_{ij},$$

so that

$$|P''(v)(h,h)|_\alpha \le \left[4|v|_{2+\alpha} + |\Delta\phi|_\alpha + \max_{i,j}|\phi_{ij}|_\alpha + 2\max_i|\phi_i|_\alpha\right]\left(|h|_{2+\alpha}\right)^2 ,$$

and

$$\|P''(v)\| \le 4|v|_{2+\alpha} + \lambda(\phi)$$

where

$$\lambda(\phi) = |\Delta\phi|_\alpha + \max_{i,j}|\phi_{ij}|_\alpha + 2\max_i|\phi_i|_\alpha .$$

Thus, for v in $S(0,r)$, we have

$$\|P''(v)\| \le 4r + \lambda(\phi).$$

We may then take

$$K = \max\left\{\frac{1}{Br} , 4r + \lambda(\phi)\right\} ,$$

in the expression

$$1/2B^2K$$

for H.

Some light can be shed on the expression for K by observing that for $r \le r_0$, where

$$r_0 = \frac{-\lambda B + \sqrt{\lambda^2B^2 + 16B}}{8B} ,$$

we have

$$\frac{1}{Br} \ge 4r + \lambda ,$$

and

$$H = \frac{r}{2B} ,$$

and for $r \ge r_0$

$$\frac{1}{Br} \le 4r + \lambda ,$$

and

$$H = \frac{1}{2B^2(4r + \lambda)} ,$$

where r_0 is the unique value of r for which

$$\frac{1}{Br} = 4r + \lambda .$$

We also note that r_0 is the optional choice for r in the sense that H has an absolute maximum for this choice of r, and that H goes to zero as r either goes to zero or to infinity.

The above theorem is stated so that the function ϕ, defined on all of R, is somehow given to us. We could turn the situation around somewhat.

Suppose we are given a boundary function ϕ_0 which can be extended to a function defined on R which is an element of $C_{2+\alpha}(R)$ and which is such that the C_α norms of the first and second derivatives of at least one of the "extensions" are bounded by some specified constant. We call such extensions admissable. Note then, that the numbers B and λ used above are determined by the constant used to define "admissable". Further, suppose we choose the value r_0 of r determined by this constant. The above theorem then guarantees existence of a solution of the boundary value problem $\left(\text{in } C_{2+\alpha}(R)\right)$ if ϕ_0 has an admissable extension ϕ which is such that:

$$|M(\phi)|_\alpha < H = r_0/2B ,$$

that is, the problem has a solution if there exists a function which is sufficiently close to being a solution in the above sense.

<u>Remark</u>. In connection with the above we mention a recent result of Jenkins an Serrin [5]. They show that for a C_2 region there is a number $\mathcal{B}$, depending on the region and the uniform norms of the first and second derivatives of the boundary data, such that if the oscillation of the boundary data is less than $\mathcal{B}$ there exists a solution. The number $\mathcal{B}$ is shown to depend in a precise way on the mean curvature of the boundary (mean curvature everywhere non-negative implies $\mathcal{B} = \infty$), and their result is also far better than ours in other ways. We would, however, like to point out that in the case of a region whose boundary does not have an everywhere non-negative mean curvature ($\mathcal{B}$ finite) our result gives a sufficient condition for existence that involves relations between norms of the first and second derivatives of an extension of the boundary function, without explicit restriction on its oscillation on the boundary.

We conclude by noting that the theorems we have presented are constructive in that by using the Kantorovich theorem they produce a sequence of functions converging to a solution, the elements of this sequence being solutions of linear problems.

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