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AN ANALYSIS OF METHODS FOR PREDICTING THE STABILITY CHARACTERISTICS OF AN ELASTIC AIRPLANE

APPENDIX A
EQUATIONS OF MOTION AND STABILITY CRITERIA

By

Members of the Aerodynamics
and Structures Research Organizations

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1. INTRODUCTION

The first document of this report, "Summary Report," lists and briefly describes various forms of the equations of motion and the stability criteria for predicting the stability characteristics of an elastic airplane and gives results of their application. This appendix is intended to give detailed developments and discussions of the material summarized in that document.

In dealing with the stability and control of elastic airplanes the engineer is confronted with three distinct problems:

(a) Equilibrium of steady-reference-state* flight conditions;

(b) Stability of steady-reference-state flight conditions;

(c) Response of the airplane to control and/or gust inputs and the behavior during unsteady maneuvers.

As specified in NASA Contract NAS 2-3662, no gust inputs will be considered. Also, since unaccelerated (interpreted as approximately constant speed) flight is the object of the study, no truly unsteady maneuver will be considered. Therefore, for the purpose of this report, problem c is defined as: response of the airplane to small control inputs.

Other fundamental restrictions that apply to this contract are:

(a) Only free-flight conditions will be investigated (no takeoff, landing, or ground effect).

(b) No thermoelastic effects will be considered.

(c) No electromagnetic effects will be considered.

Any solution of problems a, b, and c will require the definition of the shape of the deformed airplane. This implies that even in steady-reference-state situations the equations of motion must be coupled with the state of internal equilibrium.

To describe the forces acting on the elastic airplane it is necessary to know the shape of the deformed airplane. Two possibilities present themselves:

*Steady state is defined as that state for which no state variables change with time, with respect to a body-fixed axis system.

a) The airplane shape is defined in some reference flight condition. This could be a midcruise condition, where the airplane shape is defined to obtain a given lift-to-drag ratio.

b) The airplane shape is defined in the jig (jig shape). In this condition it is generally assumed that the structure is continuously supported (unloaded state, no internal stresses).

To compute the stability and control characteristics of the elastic airplane for all flight conditions it is necessary to know its jig shape. In case b) there is no problem. In case a) the jig shape must be determined first by carrying out a so-called inverse aeroelastic solution.

With this background in mind, a detailed development is presented of the equations of motion for maneuvers of elastic airplanes during unaccelerated flight. After presenting the derivation of the general equations of motion for an airplane, these equations are simplified and expanded into forms used in the solution of the following problems:

(a) Equilibrium of steady-state flight conditions;

(b) Stability of steady-state flight conditions;

(c) Response of the airplane to small control inputs.

These equations are applicable to large, flexible supersonic airplanes that may operate in a flight regime extending to Mach 5 at 30 000 meters altitude.

Emphasis has been placed on retention of as much generality as possible in the equations of motion. As the equations have been expanded into forms that allow the solution of specific problems, the assumptions and approximations made in so doing have been carefully stated. In the text each assumption is identified in the margin by the letter G, A, S, or D, followed by a number. For easy reference in reading the text all assumptions are summarized in Sec. 3.

Static and dynamic stability criteria are derived for an elastic airplane. In their mathematical formulation these criteria are the same for either elastic or rigid airplanes. The meanings of stability, stability criteria, and associated concepts are defined.

Dynamic stability criteria are developed with four methods:

(a) Characteristic equations methods;

(b) Time history method;

(c) Energy decay method;

(d) Lyapunov method.

The Lyapunov method was included to cover the case of nonlinear and/or nonautonomous equations of motion.

2. SYMBOLS

This list includes the symbols found in the Summary and appendixes. In different technologies some of the symbols have different meanings. For example, ϵ means downwash angle to an aerodynamicist, but strain to a structural engineer. In these cases the several definitions have been listed after the symbol.

General

AR	Aspect ratio, nondimensional
$[A]$	Steady aerodynamic influence coefficients matrix, meters^2/radian
$[\delta A]$	Unsteady aerodynamic influence coefficients matrix, meter^2-seconds/radian
$[A_1]$, $[A_2]$, $[A_3]$, $[A_4]$, $[A_5]$	Aerodynamic matrices, newtons, newton-meters
a	Root of characteristic equation, second^{-1}; lift curve slope, radian^{-1}
a_∞	Speed of sound, meters/second
$\bar{a}_v$	Vertical tail elastic to rigid lift ratio, nondimensional
$\bar{a}$	Acceleration, meters/second^2
b	Wingspan, meters
$C_{1/2}$	Cycles to damp to half amplitude, nondimensional
C_2	Cycles to double amplitude, nondimensional
C_D	Drag coefficient, $D/\bar{q}S$, nondimensional
C_{D_i}	Induced drag coefficient, $D_i/\bar{q}S$, nondimensional
C_L	Lift coefficient, $L/\bar{q}S$, nondimensional
C_ℓ	Rolling moment coefficient, $M_x/\bar{q}Sb$, nondimensional

C_m	Pitching moment coefficient, $M_y / \bar{q}S\bar{c}$, nondimensional
C_N	Normal pressure force coefficient, $N / \bar{q}S$, nondimensional
C_n	Yawing moment coefficient, $M_z / \bar{q}Sb$, nondimensional
C_p	Pressure coefficient, $(P - P_\infty)/q_\infty$, nondimensional
C_T	Thrust coefficient, $T / \bar{q}S$, nondimensional
C_Y, C_y	Side force coefficient, $F_y / \bar{q}S$, nondimensional
$[C]$	Flexibility matrix with reference point fixed, meters/newton
$[C_o]$	Flexibility matrix with reference point fixed and with reference point rows and columns removed, meters/newton
$[\bar{C}]$	Flexibility matrix with reference point free, meters/newton
$[\bar{C}_R]$	Residual flexibility matrix, meters/newton
c	Wing chord, meters
c_R	Root chord, meters
$\bar{c}$	Mean aerodynamic chord, meters
c_{ref}	$\bar{c}$ for the 707 and c_R for the SST, meters
D	Drag, newtons
D_i	Induced drag, newtons
$[D]$	Transformation matrix from fluid to stability axis system, nondimensional
$\bar{d}$	Elastic displacement, meters
$\{d_i\}$	Column matrix of elastic displacement components at the i^{th} element, meters
$\{d_p\}$	Matrix of elastic displacement perturbation, meters
E	Total airplane perturbation energy, newton-meters; Young's modulus, newtons/meter^2; induced drag efficiency factor, nondimensional; energy, newton-meters

e	Internal energy density, newton-meters^4/kilogram
F	Energy decay parameter, nondimensional
$\vec{F}$	Force, newtons; surface stress vector, newtons/meter^2
$\{F\}$	Total force matrix, newtons
$\{F_A\}$	Aerodynamic force matrix, newtons
$[F_d]$	Flexibility matrix relating changes in panel centroid deflections to unit loads, meters/newton
$\{F_i\}$	Generalized forces at i^{th} element, arbitrary dimensions
$\{F_T\}$	Thrust force matrix, newtons
$[F_\theta]$	Flexibility matrix relating panel slopes to unit loads, radians/newton
f_{ij}	Aerodynamic influence coefficients (subsonic), newtons/radian
$\vec{f}$	Perturbation force, newtons; perturbation surface stress vector, newtons/meter^2
$\{f\}$	Perturbation force matrix, newtons
$\{f_A\}$	Aerodynamic perturbation force matrix, newtons
$\{f_T\}$	Thrust perturbation force matrix, newtons
G	Shear modulus, newtons/meter^2
GW	Gross weight, newtons
$\bar{\bar{G}}$	Structural influence functions in diadic form with reference point free, meters^3/newton
g_{ij}	Aerodynamic influence coefficients (supersonic), newtons/radian
$\vec{g}$	Acceleration due to gravity, meters/second^2
$\vec{g}_i$	Unit base vector, nondimensional
h	Altitude, meters; specific [illegible]y, newton-meters/kilogram; center-of-gravity position, [illegible]nal

h_m	Maneuver point position, nondimensional
h_n	Neutral point position, nondimensional
$(h_n - h)$	Static margin, nondimensional
$\dot{h}_p$	Velocity of panel normal to the streamwise direction, meters/second
I_{xx}, I_{xy}, I_{xz} I_{yy}, I_{yz}, I_{zz}	Moments and products of inertia, kilogram-meters^2
$[I], \lceil I \rfloor$	Identity matrix, nondimensional
i_H	Horizontal tail deflection, degrees
$\underset{\sim}{\hat{i}}, \underset{\sim}{\hat{j}}, \underset{\sim}{\hat{k}}$ $\vec{i}, \vec{j}, \vec{k}$	Unit base vectors, nondimensional
J	Torsional constant, meters^4/radian
K	Angular deflection at the exposed horizontal tail due to a unit load at the tail, radians/newton
K_{ij}	Structural stiffness coefficient, newtons/meter
K_N	Ratio of aircraft nose lift to aircraft wing lift, nondimensional
$K_{\dot{p}}$	Effective change in vertical tail angle of sideslip due to a unit change in rolling acceleration measured at the exposed vertical tail, degrees/radian/second^2
$K_{\dot{r}}$	Effective change in vertical tail angle of sideslip due to a unit change in yawing acceleration measured at the exposed vertical tail, degrees/radian/second^2
$K_{\ddot{y}}$	Effective change in vertical tail angle of sideslip due to a unit change in side acceleration measured at the exposed vertical tail, degrees/meter/second^2
$K'_{B(W)}$	Effect of lift carryover on the body due to the wing, nondimensional
$K'_{W(B)}$	Effect of lift carryover on the wing due to the body, nondimensional
$[K]$	Stiffness mat ix with respect to fixed reference point, newtons/meter

$[K]_i$	Element stiffness matrix, newtons/meter
$[\bar{K}]$	Stiffness matrix with respect to free reference point, newtons/meter
$\lceil \bar{K} \rfloor$	Generalized stiffness matrix with free reference point, newtons/meter
k	Thermal conductivity, newton-meters/second-meter-degrees Celsius; elastic constant, newtons/meter^2; Strouhal number, nondimensional
$[K], \lceil K \rfloor$	Corrector matrix for influence coefficients, nondimensional
L	Lift, newtons
l	Moment arm, meters; characteristic length, meters; pressure difference across surface, newtons/meter^2
l_H	Wing $c_{ref}/4$ to horizontal tail $c_{ref}/4$, meters
l_v	Wing $c_{ref}/4$ to vertical tail $c_{ref}/4$, meters
l_1, l_2, l_3	Direction cosines, nondimensional
M	Mach number, nondimensional; mass of the airplane, kilograms
$\bar{M}$	Moment, meter-newtons
$[M]$	Inertial matrix, kilograms, kilogram-meters^2
$\lceil M \rfloor$	Generalized mass matrix, kilograms
m_1, m_2, m_3	Direction cosines, nondimensional
$\bar{m}$	Perturbation moment, meter-newtons
$[m]$	Mass matrix, kilograms
$\lceil m \rfloor$	Diagonal mass matrix, kilograms
N	Yawing moment, meter-newtons
$\bar{N}$	Normal force, newtons
n	Load factor, nondimensional; number of elastically connected mass elements used to represent the airplane, nondimensional

n_2, n_3	Direction cosines of the normal surface, nondimensional
$\bar{n}$	Unit vector normal to the surface, nondimensional
$\lceil n \rfloor$	Diagonal matrix of panel unit normal vectors, nondimensional
P	Period, seconds
P, Q, R	Components of the angular velocity $\bar{\omega}$ in the body axis system, radians/second
P_t	Total pressure, newtons/meter^2
$\{P\}$	Aerodynamic panel pressure forces, newtons
p	Static pressure, newtons/meter^2; roll rate, radians/second
p, q, r	Perturbation components of angular velocity $\bar{\omega}_p$ in the body axis system, radians/second
Q_i	Generalized force, arbitrary dimensions*
$\{Q\}$	Matrix of generalized aerodynamic and thrust forces, arbitrary dimensions*
q	Pitch rate, radians/second; rate of internal heat energy addition, newton-meters/second
q_i	Generalized coordinates, arbitrary dimensions*
$\bar{q}$	Dynamic pressure, newtons/meter^2
$\hat{q}$	Pitch rate, $qc_{ref}/2V_{c_1}$, nondimensional
$\{q\}$	Matrix of generalized coordinates, arbitrary dimensions*
$\{\tilde{q}\}$	Matrix of generalized coordinates of elastic free vibration, arbitrary dimensions*
$\{\bar{\tilde{q}}\}$	Cantilever eignvectors, nondimensional

*The units of a generalized force times the generalized coordinates must be newton-meters.

R	Universal gas constant, newton-meters/kilogram-degrees Kelvin; magnitude of position vector, meters; region of XY plane not covered by the airplane or wake, nondimensional
Re	Reynolds number, nondimensional
$\vec{R}$	Position vector at an initial instant of time, meters; body force per unit volume, newtons/meter^3
r	Reference distance, meters; magnitude of the position vector, meters
$\hat{r}$	Yaw rate component, $rb/2V_{c_1}$, nondimensional
$\vec{r}$	Position vector relative to the body axis system, meters; position vector relative to the fluid axis system, meters
$\vec{r}_o$	Position vector of the center of gravity relative to the fluid axis system, meters
$\vec{r}_s$	Position vector relative to the stability axis system, meters
$\vec{r}'$	Position vector relative to inertial space, meters
$\vec{r}'_o$	Position vector of the center of gravity relative to the inertial space, meters
$\vec{\bar{r}}_s$	Position vector in the undeformed airplane relative to the body axis system, meters
$\{ r'_{o_p} \}$	Matrix of airplane position and orientation perturbations, meters, radians
S	Reference area, meters^2; airplane's projection on the XY plane, nondimensional
$[S]$	Diagonal matrix of panel areas, meters^2
s	Complex frequency function, 1/seconds
T	Kinetic energy, newton-meters; thrust, newtons; time, seconds
$T_{1/2}$	Time to damp to ½ amplitude, seconds
T_2	Time to double the amplitude, seconds

$-1/T_r$	Rolling convergence mode root, 1/seconds
$-1/T_s$	Spiral mode root, 1/seconds
t .	Time, seconds; airfoil thickness, meters
t^*	Nondimensionalizing time factor, seconds
U	Potential energy, newton-meters
U, V, W	Components of velocity $\bar{V}_c$ in the body axis system, meters/second
u, v, w	Perturbation components of the velocity in the body axis system, meters/second
u_i	Generalized coordinates, nondimensional
$\hat{u}$	Forward velocity component, u/V_{c_1}, nondimensional
$\{u\}, \{u_p\}$	Generalized elastic displacements, meters
V	Lyapunov function, nondimensional; volume, meters^3
V_E	Equivalent airspeed, meters/second
$\bar{V}_c$	Velocity vector of the airplane center of gravity, meters/second
$\bar{V}$	Velocity vector, meters/second
$\bar{V}_{c_p}$	Perturbation velocity vector of the airplane center of gravity meters/second
$\{V_p\}$	Matrix of airplane linear and rotational rate perturbations, meters/second, radians/second
$\{\dot{V}_p\}$	Matrix of airplane linear and rotational acceleration perturbations, meters/second^2, radians/second^2
W	Weight, newtons; airplane's wake projection on the XY plane, nondimensional
$\{X\}$	Matrix of panel centroid distances to the reference point, meters
X, Y, Z x, y, z	Body-fixed-axis system (app. A); fluid axis system (app. B)

X_B, Y_B, Z_B; x_B, y_B, z_B	Body-fixed-axis system
X_o, Y_o, Z_o	Axis system fixed to a material point
X', Y', Z'; x', y', z'	Earth-fixed-axis system
Y	Side force, newtons
$\lceil \Delta y \rfloor$	Matrix of spanwise panel widths, meters
Z_R	Vertical displacement of structural reference point, meters
$\{Z\}$	Matrix of vertical displacements of each panel from equilibrium, meters
$[\ ]$	Square matrix
$\{\ \}$	Column matrix
$\lfloor\ \rfloor$	Row matrix
$\lceil\ \rfloor$	Diagonal matrix
$[\ ]^T, \{\ \}^T$	Transposed matrix
$[\ ]^{-1}$	Matrix inverse
$\Vert[\ ]\Vert$	Determinant of a matrix
$[0]$	All zero elements
$\{1\}$	Column matrix of ones
$[\![\ ]\!]$	"Jump" in enclosed quantity

Greek Symbols

α	Angle of attack, radians
α_R	Angular rotation of structural reference point, radians
α_{ref}	Angle between X body axis and $\bar{V}_{c_1}$, radians
$\{\alpha\}$	Matrix of panel slopes, radians

	Angle of sideslip, radians
β^2	$(M^2 - 1)$, nondimensional
Γ	Circulation, meters^2/second
$\bar{\Gamma}_o$	Structural influence functions with reference point fixed in diadic form, meters^3/newton
γ	Flight path angle, radians; ratio of specific heats for air, nondimensional
Δ	Finite change in some parameter, nondimensional
δ	Control surface deflection, radians; arbitrarily small number, nondimensional; Dirac's function, nondimensional; thickness ratio, nondimensional
$\{\delta\}$	Matrix of displacements relative to a space-fixed inertial system, meters
$\{\delta_S\}$	Matrix of flexible displacements relative to the structural axis system, meters
ϵ	Downwash angle, radians; arbitrarily small number, nondimensional; strain, meters/meter
ϵ_α	Change in downwash angle at the stabilizer per unit change in wing angle of attack, $\partial\epsilon/\partial\alpha$, radians/radian
ζ	Damping ratio, nondimensional; nondimensionalized coordinate, nondimensional; dummy variable, nondimensional
η	Efficiency factor, nondimensional; coordinate, nondimensional; dummy variable, nondimensional
Θ	Euler angle, radians
θ	Perturbed Euler angle, radians
Θ_S	Streamwise rotation of panel, radians
$\Theta_{ix},\Theta_{iy},\Theta_{iz}$	Node rotations, radians
$\dot{\Theta}$	Rate of change of Euler angle, radians/second

$\dot{\theta}_{ei}$	Rotational rate of paneled airplane about axis of rotation, radians/second
$\vec{\theta}$	Rigid-body rotation about center of gravity, radians
$[\Theta]$	Angle mode matrix, radians/meter
λ	Eigenvalue, nondimensional; taper ratio, nondimensional; bulk modulus, newtons/meter^2; Lame's constant, newtons/meter^2; sweep angle, degrees
λ_i	Roots of characteristic equation, 1/seconds
μ	Reduced mass parameter, nondimensional; Lame's constant, newtons/meter^2; extent of influence region, nondimensional
$\{\mu\}$	Cantilever mode shape matrix, nondimensional
$[\mu]$	Matrix of all cantilever modes, nondimensional
ν	Poisson's ratio, nondimensional
ξ,η,ζ	Coordinates, nondimensional; dummy variables, nondimensional
π	Constant, 3.14159. . ., nondimensional
ρ	Density, kilograms/meter^3
σ	Normal stress, newtons/meter^2; density ratio, nondimensional; real root of characteristic equation, 1/seconds
σ_R	Rotation of structural reference axis system, radians
σ_T	Rectilinear translation of structural reference axis system, meters
τ	Coefficient of viscosity, kilograms/meter-second; shear stress, newtons/meter^2; time, nondimensional
Φ	Total velocity potential, meters^2/second; Euler angle, radians
$[\Phi_n]$	Nc·malized natural free vibration modes of the airplane, nondimensional

	Perturbation velocity potential, meters; perturbed Euler angle radians
$\dot{\phi}$	Rate of change of Euler angle, radians/second
$[\phi]$	Free-vibration mode shape matrix, nondimensional
$[\bar{\phi}]$	Rigid-body mode shape matrix, nondimensional
$\overline{\overline{\varphi}}$	Stress diadic, newtons/meter^2
$\vec{\phi}_\alpha$	Normal mode of generalized coordinate, nondimensional
φ	Velocity potential, nondimensional
$\varphi(t)$	Arbitrary positive function of time, arbitrary dimension
Ψ	Euler angle, radians
ψ	Perturbed Euler angle, radians
$\dot{\psi}$	Rate of change of Euler angle, radians/second
$\overline{\overline{\psi}}$	Inertia diadic
Ω	Phase angle, radians
ω	Frequency, radians/second; imaginary part of a pair of complex roots, 1/seconds
ω_n	Undamped natural frequency, radians/second
$\vec{\omega}_p$	Perturbed angular velocity, radians/second

Subscripts

A	Aerodynamic; airplane; aileron
a	Aerodynamic
ac	Aerodynamic center
	Body reference axis system

cg	Center of gravity
cp	Center of pressure
D	Dutch roll mode
E	Equivalent elastic (Formulation II); elevator
$\bar{E}$	Equivalent elastic (Formulation I)
Eff	Effective
EqEl	Equivalent elastic
exp	Experimental
F	Flutter
HB	Handbook methods
h, ht	Horizontal tail
I	Inertia relief
ℓ	Lower surface
L.E., LE	Leading edge
ls	Lifting surface theory method
P	Phugoid mode
R	Rigid; rudder
r	Rolling convergence root mode
S	Spiral root
sp	Short period
s	Stability axis system; spiral mode

sl	Sea level
t	Tip; total
u	Upper surface
v, vert, V.T.	Vertical tail
W	Wing
WB	Wing-body
WBT	Wing-body-tail
WT	Wind tunnel
0	At $\alpha = \delta_E = i_h = 0^{\circ}$; initial state
1	Steady state motion variables; trimmed condition
	Undisturbed condition

3. ASSUMPTIONS

Assumptions used in developing the equations and methods are listed here for reference. Where appropriate in the summary report, pertinent assumptions used in obtaining a result or equation are given. However, discussions of the assumptions as they come into the developments are given in the appendixes. Further descriptions and justifications are included in those discussions.

General Assumptions

G1	Airplane mass and mass distribution are constant with time
G2	No thermoelastic effects considered
G3	No electromagnetic effects considered
G4	Symmetric airplane
G5	Variation of air density with altitude is negligible
G6	No gust effects considered
G7	Gravitational forces on the field are negligible
G8	Small perturbation theory
G9	Large perturbation theory
G10	Origin of coordinate system is at the center of mass
G11	Arbitrary perturbations

Aerodynamic Assumptions

A1	Potential flow theory
A2	Thin body
A3	Slender body
A4	High aspect ratio
A5	Prandtl boundary layer approximation
A6	Perfect gas, thermally nonconducting and chemically nonreacting
A7	Isentropic flow
A8	Steady flow

A9 Unsteady flow

A10 Inviscid flow

A11 Quasi-steady flow

A12 Aerodynamic influence coefficients for nonzero sideslip

A13 Continuum flow

A14 No finite shock waves

A15 Velocity field is irrotational

Structural Assumptions

S1 Hooke's law applies

S2 Only small strain and displacement gradients are considered

S3 Structural damping is negligible

S4 Structural perturbations can be represented by normal modes

S5 Completely elastic math model of elastic airplane

S6 Residual elastic math model of elastic airplane

S7 Equivalent elastic math model of elastic airplane

S8 Rigid math model of elastic airplane

S9 Airplane displacement vector field is such that the center of gravity does not displace or rotate

S10 X component of elastic deflection is negligible

S11 Y component of elastic deflection is negligible

S12 The structure can be adequately represented with beams

S13 Inertia of each finite mass element about its center of gravity is negligible

Dynamic Assumptions

D1	Free flight only
D2	No spinning rotors
D3	Steady-state curvilinear flight
D4	Steady-state rotation is small
D5	Zero-lag thrust derivatives
D6	$C_{L_{\ddot{\theta}}}$ is negligible
D7	$C_{Y_{\dot{p}_I}}$, $C_{Y_{\dot{r}_I}}$, $C_{\ell_{\ddot{Y}_I}}$, and $C_{n_{\ddot{Y}_I}}$ are negligible
D8	C_{D_q} is negligible
D9	Steady-state rectilinear motion
D10	Stick-fixed-and-unaugmented airplane
D11	Thrust perturbation forces are negligible
D12	Steady state, wings level, and zero sideslip
D13	Level flight (steady state)
D14	Linear aerodynamic stability derivatives
D15	Two-degree-of-freedom longitudinal motion

4. GENERAL EQUATIONS OF MOTION FOR AN ELASTIC AIRPLANE

4.1 Introduction

In formulating the general equations of motion for an elastic airplane under the ground rules of NASA Contract NAS 2-3662, the following restrictions apply:

G6 • No gust effects are considered.

D1 • Only free-flight conditions are considered (no takeoff, landing, or ground effect).

G3 • No electromagnetic effects are considered.

G2 • No thermoelastic effects are considered.

The flight domain of validity of the equations of motion will be pointed out as the analysis proceeds, but will never be less extensive than the one shown in fig. 1.

Restrictions (G6), (D1), and (G3) are clear; however, (G2) deserves some comment. In aeroelastic problems it is common to assume that changes in strain distribution have a negligible effect on the temperature distribution and the temperature magnitude. However, changes in temperature distribution and magnitude may have significant effects on the strain distribution. In light of this, restriction (G2) is interpreted to mean that the temperature distribution and temperature magnitude are known.

In fig. 2 the airplane is shown as a three-dimensional elastic body that is unrestrained in space. An inertial, rectangular cartesian coordinate system (X', Y', Z') is used to describe the motion of the airplane relative to the earth. The airplane is assumed to consist of n finite mass elements,
S5 each connected to (X', Y', Z') by a position vector $\vec{r}\,'$. The n mass elements are connected with each other through the elastic properties of the airplane structure. A mathematical description of this structural connection is given later.

Assumption S5 in no way restricts the validity of the analysis until a numerical value is associated with n . This will be done in most practical situations and leads to the familiar lumped mass representation of the airplane, whereby it is recognized that not all masses are necessarily structural.

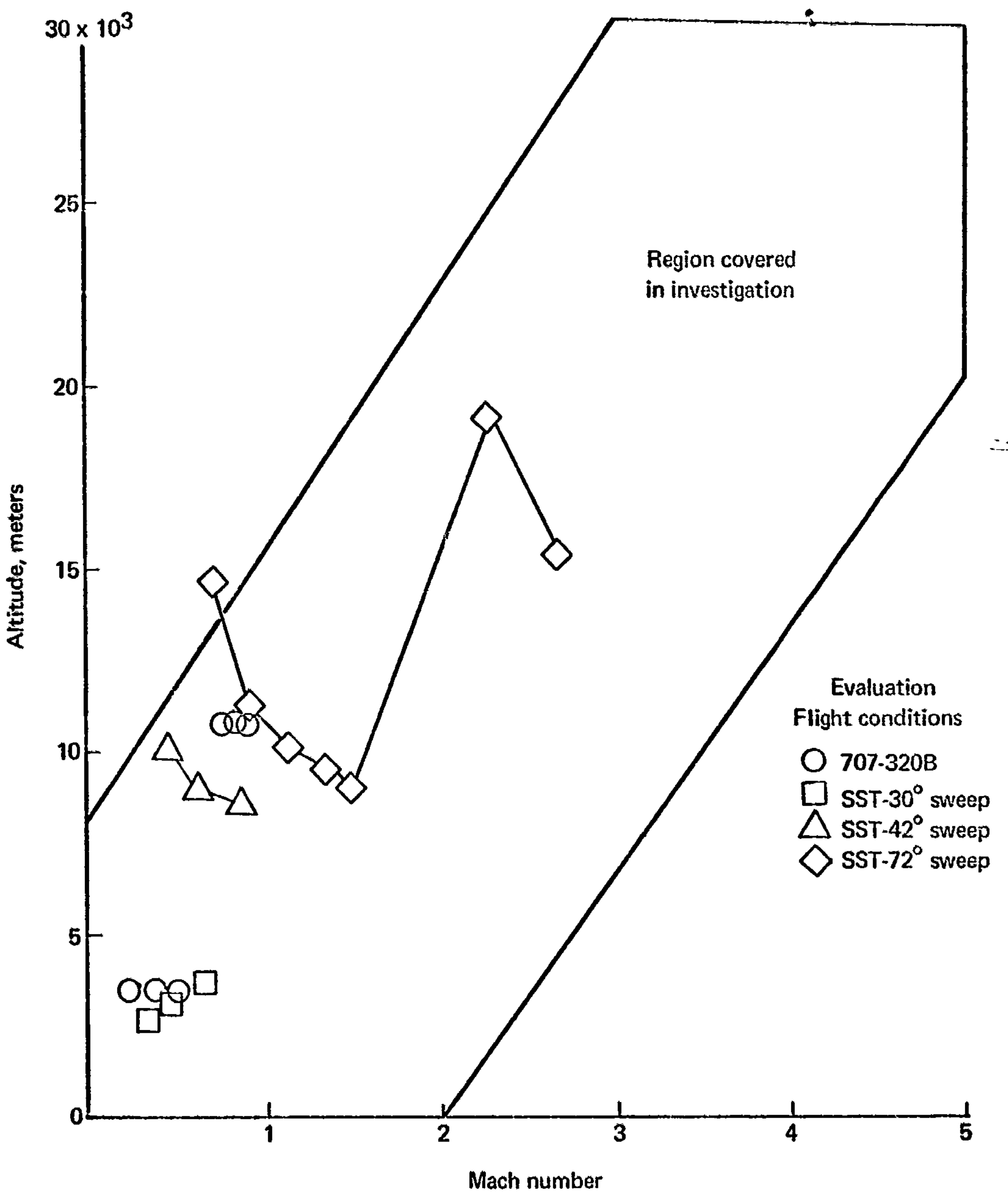


FIGURE 1. – FLIGHT ENVELOPE FOR INVESTIGATION

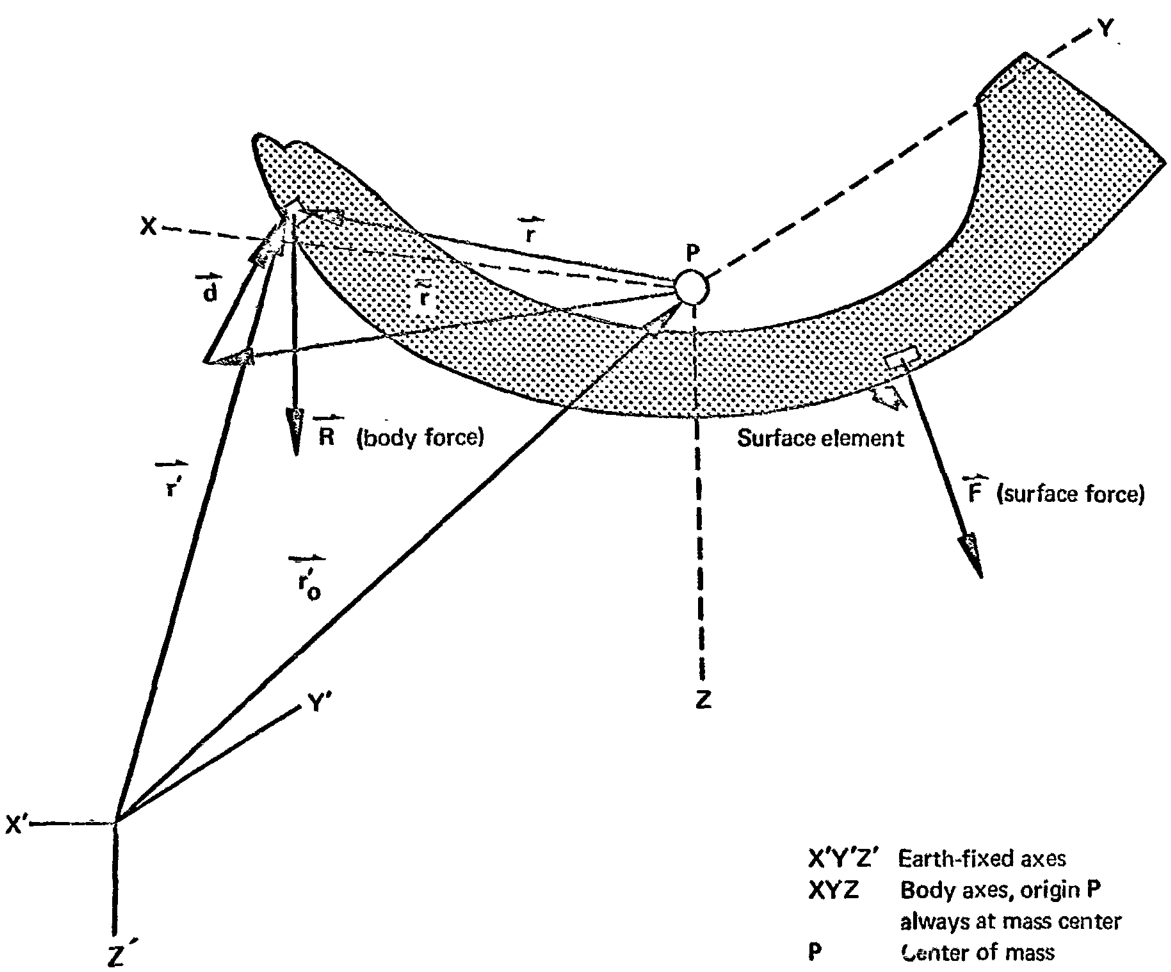


FIGURE 2. – ELASTIC AIRPLANE AXES

For the time being, n is assumed to be indefinitely large so that the analysis remains general and the equations can be maintained in the integral form.

A reference point P is selected in the airplane, and the origin of an orthogonal coordinate system (X, Y, Z) is fixed to this point. The origin at P is located in inertial space by the vector $\vec{r}_0'$. The mass elements of the elastic airplane are located with respect to P by the vector $\vec{r}'$. The following relation is now implied:

$$\vec{r}' = \vec{r}_0' + \vec{r} \tag{4.1}$$

Three components, each representing a translational degree of freedom, are required to define $\vec{r}_0'$ completely. To define the orientation of (X, Y, Z) relative to (X', Y', Z') three orientation angles are required; each angle represents an angular degree of freedom. To locate n mass elements whose positions relative to P are defined by $\vec{r}$, 3n components are required; since the components are measured relative to P, however, there are (3n - 3) translational (elastic) degrees of freedom. To define the orientation of each mass element inside (X, Y, Z) requires (3n - 3) additional degrees of freedom, and this completes the description. In total, then, the elastic airplane when represented by n finite mass elements (very large n) has 6n degrees of freedom. For a rigid airplane the position and orientation of the n finite mass points remain constant in (X, Y, Z).

The first six degrees of freedom are analogous to conventional rigid-airplane degrees of freedom. The remaining (6n - 6) degrees of freedom are the elastic degrees of freedom. A total of 6n equations are needed to describe the motion of an elastic airplane. Fortunately, there are practical ways to reduce this large number of equations without significantly affecting the solutions to the equations of motion. It may be noted in the following discussion that when n is allowed to increase without bound and the airplane is represented as a continuum, the rotational degrees of freedom of the mass points are no longer distinguishable. In fact, the concept of elastic degrees of freedom is questionable in a continuous representation.

4.2 Equations of Motion for the Rigid Degrees of Freedom

The six rigid degrees of freedom are described by force and moment equations in accordance with the laws of conservation of linear and angular momentum. These equations in vector form are:

$$\frac{d}{dt}\int_V \rho_A \frac{d\vec{r}\,'}{dt}\, dV = \int_V \vec{R}\, d\vec{V} + \int_S \vec{F}\, dS \tag{4.2}$$

$$\frac{d}{dt}\int_V \vec{r}\,' \times \rho_A \frac{d\vec{r}\,'}{dt}\, dV = \int_V \vec{r}\,' \times \vec{R}\, dV + \int_S \vec{r}\,' \times \vec{F}\, dS \tag{4.3}$$

The symbol $\vec{R}$ is defined as the body force per unit volume and is used here to represent gravity through the relation:

$$\vec{R} = \rho_A \vec{g} \tag{4.4}$$

The symbol $\vec{F}$ is defined as the surface force per unit area and represents both aerodynamic and reactive (thrust) forces. The density per unit volume of the elastic airplane is represented by ρ_A . The sum of the mass elements represented by the integral of $\rho_A dV$ is assumed to be constant with time. This yields:

G1

$$M \int_V \rho_A \, dV \quad , \quad M = \text{constant} \tag{4.5}$$

Thus the airplane mass M is constant with time. The mass distribution is also assumed to be constant with time. Hence, assumption G1 also implies that fuel slosh is not accounted for.

It is convenient to define P as being always at the center of mass of the airplane. This implies that:

G10

$$\vec{r}_o\,' = \frac{1}{M}\int_V \rho_A \vec{r}\,' \, dV \tag{4.6}$$

Assumption G10 in no way restricts the validity of the analysis, but simplifies the equations of motion.

Even though P is always taken as the center of mass, it is not always associated with the same material point on the airplane. The consequences of this fact in defining elastic deformations are discussed in par. 4.4.

Because of equation (4.1), equation (4.6) implies that:

$$\int_V \vec{r} \rho_A \, dV = 0 \tag{4.7}$$

Equations (4.4) and (4.7) allow simplification of equation (4.2) as follows:

$$M \frac{d}{dt}\left(\frac{d\vec{r}_o}{dt}\right) = M \frac{d}{dt}\left(\vec{V}_c\right) = M\vec{g} + \int_S \vec{F} \, dS \tag{4.8}$$

Substituting equations (4.1) and (4.4) into equation (4.3) yields, after some rearrangement:

$$\frac{d}{dt}\int_V \vec{r} \times \frac{d\vec{r}}{dt} \rho_A \, dV = \int_S \vec{r} \times \vec{F} \, dS \tag{4.9}$$

These results are algebraically identical to those commonly obtained for the equations of motion of a rigid airplane.

Equations (4.8) and (4.9) describe the gross motion of the elastic airplane. In particular, equation (4.8) states that the center of mass P follows the law of motion for a single mass particle equal to the total mass of the elastic airplane and under the action of the resultant of all forces. Equation (4.9) states that the rate of change of moment of momentum about P is equal to the resultant moment about P .

However, $\vec{F}$ and $\vec{r}$ depend on the shape of the elastic airplane and are therefore functions of the elastic degrees of freedom. (Just how $\vec{F}$ and $\vec{r}$ are related to the elastic degrees of freedom is the subject of later discussion.) Notice that in equation (4.8) the elastic properties of the airplane enter only into the right-hand side. In equation (4.9) the elastic properties appear to enter the left-hand side as well as the right-hand side, since $\vec{r}$ is a position vector inside the elastic airplane. How this affects the equations of motion will be come clear later.

Equations (4.8) and (4.9) say nothing about the internal equilibrium of the airplane structure and are therefore not sufficient to describe the motion of the elastic airplane. Information on the remaining (6n - 6) elastic (structural) degrees of freedom must be obtained by examining the internal equilibrium equations for an elastic airplane.

4.3 The Internal Equilibrium for an Elastic Airplane

Assume that the force vector $\vec{F}$ (aerodynamic and reactive) for an elastic airplane may be written in (X, Y, Z) as follows:

$$\vec{F} = F_X \vec{i} + F_Y \vec{j} + F_Z \vec{k} \tag{4.10}$$

At the surface of the elastic airplane the components of $\vec{F}$ are related to the internal stresses by the boundary conditions:

$$\begin{aligned} F_X &= \sigma_X \vec{n}\cdot\vec{i} + \tau_{XY}\vec{n}\cdot\vec{j} + \tau_{XZ}\vec{n}\cdot\vec{k} \\ F_Y &= \tau_{YX}\vec{n}\cdot\vec{i} + \sigma_Y \vec{n}\cdot\vec{j} + \tau_{YZ}\vec{n}\cdot\vec{k} \\ F_Z &= \tau_{ZX}\vec{n}\cdot\vec{i} + \tau_{ZY}\vec{n}\cdot\vec{j} + \sigma_Z \vec{n}\cdot\vec{k} \end{aligned} \tag{4.11}$$

where $\vec{n}$ is a unit vector normal to the surface and is positive outward. The scalar products are the components of $\vec{n}$ and are the direction cosines defined by:

$$\begin{aligned} \vec{n}\cdot\vec{i} &= \cos(X,\vec{n}) \\ \vec{n}\cdot\vec{j} &= \cos(Y,\vec{n}) \\ \vec{n}\cdot\vec{k} &= \cos(z,\vec{n}) \end{aligned} \tag{4.12}$$

The stress quantities σ and τ represent the nine components of the stress tensor $\overline{\overline{\Phi}}$. In diadic notation, equations (4.11) may be written:

$$\vec{F} = \vec{n}\cdot\overline{\overline{\phi}} \tag{4.13}$$

where $\overline{\overline{\Phi}}$ stands for the stress tensor (second order).

A convenient way to look at $\overline{\overline{\Phi}}$ is in matrix form:

$$[\overline{\overline{\phi}}] = \begin{bmatrix} \sigma_X & \tau_{XY} & \tau_{XZ} \\ \tau_{YX} & \sigma_Y & \tau_{YZ} \\ \tau_{ZX} & \tau_{ZY} & \sigma_Z \end{bmatrix} \tag{4.14}$$

Equation (4.13) then becomes a matrix equation:

$$\{F\} = [\phi]\{n\} \tag{4.15}$$

where $\{n\}$ represents the three components of $\vec{n}$.

The equations of motion (4.2) and (4.3) are also applicable to consideration of the interior of the elastic airplane. However, the local (interior) surface stresses are represented by expression (4.13). Therefore, equation (4.2) in terms of the stress tensor yields:

$$\frac{d}{dt}\int_V \rho_A \frac{d\vec{r}\,'}{dt}\, dV = \int_V \vec{R}\, dV + \int_S \vec{n}\cdot\bar{\bar{\phi}}\, dS \tag{4.16}$$ S3

Likewise, this yields for equation (4.3):

$$\int_V \vec{r}\,' \times \rho_A \frac{d^2\vec{r}\,'}{dt^2}\, dV = \int_V \vec{r}\,'\times\vec{R}\, dV + \int_S \vec{r}\,' \times (\vec{n}\cdot\phi)\, dS$$

With $\vec{r}\,' = \vec{r}_o\,' + \vec{r}$, the equation reduces to:

$$\int_V \vec{r}\times\rho_A \frac{d^2\vec{r}\,'}{dt^2}\, dV = \int_V \vec{r}\times\vec{R}\, dV + \int_S \vec{r}\times(\vec{n}\cdot\phi)\, dS \tag{4.17}$$

The surface integral in equation (4.16) may be transformed into a volume integral by:

$$\int_S \vec{n}\cdot\bar{\bar{\phi}}\, dS = \int_V \vec{\nabla}\cdot\bar{\bar{\phi}}\, dV$$

according to the Divergence Theorem. Similarly in equation (4.17):

$$\int_S \vec{r}\times(\vec{n}\,\bar{\bar{\phi}})\, dS = -\int_S (\vec{n}\cdot\bar{\bar{\phi}})\times\vec{r}\, dS = -\int_V \vec{\nabla}\cdot(\bar{\bar{\phi}}\times\vec{r})\, dV$$

Equations (4.16) and (4.17) may now be written as:

$$\int_V \left[\rho_A \frac{d^2\vec{r}\,'}{dt^2} - \vec{R} - (\vec{\nabla}\cdot\phi)\right] dV = 0 \tag{4.18}$$

and

$$\int_V \left[\vec{r} \times \rho_A \frac{d^2\vec{r}'}{dt^2} - \vec{r} \times \vec{R} + \vec{\nabla}\cdot\left(\bar{\bar{\phi}} \times \vec{r}\right)\right] dV_t = 0 \tag{4.19}$$

These equations are satisfied when theintegrandsare zero; therefore, the equations of internal equilibrium for an elastic airplane are:

$$\rho_A \frac{d^2\vec{r}'}{dt^2} - \vec{R} - \vec{\nabla}\bar{\bar{\phi}} = 0 \tag{4.20}$$

$$\vec{r} \times \rho_A \frac{d^2\vec{r}'}{dt^2} - \vec{r} \times \vec{R} + \vec{\nabla}\cdot(\bar{\bar{\phi}} \times \vec{r}) = 0 \tag{4.21}$$

The next task is to expand these equations into component form.

Introducing

$$\frac{d^2\vec{r}'}{dt^2} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \tag{4.22}$$

and

$$\vec{R} = \rho_A\vec{g} = \rho_A(g_x\vec{i} + g_y\vec{j} + g_z\vec{k}) \tag{4.23}$$

it follows, after substitution into equation (4.20), that

$$\begin{aligned}
\rho_A a_x &= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho_A g_x \\
\rho_A a_y &= \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho_A g_y \\
\rho_A a_z &= \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho_A g_z
\end{aligned} \tag{4.24}$$

The result of expanding equation (4.21) and using equation (4.20) is:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy} \tag{4.25}$$

This means that the stress tensor $\bar{\bar{\Phi}}$ is symmetric (ref. 8).

Equations (4.24) and (4.25) are not sufficient to describe fully the state of internal equilibrium of the elastic airplane. The connection between stresses and strains, as well as compatibility between strains and displacements, is lacking. Assuming that the strains and displacement gradients are small, the strains can be wri .en in terms of displacements (app. B): S2

$$
\begin{aligned}
&\varepsilon_X = \frac{\partial u}{\partial X}, \quad \varepsilon_Y = \frac{\partial v}{\partial Y}, \quad \varepsilon_Z = \frac{\partial w}{\partial Z} \\
&\gamma_{XY} = \frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y}, \quad \gamma_{XZ} = \frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z}, \quad \gamma_{YZ} = \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z}
\end{aligned}
\tag{4.26}
$$

where u, v, and w are components of elastic deformations along (X, Y, Z). A relationship between stresses and strains for homogeneous, isotropic, elastic bodies at constant temperature may be written as:

$$
\begin{aligned}
\varepsilon_X &= \frac{1}{E}\left[\sigma_X - \nu(\sigma_Y + \sigma_Z)\right] \\
\varepsilon_Y &= \frac{1}{E}\left[\sigma_Y - \nu(\sigma_X + \sigma_Z)\right] \\
\varepsilon_Z &= \frac{1}{E}\left[\sigma_Z - \nu(\sigma_X + \sigma_Y)\right] \\
\gamma_{XY} &= \frac{1}{G}\tau_{XY}, \quad \gamma_{XZ} = \frac{1}{G}\tau_{XZ}, \quad \gamma_{YZ} = \frac{1}{G}\tau_{YZ}, \quad G = \frac{E}{2(1+\nu)}
\end{aligned}
\tag{4.27}
$$

S1

Hooke's law has been introduced simultaneously with the assumption of small displacements. This, coupled with the assumption of small strains and displacement gradients, has considerable significance on the interpretation of equations (4.24) and (4.25). Those equations were developed from consideration of equilibrium of a deformed body; thus, the point where stress is evaluated in those equations differs from the point where strains are evaluated by the elastic displacement. The change in coordinates is taken to have negligible effect on the values of the stresses or strains; these are the usual approximations made in the classical theory of elasticity (ref. 8).

Equations (4.24), (4.25), (4.26), and (4.27) form a sufficient basis to determine the forces and displacements in the elastic airplane as functions of time. Just how these equations can be cast in a form useful in analyzing stability problems of elastic airplanes is the subject of par. 4.4, where the concept of influence coefficients is introduced. This concept is shown to be a direct consequence of assumptions (S1) and (S2), provided no local structural instability (buckling) occurs.

4.4 The Internal Equilibrium Equations Using Influence Coefficients

Assumptions (S1) and (S2) restrict the validity of the analysis up to this point to cases with small strains and perfect elastic behavior (i. e. linear stress-strain behavior). This is justified because the strains are actually small in the safe operating range (from a structural viewpoint) of large elastic airplanes. It follows, therefore, that linear relations exist between
(S1) forces and deflections. Moreover, from the "perfect elastic" assumption (S1), it follows that when external forces are removed the structure assumes its initial form. To illustrate the physical significance of this, consider the following example.

The linear relation between force and deformation can be written as:

$$d = CF \tag{4.28}$$

where d is a deflection, F is a force along d, and C is a constant of proportionality that will be called an influence coefficient. Figure 3 illustrates the meaning of equation (4.28).

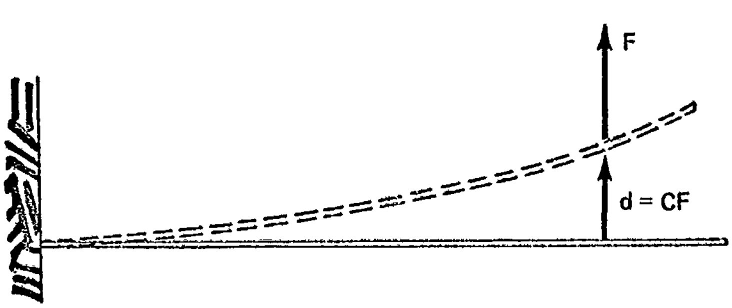


FIGURE 3. – LINEAR FORCE-DEFORMATION RELATION

In a more general sense it is possible to write equation (4.28) in diadic form:

$$\vec{d} = \bar{\bar{C}} \cdot \vec{F} \tag{4.29}$$

This type of formulation of elastic deformations is extensively discussed in ref. 42.

In this case $\bar{\bar{C}}$ is a second-order tensor, the components of which are called influence coefficients. In the more familiar matrix form, the components of $\vec{d}$ are related to the components of $\vec{F}$ by:

$$\{d_i\} = [C_{ij}]\{F_i\} \qquad \begin{array}{l} i = 1, 2, 3, \cdots, n \\ j = 1, 2, 3, \cdots, n \end{array} \tag{4.30}$$

The physical significance of the elements of matrix $[C_{ij}]$ can be readily deduced from equation (4.30).

The introduction of the concept of influence coefficients represents no new assumptions or restrictions of the derivation. The influence coefficient concept is an automatic result of assumptions S1 and S2 made in par. 4.3.

Even though local strains are assumed to be small (assumption S2), a deflection of the structure can still be large. This is a well-known fact; typical airplane examples are the wings on Boeing models B-47, B-52, and 707.

In airplane structures, elastic buckling (nonlinear relation between d_i and F_i) can occur even when local strains are small. In such cases it is theoretically possible to rewrite equation (4.30) in a nonlinear form by allowing $[C_{ij}]$ to vary with d_i, or to apply equation (4.30) locally, thereby constantly re-evaluating C_{ij}. The consequence of such buckling on stability analyses is not considered in this report.

The concept of influence coefficients may be used to modify equations (4.24) through (4.27) and bring them into a more useful form. To do this, it is first necessary to express these equations in terms of displacements u, v, and w rather than in terms of stresses. This may be done by inverting equations (4.27):

$$
\begin{aligned}
\sigma_X &= \lambda\left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z}\right) + 2G\,\frac{\partial u}{\partial X}; \\
\sigma_Y &= \lambda\left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z}\right) + 2G\,\frac{\partial v}{\partial Y} \\
\sigma_Z &= \lambda\left(\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z}\right) + 2G\,\frac{\partial w}{\partial Z}
\end{aligned}
\tag{4.31}
$$

where

$$
\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{4.32}
$$

and

$$
\begin{aligned}
\tau_{XY} &= G\left(\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y}\right) \\
\tau_{YZ} &= G\left(\frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z}\right) \\
\tau_{ZX} &= G\left(\frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X}\right)
\end{aligned}
\tag{4.33}
$$

For more detailed discussions on these equations, see ref. 26.

Substituting equations (4.31) through (4.33) into equation (4.24) yields:

$$
\begin{aligned}
(\lambda + G)\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 v}{\partial X\,\partial Y} + \frac{\partial^2 w}{\partial X\,\partial Z}\right) + G\,\vec{\nabla}^2 u + \rho_A g_X &= \rho_A a_X \\
(\lambda + G)\left(\frac{\partial^2 u}{\partial X\,\partial Y} + \frac{\partial^2 v}{\partial Y^2} + \frac{\partial^2 w}{\partial Y\,\partial Z}\right) + G\,\vec{\nabla}^2 v + \rho_A g_Y &= \rho_A a_Y \\
(\lambda + G)\left(\frac{\partial^2 u}{\partial X\,\partial Z} + \frac{\partial^2 v}{\partial Y\,\partial Z} - \frac{\partial^2 w}{\partial Z^2}\right) + G\,\vec{\nabla}^2 w + \rho_A g_Z &= \rho_A a_Z
\end{aligned}
\tag{4.34}
$$

These are recognizable as Navier's equations. They are adjoined by a set of boundary equations (conditions) that relate the surface traction $\vec{F}$ to the surface displacements. These boundary conditions follow by substitution of equations (4.31) through (4.33) into equation (4.11):

$$F_X = \lambda \Delta \vec{n}\cdot\vec{i} + G\left[2\frac{\partial u}{\partial X}\vec{n}\cdot\vec{i} + \left(\frac{\partial v}{\partial X}+\frac{\partial u}{\partial Y}\right)\vec{n}\cdot\vec{j} + \left(\frac{\partial u}{\partial Z}+\frac{\partial w}{\partial X}\right)\vec{n}\cdot\vec{k}\right]$$
$$F_Y = \lambda \Delta \vec{n}\cdot\vec{j} + G\left[2\frac{\partial v}{\partial Y}\vec{n}\cdot\vec{j} + \left(\frac{\partial v}{\partial X}+\frac{\partial u}{\partial Y}\right)\vec{n}\cdot\vec{i} + \left(\frac{\partial w}{\partial Y}+\frac{\partial v}{\partial Z}\right)\vec{n}\cdot\vec{k}\right] \quad (4.35)$$
$$F_Z = \lambda \Delta \vec{n}\cdot\vec{k} + G\left[2\frac{\partial w}{\partial Z}\vec{n}\cdot\vec{k} + \left(\frac{\partial w}{\partial Y}+\frac{\partial v}{\partial Z}\right)\vec{n}\cdot\vec{j} + \left(\frac{\partial u}{\partial Z}+\frac{\partial w}{\partial X}\right)\vec{n}\cdot\vec{i}\right]$$

where

$$\Delta = \frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z} \qquad (4.36)$$

Equations (4.34) may be written in vector form after introducing:

$$\vec{d} = u\vec{i} + v\vec{j} + w\vec{k} \qquad (4.37)$$

This yields Navier's equations:

$$(\lambda + G)\vec{\nabla}(\vec{\nabla}\cdot\vec{d}) + G\vec{\nabla}^2\vec{d} + \vec{R} = \rho_A\,\vec{a} \qquad (4.38)$$

where

$\vec{\mathbf{d}}$ = a displacement

$\vec{\mathbf{R}}$ = a body force (gravity)

$\vec{\mathbf{a}}$ = an acceleration

Equation (4.38) can be written in symbolic form as follows:

$$\vec{R} - \rho_A\vec{a} = \tilde{\mathcal{L}}(\vec{d}) \qquad (4.39)$$

where $\tilde{\mathcal{L}}$ is now defined as a differential operator given by:

$$\tilde{\mathcal{L}} = -[(\lambda + G)\,GRAD\ DIV + G\vec{\nabla}^2] \qquad (4.40)$$

It is now feasible to think of a new operator $\tilde{\mathcal{L}}^{-1}$, inverse of $\tilde{\mathcal{L}}$, which has the property such that:

$$\vec{d} = \tilde{\mathcal{L}}^{-1}(\vec{R} - \rho_A\vec{a}) \qquad (4.41)$$

The attractive feature of this inverse operator is that it allows a symbolic explicit solution for $\vec{d}$ that represents the deformed shape of the elastic airplane as a function of (X, Y, Z). In addition, it allows the introduction of the influence coefficient function concept as a further generalization of equation (4.30). This is done by expressing $\tilde{\mathcal{L}}^{-1}$ in integral form as:

$$\tilde{\mathcal{L}}^{-1}(\;) = \int_V \tilde{\Gamma}(\;)\, d\xi\, d\eta\, d\zeta \tag{4.42}$$

where $\tilde{\Gamma}$ is a second-order influence coefficient function tensor, defined in matrix form by:

$$\tilde{\Gamma} = \begin{matrix} C_{XX}(X,Y,Z;\xi,\eta,\zeta) & C_{XY}(X,Y,Z;\xi,\eta,\zeta) & C_{XZ}(X,Y,Z;\xi,\eta,\zeta) \\ C_{YX}(X,Y,Z;\xi,\eta,\zeta) & C_{YY}(X,Y,Z;\xi,\eta,\zeta) & C_{YZ}(X,Y,Z;\xi,\eta,\zeta) \\ C_{ZX}(X,Y,Z;\xi,\eta,\zeta) & C_{ZY}(X,Y,Z;\xi,\eta,\zeta) & C_{ZZ}(X,Y,Z;\xi,\eta,\zeta) \end{matrix} \tag{4.43}$$

Applying the Betti reciprocal theorem (ref. 8), it is found that the tensor is symmetric:

$$C_{XY} = C_{YX}\,, \qquad C_{XZ} = C_{ZX}\,, \quad C_{YZ} = C_{ZY} \tag{4.44}$$

The following physical interpretation can be given to C_{rs}. For example, the function $C_{XY}(X, Y, Z; \xi, \eta, \zeta)$ stands for the displacement of the structure in the X-direction at point (X, Y, Z) due to a unit load in the Y-direction at point (ξ, η, ζ). These influence coefficient functions can actually be computed or determined experimentally when there are a finite number of points (X, Y, Z). A detailed discussion is found in ref. 26.

The solution for the elastic deformation vector $\vec{d}$ can now be written as:

$$\vec{d} = \int_V \tilde{\Gamma}(\qquad)\, d\xi\, d\eta\, d\zeta \tag{4.45}$$

where the parentheses contain the applied forces. Equation (4.45) may be written in terms of the inertial and body forces as well as the forces at the boundary. Thus the surface force $\vec{F}$ should be added to $(\vec{R} - P\vec{a})$ at the surface. By introducting the concept of Dirac's function it is possible to write equation (4.38) as

$$\vec{d} = \int_V \tilde{\Gamma}\cdot(\vec{R} - \rho_A \vec{a})\, d\xi\, d\eta\, d\zeta + \int_V \tilde{\Gamma}\cdot\vec{F}\,\delta(\vec{r} - \vec{r}_s)\, d\xi\, d\eta\, d\zeta \tag{4.46}$$

where $\delta(\vec{r} - r_s)$ is Dirac's function. The vector $\vec{r}_s$ is a position vector at the surface. Dirac's function $\delta(\vec{r} - \vec{r}_s)$ is defined as follows:

$$\left.\begin{aligned} &\int_V g(X,Y,Z)\,\delta(\vec{r} - \vec{r}_s)\,dV = 0 && \text{For } r \neq r_s \\ &\int_V g(X,Y,Z)\,\delta(\vec{r} - \vec{r}_s)\,dV = \int_S g(X,Y,Z)\,dS && \text{For } r = r_s \end{aligned}\right\} \tag{4.47}$$

The airplane is unrestrained; hence the boundary conditions for the structures problem are entirely in terms of forces. There are no displacement boundary conditions specified in the usual sense, i.e. that kinematically constrain the airplane. The surface aerodynamic forces obviously depend on the displacements, but this dependence is of little assistance in the derivation of the influence coefficient tensor $\overline{\overline{\Gamma}}$. As noted in the discussion on equation (4.43), the elements of $\overline{\overline{\Gamma}}$ give the components of displacement at X, Y, and Z due to unit components of force at ξ , η , ζ . To carry out the computation of these elements determinantly, some point (the reference point) is held fixed. However, this is not consistent with the unrestrained airplane, since in that case the reference point does not remain fixed.

The influence function tensor $\overline{\overline{\Gamma}} = \overline{\overline{\Gamma}}_0$ is computed with the airplane clamped at a reference point in the jig shape . The airplane jig shape is defined as the completely unloaded shape, i.e. without aerodynamic or gravitational loading. A jig shape coordinate system is selected as (X_0, Y_0, Z_0) with origin at P_0 , which is the reference point and is also the material point that represents the center of gravity (c.g.) of the jig shape. The geometric relation between coordinate systems (X, Y, Z) and (X_0, Y_0, Z_0) is presented in fig. 4. The center of mass of the jig shape and the center of mass of the deformed shape are coincident in fig. 4 as seen by an observer in (X', Y', Z'). The reason for this is obvious: the center of mass does not change its position in space when an equilibrium loading system is taken away from the airplane. In fig. 4, then, P is both the center of mass of the jig shape and the center of mass of the deformed shape, whereas P_0 is the material point in the deformed shape that becomes P in the jig shape.

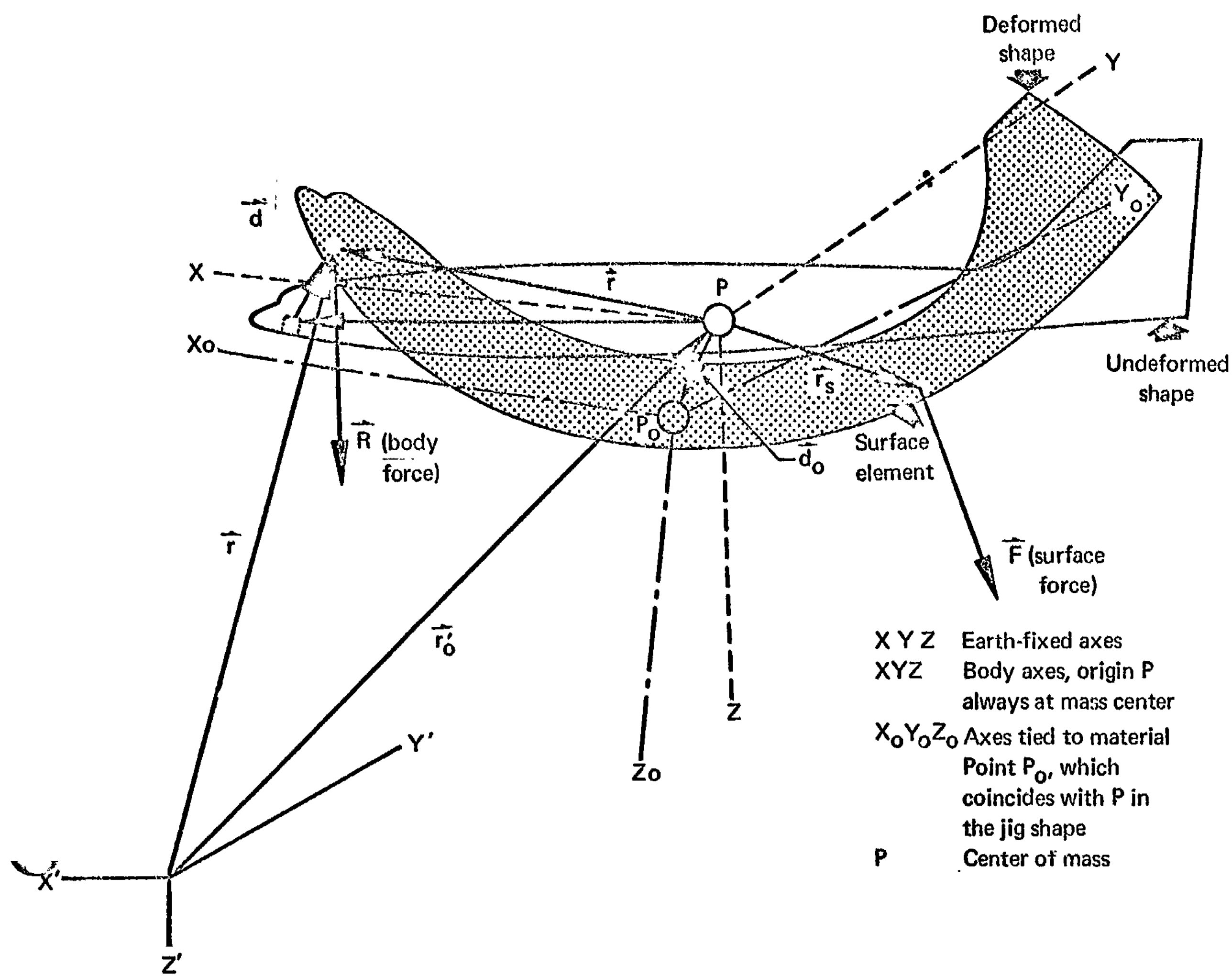


FIGURE 4. – AXIS SYSTEMS FOR THE DEFORMED SHAPE AND THE UNDEFORMED SHAPE OF AN ELASTIC AIRPLANE

A consequence of the computation of $\vec{\vec{\Gamma}}$ relative to (X_o, Y_o, Z_o) is that a transformation must be made in equation (4.46):

$$\vec{d}_{X_oY_oZ_o} = \vec{d}_{XYZ} - \vec{d}_o - \frac{1}{2}\left(\vec{\nabla}\times\vec{d}\right)_o \times \vec{\vec{r}} \qquad (4.48)$$

Applying the transformation equation (4.48) to equation (4.46) yields:

$$\vec{d} - \vec{d}_o - \frac{1}{2}\left(\vec{\nabla}\times\vec{d}\right)_o \times \vec{\vec{r}} = \int_V \vec{\vec{\Gamma}}_o\cdot\left(\vec{R} - \rho_A\vec{a}\right)dV + \int_V \vec{\vec{\Gamma}}_o\cdot\vec{F}\,\delta\left(\vec{r}-\vec{r}_s\right)dV \qquad (4.49)$$

This equation defines the elastic deformation $\vec{d}$ in (X, Y, Z) but allows $\vec{\vec{\Gamma}}_o$ to be specified in (X_o, Y_o, Z_o) so that it becomes a unique property of the structure rather than a function of the flight condition.

4.5 Summary and Interpretation of General Elastic Airplane Equations of Motion

The general equations of motion of an elastic airplane as derived in the previous sections may be summarized as follows:

$$M\frac{d}{dt}\left(\frac{d\vec{r}_0}{dt}\right) = M\frac{d}{dt}(\vec{V}_c) = M\vec{g} + \int_S \vec{F}\, dS \tag{4.8}$$

$$\frac{d}{dt}\int_V \vec{r} \times \frac{d\vec{r}}{dt}\rho_A\, dV = \int_S \vec{r} \times \vec{F}\, dS \tag{4.9}$$

$$\vec{d} - \vec{d}_0 - \frac{1}{2}(\vec{\nabla} \times \vec{d})_0 \times \vec{r} = \int_V \vec{\vec{\Gamma}}_0 \cdot (\vec{R} - \rho_A \vec{a})\, dV + \int_V \vec{\vec{\Gamma}}_0 \cdot \vec{F}\delta(\vec{r} - \vec{r}_S)\, dV \tag{4.49}$$

All equations are written relative to an earth-fixed coordinate system (X', Y', Z'), which is assumed to be inertial. The motion vector $\vec{r}\,'_0$ describes the motion of the center of mass of the elastic airplane. The deformation vectors $\vec{r}$ and $\vec{d}$ are defined relative to a body-fixed axis system (X, Y, Z) with origin at the center of mass of the elastic airplane.

A complete list of assumptions on which equations (4.8), (4.9), and (4.49) are based is given below:

- No gust effects are considered. (Strictly speaking, this is not true at this point because $\vec{F}$ could contain forcing functions if desired.) G6
- Only free-flight conditions are considered. (No takeoff, landing, or ground effect.) D1
- No electromagnetic effects are considered. G3
- No thermoelastic effects are considered. G2
- The airplane consists of n finite mass points, where n is to be large if accurate approximations to continuous solutions are desired. G1
- Airplane mass is constant with time, and no fuel slosh is accounted for
- The origin of system (X, Y, Z) is at the center of mass. G10
- Strains and displacement gradients are small. S2

- S2 Hoc [illegible] aw is valid and displacements are small.
- S3 Structural damping is negligible.

Coupling between elastic and rigid degrees of freedom enters the equations through the vectors $\vec{r}$ and $\vec{F}$, where $\vec{F}$, in general, is a function of $\vec{r}$. From an analytical viewpoint, the problem of describing the static and dynamic behavior of the elastic airplane is now solved. From the stability and control engineer's viewpoint, this is only the beginning.

The vector quantity $\vec{\theta}_0$ may be introduced to represent the rotation of (X_0, Y_0, Z_0) relative to (X, Y, Z):

$$\vec{\theta}_0 = \frac{1}{2}(\vec{\nabla} \times \vec{d})_0 \tag{4.50}$$

Equation (4.49) describes the elastic desplacement field at the surface of the airplane. With this substitution, equation (4.49) becomes:

$$\vec{d} - \vec{d}_0 - \vec{\theta}_0 \times \vec{r} = \int_V \vec{\vec{\Gamma}}_0 \cdot (\vec{R} - \rho_a \vec{a})\, dV + \int_V \vec{\vec{\Gamma}}_0 \cdot \vec{F}\, \delta(\vec{r} - \vec{r}_s)\, dV \tag{4.51}$$

4.6 Use and Specialization of the General Equations of Motion

Equations (4.8), (4.9), and (4.49) will be specialized to steady-state flight equations. The determination of motion variables and airplane shape in steady-state flight is needed to solve most dynamic stability and control problems.

As will be shown in Sec. 6, where the equations are specialized to form stability (perturbed motion) equations, the steady-state motion variables, inertial properties, and shape enter into the stability equations, making it necessary first to establish the steady state completely.

Solutions to the force equations can be obtained by adding to $\vec{F}$ the control deflections or other forcing functions. For linear equations, it is then possible to obtain explicit solutions by means of Laplace transforms. Nonlinear equations can be solved numerically.

5. STEADY-STATE EQUATIONS OF MOTION

5.1 Introduction and Definitions

The purpose of this section is to present equations from which the equilibrium of steady-state flight can be determined. This determination is needed because the perturbed equations of motion, developed in Sec. 4, are so written that the perturbations take place about the steady-state condition. Several of the steady-state characteristics such as inertia, deformations, and aerodynamic forces affect the perturbed state, and knowledge of these steady-state quantities is therefore required.

Steady-state flight is defined as a flight condition for which all motion variables remain constant with time relative to a body fixed axis system. The three vectors defining the elastic airplane flight condition are $\vec{V}_c$, $\vec{\omega}$, and $\vec{d}$, the velocity of the center of mass, the angular velocity about the center of mass, and the elastic deformation, respectively. This definition of steady-state flight implies that $\vec{V}_c$, $\vec{\omega}$, and $\vec{d}$ are constant in time.

If the atmosphere is inhomogeneous because of variations with altitude, steady-state flight implies that $\vec{V}_c$ is horizontal. This follows because ρ will change in time if $\vec{V}_c$ is not horizontal; thus the aerodynamic forces will change in time, causing the deformed shape $\vec{d}$ to change, and so on. However, for shallow flight path angles, ρ may be taken as a constant for reasonable lengths of time. Hence assumption of constant air density ρ is introduced at this point. A.1
This assumption must not be made, however, without careful checking.

Steady-state flight also implies that $\vec{\omega}$ is vertical (or approximately so for shallow flight path angles); otherwise unsteady flight would result. However, to include the important steady-state pullup maneuver in the equations (the steady state refers here to speed only), an exception will be made.

Two important types of steady-state flight must be considered: steady-state rectilinear flight and steady-state curvilinear flight. Steady-state curvilinear flight will be considered in two parts: steady-state (approximately level) turning flight and steady-state symmetrical pullup. The following assumptions apply to these conditions:

D9 • Steady-state rectilinear flight.

  - C   nt air density
  - $\vec{V}_c$ and $\vec{d}$ constant
  - $\vec{\omega} = 0$

D3 • Steady-state curvilinear flight

  - Constant air density
  - $\vec{V}_c$ and $\vec{d}$ constant
  - $\vec{\omega} \neq 0$ but constant
  - (a) Level turn: $P_1 = 0$, Q_1 and R_1 constant

    (b) Pullup: $P_1 = R_1 = 0$, Q_1 constant

The subscript 1 is used from here on to indicate steady-state motion variables.

Equations for both types of steady-state flight are presented in pars. 5.2 and 5.3. In par. 5.4 the expressions for aerodynamic forces and moments are developed in terms of steady-state motion variables. Finally, in par. 5.5 the practical problems that can be solved with the results of pars. 5.2 through 5.4 are discussed. A summary of assumptions is also provided in par. 5.5.

5.2 Steady-State Rectilinear Flight

For rectilinear flight, the conditions of steady rectilinear motion must be applied to the equations of motion (4.8), (4.9), and (4.50). This yields:

$$
\begin{aligned}
&M\vec{g} + \int \vec{F}\, dS = 0 \\
&\int_S \vec{r} \times \vec{F}\, dS = 0 \qquad\qquad (5.1)\\
&\vec{d} - \vec{d}_0 - \vec{\theta}_0 \times \vec{r} = \int_S \vec{\Gamma}_0 \cdot (\rho_A \vec{g} + \vec{F})\, dS
\end{aligned}
$$

At this point it is necessary to expand equations (5.1) into cartesian form. They are expanded one by one below.

5.2.1 Momentum equation. —The momentum equation for steady-state rectilinear flight is:

$$M\vec{g} + \int \vec{F}\, dS = 0 \qquad (5.2)$$

The expansion of this equation will allow for shallow climbs and dives, with the stipulation that assumption G5 (constant air density) is not grossly violated.

The components of $\vec{g}$ in the body axis system (X, Y, Z) can be identified only if the orientation of (X, Y, Z) relative to the earth axis system (X', Y', Z') is given (fig. 2). To do this the conventional system of Euler angles, Ψ, θ, ϕ, is used (fig. 5).

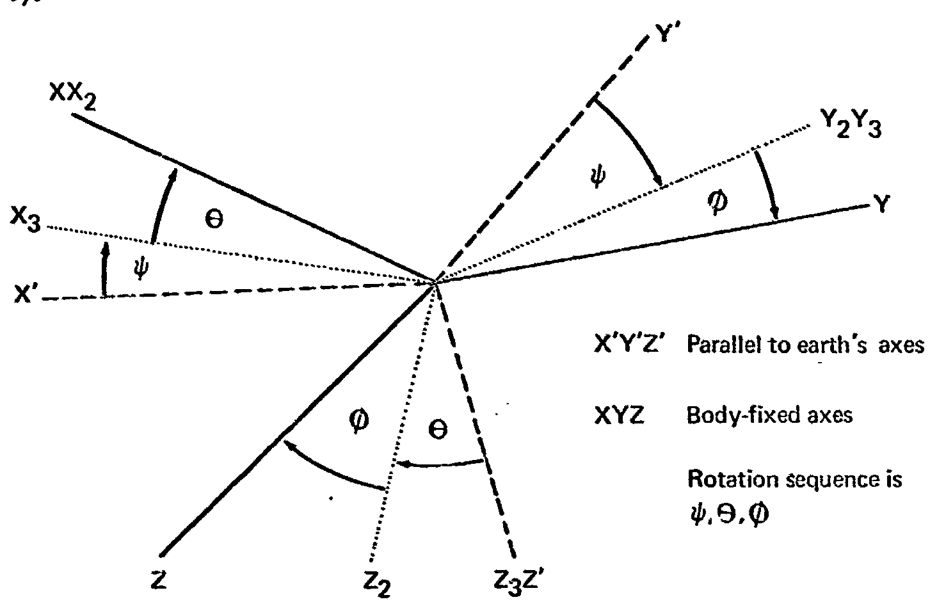


FIGURE 5. – EULER ANGLES

It is readily seen that $M\vec{g}$ can be resolved as follows:

$$M\vec{g} = (\vec{i}\, g_x + \vec{j}\, g_y + \vec{k}\, g_z) \qquad (5.3)$$

where:

$$g_x = -g \sin \theta_1$$

$$g_y = g \cos \theta_1 \sin \phi_1 \qquad (5.4)$$

$$g_z = g \cos \theta_1 \cos \phi_1$$

The force per unit area $\vec{F}$ consists of an aerodynamic component and a thrust component. For convenience of notation, it is now defined that:

$$\int_S \vec{F} dS = \vec{F_A} + \vec{F_T} \tag{5.5}$$

where $\vec{F}_A$ and $\vec{F}_T$ have the dimension of force, not force per unit area.

Using subscripts again to designate (X, Y, Z) components, equations (5.2) in steady-state notation are:

$$\begin{aligned} F_{AX_1} + F_{TX_1} - Mg \sin\theta_1 &= 0 \\ F_{AY_1} + F_{TY_1} + Mg\cos\theta_1 \sin\phi_1 &= 0 \\ F_{AZ_1} + F_{TZ_1} + Mg\cos\theta_1\cos\phi_1 &= 0 \end{aligned} \tag{5.6}$$

5.2.2 Moment of momentum equation. — The moment of momentum equation for steady-state rectilinear flight is:

$$\int_S \vec{r} \times \vec{F} dS = 0 \tag{5.7}$$

by introducing

$$\vec{M} = \int_S \vec{r} \times \vec{F} dS = \vec{M_A} + \vec{M_T} \tag{5.8}$$

it is easily seen that this equation, when written in component form, yields in steady-state notation:

$$\begin{aligned} M_{AX_1} + M_{TX_1} &= 0 \\ M_{AY_1} + M_{TY_1} &= 0 \\ M_{AZ_1} + M_{TZ_1} &= 0 \end{aligned} \tag{5.9}$$

5.2.3 Internal equilibrium equation. — The equation for internal equilibrium for steady-state rectilinear flight is:

$$\vec{d} - \vec{d_o} - \vec{\theta_o} \times \vec{r} = \int_S \overline{\overline{\Gamma_o}} \cdot (\rho_A \vec{g} + \vec{F}) dS \tag{5.10}$$

This equation represents a solution in continuous form and is written in terms of the elastic deformation vector $\vec{d}$. It is now necessary to specify the structural properties represented by the influence function tensor $\overline{\Gamma}_0$ in practical (state-of-the-art) terms. To this end, the airplane is divided into a large number (n) of panels as in fig. 6. A control point is located on each panel. The structural properties are represented at each control point by a matrix of influence coefficients $[C_{ij}]$ such as described in Sec. 6 and such that equation (5.10) is written in matrix form. Equation (5.10) must now be written in matrix form because once a finite number is assigned to n , the integral formulation is no longer meaningful.

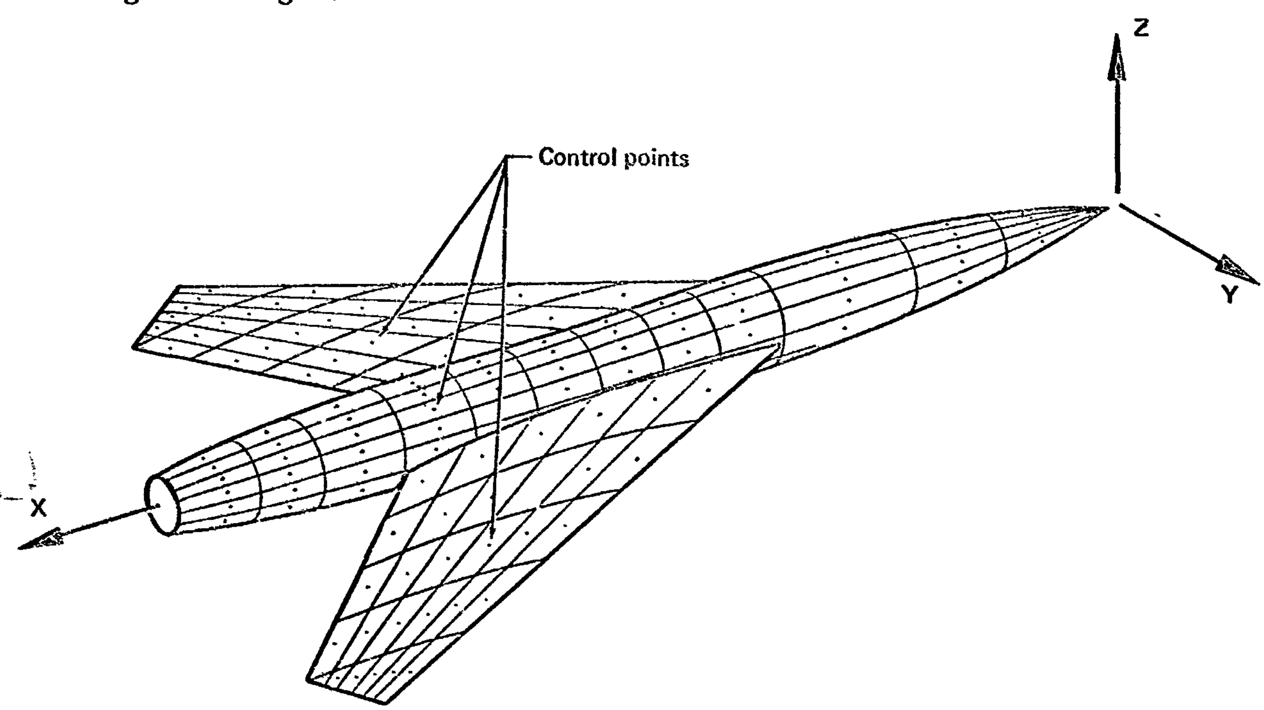


FIGURE 6. – EXAMPLE OF A PANELED AIRPLANE

The use of structural influence coefficients is based on assumptions S1, S2, G2, G3, S5, and S9. In addition, an assumption must be made as to how the structure itself is broken down. The various possibilities available to the analyst are discussed in app. B. The mathematical form of the equation is not affected by this choice.

The general form of $[C_{ij}]$ relates the six degrees of freedom of each control point to the six possible loads at each control point, namely, three local moments and three local translational forces. The general form of the load-deflection relation in analogy to equation (5.10) is as follows:

$$\begin{Bmatrix} d_{xo_i} \\ d_{yo_i} \\ d_{zo_i} \\ \theta_{xo_i} \\ \theta_{yo_i} \\ \theta_{zo_i} \end{Bmatrix} = \begin{Bmatrix} d_{ox} + \theta_{oy} z_i + \theta_{oz} y_i \\ d_{oy} + \theta_{ox} z_i + \theta_{oz} x_i \\ d_{oz} + \theta_{oy} x_i + \theta_{ox} y_i \\ \theta_{ox} \\ \theta_{oy} \\ \theta_{oz} \end{Bmatrix} + \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} & C_{\theta_x x} & C_{\theta_y x} & C_{\theta_z x} \\ C_{xy} & C_{yy} & C_{zy} & C_{\theta_x y} & C_{\theta_y y} & C_{\theta_z y} \\ C_{xz} & C_{yz} & C_{zz} & C_{\theta_x z} & C_{\theta_y z} & C_{\theta_z z} \\ C_{x\theta_x} & C_{y\theta_x} & C_{z\theta_x} & C_{\theta_x\theta_x} & C_{\theta_y\theta_x} & C_{\theta_z\theta_x} \\ C_{x\theta_y} & C_{y\theta_y} & C_{z\theta_y} & C_{\theta_x\theta_y} & C_{\theta_y\theta_y} & C_{\theta_z\theta_y} \\ C_{x\theta_z} & C_{y\theta_z} & C_{z\theta_z} & C_{\theta_x\theta_z} & C_{\theta_y\theta_z} & C_{\theta_z\theta_z} \end{bmatrix} \begin{Bmatrix} m_i g_{x_o} + F_{Axo_i} + F_{Txo_i} \\ m_i g_{y_o} + F_{Ayo_i} + F_{Tyo_i} \\ m_i g_{z_o} + F_{Azo_i} + F_{Tzo_i} \\ M_{xo_i} \\ M_{yo_i} \\ M_{zo_i} \end{Bmatrix}$$

(5.11)

Note that $i = 1, 2, \ldots, n$ in equation (5.11). A two-dimensional interpretation of equation (5.11) is provided in fig. 7. The transformation quantities $\vec{d}_o$ and $\vec{\theta}_o$ are clearly indicated in fig. 7; it will be seen, however, that these transformation quantities are not essential in a practical analysis.

Equation (5.11) is the most ambitious static aeroelastic analysis that can be undertaken in the current state of the art of structural analysis. Notice that each element in the matrix $[C_{ij}]$ is itself an $(n \times n)$ matrix. There are two reasons why simplifications must be introduced:

A12

- Aerodynamic theory has not reached the point where meaningful matrix expressions can be generated for forces in the Y-direction. This is further discussed in app. B.
- Computer time and space are limited.

S10 It is generally assumed that the airplane does not deform significantly in the X-direction or as a consequence of forces in the X-direction. This results in zero matrices for all terms with subscript X in equation (5.11).

A2 A3 A4 A12 Because of the state of the art of aerodynamic theory, no meaningful matrix equations can be presented for steady-state deformations involving displacements along Y. This results in zero matrices for all terms associated with displacements with subscript Y in equation (5.11). Thus the following analysis is restricted to zero-sideslip conditions.

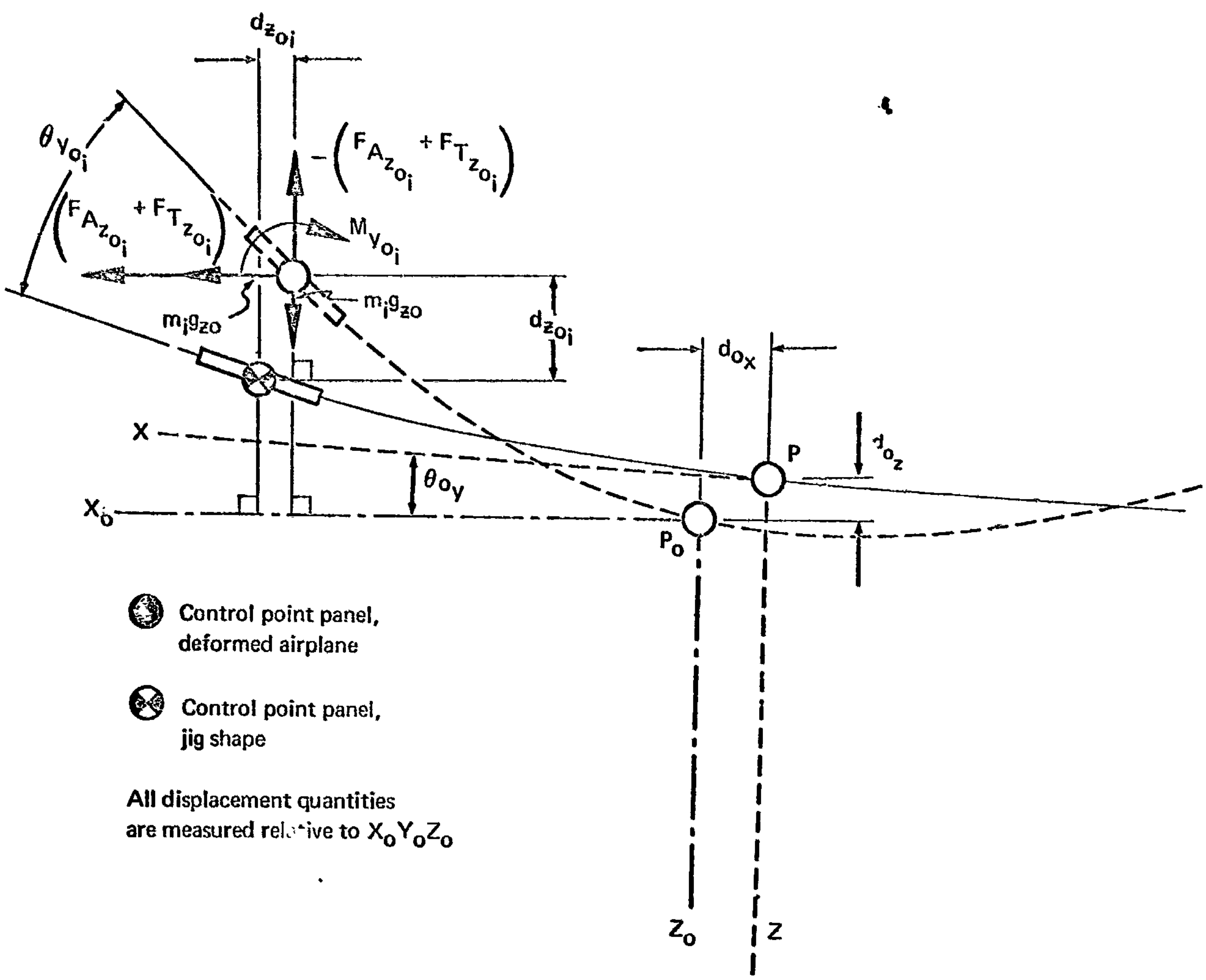


FIGURE 7. – GEOMETRY OF PANEL DISPLACEMENT AND ROTATION

Furthermore, because of assumptions A2, A3, A4 and A12, it is not necessary to carry influence coefficients involving θ_X and θ_Z. This reduces equation (5.11), for practical purposes, to:

$$\begin{Bmatrix} d_{z_{o_i}} \\ \theta_{y_{o_i}} \end{Bmatrix} = \begin{Bmatrix} d_{o_z} + \theta_{o_y} x_i \\ \theta_{o_y} \end{Bmatrix} + \begin{bmatrix} C_{zz} & C_{z\theta_y} \\ C_{z\theta_y} & C_{\theta_y\theta_y} \end{bmatrix} \begin{Bmatrix} m_i g_{z_o} + F_{A_{z_{o_i}}} + F_{T_{z_{o_i}}} \\ M_{y_{o_i}} \end{Bmatrix} \tag{5.12}$$

Finally, it is generally assumed that $\{M_{Y_0}\} = 0$, which really means that the control points i are selected so that at no time does a moment M_y occur. Equation (5.12) therefore reduces to:

$$\{\alpha_{z_{oi}}\} = (\alpha_{o_z} + \theta_{o_y} x_i)\{1\} + [C_{zz}]\{m_i g_{z_o} + F_{A_{z_{oi}}} + F_{T_{z_{oi}}}\} \tag{5.13}$$

and

$$\{\theta_{y_{oi}}\} = \theta_{o_y}\{1\} + [C_{z\theta_y}]\{m_i g_{z_o} + F_{A_{z_{oi}}} + F_{T_{z_{oi}}}\} \tag{5.14}$$

Note that the axis system used above is the body fixed axis system (X_o, Y_o, Z_o). Equation (5.14) relates the streamwise elastic change of the angle of attack to the forces applied normal to the X_oY_o plane.

Equations (5.13) and (5.14) represent the expanded form of the internal equilibrium equation of an elastic airplane in steady-state rectilinear flight with zero sideslip.

5.3 Steady-State Curvilinear Flight

For curvilinear flight, the conditions of steady curvilinear motion must be applied to the equations of motion (4.8), (4.9), and (4.50). Because the angular velocity vector $\vec{\omega}$ is nonzero, complications arise in the expansion of the equations of motion (4.8), (4.9), and (4.50). The differential operator $\frac{d}{dt}(\ )$ can be written in body axes as:

$$\frac{d}{dt}(\quad) = \frac{\partial}{\partial t}(\quad) + \vec{\omega} \times (\quad) \tag{5.15}$$

where $\partial/\partial t$ indicates a time differentiation with respect to body axes. Because in steady-state flight $\partial/\partial t = 0$, this yields:

$$\frac{d}{dt}\left(\frac{d\vec{r}_c'}{dt}\right) = \frac{d}{dt}\vec{V}_{c_1} = \vec{\omega}_1 \times \vec{V}_{c_1} \tag{5.16}$$

The momentum equation (4.8) may now be written:

$$M\vec{\omega}_1 \times V_{c_1} = M\vec{g}_1 + \int_S \vec{F}_1\, dS \tag{5.17}$$

For the moment of momentum equation (4.9), it follows that

$$\frac{d}{dt}\int_V \vec{r}_1 \times \frac{d\vec{r}_1}{dt}\rho_A\, dV = \int_S \vec{r}_1 \times \vec{F}_1\, dS \;, \tag{5.18}$$

where steady-state notation is used. Employing equation (5.16), equation (5.18) may be written:

$$\frac{d}{dt}\int_V \vec{r}_1 \times (\vec{\omega}_1 \times \vec{r}_1)\rho_A\, dV = \vec{\omega}_1 \times (\overline{\overline{\Psi}}_1 \cdot \vec{\omega}_1) \tag{5.19}$$

where:

$$\overline{\overline{\Psi}}_1 \cdot \vec{\omega}_1 = \int_V \vec{r}_1 \times (\vec{\omega}_1 \times \vec{r}_1)\rho_A\, dV$$

so that $\overline{\overline{\Psi}}_1$ is the inertia tensor. In matrix form, the inertia tensor for a symmetrical airplane is written:

$$\overline{\overline{\Psi}}_1 = \begin{bmatrix} I_{xx_1} & 0 & I_{xz_1} \\ 0 & I_{yy_1} & 0 \\ I_{xz_1} & 0 & I_{zz_1} \end{bmatrix} \tag{5.20}$$

The moment of momentum equation (5.18) may now be written:

$$\vec{\omega}_1 \times (\overline{\overline{\Psi}}_1 \cdot \vec{\omega}_1) = \int \vec{r}_1 \times \vec{F}_1\, dS \tag{5.21}$$

The significance of the subscript 1 in the equation for the inertia tensor (5.20) is simply that its components are a function of the elastic equilibrium shape of the airplane in the particular steady-state flight being considered. This is a result different from rigid-airplane steady flight conditions. It is found here that with elastic airplanes, the inertia tensor is a function of the elastic properties of the airplane.

The equation of internal equilibrium at this time is best left in the form of equation (4.50). Equations (5.17), (5.21), and (4.50) will now be expanded one by one.

5.3.1 Momentum equation. — The momentum equation for steady-state curvilinear flight is equation (5.17):

$$M \vec{\omega}_1 \times \vec{V}_{c_1} = M \vec{g}_1 + \int_S \vec{F}_1 \, dS \tag{5.17}$$

Allowing for shallow climbs and dives and using the $\vec{g}$ and $\vec{F}$ of equations (5.4) and (5.5) yields in body axes:

$$\begin{aligned} M(Q_1 W_1 - R_1 V_1) + Mg \sin \theta_1 &= F_{A_{x_1}} + F_{T_{x_1}} \\ M(R_1 U_1 - P_1 W_1) - Mg \cos \theta_1 \sin \phi_1 &= F_{A_{y_1}} + F_{T_{y_1}} \\ M(P_1 V_1 - Q_1 U_1) - Mg \cos \theta_1 \cos \phi_1 &= F_{A_{z_1}} + F_{T_{z_1}} \end{aligned} \tag{5.22}$$

However, since stability axes will be used to describe the motion, $W_1 = 0$. For shallow climbs and dives, since the vector $\vec{\omega}_1$ is directed approximately vertically, $P_1 \approx 0$ so that equations (5.22) reduce to:

$$\begin{aligned} (-M R_1 V_1) + Mg \sin \theta_1 &= F_{A_{x_1}} + F_{T_{x_1}} \\ (M R_1 U_1) - Mg \cos \theta_1 \sin \phi_1 &= F_{A_{y_1}} + F_{T_{y_1}} \\ (-M Q_1 U_1) - Mg \cos \theta_1 \cos \phi_1 &= F_{A_{z_1}} + F_{T_{z_1}} \end{aligned} \tag{5.23}$$

5.3.2 Momentum of momentum equation. — The momentum equation for steady-state curvilinear flight is equation (5.21):

$$\vec{\omega}_1 \times (\overline{\overline{I}}_1 \cdot \vec{\omega}_1) = \int_S \vec{r} \times \vec{F}_1 \, dS \tag{5.21}$$

Expansion, using the steady-state notation and employing equation (5.8), yields:

$$\begin{aligned} -I_{xz_1} P_1 Q_1 + (I_{zz_1} - I_{yy_1}) R_1 Q_1 &= M_{A_{x_1}} + M_{T_{x_1}} \\ (I_{xx_1} - I_{zz_1}) P_1 R_1 + I_{xz_1}(P_1^2 - R_1^2) &= M_{A_{y_1}} + M_{T_{y_1}} \\ (I_{yy_1} - I_{xx_1}) P_1 Q_1 + I_{xz_1}(R_1 Q_1) &= M_{A_{z_1}} + M_{T_{z_1}} \end{aligned} \tag{5.24}$$

Using the shallow climb and dive condition $P_1 = 0$, equations (5.25) reduce to:

$$(I_{zz_1} - I_{yy_1}) R_1 Q_1 = M_{Ax_1} + M_{Tx_1}$$
$$-I_{xz_1} R_1^2 = M_{Ay_1} + M_{Ty_1} \qquad (5.25)$$
$$I_{xz_1} R_1 Q_1 = M_{Az_1} + M_{Tz_1}$$

5.3.3 Internal equilibrium equation. — from equation (4.50) the internal equilibrium equation for steady-state curvilinear flight is:

$$\vec{d} - \vec{d}_o - \frac{1}{2} (\vec{\nabla} \times \vec{d})_o \times \vec{\vec{r}} = \int_S \vec{\vec{\Gamma}} \cdot (\rho_A \vec{G} - \rho_A \vec{a} + \vec{F}) dS \qquad (5.26)$$

The acceleration $\vec{a}$ may be expanded as follows, referring to fig. 4 and using the steady-state condition $\partial/\partial t(\,) = 0$:

$$\vec{a} = \frac{d^2 \vec{r}_o'}{dt^2} + \frac{d^2 \vec{r}_1}{dt^2} = \vec{\omega}_1 \times \vec{V}_{c_1} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_1) \qquad (5.27)$$

The second term of equation (5.27) vanishes upon substitution into equation (5.26), as a consequence of the definition of the center of mass. The first term of equation (5.27) is nonzero. However, since $\vec{\omega}_1 = \vec{j}Q_1 + \vec{k}R_1$ because $P_1 = 0$ and since $\vec{V}_{c1} = \vec{i}U_1 + \vec{j}V_1$ because of the use of stability axes ($W_1 = 0$), equation (5.29) after expansion is:

$$a_x = -R_1 V_1$$
$$a_y = R_1 U_1 \qquad (5.28)$$
$$a_z = -Q_1 U_1$$

Since only a_z is retained in the internal equilibrium equations (following the development of par. 5.2.3), it follows that these equations reduce to:

$$\{d_{z_{oi}}\} = d_{o_z}\{1\} + [C_{zz}]\{m_i g_{z_o} + F_{Az_{oi}} + F_{Tz_{oi}} + m_i Q_1 U_1\} \qquad (5.29)$$

$$\{\theta_{y_{oi}}\} = \theta_{o_y}\{1\} + [C_{z\theta_y}]\{m_i g_{z_o} + F_{Az_{oi}} + F_{Tz_{oi}} + m_i Q_1 U_1\} \qquad (5.30)$$

Notice the slight difference between equations (5.13), (5.14), and equations (5.29), (5.30). The latter are written in stability axes, the former not necessarily.

The only complication, then, is that in steady-state curvilinear flight the elastic shape is also a function of Q_1. If an estimate is made of the steady-state bank angle in a level turn, a reasonable estimate of this acceleration term results from:

$$\vec{\omega} = \frac{\vec{g}\tan\phi}{V_c} \tag{5.31}$$

so that:

$$Q_1 = \frac{g\tan\phi_1}{V_{c_1}}\sin\phi_1 \quad , g = |\vec{g}| \tag{5.32}$$

and:

$$Q_1 U_1 = \frac{g\sin^2\phi_1}{V_{c_1}\cos\phi_1} U_1 \tag{5.33}$$

But, since $V_{c_1} \approx U_1$,

$$Q_1 U_1 = g\frac{\sin^2\phi_1}{\cos\phi_1} \tag{5.34}$$

Notice that this term is certainly not negligible because even in a mild 30-degree bank turn $g_{z_0} = 0.87g$ and $Q_1U_1 \approx 0.29g$ so that $g_{z_0} + Q_1U_1 = 1.16g$.

For symmetrical pullup, the value of Q_1U_1 may be found from:

$$Q_1 U_1 = (n-1)g \tag{5.35}$$

Depending on the load factor n being applied, Q_1U_1 can be very significant.

5.4 Representation of Aerodynamic and Thrust Forces and Moments

In this section the functional dependence of aerodynamic and thrust forces and moments is discussed. First, the parameters needed to define these forces and moments are identified. Second, the mathematical forms relating forces and moments to motion variables are presented. Restrictions and assumptions made in doing so are carefully listed.

5.4.1 Functional dependence of aerodynamic and thrust forces and moments. — The aerodynamic force $\vec{F}_A$ is a function of:

- The geometry of the elastic airplane, $(\vec{d} + \vec{r})$
- The altitude h, which defines the air density ρ
- The Mach number M and the airplane speed V_c
- The angle of attack α
- The angle of sideslip β
- The Reynolds number Re
- The angular velocity $\vec{\omega}$
- The control surface angles δ_A, δ_E, δ_R

Airplane speed and air density (altitude) define the dynamic pressure:

$$\bar{q} = \frac{1}{2} \rho V_c^2 \tag{5.36}$$

The thrust force $\vec{F}_T$ is a function of:

- Engine control parameters
- The airplane speed V_c
- The altitude h, which defines the air density ρ
- The Mach number M
- The angle of attack α
- The angle of sideslip β
- The angular velocity $\vec{\omega}$
- The geometry of the elastic airplane, $(\vec{d} + \vec{r})$, in certain configurations

The aerodynamic moment $\vec{M}_A$ depends on the same parameters as the aerodynamic force: $(\vec{d} + \vec{r})$, ρ, M, α, β, Re, $\vec{\omega}$, δ_A, δ_E, δ_R.

The thrust moment M_T depends on the same parameters as the thrust force: V_0, ρ , M , α , β , $\vec{\omega}$, $(\vec{d} + \vec{f})$, and engine control parameters.

The mathematical form of the functional dependence described above depends on the type of aerodynamic theory used. The most commonly used theories for describing aerodynamic forces and moments are discussed in app. B An interaction exists between the type of structural representation used and the type of aerodynamic representation used. Since it is not feasible to include all possible combinations of aerodynamic representation and structural representation, only four possibilities are developed herein:

5.4.2 Rigid airplane with aerodynamic derivative theory

5.4.3 Rigid airplane with aerodynamic influence coefficient theory

5.4.4 Elastic airplane with equivalent elastic derivative theory

5.4.5 Elastic airplane with aerodynamic and structural influence coefficient theory

Specific mathematical relations for the thrust forces and moments are not developed in this report. It is assumed that data are always available to properly account for thrust effects in the equations for practical situations. In steady-state solutions it is common to estimate steady-state values of the thrust forces and moments and, if necessary, to iterate to the correct solution.

5.4.2 Rigid airplane with aerodynamic derivative theory. — The derivatives are commonly computed in a so-called stability axis system. Stability axes are defined as body fixed axes with the X-axis pointing in the direction opposite to that of the steady-state velocity vector, provided no sideslip exists ($\beta = 0$). If sideslip does exist, the stability X-axis is defined as pointing in the direction opposite to that of the projection of the steady-state velocity vector on the XZ plane of symmetry.

In steady-state flight the expressions for the aerodynamic forces and moments are as follows:

$$F_{A_{Z_S}} = -C_{D_1} \bar{q} S_W$$

$$= -\{\underbrace{C_{D_0} + C_{D_\alpha}\alpha_1 + [C_{D_{\alpha^2}}\alpha_1^2]}_{C_{D_{\alpha_1}}\alpha_1} + [C_{D_\beta}|\beta_1|] + \Sigma C_{D_{\delta_i}}\delta_{i_1}\}\bar{q} S_W \tag{5.37}$$

where $i = A$, E, or R, signifying lateral, longitudinal, or directional control, respectively;

$$
\begin{aligned}
F_{A_{z_S}} &= -C_{L_1}\bar{q}S_w \\
&= -\left\{C_{L_0} + C_{L_\alpha}\alpha_1 + C_{L_q}\frac{Q_1\bar{c}}{2V_{c_1}} + [C_{L_\beta}|\beta_1|] + \Sigma C_{L_{\delta_i}}\delta_{i_1}\right\}\bar{q}S_w
\end{aligned}
\tag{5.38}
$$

where Q_1 is the steady rotational velocity about the Y_S-axis;

$$
\begin{aligned}
F_{A_{y_S}} &= C_{y_1}\bar{q}S_w \\
&= \left\{C_{y_\beta}\beta_1 + C_{y_r}\frac{R_1 b}{2V_{c_1}} + \Sigma C_{y_{\delta_i}}\delta_{i_1}\right\}\bar{q}S_w
\end{aligned}
\tag{5.39}
$$

where R_1 is the steady rotation velocity about the Z_S-axis;

$$
\begin{aligned}
M_{A_{x_S}} &= C_{l_1}\bar{q}S_w b \\
&= \left\{C_{l_\beta}\beta_1 + C_{l_r}\frac{R_1 b}{2V_{c_1}} + \Sigma C_{l_{\delta_i}}\delta_{i_1}\right\}\bar{q}S_w b
\end{aligned}
\tag{5.40}
$$

$$
\begin{aligned}
M_{A_{y_S}} &= C_{m_1}\bar{q}S_w\bar{c} \\
&= \left\{C_{m_0} + C_{m_\alpha}\alpha_1 + C_{m_q}\frac{Q_1\bar{c}}{2V_{c_1}} + [C_{m_\beta}|\beta_1|] + \sum_i C_{m_{\delta_i}}\delta_{i_1}\right\}\bar{q}S_w\bar{c}
\end{aligned}
\tag{5.41}
$$

$$
\begin{aligned}
M_{A_{z_S}} &= C_{n_1}\bar{q}S_w\bar{c} \\
&= \left\{C_{n_\beta}\beta_1 + C_{n_r}\frac{R_1 b}{2V_{c_1}} + \Sigma C_{n_{\delta_i}}\delta_i\right\}\bar{q}S_w b
\end{aligned}
\tag{5.42}
$$

In setting up these expressions for the steady-state aerodynamic forces and moments, use has been made of the development in app. B. However, terms not belonging to the steady state have been omitted. Conversely, some terms not appearing in perturbed forces and moments have been added. Examples of the latter are the C_{D_0}, C_{L_0}, and Cm_0 coefficients and $C_{D_\alpha}2$. The coefficients C_{D_α} and $C_{D_\alpha}2$ are included to indicate the quadratic relationship between C_D and α.

There are many forms in which the drag dependence on α can be written. It is only important to determine $C_{D_0} + C_{D\alpha}\alpha_1 + C_{D_{\alpha^2}}\alpha_1^2 = CD(\alpha_1)$ such that at the steady-state angle of attack α_1 the value for C_D comes out correctly. To make the equations linear it is of course possible to write the first three terms as $C_{D_\alpha}\alpha_1$. The term $C_{D\alpha}$ here is a predetermined but artificial function of α_1, determined to match the drag at $\alpha = \alpha_1$ and $\beta = \delta_i = 0$.

It should be kept in mind that all motion variables in equations (5.37) through (5.42) are steady-state motion variables. The derivatives are employed here to describe total forces and moments; they are not merely perturbed forces and moments, as is common in perturbation equations.

5.4.3 Rigid airplane with aerodynamic influence coefficient theory. — For the calculation of aerodynamic forces and moments under zero sideslip conditions, the aerodynamic force field can be represented by an influence coefficient matrix A_{ij}.

Aerodynamic influence coefficient theory is discussed in App. B, where the assumptions on which this theory are based are identified as A2, A3, A1, A7, and A8. The application of this concept is worked out below.

Figure 8 presents the airplane in its rigid or jig shape. All angles are exaggerated for clarity. The attitude of the airplane is defined by the attitude of (X, Y, Z), the body fixed coordinate system with origin at the center of mass P. In the jig shape P is identified as some material point. A stability axis system (X_S, Y_S, Z_S) is also shown. The axis $Y = Y_S$, and neither of these is shown. Built-in twist and camber may be present (X, Y, Z). In addition, the existence of control angles is recognized by introducing the scalar column $\{a_i\}$. The scalar column $\{a_i\}$ has zeros for all panels that are outside the boundaries of control surfaces, and numbers between zero and 1 for all panels within. The term $Q_1(X_i/V_{c1})$ is added to represent the panel angles of attack induced by the rotational velocity Q_1, as in a steady symmetrical pullup.

The aerodynamic forces on all panels i, using aerodynamic influence coefficient theory, are given by:

$$\{F_{A_i}\} = \bar{q}\,[A_{ij}]\{\alpha_{j_i}\} \tag{5.43}$$

where:

$$\{\alpha_{J_i}\} = \{-\gamma + \Theta_J + \theta_{J_i} + \delta_E a_i + Q_i \frac{x_i}{V_{c_{1s}}}\} \qquad (5.44)$$

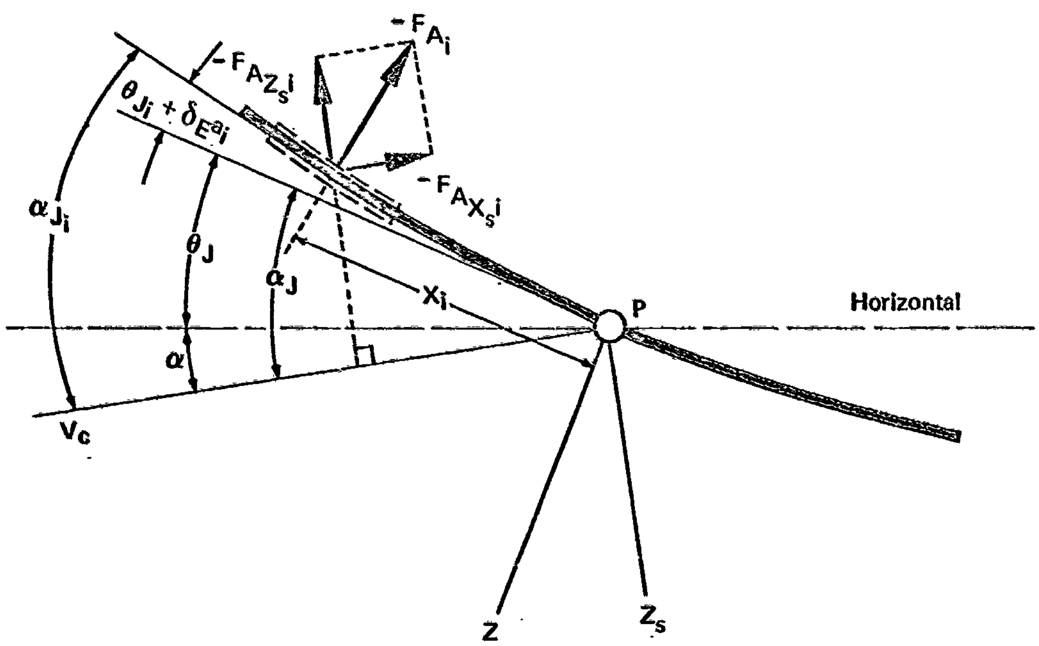


FIGURE 8. – ANGLES AND AXES FOR RIGID AIRPLANE AERODYNAMIC INFLUENCE COEFFICIENT THEORY

Note that the aerodynamic influence coefficient matrix $[A_{ij}]$ relates panel angles of attack to loads perpendicular to each panel. To obtain the force and moment components along stability axes, it is necessary to employ the angle α_{Ji}:

$$\{F_{A_{x_{s_i}}}\} = \bar{q}\,[A_{ij}]\,\left[\sin\alpha_{J_i}\right]\{\alpha_{J_i}\} \qquad (5.45)$$

$$\{F_{A_{z_{s_i}}}\} = \bar{q}\,[A_{ij}]\,\left[\cos\alpha_{J_i}\right]\{\alpha_{J_i}\} \qquad (5.46)$$

$$\{M_{A_{y_{S_i}}}\} = \bar{q} \lceil x_i \rfloor [A_{ij}] \{\alpha_{J_i}\} \qquad (5.47)$$

where it is assumed that F_{A_i} is approximately perpendicular to the X-axis.

To summarize, for zero sideslip, the aerodynamic forces and moment of a rigid airplane paneled into n panels can be written:

$$F_{A_{X_S}} = \bar{q} \lfloor 1 \rfloor [A_{ij}] \lceil \sin\alpha_{J_i} \rfloor \{\alpha_{J_i}\} \qquad (5.48)$$

$$F_{A_{Z_S}} = \bar{q} \lfloor 1 \rfloor [A_{ij}] \lceil \cos\alpha_{J_i} \rfloor \{\alpha_{J_i}\} \qquad (5.49)$$

$$M_{A_{Y_S}} = \bar{q} \lfloor x_i \rfloor [A_{ij}] \{\alpha_{J_i}\} \qquad (5.50)$$

For nonzero sideslip, it is not possible in the current state of the art to write meaningful expressions for $F_{A_{Y_S}}$, $M_{A_{X_S}}$, and $M_{A_{Z_S}}$ in a way analogous to equations (5.48) through (5.50).

5.4.4 Elastic airplane with equivalent elastic derivative theory. — Equivalent elastic derivative theory is based on the assumption that the rigid airplane derivatives of par. 5.4.2 can be modified for the effects of elasticity by multiplication or addition of constants that account for the flexibility of the airplane. In app. B this is said to apply if a reasonable natural frequency separation exists between rigid-degrees-of-freedom motions and elastic-degrees-of-freedom motion. The assumptions on which equivalent elastic derivative theory is based are identified in app. B.

This theory has particular value as a preliminary design tool because it does not require closed aeroelastic solutions based on extensive paneling of the entire airplane. Aeroelastic correction factors can be evaluated on the basis of "large scale" influence coefficients. For example, the elastic deformation of the body can be represented by displacements and rotations at the tail surfaces. This approach can be carried out for various sections of the airplane, such as wing, body, or tail. The elastic influence appears as a much smaller package than when the airplane is fully paneled. In practice, it is almost as fast and certainly more accurate to compute the aeroelastic correction constants with matrix methods (flexibility influence coefficient), but instead of having to find total airplane solutions, it is possible to apply these matrix methods to the major airplane components separately.

In equivalent elastic derivative theory, therefore, the conventional "building block" method of considering each derivative as the sum of wing-body and tail contributions is usually followed. This means now that lateral and sideslip derivatives can also be evaluated and that the $\beta = 0$ restriction does not apply. In the equations of motion the derivative notation used for the rigid airplane is employed, but the subscript E is usually added to indicate that a derivative corrected for aeroelastic effects is implied. The aerodynamic forces and moments, therefore, have basically the same form as those of equations (5.37) through (5.42), but with the subscript E added.

Detailed expressions for and derivations of the equivalent elastic derivatives are presented in app. B. One example is given here to indicate the form of these equivalent elastic derivatives. The equivalent elastic lift curve slope of a conventional wing-body-tail airplane can be shown to be:

$$C_{L_{\alpha_E}}\Big|_{V_c} = \frac{a_{TO}\big|_{n=0} + \frac{S_H}{S_W} K\, a_H\big|_{n=0}\left(1 - \frac{d\epsilon}{d\alpha}\right)}{1 - \frac{\partial C_{L_{TO}}}{\partial n}\frac{1}{C_{L_{TRIM}}} - \frac{S_H}{S_W} K \frac{1}{C_{L_{TRIM}}}\left(a_H\big|_{n=0}\frac{\partial \alpha_H}{\partial n} + \frac{\partial C_{L_H}}{\partial n}\right)} \tag{5.51}$$

where:

$A_{T_O}/n = 0$	is the tail-off lift curve slope for zero load factor
$k =$	$(1 - S_H A_H/n = 0\,\bar{K}_{\bar{q}})^{-1}$
$\bar{k}$	is the angular deflection at the tail due to a unit upload at the tail ($k<0$)
$\partial C_{L_{T_0}}/\partial n$	is the increment in tail-off lift coefficient per unit load factor
$\partial\alpha_H/\partial n$	is the change in horizontal tail angle of attack per unit load factor

$C_{L_{trim}}$ is the trim or equilibrium lift coefficient

Methods for computing the elastic constant in equation (5.51) are given in app. B.

5.4.5 Elastic airplane with aerodynamic and structural influence coefficient theory. — The developments represented by equations (5.14) and (5.43) will be used to show how the aerodynamic forces and moments are formulated for the completely elastic airplane. It was already explained in par. 5.4.3 that, because of the state of the art of aerodynamic influence coefficient theory, only the case of zero sideslip will be presented. Assumptions S7, S8, and D12 apply to the developments in this section.

It is now necessary to be more precise about the relationship between panel angles, flow angles, and the coordinate systems (X, Y, Z) and (X_0, Y_0, Z_0). See fig. 9, where all angles are exaggerated for clarity.

For the deformed airplane the aerodynamic panel force matrix can be written:

$$\{F_{A_{E_i}}\} = \bar{q}[A_{ij}]\{\alpha_{E_i}\} \tag{5.52}$$

in analogy with equation (5.43) and fig. 8. The elastic airplane panel angle may be written:

$$\{\alpha_{E_i}\} = \{-\gamma + \theta + \theta_{J_i} + \delta_E a_i + \Delta\alpha_{E_i}\} \tag{5.53}$$

where:

$\Delta\alpha_{E_i}$ = elastic deformation angle of panel i

δ_E = elevator reference deflection angle (not shown)

$\{a_i\}$ = scalar column consisting of zeros for panels outside the control surface boundary and of numbers between zero and 1 for panels inside the boundary

The deformation vector $\{\theta_{y_{0_i}}\}$ was solved for in equation (5.14):

$$\{\theta_{y_{0_i}}\} = \theta_{0_y}\{1\} + [C_{z\theta_y}]\{m_i g \cos\theta_{REF} + F_{A_{E_i}}\cos(\theta_{J_i} + \delta_E a_i + \Delta\alpha_{E_i}) + F_{T_{z0_i}}\} \tag{5.54}$$

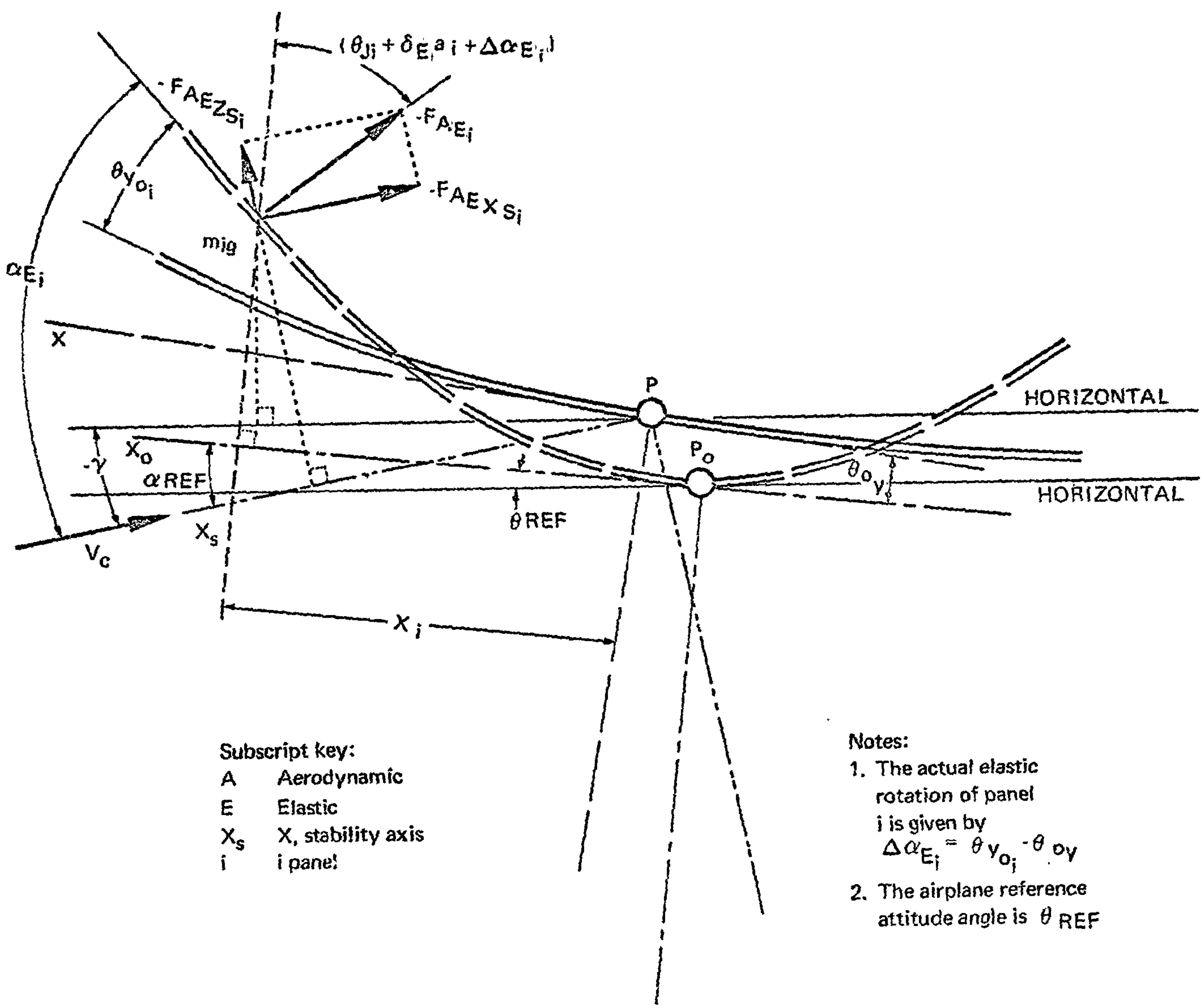


FIGURE 9. – ANGLES AND AXES FOR ELASTIC AIRPLANE WITH AERODYNAMIC AND STRUCTURAL INFLUENCE COEFFICIENT THEORY

where $F_{T_{Z_{oi}}}$ represents the thrust components along Z_o. Therefore, the expression for the aerodynamic panel forces on the deformed airplane, keeping in mind that $\{\theta_{y_{oi}}\} = \theta_{o_y}\{1\} + \{\Delta\alpha_{E_i}\}$, and using equations (5.52) and (5.53), is:

$$\{F_{AE_i}\} = \bar{q}[A_{ij}]\Big\{\{-\gamma + \theta_{REF} + \theta_{J_i} + \delta_E a_i\} + [C_{z\theta_y}]\{m_i g \cos\theta_{REF} + F_{AE_i}\cos(\theta_{J_i} + \delta_E a_i + \Delta\alpha_{E_i}) + F_{T_{zo_i}}\}\Big\}. \tag{5.55}$$

The substitution for $\Delta\alpha_{E_i} = \theta_{y_i} - \theta_o$ can be carried out indefinitely. However, for practical purposes it is sufficient to assume that $\Delta\alpha_{E_i}$ is small. In addition, it is assumed that the built-in twist and camber θ_{J_i} as well as the control angle δ_E are small, so that:

$$\cos(\theta_{J_i} + \delta_E a_i + \Delta\alpha_{E_i}) \approx 1 \tag{5.56}$$

Thus:

$$\{F_{AE_i}\} \approx \bar{q}[A_{ij}]\Big\{\{-\gamma + \theta_{REF} + \theta_{J_i} + \delta_E a_i\} + [C_{z\theta_y}]\{m_i g \cos\theta_{REF} + F_{AE_i} + F_{T_{zo_i}}\}\Big\} \tag{5.57}$$

Notice that at this point θ_{ref} is not necessarily small. From this equation it can be verified that a solution for $\{F_{A_{E_i}}\}$ is:

$$\{F_{AE_i}\} \approx \Big[[I] - \bar{q}[A_{ij}][C_{z\theta_y}]\Big]^{-1}\bar{q}[A_{ij}]\Big\{\{-\gamma + \theta_{REF} + \theta_{J_i} + \delta_E a_i\} + [C_{z\theta_y}]\{m_i g \cos\theta_{REF} + F_{T_{zo_i}}\}\Big\} \tag{5.58}$$

The stability axis components of the $F_{A_{E_i}}$ forces and moments may now be found by analogy from equations (5.48) through (5.50):

$$F_{AE_{xs}} \approx \lfloor \sin\alpha_{E_i} \rfloor \{F_{AE_i}\} \tag{5.59}$$

$$F_{AE_{zs}} \approx \lfloor 1 \rfloor [\cos\alpha_{E_i}] \{F_{AE_i}\} \tag{5.60}$$

$$M_{AE_{ys}} \approx \lfloor x_{o_i} \rfloor \{F_{AE_i}\} \tag{5.61}$$

If desired, the effect of initial pitch rotation Q_1 can be accounted for by adding the term $x_i Q_i$ to equation (5. 53) and following through.

5.5 Summary of Equations for Steady-State Flight

5.5.1 Rigid and equivalent elastic airplane equations of motion. — By substituting the expressions for aerodynamic forces and moments, equations (5. 37) through (5. 42), into equations of motion (5. 23) and (5. 25), the complete steady-state equations of motion are obtained. These equations are presented in table 1 for rectilinear as well as curvilinear flight. Those parts of the equations that apply only to curvilinear flight are clearly indicated.

The steady-state equations of table 1 are valid also for the equivalent elastic airplane, provided that all derivatives and inertial constants are evaluated to account for quasi-static elastic effects. A complete summary of assumptions used in deriving the equations of table 1 is presented in table 2.

The equations in table 1 represent a set of six equations in twelve variables ρ_1, α_1, β_1, Q_1, R_1, δA_1, δE_1, δR_1, Θ_1, Φ_1, U_1, and F_{T_1}. A total of six variables must therefore be specified before a solution can be obtained. It is assumed that the mass M of the airplane is always known.

In general, the preselected variables are:

(a) For steady-state rectilinear flight:

- Thrust $\vec{F}_{T_1}$
- Rotational velocities $P_1 = Q_1 = R_1 = 0$
- Speed U_1
- Altitude h_1 (density = ρ_1)

(b) For steady-state curvilinear flight:

(1) Level turn:

- Bank angle Φ_1
- Sideslip angle $\beta_1 = 0$ (coordinated turn)
- Speed U_1

TABLE 1.–STEADY-STATE EQUATIONS OF MOTION FOR RIGID AND EQUIVALENT ELASTIC AIRPLANES

$$-MR_1V_1 + Mg\sin\Theta_1 = F_{T_{x_S}} - \{C_{D_{\alpha}}\alpha_1 + [C_{D_\beta}|\beta_1|] + \sum C_{D_{\delta_i}}\delta_{i_1}\}\bar{q}S_w \tag{5.62}$$

$$MR_1U_1 - Mg\cos\Theta_1\sin\Phi_1 = F_{T_{y_S}} - \{C_{L_0} + C_{L_\alpha}\alpha_1 + C_{L_q}\frac{Q_1\bar{c}}{2V_{T_1}} + [C_{L_\beta}|\beta_1|] + \sum C_{L_{\delta_i}}\delta_{i_1}\}\bar{q}S_w \tag{5.63}$$

$$-MQ_1U_1 - Mg\cos\Theta_1\cos\Phi_1 = F_{T_{z_S}} + \{C_{y_\beta}\beta_1 + C_{y_r}\frac{R_1 b}{2V_{T_1}} + \sum C_{y_{\delta_i}}\delta_{i_1}\}\bar{q}S_w \tag{5.64}$$

$$(I_{zz_1} - I_{yy_1})R_1Q_1 - = M_{T_{x_S}} + \{C_{l_\beta}\beta_1 + C_{l_r}\frac{R_1 b}{2V_{T_1}} + \sum C_{l_{\delta_i}}\delta_{i_1}\}\bar{q}S_w b \tag{5.65}$$

$$-I_{xz_1}R_1^2 = M_{T_{y_S}} + \{C_{m_0} + C_{m_\alpha}\alpha_1 + C_{m_q}\frac{Q_1\bar{c}}{2V_{T_1}} + [C_{m_\beta}|\beta_1|] + \sum C_{m_{\delta_i}}\delta_{i_1}\}\bar{q}S_w\bar{c} \tag{5.66}$$

$$-I_{xz_1}R_1Q_1 = M_{T_{z_S}} + \{C_{n_\beta}\beta_1 + C_{n_r}\frac{R_1 b}{2V_{T_1}} + \sum C_{n_{\delta_i}}\delta_{i_1}\}\bar{q}S_w b \tag{5.67}$$

NOTE:

1. For zero initial sideslip, substitute $V_1 = \beta_1 = 0$
2. For equivalent elastic airplane, add subscript E to all derivatives and inertias.
3. The subscripts 1 and S are interchangeable in the stability axis system.

TABLE 2.–ASSUMPTIONS FOR EQUATIONS (5.62) THROUGH (5.67) OF TABLE 1

RIGID AIRPLANE	
General Assumptions	D1 D3 G1 G2 G3 G10 G7 G5
Rectilinear Flight	D4
Curvilinear Flight	G11
EQUIVALENT ELASTIC AIRPLANE	
General Assumptions	G1 - G5 G7 G9 G10 S1 S2 S3 D1 D2 D3
Rectilinear Flight	D4
Curvilinear Flight	G11

- Altitude h_1 (= density ρ_1)

(Equation (5.31) relates R_1, Q_1, and Φ_1.)

(2) Steady symmetrical pullup (only lift and moment equations needed):

- Bank angle $\Phi_1 = 0$
- Sideslip angle $\beta_1 = 0$
- Speed U_1 ($\delta_A = \delta_R = 0$, $R_1 = P_1 = 0$)
- Altitude h_1 (= density ρ_1)
- Load factor n (determines Q_1 through equation (5.35))
- Thrust $\vec{F}_{T_1}$

It is assumed that none of the six thrust components are variables. In general, these quantities are estimated so that they are essentially treated as knowns. If necessary, iteration can be used to refine the solution. The reason for this procedure is the complicated dependence of the thrust components on the motion variables. This relation is generally not given explicitly, but is provided in the form of graphs and tables by the engine manufacturer.

The problems that can be solved with the equations of table 1 are as follows:

(a) Determine the steady-state motion variables when certain variables are preselected (cases (a) and (b) above).

(b) Along with (a) comes the possibility of determining trim characteristics. For example, it is possible to compute control deflections as a function of the preselected variables such as speed or Mach number.

In solving steady-state flight problems it is not uncommon to solve the drag equation by intuition and thereby reduce the problem to five equations. Furthermore, the lateral degrees of freedom are often not desired, so the steady-state problem reduces to solving the lift and pitching moment equations. In this form, by using equation (5.35), these equations are also used to solve such problems as longitudinal control required per gravitational load (g).

5.5.2 Elastic airplane equations of motion. — As in par. 5.5.1, equations are presented here for steady-state rectilinear as well as for steady-state curvilinear flight; however, no sideslip or Y-deformations are considered. Steady-state curvilinear flight is therefore reduced to a consideration of steady-state symmetrical pullup only.

Table 3 presents the equations of motion for the elastic airplane and the assumptions on which they are based. These equations are obtained by substituting equations (5.59) through (5.61) into equations (5.23) and (5.25) and by using equation (5.58). Equations (5.68) through (5.71) represent a total of (n + 3) equations. Notice that there are (7 + n) variables: ρ_1, α_{ref}, θ_{ref}, δ_{E_1}, Q_1, V_{c_1}, F_{T_1}, plus n elastic deformations Θ_{y_i}, which, however, are represented by the equation for the elastic aerodynamic panel forces $F_{A_{E_i}}$. The flight path angle γ_1 is not listed as a separate variable, since γ_1 is determined by:

$$-\gamma_1 + \theta_{ref} = \alpha_{ref} \tag{5.72}$$

TABLE 3.–STEADY-STATE EQUATIONS OF MOTION FOR THE ELASTIC AIRPLANE

$$Mg \sin\theta_1 = F_{T_{X_S}} + \lfloor \sin\alpha_{E_i} \rfloor \{F_{A_{E_i}}\} \tag{5.68}$$

$$-MQ_1 U_1 - Mg \cos\theta_1 = F_{T_{Z_S}} + \lfloor 1 \rfloor \lceil \cos\alpha_{E_i} \rfloor \{F_{A_{E_i}}\} \tag{5.69}$$

$$0 = M_{T_{Y_S}} + \lfloor x_{0_i} \rfloor \{F_{A_{E_i}}\} \tag{5.70}$$

$$\{F_{A_{E_i}}\} \approx \left[\lceil E_1 \rfloor - \bar{q}[A_{ij}][C_{z\theta_{ij}}]\right]^{-1} \bar{q}[A_{ij}]\Big\{-\gamma + \theta_{REF} + \theta_{J_i} + \delta_E a_i + Q\frac{x_i}{V_{C_1}} + [C_{z\theta_{ij}}]\{m_i g \cos\theta_{REF} + F_{T_{Z_{0_i}}}\}\Big\} \tag{5.71}$$

NOTE: Friction drag left out.

ASSUMPTIONS: D1 - D3 D5 S1 - S3 G11 $R_1 = 0$ G_1 - G_5 G7 G8 G10

Four variables must be specified before a solution can be obtained. It is assumed that the mass M and the mass distribution m_i of the airplane are always known. G1

In general, the preselected variables are:

(a) For steady-state rectilinear flight:

- Altitude h_1 (= density ρ_1)
- Speed V_{C_1}
- Thrust $\vec{F}_{T_1}$
- Rotational velocity $Q_1 = 0$

(b) For steady symmetrical pullup:

- Altitude h_1 (= density ρ_1)
- Speed V_{c_1}
- Thrust $\vec{F}_{T_1}$
- Load factor n (determines Q_1 through equation (5.35))

The problems that can be solved with the equations of table 3 are as follows:

(a) Determine the steady-state motion variables (including the equilibrium shape of the elastic airplane) when certain variables are preselected (cases (a) and (b) above).

(b) Determine the jig shape of the elastic airplane if the elastic shape is given in some reference steady-state flight condition. (This problem is further discussed in par. 5.5.3.)

(c) Determine the trim characteristics of the elastic airplane.

(d) Finally, the equations of table 3 can be used to compute elastic correction factors for rigid-airplane wind tunnel model data. This is necessary on very elastic airplanes because, in general, only one rigid tunnel model shape is tested throughout the Mach range of the airplane.

Owing to the complicated matrix relations between the elastic airplane equations in table 3, an explicit solution can be obtained only by linearizing these equations, which means restricting the problem to small angles of attack. This is consistent with the current state of the art of aerodynamic theory. Solutions for high angles of attack are possible, however, provided that the analysis:

- Programs iterative solutions.
- Determines a way to relate aerodynamic influence coefficients to the angle of attack α_{E_i} in a nonlinear manner.

5.5.3 The jig shape problem. — An important problem that arises in stability and control calculations for completely elastic airplanes is that of the jig shape. The jig shape is defined as the undeformed shape of the airplane and comes about by removing all inertial and aerodynamic loading from the airplane. The term "jig shape" is used because it is the shape in which the airplane structure is assembled in the jigs, where it is nearly continuously supported.

If a certain shape is desired for an elastic airplane in some flight condition (for example, to achieve an optimum lift/drag ratio), an accurate knowlege of the jig shape is important to ensure that the desired shape is obtained. Four items are needed to determine the jig shape of an elastic airplane:

- The aerodynamic loading in the "design" flight condition
- The desired shape in the design flight conditi
- The mass distribution in the design flight condition (structural as well as nonstructural mass)
- The structural properties

With the aid of these items an "inverse" aeroelastic solution can be obtained. The term "inverse" is used because in most aeroelastic problems the task is to find the equilibrium (loaded) shape. In the jig shape problem, the task is to find the unloaded shape that when loaded in a known manner results in a known shape.

Another important application of the jig shape should be mentioned. In testing rigid wind tunnel models, the characteristics of only one shape are found over a range of flight conditions (Mach number and angle of attack). By knowing the jig shape it is possible to compute the shape in other than the design flight condition, and with this the stability and control properties. By also applying the theory to the rigid shape at the test flight conditions, it is possible to find the correction factors that should be applied to the rigid wind tunnel model data.

A simplified approach to finding the jig shape of an elastic airplane is outlined in the remainder of this section.

Applying approximation (5.56) to equation (5.54), the result is:

$$\{\Delta\alpha_{E_i}\} = [C_{z\theta_y}]\{m_i g \cos\theta_{REF} + F_{AE_i} + F_{TZ_{0_i}}\} \qquad (5.73)$$

Substituting equations (5.52) and (5.53) into equation (5.73) yields:

$$
\begin{aligned}
&\{\Delta\alpha_{E_i}\} \qquad\qquad (5.74)\\
&= [C_{z\theta_{ij}}]\{m_i g \cos\theta_{REF} + \bar{q}\,[A_{ij}]\{-\gamma + \theta_{REF} + \theta_{J_i} + \delta_E a_i \\
&\qquad + \Delta\dot{\alpha}_{E_i}\} + F_{T_{Z_{o_i}}}\}
\end{aligned}
$$

If the equilibrium shape is known, this means that $\{\Delta\alpha_{E_i}\}$ for the design flight condition is known. Equation (5.74) can then be used to solve for $\{\theta_{J_i}\}$, since in the design flight condition the other variables γ, θ_{ref}, δ_E, and $F_{T_{Z_{o_i}}}$ are also known. The column vector $\{\theta_{J_i}\}$ represents the desired jig shape.

6. PERTURBED EQUATIONS OF MOTION

6.1 Introduction and Definitions

The purpose of this section is to present equations from which the stability of steady-state flight and the response to control deflections can be determined.

To study the stability of an airplane in any steady-state flight condition, it has been found useful to write the state variables as the sum of a steady-state quantity and a disturbance (or perturbation) quantity. The algebraic manipulation needed to do so is called the perturbation substitution. In this manner, the steady-state equations are recovered when the disturbance quantities are set equal to zero.

In the past the reason for using the perturbation substitution was to derive small disturbance equations of motion. Those equations are based on the "small disturbance assumption," which implies that products and cross products of disturbance quantities are negligible. As a result the equations of motion become linear, but have only limited application. The developments of refs. 4 and 36 are typical examples. It should be noted that the perturbation substitution itself in no way restricts the validity of the analysis. Only when certain assumptions regarding the size of the perturbations have been made does the analysis become restricted in its validity.

The following typical perturbation substitutions are made:

$$\left.\begin{aligned} \vec{F} &\Rightarrow \vec{F}_1 + \vec{f} \\ P &\Rightarrow P_1 + p \\ U &\Rightarrow U_1 + u \end{aligned}\right\} \qquad (6.1)$$

To assist in the identification of perturbation quantities either a lowercase symbol or a subscript p is used. Note that in par. 5.2 all motion variables were total variables, whereas in par. 5.3 all motion variables were steady-state variables.

In dealing with perturbations, the following philosophy regarding their size is introduced. Distinctions will be made between:

- Small perturbations
- Large perturbations
- Arbitrary perturbations.

6.1.1 Small perturbations. - Small perturbations are defined as having such a size that their products and cross products can be neglected. It is not possible to associate any definite numerical values with such a definition. However, for purposes of discussion the following is suggested as a typical bracket for small disturbance variables:

$$\left.\begin{array}{l} |\alpha, \beta, \theta, \Phi| < 2.5^\circ \\ |u, v, w| < .045\, U_1 \end{array}\right\} \qquad (6.2)$$

The value 0.045 was selected to satisfy the first inequality. However, the value 2.5 degrees is arbitrary.

Lyapurov's stability theory can be used to compute the size of disturbance under which the small perturbation assumption can be made, as is shown in ref. 23. Classical stability theory cannot cope with this problem. Detailed discussion of this subject is deferred to Sec. 9.

6.1.2 Large perturbations. - Large perturbations are defined as having such a size that their products and cross products cannot be neglected. However, it is still assumed that angular quantities satisfy the condition:

$$\left.\begin{array}{l} \sin\theta \approx \theta \\ \tan\theta \approx \theta \\ \cos\theta \approx \theta \end{array}\right\} \qquad (6.3)$$

This means that the trigonometric relations involved in the perturbation components of $\vec{g}$ along X, Y, Z can still be written as in equation (6.3). The following arbitrary size brackets are suggested:

$$|\alpha, \beta, \theta, \phi| < 7.5^\circ \tag{6.4}$$
$$|u, v, w| < .13\, U_1$$

Aerodynamic theories used in stability equations for large disturbances should be able to account for large perturbation effec. in order to provide realism to large perturbation equations.

6.1.3 Arbitrary perturbations. - In this case, no limits exist for the perturbed motion variables. Equations of motion involving arbitrary perturbations should be used in such flight conditions as severe upsets and airplane spin. Because such maneuvers can hardly be classified as unaccelerated (in the forward speed sense), no detailed equations have been developed to cover these cases.

The equations of motion are developed in perturbation form in pars. 6.2 and 6.3. Paragraph 6.3 makes use of the lumped parameter representation of the airplane. The aerodynamic theory used in the perturbation expansions is discussed briefly in this appendix and in detail in app. B.

6.2 Expansion of the Momentum and Moment of Momentum Equations for an Elastic Airplane

6.2.1 Momentum equation. - The general form of the momentum equation (4.8) is:

$$M \frac{d}{dt}\left(\vec{V}_c\right) = M\vec{g} + \int_S \vec{F}\, dS \tag{6.5}$$

To write this equation in body axes it is necessary to substitute

$$\frac{d\vec{V}_c}{dt} = \frac{\partial \vec{V}_c}{\partial t} + \vec{\omega} \times \vec{V}_c \tag{6.6}$$

After carrying out this substitution, equation (6.5) in expanded form is:

$$
\begin{aligned}
M\dot{U} + M(QW - RV) &= -Mg \sin\Theta + F_{A_X} + F_{T_X} \\
M\dot{V} + M(RU - PW) &= Mg \cos\Theta \sin\Phi + F_{A_Y} + F_{T_Y} \\
M\dot{W} + M(PV - QU) &= Mg \cos\Theta \cos\Phi + F_{A_Z} + F_{T_Z}
\end{aligned}
\tag{6.7}
$$

The dot (.) indicates a rate of change apparent to an observer fixed to the body axis system.

Typical perturbation substitutions are introduced as follows:

$$
\begin{aligned}
Q &\Rightarrow Q_1 + q \\
\Theta &\Rightarrow \Theta_1 + \theta \\
U &\Rightarrow U_1 + u \\
\vec{F}_A &\Rightarrow \vec{F}_{A_1} + \vec{f}_A \\
\vec{F}_T &\Rightarrow \vec{F}_{T_1} + \vec{f}_T
\end{aligned}
\tag{6.8}
$$

Substitution of equations (6.8) into equations (6.7) yields:

$$
\begin{aligned}
&M\dot{u} + M[(Q_1 + q)(W_1 + w) - (R_1 + r)(V_1 + v)] \\
&\qquad = -Mg \sin(\Theta_1 + \theta) + F_{A_{X_1}} + f_{A_X} + F_{T_{X_1}} + f_{T_X} \\
&M\dot{v} + M[(R_1 + r)(U_1 + u) - (P_1 + p)(W_1 + w)] \\
&\qquad = Mg \cos(\Theta_1 + \theta) \sin(\Phi + \phi) + F_{A_{Y_1}} + f_{A_Y} + F_{T_{Y_1}} + f_{T_Y} \\
&M\dot{w} + M[(P_1 + p)(V_1 + v) - (Q_1 + q)(U_1 + u)] \\
&\qquad = Mg \cos(\Theta_1 + \theta) \cos(\Phi + \phi) + F_{A_{Z_1}} + f_{A_Z} + F_{T_{Z_1}} + f_{T_Z}
\end{aligned}
\tag{6.9}
$$

The steady-state equilibrium condition (subscript 1) is coupled with the perturbed state through trigonometric functions and products with perturbation variables. In the form of equation (6.9), the momentum equations are valid for arbitrary disturbances.

For most practical purposes it will suffice to make the large disturbance assumption: G9

$$
\begin{aligned}
&\cos\,(\text{perturbed angle}) = 1 \\
&\sin\,(\text{perturbed angle}) = \text{perturbed angle}
\end{aligned}
\tag{6.10}
$$

Thus the steady-state form of the equations may be used to simplify the perturbed equations to:

$$
\begin{aligned}
&M\dot{u} + M\left[Q_1 w + qW_1 + qw - R_1 v - rV_1 - rv\right] \\
&\qquad = -Mg\,\theta\cos\Theta_1 + f_{A_x} + f_{T_x} \\
&M\dot{v} + M\left[R_1 u + rU_1 + ru - P_1 w - pW_1 - pw\right] \\
&\qquad = Mg\left\{\phi\cos\Theta_1\cos\Phi_1 - \theta\sin\Theta_1\sin\Phi\right\} + f_{A_y} + f_{T_y} \qquad (6.11) \\
&M\dot{w} + M\left[P_1 v + pV_1 + pv - Q_1 u - qU_1 - qu\right] \\
&\qquad = Mg\left\{-\theta\sin\Theta_1\cos\Phi_1 - \phi\cos\Theta_1\sin\Phi_1\right\} + f_{A_z} + f_{T_z}
\end{aligned}
$$

Note that $\vec{f}_A$ and $\vec{f}_T$ are perturbation forces.

Equations (6.11) are dynamically nonlinear, which means that products and cross products of perturbed translational and rotational velocities occur in the equations of motion. Whether or not the aerodynamic and thrust terms are nonlinear in the motion variables depends on the theory used to describe them.

The perturbed momentum equations are not dynamically coupled with elastic degrees of freedom. Aerodynamic and thrust coupling with elastic degrees of freedom exists, however, because these forces in general depend on the deformed shape. The extent of this coupling depends on the mathematical models used to describe $\vec{f}_A$ and $\vec{f}_T$. In this section rigid and equivalent elastic derivative theory will be used to eliminate elastic degrees of freedom from the equations.

Equations (6.11) will now be written in stability axes. At $t = 0$ (before the perturbations are introduced) the X_S-axis is aligned with the projection of $\vec{V}_C$ on the XZ plane. This implies $W_1 = 0$ by definition. Note that V_1 may be nonzero, which implies that steady-state sideslip is admissible. In stability axes, equations (6.11) yield:

$$
\begin{aligned}
&M\dot{u} + M(Q_1 w + qw - R_1 v - rV_1 - rv) \\
&\qquad = -Mg\,\theta\cos\Theta_1 + f_{A_{x_s}} + f_{T_{x_s}} \\
&M\dot{v} + M(R_1 u + rU + ru - P_1 w - pw) \qquad (6.12) \\
&\qquad = Mg(\phi\cos\Theta_1\cos\Phi_1 - \theta\sin\Theta_1\sin\Phi_1) + f_{A_{y_s}} + f_{T_{y_s}} \\
&M\dot{w} + M(P_1 v + pV_1 + pv - Q_1 u - qU_1 - qu) \\
&\qquad = Mg(-\theta\sin\Theta_1\cos\Phi_1 - \phi\cos\Theta_1\sin\Phi_1) + f_{A_{z_s}} + f_{T_{z_s}}
\end{aligned}
$$

These equations are still dynamically nonlinear.

The following two cases are most frequently encountered in solving stability and response problems:

- Steady-state rectilinear flight:

  In steady-state rectilinear flight the following conditions hold:

$$P_1 = Q_1 = R_1 = 0$$

No sideslip	Small perturbations
$V_1 = 0$	$\Theta\phi = 0$
	$qw = 0$ (cyclic)

Equations (6.12) are reduced to equations (6.13) and are presented in table 4. They have become dynamically linear.

No sideslip	Large perturbations
$V_1 = 0$	$\Theta\phi = 0$
	$qw \neq 0$ (cyclic)

Equations (6.12) are reduced to equations (6.14) and are presented in table 4. They remain dynamically nonlinear.

Sideslip	Small perturbations
$V_1 \neq 0$	$\Theta\phi = 0$
	$qw = 0$ (cyclic)

Equations (6.12) are reduced to equations (6.15) and are presented in table 4. They have become dynamically linear.

Sideslip	Large perturbations
$V_1 \neq 0$	$\Theta\phi = 0$
	$qw \neq 0$ (cyclic)

Equations (6.12) are reduced to equations (6.16) and are presented in table 4. They remain dynamically nonlinear.

- Steady-state curvilinear flight:

  In steady-state curvilinear flight the following conditions hold:

$$P_1 = 0$$

TABLE 4.–PERTURBED MOMENTUM EQUATIONS OF MOTION FOR AN ELASTIC AIRPLANE–RECTILINEAR FLIGHT

No Sideslip in Steady State	Sideslip in Steady State
A1.1 Small Perturbations (6.13) $M\dot{u} + Mg\theta\cos\Theta_1 = f_{A_{X_s}} + f_{T_{X_s}}$ $M\dot{v} + MU_1 r + Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1) = f_{A_{Y_s}} + f_{T_{Y_s}}$ $M\dot{w} - MU_1 q + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1) = f_{A_{Z_s}} + f_{T_{Z_s}}$	A2.1 Small Perturbations (6.15) $M\dot{u} - MV_1 r + Mg\theta\cos\Theta_1 = f_{A_{X_s}} + f_{T_{X_s}}$ $M\dot{v} + MU_1 r + Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1) = f_{A_{Y_s}} + f_{T_{Y_s}}$ $M\dot{w} + M(V_1 p - U_1 q) + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1) = f_{A_{Z_s}} + f_{T_{Z_s}}$
A1.2 Large Perturbations (6.14) $M\dot{u} + M(qw - rv) + Mg\theta\cos\Theta_1 = f_{A_{X_s}} + f_{T_{X_s}}$ $M\dot{v} + M(U_1 r + ru - pw) + Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1) = f_{A_{Y_s}} + f_{T_{Y_s}}$ $M\dot{w} + M(-U_1 q + pv - qu) + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1) = f_{A_{Z_s}} + f_{T_{Z_s}}$	A2.2 Large Perturbations (6.16) $M\dot{u} + M(-V_1 r + qw - rv) + Mg\theta\cos\Theta_1 = f_{A_{X_s}} + f_{T_{X_s}}$ $M\dot{v} + M(U_1 r + ru - pw) + Mg(\theta\sin\Theta_1\sin\Phi_2 - \phi\cos\Theta_1\cos\Phi_1) = f_{A_{Y_s}} + f_{T_{Y_s}}$ $M\dot{w} + M(V_1 p - U_1 q + pv - qu) + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1) = f_{A_{Z_s}} + f_{T_{Z_s}}$

No sideslip | Small perturbations

$V_1 = 0$ | $\Theta\phi = 0$

$qw = 0$ (cyclic)

Equations (6.12) are reduced to equations (6.17) and are presented in table 5. They have become dynamically linear.

No sideslip | Large perturbations

$V_1 = 0$ | $\Theta\phi = 0$

$qw \neq 0$ (cyclic)

Equations (6.12) are reduced to equations (6.18) and are presented in table 5. They remain dynamically nonlinear.

Sideslip | Small perturbations

$V_1 \neq 0$ | $\Theta\phi = 0$

$qw = 0$ (cyclic)

Equations (6.12) are reduced to equations (6.19) and are presented in table 5. They have become dynamically linear.

Sideslip | Large perturbations

$V_1 \neq 0$ | $\Theta\phi = 0$

$qw \neq 0$ (cyclic)

Equations (6.12) are reduced to equations (6.20) and are presented in table 5. They remain dynamically nonlinear.

Combinations of these momentum equations with the moment of momentum equations to form solutions of stability problems are discussed in par. 6.3, as is the development of expressions for $\vec{f}_A$ and $\vec{f}_T$.

6.2.2 Moment of momentum equation. - The general form of the moment of momentum equation (4.9) is:

$$\frac{d}{dt}\int_V \vec{r} \times \frac{d\vec{r}}{dt}\rho_A \, dV = \int_S \vec{r} \times \vec{F} \, dS \qquad (6.21)$$

TABLE 5.–PERTURBED MOMENTUM EQUATIONS OF MOTION FOR AN ELASTIC AIRPLANE–CURVILINEAR FLIGHT

No Sideslip in Steady State	Sideslip in Steady State
B1.1 Small Perturbations (6.17) $M\dot{u} + M(Q_1 w - R_1 v) + Mg\theta\cos\Theta_1 = f_{A_{X_S}} + f_{T_{X_S}}$ $M\dot{v} + M(U_1 r + R_1 u) + Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1)$ $= f_{A_{Y_S}} + f_{T_{Y_S}}$ $M\dot{w} + M(-Q_1 u - U_1 q) + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1)$ $= f_{A_{Z_S}} + f_{T_{Z_S}}$	B2.1 Small Perturbations (6.19) $M\dot{u} + M(Q_1 w - R_1 v - V_1 r) + Mg\theta\cos\Theta_1 = f_{A_{X_S}} + f_{T_{X_S}}$ $M\dot{v} + M(R_1 u + U_1 r) + Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1)$ $= f_{A_{Y_S}} + f_{T_{Y_S}}$ $M\dot{w} + M(V_1 p - Q_1 u - U_1 q) + Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1)$ $= f_{A_{Z_S}} + f_{T_{Z_S}}$
B1.2 Large Perturbations (6.18) $M\dot{u} + M(Q_1 w - R_1 v + qw - rv) + Mg\theta\cos\Theta_1 = f_{A_{X_S}} + f_{T_{X_S}}$ $M\dot{v} + M(R_1 u + U_1 r + ru - pw)$ $+ Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1)$ $= f_{A_{Y_S}} + f_{T_{Y_S}}$ $M\dot{w} + M(-Q_1 u - U_1 q + pv - qu)$ $+ Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1)$ $= f_{A_{Z_S}} + f_{T_{Z_S}}$	B2.2 Large Perturbations (6.20) $M\dot{u} + M(Q_1 w - R_1 v - V_1 r + qw - rv) + Mg\theta\cos\Theta_1 = f_{A_{X_S}} + f_{T_{X_S}}$ $M\dot{v} + M(R_1 u + U_1 r + ru - pw)$ $+ Mg(\theta\sin\Theta_1\sin\Phi_1 - \phi\cos\Theta_1\cos\Phi_1)$ $= f_{A_{Y_S}} + f_{T_{Y_S}}$ $M\dot{w} + M(V_1 p - Q_1 u - U_1 q + pv - qu)$ $+ Mg(\theta\sin\Theta_1\cos\Phi_1 + \phi\cos\Theta_1\sin\Phi_1)$ $= f_{A_{Z_S}} + f_{T_{Z_S}}$

The perturbation substitution is to be made in this equation. In part, that substitution involves the relative position vector $\vec{r}$ such that

$$
\begin{aligned}
\vec{r} &= \vec{\tilde{r}} + \vec{d}_1 + \vec{d}_P \\
&= \vec{r}_1 + \vec{d}_P
\end{aligned}
\tag{6.22}
$$

where $\vec{d}_p$ is the perturbed elastic deformation vector and $\vec{r}_1$ is the relative position containing the steady-state elastic deformation vector.

Substituting equation (6.22) into equation (6.21) yields:

$$
\frac{d}{dt}\int_V (\vec{r}_1 + \vec{d}_P) \times \frac{d}{dt}(\vec{r}_1 + \vec{d}_P)\, \rho_A \, dV = \int_S \vec{r} \times \vec{F}\, dS
$$

In the body axis system the operator (d/dt) is replaced by $(d/dt + \vec{\omega}\times)$ so that:

$$
\frac{d}{dt}\int_V (\vec{r}_1 + \vec{d}_P) \times \left(\vec{\omega} \times \vec{r}_1 + \frac{\partial \vec{d}_P}{\partial t} + \vec{\omega} \times \vec{d}_P\right) \rho_A \, dV = \int_S \vec{r} \times \vec{F}\, dS
$$

or after expanding:

$$
\begin{aligned}
\frac{d}{dt}\int_V \Big\{ &\vec{r}_1 \times (\vec{\omega} \times \vec{r}_1) + \vec{d}_P \times (\vec{\omega} \times \vec{r}_1) + \vec{r}_1 \times \frac{\partial \vec{d}_P}{\partial t} + \overbrace{\cancel{d_P \times \frac{\partial \vec{d}_P}{\partial t}}}^{\approx 0} \\
&+ \vec{r}_1 \times (\vec{\omega} \times \vec{d}_P) + \overbrace{\cancel{d_P \times (\vec{\omega} \times \vec{d}_P)}}^{\approx 0} \Big\} \rho_A \, dV = \int_S \vec{r} \times \vec{F}\, dS
\end{aligned}
\tag{6.23}
$$

An assumption that was made in expanding the left-hand side of equation (6.21) is that the effect of spinning rotors on perturbed airplane motions is negligible.

Because of assumption S1, whereby all structural perturbations are defined as small, the fourth and sixth terms in equation (6.23) are negligible. Using normal modes to describe the structural perturbations, it can be shown that the second, third, and fifth terms in equation (6.23) are not coupled to dynamic elastic motion by virtue of the normal mode properties to be introduced by equations (6.45) and (6.46); these terms, then, will be ignored. The first term is the dot product of the inertia tensor and the rotational velocity. Equation (6.23) therefore reduces to:

$$\frac{d}{dt}\left(\bar{\bar{\Psi}}_1 \cdot \vec{\omega}\right) = \int_S \vec{r} \times \vec{\bar{F}}\, dS = M_A + M_T \qquad (6.24)$$

where $\bar{\bar{\psi}}_1$ is the inertia tensor relative to the steady-state shape and where the right-hand side has been replaced by a thrust and aerodynamic moment.

In common matrix notation, the inertia tensor, which is symmetrical, is written:

$$\bar{\bar{\Psi}}_1 = \begin{bmatrix} I_{xx_1} & -I_{xy_1} & -I_{xz_1} \\ -I_{xy_1} & I_{yy_1} & -I_{yz_1} \\ -I_{xz_1} & -I_{yz_1} & I_{zz_1} \end{bmatrix} \qquad (6.25)$$

where $\bar{\bar{\psi}}_1$ is defined in steady-state notation by:

$$\bar{\bar{\Psi}}_1 \cdot \vec{\omega} = \int_V \vec{r}_1 \times (\vec{\omega} \times \vec{r}_1)\, \rho_A\, dV \qquad (6.26)$$

If the commonly used symmetric airplane assumption is used, equation (6.25) reduces to:

$$\bar{\bar{\Psi}}_1 = \begin{bmatrix} I_{xx_1} & 0 & -I_{xz_1} \\ 0 & I_{yy_1} & 0 \\ -I_{xz_1} & 0 & I_{zz_1} \end{bmatrix} \qquad (6.27)$$

The important conclusion to be drawn from equation (6.29) is that the moment of momentum equation, like the momentum equation, is dynamically uncoupled from elastic degrees of freedom. Notice, however, that the left-hand side of the momentum equation (6.5) uncouples naturally from the elastic degrees of freedom. The left-hand side of the moment of momentum equation (6.23) uncouples from the elastic degrees of freedom only because of assumption S2. The elastic degrees of freedom are coupled with the rigid degrees of freedom in the aerodynamic and thrust moment term on the right-hand side of equation (6.24).

Expansion of equation (6.24) in cartesian form yields:

$$
\begin{aligned}
&I_{xx_{1_S}}\dot{P} - I_{xz_{1_S}}\dot{R} - I_{xz_{1_S}}PQ + \left(I_{zz_{1_S}} - I_{yy_{1_S}}\right)QR = M_{A_{x_S}} + M_{T_{x_S}} \\
&I_{yy_{1_S}}\dot{Q} + I_{xz_{1_S}}\left(P^2 - R^2\right) + \left(I_{xx_{1_S}} - I_{zz_{1_S}}\right)PR = M_{A_{y_S}} + M_{T_{y_S}} \\
&I_{zz_{1_S}}\dot{R} - I_{xz_{1_S}}\dot{P} + I_{xz_{1_S}}QR + \left(I_{yy_{1_S}} - I_{xx_{1_S}}\right)PQ = M_{A_{z_S}} + M_{T_{z_S}}
\end{aligned}
\qquad (6.28)
$$

Equations (6.28) are valid for arbitrary perturbations of the motion variables except, of course, the structural perturbations.

Using the typical perturbation substitution, equation (6.8) results in:

$$
\begin{aligned}
&I_{xx_{1_S}}\dot{P} - I_{xz_{1_S}}\dot{r} - I_{xz_{1_S}}(P_1+p)(Q_1+q) + \left(I_{zz_{1_S}} - I_{yy_{1_S}}\right)(Q_1+q)(R_1+r) \\
&\qquad = M_{A_{x_{1_S}}} + m_{A_{x_S}} + M_{T_{x_{1_S}}} + m_{T_{x_S}} \\
&I_{yy_{1_S}}\dot{q} + I_{xz_{1_S}}\left[(P_1+p)^2 - (R_1+r)^2\right] + \left(I_{xx_{1_S}} - I_{zz_{1_S}}\right)(P_1+p)(R_1+r) \\
&\qquad = M_{A_{y_{1_S}}} + m_{A_{y_{1_S}}} + M_{T_{y_{1_S}}} + m_{T_{y_S}} \\
&I_{zz_{1_S}}\dot{r} - I_{xz_{1_S}}\dot{p} + I_{xz_{1_S}}(Q_1+q)(R_1+r) + \left(I_{yy_{1_S}} - I_{xx_{1_S}}\right)(P_1+p)(Q_1+q) \\
&\qquad = M_{A_{z_{1_S}}} + m_{A_{z_S}} + M_{T_{z_{1_S}}} + m_{T_{z_S}}
\end{aligned}
\qquad (6.29)
$$

After eliminating the steady-state form of these equations, the moment of momentum equations in perturbation form are:

$$
\begin{aligned}
&I_{xx_{1_S}}\dot{p} - I_{xz_{1_S}}\dot{r} - I_{xz_{1_S}}(P_1 q + Q_1 p + pq) + \left(I_{zz_{1_S}} - I_{yy_{1_S}}\right)(Q_1 r + R_1 q + qr) \\
&\qquad = m_{A_{x_S}} + m_{T_{x_S}} \\
&I_{yy_{1_S}}\dot{q} + I_{xz_{S_1}}\left(2P_1 p + p^2 - 2R_1 r - r^2\right) + \left(I_{xx_{1_S}} - I_{zz_{1_S}}\right)(P_1 r + R_1 p + pr) \\
&\qquad = m_{A_{y_S}} + m_{T_{y_S}} \\
&I_{zz_{1_S}}\dot{r} - I_{xz_{1_S}}\dot{p} + I_{xz_{1_S}}(Q_1 r + R_1 q + qr) + \left(I_{yy_{1_S}} - I_{xx_{1_S}}\right)(P_1 q + Q_1 p + pq) \\
&\qquad = m_{A_{z_S}} + m_{T_{z_S}}
\end{aligned}
\qquad (6.30)
$$

where $\vec{m}_A$ and $\vec{m}_T$ are perturbation moments. These equations are dynamically nonlinear. Whether or not the aerodynamic and thrust terms are nonlinear depends on the theory used to describe them.

Equations (6.30) are normally expressed in stability axes. Doing so does not change the form of equations (6.30). It does imply, however, that all inertial constants become a function of the flight condition (in particular the angle of attack) selected for the steady reference state, even for the rigid airplane. This is a well-known fact in rigid airplane stability analysis.

The following two cases are most frequently encountered in solving stability and response problems.

- Steady-state rectilinear flight:
  In steady-state rectilinear flight the following conditions hold:

$$P_1 = Q_1 = R_1 = 0$$

<u>Small perturbations</u>

$$pr = 0 \text{ (cyclic)}$$

Equations (6.30) are reduced to equations (6.31) and are presented in table 6. They have become dynamically linear.

<u>Large perturbations</u>

$$pr \neq 0 \text{ (cyclic)}$$

Equations (6.30) are reduced to equations (6.32) and are presented in table 6. They remain dynamically nonlinear.

- Steady-state curvilinear flight:
  In steady-state flight the following condition holds:

$$P_1 \approx 0$$

<u>Small perturbations</u>

$$pr = 0 \text{ (cyclic)}$$

Equations (6.30) are reduced to equations (6.33) and are presented in table 6. They have become dynamically linear.

TABLE 6.–PERTURBED MOMENT OF MOMENTUM EQUATIONS OF MOTION FOR AN ELASTIC AIRPLANE

RECTILINEAR FLIGHT	CURVILINEAR FLIGHT
A1. Small Perturbations	B1. Small perturbations
$I_{xx_{1s}}\dot{p} - I_{xz_{1s}}\dot{r} = m_{A_{x_s}} + m_{T_{x_s}}$ $I_{yy_{1s}}\dot{q} = m_{A_{y_s}} + m_{T_{y_s}}$ (6.31) $I_{zz_{1s}}\dot{r} - I_{xz_{1s}}\dot{p} = m_{A_{z_s}} + m_{T_{z_s}}$	$I_{xx_{1s}}\dot{p} - I_{xz_{1s}}\dot{r} - I_{xz_{1s}}Q_1 p + (I_{zz_{1s}} - I_{yy_{1s}})(Q_1 r + R_1 q) = m_{A_{x_s}} + m_{T_{x_s}}$ (6.33) $I_{yy_{1s}}\dot{q} - I_{xz_{1s}}2R_1 r + (I_{xx_{1s}} - I_{zz_{1s}})R_1 p = m_{A_{y_s}} + m_{T_{y_s}}$ $I_{zz_{1s}}\dot{r} - I_{xz_{1s}}\dot{p} + I_{xz_{1s}}(Q_1 r + R_1 q) + (I_{yy_{1s}} - I_{xx_{1s}})Q_1 p = m_{A_{z_s}} + m_{T_{z_s}}$
A2. Large Perturbations	B2. Large Perturbations
$I_{xx_{1s}}\dot{p} - I_{xz_{1s}}\dot{r} - I_{xz_{1s}}pq + (I_{zz_{1s}} - I_{yy_{1s}})qr = m_{A_{x_s}} + m_{T_{x_s}}$ (6.32) $I_{yy_{1s}}\dot{q} + I_{xz_{1s}}(p^2 - r^2) + (I_{xx_{1s}} - I_{zz_{1s}})pr = m_{A_{y_s}} + m_{T_{y_s}}$ $I_{zz_{1s}}\dot{r} - I_{xz_{1s}}\dot{p} + I_{xz_{1s}}qr + (I_{yy_{1s}} - I_{xx_{1s}})pq = m_{A_{z_s}} + m_{T_{z_s}}$	$I_{xx_{1s}}\dot{p} - I_{xz_{1s}}\dot{r} - I_{xz_{1s}}(Q_1 p + pq) + (I_{zz_{1s}} - I_{yy_{1s}})(Q_1 r + R_1 q + qr) = m_{A_{x_s}} + m_{T_{x_s}}$ (6.34) $I_{yy_{1s}}\dot{q} + I_{xz_{1s}}(-2R_1 r + p^2 - r^2) + (I_{xx_{1s}} - I_{zz_{1s}})(R_1 p + pr) = m_{A_{y_s}} + m_{T_{y_s}}$ $I_{zz_{1s}}\dot{r} - I_{xz_{1s}}\dot{p} + I_{xz_{1s}}(Q_1 r + R_1 q + qr) + (I_{yy_{1s}} - I_{xx_{1s}})(Q_1 p + pq) = m_{A_{z_s}} + m_{T_{z_s}}$

Large perturbations

pr $\neq$ 0 (cyclic)

Equations (6.30) are reduced to equation (6.34) and are presented in table 6. They remain dynamically nonlinear.

Notice that the distinction between sideslip or no sideslip in the steady-state condition has no effect on the form of the equations in table 6. Combinations of the moment of momentum equations with the momentum equations are discussed in par. 6.3, as is the development of expressions for m_A and m_T.

6.2.3 Internal equilibrium equation. - The general form of the internal equilibrium equation (4.49) is

$$\vec{d} - \vec{d}_o - \vec{\theta}_o \times \vec{\vec{r}} = \int_V \vec{\vec{\Gamma}}_o \cdot \left(\vec{R} - \rho_A \vec{a}\right) dV + \int_V \vec{\vec{\Gamma}}_o \cdot \vec{F}\, \delta\left(\vec{r} - \vec{r}_s\right) dV \qquad (6.35)$$

The perturbation substitution is to be made into this equation. But first it must be noted that the body force is solely due to gravity, i.e.,

$$\vec{R} = \rho_A \vec{g} = \rho_A g\left(-\sin\theta\, \vec{i} + \cos\theta \sin\phi\, \vec{j} + \cos\theta\cos\phi\, \vec{k}\right) \qquad (6.36)$$

and that the acceleration is:

$$\vec{a} = \frac{d^2 \vec{r}\,'}{d t^2} \qquad (6.37)$$

where:

$$\vec{r}\,' = \vec{r}_o\,' + \vec{\vec{r}} + \vec{d} \qquad (6.38)$$

so that:

$$\vec{a} = \frac{d^2 \vec{r}\,'}{d t^2} + \frac{d^2}{d t^2}\left(\vec{\vec{r}} + \vec{d}\right) \qquad (6.39)$$

with $\vec{\bar{r}}$ a constant and $d\vec{r}_0'/dt = \vec{V}_c$. Expanding the acceleration gives:

$$\vec{a} = \frac{\partial \vec{V}_c}{\partial t} + \vec{\omega} \times \vec{V}_c + \frac{\partial^2 \vec{d}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{d}}{\partial t} + \frac{\partial \vec{\omega}}{\partial t} \times (\vec{\bar{r}} + \vec{d}) + \vec{\omega} \times [\vec{\omega} \times (\vec{\bar{r}} + \vec{d})] \tag{6.40}$$

In the perturbation substitution:

$$\begin{aligned} \vec{d} &= \vec{d}_1 + \vec{d}_P, \quad \frac{\partial \vec{d}_1}{\partial t} = 0 \\ \vec{\omega} &= \vec{\omega}_1 + \vec{\omega}_P, \quad \frac{\partial \vec{\omega}_1}{\partial t} = 0 \\ \vec{V}_c &= \vec{V}_{c_1} + \vec{V}_{cP}, \quad \frac{\partial \vec{V}_{c_1}}{\partial t} = 0 \end{aligned} \tag{6.41}$$

Thus the acceleration vector is written:

$$\vec{a} = \frac{\partial \vec{V}_{cP}}{\partial t} + (\vec{\omega}_1 + \vec{\omega}_P) \times (\vec{V}_{c_1} + \vec{V}_{cP}) + \frac{\partial^2 \vec{d}}{\partial t^2} + 2(\vec{\omega}_1 + \omega_P) \times \frac{\partial \vec{d}_P}{\partial t} + \frac{\partial \vec{\omega}_P}{\partial t} \times (\vec{r}_1 + \vec{d}_P) + (\vec{\omega}_1 + \vec{\omega}_P) \times [(\vec{\omega}_1 + \vec{\omega}_P) \times (\vec{r}_1 + \vec{d}_P)] \tag{6.42}$$

On separating into steady-reference and perturbation quantities, it follows that:

$$\vec{a}_1 = \vec{\omega}_1 \times \vec{V}_{c_1} + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{r}_1) \tag{6.43}$$

and:

$$\vec{a}_P = \frac{\partial \vec{V}_{cP}}{\partial t} + \vec{\omega}_P \times (\vec{V}_{c_1} + \vec{V}_{cP}) + \vec{\omega}_1 \times \vec{V}_{cP} + \frac{\partial^2 \vec{d}_P}{\partial t^2} + 2(\vec{\omega}_1 + \vec{\omega}_P) \times \frac{\partial \vec{d}_P}{\partial t} + \frac{\partial \vec{\omega}_P}{\partial t} \times (\vec{r}_1 + \vec{d}_P) + \vec{\omega}_P \times [(\vec{\omega}_1 + \vec{\omega}_P) \times (\vec{r}_1 + \vec{d}_P)] + \vec{\omega}_1 \times [\vec{\omega}_P \times (\vec{r}_1 + \vec{d}_P)] + \vec{\omega}_1 \times (\vec{\omega}_1 \times \vec{d}_P) \tag{6.44}$$

Hence for arbitrary perturbations the internal equilibrium equations become:

$$
\begin{aligned}
&\left(\vec{d}_1 - \vec{d}_{o_1} - \vec{\theta}_{o_1} \times \vec{\vec{r}}\right) + \left(\vec{d}_p - \vec{d}_{op} - \vec{\theta}_{op} \times \vec{\vec{r}}\right) \\
&\quad = \int_V \bar{\bar{\Gamma}}_o \cdot \left(\vec{g} - \vec{a}_1 - \vec{a}_p\right) \rho_A \, dV + \int_V \bar{\bar{\Gamma}}_o \cdot \left(\vec{F}_1 + \vec{F}\right) \delta\left(\vec{r} - \vec{r}_s\right) dV
\end{aligned}
\tag{6.45}
$$

In the steady-reference flight condition the internal equilibrium is given by:

$$
\begin{aligned}
&\vec{d}_1 - \vec{d}_{o_1} - \vec{\theta}_{o_1} \times \vec{\vec{r}} \\
&\quad = \int_V \bar{\bar{\Gamma}}_o \cdot \left(\vec{g}_1 - \vec{a}_1\right) \rho_A \, dV + \int_V \bar{\bar{\Gamma}} \cdot \vec{F}_1 \, \delta\left(\vec{r} - \vec{r}_s\right) dV
\end{aligned}
\tag{6.46}
$$

For arbitrary perturbations, the equations for the steady-reference flight condition cannot be used to simplify the perturbed equations. This complication arises from the gravitational term $\vec{g}$ as it did in obtaining the momentum equations (6.12). The substitution of perturbed Euler angles into equation (6.36) does not admit this type of separation because of the trigonometric terms. The separation becomes possible when the large perturbation approximation is introduced.

Free vibrations of the airplane are governed by t[illegible]al equilibrium equations with all applied forces set equal to zero and the airplane moving at a constant velocity without rotation. Thus,

$$
\begin{aligned}
\vec{F} &= 0 \\
\vec{R} &= 0 \\
\vec{V}_c = \vec{\omega} &= 0
\end{aligned}
\tag{6.47}
$$

Under these conditions the internal equilibrium equations become

$$
\vec{d} - \vec{d}_o - \vec{\theta}_o \times \vec{\vec{r}} = - \int_V \bar{\bar{\Gamma}}_o \cdot \frac{\partial^2 \vec{d}}{\partial t^2} \rho_A \, dV \tag{6.48}
$$

In addition, the momentum and moment of momentum equations become

$$
\frac{d}{dt} \int_V \rho_A \frac{d\vec{r}'}{dt} \, dV = 0 \tag{6.49}
$$

and

$$\frac{d}{dt}\int_V \vec{r} \times \frac{d\vec{r}}{dt}\rho_A \, dV = 0 \tag{6.50}$$

Because $\vec{r}' = \vec{r}_o + \vec{\bar{r}} + \vec{d}$ and $d^2\vec{r}'_o/dt^2 = 0$, the momentum equation becomes

$$\int_V \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV = 0 \tag{6.51}$$

Further, because $\vec{\omega} = 0$, the moment of momentum equation becomes

$$\int_V (\vec{\bar{r}} + \vec{d}) \times \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV = 0$$

Neglecting products of the elastic displacement vector $\vec{d}$, this reduces to:

$$\int_V \vec{\bar{r}} \times \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV = 0 \tag{6.52}$$

Thus the equations of motion for the freely vibrating airplane are given by:

$$\begin{aligned} \vec{d} - \vec{d}_o - \vec{\theta}_o \times \vec{\bar{r}} &= -\int_V \bar{\Gamma}_o \cdot \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV \\ \int_V \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV &= 0 \\ \int_V \vec{\bar{r}} \times \frac{\partial^2 \vec{d}}{\partial t^2}\rho_A \, dV &= 0 \end{aligned} \tag{6.53}$$

A solution to equations (6.53) may be obtained by writing the time-varying vector field $\vec{d}(X, Y, Z, t)$ as a product function, i.e.,

$$\vec{d} = \vec{\phi}(x, y, z)\, T(t) \tag{6.54}$$

Substituting into the equations of motion yields:

$$\begin{aligned} T\left[\vec{\phi} - \vec{\phi}(0) - \tfrac{1}{2}(\vec{\nabla} \times \vec{\phi})_{r=0} \times \vec{\bar{r}}\right] &= -\ddot{T}\int_V \bar{\Gamma}_o \cdot \vec{\phi}\,\rho_A \, dV \\ \int_V \vec{\phi}\,\rho_A \, dV = 0 \qquad & \int_V \vec{\bar{r}} \times \vec{\phi}\,\rho_A \, dV = 0 \end{aligned} \tag{6.55}$$

A separation of variables may be carried out with the introduction of a separation constant ω^2. It follows that:

$$\ddot{T} + \omega^2 T = 0 \tag{6.56}$$

and

$$\vec{\phi} - \vec{\phi}(0) - \frac{1}{2}(\vec{\nabla} \times \vec{\phi})_{r=0} \times \vec{\vec{r}} = \omega^2 \int_V \vec{\vec{r}}_o \cdot \vec{\phi}\, \rho_A \, dV \tag{6.57}$$

It is seen from the form of equation (6.56) that the dependence on time is simple harmonic with frequency ω. The vector $\vec{\phi}$ is termed the free-vibration mode shape. Equation (6.57) is an eigenvalue problem having an infinity of solutions consisting of eigenvectors $\vec{\phi}_i$, which correspond to eigenvalues $\omega^2{}_i$.

The equations of motion for this case are linear, so it is possible to form an infinite sum of solutions as:

$$\vec{d}(x, y, z, t) = \sum_{i=1}^{\infty} \vec{\phi}_i (x, y, z)\, u_i(t) \tag{6.58}$$

This result is also a solution if the infinite set of eigenvectors (mode shapes) $\vec{\phi}_i$ can be generated.

The equations of motion may now be written in terms of the free vibration modes as:

$$\int_V \rho_A \vec{\phi}_i \, dV = 0 \qquad \int_V \rho_A \vec{\vec{r}} \times \vec{\phi}_i \, dV = 0 \tag{6.59a}$$

$$\ddot{u}_i + \omega_i^2 u_i = 0 \tag{6.59b}$$

$$\vec{\phi}_i - \vec{\phi}_i(0) - \frac{1}{2}(\vec{\nabla} \times \vec{\phi}_i)_{\vec{r}=0} \times \vec{\vec{r}} = \omega_i^2 \int_V \rho_A \vec{\vec{r}}_o \cdot \vec{\phi}_i \, dV \tag{6.59c}$$

The constant vectors $\vec{\phi}_i(0)$ and $1/2(\vec{\nabla} \times \vec{\phi}_i)_o \vec{\vec{X}}r$ may be evaluated and eliminated from the formulation. Multiplying the internal equilibrium equations by ρ_A and integrating over the volume, it follows that since

$$\int_V \rho_A \vec{\phi}_i \, dV = 0 \quad , \qquad \int_V \rho_A \vec{\vec{r}} \times \vec{\phi}_i \, dV = 0 \tag{6.60}$$

then:

$$\vec{\phi}_i(0) = -\frac{\omega_i^2}{M}\int_V \rho_A \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\phi}_i \, dV\, dV \tag{6.61}$$

Forming the cross product of the internal equilibrium equation with $\rho_A \vec{\tilde{r}}$ and integrating over the volume results in:

$$\frac{1}{2}\int_V \rho_A \vec{\tilde{r}} \times [(\vec{\nabla} \times \vec{\phi}_i)_{\vec{r}=0} \times \vec{\tilde{r}}]\, dV = \omega_i^2 \int_V \rho_A \vec{\tilde{r}} \times \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\phi}_i \, \rho_A \, dV dV \tag{6.62}$$

But the left-hand member of this expression may be written in terms of the inertia tensor such that:

$$-\frac{1}{2}\bar{\bar{\Psi}} \cdot (\vec{\nabla} \times \vec{\phi}_i)_{\vec{r}=0} = \omega_i^2 \int_V \rho_A \vec{\tilde{r}} \times \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\phi}_i \, \rho_A \, dV dV \tag{6.63}$$

where:

$$\bar{\bar{\Psi}} \sim \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

so that:

$$-\frac{1}{2}(\vec{\nabla} \times \vec{\phi}_i)_{\vec{r}=0} = \omega_i^2 \, \bar{\bar{\Psi}}^{-1} \cdot \int_V \rho_A \vec{\tilde{r}} \times \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\phi}_i \, \rho_A \, dV\, dV \tag{6.64}$$

Milne (ref. 43) noted that equation (6.55) leads to elastic deflections $\vec{dp}_i$, which are measured relative to the mean coordinate axis system. The mean axis may rotate relative to the directions of the axis of principal moments of inertia of the airplane. If the second condition of equation (6.60) is replaced by the conditions

$$\begin{aligned} &\int_V (\tilde{y}\,\phi_z + \tilde{z}\,\phi_y)\rho_A \, dV = 0 \\ &\int_V (\tilde{z}\,\phi_x + \tilde{x}\,\phi_z)\rho_A \, dV = 0 \\ &\int_V (\tilde{x}\,\phi_y + \tilde{y}\,\phi_x)\rho_A \, dV = 0 \end{aligned} \tag{6.65}$$

then the elastic deflections are measured relative to a principal axis system. For most airplane configurations in which the dominant elastic deflections are transverse to a radius vector from the c.g. the mean and principal axis systems are identical within the order of approximation of this analysis.

By defining a new structural influence function as

$$\bar{\bar{G}}(x,y,z;\xi,\eta,\zeta) = \bar{\bar{\Gamma}}_o(x,y,z;\xi,\eta,\zeta) - \frac{1}{M}\int_V \bar{\bar{\Gamma}}_o(r,s,t;\xi,\eta,\zeta)\rho_A(r,s,t)\,dr\,ds\,dt + \vec{r}(x,y,z)\times\left[\bar{\bar{\psi}}^{-1}\cdot\int_V \vec{r}(r,s,t)\times\bar{\bar{\Gamma}}_o(r,s,t;\xi,\eta,\zeta)\rho_A(r,s,t)\,dr\,ds\,dt\right] \tag{6.66}$$

the internal equilibrium equation for free vibration becomes:

$$\vec{\phi}_i(x,y,z) = \omega_i^2\int_V \bar{\bar{G}}(x,y,z;\xi,\eta,\zeta)\cdot\vec{\phi}_i(\xi,\eta,\zeta)\rho_A(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta \tag{6.67}$$

where the symmetry of $\bar{\Gamma}_o$ in (X, Y, Z) and (ξ, η, ζ) has been used to write:

$$\int_V \rho_A \int_V \bar{\bar{\Gamma}}_o\cdot\vec{\phi}_i\,\rho_A\,dV\,dV = \int_V \rho_A\,\vec{\phi}_i\cdot\int_V \bar{\bar{\Gamma}}_o\,\rho_A\,dV\,dV \tag{6.68}$$

and

$$\int_V \rho_A\,\vec{r}\times\int_V \bar{\bar{\Gamma}}_o\cdot\vec{\phi}_i\,\rho_A\,dV\,dV = -\int_V \rho_A\,\vec{\phi}_i\cdot\int_V(\vec{r}\times\bar{\bar{\Gamma}}_o)\,\rho_A\,dV\,dV \tag{6.69}$$

The above result represents an infinite set of integral equations of the homogeneous, Fredholm type in which the components of the tensor $\bar{G}$ are the kernel functions. The equations are satisfied if $\vec{\phi}_i$ is a zero vector, but this is a trivial solution. An infinite number of nontrivial solutions are found by solving the eigenvalue problem represented by equation (6.67). These non-trivial solutions are termed characteristic solutions $\vec{\phi}_i$ (eigenvectors) and correspond with values of ω_i termed characteristic values (eigenvalues). These eigenvectors have properties of orthogonality, which are found in the following manner. Forming the scalar product of $\rho_A \vec{\phi}_j$ with the internal equilibrium equation as given by equation (6.59) and integrating over the volume, the symmetry of $\overline{\Gamma}_0$ may be used to write:

$$\int_V \rho_A \vec{\phi}_j \cdot \vec{\phi}_i \, dV = \omega_i^2 \int_V \rho_A \vec{\phi}_j \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_i \rho_A \, dV dV$$
$$= \omega_i^2 \int_V \rho_A \vec{\phi}_i \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV dV \quad (6.70)$$

But this is precisely the result that would be achieved by starting with equation (6.59) written in terms of the j^{th} eigenvector as:

$$\vec{\phi}_j - \vec{\phi}_j(0) - \frac{1}{2}(\vec{\nabla} \times \vec{\phi}_j)_{\vec{r}=0} \times \vec{r} = \omega_j^2 \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV \quad (6.71)$$

Integrating the scalar product with $\rho_A \vec{\phi}_i$ then results in

$$\int_V \rho_A \vec{\phi}_i \cdot \vec{\phi}_j \, dV \equiv \int_V \rho_A \vec{\phi}_j \cdot \vec{\phi}_i \, dV = \omega_j^2 \int_V \rho_A \vec{\phi}_i \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV dV$$

(6.72)

Combining the above two expressions leads to:

$$0 = (\omega_i^2 - \omega_j^2) \int_V \rho_A \vec{\phi}_i \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV dV \quad (6.73)$$

If $\omega_i \neq \omega_j$, it follows that:

$$\int_V \rho_A \vec{\phi}_i \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV dV = 0, \qquad i \neq j$$

and conversely, (6.74a)

$$\int_V \rho_A \vec{\phi}_i \cdot \int_V \overline{\Gamma}_0 \cdot \vec{\phi}_j \rho_A \, dV dV = \Gamma_i, \qquad i = j$$

where Γ_i is the generalized flexibility associated with the i^{th} mode. Similar reasoning leads to the orthogonality property

$$\int_V \rho_A \vec{\phi}_i \cdot \vec{\phi}_j \, dV = 0, \qquad i \neq j \quad (6.75a)$$
$$= \overline{m}_i, \qquad i = j$$

where $\bar{m}_i$ is the generalized mass associated with the i^{th} mode.

The vectors $\vec{\phi}_i$ are the free-vibration mode shapes. Dividing the mode shapes by $\sqrt{\bar{m}_i}$, one may define normal mode shapes $\vec{\Phi}_i$ so that:

$$\vec{\Phi}_i = \frac{1}{\sqrt{\bar{m}_i}} \phi_i \tag{6.76}$$

The orthogonality expressions, equations (6.74a) and (6.75a), may be written in terms of the normalized modes:

$$\begin{aligned} \int_V \rho_A \vec{\Phi}_i \cdot \int_V \bar{\Gamma}_o \cdot \vec{\Phi}_j \rho_A \, dV \, dV &= \Gamma_i^2 \bar{m}_i , \quad & i = j \\ &= 0 , \quad & i \neq j \end{aligned} \tag{6.74b}$$

and

$$\begin{aligned} \int_V \rho_A \vec{\Phi}_i \cdot \vec{\Phi}_j \, dV &= 1 , \quad & i = j \\ &= 0 , \quad & i \neq j \end{aligned} \tag{6.75b}$$

Returning to the internal equilibrium equation with the arbitrary perturbation substitution, that expression may be written in terms of the new structure influence functions $\bar{G}$ as

$$\vec{d}_1 + \vec{d}_P = \int_V \bar{G} \cdot (\vec{g} - \vec{a}_1 - \vec{a}_P) \rho_A \, dV + \int_V \bar{G} \cdot (\vec{F} + \vec{F}) \delta (\vec{r} - \vec{r}_S) \, dV \tag{6.77}$$

Introducing the large perturbation approximation, it follows that

$$\vec{g} \approx \vec{g}_1 + \vec{g}_P \tag{6.78}$$

so that the steady-reference motion equation (6.77) becomes

$$\vec{d}_1 = \int_V \bar{G} \cdot (\vec{g}_1 - \vec{a}_1) \rho_A \, dV + \int_V \bar{G} \cdot \vec{F}_1 \delta (\vec{r} - \vec{r}_S) \, dV \tag{6.79}$$

The internal equilibrium equations in the perturbation form may now be written:

$$\vec{d}_P = \int_V \bar{\bar{G}} \cdot (\vec{g}_P - \vec{a}_P)\rho_A \, dV + \int_V \bar{\bar{G}} \cdot \vec{F} \, \delta(\vec{r} - \vec{r}_S) \, dV \tag{6.80}$$

or, in terms of $\bar{\bar{\Gamma}}_0$:

$$\vec{d}_P - \vec{d}_{P_0} - \theta_{P_0} \times \vec{r} = \int_V \bar{\bar{\Gamma}}_0 \cdot (\vec{g}_P - a_P)\rho_A \, dV + \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{F} \, \delta(\vec{r} - \vec{r}_S) \, dV \tag{6.81}$$

The perturbation elastic displacements may now be written in terms of the normal modes as:

$$\vec{d}_P = \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \tag{6.82}$$

Recalling the form of the perturbation acceleration from equation (6.44), that vector field may be written in terms of the normal modes as

$$\begin{aligned} \vec{a}_P = {} & \frac{\partial \vec{V}_{c_P}}{\partial t} + \vec{\omega}_P \times (\vec{V}_{c_1} + \vec{V}_{c_P}) + \vec{\omega}_1 \times \vec{V}_{c_P} + \frac{\partial \vec{\omega}_P}{\partial t} \times \vec{r}_1 \\ & + \vec{\omega}_P \times [(\vec{\omega}_1 + \vec{\omega}_P) \times \vec{r}_1] + \vec{\omega}_1 \times (\vec{\omega}_P \times \vec{r}_1) + \sum_{i=1}^{\infty} \vec{\Phi}_i \ddot{U}_i \\ & + 2(\vec{\omega}_1 + \vec{\omega}_P) \times \sum_{i=1}^{\infty} \vec{\Phi}_i \dot{U}_i + \frac{\partial \vec{\omega}_P}{\partial t} \times \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \\ & + (\vec{\omega}_1 + \vec{\omega}_P) \times \left[(\vec{\omega}_1 + \vec{\omega}_P) \times \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \right] \end{aligned} \tag{6.83}$$

The first six terms of equation (6.83) do not contain the mode shapes and are essentially rigid-body perturbation accelerations. They will be denoted by:

$$\begin{aligned} \tilde{\vec{a}}_P = {} & \frac{\partial \vec{V}_{c_P}}{\partial t} + \vec{\omega}_P \times (\vec{V}_{c_1} + \vec{V}_{c_P}) + \vec{\omega}_1 \times \vec{V}_{c_P} + \frac{\partial \vec{\omega}_P}{\partial t} \times \vec{r}_1 \\ & + \vec{\omega}_P \times [(\vec{\omega}_1 + \vec{\omega}_P) \times \vec{r}_1] + \vec{\omega} \times (\vec{\omega}_P \times \vec{r}_1) \end{aligned} \tag{6.84}$$

The final three terms of equation (6.83) enter scalar products with $\vec{r}_0$ when equation (6.83) is substituted into (6.81). These may be written:

$$\begin{aligned}
&\vec{r}_0 \cdot \Big\{ 2(\vec{\omega}_1 + \vec{\omega}_P) \times \sum_{i=1}^{\infty} \vec{\Phi}_i \dot{U}_i + \frac{\partial \vec{\omega}_P}{\partial t} \times \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \\
&\qquad + (\vec{\omega}_1 + \vec{\omega}_P) \times \Big[(\vec{\omega}_1 + \vec{\omega}_P) \times \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \Big] \Big\} \\
&= -2 \left[(\vec{\omega}_1 + \vec{\omega}_P) \times \vec{r}_0 \right] \cdot \sum_{i=1}^{\infty} \vec{\Phi}_i \dot{U}_i - \left[\frac{\partial \vec{\omega}_P}{\partial t} \times \vec{r}_0 \right] \cdot \sum_{i=1}^{\infty} \vec{\Phi}_i U_i \\
&\qquad - \left[(\vec{\omega}_1 + \vec{\omega}_P) \times \vec{r}_0 \right] \cdot \Big[\sum_{i=1}^{\infty} \vec{\Phi}_i U_i \times (\vec{\omega}_1 + \vec{\omega}_P) \Big]
\end{aligned} \tag{6.85}$$

Equation (6.85) introduces the coupling between elastic deformation and airplane rotation. If the properties of the normal modes developed above are used, all these terms vanish. This cannot be taken to mean that this coupling does not exist. It means that it is ignored when the normal modes are generated from the eigenvalue problem represented by equation (6.59) and are introduced into the analysis. This tacit assumption is always included in elastic airplane analyses in the literature, but is not always described. The assumption of small rotation rates and rotational accelerations, which is taken to be valid in these reports, justifies the deletion of these terms for the present analysis. The rotational accelerations and rotation rates of large airplanes are indeed small. Hence the products of these quantities with the perturbation elastic deflections and perturbation elastic deflection rates are ignored. The terms in equation (6.84) represented by equation (6.85) are dropped from the analysis, and the vector $\vec{a}_p$ is reduced to:

$$\vec{a}_P = \sum_{i=1}^{\infty} \vec{\Phi}_i \ddot{U}_i + \vec{\bar{a}}_P \tag{6.86}$$

With these approximations the perturbation form of internal equilibrium may be written:

$$\sum_{i=1}^{\infty} \vec{\Phi}_i U_i - \vec{d}_{P_0} - \vec{\theta}_{P_0} \times \vec{r} = -\int_V \bar{\bar{\Gamma}} \cdot \left(\sum_{i=1}^{\infty} \vec{\Phi}_i\right) \rho_A \, dV \, \ddot{U}_i + \int_V \bar{\bar{\Gamma}}_0 \cdot \left[(\vec{g}_P - \vec{a}_P)\rho_A + \vec{F}\,\delta(\vec{r} - \vec{r}_s)\right] dV \tag{6.87}$$

This result is valid for large and small perturbations of large elastic airplanes having small rotational rates and rotational accelerations.

Equation (6.87) may be formed in a scalar product with $\rho_A \vec{\phi}_j$ and integrated over the volume V. Using the orthogonality represented by equations (6.74b) and (6.75b) as well as equation (6.59a), it follows that:

$$U_j = -\Gamma_j \bar{m}_j \ddot{U}_j + \int_V \rho_A \vec{\Phi}_j \cdot \int_V \bar{\bar{\Gamma}}_0 \cdot (\vec{g}_P - \vec{a}_P)\rho_A \, dV dV + \int_V \rho_A \vec{\Phi}_j \cdot \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{F}\,\delta(\vec{r} - \vec{r}_s) \, dV dV \tag{6.88}$$

The symmetry of $\bar{\bar{\Gamma}}_0$ may be used to write equation (6.88) as:

$$U_j + \Gamma_i \bar{m}_i \ddot{U}_j = \int_V \left[(\vec{g}_P - \vec{a})\rho_A + \vec{F}\delta(\vec{r} - \vec{r}_s)\right] \cdot \int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\Phi}_j \rho_A \, dV dV \tag{6.89}$$

Recalling equation (6.59c), it follows that:

$$\int_V \bar{\bar{\Gamma}}_0 \cdot \vec{\Phi}_j \rho_A \, dV = \frac{1}{\omega_j^2}\left[\vec{\Phi}_j - \vec{\Phi}_j(0) - \frac{1}{2}(\vec{\nabla} \times \vec{\Phi}_j)_{\vec{r}=0} \times \vec{r}\right] \tag{6.90}$$

Substituting (6.90) into (6.89) leads to:

$$U_j + \Gamma_j \bar{m}_j \ddot{U}_j = \frac{1}{\omega_j^2} \int_S \vec{F} \cdot \vec{\Phi}_j \, dS - \frac{1}{\omega_j^2} \left\{ \int_V \left[\left(\vec{q}_P - \vec{\alpha} \right) \rho_A \right. \right.$$

$$\left. \left. + \vec{F} \delta (\vec{r} - \vec{r}_S) \right] \cdot \left[\vec{\Phi}_j(0) + \frac{1}{2} \left(\vec{\nabla} \times \Phi_j \right)_{\vec{r}=0} \times \vec{r} \right] \right\} dV$$

The last term of this expression vanishes as a consequence of perturbation form of the conservation of linear and angular momentum. Hence the internal equilibrium equations in terms of the normal modes are given by:

$$U_j + \Gamma_j \bar{m}_j \ddot{U}_j = \frac{1}{\omega_j^2} \int_S \vec{F} \cdot \vec{\Phi}_j \, dS \tag{6.91}$$

Multiplying equation (6.90) by $\rho_A \vec{\Phi}_j$, integrating, and using the orthogonality of equations (6.74b) and (6.75b) leads to a determination of ω^2_j as

$$\frac{1}{\omega_j^2} = \Gamma_j \bar{m}_j \tag{6.92}$$

so that equation (6.92) may be written:

$$U_j + \Gamma_j \bar{m}_j \ddot{U}_j = \Gamma_j \bar{m}_j \int_S \vec{F} \cdot \vec{\Phi}_j \, dS \tag{6.93}$$

Finally, defining generalized aerodynamic and thrust perturbation forces as

$$Q_j \equiv \bar{m}_j \int_S \vec{F} \cdot \vec{\Phi}_j \, dS \tag{6.94}$$

the internal equilibrium equations may be written:

$$U_j + \Gamma_j \bar{m}_j \ddot{U}_j = \Gamma_j Q_j \tag{6.95}$$

The quantities Γ_j may be denoted as generalized flexibilities, and $\bar{m}_j$ may be denoted as generalized masses. Equations (6.95) represent an infinity of ordinary differential equations. They are coupled only through the generalized forces Q_j. In general, they are coupled not only with one another, but also aerodynamically with the rigid-body motion of the airplane, The elastic perturbation displacements $\vec{d}_P$, which are related to the generalized displacements u_j, are not changes in position relative to inertial space. They are changes in position relative to a mean axis system when equation (6.60) is used as a condition in the formulation or relative to a principal axis system when equation (6.65) is used. It is clear that equations (6.95) no longer contain the perturbation acceleration and gravity forces. However, the generalized forces Q_j also do not contain the entire aerodynamic and thrust perturbation forces. The operations leading from equation (6.89), containing $(\vec{g}_p - \vec{\vec{a}}_p)\ \rho_A$, to equation (6.91) remove a distribution of aerodynamic and thrust forces that balance the perturbation inertia and gravity forces $(\vec{g}_p - \vec{\vec{a}}_p)\ \rho_A$ at every point of the airplane.

If the dependence of $\vec{\vec{\Gamma}}_0$ on the coordinate through the thickness of the structure is eliminated in the above formulation, as it usually is in practice, the above assertion is more acceptable. Aerodynamic and thrust forces that just balance $(\vec{g}_p - \vec{\vec{a}}_p)\ \rho_A$ x (thickness) are subtracted. The computation $\int_{S_M} (\vec{f}_u - \vec{f}_l)\cdot\vec{\Phi}dS$, which is the difference between the upper and lower surface pressures integrated over a middle surface of the structure S_M, is calculated from the deviation of $(\vec{f}_u - \vec{f}_l)$ from the pressure difference that just balances $(\vec{g}_p - \vec{\vec{a}}_p)\ \rho_A$ x (thickness) at every point of the surface.

These considerations are vital to an appropriate formulation of residual flexibility. In that formulation, some of the normal modes are eliminated from the problem. The deflections of the structure associated with the eliminated modes are treated as quasi-static. Inertia relief and gravity force perturbation must be retained in the formulation.

Consider the structure to be platelike, so that the component of elastic displacement in the direction of thickness is independent of the coordinate in that direction. Let the direction of thickness (normal to the plate) be denoted by the unit vector $\vec{\lambda}$. Also introduce a coordinate system r, s, t, with t in the direction of $\vec{\lambda}$. Then internal equilibrium may be expressed as

$$d_\lambda - d_{\lambda_o} - \frac{1}{2}\left[(\vec{\nabla}\times\vec{d}_P)_{\vec{r}=0}\times\vec{\vec{r}}\right]\cdot\vec{\lambda} = -\int_{S_M}\vec{\lambda}\cdot\Big(V_P + \dot{\vec{\omega}}_P\times\vec{\vec{r}} + \vec{\omega}_P\times\vec{V}_{c_1} + \ddot{\vec{d}}_P - \vec{g}_P\Big)\rho_A\,dS + \int_{S_M}\vec{\lambda}\cdot\vec{\vec{\Gamma}}\cdot\vec{\ell}\,dS \qquad (6.96)$$

where $\overline{\rho}_A$ is the mass distribution per unit area of S_M, and ℓ is the surface aerodynamic and thrust stress difference across S_M.

Ignoring the influence of forces in directions other than the direction of $\vec{\lambda}$, the influence function

$$\Gamma(r, s; \rho, \sigma) \approx \vec{\lambda} \cdot \bar{\Gamma} \cdot \vec{\lambda} \tag{6.97}$$

may be introduced so that equation (6.96) becomes:

$$d_\lambda - d_{\lambda_0} - \frac{1}{2}\left[(\vec{\nabla} \times \vec{d}_P)_{\vec{r}=0} \times \vec{\tilde{r}}\right] \cdot \vec{\lambda} = -\int_{S_M} \Gamma \vec{\lambda} \cdot (\dot{\vec{V}}_P + \dot{\vec{\omega}}_P \times \vec{\tilde{r}}$$

$$+ \vec{\omega}_P \times \vec{V}_{c_1} - g_P) \overline{\rho}_A \, dS - \int_{S_M} \Gamma \ddot{d}_\lambda \overline{\rho}_A \, dS + \int_{S_M} \Gamma \ell_\lambda \, dS \tag{6.98}$$

where: $\qquad \ell_\lambda \equiv \vec{\lambda} \cdot \vec{\ell}$

For free vibrations equation (6.98) reduces to

$$d_\lambda - d_{\lambda_0} - \frac{1}{2}\left[(\vec{\nabla} \times \vec{d}_P)_{\vec{r}=0} \times \vec{\tilde{r}}\right] \cdot \vec{\lambda} = -\int_{S_M} \Gamma \ddot{d}_\lambda \overline{\rho}_A \, dS$$

$$\Phi_{\lambda_i} - \Phi_{\lambda_i}(0) - \frac{1}{2}\left[(\vec{\nabla} \times \vec{\Phi}_i)_{\vec{r}=0} \times \vec{\tilde{r}}\right] \cdot \vec{\lambda} = \omega_i^2 \int_{S_M} \Gamma \Phi_{\lambda_i} \overline{\rho}_A \, dS$$

This result may be developed into an eigenvalue problem such that

$$\ddot{u}_i + \omega_i^2 u_i = 0 \tag{6.99a}$$

$$\int_{S_M} \Phi_{\lambda_i} \rho_A \, dS = 0 \tag{6.99b}$$

$$\int_{S_M} (\vec{\tilde{r}} \times \vec{\Phi}_i) \cdot \vec{\lambda} \rho_A \, dS = 0 \tag{6.99c}$$

(6.99d)

Letting
$$d_\lambda = \sum_{i=1}^{\infty} \Phi_{\lambda_i} u_i \; , \tag{6.100}$$

this expression may be introduced into equation (6.98) to find :

$$u_i = -\Gamma_i \bar{m}_i \ddot{u}_i - \int_{S_M} \bar{\rho}_A \Phi_{\lambda_i} \int_{S_M} \Gamma \vec{\lambda} \cdot \left(\dot{\vec{V}}_P + \dot{\vec{\omega}}_P \times \vec{\bar{r}} + \vec{\omega}_P \times \vec{V}_{c_1} - \vec{g}_P \right) \bar{\rho}_A \, dS \, dS + \int_{S_M} \bar{\rho}_A \Phi_{\lambda_i} \int_{S_M} \Gamma \ell_\lambda \, dS \, dS \tag{6.101}$$

As previously done, use the symmetry of $\vec{\Gamma}_0$ to write the final two integral terms as

$$-\int_{S_M} \vec{\lambda} \cdot \left(\dot{\vec{V}}_P + \dot{\vec{\omega}}_P \times \vec{\bar{r}} + \vec{\omega}_P \times \vec{V}_{c_1} - \vec{g}_P \right) \rho_A \int_{S_M} \Gamma \Phi_{\lambda_i} \bar{\rho}_A \, dS \, dS + \int_{S_M} \ell_\lambda \int_{S_M} \Gamma \Phi_{\lambda_i} \bar{\rho}_A \, dS \, dS$$

Introducing equation (6.99a) written as:

$$\int_{S_M} \Gamma \Phi_{\lambda_i} \bar{\rho}_A \, dS = \frac{1}{\omega_i^2} \left\{ \Phi_{\lambda_i} - \Phi_{\lambda_i}(0) - \frac{1}{2} \left[(\vec{\nabla} \times \vec{\Phi})_{\vec{r}=0} \times \vec{\bar{r}} \right] \cdot \vec{\lambda} \right\} ,$$

these two terms become, as a consequence of the last two expressions of equation (6.100),

$$\frac{1}{\omega_i^2} \int_{S_M} \left\{ \vec{\lambda} \cdot \left(\dot{\vec{V}}_P + \dot{\vec{\omega}}_P \times \vec{\bar{r}} + \vec{\omega}_P \times \vec{V}_{c_1} - \vec{g}_P \right) \rho_A - \ell_\lambda \right\} \left\{ \Phi_{\lambda_i}(0) + \frac{1}{2} \left[(\vec{\nabla} \times \vec{\Phi})_{\vec{r}=0} \times \vec{\bar{r}} \right] \cdot \vec{\lambda} \right\} dS + \frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} \ell_\lambda \, dS \tag{6.102}$$

The first integral may be made to vanish as a consequence of the conservation of linear and angular momentum. But one may also let

$$\ell_\lambda = \ell_\lambda^* + \bar{\ell}_\lambda$$

such that:

$$\int_{S_M} \ell_\lambda^* \, dS = 0$$

and:

$$\int_{S_M} (\vec{r} \times \ell^*) \cdot \vec{\lambda} \, dS = 0$$

while:

$$\vec{\lambda} \cdot \left(\dot{\vec{V}}_P + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega}_P \times \vec{V}_{c_1} - \vec{g}_P \right) \bar{\rho}_A = \bar{\ell}_\lambda$$

With these substituted into expression (6. 102), the first integral vanishes identically while the second integral becomes

$$\frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} \ell_\lambda \, dS = \frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} (\ell_\lambda^* + \bar{\ell}_\lambda) \, dS$$

Clearly, this must reduce to:

$$\frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} \ell_\lambda \, dS = \frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} \ell_\lambda^* \, dS$$

so that:

$$\frac{1}{\omega_i^2} \int_{S_M} \Phi_{\lambda_i} \bar{\ell}_\lambda \, dS = 0$$

The portion of the surface load represented by $\bar{\ell}_\lambda$ is orthogonal to Φ_{λ_i}. The resulting internal equilibrium equations are given by

$$u_i + \Gamma_i \bar{m}_i \ddot{u}_i = \Gamma_i \bar{m}_i \int_{S_M} \Phi_{\lambda_i} \ell_\lambda \, dS \tag{6. 103}$$

This expression represents the appropriate form of internal equilibrium. The generalized aerodynamic and thrust forces represented by

$$Q_i = \bar{m}_i \int_{S_M} \Phi_{\lambda_i} \ell_\lambda \, dS$$

have been shown to be computed from surface aerodynamic and thrust pressures that deviate from those that just balance the perturbation inertia and gravity forces. Thus, when internal equilibrium is expressed in terms of free-vibration mode shapes, the effects of inertia and gravity forces are eliminated.

6.3 Lumped Parameter Representation of Equations of Motion

In the preceding sections the momentum equations and the internal equilibrium equation were formulated in terms of perturbation quantities. For the approximations called large perturbations or small perturbations, the momentum equations were given by tables 4, 5, and 6.

Internal equilibrium has been formulated as a system of integral equations that treat the airplane as a continuous body. The kernel functions for these integral equations cannot be found except for very simple structural forms. For complicated structures such as that of an airplane another, but equivalent, approach must be used. This alternate approach is based on the lumped parameter formulation of the equations of motion.

6.3.1 Lumped parameters. - The airplane is divided into a large number n of volume elements so that its total volume is given by:

$$V = \sum_{i=1}^{n} V_i \qquad (6.104)$$

The mass associated with the i^{th} element is then

$$m_i = \int_{V_i} \rho_A \, dV \qquad (6.105)$$

and is termed the i^{th} lumped mass. Its position relative to the airplane c.g. is

$$\vec{r}_i = \frac{1}{m_i}\int_{V_i} \rho_A \, \vec{r} \, dV \qquad (6.106)$$

If the c.g. undergoes the virtual displacement $\delta\vec{r}'_0$ and virtual rotation $\delta\vec{\Omega}_0$, the position of the i^{th} lumped mass relative to inertial space, i.e.,

$$\vec{r}_i{}' = \frac{1}{m_i}\int_{V_i} \rho_A \, \vec{r}\,' dV \ , \qquad (6.107)$$

undergoes the virtual change in position. The virtual change is

$$\left.\begin{aligned} \delta\vec{r}_i{}' &= \frac{1}{m_i}\int_{V_i} \rho_A (\delta\vec{r}_o{}' + \delta\vec{\Omega}_o \times \vec{r}) \, dV \\ &= \delta\vec{r}_o{}' + \delta\vec{\Omega}_o \times \vec{r}_i \ . \end{aligned}\right\} \qquad (6.108)$$

Now, if the components of $\delta\vec{r}'_i$ are denoted by the column matrix

$$\delta\vec{r}_i{}' = \begin{Bmatrix} \delta x_i' \\ \delta y_i' \\ \delta z_i' \end{Bmatrix} \qquad (6.109)$$

and the components of the other vectors are similarly written in matrix form:

$$\vec{r}_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \quad \delta\vec{r}_o{}' = \begin{Bmatrix} \delta x_o' \\ \delta y_o' \\ \delta z_o' \end{Bmatrix} \quad \delta\vec{\Omega}_o = \begin{Bmatrix} \delta \phi_o \\ \delta \theta_o \\ \delta \psi_o \end{Bmatrix} \qquad (6.110)$$

the above expression for the virtual displacement relative to inertial space becomes, in matrix form,

$$\begin{Bmatrix} \delta x_i' \\ \delta y_i' \\ \delta z_i' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & z_i & -y_i \\ 0 & 1 & 0 & -z_i & 0 & x_i \\ 0 & 0 & 1 & y_i & -x_i & 0 \end{bmatrix} \begin{Bmatrix} \delta x_o' \\ \delta y_o' \\ \delta z_o' \\ \delta \phi_o \\ \delta \theta_o \\ \delta \psi_o \end{Bmatrix} \qquad (6.111)$$

The rectangular (3 x 6) matrix is termed the rigid-body mode matrix of the i^{th} lumped mass. It is denoted by

$$[\bar{\phi}_i] = \begin{bmatrix} 1 & 0 & 0 & 0 & z_i & -y_i \\ 0 & 1 & 0 & -z_i & 0 & x_i \\ 0 & 0 & 1 & y_i & -x_i & 0 \end{bmatrix} \tag{6.112}$$

and for all masses an airplane rigid-body mode matrix is defined by the (3n x 6) matrix:

$$[\bar{\phi}] = \begin{bmatrix} [\bar{\phi}_1] \\ [\bar{\phi}_2] \\ \cdot \\ \cdot \\ \cdot \\ [\bar{\phi}_n] \end{bmatrix} \tag{6.113}$$

In conjunction with this definition a (3n x 3n) diagonal mass matrix is defined as:

$$\lceil m \rfloor = \begin{bmatrix} m_1 & & & & & & & & \\ & m_1 & & & & & & & \\ & & m_1 & & & & & & \\ & & & m_2 & & & & & \\ & & & & m_2 & & & & \\ & & & & & m_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & m_n & \\ & & & & & & & & m_n \end{bmatrix} \tag{6.114}$$

The matrices defined by equations (6.113) and (6.114) may be combined into

G4 $$[M] \equiv [\bar{\phi}]^T \lceil m \rfloor [\bar{\phi}] = \begin{bmatrix} M & 0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & 0 & -I_{xz} \\ 0 & 0 & 0 & 0 & I_Y & 0 \\ 0 & 0 & 0 & -I_{xz} & 0 & I_z \end{bmatrix} \tag{6.115}$$

where the moments and products of inertias are only approximate because the lumped masses are finite and their inertias relative to their own centers of gravity are neglected. They become exact only as the lumped masses become infinitesimals. Also, the moments and products inertias differ for the reference and perturbation flight conditions because of the perturbation elastic deformation. This difference is ignored as small, so that $\vec{r}_i$ in equation (6.106) is taken to be the position in the reference flight condition. This approximation is used throughout the following development.

Now introduce the further definitions

$$\{V_P\}^T = \lfloor u, v, w, p, q, r \rfloor \tag{6.116}$$

as the c.g. velocity matrix and

$$\{r'_{oP}\}^T = \lfloor x'_{oP}, y'_{oP}, z'_{oP}, \phi_P, \theta_P, \psi_P \rfloor \tag{6.117}$$

as the c.g. position matrix.

In terms of these definitions the momentum and moment of momentum equations for general curvilinear flight, $(P_1, Q_1, R_1) \neq 0$, and small rotation rates (so products with perturbation elastic displacements can be ignored) may be written in matrix form as

$$[M]\left(\frac{\partial}{\partial t}\{V_P\} + [M_1]\{V_P\} + [M_2]\{r'_{oP}\}\right) = [\bar{\phi}]^T\{F\} \tag{6.118}$$

where

$$\{F\}^T = \lfloor f_{x_1}, f_{y_1}, f_{z_1}, \cdots, f_{x_n}, f_{y_n}, f_{z_n} \rfloor$$

and where f_{x_i}, f_{y_i}, and f_{z_i} are the components of aerodynamic and thrust forces acting on the i^{th} lumped mass.

For large perturbations:

$$[M_1] = \begin{bmatrix} 0 & -(R_1+r) & (Q_1+q) & 0 & W_1 & -V_1 \\ (R_1+r) & 0 & -(P_1+p) & -W_1 & 0 & U_1 \\ -(Q_1+q) & (P_1+p) & 0 & V_1 & -U_1 & 0 \\ 0 & 0 & 0 & A & B & C \\ 0 & 0 & 0 & D & E & F \\ 0 & 0 & 0 & G & H & I \end{bmatrix} \qquad (6.119a)$$

$$[M_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & g\cos\theta_1 & 0 \\ 0 & 0 & 0 & -g\cos\Phi_1\cos\theta_1 & g\sin\theta_1\sin\Phi_1 & 0 \\ 0 & 0 & 0 & g\cos\theta_1\sin\Phi_1 & g\sin\theta_1\cos\Phi_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(6.119b)

where in equation (6.119a):

$$A = \left[\frac{-I_z I_{xz} + I_{xz}(I_Y - I_X)}{I_X I_Z - I_{XZ}^2}\right](Q_1 + q)$$

$$B = \left[\frac{-I_z I_{xz} + I_{xz}(I_Y - I_x)}{I_x I_z - I_{xz}^2}\right] P_1 + \left[\frac{I_z(I_z - I_Y) + I_{xz}^2}{I_x I_z - I_{xz}^2}\right](R_1 + r)$$

$$C = \left[\frac{I_z(I_z - I_Y) + I_{xz}^2}{I_x I_z - I_{xz}^2}\right] Q_1$$

$$D = \frac{I_{xz}(2P_1 + p)}{I_Y} + \frac{(I_x - I_z)}{I_Y} R_1$$

$$E = 0$$

$$F = -\frac{I_{xz}}{I_Y}(2R_1 + r) + \frac{(I_x - I_z)}{I_Y}(P_1 + p) \quad ;$$

$$G = \left[\frac{-I_{xz}^2 + I_x(I_Y - I_x)}{I_x I_z - I_{xz}^2}\right](Q_1 + q)$$

$$H = \left[\frac{-I_{xz}^2 + I_x(I_Y - I_x)}{I_x I_z - I_{xz}^2}\right]P_1 + \left[\frac{I_{xz}(I_z - I_Y) + I_x I_{xz}}{I_x I_z - I_{xz}^2}\right](R_1 + r)$$

$$I = \left[\frac{I_{xz}(I_z - I_Y) + I_x I_{xz}}{I_x I_z - I_{xz}^2}\right] Q_1$$

and, for small perturbations,

$$[M_1] = \begin{bmatrix} 0 & -R & Q_1 & 0 & W_1 & -V_1 \\ R_1 & 0 & -P_1 & -W_1 & 0 & U_1 \\ -Q & P_1 & 0 & V_1 & -U_1 & 0 \\ 0 & 0 & 0 & A & B & C \\ 0 & 0 & 0 & D & E & F \\ 0 & 0 & 0 & G & H & I \end{bmatrix} \tag{6.120a}$$

$$[M_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & g\cos\theta_1 & 0 \\ 0 & 0 & 0 & -g\cos\Phi_1\cos\theta_1 & g\sin\theta_1\sin\Phi_1 & 0 \\ 0 & 0 & 0 & g\cos\theta_1\sin\Phi_1 & g\sin\theta_1\cos\Phi_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(6.120b)

where in equation (6.120a):

$$A = \left[\frac{-I_z I_{xz} + I_{xz}(I_Y - I_X)}{I_X I_z - I_{xz}^2}\right](Q_1$$

$$B = \left[\frac{-I_z I_{xz} + I_{xz}(I_Y - I_x)}{I_x I_z - I_{xz}^2}\right]P_1 + \left[\frac{I_z(I_z - I_Y) + I_{xz}^2}{I_x I_z - I_{xz}^2}\right]R_1$$

$$C = \left[\frac{I_z(I_z - I_Y) + I_{xz}^2}{I_x I_z - I_{xz}^2}\right]Q_1$$

$$D = \frac{I_{xz}(2P_1 + p)}{I_Y} + \frac{(I_x - I_z)}{I_Y}R_1$$

$$E = 0$$

$$F = -\frac{I_{xz}}{I_Y}(2R_1 + r) + \frac{(I_x - I_z)}{I_Y}P_1$$

$$G = \left[\frac{-I_{xz}^2 + I_x(I_Y - I_x)}{I_x I_z - I_{xz}^2}\right]Q_1$$

$$H = \left[\frac{-I_{xz}^2 + I_x(I_Y - I_x)}{\qquad}\right]P_1 + \left[\frac{I_{xz}(I_z - I_Y) + I_x I_{xz}}{I_x I_z - I_{xz}^2}\right]R_1$$

$$I = \left[\frac{I_{xz}(I_z - I_Y) + I_x I_{xz}}{I_x I_z - I_{xz}^2}\right]Q_1$$

Finally, equations (6.118) constitute the laws of conservation of linear and angular momentum. They are in the desired lumped parameter formulation as a single matrix equation.

6.3.2 Internal equilibrium equation in lumped parameter form. - The general equations of motion for the airplane as a body with six degrees of freedom have been derived in the preceding sections of this appendix. In addition, internal equilibrium equations were derived using the laws of conservation of momentum, the concept of internal stress, and Hooke's law. The internal equilibrium equations are essentially equations of motion governing the elastic deformation motion of the airplane.

Those equations are given by:

$$\vec{d} - \vec{d}_o - \vec{\theta}_o \times \vec{\bar{r}} = \int_V \bar{\bar{\Gamma}}_o \cdot (\vec{R} - \rho_A \vec{a})\, dV + \int_V \bar{\bar{\Gamma}}_o \cdot \vec{F}\, \delta(\vec{r} - \vec{r}_s)\, dV \tag{6.121}$$

where the acceleration is

$$\vec{a} = \frac{d^2 \vec{r}_o'}{dt^2} + \frac{d^2 \vec{\bar{r}}}{dt^2} + \frac{d^2 \vec{d}}{dt^2} \tag{6.122}$$

The dynamic effect of elastic motion enters from the acceleration component $d^2\vec{d}/dt^2$.

Internal equilibrium may be expressed in lumped parameter form by introducing the following definitions.

The displacement vector field is replaced by the mean value of displacement of the lumped masses given by:

$$\vec{d}_i = \frac{1}{m_i} \int_{V_i} \rho_A \vec{d}\, dV \tag{6.123}$$

Then the column matrix of displacement components may be introduced as:

$$\{d\}^T = \lfloor dx_1, dy_1, dz_1, \cdots, dx_n, dy_n, dz_n \rfloor \tag{6.124}$$

The structu[illegible] fluence function $\bar{\Gamma}_0$ is replaced by the matrix $[C]$. The c.g. of the airplane is clamped for this matrix to be consistent with the definition of $\bar{\Gamma}_0$. The matrix $[C]$ is of size $(3n + 3)x(3n + 3)$ and is composed of an $(n + 1)x(n + 1)$ array of (3×3) submatrices. The i,j^{th} submatrix is

$$\begin{bmatrix} C_{XX} & C_{XY} & C_{XZ} \\ C_{YX} & C_{YY} & C_{YZ} \\ C_{ZX} & C_{ZY} & C_{ZZ} \end{bmatrix}_{i,j} \tag{6.125}$$

A typical element, C_{XY}, gives the component of mean displacement in the X-direction at the i^{th} lumped mass due to the component of force in the Y-direction at the j^{th} lumped mass. The three rows and colums of $[C]$ corresponding to the clamped point at the c.g. contain zeros. Thus the matrix $[C]$ is singular. The reduced matrix obtained by deleting those three rows and columns is nonsingular. This will be true unless the structure is a mechanism. The reduced matrix is denoted by $[C_0]$ and is $(3n \times 3n)$ in size.

The matrix $[C_0]$ has an inverse denoted by

$$[C_0]^{-1} = [K_{11}] \tag{6.126}$$

and is the stiffness matrix for the airplane clamped at its c.g. The stiffness matrix for the free airplane is

$$[K] = \begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \tag{6.127}$$

The matrix represented by equation (6.127) is singular, and the defect is removed by deleting the submatrices $[K_{12}]$, $[K_{21}]$, and $[K_{22}]$. The submatrix $[K_{22}]$ is the force at the reference point due to a unit displacement at the reference point, $[K_{21}]$ is the force at the reference point due to a unit displacement at the i^{th} lumped mass with all other lumped masses held fixed, and $[K_{12}]$ is the force at the i^{th} lumped mass due to a unit displacement of the reference point.

Further, introduce the matrix

$$\{B\}^T = \lfloor dx_o , dy_o , dz_o , \theta x_o , \theta y_o , \theta z_o \rfloor \tag{6.128}$$

which is the displacement of the reference point (c.g.of the airplane before loading) relative to the c.g. of the deformed airplane.

With these definitions plus those of par. 6.2.3, the expression for internal equilibrium, equation (6.35), is written in lumped parameter form as:

$$\{d_P\} - [\bar{\phi}]\{B\} = -[C_o]\Big[[m]\Big(\frac{\partial^2}{\partial t^2}\{d_P\} + [\bar{\phi}]\frac{\partial}{\partial t}\{V_P\} + [\bar{\phi}][M_1]\{V_P\} + [\bar{\phi}][M_2]\{r'_{oP}\}\Big) - \{F\}\Big] \tag{6.129}$$

This result is written in perturbation form and holds for large and small perturbations, depending on the choice of the matrices $[M_1]$ or $[M_2]$.

The terms on the right-hand side of equation (6.129) are a self-equilibrating system of forces. Hence the total deflection of the structure cannot give rise to a change in position of the center of mass or to a rotation of the airplane about its center of mass. Thus, if the perturbation displacements of equation (6.129) are multiplied by the matrix consisting of $[\bar{\phi}]^T[m]$, the result must be:

$$[\bar{\phi}]^T [m] \{d_P\} = 0 \tag{6.130}$$

Using the definition of equation (6.128), the matrix $\{B\}$ may be determined to be:

$$\{B\} = [M]^{-1}[\bar{\phi}]^T [m] [C_o]\Big[[m]\Big(\frac{\partial^2}{\partial t^2}\{d_P\} + [\bar{\phi}]\frac{\partial}{\partial t}\{V_P\} + [\bar{\phi}][M_1]\{V_P\} + [\bar{\phi}][M_2]\{r'_{oP}\}\Big) - \{F\}\Big] \tag{6.131}$$

This result substituted into equation (6.129) gives

$$\{d_P\} = -[\bar{C}]\Big[[m]\Big(\frac{\partial^2}{\partial t^2}\{d_P\} + [\bar{\phi}]\frac{\partial}{\partial t}\{V_P\} + [\bar{\phi}][M_1]\{V_P\} + [\bar{\phi}][M_2]\{r'_{oP}\}\Big) - \{F\}\Big] \tag{6.132}$$

where:

$$[\bar{C}] = \Big[[I] - [\bar{\phi}]\,[M]^{-1}[\bar{\phi}]^{T}[m]\Big]\,[C_o] \tag{6.133}$$

The flexibility matrix $[\bar{C}]$ relates the displacements of the lumped masses to the airplane's c.g. The result of multiplying this flexibility matrix by an arbitrary set of self-equilibrating forces is a set of displacements that do not give rise to a rigid-body motion of the airplane. A similar analysis is carried out for the continuous airplane in par. 6.2.3 and by Bisplinghoff and Ashley in Chapter 9 of ref. 1. They arrive at the following expression:

$$\begin{aligned}
\bar{\bar{G}}(x,y,z;\xi,\eta,\zeta) &= \bar{\bar{\Gamma}}(x,y,z;\xi,\eta,\zeta) \\
&- \frac{1}{M}\int_V \bar{\bar{\Gamma}}_o(r,s,t;\xi,\eta,\zeta)\,\rho_A(r,s,t)\,dr\,ds\,dt \\
&+ \vec{r}(x,y,z)\times\Big\{\bar{\bar{\Psi}}^{-1}\Big[\int_V \vec{r}(r,s,t)\times\bar{\bar{\Gamma}}_o(r,s,t;\xi,\eta,\zeta)\,\rho_A(r,s,t)\,dr\,ds\,dt\Big]\Big\}
\end{aligned} \tag{6.134}$$

However, as pointed out by Milne (ref. 43), this result describes elastic deformation relative to mean axes and not relative to principal axes as implied above. However, the approximations in equation (6.133) lead to coincidence of the mean and principal axes.

The similarity to equation (6.133) is apparent. The structural influence function $[\bar{\bar{G}}]$ is the continuous analog of the matrix $[\bar{C}]$.

6.3.3 Free-vibration normal mode shapes. - The form for the intern equilibrium equations given by equation (6.132) is not the most convenient. These are 3n coupled equations of motion for the n lumped masses. They may be uncoupled by introducing a change of variables, and the mathematical process for doing this for the continuous airplane is contained in the preceding subsection. An analogous development is presented here for the lumped parameter form of equation (6.35).

Let the airplane be vibrating freely in empty space. The internal equilibrium equations then reduce to

$$\{u_p\} = -[\bar{C}]\,[m]\,\frac{\partial^2}{\partial t^2}\,\{d_p\} \tag{6.135}$$

If the displacement matrix is written as

$$\{d_P\} = \{\phi\} u \tag{6.136}$$

where u is a function time alone and $\{\phi\}$ is a matrix of constants, then equation (6.135) becomes

$$\{\phi\} u = -[\bar{C}][m]\{\phi\}\ddot{u} \tag{6.137}$$

This result separates into two equations:

$$\{\phi\} = \omega^2 [\bar{C}][m]\{\phi\} \tag{6.138}$$

and

$$\ddot{u} + \omega^2 u = 0 \tag{6.139}$$

where ω^2 is the separation constant. Physically, ω represents a natural frequency of the free vibration. Equation (6.138) represents an eigenvalue problem. The eigenvalues are denoted by ω^2_i, and the eigenvectors by $\{\phi_i\}$.

The matrix $[\bar{C}]$ is singular because

$$[\bar{C}][\bar{\phi}] = 0 \tag{6.140}$$

The defect in $[\bar{C}]$ is of order six. Hence the number of independent eigenvalues obtainable from equation (6.138) is $3n - 6$. This is equal to the number of elastic degrees of freedom of an airplane represented by n lumped masses that are granted only translation degrees of freedom relative to the center of mass. These results do not lead readily to a derivation of the orthogonality properties of $\{\phi_i\}$. The problem must be reformulated in terms of stiffness.

To simplify writing, let

$$\{R\} = \{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{oP}\} \tag{6.141}$$

so that equation (6.129) may be written as

$$\{d_P\} - [\phi]\{B\} = -[C_o][m]\left(\{\ddot{d}_P\} + [\bar{\phi}]\{R\}\right) + [C_o]\{F\} \tag{6.142}$$

and equation (6.118) may be written as

$$[M]\{R\} = [\bar{\phi}]^T\{F\} \tag{6.143}$$

Equation (6.142) represents internal equilibrium, while equation (6.143) represents conservation of linear and angular momentum.

The internal equilibrium may be expressed in terms of stiffness by multiplying equation (6.142) by $[K_{11}] = [C_0]^{-1}$ to find

$$[K_{11}]\left(\{d_P\} - [\bar{\Phi}]\{B\}\right) = -[m]\left(\{\ddot{d}_P\} + [\bar{\Phi}]\{R\}\right) + \{F\} \qquad (6.144)$$

If this result is multiplied by $[\bar{\Phi}]^T$, the right-hand member vanishes as a consequence of equations (6.130) and (6.143). This leads to

$$\{B\} = \left([\bar{\Phi}]^T[K_{11}][\bar{\Phi}]\right)^{-1}[\bar{\Phi}]^T[K_{11}]\{d_P\} \qquad (6.145)$$

so that equation (6.144) may be written:

$$[K]\{d_P\} = -[m]\{\ddot{d}_P\} - [m][\bar{\Phi}]\{R\} + \{F\} \qquad (6.146)$$

where:

$$[K] \equiv \left[[K_{11}] - [K_{11}][\bar{\Phi}]\left([\bar{\Phi}]^T[K_{11}][\bar{\Phi}]\right)^{-1}[\bar{\Phi}]^T[K_{11}]\right] \qquad (6.147)$$

The matrix $[K]$ is singular because

$$[K][\bar{\Phi}] = 0 \qquad (6.148)$$

but it is a symmetric matrix. The defect in $[K]$ is of order six. Assume that the elastic motion is simple harmonic, such that

$$\{\ddot{d}_P\} = -\omega^2 \{d_P\} \qquad (6.149)$$

Equation (6.146) becomes

$$[K]\{d_P\} = \omega^2[m]\{d_P\} + \{F\} - [m][\bar{\Phi}]\{R\} \qquad (6.150)$$

For free vibrations, $\{F\} - [m][\bar{\Phi}]\{R\} = 0$ and internal equilibrium is given by

$$[K]\{d_P\} = \omega^2[m]\{d_P\} \qquad (6.151)$$

As previously noted, [K] has a defect of order six. Equations (6.148) and 6.151) imply that there are six vectors that satisfy equation (6.151) with $\omega^2 = 0$. These are the six columns of the rigid-body mode matrix $[\bar{\phi}]$. Further, letting

$$\{dp\} = \{\phi\} u \tag{6.152}$$

where u is a function of time alone, there are (3n - 6) vectors (eigenvectors $\{\phi_i\}$) that satisfy equation (6.138) and correspond to (3n - 6) distinct eigenvalues ω_i^2. Thus there are (3n - 6) solutions to equation (6.151) that satisfy

$$[K]\{\phi_i\} = \omega_i^2 [m]\{\phi_i\} \tag{6.153}$$

The stiffness and the mass matrices are symmetric. It follows from this that

$$\{\phi_j\}^T [K]\{\phi_i\}\left(\frac{1}{\omega_i^2} - \frac{1}{\omega_j^2}\right) = 0$$

and

$$\{\phi_j\}^T [m]\{\phi_i\}\left(\omega_i^2 - \omega_j^2\right) = 0$$

Define $\bar{k}_i$ and $\bar{m}_i$ such that

$$\left.\begin{aligned}\{\phi_j\}^T [K]\{\phi_i\} &= \bar{K}_i \qquad \text{For} \quad i = j \\ &= 0 \qquad \text{For} \quad i \neq j\end{aligned}\right\} \tag{6.154}$$

and

$$\left.\begin{aligned}\{\phi_j\}^T [m]\{\phi_i\} &= m_i \quad , \quad i = j \\ &= 0 \quad , \quad i \neq j\end{aligned}\right\} \tag{6.155}$$

These are termed the orthogonality properties of the free-vibration mode shapes $\{\phi_i\}$. A free-vibration mode shape matrix may be defined as:

$$[\phi] = \left[\{\phi_1\}, \{\phi_2\}, \cdots, \{\phi_{3n-6}\}\right] \tag{6.156}$$

With this definition, equations (6.154) and (6.155) may be written:

$$[\phi]^T [K] [\phi] = [\bar{K}] \qquad (6.157)$$

and

$$[\phi]^T [m] [\phi] = [\bar{m}] \qquad (6.158)$$

The elastic displacements $\{dp\}$ may now be written:

$$\{dp\} = [\phi] \{u\} \qquad (6.159)$$

where $\{u\}$ is the column matrix of $(3n - 6)$ generalized elastic displacements. These are linearly independent, while the physical displacements $\vec{dp}$ are $3n$ in number and are seen to be linearly dependent in consequence of equation (6.130).

The formulation above must be distinguished from an alternate formulation, appearing often in the literature, that does not distinguish between internal equilibrium and conservation of linear and angular momentum. The equations of motion are initially written in terms of the positions of the lumped masses relative to an inertia reference system. The connection between the two formulations may be demonstrated by noting that $\{r'_p\}$ defines $3n$ components of displacement relative to inertial space; hence,

$$\{r'_p\} = [\bar{\phi}] \{r'_{op}\} + \{dp\} \qquad (6.160)$$

and it follows from equation (6.159) that:

$$\left.\begin{aligned} \{r'_p\} &= [\bar{\phi}] \{r'_{op}\} + [\phi] \{u\} \\ &= \left[[\bar{\phi}] \,\vdots\, [\phi] \right] \begin{Bmatrix} \{r'_{op}\} \\ \{u\} \end{Bmatrix} \\ &= [\phi_r] \{\xi_r\} \end{aligned}\right\} \qquad (6.161)$$

where:

$$[\phi_r] \equiv \left[[\bar{\phi}] \,\vdots\, [\phi] \right] \tag{6.162}$$

$$\{\xi_r\} \equiv \begin{Bmatrix} \{r_{op}\} \\ \{u\} \end{Bmatrix} \tag{6.163}$$

From the foregoing it follows that the eigenvalue problem represented by equation (6. 151) may be replaced by

$$[K_r]\{\xi_r\} = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & [K] \end{array}\right]\{\xi_r\} = \omega^2 \left[\begin{array}{c|c} [M] & \\ \hline & [m] \end{array}\right]\{\xi_r\} \tag{6.164}$$

This formulation introduces the rigid-body degrees of freedom into the problem. The matrix $[K_r]$ has zeros in its first six rows and columns. The first six scalar equations contained in equation (6. 164) may be used to write

$$\omega^2 [M]\{r'_{op}\} = 0$$

This represents the perturbation equations of motion of the airplane as a rigid body in free space without reference motion. If appropriate forces are added to the right-hand members of these equations, the result may be made to correspond to small perturbation equations of motion for an airplane in steady, rectilinear, level flight.

Equation (6. 164) was introduced to emphasize the distinction between the formulation of internal equilibrium, equation (6. 142), and flutter problem formulations. In a flutter problem formulation the displacements $\{dp\}$ in equation (6. 142) are directly replaced by $\{r'_p\}$, i. e. perturbation displacements relative to inertial space. Also, the stiffness matrix $[K_{11}]$ is replaced by the stiffness matrix $[K']$ given by equation (6. 127). This distinction must be clearly recognized to avoid confusion. Rigid-body motions are not usually included in a flutter analysis. The defect of order six in $[K']$ gives rise to six zero values for eigenvalves corresponding to the rigid modes $[\bar{\phi}]$; the corresponding generalized coordinates, however, are given by $\{B\}$. Motion which might be te[illegible] rigid-body motion is the motion of the structural reference point relative to the airplane's c.g. In many cases the characteristics of that motion are nearly identical to the characteristics of the airplane's rigid-body motion. This tends to obscure the distinction between the two formulations.

Internal equilibrium equations appropriate to the completely elastic airplane may now be obtained by introducing equation (6.159) into equation (6.146) to find:

$$[K][\phi]\{u\} = -[\bar{m}][\phi]\{\ddot{u}\} - [\bar{m}][\bar{\phi}]\{R\} + \{F\} \quad (6.165)$$

This expression may be premultiplied by $[\phi]^T$, and equations (6.157) and (6.158) may be used to find

$$[\bar{m}]\{\ddot{u}\} + [\bar{K}]\{u\} = [\phi]^T\{F\} \quad (6.166)$$

These internal equilibrium equations are uncoupled in the left-hand member. The right-hand member represents generalized aerodynamic and thrust perturbation forces:

$$\{Q\} \equiv [\phi]^T\{F\} \quad (6.167)$$

Thus it is seen that Q_i is a "force" acting on the i^{th} free-vibration mode shape.

6.3.4 Generalized perturbation aerodynamic forces. - In the preceding subsection the internal equilibrium equations were transformed by introducing free vibration modes. The result given by equations (6.166) and (6.167) is

$$[\bar{K}]\{u\} = -[\bar{m}]\{\ddot{u}\} + \{Q\} \quad (6.168)$$

The generalized aerodynamic forces were obtained from the aerodynamic forces on the lun· ..: masses, i. e. $\{f\}$. The aerodynamic pressure at the surface of the airplane is represented by the vector $\vec{F}$ in the continuous representation of the internal equilibrium equations. Considering only the perturbation part, the components of the perturbation aerodynamic pressure force on the i^{th} lumped mass are given by

$$\left.\begin{aligned} f_{x_i} &= \int_{V_i} \vec{i} \cdot \vec{F}\, \delta(\vec{r} - \vec{r}_s)\, dV \\ f_{y_i} &= \int_{V_i} \vec{j} \cdot \vec{F}\, \delta(\vec{r} - \vec{r}_s)\, dV \\ f_{z_i} &= \int_{V_i} \vec{k} \cdot \vec{F}\, \delta(\vec{r} - \vec{r}_s)\, dV \end{aligned}\right\} \tag{6.169}$$

For an inviscid fluid the pressure is in the direction normal to the surface denoted by $\vec{n}$. A mean normal to the surface may be defined by:

$$\vec{n}_i = \frac{1}{S_i} \int_{V_i} \vec{n}\, \delta(\vec{r} - \vec{r}_s)\, dV \tag{6.170}$$

Hence the column matrix $\{f\}$ may be written:

$$\{F\} = -\{pn\}$$

and the generalized aerodynamic perturbation forces are:

$$\{Q\} = -[\phi]^T \{pn\} \tag{6.171}$$

As a consequence, the generalized forces may be obtained from aerodynamic influence coefficients. It is shown in app. B that

$$\begin{aligned} \{F\} &= -\{pn\} \\ &= [A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3][\phi]\{u\} \\ &\quad + [A_4][\phi]\{\dot{u}\} + [A_5][\phi]\{\ddot{u}\} \end{aligned} \tag{6.172}$$

where the matrices $[A_1]$, $[A_2]$, $[A_3]$, $[A_4]$, and $[A_5]$ contain the aerodynamic influence coefficients and certain aspects of the airplane geometry in lumped parameter form.

6.3.5 Equations of motion for completely and equivalent elastic airplanes in lumped parameters. - This subsection is essentially a summary and compilation of the preceding. All the formulation of the equations of motion for completely and equivalent elastic airplanes has been carried out. The central results are the momentum equations and the internal equilibrium equations.

The momentum equations were combined into a single matrix expression for either small or large perturbations:

$$[M]\left(\{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{oP}\}\right) = [\bar{\phi}]^T\{F\} \tag{6.173}$$

As shown in app. B, the perturbation aerodynamic forces are:

$$\{F\} = [A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3]\{d_P\} + [A_4]\{\dot{d}_P\} + [A_5]\{\ddot{d}_P\} \tag{6.174}$$

Also, the internal equilibrium equations were given by equation (6.132), i.e.,

$$\begin{aligned}\{d_P\} = -[\bar{C}]\Big[[m]\Big(&\{\ddot{d}_P\} + [\bar{\phi}]\{\dot{V}_P\} + [\bar{\phi}][M_1]\{V_P\} \\ &+ [\bar{\phi}][M_2]\{r'_{oP}\}\Big) - \{F\}\Big]\end{aligned} \tag{6.175}$$

This set of equations represents the equations of motion for the completely elastic airplane. The airplane is taken as n lumped masses, and the effects of rotations of the lumped masses about their centers of gravity are ignored. Thus each lumped mass has three degrees of freedom; these are three translations relative to the c.g. of the airplane and the mean or principal axis system. These are (3n - 6) degrees of freedom (termed elastic degrees of freedom) that enter the problem through the elastic displacements {dp}.

There are three translational and three rotational degrees of freedom for the airplane as a whole. Thus, in total, there are 3n degrees of freedom, and the above equations are 3n in number. They are therefore a determinant set of equations.

The equivalent elastic airplane formulation results by neglecting the generalized inertial forces $[m]\{\ddot{dp}\}$ and the aerodynamic damping and inertia of the structural motion $[\phi]^T[A_4]\{\dot{dp}\}$ and $[\phi]^T[A_5]\{\ddot{dp}\}$. Equation (6.173) remains unchanged except in the expression for aerodynamic perturbation forces. The internal equil brium equation becomes

$$\{dp\} = -[\bar{c}]\Big[[m][\bar{\phi}]\big(\{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{oP}\}\big) - \{F\}\Big] \qquad (6.176)$$

Combining equations (6.173), (6.174), and (6.176) results in the equation of motion for the equivalent elastic airplane:

$$\begin{aligned}
[M]\big(\{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{oP}\}\big) &= [\bar{\Phi}]^T\big[[I] - [A_3][\bar{c}]\big]^{-1}[A_1]\{V_P\} \\
&+ [\bar{\Phi}]^T\big[[I] - [A_3][\bar{c}]\big]^{-1}[A_2]\{\dot{V}_P\} - [\bar{\Phi}]^T\big[[I] - [A_3][\bar{c}]\big]^{-1}[A_3][\bar{c}][m][\bar{\phi}]\{\dot{V}_P\} \\
&- [\bar{\Phi}]^T\big[[I] - [A_3][\bar{c}]\big]^{-1}[A_3][\bar{c}][m][\bar{\phi}][M_1]\{V_P\} \qquad (6.177) \\
&- [\bar{\Phi}]^T\big[[I] - [A_3][\bar{c}]\big]^{-1}[A_3][\bar{c}][m][\bar{\phi}][M_2]\{r'_{oP}\}
\end{aligned}$$

The first two terms on the right of equation (6.177) contain the aerodynamic stability derivatives. The effect of elasticity is introduced into those stability derivatives by including the factor

$$\big[[I] - [A_3][\bar{c}]\big]^{-1}$$

If the airplane is taken to be rigid, the flexibility vanishes and the first two terms of equation (6.177) reduce to the stability derivatives for a rigid airplane. The final terms of equation (6.177) contain stability derivatives related to inertia and gravity force perturbations. These terms vanish for a rigid airplane, e.g. for a rigid airplane or a completely elastic airplane the stability derivative C_{L_θ} does not exist.

This stability derivative is present, however, in the equations of motion of an equivalent elastic airplane. It arises as a result of the change in direction of the gravity force vector relative to the airplane in going from steady-reference flight to perturbed flight. This introduces a change in airplane shape, hence a change in aerodynamic forces. This does not occur in a rigid airplane. In a completely elastic airplane formulation, all elastic shape parameters are held constant during a θ perturbation.

6.3.6 Residual flexibility. - The equivalent elastic airplane has six degrees of freedom, and the perturbation equations of motion are very similar to those used to describe the perturbation motion of a rigid airplane. The sole difference between perturbation equations of motion representing a rigid airplane and those representing an equivalent elastic airplane is in the stability derivatives. Those for the equivalent elastic airplane contain a correction that adjusts the stability derivatives of the rigid airplane to account for the quasi-static elastic deflections of the equivalent elastic airplane. For the completely elastic airplane, additional degrees of freedom are introduced to describe the elastic motion arising from structural dynamics. The method of residual flexibility leads to perturbation equations of motion that combine features of both the equivalent elastic and completely elastic representations. The result is a set of perturbation equations of motion that lead to a more accurate evaluation of the motion of elastic airplanes than may be achieved by either the equivalent elastic or the completely elastic perturbation equations of motion.

Consider the manner in which the elastic airplane is represented. Although it is a continuous body, for computation it is approximated by a large number of lumped masses subject to aerodynamic forces and connected by the elastic structure.

In the equivalent elastic airplane, the only inertial forces considered to act on the lumped masses arise as a consequence of accelerations of the airplane's c.g. Thus the deflected shape of the airplane at any instant is due entirely to the applied aerodynamic forces and the inertial forces resulting from motion of the airplane as a rigid body. This representation neglects all structural dynamics, i.e. the inertial and damping forces due to motion in the elastic degrees of freedom.

The differences between the equivalent elastic and completely elastic airplane representations may be illustrated by a simple example. Consider an airplane clamped at its plane of symmetry, as shown in fig. 10. The airplane is subjected to a sinusoidally varying force of frequency ω at its wingtip. The deflection of the tip for the equivalent elastic airplane is given by

$$\Delta = \frac{1}{K} P_o \sin \omega t \tag{6.178}$$

where K is an elastic constant representing the effective stiffness of the wing. Under these conditions, the deflection is always in phase with the load and in constant proportion with the load. When the load is aerodynamic, the problem is complicated by the fact that the load is dependent on the deflection. However, this complication does not change the essential features illustrated by the example.

When the airplane is completely elastic, the inertia force M must be included (where M is an effective mass of the wing). A differential equation now governs the elastic deflection, i.e.,

$$M\ddot{\Delta} + K\Delta = P_o \omega t \tag{6.179}$$

This differential equation is solved by

$$\Delta = \frac{1}{K} P_o \frac{1}{1 - \frac{\omega^2}{\omega_o^2}} \left(\sin \omega t - \frac{\omega}{\omega_o} \sin \omega_o t\right) \tag{6.180}$$

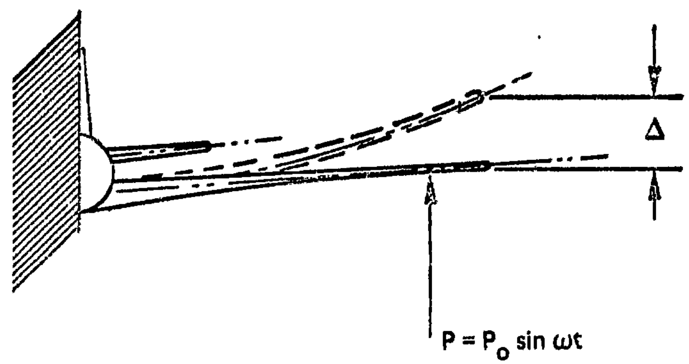
where $\omega_o = \sqrt{\frac{K}{M}}$, the natural frequency of the wing.

Note that, as $M \rightarrow \infty$, $\omega_0 \rightarrow 0$ and the solution tends toward that of the equivalent elastic airplane. Consider what happens when $\omega_0 = 2\omega$. In this case,

$$\Delta = 1.33 \frac{1}{K} P_o \left(\sin \omega t - 2 \sin 2\omega t\right) \tag{6.181}$$

The deflection now exceeds that of the equivalent elastic airplane and is no longer in phase with the applied load. The excessive deflection is referred to as "dynamic overshoot."

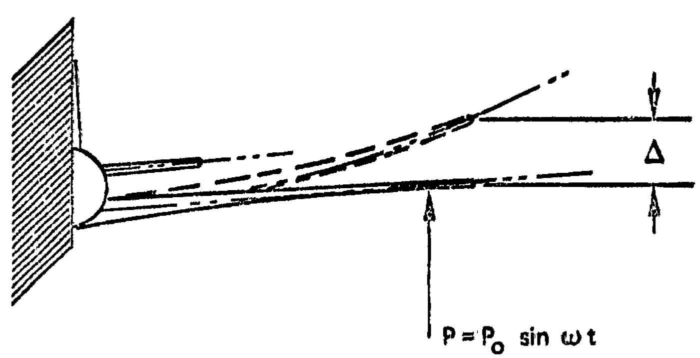
Equivalent elastic airplane



$\Delta = \frac{1}{K} P_o \sin \omega t$

K: Elastic constant

Completely elastic airplane



$M\ddot{\Delta} + \Delta K = P_o \sin \omega t$

$\Delta = \frac{1}{K} P_o \quad \frac{1}{1 - \frac{\omega^2}{\omega_o^2}} \quad (\sin \omega t - \frac{\omega}{\omega_o} \sin \omega_o t)$

ω_o: Natural frequency

FIGURE 10. – SIMPLIFIED MODELS FOR AN EQUIVALENT ELASTIC AND A COMPLETELY ELASTIC AIRPLANE

The effective mass and effective stiffness of the wing depend on the deflected shape. For a continuous wing there is an infinite number of free vibration shapes and associated natural frequencies. However, with the wing represented by n lumped masses and considering motion only in the direction of the applied load, the number of free-vibration mode shapes and frequencies is n. These may be denoted by

$$\omega_i = \sqrt{\frac{K_i}{M_i}} \tag{6.182}$$

where: ω_i = natural frequency of i^{th} free-vibration mode shape

K_i = effective stiffness of i^{th} free vibration mode shape

M_i = effective mass of i^{th} free vibration mode shape.

The deflection of the i^{th} mode shape is given by

$$\Delta_i = \frac{1}{K_i} P_o \frac{R_i}{1 - \frac{\omega^2}{\omega_i^2}} \left(\sin \omega t - \frac{\omega}{\omega_i} \sin \omega_i t \right) \tag{6.183}$$

where R_i is the participation factor for the i^{th} mode with P_o and the total deflection is

$$\Delta = \sum_{i=1}^{n} \Delta_i \tag{6.184}$$

Now assume that the first five natural frequencies are of the order of magnitude of the frequency of the applied load and that all the remaining natural frequencies are higher order frequencies. Then one may write

$$\begin{aligned} \Delta \approx & \sum_{i=1}^{5} \frac{1}{K_i} P_o \frac{R_i}{1 - \frac{\omega^2}{\omega_i^2}} \left(\sin \omega t - \frac{\omega}{\omega_i} \sin \omega_i t \right) \\ & + \sum_{i=6}^{n} \frac{R_i}{K_i} P_o \sin \omega t \end{aligned} \tag{6.185}$$

That is, the total deflection is the sum of a dynamic part and an equivalent elastic part. The latter portion, due to the higher order free-vibration mode shapes, is referred to as the deflection due to residual flexibility.

Recall equation (6.146):

$$[K]\{d_P\} = -[m]\{\ddot{d}_P\} + \{F\} - [m][\phi]\{R\} \tag{6.146}$$

Introducing the free-vibration modes from equation (6.152) on the left-hand side of equation (6.146), premultiplying by $[\phi]^T$, and using the orthogonality relation equation (6.156), results in

$$[\phi]^T[K][\phi]\{u\} = [\phi]^T\left(-[m]\{\ddot{d}_P\} + \{F\}\right) \tag{6.186}$$

The diagonal generalized stiffness, equation (6.157), may be introduced to find:

$$[\bar{K}]\{u\} = [\phi]^T\left(-[m]\{\ddot{d}_P\} + \{F\}\right) \tag{6.187}$$

To introduce the concept of residual flexibility consider the partitioning:

$$[\phi] = \left[[\phi_1] \,\vdots\, [\phi_2]\right] \tag{6.188}$$

It then follows that:

$$[\bar{K}] = \begin{bmatrix} [\phi_1]^T \\ \hline [\phi_2]^T \end{bmatrix} [K]\left[[\phi_1] \,\vdots\, [\phi_2]\right] = \begin{bmatrix} [\bar{K}_1] & [0] \\ [0] & [\bar{K}_2] \end{bmatrix} \tag{6.189}$$

and

$$\{d_P\} = \left[[\phi_1] \,\vdots\, [\phi_2]\right]\begin{Bmatrix} u_1 \\ \hline u_2 \end{Bmatrix} = [\phi_1]\{u_1\} + [\phi_2]\{u_2\} \tag{6.190}$$

In terms of these partition matrices equation (6.187) may be written:

$$\{u\} = \begin{Bmatrix} u_1 \\ \hline u_2 \end{Bmatrix} = \begin{bmatrix} [\bar{K}_1]^{-1} & [0] \\ [0] & [\bar{K}_2]^{-1} \end{bmatrix}\begin{bmatrix} [\phi_1]^T \\ \hline [\phi_2]^T \end{bmatrix}\left(-[m]\{\ddot{d}_P\} + \{F\}\right) \tag{6.191}$$

Premultiplying equation (6.191) by $[\phi]$ leads to:

$$\{d_P\} = \left([\phi_1][\bar{K}_1]^{-1}[\phi_1]^T + [\phi_2][\bar{K}_2]^{-1}[\phi_2]^T\right)\left(-[m]\{\ddot{d}_P\} + \{F\}\right) \tag{6.192}$$

The equations of motion in terms of the flexibility matrix $[\bar{C}]$ were given by equation (6.132). Comparing equations (6.132) and (6.192) we see that

$$[\phi_2][K_2]^{-1}[\phi_2]^T\left(-[m]\{\ddot{d}_P\} + \{F\}\right) = \left([\bar{C}] - [\phi_1][\bar{K}_1]^{-1}[\phi_1]^T\right)\left(-[m]\{\ddot{d}_P\} + \{F\}\right) - [\bar{C}][m][\phi]\{R\} \tag{6.193}$$

Introducing the definition

$$[\bar{C}_R] \equiv [\bar{C}] - [\phi_1][\bar{K}_1]^{-1}[\phi_1]^T \qquad (6.194)$$

it follows that:

$$[\phi_2][\bar{K}_{22}]^{-1}[\phi_2]^T(-[m]\{\ddot{d}_p\} + \{f\}) = [\bar{C}_R](-[m]\{\ddot{d}_p\}$$

$$+ \{f\}) - [\bar{C}][m][\bar{\Phi}]\{R\} \qquad (6.195)$$

This result is used later in the discussion to avoid the necessity of determining the free-vibration modes $[\phi_2]$.

The term "residual flexibility" follows from the definition of equation (6.194). The "residual flexibility" of the structure is found when the flexibility associated with the generalized elastic displacements of the dynamically included modes, $\{u_1\}$, is subtracted from the total flexibility of the structure $[\bar{C}]$. However, it is important to recall that the flexibility represented by $[\bar{C}]$ is different from that usually termed the flexibility of the structure. It is defined by equation (6.133) as

$$[\bar{C}] = \Big[[I] - [\bar{\phi}][M]^{-1}[\bar{\phi}]^T[m]\Big][C_0] \qquad (6.196)$$

Even through the matrix $[C_0]$ has an inverse defined as $[K_{11}]$, the flexibility matrix $[\bar{C}]$ does not possess an inverse. When multiplied into a set of self-equilibrating froces, $[\bar{C}]$ yields a set of meaningful deflections. However, the first column of $[\bar{C}]$ cannot be regarded as the structural deflections resulting from a unit load applied at the first structural mode, as in the case of $[C_0]$.

The formulation of the equations of motion for residual flexibility results by assuming that the inertial loads represented by $[\bar{M}_2]\{u_2\}$ are small enough to be ignored. The only intrinsic structural effect that resists the deflections $\{u_2\}$ will be that represented by the residual flexibility.

The above result is incomplete because no consideration has been given to the dependence of the aerodynamic and thrust peturbation forces $\{f\}$ on the elastic displacements. Recall equation (6.172), i.e.,

$$\{F\} = [A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3][\phi]\{u\}$$

$$+ [A_4][\phi]\{\dot{u}\} + [A_5][\phi]\{\ddot{u}\} \qquad (6.197)$$

Introduce the partitioned mode shape matrix and write equation (6.197) as

$$\{F\} = [A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3]\Big[[\phi_1]\{u_1\} + [\phi_2]\{u_2\}\Big]$$
$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_4][\phi_2]\{\dot{u}_2\}$$
$$+ [A_5][\phi_1]\{\ddot{u}_1\} + [A_5][\phi_2]\{\ddot{u}_2\} \tag{6.198}$$

For the residual flexibility formulation, let

$$[A_4][\phi_2]\{\dot{u}_2\} \simeq 0 \tag{6.199}$$

and

$$[A_5][\phi_2]\{\ddot{u}_2\} \simeq 0 \tag{6.200}$$

so that equation (6.198) may be replaced by the approximate relationship

$$\{F\} = [A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3][\phi_1]\{u_1\} + [A_3][\phi_2]\{u_2\}$$
$$+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\} \tag{6.201}$$

Expand equation (6.191) for $\{u_2\}$ using equation (6.154) and

$$[m_2]\{\ddot{u}_2\} \simeq 0$$

with the result

$$\{u_2\} = [\bar{K}_2]^{-1}[\phi_2]^T\{F\} \tag{6.202}$$

Similarly, using the same approximation as above, equation (6.195) becomes

$$[\phi_2][\bar{K}_2]^{-1}[\phi_2]^T\{F\} = [\bar{C}_R]\Big[-[m][\phi_1]\{\ddot{u}_1\} + \{F\}\Big]$$
$$-[\bar{C}][m][\bar{\phi}]\{R\} \tag{6.203}$$

Combining equations (6.201) through (6.203) results in

$$
\begin{aligned}
\{F\} = \Big[[I] - [A_3][\bar{C}_R]\Big]^{-1} \Big[&[A_1]\{V_P\} + [A_2]\{\dot{V}_P\} + [A_3][\phi_1]\{u_1\} \\
&- [A_3]\Big([\bar{C}][m][\bar{\phi}]\{R\} + [\bar{C}_R][m][\phi_1]\{\ddot{u}_1\}\Big) \\
&+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}\Big]
\end{aligned}
\tag{6.204}
$$

Equation (6.204) describes the perturbation aerodynamic forces acting on the airplane in the residual flexibility formulation. It is now necessary to obtain the appropriate form for the internal equilibrium equations. Recalling equation (6.146), one may introduce the partitioned mode shape matrix of equation (6.188) to write:

$$
[K]\Big[[\phi_1] \,\vdots\, [\phi_2]\Big]\begin{Bmatrix}\{u_1\}\\ \{u_2\}\end{Bmatrix} = -[m]\Big[[\phi_1] \,\vdots\, [\phi_2]\Big]\begin{Bmatrix}\{\ddot{u}_1\}\\ \{u_2\}\end{Bmatrix} - [m][\bar{\phi}]\{R\} + \{F\}
\tag{6.205}
$$

Premultiplying this expression by the transpose of the partitioned mode matrix, it follows from equation (6.155) and equation (6.167) that:

$$
[\bar{K}_1]\{u_1\} = -[\bar{m}]\{\ddot{u}_1\} + \{Q_1\}
\tag{6.206}
$$

where:

$$
\{Q_1\} = [\phi_1]^T\{F\}
\tag{6.207}
$$

Equations (6.118), (6.204), and (6.206) represent the equations of motion for the residual flexibility representation of the airplane. Their expanded forms are:

$$
\begin{aligned}
[M]\Big(\{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{OP}\}\Big) = [\bar{\phi}]^T \Big[[I] - [A_3][\bar{C}_R]\Big]^{-1}\Big[&[A_1]\{V_P\} \\
&+ [A_2]\{\dot{V}_P\} + [A_3][\phi_1]\{u_1\} - [A_3][\bar{C}_R][m][\phi_1]\{\ddot{u}_1\} \\
&- [A_3][\bar{C}][m][\bar{\phi}]\Big([\dot{V}_P] + [M_1]\{V_P\} + [M_2]\{r'_{OP}\}\Big) \\
&+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}\Big]
\end{aligned}
\tag{6.208}
$$

$$
\begin{aligned}
[\bar{m}_1]\{\ddot{u}_1\} + [\bar{K}_1]\{u_1\} = [\phi_1]^T\Big[[I] - [A_3][\bar{C}_R]\Big]^{-1}\Big[&[A_1]\{V_P\} \\
&+ [A_2]\{\dot{V}_P\} + [A_3][\phi_1]\{u_1\} - [A_3][\bar{C}_R][m][\phi_1]\{\ddot{u}_1\} \\
&- [A_3][\bar{C}][m][\bar{\phi}]\Big(\{\dot{V}_P\} + [M_1]\{V_P\} + [M_2]\{r'_{OP}\}\Big) \\
&+ [A_4][\phi_1]\{\dot{u}_1\} + [A_5][\phi_1]\{\ddot{u}_1\}\Big]
\end{aligned}
\tag{6.209}
$$

The similarity of the rigid-body equations of motion for residual flexibility, equation (6.208), to those for equivalent elasticity, equation (6.177), is apparent. If all elastic degrees of freedom are treated as quasi-static, then

$$[\bar{C}_R] \rightarrow [\bar{C}]$$

and equations (6.177) and (6.208) are identical. Also, since the rigid and completely elastic airplane formulations may be obtained from equations (6.208) and (6.209), this set of equations is clearly the most general form. Further, if $[\bar{C}] = 0$,

$$[M]\left(\frac{\partial}{\partial t}\{V_P\} + [M_1]\{V_P\} + [M_2]\{\gamma_P'\}\right) = [\bar{\Phi}]^T\left([A_1]\{V_P\} + [A_2]\frac{\partial}{\partial t}\{V_P\}\right) \tag{6.210}$$

These are the equations of motion for a rigid airplane.

6.3.7 Potential application of residual flexibility theory. - The potential application of residual flexibility theory by stability and control engineers may be discerned by considering certain practical aspects of the preceding analysis. This subsection points out those considerations.

The completely elastic airplane representation is the most precise mathematical model for assessing the dynamic stability of an elastic airplane. However, the limitations of computers that will be available in the foreseeable future for carrying out the numerical computations forbid its use. These limitations are a consequence of the large number of elastic degrees of freedom involved in adequately describing the elastic airplane.

A discussion of the number of free-vibration mode shapes required for the airplane's representation is included in app. C. In that discussion it is pointed out that the stability and control engineer is usually concerned with dynamic participation of only a small number of vibration modes. This follows from the fact that he is primarily concerned with the six-degree-of-freedom motion of the airplane c.g. A free vibration mode participates dynamically with the motion of the airplane c.g. if the natural frequency of the free vibration mode is nearly equal to the frequency of the c.g. motion. The stiffness and mass distribution of most airplanes is such that only a very few free vibration modes have natural frequencies low enough to participate dynamically. However, there is quasi-static motion, due to the higher frequency modes, that can have a significant effect on airplane stability and control.

Without the residual flexibility formulation of the equations of motion, the stability and control engineer is faced with two alternatives. He may include the free vibration modes that contribute the major quasi-static elastic deflections as dynamically participating. Or he may ignore all structural dynamics and base his stability and control analysis on the equivalent elastic airplane representation. Either choice carries a penalty. In the first case, numerical accuracy is lost because of the complexity of the equations of motion. In the second case, the mathematical model does not accurately represent the airplane. Residual flexibility theory provides a middle ground between these two alternatives, including the quasi-static deflections of all elastic modes that do not participate dynamically. Thus residual flexibility theory may be expected to give optimal accuracy in predicting dynamic stability of elastic airplanes.

The sole difference in the equations of motion introduced by including residual flexibility is seen by examining equations (6.208) and (6.209). This difference is represented by inclusion of the square matrix as a factor:

$$\left[[I] - [A_3]^{-1}\left([\bar{C}] - [\phi_1]\,[\bar{K}_1]^{-1}[\phi_1]\right)\right]^{-1} \tag{6.211}$$

where in accordance with equation (6.189):

$$[K_1] = [\phi_1]^T\,[\bar{K}]\,[\phi_1] \tag{6.212}$$

All the matrices contained in this factor must be available for the analysis neglecting residual flexibility. No new information is required. However, the computation of the factor involves the inversion of a matrix of large order. This appears to be the only drawback associated with residual flexibility theory but it seems to be adequately offset by the advantages.

6.3.8. Connection between equivalent elastic airplane stability derivatives and completely elastic airplane stability derivatives. - This section discusses the difference between the stability derivatives appearing in the equations of motion for completely elastic and equivalent elastic airplanes. The theoretical basis of this difference appears in this section and in app. B.

A previous discussion at the beginning of par. 6.3.6 illustrates the motion due to elastic deformation of an airplane for simple loading. This motion was shown to be different for the two airplane representations. It is this difference in the elastic motion which leads to the differences in the stability derivatives appearing in the equations of motion for the two cases. The difference in the elastic motions arises as a consequence of the inertial and damping forces generated by the elastic motions. These forces are neglected in the case of the equivalent elastic airplane. In that case, the elastic deformations are in phase with and in constant proportion with the loads causing the deflections. For the completely elastic airplane, the inertial and damping forces generated by the elastic motion lead to dynamic overshoot (nonconstant proportionality) and a phase difference between the time of maximum deflection and maximum applied load.

Recall the general form of the equations of motion given by equation (6.118):

$$[M]\frac{\partial}{\partial t}\{V_P\} + [M][M_1]\{V_P\} + [M][M_2]\{\gamma_P'\} = [\bar{\phi}]^T\{F\} \tag{6.213}$$

The motion variables are the elements of the column matrix $\{V_p\}$ as given by equation (6.116). They consist of the three components of the perturbation translational velocity of the c.g. and the three components of the perturbation rotational velocity of the airplane about its c.g. The first two terms on the left-hand side represent the perturbation inertial forces on the airplane. The final term on the left is the perturbation gravity force. The right-hand term is the perturbation aerodynamic force.

For a rigid airplane, equation (6.213) may be brought directly into the form of the equations of motion given by Etkin (equations (4.15,7) and (4.15,8), ref. 4). This is done by introducing the stability derivatives for the rigid airplane and neglecting some of them in accordance with the development of ref. 4.

For the completely elastic airplane, additional motion variables must be introduced to include the elastic motion. These enter equation (6.213) through the right-hand term. To see how that occurs, note that the perturbation aerodynamic forces on the airplane panels are given by equation (94) of the summary report as:

$$\{F_A\} = \bar{q}_1 [A_1] \{\psi_P\} + \bar{q}_1 [\delta A] \{\dot{\psi}_P\} + \bar{q}_1 \Bigg\{ \Bigg[2[A_1] + M_1 \frac{\partial A}{\partial M}\bigg|_1 \Bigg] \{\psi_1\} + \Bigg[2[A_{T_1}] + M_1 \left[\frac{\partial A_T}{\partial M}\bigg|_1 \right] \Bigg] \{\psi_T\} \Bigg\} \frac{u}{U_1} \tag{6.214}$$

In keeping with the concept of a stability derivative, all the perturbation variables except one must be set equal to zero in equation (6.214) in order to compute the "stability derivative" corresponding to the nonzero variable. Thus the appropriate equation for the quasi-steady aerodynamic perturbation forces due to elastic perturbation deformation is obtained with $u/U_1 = 0$ and is given by:

$$\{F_A\} = \bar{q}_1 [A_1] \{\psi_P\} \tag{6.215}$$

where the perturbation flow incidence angles are functions of the perturbation elastic deformation only. Following the development of the preceding and the summary report, the perturbation elastic deformation is represented in terms of the free-vibration mode shapes of the airplane. The components of elastic rotation at each panel, ϕ_{E_i}, ϕ_{E_i}, and ψ_{E_i}, defined in app. B, may be computed from each free-vibration mode shape. Denoting some arbitrary free-vibration mode shape by α, the elastic rotations due to a unit amplitude for that mode shape are denoted as $\phi_{E_{i_\alpha}}$, $\theta_{E_{i_\alpha}}$, and $\psi_{E_{i_\alpha}}$. One may then construct column matrices $\{\phi_E\}_\alpha$, $\{\theta_E\}_\alpha$, $\{\psi_E\}_\alpha$, which may be regarded as the mode shapes themselves. Further, in accordance with app. B, it follows that

$$\{\psi_E\}_\alpha = [\psi_\phi] \{\phi_E\}_\alpha + [\psi_\theta] \{\theta_E\}_\alpha + [\psi_\psi] \{\psi_E\}_\alpha \tag{6.216}$$

Letting the amplitudes of the free-vibration mode shape be given by u_α for the arbitrary mode shape denoted by $\{\psi_E\}_\alpha$, the flow incidence angle changes for that mode shape are given by $\{\psi_E\}_\alpha$ u_α. Now, in accordance with equation (6.215), the perturbation aerodynamic forces on the panels due to u_α are:

$$\{f_A\} = \bar{q}_1 [A_1]\{\psi_E\}_\alpha u_\alpha \tag{6.217}$$

These forces are directed along the normals to the panels. A matrix $[n]$ may be defined as:

$$[n] = \begin{bmatrix} n_{x_1} & & & & & \\ & n_{y_1} & & & & \\ & & n_{z_1} & & & \\ & & & n_{x_2} & & \\ & & & & n_{y_2} & \\ & & & & & \ddots \end{bmatrix} \tag{6.218}$$

so that:

$$\{F\} = [n]\{f_A\} \tag{6.219}$$

where $\{f\}$ is the matrix of components of panel perturbation aerodynamic forces defined by equation (6.118). It follows that:

$$\{f\} = \bar{q}_1 [n][A_1]\{\psi_E\}_\alpha u_\alpha \tag{6.220}$$

Finally, in accordance with the right-hand term of equation (6.213), the perturbation aerodynamic forces and moments on the airplane due to u are found as:

$$\bar{q}_1 [\psi]^T [n][A_1]\{\psi_E\}_\alpha u_\alpha$$

where, for example,

$$f_x = \bar{q}_1 \{1\}^T [n_x][A_1]\{\psi_E\}_\alpha u_\alpha \tag{6.221}$$

and

$$m_y = \bar{q}_1 \left(\{z\}^T [n_x] - \{x\}^T [x_y] \right) [A_1] \{\psi_E\}_\alpha u_\alpha \tag{6.222}$$

The "stability derivatives" consist of the multipliers of u_α in a non-dimensional form. Thus, continuing with the examples,

$$C_{x_{u_\alpha}} = \frac{1}{\bar{q}_1 S_W} \frac{\partial F_x}{\partial u_\alpha} = \frac{1}{S_W} \{1\}^T [n_x] [A_1] \{\psi_E\}_\alpha \tag{6.223}$$

and

$$C_{m_{u_\alpha}} = \frac{2}{\bar{q} S_W \bar{C}} \frac{\partial m_y}{\partial u_\alpha} = \frac{2}{S_W \bar{C}} \left(\{z\}^T [n_x] - \{x\}^T [n_z] \right) [A_1] \{\psi_E\} \tag{6.224}$$

Now consider the equations of motion associated with the elastic motion that were derived in the preceding. These are:

$$[\bar{M}] \frac{\partial^2}{\partial t^2} \{u\} + [\bar{K}] \{u\} = - [\phi]^T [m] [\bar{\phi}] \left(\frac{\partial}{\partial t} \{V_P\} + [M_1] \{V_P\} + [M_2] \{r'_{oP}\} \right) + [\phi]^T \{F\} \tag{6.225}$$

where:

$$[\phi]^T [m] [\bar{\phi}] = 0 \tag{6.226}$$

because of orthogonality of the mode shapes. Thus,

$$[\bar{M}] \frac{\partial^2}{\partial t^2} \{u\} + [\bar{K}] \{u\} = [\phi]^T \{F\} \tag{6.227}$$

The first term represents the generalized inertial forces associated with each motion variable representing the elastic motion. The term "generalized force" is appropriate because the use of free-vibration mode shapes has introduced a transformation to the generalized coordinates u_α. The second term involves the generalized stiffness of the airplane structure. The right-hand term gives the generalized aerodynamic forces.

The stability derivatives give the change in the aerodynamic forces and moments acting on the airplane due to small changes in the motion of the airplane. Different stability derivatives must be used, depending on whether the airplane is considered completely or equivalently elastic.

The matrix $[\phi]$ is defined by equation (6.149) as the matrix of free-vibration mode shapes. If δu_α is taken to be a virtual change in the α^{th} elastic generalized coordinate (elastic motion variable), premultiplication of the right-hand member of equation (6.159) by $\{\delta u\}^T$ gives the virtual work of the aerodynamic forces in a virtual elastic deformation of the airplane, i.e.,

$$\text{virtual work} = \{\delta u\}^T[\phi]^T\{f\} \tag{6.228}$$

The first column of $[\phi]$, denoted as $\{\phi\}_1$ of equation (6.149), gives the displacements of the panels due to a unit value of u_1. Thus $\{\phi\}_1{}^T\{F\}$ is the force that does work in the displacement of the first mode. It appears appropriate to define

$$Q_\alpha = \{\phi\}_\alpha^T\{f\} \tag{6.229}$$

as the force associated with an arbitrary free-vibration mode shape.

Recalling equation (6.220), the aerodynamic force component matrix due to the elastic deformation is:

$$\{f\} = \bar{q}_1[n][A_1][\psi_E]\{u\} \tag{6.230}$$

where the rectangular matrix $[\psi_E]$ has columns in accordance with equation (6.219):

$$[\psi_E] = [\{\psi_E\}_1\{\psi_E\}_2 \cdots] \tag{6.231}$$

Also recall equation (152) from the summary report, which gives the aerodynamic pressure forces due to motion of the airplane:

$$\{F_A\} = [A_1]\{V_P\} + [A_2]\frac{\partial}{\partial t}\{V_P\} + [A_3]\{U_P\} + [A_4]\frac{\partial}{\partial t}\{U_P\} + [A_5]\frac{\partial^2}{\partial t^2}\{U_P\} \tag{6.232}$$

Consider only the first and third terms for simplicity. This expression can then be written:

$$\{F_A\} = \bar{q}_1 [A_1][G_1]\{V_P\} + \bar{q}_1 [A_1][G_2]\{U_P\} \tag{6.233}$$

using the notation of equations (151) in the summary report.

The matrix of aerodynamic force components arising from perturbation elastic deformation as well as perturbation motion in the rigid-body degrees of freedom is given by equation (6.233). Comparing the matrices appearing in equation (6.233) with those of equation (6.230), it is possible to write:

$$\{F\} = \bar{q}_1 [n]\Big([A_1][\Psi_E]\{U\} + [A_1][G_2]\{V_P\}\Big) \tag{6.234}$$

Finally, this result may be introduced into equation (6.229) to find the component of force in the "direction" of the arbitrary free-vibration mode shape:

$$Q_\alpha = \bar{q}_1 \{\phi\}_\alpha^T [n]\Big([A_1][\Psi_E]\{U\} + [A_1][G_2]\{V_P\}\Big) \tag{6.235}$$

The "stability derivatives" for elastic motion may now be found as

$$C_{Q_{\alpha u_\beta}} = \frac{1}{\bar{q}_1 S_W}\frac{\partial Q_\alpha}{\partial U_\beta}$$

$$C_{Q_{\alpha \alpha}} = \frac{1}{\bar{q}_1 S_W}\frac{\partial Q_\alpha}{\partial \alpha} = \frac{U_1}{\bar{q}_1 S_W}\frac{\partial Q_\alpha}{\partial w} \tag{6.236}$$

and so on.

Using equation (6.229), the equations of motion for elastic deformation are given by

$$[M]\frac{\partial^2}{\partial t^2}\{u\}+[K]\{u\}=\{Q\} \tag{6.237}$$

However, it has been shown by equation (6.235) that this may be written:

$$[M]\frac{\partial^2}{\partial t^2}\{u\}+[K]\{u\}=\bar{q}_1 S_w\left([C_{Qu}]\{u\}+[C_{QV_p}]\{V_p\}\right) \tag{6.238}$$

Unsteady aerodynamics and aerodynamic effects of displacement rates have been neglected so that there are no generalized damping forces contained in equation (6.238).

For the equivalent elastic airplane, the generalized damping and inertial forces are set equal to zero in equation (6.238). The resulting expression may be solved for column matrix $\{u\}$ to find:

$$\{u\}=\left[[K]-\bar{q}_1 S_w[C_{Qu}]\right]^{-1}[C_{QV_p}]\{V_p\} \tag{6.239}$$

This result may be used in equation (6.235) to find:

$$\{f\}=\bar{q}_1[n]\left[[A_1]\left[[K]-\bar{q}_1 S_w[C_{Qu}]\right]^{-1}[C_{QV_p}]+[A_1][G_2]\right]\{V_p\} \tag{6.240}$$

This result may finally be introduced into equation (6.213) to find the equations of motion for the equivalent elastic airplane.

7. STATIC STABILITY CRITERIA

7.1 Introduction

The purpose of this section is to derive and discuss the static stability criteria for an elastic airplane. These criteria will be shown to follow logically by examining the definitions of the words "static stability" and "static stability criterion" and then applying these definitions. The definitions are intended to provide a precise basis for the ensuing discussion:

- Static stability is here defined as the tendency of the airplane to develop forces or moments that directly oppose an instantaneous disturbance of a motion variable from a steady-state (i.e., equilibrium or trim-state) flight condition.

  For example, when the nose of an airplane is raised relative to the flight path and as a result the airplane develops a nose-down moment, the airplane is said to be statically stable for such a disturbance.
- Static stability criterion is here defined as a rule by which steady-state (i.e., equilibrium or trim-state) flight conditions are separated into the categories of stable, unstable, and neutrally stable.
  In another context, the term has been used as a requirement for an arbitrary minimum static margin. For example, the military specification for flying qualities (ref. 10, par. 3.3.1.1.) requires a negative value of C_{m_α} at all times, which implies a positive static margin. In still another interpretation, the civil airworthiness requirements (ref. 44, articles 4b.151-155) associate stability criteria with stick force versus speed behavior.

The reasons for defining static stability criteria in the form given are these:

- The definitions are clear. Judgment and opinion are eliminated as factors.
- The definitions lead directly to important aerodynamic derivatives and show how these are related to the static stability behavior of the elastic airplane.

Note that these definitions are largely independent of notions of stability and stability criteria associated with control force or control surface displacement. Specifically, this report does not deal with:

- Stick-force stability involving surface hinge moments
- Stability as affected by the feel system, including bobweights
- Stability augmentation systems in general.

It is recognized that when control surfaces are allowed to float or when springs or other devices are added, the longitudinal stability derivatives and associated control characteristics can be significantly affected. Such effects have not been discussed directly in this report; however, the discussions on the effects of derivatives are applicable.

The steady-state motion of an airplane was defined as that motion for which speed $\vec{V}_c$, rotational velocity $\vec{\omega}$, and elastic displacement field $\vec{d}$ (exterior shape of the airplane) remain constant with time in a body-fixed axis system (X,Y,Z.). (See fig. 4). Relative to the inertial reference system (X',Y',Z'), the steady-state motion at any time t is completely described by the quantities $\vec{r}_{o_1}'$, $\vec{V}_{c_1}$, $\vec{\omega}_1$, and $\vec{d}_1$

In more common language, steady-state flight is defined as having constant speed, constant rotational velocities, and constant load factor. This type of flight is frequently encountered in straight and level cruise and in steady turns. For the elastic airplane, it is also required that the exterior shape remain constant in steady-state flight.

The momentary position of the center of mass in inertial space, $\vec{r}_{o_1}'$, is not important for calculation of stability behavior.* The state vector components, X_1', Y_1', and Z_1', will therefore not be included in stability considerations, and neutral stability with respect to changes in these motion variables is accepted. A similar statement can be made with respect to heading angle, Ψ_1, attitude angle θ_1, and bank angle Φ_1, which are needed to describe the steady state for zero rotational velocity $\vec{\omega}_1$. In other words, neutral stability is accepted also with respect to Ψ, θ, and Φ.

*There is a small exception in that the atmosphere is not homogeneous, which means that density is a function of Z'. In discussions of static stability, this fact will be ignored.

The state vector (i.e. the steady-state description) of the airplane for discussions of stability is therefore defined as having the components U_1, V_1, W_1, P_1, Q_1, R_1, and components of $\vec{d}_1$. Disturbances from the steady state are described by the components u, v, w, p, q, r, and components of $\vec{d}_p$.

Kolk (ref. 16, p. 2) states that stability "can be defined along and about all axes, and in respect to any parameter one may choose." In applying the definition of static stability here, the "tendency to oppose disturbances" is judged in terms of the instantaneous force and moment behavior of the airplane to disturbances from a steady-state flight condition. In determining which combinations of forces, moments, and disturbances are to be singled out, the following arbitrary rules have been followed:

- Velocity disturbances are initially opposed only by forces.
- Rotational velocity disturbances are initially opposed only by moments.
- Angles of attack and sideslip disturbances obtained by interpreting the velocity disturbances v and w as $\beta \approx v/V_{c_1}$ and $\alpha = w/V_{c_1}$ are initially opposed by moments.

By consistently applying these rules and the definition of static stability to the instantaneous force and moment behavior of an airplane, criteria for static stability evolve. The results are stated in table 7. An airplane will be considered statically stable in a motion variable if it satisfies the corresponding criterion of table 7.

Neutral and unstable criteria follow by deduction. For convenience, each static stability statement in table 7 is accompanied by a statement involving the most important derivative in each case.

Note that the criteria of table 7 are equally valid for both rigid and elastic airplanes. In the formulation of the inequalities of table 7, the behavior of the structure is not important except that structural stability is implied.

Notice also that the criteria expressed in table 7 are expressions of local slope behavior. For that reason they also apply to situations where aerodynamic forces behave in a nonlinear manner. This is important because airplanes in many instances do behave in a nonlinear fashion. Typical examples are stall and pitchup.

TABLE 7.–CRITERIA FOR STATIC STABILITY (STABLE CASE)

FORCES AND MOMENTS \ MOTION DISTURBANCES	u	v	w	β	α	p	q	r
F_x	$\frac{\partial F}{\partial u} < 0$ $C_{D_u} > 0$							
F_Y		$\frac{\partial F_Y}{\partial v} < 0$ $C_{Y_\beta} < 0$						
F_Z			$\frac{\partial F}{\partial w} < 0$ $C_{L_\alpha} > 0$					
M_x				$\frac{\partial M_x}{\partial \beta} < 0$ * $C_{l_\beta} < 0$		$\frac{\partial M_x}{\partial p} < 0$ $C_{l_p} < 0$		
M_Y	$\frac{\partial M_Y}{\partial u} > 0$ * $C_{m_u} > 0$				$\frac{\partial M_Y}{\partial \alpha} < 0$ $C_{m_\alpha} < 0$		$\frac{\partial M_Y}{\partial q} < 0$ $C_{m_q} < 0$	
M_Z				$\frac{\partial M_Z}{\partial \beta} > 0$ $C_{n_\beta} > 0$				$\frac{\partial M_Z}{\partial r} < 0$ $C_{n_r} < 0$

*These stability criteria are important, but do not follow from the definition of static stability as used in this report.

Even though the static stability criteria of table 7 evolve from the definitions and rules selected here, there is a wide variation in importance among the ten criteria tabulated. For example, $\partial M_Y/\partial \alpha$ $(\sim Cm_\alpha)$ is of much greater practical importance than $\partial F_Y/\partial \nu$ $(\sim C_{Y_\beta})$

Note that under the adopted definition of static stability, the partials $\partial M_Y/\partial u$ $(\sim Cm_u)$ and $\partial M_X/\partial \nu$ $(\sim C_{\ell_\beta})$ do not belong in table 7. This implies that for static stability under the current definition, the signs of C_{m_u} and C_{ℓ_β} are not important. However, these derivatives are important in the practical case and will therefore be discussed.

An unusual feature of table 7 is that it includes moment derivatives with respect to rotational velocities. Such derivatives are normally associated with dynamic stability and not with static stability. The reason for their appearance in table 7 must be found in the definition of static stability. The physical justification for including these moment derivatives in static stability considerations is that steady-state flight can actually involve constant rotational velocities.

An important point is the following: table 7 merely states the conditions necessary for static stability as defined herein. This does not imply that static stability is or should be required. Whether or not static stability with respect to a particular motion variable is desirable is a question of handling qualities. It is not the purpose of this report to deal directly with this question. However, there are significant connections between handling quality parameters and static stability criteria which will be pointed out.

7.2 Static Stability Criteria for Speed Disturbances

7.2.1 Forward speed disturbance. -

Criterion 7.1

From table 7, an airplane is statically stable for a forward speed disturbance u if: $\frac{\partial F_x}{\partial u} < 0$

The physical meaning of criterion 7.1 is that, as a consequence of an increase in forward speed u (along the X-axis), a force must be generated that opposes the increase in speed.

The consequences and significance of criterion 7.1 will now be examined in detail. In stability axes:

$$F_{X_S} = F_{A_{X_S}} + F_{T_{X_S}} = C_{X_S} \bar{q} S_W$$
$$= \left(C_{T_{X_S}} - C_D\right) \bar{q} S_W \qquad (7.1)$$

Application of criterion 7.1 yields:

$$\left(C_{T_{X_{S_u}}} - C_{D_u}\right) + \left(C_{T_{X_{S_1}}} - C_{D_1}\right) \frac{2}{V_{c_1}} < 0 \qquad (7.2)$$

If the steady-state flight condition is level, it follows that:

$$\left(C_{T_{X_{S_1}}} - C_{D_1}\right) = 0$$

In that case, the static speed stability criterion reduces to:

$$C_{X_{S_u}} = \left(C_{T_{X_{S_u}}} - C_{D_u}\right) < 0 \qquad (7.3)$$

The subject of static speed stability is treated in many different forms. Examples are Seckel (ref. 13, p. 120) and Etkin (ref. 4, p. 148).

Intuitively, it seems that $C_{X_{S_u}} < 0$ is a desirable characteristic in an airplane. When $C_{X_{S_u}} < 0$ is satisfied, the airplane tends to maintain its speed. In addition, in approximating the phugoid behavior of an airplane, Etkin (ref. 4, p. 148) has shown that $C_{X_{S_u}} < 0$ is needed to ensure a stable phugoid.

An unstable sign of $\partial F_X / \partial u$ is considered undesirable in approach flight. The reason is illustrated in fig. 11, where induced drag is the primary cause of the behavior of F_{X_S} versus speed.

At a fixed throttle setting, the airplane has speed stability in steady-state flight (trimmed) at point B . (Note that the slope of F_{X_S} versus speed is not treated as a partial derivative here.) An increase in speed u leads to a force which tends to slow the airplane down again. Also, an increase in thrust is needed to increase speed, a decrease in thrust to decrease speed.

However, at point A, where many airplanes fly in the approach, the airplane is unstable with respect to speed changes. A decrease in speed leads to a force that tends to slow the airplane down even more. If the airplane has a limited thrust margin at point A (i.e. the difference between thrust available and thrust required is small) or if the throttle response is slow, it is possible to get into a situation from which recovery is possible only by diving. This is of course not reasonable in an approach, and the result may well be a crash. However, Boeing experience has shown that speed stability is not required if good thrust response and pitch control are provided. In particular, when an autothrottle system is provided, speed instability can be artificially masked. As indicated on the figure, this discussion has been concerned with the special case of 1g flight. The balance of pitching moments is usually of great importance in cases where $\partial F_X/\partial u$ produces significant effects. This is especially true in the case of an autothrottle involving pitch effects due to thrust modulation.

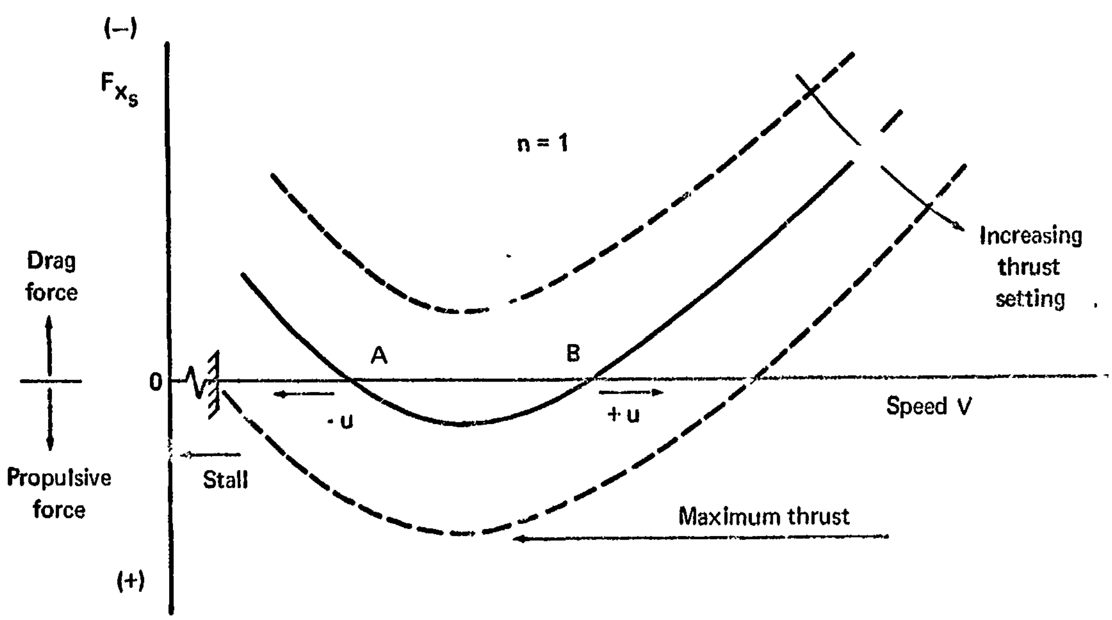


FIGURE 11. – EXAMPLE OF STABLE AND UNSTABLE SPEED BEHAVIOR

On fig. 11, airplanes "stall" at speeds to the left of the minimum point. If the unstable branch of the maximum t ust curve intersects the $F_{X_S} = 0$ line at a speed greater than the stall, the airspeed will diverge and result in a stall unless the pilot dives. The instability accompanying the divergence is usually considered a performance factor and will not be further discussed here.

In cruise flight at high speeds, Mach number effects become important; it is possible to have an adverse sign of $C_{X_{S_u}}$ in the transonic speed range. If it is necessary to fly in this speed range for long periods, an automatic Mach trim compensator and/or an autothrottle system may be used to obtain de facto stability. The effects of elasticity on $C_{X_{S_u}}$ are thought to be very small.

7.2.2 Side speed disturbance. -

Criterion 7.2 From table 7, the airplane is statically stable for a side speed disturbance v if: $\dfrac{\partial F_Y}{\partial v} < 0$

The physical meaning of this criterion is that as a consequence of a side speed disturbance v (along the Y-axis) a force is generated that tends to oppose v. The approximation $v \approx \beta V_{c_1}$ will be used.

Starting from a symmetrical flight condition (zero sideslip) and using stability axes,

$$F_{y_S} = F_{A_{y_S}} = C_y \bar{q} S_W \tag{7.4}$$

This relation assumes that side force effects due to thrust are negligible. Assuming that the side speed disturbance v does not affect dynamic pressure, application of criterion 7.2 yields:

$$C_{y_\beta} < 0 \tag{7.5}$$

Therefore, a requirement for static stability is that the sideforce coefficient be negative. This condition is satisfied by current configurations for angles of sideslip below that where flow separation is important. The military airworthiness requirements of ref. 10 (par. 3.4.8) require inequality (7.5) to be satisfied. The sideforce derivative C_{Y_β} is generally thought to be unimportant in affecting static stability.

However, C_{Y_β} does produce two practical static effects. Sideslip angle is very difficult for the pilot to perceive, and $C_{Y_\beta} < 0$ increases its "visibility" by forcing symmetrical airplanes to bank in steady sideslips (ref. 10, par. 3.4.8 and ref. 44, par. 25.177b and c). It also allows the pilot to perform skidding turns at very low altitude, where bank angle restrictions may have to be observed because of terrain.

In its effect on dynamic stability C_{Y_β} is frequently neglected, as stated by Etkin (ref. 4, p. 167). The derivative in some cases affects damping of lateral oscillations; its capability in dissipating lateral kinetic energy has been illustrated by Roskam (ref. 23, pp. 65-75).

Effects of elasticity on sideslip enter mainly through the vertical tail and the fuselage. Even though the derivative C_{Y_β} is of little importance to basic airplane stability, effects of elasticity are usually accounted for because the data required for so doing are also required for correcting C_{n_β}, as will be discussed in par. 7.3.

7.2.3 Vertical speed disturbance. -

Criterion 7.3

From table 7, the airplane is statically stable for a vertical speed disturbance w if:

$$\frac{\partial F_z}{\partial w} < 0$$

The physical meaning of criterion 7.3 is that as a consequence of a positive velocity disturbance w (along the Z-axis), a force is generated that tends to oppose w. The approximation $w \approx \alpha V_{c_1}$ will be used.

In stability axes:

$$F_{z_s} = F_{A_{z_s}} + F_{T_{z_s}} = C_{z_s}\,\bar{q}\,S_W$$
$$= \left(-C_L + C_{T_{z_s}}\right)\bar{q}\,S_W \tag{7.6}$$

Applying criterion 7.3 and using the fact that $w \approx \alpha V_{c_1}$, while neglecting the effect of w on dynamic pressure, yields:

$$\frac{1}{V_{c_1}}\left(-\frac{\partial C_L}{\partial \alpha} + \frac{\partial C_{T_{z_s}}}{\partial \alpha}\right)\bar{q}\,S_W < 0 \tag{7.7}$$

The variation of $C_{T_{z_s}}$ and α reflects the behavior of the normal force at the inlet of a jet engine or at the propeller disk, as well as the basic change of thrust with angle of attack.

Etkin (ref. 4, p. 69) has shown that for the normal force:

$$C_{T_{z_{s_{NORMAL}}}} = \frac{m'^2}{A_i \rho_i \bar{q} S_W}\left(\alpha_j + \epsilon_j\right) \tag{7.8a}$$

so that, neglecting the change of basic thrust with α,

$$\frac{\partial C_{T_{z_s}}}{\partial \alpha} = \frac{m'^2}{A_i \rho_i \bar{q} S_W}\left(1 + \frac{d\epsilon_j}{d\alpha}\right) \tag{7.8b}$$

Since $d\epsilon_j/d\alpha$ is normally small but positive, criterion 7.3 is certainly satisfied if:

$$C_{L_\alpha} > 0 \tag{7.9}$$

Thus lift curve slope C_{L_α} must be positive for static stability against a disturbance w. This condition is always satisfied for angles of attack below stall. Mach number does affect C_{L_α} strongly, but as long as the flow remains attached, the condition set forth in equation (7.9) is always satisfied. Usually C_{L_α} increases with Mach number in the subsonic speed range and decreases with Mach number in the supersonic speed range. In the transonic speed range C_{L_α} can behave erratically, depending on the configuration.

Lift curve slope has always been recognized as an important derivative. It directly affects the handling qualities of an airplane in two ways: first, in determining the load factor response due to angle of attack (this also has strong implications on the ride qualities of an airplane) and second, in damping the short-period oscillations. The first effect is obvious, as indeed C_{L_α} provides the fundamental means of controlling the flight path in conventional aircraft. The second effect may be seen by inspection from the approximation of short-period damping ratio given by Etkin (ref. 4, p. 211). Replacing C_{L_α} by $V_{c_1} C_{L_w}$, the interpretation as a damping factor is physically clear.

Aeroelastic effects on C_{L_α} can be very large, even to the point of reversing the sign, which is of course undesirable. Because of this, the structure must be such that C_{L_α} sign reversal does not occur inside the flight envelope. Aeroelastic effects generally tend to decrease C_{L_α} for high aspect ratio and highly swept configurations. On delta configurations, aeroelastic effects tend to be weaker and in fact can sometimes cause C_{L_α} to increase rather than decrease.

7.3 Weathercock (β and α) Stability Criteria

7.3.1 Static directional stability. -

Criterion 7.4

From table 7, the airplane is statically (directionally) stable for a sideslip disturbance β if:

$$\frac{\partial M_z}{\partial \beta} > 0$$

The physical meaning of criterion 7.4 is that as a result of an angle of sideslip disturbance β the airplane weathercocks into the new relative wind.

The term "static directional stability" is used because it agrees with conventional usage of the word. Strictly speaking, this usage is not correct because the word "directional" implies heading, but heading stability ($\partial M_Z/\partial\psi < 0$) is not needed in an airplane; in fact, all airplanes have neutral heading stability.

In stability axes:

$$M_{z_S} = M_{A_{z_S}} + M_{T_{z_S}}$$
$$= (C_n + C_{T_n})\,\bar{q}\,S_w b \qquad (7.10)$$

Applying criterion 7.4:

$$C_{n_\beta} + C_{T_{n_\beta}} > 0 \qquad (7.11)$$

If the thrust dependence on sideslip is negligible, then:

$$C_{n_\beta} > 0 \qquad (7.12)$$

It is generally felt that static directional stability is desirable because it gives the airplane the tendency to return to a straight flight path. When the airplane is flying a straight flight path with initial sideslip, the steady-state yawing moment coefficient is nonzero, $C_{n_1} \neq 0$. In that case, the requirement for directional stability is:

$$\left.\frac{\partial C_n}{\partial \beta}\right|_1 > 0 \qquad (7.13)$$

This means that the local slope of C_n versus β must be positive. This is what is required in the military airworthiness requirements (ref. 10, par. 3.4.3-3.4.5), which state the requirement for directional stability in terms of characteristics involving rudder position and rudder force. For conventional rudder control arrangements and effectiveness, this implies inequality (7.12). The civil airworthiness requirements of ref. 44 take a similar position in par. 25.177 but, in addition, require criterion inequality (7.12) to be satisfied.

Inequality (7.13) specifically covers situations involving nonlinear variation of yawing moment with sideslip angle. Such nonlinear variations occur quite often. A typical example is the XB-70A.

Mach number has a strong effect on C_{n_β}. For SST-type configurations, a high Mach number and a large angle of attack can combine to seriously deteriorate C_{n_β}. In such cases, the requirement $C_{n_\beta} > 0$ can be a serious design problem. In recent years it has become a custom to specify a minimum value for some unfavorable combination of Mach number and angle of attack. It is not clear at present whether or not such a requirement should be replaced by a requirement for certain minimum acceptable dynamic response characteristics. One approach, suggested in ref. 13, p. 62, is to specify a minimum rate of dissipation of lateral-directional kinetic energy. This idea is further discussed in par. 7.4 of this report because it is more a matter of dynamic than static stability.

Aeroelastic effects on C_{n_β} can be quite significant. In fact, several current configurations can fly at values of dynamic pressure close to local structural reversal of the vertical tail.

7.3.2 Static longitudinal stability. -

> Criterion 7.5
>
> From table 7, the airplane is statically (longitudinally) stable for an angle-of-attack disturbance α if:
>
> $$\frac{\partial M_y}{\partial \alpha} < 0$$

The physical significance of this criterion is that as a result of an angle-of-attack disturbance α the airplane weathercocks into the new relative wind.

In stability axes:

$$\begin{aligned} M_{y_s} &= M_{A_{y_s}} + M_{T_{y_s}} \\ &= \left(C_m + C_{T_m}\right) \bar{q}\, S_w \bar{c} \end{aligned} \tag{7.14}$$

Applying criterion 7.5 yields:

$$C_{m_\alpha} + C_{T_{m_\alpha}} < 0 \tag{7.15}$$

The sign of $C_{T_{m_\alpha}}$ depends not only on the basic variation of thrust and normal forces with angle of attack, but also on where the engines are located relative to the center of mass.

For several current subsonic transport configurations the effect of power on static longitudinal stability is significant. On current SST configurations the engines are located directly beneath a large lifting surface. Therefore, the effect of α on thrust will be very small so that criterion 7.5 reduces to:

$$C_{m_\alpha} < 0 \tag{7.16}$$

This is the familiar condition for static longitudinal stability.

It is generally felt that static longitudinal stability is desirable because it implies that an airplane, once disturbed from a trim angle of attack, tends to return to its trim angle of attack. A common feeling about a stability criterion such as $C_{m_\alpha} < 0$ is that something disastrous happens in a stepwise manner as the "forbidden" boundary of $C_{m_\alpha} = 0$ is crossed. Such is not the case. Instead, it has been found from both flight and simulator tests that the precision of control and the "forgiveness" of the total system steadily decrease as static longitudinal stability is decreased and goes positive. The degree of pilot attention required increases, and the pilot generally must add lead with positive C_{m_α}, thereby increasing his workload until finally he is no longer able to control the svstem. Boeing SST simulator studies have shown that flight at positive values of C_{m_α} is possible. In this connection it is interesting to observe that the British airworthiness requirements of ref. 11, par. 2.1, specify a maximum allowable negative (unstable) static margin of -0.05.

The tie-in between C_{m_α} and some other important static longitudinal handling qualities parameters is discussed in par. 7.7.

Mach number has a strong effect on C_{m_α}; increasing Mach number generally results in an aft shift of the center of pressure (increases C_{m_α} negatively) in the subsonic speed range. In the supersonic speed range, the variation of C_{m_α} with Mach number is such that it can either decrease or increase depending on configuration. In the transonic speed range, C_{m_α} can behave erratically, again depending on configuration. Aeroelastic effects on C_{m_α} can be quite important and, in fact, can be useful as a design tool in counteracting the effect of Mach number on C_{m_α} on some configurations. Another important consequence of aeroelasticity which has been observed from wind tunnel tests of elastic models is a "straightening out" of C_m versus α curves: when rigid models exhibit nonlinear C_m versus α behavior, corresponding elastic models have almost linear C_m versus α behavior.

7.4 Static Stability Criteria for Rotational Velocity Disturbances

7.4.1 Roll rate disturbance. -

> Criterion 7.6
>
> From table 7, the airplane is statically stable for a disturbance in roll velocity p if:
>
> $$\frac{\partial M_x}{\partial p} < 0$$

The physical meaning of this criterion is that as a result of an increase in rolling velocity p a moment is generated which tends to oppose the increase in rolling velocity.

In stability axes:

$$M_{x_S} = M_{A_{x_S}} + M_{T_{x_S}} \tag{7.17}$$

Neglecting any roll effects on power and noting that:

$$M_{A_{x_S}} = C_l \bar{q} S_w b \tag{7.18}$$

it follows that criterion 7.6 implies that:

$$C_{l_p} < 0 \tag{7.19}$$

The derivative C_{l_p} is recognized as the conventional roll damping derivative. For a rigid airplane without significant flow separation, the condition indicated by equation (7.19) is always satisfied.

Roll damping is an important handling qualities parameter, particularly in rolls and in Dutch roll. The airworthiness requirements of refs. 10 and 44 do not specify sign or minimum values for C_{l_p} directly. Reference 10 does, however, specify roll performance and Dutch roll response requirements.

Mach number can have a fairly strong effect on roll damping, but more so for low sweep angles than for high sweep angles. Aeroelastic effects on C_{l_p} can be strong, particularly in high-aspect-ratio structures.

Roll damping is affected primarily by the planform and in particular the wing, although the vertical tail can also make a significant contribution.

7.4.2 Pitch rate disturbance. -

> Criterion 7.7
>
> From table 7, the airplane is statically stable for a disturbance in pitching velocity q if:
>
> $$\frac{\partial M_y}{\partial q} < 0$$

The physical meaning of this criterion is that as a result of an increase in pitching velocity q a moment is generated which tends to oppose the increase in pitching velocity.

In stability axes:

$$\begin{aligned} M_y &= M_{A_y} + M_{T_y} \\ &= (C_m + C_{T_m})\,\bar{q}\,S_w\,\bar{c} \end{aligned} \tag{7.20}$$

Application of criterion 7.7 therefore yields:

$$C_{m_q} + C_{T_{m_q}} < 0 \tag{7.21}$$

The derivative $C_{T_{m_q}}$ pitch damping due to thrust effects (inlet or propeller disk normal force or jet damping) is normally neglected. This is conservative, since it is seen by equation (7.8a) that $C_{T_{m_q}}$ is usually negative. Neglecting $C_{T_{m_q}}$, inequality (7.21) reduces to:

$$C_{m_q} < 0 \tag{7.22}$$

The derivative C_{m_q} is, of course, the conventional pitch damping derivative. It is very important to handling qualities because together with C_{L_α} it determines the damping of the short-period mode.

Unless flow separation is a factor, condition (7.22) is always satisfied. Pitch damping is affected by Mach number as well as by aeroelastic effects. In both cases the effects are very much configuration-dependent.

7.4.3 Yaw rate disturbance. -

Criterion 7.8

From table 7, the airplane is statically stable for a disturbance in yawing velocity r if:

$$\frac{\partial M_z}{\partial r} < 0$$

The physical meaning of this criterion is that, as a result of an increase in yawing velocity r , a moment is generated which tends to oppose the increase in yawing velocity.

In stability axes:

$$\begin{aligned} M_{z_s} &= M_{A_{z_s}} + M_{T_{z_s}} \\ &= (C_n + C_{T_n})\,\bar{q}\,S_w b \end{aligned} \tag{7.23}$$

Neglecting the effect of thrust, application of criterion 7.8 therefore, yields:

$$C_{n_r} < 0 \tag{7.24}$$

The derivative C_{n_r} is the conventional yaw damping derivative. It is very important in handling qualities because it strongly affects Dutch roll damping. The main contribution to C_{n_r} comes from the vertical tail. The magnitude of C_{n_r} depends strongly on Mach number, angle of attack, and aeroelastic effects. In general, as long as no serious flow separation takes place, condition (7.24) will be satisfied. For high Mach numbers coupled with high angles of attack, it is possible that C_{n_r} deteriorates seriously.

7.5 Discussion of C_{m_u} and C_{ℓ_β}

7.5.1 Pitching moment due to forward speed, C_{m_u}. — Under the definition of static stability used in this report, the partial differential $\partial M_Y/\partial u$ ($\sim C_{m_u}$) does not qualify as a static stability parameter. However, as will be shown, C_{m_u} has important consequences to longitudinal stability from the viewpoint of the pilot. In addition, in much of the literature this parameter is identified with longitudinal stability.

A positive sign of $\partial M_Y/\partial u > 0$ means physically that as a result of an increase in forward speed, the airplane noses up. This tends to slow the airplane down because of the resulting drag increase plus the increase in gravitational pull along the body X-axis. Therefore, an airplane will have stable pitch moment versus speed behavior if:

$$\partial M / \partial u > 0 \tag{7.25}$$

In stability axes:

$$\left.\begin{aligned} M_{Y_S} &= M_{A_{Y_S}} + M_{T_{Y_S}} \\ &= (C_m + C_{T_m})\, \bar{q}\, S_w \bar{c} \end{aligned}\right\} \tag{7.26}$$

Application of inequality (7.25) to equation (7.26) yields:

$$(C_{m_u} + C_{T_{m_u}}) + (C_{m_1} + C_{T_{m_1}})\frac{2}{V_{C_1}} > 0$$

If the steady flight condition is such that $Q_1 = 0$, so that $(C_{m_1} + C_{T_{m_1}}) = 0$, this reduces to:

$$(C_{m_u} + C_{T_{m_u}}) > 0 \tag{7.27}$$

If the thrust passes through the center of mass or if $C_{T_{m_u}}$ is negligible, the condition becomes:

$$C_{m_u} > 0 \tag{7.28}$$

Whether nor not $C_{T_{m_u}}$ is negligible depends strongly on the configuration. For example, $C_{T_{m_u}}$ is not negligible on the 707 series of transports, whereas on the 727 series it is.

The sign and magnitude of the derivative C_{m_u} depend strongly on planform and on Mach number. Aeroelastic effects can also be significant but, in general, no specific trend can be given. In current transport configurations, condition (7.28) is frequently violated because of the aft shift in center of pressure with increasing subsonic Mach number. In that case, the airplane is said to have tuck-under. This characteristic ($C_{m_u} < 0$) causes the airplane to tend toward a dive. If the accompanying consequence is a loss in longitudinal control effectiveness (such as might be the case due to the resulting higher Mach number or aeroelastic effects), the pilot may have difficulty recovering. Whether or not an airplane has satisfactory handling qualities in pitch does <u>not</u> necessarily depend on meeting inequality (7.27), because the behavior of C_{D_u} interacts strongly with C_{m_u}. For example, an unstable C_{m_u} may be acceptable if its effect is checked by a large drag rise.

Most of the current family of transports have rather mild tuck-under. Certifying agencies have significantly differing opinions about this characteristic. The FAA requires complete stick-force speed stability, and this generally leads to incorporation of Mach trim compensators to hide unstable C_{m_u} characteristics from the pilot. The military authorities do not require complete stick-force speed stability (ref. 10, par. 3.3.3). As a consequence, commercial 707 airplanes are equipped with Mach trim compensators, while the military versions (KC-135) do not have these devices. Experience has shown that the KC-135 airplanes handle well in the transonic speed regime.

It may be concluded that mild violations of $C_{m_u} > 0$ are acceptable. Just what is meant by "mild" can only be settled through flight testing.

<u>7.5.2 Dihedral effect (lateral stability) C_{ℓ_β}</u>. — Under the definition of static stability used in this report, the partial differential $\partial M_X/\partial v$ ($\sim C_{\ell_\beta}$) does not qualify as a static stability parameter. Nevertheless, this derivative has an important effect on stability and handling qualities.

In stability axes:

$$
\begin{aligned}
M_{X_S} &= M_{AX_S} + M_{TX_S} \\
&= (C_\ell + C_{T_\ell})\,\bar{q}\,S_w b
\end{aligned}
\qquad (7.29)
$$

Neglecting the effect of thrust and considering $\partial M_X / \partial v < 0$ with $v \approx \beta V_{c_1}$ to be the condition for stability, it follows that

$$C_{\ell_\beta} < 0 \tag{7.30}$$

must be satisfied for stability. The derivative C_{ℓ_β} is sometimes called lateral stability, sometimes dihedral effect.

It has been a long-standing practice to design airplanes with negative dihedral effect, i.e.,

$$C_{\ell_\beta} < 0$$

The physical significance of this is that for a positive sideslip disturbance (nose left), the airplane tends to roll away from the disturbance, i.e. to the left. If the airplane rolls about its stability X-axis as a result of this, this tends to diminish the effective sideslip angle. For this reason some investigators identify C_{ℓ_β} as a lateral stability parameter even though strictly speaking the derivative should not be considered as such. The military flying quality requirement (ref. 10, par. 3.4.7) states that the left aileron force shall be required for left sideslip. For conventional control arrangements this implies that $C_{\ell_\beta} < 0$ must be satisfied.

The derivative C_{ℓ_β} is strongly affected by Mach number, sweep angle, lift coefficient, and configuration. The effect of aeroelasticity is little known, and much research is needed in this area.

It has been found that large negative values of C_{ℓ_β} can be very detrimental to damping of the lateral response characteristics of an airplane. Large sweep angles and large wing dihedrals contribute to negative values of C_{ℓ_β}. It will be shown that under certain simplifying assumptions, $C_{\ell_\beta} < 0$ is needed to keep the spiral mode from being divergent.

7.6 Connections Between Static Stability Parameters and Handling Qualities

The handling qualities aspects to be discussed in this section are those associated with longitudinal control only -- in particular, with C_{m_α} and C_{m_u}.

To the pilot such relationships appear through stick-force-versus-speed and stick-force-per-g behavior. In the discussion that follows, it is assumed that a change in stick force automatically leads to a change in control surface position of the same sign. This makes it possible to eliminate the feel system characteristics from the ensuing discussion.

7.6.1 Control displacement versus speed (constant load factor). - At constant load factor and zero pitch rate, the following expression can be written for moment coefficient:

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_\theta}\theta + C_{T_m} + C_{m_{\delta_E}}\delta_E$$

The quantity C_{m_θ} symbolizes the effect of fuel shifts on C_m due to a change in attitude, while C_{T_m} is the thrust moment coefficient. Even though these quantities are of considerable importance for several subsonic transports, they will be neglected in the following discussion. However, the restriction imposed on the discussion by this simplification should be kept in mind. With the simplification, the moment coefficient can be written:

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_{\delta_E}}\delta_E \tag{7.31}$$

For a trimmed flight condition $C_m = 0$, so that:

$$\delta_{E_{TRIM}} = -\frac{C_{m_0} + C_{m_\alpha}\alpha}{C_{m_{\delta_E}}} \tag{7.32}$$

The parameter of interest, control displacement versus speed, is obtained by differentiating equation (7.32):

$$\left.\frac{d\delta_{E_{TRIM}}}{dV}\right|_{n=1} = -\left\{\frac{\left.\frac{dC_{m_0}}{dV}\right|_{n=1} + C_{m_\alpha}\left.\frac{d\alpha_{TRIM}}{dV}\right|_{n=1} + \alpha_{TRIM}\left.\frac{dC_{m_\alpha}}{dV}\right|_{n=1}}{C_{m_{\delta_E}}}\right\}$$

$$+\left(\frac{C_{m_0} + C_{m_\alpha}\alpha_{TRIM}}{C_{m_{\delta_E}}^2}\right)\left.\frac{dC_{m_{\delta_E}}}{dV}\right|_{n=1} \tag{7.33}$$

For a rigid airplane and negligible Mach effects,

$$\left.\frac{dC_{m_0}}{dV}\right|_{n=1} = \left.\frac{dC_{m_\alpha}}{dV}\right|_{n=1} = \left.\frac{dC_{m_{\delta_E}}}{dV}\right|_{n=1} = 0$$

so that the expression reduces to:

$$\left.\frac{d\delta_{E_{TRIM}}}{dV}\right|_{n=1} \approx -\frac{C_{m_\alpha}}{C_{m_{\delta_E}}} \left.\frac{d\alpha_{TRIM}}{dV}\right|_{n=1} \tag{7.34}$$

Since in a trimmed flight condition, also approximately,

$$L = W = C_L \bar{q} S_W$$
$$= (C_{L_0} + C_{L_\alpha}\alpha + C_{L_{\delta_E}} \delta_E)\,\bar{q}\, S_W \tag{7.35}$$

which yields:

$$\alpha_{TRIM} = \frac{W - (C_{L_0} + C_{L_{\delta_E}} \delta_{E_{TRIM}})\bar{q} S_W}{C_{L_\alpha} \bar{q}\, S_W} \tag{7.36}$$

$$= \frac{W}{\bar{q} S_W C_{L_\alpha}} - \frac{C_{L_0} + C_{L_{\delta_E}} \delta_{E_{TRIM}}}{C_{L_\alpha}}$$

it is found that:

$$\left.\frac{d\alpha_{TRIM}}{dV}\right|_{n=1} \approx \frac{-4W}{\rho S_W C_{L_\alpha} V_{C_1}^3} = \frac{-2C_{L_{TRIM}}}{C_{L_\alpha} V_{C_1}} \tag{7.37}$$

Substituted into equation (7.34), this yields:

$$\left.\frac{d\delta_{E_{TRIM}}}{dV}\right|_{n=1} \approx \frac{C_{m_\alpha}}{C_{m_{\delta_E}}} \cdot \frac{2C_{L_{TRIM}}}{C_{L_\alpha} V_{C_1}} \tag{7.38}$$

Definition

A stable gradient of elevator displacement versus speed is one for which

$$\left.\frac{d\delta_{E_{TRIM}}}{dV}\right|_{n=1} > 0$$

Because control power $C_{m_{\delta_E}}$ is usually arranged to be negative, it follows that $(d\delta_{E_{trim}}/dV)\,|n = 1$ will have a "stable" gradient if $C_{m_\alpha} < 0$. Figure 12 illustrates such a stable gradient. It is seen that these simplified relations connect the static longitudinal stability parameter C_{m_α} directly to the handling quality parameter $(d\delta_E/dV)\ |\ n = 1$. At least in smooth air, this is one way for the pilot to judge the stability of an airplane. Note that for $C_{m_\alpha} = 0$, which means that the c.g. and aerodynamic c nter coincide, no elevator change is required for a change in speed. For this reason $(d\delta_E/dV)\ |$ n = 1 (as evidenced to the pilot through $(dF_S/dV)\ |$ n = constant) has been strongly identified with longitudinal stability.

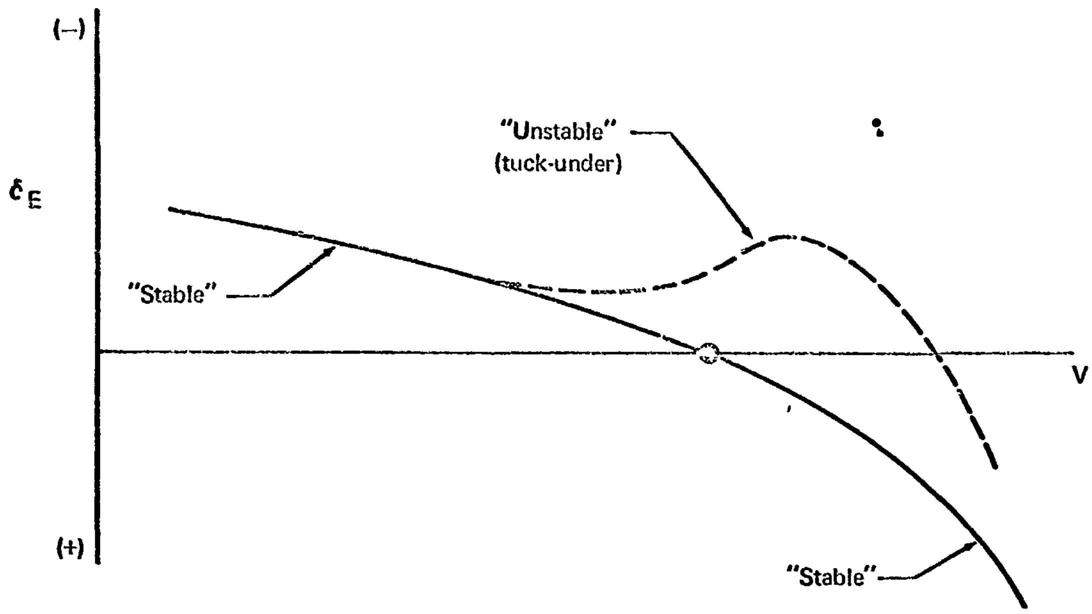


FIGURE 12. – EXAMPLES OF ELEVATOR DEFLECTION VERSUS SPEED GRADIENTS

For an elastic airplane or an airplane flying at transonic speed the variations of C_{m_o}, C_{m_α}, $C_{m_{\delta_E}}$ with speed can be very large and should no longer be neglected. In such cases the complete relation (7.33) should be used.* However, the relation between elevator δ_E and speed V for an elastic airplane is a very complicated one.

For an elastic airplane the derivative C_{m_α} evaluated at constant load factor is not the same as C_{m_α} evaluated at constant speed. The difference is caused by inertial effects (called inertia relief in the case of airplanes with conventional tail arrangements). For calculating the elevator-versus-speed relation at constant load factor, the inertia relief must be left out. A similar comment applies to $C_{m_{\delta_E}}$, although experience has shown that the effect of inertia on $C_{m_{\delta_E}}$ is very small. This is the case even on the B-2707 (SST).

*In fact, the effects of thrust and fuel displacement should also be accounted for.

It can be shown that in equation (7.33) the expression

$$\left\{ \frac{d}{dV}(C_{m_o}) \Big|_{n=1} + \alpha \frac{d}{dV}(C_{m_\alpha}) \Big|_{n=1} \right\};$$

approximately represents the speed derivative C_{m_u}. A positive sign of C_{m_u} has a stabilizing effect on $(d\delta_E/dV) \mid n = 1$. The converse is also true in that a significant sign reversal in C_{m_u} (tuck-under), can cause $(d\delta_E/dV) \mid n = 1$ to change sign. This is illustrated in fig. 12 by the dotted line. Equation (7.33) also shows that a positive sign of $(dC_{m_{\delta_E}}/dV) \mid n = 1$ is detrimental to a stable gradient $(d\delta_E/dV) \mid n = 1$, but this effect is masked by a decrease in $C_{m_{\delta_E}}$ itself. A typical relationship between control power $C_{m_{\delta_E}}$ and speed V is illustrated in fig. 13. Largely owing to aeroelastic effects, $C_{m_{\delta_E}}$ tends toward zero, resulting in steepening of the gradient $(d\delta_E/dV) \mid n = 1$.

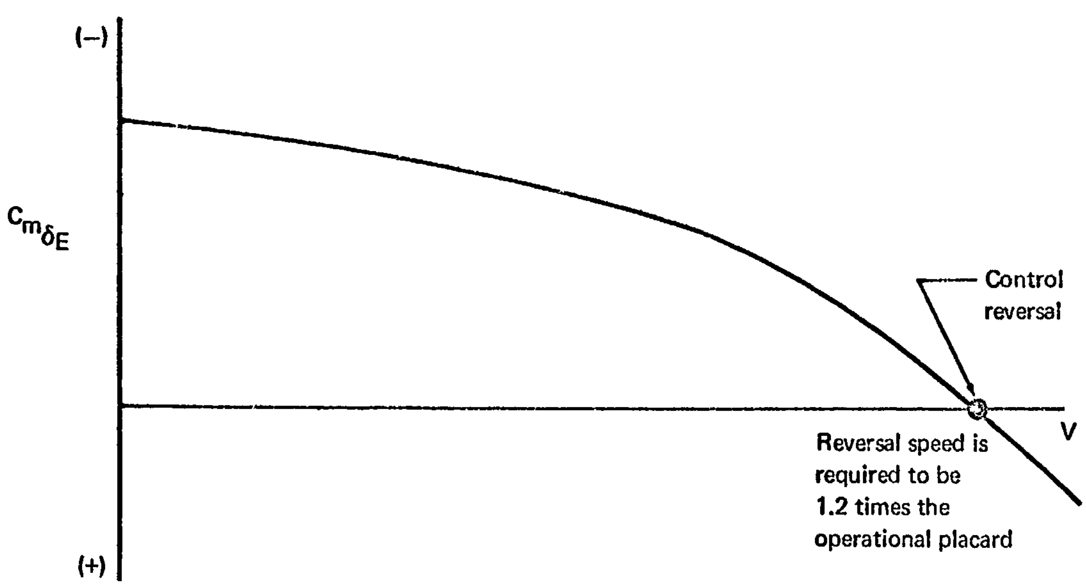


FIGURE 13. – EXAMPLE OF AEROELASTIC DEGRADATION OF CONTROL EFFECTIVENESS

The discussi[illegible] stability and control at constant load factor. A paramet[illegible] e considered in conjunction with $(d\delta_E/dV) \mid n = 1$ is [illegible] control required per g.

The pilot may not object to a mild sign change in $(d\delta_E/dV) \mid n = 1$, provided the airplane retains the correct (stable) gradient of control displacement per g. Some aspects of the latter parameter are discussed in the next paragraph.

7.6.2 Control displacement versus load factor (constant speed). - Using the same simplified relationship between moment coefficient and control angle as in par. 7.6.1, but accounting for pitch damping C_{m_q}, it is found that:

$$\delta_{E_{TRIM}} = - \frac{C_{m_o} + C_{m_\alpha}\alpha + C_{m_q}\frac{Q_1 \bar{c}}{2V_{C_1}}}{C_{m_{\delta_E}}} \tag{7.39}$$

In a steady symmetrical pullup the following relationship is found between pitching velocity Q_1 and load factor n :

$$Q_1 = \frac{(n-1)g}{V_{C_1}} \tag{7.40}$$

From equations (7.39) and (7.40), the control displacement versus load factor gradient at constant speed is found by differentiation:

$$\frac{d\delta_{E_{TRIM}}}{dn} = \left\{ \frac{\left.\frac{dC_{m_o}}{dn}\right|_{V_{C_1}} + C_{m_\alpha}\left.\frac{d\alpha_{TRIM}}{dn}\right|_{V_{C_1}} + \alpha\left.\frac{dC_{m_\alpha}}{dn}\right|_{V_{C_1}} + C_{m_q}\frac{g\bar{c}}{2V_{C_1}^2}}{C_{m_{\delta_E}}} \right\}$$

$$+ \left\{ \frac{C_{m_o} + C_{m_\alpha}\alpha_{TRIM} + C_{m_q}\frac{(n-1)g\bar{c}}{2V_{C_1}^2}}{C_{m_{\delta_E}}^2} \right\} \left.\frac{dC_{m_{\delta_E}}}{dn}\right|_{V_{C_1}} \tag{7.41}$$

For a rigid airplane the coefficients do not change with load factor, and this results in:

$$\left.\frac{d\delta_{E_{TRIM}}}{dn}\right|_{V_{C_1}} = - \left(\frac{C_{m_\alpha}\left.\frac{d\alpha_{TRIM}}{dn}\right|_{V_{C_1}} + C_{m_q}\frac{g\bar{c}}{2V_{C_1}^2}}{C_{m_{\delta_E}}} \right) \tag{7.42}$$

Assuming that the following relation holds approximately:

$$nW = C_L \bar{q} S_W \tag{7.43}$$

it is seen that:

$$\left.\frac{dC_L}{dn}\right|_{V_{c_1}} = \frac{W}{\bar{q} S_W} = C_{L_{TRIM}} \tag{7.44}$$

Substitution into equation (7.42) yields:

$$\left.\frac{d\delta_{E_{TRIM}}}{dn}\right|_{V_{c_1}} = \left(\frac{C_{m_\alpha} \left.\frac{C_{L_{TRIM}}}{C_{L_\alpha}}\right|_{V_{c_1}} + C_{m_q} \frac{g\bar{c}}{2V_{c_1}^2}}{C_{m_{\delta_E}}} \right) \tag{7.45}$$

Definition

A stable gradient of elevator displacement versus load factor is one that satisfies:

$$\left.\frac{d\delta_{E_{TRIM}}}{dn}\right|_{V_{c_1}} < 0 \tag{7.46}$$

From equation (7.45) it follows that the above definition is satisfied when

$$C_{m_\alpha} < \frac{-C_{m_q} \frac{g\bar{c}}{2V_{c_1}^2} \left. C_{L_\alpha}\right|_{V_{c_1}}}{C_{L_{TRIM}}} = -\frac{C_{m_q} \left. C_{L_\alpha}\right|_{V_{c_1}} \bar{c} \rho S_W}{4M} \tag{7.47}$$

The center of mass for which the elevator per g is zero is called the maneuver point. It coincides with the aerodynamic center for negligible pitch damping. From relation (7.47) it can be seen that at altitude the term on the right becomes less important, indicating that maneuver point and neutral point approach each other. Thus, once more a direct relation is established between a handling quality parameter and static longitudinal stability C_{m_α}.

Note that:

- For a rigid airplane without pitch damping, a stable gradient $(d\delta_E/dn) \mid V_{c_1}$ implies a positive static margin. With pitch damping, it implies a positive maneuver margin.
- For a rigid airplane with negligible pitch damping term or flying at high altitude, $C_{m_\alpha} < 0$ is both necessary and sufficient as a requirement for the control gradients $(d\delta_E/dn) \mid V_{c_1}$ and $(d\delta_E/dV) \mid n = 1$ to be stable.
- For a rigid airplane in the transonic range, $C_{m_\alpha} < 0$ assures the gradient $(d\delta_E/dn) \mid V_{c_1}$ to be stable, but for the gradient $(d\delta_E/dV) \mid n = 1$ to be stable, the additional requirement $C_{m_u} > 0$ is needed.
- For an elastic airplane, the situation is more complicated. The gradient elevator per g can depend strongly on the elasticity of the structure, which determines the values of $(dC_{m_o}/dn) \mid V_{c_1}$ and $(dC_{m_\alpha}/dn) \mid V_{c_1}$. It is evident from equation (7.41) that when aeroelastic effects cause $C_{m_{\delta_E}}$ to approach zero, the elevator per g gradient steepens considerably.

7.7 Summary of Static Stability Criteria

In this section the static stability criteria of an airplane have been shown to evolve logically from the definition of static stability and static stability criteria given at the beginning of par. 7.1.

The question of whether or not certain forms of static stability are desirable was referred to handling qualities, but some aspects of this question were briefly discussed. The physical significance of stability derivatives appearing in the static stability criteria has been discussed. A summary of static stability criteria is presented in table 8. The viewpoints expressed as requirements (criteria) by military and civil flying quality specifications (refs. 10, 11, and 44) are included in this table.

TABLE 8.-SUMMARY OF STATIC STABILITY CRITERIA

GENL. FORM OF STATIC STAB. CRITERION	APPROXIMATE OR ALTERNATE FORM	IMPORTANCE TO HANDLING QUALITIES	MIL-F-8785, REF. 10
$\frac{\partial F_x}{\partial u} < 0$	$C_{x_{s_u}} < 0$ or $C_{D_u} > 0$ (No thrust effect)	Needed for stable phugoid. Not important if throttle response is good.	Par. 3.3.6 limits phugoid divergence. No direct requirement.
$\frac{\partial F_Y}{\partial v} < 0$	$C_{Y_\beta} < 0$ (No thrust effect)	Helps pilot perceive sideslip. Allows skidding turns at low altitude (wings level).	Pars. 3.4.3 and 3.4.8 interpreted to mean $C_{Y_\beta} < 0$.
$\frac{\partial F_Z}{\partial w} < 0$	$C_{L_\alpha} > 0$	Primary means for flight path control. Significant to short period. Always satisfied before stall.	Par. 3.3.3* specifies short period requirements. No direct requirement.
$\frac{\partial M_Z}{\partial \beta} > 0$	$C_{n_\beta} > 0$	Needed to maintain straight flight path.	Pars. 3.4.3, 3.4.4, 3.4.5 interpreted to mean $C_{n_\beta} > 0$.
$\frac{\partial M_Y}{\partial \alpha} < 0$	$C_{m_\alpha} < 0$	Affects time history of pitch response. $C_{m_\alpha} \simeq 0$ can be tolerated on large A/P. Affects stick force behavior.	Par. 3.3.1 interpreted to mean $C_{m_\alpha} < 0$
$\frac{\partial M_X}{\partial p} < 0$	$C_{l_p} < 0$	Affects time history of roll response. Affects Dutch roll damping.	Par. 3.4.1* specifies Dutch roll requirement. Par. 3.4.16 specifies roll performance.
$\frac{\partial M_Y}{\partial q} < 0$	$C_{m_q} < 0$	Affects damping of short period (increases pitch stiffness).	Par. 3.3.5* specifies short period requirements. No direct requirement.
$\frac{\partial M_Z}{\partial r} < 0$	$C_{n_r} < 0$	Affects Dutch roll damping (increases yaw stiffness).	Par. 3.4.1* specifies Dutch roll requirements. No direct requirement.

*MIL-F-8785 recognizes augmentation-on and -off cases. This document deals only with unaugmented cases.

TABLE 8.--SUMMARY OF STATIC STABILITY CRITERIA (CONTINUED)

GENL. FORM OF STATIC STAB. CRITERION	APPROXIMATE OR ALTERNATE FORM	IMPORTANCE TO HANDLING QUALITIES	MIL-F-8785, REF. 10
$\frac{\partial M_y}{\partial u} > 0$	$C_{m_u} > 0$	Improves speed control. Warns of inadvertent over (under) speed. Affects stick force behavior.	No direct requirement, but par. 3.3.3 implies that violation is allowed transonically.
$\frac{\partial M_x}{\partial v} < 0$	$C_{l_\beta} < 0$	Warns of sideslip, Allows emergency roll control. Affects Dutch roll.	Par. 3.4.3, 3.4.6, and 3.4.7 interpreted to mean $C_{l_\beta} < 0$.

TABLE 8.–SUMMARY OF STATIC STABILITY CRITERIA (CONCLUDED)

GENL. FORM OF STATIC STAB. CRITERION	APPROXIMATE OR ALTERNATE FORM	FAR—PART 25, REF. 44	BRITISH CAR, SECTION D, REF. 11
$\frac{\partial F_x}{\partial u} < 0$	$C_{X_{S_u}} < 0$ or $C_{D_u} > 0$ (No thrust effect)	No direct requirement.	No direct requirement.
$\frac{\partial F_Y}{\partial v} < 0$	$C_{Y_\beta} < 0$ (No thrust effect)	Par. 25.177 (c) interpreted to mean $C_{Y_\beta} < 0$.	Par. 7.3 interpreted to mean $C_{Y_\beta} < 0$.
$\frac{\partial F_z}{\partial w} < 0$	$C_{L_\alpha} > 0$	No direct requirement.	No direct requirement.
$\frac{\partial M_z}{\partial \beta} > 0$	$C_{n_\beta} > 0$	Par. 25.177 (a) interpreted to mean $C_{n_\beta} > 0$.	Par. 7.2 interpreted to mean $C_{n_\beta} > 0$.
$\frac{\partial M_Y}{\partial \alpha} < 0$	$C_{m_\alpha} < 0$	No direct requirement, but pars. 25.173 & 25.175 interpreted to mean $C_{m_\alpha} < 0$.	Par. 2.1 requires $\frac{dC_m}{dC_L} < +0.05$.
$\frac{\partial M_x}{\partial p} < 0$	$C_{l_p} < 0$	No direct requirement.	No direct requirement.
$\frac{\partial M_Y}{\partial q} < 0$	$C_{m_q} < 0$	No direct requirement, but par. 25.181 requires all short periods to be heavily damped.	No direct requirement, but par. 8.1 requires all short periods to be heavily damped.
$\frac{\partial M_z}{\partial r} < 0$	$C_{n_r} < 0$	No direct requirement, but par. 25.181 requires all short periods to be heavily damped.	No direct requirement, but par. 8.1 requires all short periods to be heavily damped.
$\frac{\partial M_Y}{\partial u} > 0$	$C_{m_u} > 0$	P 25.175 (c) implies that violation is not allowed.	Par. 31.2 implies that violation is not allowed.
$\frac{\partial M_x}{\partial v} < 0$	$C_{l_\beta} < 0$	Par. 25.177 (b) interpreted to mean $C_{l_\beta} < 0$.	Par. 7.1 interpreted to mean $C_{l_\beta} < 0$.

8. DYNAMIC STABILITY CRITERIA

8.1 Introduction

This section presents dynamic stability criteria for rigid and elastic airplanes. The majority of current airplane dynamic stability analyses are for controlled airplanes and employ the root locus method of analysis, based on linear theory (i.e., linear approximation of the equations of motion). This report, however, deals only with the uncontrolled (controls fixed) airplane. The subject of dynamic stability is here treated from a general viewpoint. This means that methods of dynamic stability analysis other than those based on linear theory will be examined. The definitions of dynamic stability and dynamic stability criteria to be used are stated below.

<u>Definition</u>

Dynamic stability is the tendency of the amplitudes of the perturbed motion of an airplane to decrease to zero or to values corresponding to a new steady state at some time after the disturbance has stopped.

For example, when the airplane is disturbed in pitch from steady-state flight and the resulting perturbed motion is damped out after some time, although the new steady state is not significantly different from the original one, the airplane is called dynamically stable. The example and the definition indicate that the subject of dynamic stability deals with the behavior of the perturbed motion of an airplane about some steady-state flight path.

<u>Definition</u>

A dynamic stability criterion is a rule by which perturbed motions are separated into the categories of stable, neutrally stable, or unstable.

In other context dynamic stability criteria have been interpreted as requirements for specific response characteristics or for meeting specific frequency damping relations. This type of interpretation is embodied in the military specification for flying qualities (ref. 10) and its proposed revision as documented in ref. 12. The flying qualities specifications of ref. 10 and

12 are here viewed as handling qualities criteria; as such, they are beyond the scope of this report. However, there are important connections between dynamic stability criteria (viewed as mathematical statements of stability) and the handling quality criteria of refs. 10 and 12. Therefore, where needed for physical interpretation of the stability criteria established in this report, the connections with handling qualities are pointed out and discussed.

The static stability criteria evolve from application of the definition of static stability to the instantaneous forces and moments. For dynamic stability criteria, such a development is not possible. Dynamic stability is associated with the response behavior of an airplane as a result of disturbances. Because this response behavior is expressed by differential equations of motion, the study of dynamic stability behavior of airplanes relies heavily on the theory of stability of differential equations. The differential equations of motion of an airplane can be cast in many different forms, and the form selected in a particular case depends on the similarity of the mathematical model to the real physical problem. Differential equations of motion of an airplane can be linear, nonlinear, autonomous, or nonautonomous. In each case, the corresponding theory of stability is, or can be, different as will become clear from the developments to follow.

Experience has shown that in many cases the dynamic behavior of airplanes can be satisfactorily represented by assuming that perturbations away from steady-state flight are small. In that case, the equations of motion can be approximated by a set of linear second-order differential equations with constant coefficients. These equations are called small perturbation equations. The stability theories most commonly associated with these equations are called characteristic equation methods.

In general, it can be said that linear approximation methods have given satisfactory results in representing airplane dynamic behavior. In other words, it has been found that when airplanes satisfy stability and/or handling qualities criteria based on such approximations, their real-life dynamic characteristics are roughly as predicted. There are important exceptions, however. For example, the mildly divergent Dutch roll behavior of the

Boeing 727 at altitudes above 26,000 feet was not predicted by the linear theory. Also, as shown in ref. 23, it is possible for certain slender aircraft configurations to exhibit significant nonlinear behavior even when the aerodynamic forces are assumed to be linear. Finally, there are cases where nonlinear aerodynamic behavior is important. An example is the nonlinear variation of directional stability with sideslip, as found on the XB-70A.

It is apparent that there is an increasing number of cases where linearization of the equations of motion is no longer permissible. For that reason, it was felt necessary to include in this report several more generally valid stability criteria.

Perhaps the most general way of determining stability behavior of airplanes with significant nonlinear effects is a brute-force integration of the complete (nonlinear) equations of motion. Such integration results in time histories of motion.

A general description of time history generation (integration) and the corresponding stability criteria are presented in par. 8.3.

Some dynamic stability criteria which are based on an energy decay method are presented in par. 8.4. In this case no particular form is required for the equations of motion, although the perturbed form (linear or nonlinear) seems to be preferred.

Paragraph 8.5 presents a dynamic stability criterion, based on Lyapunov's stability theory, which applies to nonlinear as well as linear differential equations of motion.

Finally, par. 8.6 provides a summary of dynamic stability criteria.

8.2 Dynamic Stability Criteria Based on Characteristic Equation Methods

When airplane dynamic behavior can be approximated by assuming that motion perturbations (excursions) relative to a steady state are small, it is possible to reduce the equations of motion to a set of linear, second-order differential equations with constant coefficients. These equations can be reduced to the following general form:

$$\{\dot{X}\} = [A]\{X\} \qquad (8.1)$$

where $[A]$ is a matrix of constant coefficients and $\{x\}$ represents a column matrix, the elements of which are the motion variables. For example, in the case of rigid-airplane longitudinal small perturbations:

$$\{X\}^T = \lfloor u, \alpha, q, \theta \rfloor$$

The purpose of this section is to establish dynamic stability criteria for airplanes in cases where the equations of motion can be brought into the form of equation (8.1).

The basic form of equation (8.1) applies to the rigid*, the equivalent elastic*, and the completely elastic* airplanes. For that reason, stability criteria deduced from equations of motion to the form of equation (8.1) apply to the rigid as well as to the equivalent elastic and the completely elastic airplanes.

The expanded forms of equation (8.1) for the rigid and equivalent elastic airplanes are given in tables 4, 5, and 6. For the completely elastic airplane equations 6.166 and 6.168 are representative of the form of equation (8.1).

Stability of equations of the type (8.1) can be determined with the aid of their characteristic equation. The following development shows how this characteristic equation can be obtained.

*As shown in Sec. 9, the rigid airplane has six degrees of freedom. The same is true for the equivalent elastic airplane, but now the aerodynamic derivatives are corrected for static effects of elasticity. In the completely elastic airplane, dynamic response of the structure is accounted for by separate equations of motion. Thus there are 6 + n degrees of freedom, where n is the number of structural degrees of freedom accounted for.

Taking the Laplace transform* of equation (8.1), it follows that:

$$s\{X_s(S)\} - \{X_t(t_o^+)\} = [A]\{X_s(S)\} \tag{8.2}$$

where

$s = \sigma \pm j\omega$ = complex frequency variable

and the subscripts s and t are used to distinguish between the functional relationships $\{x_s\}$ and $\{x_t\}$.

Solving equation (8.2) for $\{X_s(S)\}$,

$$\{X_s(S)\} = \left[[S] - [A]\right]^{-1}\{X_t(t_o^+)\} \tag{8.3a}$$

Equation (8.3a) forms the frequency-domain (Laplace domain) solution to equation (8.1). It can be shown (refs. 4 and 45) that time-domain solutions to equation (8.1) are obtained by applying the inverse Laplace transform to equation (8.3a) with the following results:

$$X_i(t) = \sum_{j=1}^{k} D_{ij}\, e^{\sigma_j t} \cos(\omega_j t + \Omega_j) \tag{8.3b}$$

where $X_i(t)$ are the components of $\{X_t(t)\}$

D_{ij} = constants determined by the initial conditions

Ω_j = constant phase angles to be determined from initial conditions

The quantities σ_j and ω_j are respectively the real and the imaginary parts of the roots S_j of the characteristic equation

$$\left\| [S] - [A] \right\| = 0 \tag{8.4}$$

Since ω_j, the motion frequency, is always positive or zero and since all D_{ij} and Ω_j are constants, it follows that the motions X_i (t) are governed by the real parts σ_j of the roots S_j in the following manner:

*For discussion of the Laplace transform, see ref. 45.

a) If each $\sigma_j < 0$, the amplitudes $D_{ij}e^{\sigma_j t}$ will decay exponentially with time (fig. 14a).

b) If at least one $\sigma_j = 0$, say σ_ℓ, while all other $\sigma_j < 0$, at least one residual perturbation of constant amplitude will be observed (fig. 14b).

c) If at least one $\sigma_j > 0$, the corresponding amplitude will grow exponentially with time (fig. 14c).

Thus, the behavior of the motion is seen to be governed largely by the roots of the characteristic equation (8.4). Expanding equation (8.4) yields a polynomial in S of the form:

$$\sum_{i=0}^{n} S^i A_i = 0 \tag{8.5}$$

where the A_i are constant coefficients and n is the order of the matrix [A]. Sometimes equation (8.5) is also called the characteristic equation.

There are three basic techniques that can be used to determine airplane stability from equation (8.5). Application of these techniques leads directly to a corresponding dynamic stability criterion, as will become clear from the development that follows.

The most widely used technique is to solve for the roots S_i of equation (8.5) and discuss their significance to the motion. A dynamic stability criterion based on this technique is presented in par. 8.2.1.

The second technique deals directly with the coefficients A_i of equation (8.5). It leads to a dynamic stability criterion known as Routh's criterion, and is presented in par. 8.2.2. Routh's criterion is not widely used in practice, but it provides a logical connection between static and dynamic stability. This connection is important and is discussed in detail.

The third technique consists of a collection of methods that are largely based on linear control theory. Because a detailed discussion of these methods is beyond the scope of this report, only a brief summary is presented in par. 8.2.3, with several references where detailed discussions may be found.

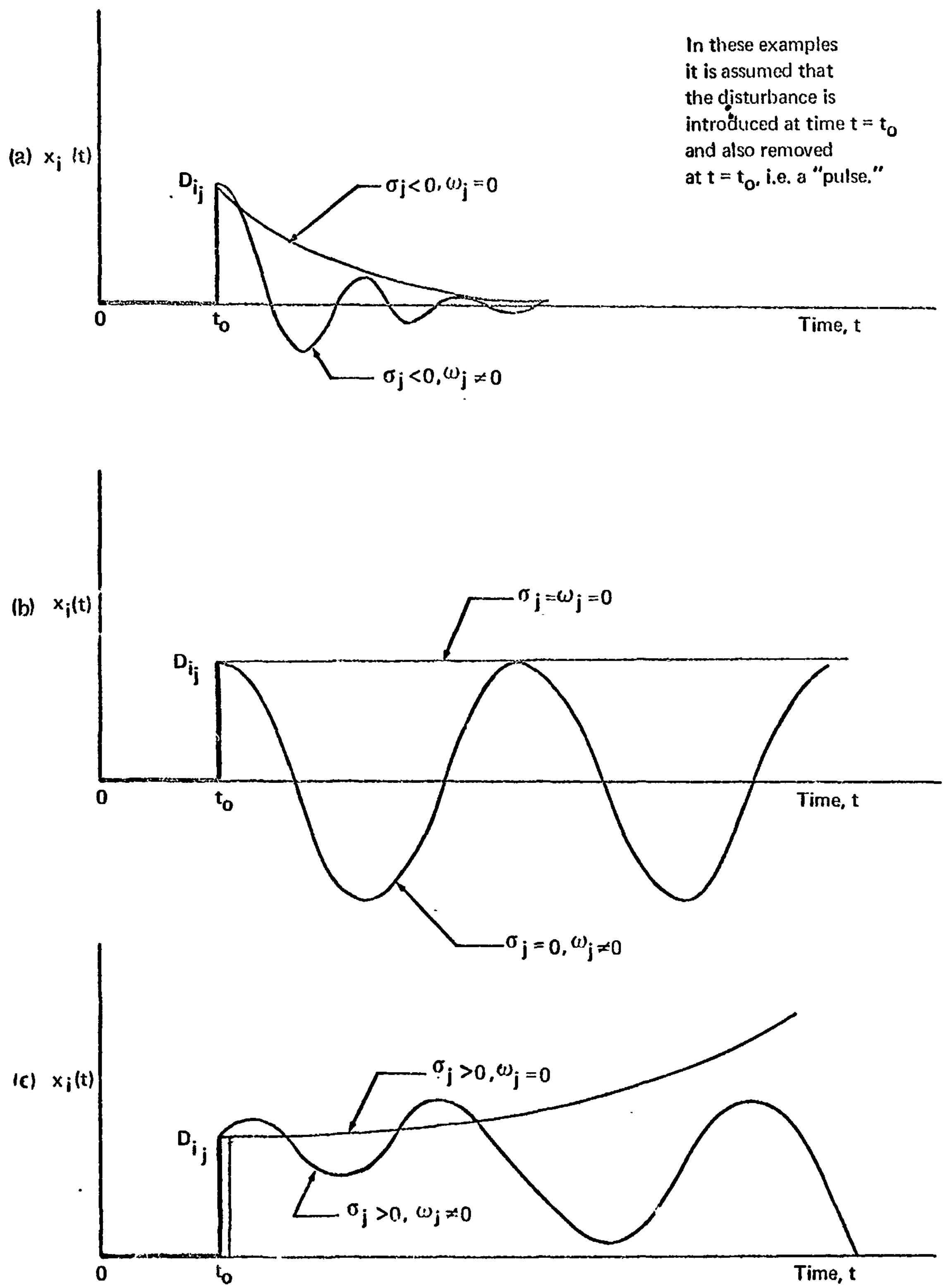


FIGURE .4. – EXAMPLES OF SMALL PERTURBATION MOTION BEHAVIOR

8.2.1 Dynamic stability criteria based on the roots of the characteristic equation. -- The roots of equation (8.5), $S_i = \sigma_i \pm j\omega_i$ ($i = 1, 2, \ldots, n$) can be obtained by classical techniques. For $n < 4$, this can be done by hand -- for example, by the methods of ref. 46, pp. 22-24, or ref. 16, pp. 271-273. For higher order equations the roots are generally calculated with digital computers, using methods such as presented in ref. 47. The roots of equation (8.5) are identical with the eigenvalues* of [A] for linear equations with constant coefficients. They determine airplane stability by virtue of the signs of σ_i, the real part of S_i as discussed above. The following dynamic stability criteria can now be formulated.

Criteria 8.1

If the airplane equations of motion are linear and autonomous, then the airplane stability behavior is said to be:

a) Stable, if the real parts of the roots of the characteristic equation are all negative,

b) Neutrally stable, if there are one or more roots of the characteristic equation with zero real parts and the remaining roots have all negative real parts, and

c) Unstable, if there is at least one root of the characteristic equation with a positive real part.

A simple proof of criterion 8.1a is given in app. C. The dynamic stability criteria 8.1 are both necessary and sufficient. These criteria have formed the basis for most dynamic stability work during the past decades. In most of the standard literature (refs. 4, 13, 14, 15, and 16) dynamic stability of airplanes is treated from this viewpoint, which finds its justification in the assumption that airplane dynamic behavior can be described by a set of linear second-order differential equations with constant coefficients. The handling qualities criteria (specifications) of refs. 10, 11, and 44 also rely heavily on this assumption and consequently on criteria 8.1.

*For a discussion of eigenvalues, see ref. 12.

The application and interpretation of criteria 8.1 to the rigid, equivalent elastic, and completely elastic airplanes is discussed in more detail below.

It is shown in Sec. 6 that the possibility exists for [A] in equation (8.1) to have elements that are known functions of time. This occurs in steady climbs and dives when dynamic pressure is allowed to vary. A typical example is discussed in app. A, where it is shown that the SST in certain areas of the flight envelope violates the constant air density assumption made in deriving the equations of motion. In such a case, equation (8.1) is still linear but is called nonautonomous, and the equations assume the form:

$$\{\dot{X}\} = [A(t)]\{X\} \tag{8.6}$$

For this type of equation, no simple stability theory is known. The stability theory of Lyapunov, discussed in par. 8.5 and Sec. 9, could be used. The writers of this report feel that the following simple approach to stability determination of equation (8.6) is valid; however, they have not found a proof. The approach consists of applying the characteristic equation method to equation (8.6) with the following modifications. The characteristic equation considered is:

$$\left\| [A(t)] - \lambda [I] \right\| = 0 \tag{8.7}$$

The time-variable coefficients in [A(t)] are bounded by the physical aspects of the problem as discussed in Sec. 6. The following dynamic stability criterion is postulated.

Criterion 8.1d When the real parts of the roots of the characteristic equation (8.7), are negative for $t = 0$ as well as for $t = t_1$, where t_1 is the practical limit* of the time interval considered, the airplane is stable in that time interval.

*The practical limit is, for example, determined by the time to reach the ceiling of the airplane or the time to reach the ground.

As stated, this criterion needs proof. To make it feasible, the following considerations are offered. Define the quantity E as equal to the total kinetic energy of the airplane in the perturbed state and also define $\dot{E} = \frac{dE}{dt}$. In that case, $\dot{E}$ takes the place of the Lyapunov function in theorem 1, Sec. 9. If criterion 8.1d is satisfied, it can be interpreted to mean, according to Lyapunov's theorem 1 (Sec. 9), that $\dot{E}$ is negative at the beginning and end of the time interval. This indicates that energy is being dissipated at $t = 0$ and at $t = t_1$. The writers of this report feel that the proof of criterion 8.1d hinges on the sign behavior of $\dot{E}$ for $0 < t < t_1$.

Whether or not criteria 8.1 are satisfied in a practical case can be determined by solving directly for the roots of the characteristic equation. A technique for determining stability behavior from the characteristic equation without solving for the roots, known as Routh's criterion, is discussed in par. 8.2.2.

8.2.2 Routh's criterion (dynamic stability). -- Routh's criterion can be used to determine whether or not criteria 8.1 are met without solving for the roots of the characteristic equation. This should not be confused with what is sometimes called the Routh-Hurwitz criterion. Routh and Hurwitz developed similar but not identical criteria. However, from the standpoint of calculations, Routh's is the more direct approach.*

The result of expanding equation (8.4) is a polynomial in S of the following form:

$$\sum_{i=0}^{n} S^i A_i = 0 \tag{8.8}$$

where n is the order of the matrix [A]. Routh's criterion is stated as a series of conditions involving the coefficients A_i. Before stating the necessary and sufficient form of this criterion, it is necessary to develop the relations between the coefficients A_i that are used in the formulation of this criterion. These relations are called test functions.

*For a discussion of the Hurwitz criterion, see ref. 48.

The test functions are constructed by first writing down the coefficients of the polynomial as follows:

$$
\begin{array}{llll}
A_n & A_{n-2} & A_{n-4} & \cdots \\
A_{n-1} & A_{n-3} & A_{n-5} & \cdots
\end{array}
$$

A necessary but not sufficient condition for stability is that all of these coefficients have the same sign. Next, additional rows and columns are determined by the following scheme:

$$
\begin{array}{llllll}
P_{31} & P_{32} & P_{33} & \cdot & \cdot & \cdot \\
P_{41} & P_{42} & P_{43} & \cdot & \cdot & \cdot \\
P_{51} & P_{52} & P_{53} & \cdot & \cdot & \cdot \\
\cdot & & & & & \\
\cdot & & & & & \\
\cdot & & & & & \\
P_{n+1,1} & 0 & 0 & \cdot & &
\end{array}
$$

where

$$P_{31} = \frac{A_{n-1} A_{n-2} - A_n A_{n-3}}{A_{n-1}} \tag{8.9}$$

$$P_{32} = \frac{A_{n-1} A_{n-4} - A_n A_{n-5}}{A_{n-1}}$$

$$P_{41} = \frac{P_{31} A_{n-1} - P_{33} A_{n-1}}{P_{31}}$$

$$P_{42} = \frac{P_{31} A_{n-5} - P_{33} A_{n-1}}{P_{31}} \quad , \quad \text{etc.}$$

$$P_{51} = \frac{P_{41} P_{32} - P_{31} P_{42}}{P_{41}} \quad ; \quad \text{etc.}$$

If the test functio A_{n-1}, and $P_{i,1}$ for $(i = 3, 4, \ldots, n+1)$ all have the same sign, no unstable root occurs. This is both necessary and sufficient. The rows will be found to become shorter by one element every two rows and to be $(n + 1)$ in number.

It has been shown by Duncan (ref. 49) that the vanishing of A_o and $P_{n-1,1}$ represent significant critical cases. When a design parameter for a stable airplane is altered so as to cause instability, the following conditions hold:

a) If only A_o changes sign, one real root S_j of equation (8.8) changes its sign from negative to positive. This implies that a pure divergence (instability) occurs in the solution. This is called flight path divergence.

b) If only $P_{n-1,1}$ changes sign, the real part of one complex pair of roots changes from negative to positive. This implies that a divergent (unstable) oscillation occurs in the solution. This is called dynamic or oscillatory flight path divergence.

The type of divergence in a) is not to be confused with the type of static instability discussed in Sec. 7. The instability in Duncan's sense implies divergence of the perturbed flight path away from the steady-state flight path. Static instability in the sense of Sec. 7 merely means the tendency of not instantaneously opposing a disturbance of the steady-state flight path. Figure 15 graphically illustrates four extreme examples. There are some strong connections between the two types of instability; these are discussed in Sec. 6.

The interesting result according to Duncan is that $A_o = 0$ represents a boundary between divergence and convergence of the flight path, while $P_{n-1,1} = 0$ represents a boundary between oscillatory stability and instability.*

*It is noted in ref. 48 that a change in sign of any $P_{i,1}$ implies that a complex pair of roots crosses the imaginary axis if:

a) When $P_{i,1} = 0$, the two preceding rows $P_{i-2,j}$ and $P_{i-1,j}$ have the same number of nonzero elements j and are such that the ratio of corresponding elements in the two rows is a constant, i.e. $P_{i-2,j}/P_{i-1,j}$ = a constant for each j . Also, in ref. 48 , if a) is not satisfied, then:

b) There is at least one root in the right half plane, and the airplane is dynamically "unstable" and not neutrally stable.

This constitutes a contradiction of Duncan's work. The writers of this report have not resolved the discrepancy.

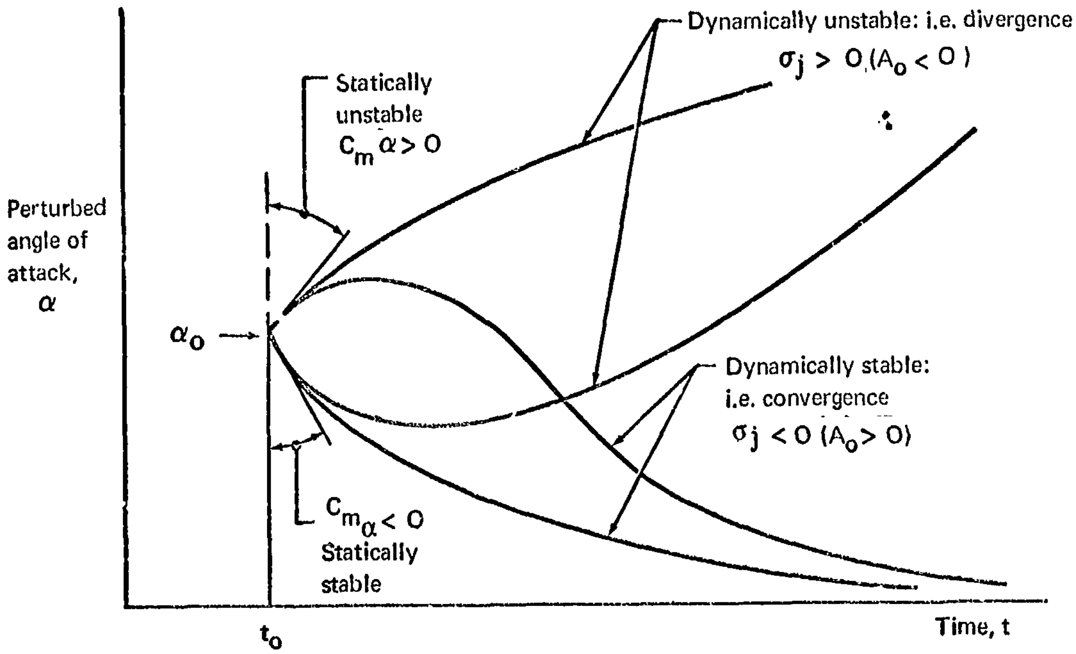


FIGURE '5. – EXTREME EXAMPLES OF THE DIFFERENCE BETWEEN STATIC STABILITY AND FLIGHT PATH STABILITY

From the preceding discussion, the following necessary and sufficient dynamic stability criterion is deduced.

Criterion 8.2a If all A_i in the characteristic equation are positive and if the test functions $P_{i,1}$ for $i = 3, 4, \ldots, n+1$ are all positive, the airplane is dynamically stable.

Two different conditions indicate dynamic neutral stability:

Criteria 8.2b

a) If $A_0 = 0$ and the reduced equation

$$\sum_{i=0}^{n} \frac{S_i A_i}{S} = 0$$

satisfies criteria 8.2a or,

b) If all A_1 are positive, and condition a of the footnote following equation (8.9) is satisfied,

then the airplane is dynamically neutrally stable.

There are three different conditions that indicate dynamic instability:

> Criteria 8.2c
>
> a) If there is a s[illegible]nge in A_i, $(i = 1, 2, \ldots, n,)$ or
>
> b) If all A_i are positive but one $P_{i,1} = 0$ or
>
> c) If all A_i are positive and one $P_{i,1}$ is negative,
>
> the airplane is dynamically unstable.

The relationships $A_0 = 0$ and $P_{n-1,1} = 0$ have already been identified as stability boundaries. It is possible to construct relations between two or more stability derivatives (or inertial parameters) for which $A_0 = 0$ and/or $P_{n-1,1} = 0$ are satisfied. Such relations are also called stability boundaries because they identify combinations of values of derivatives (or inertial parameters) for which instabilities occur.

It is unfortunate that the expressions for $P_{i,1}$ are usually so complicated; they can be used in practice only in conjunction with a computer. Expressions for A_0 are more easily handled; because they deal with the important relationship between static and dynamic stability, they are discussed in Sec. 6.

Routh's criterion can be applied to equation (8.7) as well as to equation (8.4). This would lead to an [illegible]ray like equations (8.9), where the test functions would now be polynomials in t. It would then be possible to solve for the values of t that would allow any $P_{i,1} = 0$. If these values were such that $t < 0$ or $t > t_1$, the eigenvalues $\lambda_i(t)$ would have negative real parts for all t of interest. It may be possible to develop the needed proof for criterion 8.1d along these lines.

In addition to the techniques described so far, there are other approaches to the problem of determining stability behavior of the small perturbation equations of motion of the form of equation (8.1). These are briefly discussed in the next paragraph.

8.2.3 Other techniques associated with characteristic equations. -- Many techniques used in systems analyses and synthesis techniques (control theory) may be applied to the perturbed airplane equations of motion of the form of equation (8.1). Some of the more widely used -- for example, Bode diagrams, Nichol's charts, Nyquist criterion, root locus plots, phase trajectories, etc.-- can be found in literature such as refs. 18, 19, 20, 21, and 22. Most of these techniques were generated for special types of problems, and their use is restricted because of limitations imposed by assumptions and/or effort required in their application. However, they are generally quite useful in approaching the problems of handling qualities, ride qualities, and control system (closed loop) analyses.

An adequate description of any of these techniques would require much more space than can be given here. However, some of the useful applications of these techniques should be noted. An example is the process of varying design parameters to stabilize an airplane which is unstable for certain flight conditions. For another example, the effect of "closing the loop" when adding an augmentation system can be assessed by using the Bode diagram, Nichol's chart, Nyquist criterion, or root locus.

Many of the special techniques involved in nonlinear analyses are just more sophisticated linearizing techniques that allow the engineer to apply linear techniques to approximate transfer functions.

Linearized or quasi-linearized airplane and system models are usually described by transfer functions, i.e. outputs ÷ inputs, where output = variable behavior and input = disturbance behavior. It is assumed that the transfer function approach to the relationship of rigid and elastic degrees of freedom for the small perturbation equations could become a valuable tool. In this sense, the transfer functions could be:

$$\frac{X_i(S)}{X_j(S)} = G(S)$$

where $X_i(S)$ = rigid degree of freedom

$X_j(S)$ = elastic degree of freedom

$G(S)$ = transfer function

The amplitude and phase relationships obtained by applying some of the above techniques to G(s) would lead to a more enlightened viewpoint of the influence of the elastic degrees of freedom on the rigid-body degrees of freedom and vice versa. This approach is used in conjunction with root loci in ref. 50 for considering the problem of flutter.

Many applications of the various systems analysis techniques mentioned above have not been discussed here. However, most of them would probably lead into the areas of handling qualities, ride qualities, or control system analysis or synthesis. Therefore, the pursuit of knowledge in this area is left to the reader while subjects more pertinent to pure stability behavior analyses are pursued. The next section deals with stability criteria based on the time history approach, which is of particular interest for flight situations where the small perturbation assumption is not valid.

8.3 Dynamic Stability Criteria Based on Time Histories

It was stated in par. 8.1 that today there are several practical cases where nonlinearities in the equations of motion (dynamic or aerodynamic) are too large to be neglected. When the equations of motion of an airplane are nonlinear, it is not possible to apply the characteristic equation methods described in par. 8.2. It has been common practice in such cases to base judgment of stability behavior on time history solutions of the equation of motion. A time history is a set of data that describes airplane motions as a function of real time, i.e. $\{X\} = \{X(t)\}$.

Time histories have the advantage of providing a clear physical picture of the motion of the airplane. In addition, they allow a direct comparison of analytical with experimental data.

Time histories can be generated by integrating, with respect to time, the complete airplane equations of motion or, for that matter, any of the equations of motion shown in Sec. 6. The integration technique may vary, but the approach is generally the same for any type of computer. The airplane (equations) must be trimmed (equilibrated) either exterior to or in conjunction with the problem to be solved, i.e. the solutions $\{X_1\}$ of the algebraic steady-state equations must be obtained and used as initial conditions. The program is executed (started) with $t = 0$. At some time $t_o \geq 0$, a disturbance $\{\Delta X\}$ is introduced and the response $\{X(t)\}$ calculated for $t_o < t < t_1$, where $t_1 - t_o$ is usually a time interval long enough to establish stability behavior but not so long as to involve mass or other changes that would significantly affect assumptions made in deriving the airplane equations of motion.

In this fashion, stability behavior can be determined by observation, i.e. by "judging" the behavior of the variables of the resulting time history.

The judgment of stability behavior through the use of time histories will be referred to as stability criteria 8.3. These criteria are formulated as follows:

<u>Criteria 8.3</u>

If the motions of an airplane following a disturbance from steady-state flight are determined by a time history (integration), then the stability behavior is said to be:

a) Stable if the motions remain in proximity to the steady-state
b) Neutrally stable if the motions are undamped and oscillatory about some steady state
c) Unstable if the motions diverge from the steady state either linearly, exponentially, oscillatorily, or in any combination thereof.

If the disturbance is temporary, the reference steady state is the initial steady state. If it is permanent, e.g. a step elevator change, the reference steady state is a different one determined by the new equilibrium flight conditions.

The following observation is important. For nonlinear equations of motion such as those of tables 4, 5, and 6, several different cases involving different disturbances, both in kind and magnitude, must be run to obtain enough information to establish the stability behavior. The reason for this is found in the property of nonlinear differential equations whereby their response behavior can be a function of the initial disturbance. Therefore, one stable case does not imply airplane stability for these nonlinear equations. Here is where engineering experience and judgment play an important role. For any problem there is an infinite number of combinations of different magnitudes of initial disturbances. Setting physically realizable limits (positive and negative) on the disturbances and choosing a "representative set" within these limits is a job for the experienced stability and control engineer. This representative set of disturbances can then be used to generate a set of time histories.

For the linearized, uncoupled, small perturbation equations of motion of tables 4 and 6, only one arbitrary disturbance is required for each mode (longitudinal or lateral direction). Linearity implies that the response behavior is independent of the size or type of disturbance in that mode. However, time history generation for the linear small-perturbation equations is not necessarily the most efficient approach to stability analysis.

The major advantage to using the time history (integration) approach is that it is in terms of real time. The analyst can experience more of a "physical feel" for the problem, since he is observing motions similar to those which the airplane would be experiencing in flight under the same conditions. Most of the disadvantages of the time history method are not really pertinent to the problem of stability behavior. Instead, they are of an economic nature, such as hardware and facility acquisition, upkeep, availability, man-hour expenditures in programming, data preparation and reduction, etc. The advantages and disadvantages involved in choosing an analog, a digital, or a hybrid computer are not pertinent to the discussion and will not be considered here. For detailed discussions of numerical integration techniques, see ref. 47. For an extensive discussion of analog computation methods, see ref. 51.

The next section deals with stability analysis techniques called "energy decay methods."

8.4 Dynamic Stability Criteria Based on Energy Decay Methods

A relatively new and unknown area of stability analysis is the energy decay method. This approach is discussed in refs. 23 and 24.

The fundamental idea behind energy decay methods is that energy E is dissipated in dynamically stable systems. For linear differential equations with constant coefficients, it is possible to show the inverse; that is, if energy is being dissipated, the corresponding system is dynamically stable. Extension of this idea to nonlinear equations of motion can be justified by applying the Lyapunov stability theory of Sec. 9.

It is possible to formulate stability criteria based on this idea of energy dissipation. Two examples of such criteria are discussed below. The first one deals with the linear equations of motion of the type used in par. 8.2, while the second deals with nonlinear equations of motion of the types used in par. 8.3.

For equations of motion of the type of equation (8.1), the approach can be stated in the following steps:

a) Derive expressions for the total perturbed energy, E, of the airplane.
b) From a) derive the ΔE required to make the airplane appear to be a conservative system in the first half-cycle of oscillation, i.e. neutrally stable.

The following dynamic stability criteria can now be formulated.

<u>Criteria 8.4</u>

If: a) $\Delta E > 0$, the airplane is stable.
b) $\Delta E = 0$, the airplane is neutrally stable or not disturbed.
c) $\Delta E < 0$, the airplane is unstable.

A theoretical approach to applying these criteria is given in ref. 24.

Because of algebraic complexities, it is not considered practical to use criteria 8.4 in cases involving nonlinear equations of motion. For nonlinear equations of motion, Hahn (ref. 25) suggests an energy decay method based on an idea by Lebedev. This idea is further developed by Roskam (ref. 23, pp. 55-72). There, stability is connected with energy decay through the parameter:

$$F = \frac{\int_{t_2}^{t_3} T\,dt}{\int_{t_1}^{t_2} T\,dt}$$

where T is the perturbed kinetic energy, t_1 is the beginning of a time interval during which the motion of the airplane is being studied, t_3 is the end of that time interval, and t_2 is the midpoint of that time interval. The criteria for stability in this case would be as follows:

Criteria 8.5
If: a) $F < 1$, the airplane is stable, b) $F = 1$, the airplane is neutrally stable, c) $F > 1$, the airplane is unstable.

It is shown in ref. 23 that $F < 1$, indicating stability, is satisfied in the case of stable, linear small perturbation equations of motion. The advantage of criterion 8.5 is that they apply to nonlinear equations of motion. A disadvantage is that considerable numerical work or a computer program is required.

The potential application of energy decay stability criteria is believed to be in the area of stability in limited time intervals. From the discussion at the beginning of this sub-section it is seen that for linear and autonomous small perturbation equations of motion, stability according to the characteristic equation method implies $\dot{E} < 0$ and therefore $\Delta E > 0$ and $F < 1$. The condition $\dot{E} < 0$ follows straightforwardly from Lyapunov's theorem 1 (Sec. 9) by using the total perturbation energy E as the Lyapunov function. Because of the analytical difficulties involved in treating the problem in general and because of lack of time, this approach is left as a suggested area for future research.

An area of stability analysis that is relatively unknown to airplane stability and control engineers is based on Lyapunov's stability theory. This is discussed in the par. 8.5.

8.5 Dynamic Stability Criteria Based on Lyapunov's Method

In par. 8.3 the time history method was suggested as a way to determine the stability behavior of the airplane when the equations of motion are nonlinear. However, with the time history method, it is necessary to solve the equations of motion. Lyapunov has devised a stability theory for both linear and nonlinear perturbed differential equations of motion that obviates the necessity to solve these equations.

Lyapunov's stability theory is an approach to determination of stability behavior that has received little attention from airplane stability and control engineers. For this reason, an introduction to this theory and some pertinent definitions and theorems are given in Sec. 9. The potential applications of the analysis techniques devised by Lyapunov and those who have followed his approach are virtually unlimited. The reason for this is the generality of the approach. Rather than solving any particular problem, Lyapunov realized that the stability of dynamic systems (moving bodies, etc.) could be approached by studying the behavior of differential equations in general. He devised two classes of approach, one for equations whose solutions are known functions of time and another for the equations of motion written in perturbation form. The first approach (known solutions) is similar to the stability criterion for time histories given in par. 8.3.

The second approach is called the "direct" or "second" method of Lyapunov. This method, the essential details of which are discussed in Sec. 9 requires choosing a "Lyapunov function" and relating its behavior to the behavior of the differential equations of motion. A particularly attractive approach to the problem of nonlinear airplane stability behavior using Lyapunov's direct method derives from a theorem attributed to Zubov. Because of similarity, it is felt that Zubov's theorem should appear as a logical extension of the more familiar characteristic equation approach. In fact, as shown in Sec. 9, it is possible to prove criterion 8.4a (stable roots for characteristic equations) using Zubov's theorem for linear, autonomous equations. However, the application of Zubov's theorem would be more useful for nonlinear equations.

It is shown in Sec. 6 that the large perturbation equations of motion of an airplane can be written in the form:

$$\{\dot{x}\} = \left[F(\{x\}, t)\right]\{x\} \tag{8.10}$$

Nonlinear small perturbation equations, with nonlinear aerodynamic cross-coupling terms, can also be written in this form. Before stating the stability criterion for these nonlinear equations, the following definitions are required:

The equation

$$\left\| \frac{1}{2}\left[[F(\{X_R\}, t_R)]^T + [F(\{X_R\}, t_R)] \right] - \lambda [I] \right\| = 0 \qquad (8.11)$$

will be called the "quasi-characteristic equation" where $\{X_R\}$ and t_R are defined as values belonging to a "representative set" of X and t. By a representative set, the following is meant: For given initial disturbances, the solutions to the equations of motion (8.11) yield a time sequence of values of the motion variables $\{X\}$. In most practical cases the engineer will have an idea of the practical limits of the perturbed motions that his airplane can experience. In other words, the engineer can make a reasonable estimate of the "cylindrical neighborhood" surrounding the time axis, within which the motion takes place. Figure 16 illustrates such a cylindrical neighborhood for a case with only two motion variables. The idea is readily extended to cases involving more motion variables. Combinations of values of time t and values of the motion variables in and inside this cylindrical neighborhood are called a representative set.

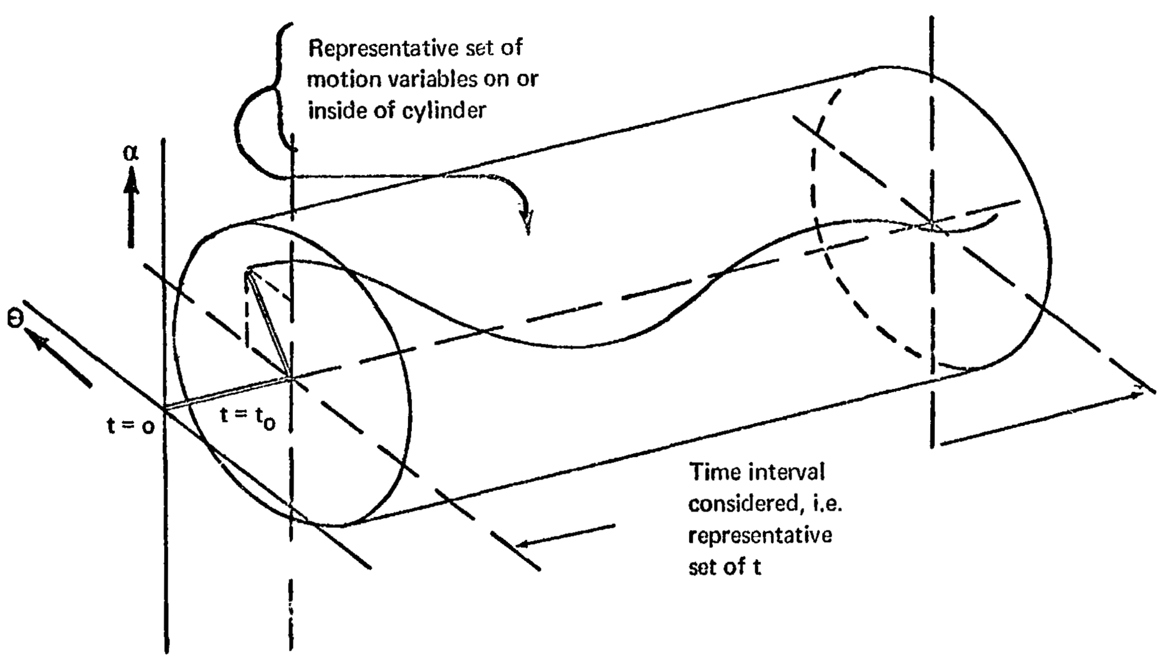


FIGURE 16. – ILLUSTRATION OF A REPRESENTATIVE SET OF TIME AND MOTION VARIABLES

In addition to limitations on the size of motion variables, there is a limitation to the time interval during which motion behavior is considered. It is shown in Sec. 5 and in the summary report that there are definite limits on time because of the assumption of constant airplane mass. For example, it is shown that the constant-mass assumption for the SST can become questionable for time intervals beyond 150 seconds.

Choosing discrete values of $\{X\}$ and t, called $\{X_R\}$ and t_R, within practical limits related to the steady-state flight condition in accordance with these ideas generates a "representative set" of $\{X\}$ and t. An analogy to the representative set is the selection of combinations of Mach numbers, dynamic pressures, angles of attack, and angles of sideslip for which wind tunnel data are to be obtained or for which stability is to be assessed in the usual analysis approach.

The eigenvalues λ that will satisfy equation (8.11) are called the eigenvalues of the quasi-characteristic equation.

Using the above definitions, the application of Zubov's theorem as a dynamic stability criterion is postulated as follows:

> Criterion 8.6
>
> If the eigenvalues of the quasi-characteristic equation are non-positive (≤ 0) for each $\{X_R\}$ and t_R in a representative set of $\{X\}$ and t, the airplane is considered stable.

As opposed to the other stability criteria presented in this section, this criterion has no neutral or unstable counterparts. It is shown in Sec. 9 that this technique has its limitations and the existence of positive eigenvalues does not necessarily imply instability. In fact, in a numerical example, it is shown in Sec. 9 that applying Zubov's theorem directly to a set of linear, autonomous equations yields no conclusive information about the equations and that applying the characteristic-equation approach shows conclusively that the equations have stable behavior. In other words, criterion 8.6 is necessary but not sufficient.

It is emphasized here and in Sec. 9 that using Zubov's theorem as a basis for determining stability has its limitations and disadvantages. Particularly important is the consideration that proving Zubov's theorem requires the use of a particular Lyapunov function, which may lead to very rough stability analyses. This is further discussed in Sec. 9.

Another disadvantage of this approach is the loss of "physical feel" for the problem until familiarity with and understanding of the direct method are achieved. This is one aspect where the time history approach has a distinct advantage, because the engineer can "see" the predicted motions.

The question of when it is valid to use the linearized form of the equations of motion is raised in ref. 23 and is also discussed in Sec. 9.

It is obvious that the application of Lyapunov stability theory to airplane stability problems is an area where further research is needed.

8.6 Summary of Dynamic Stability Criteria

Dynamic stability criteria were established covering the linear and nonlinear equations of motion of an airplane. These criteria apply to rigid, equivalent elastic, and completely elastic descriptions of airplanes, provided the corresponding equations of motion are written in the form required by the criteria. Table 9 presents a summary of dynamic stability criteria and their relations to the various forms of the equations of motion. The arrangement of the equations of motion into the required forms is discussed in Sec. 9. The combinations of criteria and equations that are most commonly used in airplane stability analysis are identified with heavy lines in table 9.

The question of whether or not dynamic stability is required has not been discussed in this chapter. The handling quality criteria of refs. 10, 11, 12, and 44 require dynamic stability of all short period oscillations. References 11 and 12 do not specify requirements for long period oscillations or for divergences or convergences. References 10 and 12, however, do specify maximum allowable times to double for such cases. It is the opinion of the writers of this report that dynamic stability should certainly be required of the airplane when considered as a controlled system, whether control is exercised by the human pilot or by an automatic system. Whether or not this means that the uncontrolled airplane should have dynamic stability and to what extent is largely a matter of opinion and depends on such factors as airplane mission, configuration, flight condition, and the reliability and

TABLE 9.–SUMMARY OF DYNAMIC STABILITY CRITERIA

EQUATION TYPE / STABILITY CRITERIA	SMALL PERTURBATIONS		LARGE PERTURBATIONS	ARBITRARY PERTURBATIONS	ARBITRARY MOTIONS
	LINEAR	NONLINEAR	NONLINEAR	NONLINEAR	
Characteristic equation methods criteria	Applicable	Not Applicable	Not Applicable	Not Applicable	Not Applicable
Time histories criterion	Applicable	Applicable	Applicable	Applicable	Applicable
Energy decay methods criteria	Applicable*	Applicable*	Applicable--Research Required	Applicable--Research Required	Applicable--Research Required
Lyapunov theory criterion	Applicable*	Applicable*--Research Required	Applicable--Research Required	Applicable--Research Required	Not Applicable

*Additional research required for nonautonomous systems.

capability attached to its controller, whether human or automatic. It is not the purpose of this report to deal with this matter in any detail.

Because of the analytical complexities involved in the application of energy decay criteria and Lyapunov criteria, no interpretations of the significance of individual aerodynamic or inertial terms in relation to these criteria have been presented. For time history criteria, such interpretations can only be given after carrying out specific numerical integrations. Such interpretations can be more easily given for criteria based on characteristic equation methods.

Paragraph 8.6.1 summarizes the advantages, disadvantages, and limitations of the various stability criteria established in this chapter. Areas for further research are suggested in par. 8.6.2.

8.6.1 Limitations, advantages, and disadvantages. -

8.6.1.1 Characteristic equations method: Applications of the characteristic equations method are limited to linear differential equations of motion. This imposes a restriction because it is expected that significant nonlinearities will be encountered in future designs. Examples of nonlinear cases were pointed out in par. 8.1. However, when the equations of motion can be linearized, the characteristic equations method represented by criterion 8.1 is a most efficient technique for determining airplane stability behavior. In addition to determining the stability behavior, the roots of the characteristic equations can be used for other analyses. For example, the frequency and damping characteristics, imaginary and real parts of the roots, are used extensively in handling qualities analyses and stability augmentation systems design. References 4, 12 through 16, 36 and 49 are typical examples of such cases.

The application of the characteristic equations method to the linear nonautonomous equations was presented as a valid extension of this approach. Routh's criterion, 8.2, permits a connection between static and dynamic stability considerations, and this was discussed in par. 8.2.2.

8.6.1.2 Time history method: There are no conceptual restrictions to the time history approach to any of the equations of motion. The only restrictions are those imposed by the assumptions used in deriving the equations to be considered. A particular advantage to this approach is that it presents a physical picture of the motions involved. Another advantage is that it allows a comparison of analytical and experimental data. Most of the disadvantages of time history method are inherent in the computer itself (storage space, etc.) or the integration techniques used.

8.6.1.3 Energy decay methods: Energy decay methods have not been widely applied. As a result, the limitations, advantages, and disadvantages have not been assessed. It is felt, however, that there should be few limitations because of the general nature of the approach. For linear, autonomous, small perturbation equations of motion, this approach will probably prove less efficient than the characteristic equations method. However, it may lead to a better insight into the effect of certain stability derivatives on stability behavior.

8.6.1.4 Lyapunov stability method: The particular method presented here (Zubov's theorem) has no restrictions with regard to the types of perturbed equations to which it can be applied. However, the criterion only pertains to dynamic stability, and there are no neutral or unstable counterparts. Also, it will not always predict stability for stable airplanes. Here again, this approach has not been sufficiently explored to truly assess its value.

8.6.2 Suggested areas for further research. --

8.6.2.1 Characteristic equations: A mathematical proof of criterion 8.1d is required. Whether it is proved or disproved, the work involved should result in more insight into the dynamics of airplanes requiring non-autonomous mathematical modes.

8.6.2.2 Time histories: The research aspects for time histories are involved in computer sophistication and are not pertinent to this discussion.

8.6.2.3 Energy decay: Further investigation in this area is required. The main objective would be to develop the necessary energy expressions and apply the criteria to practical cases. It is worthwhile to note that the "first half-cycle" may not be a sufficient time interval for stability determination using criterion 8.4. This may require validation using the direct method of Lyapunov for limited time intervals (ref. 25).

The separation into easily identifiable modes of motion (phugoid, Dutch roll, etc.) is a property of the linear small perturbation equations of motion. For nonlinear equations, such a separation does not occur. Because nonlinear behavior is expected to be dominant for future designs, the question of how to specify dynamic stability requirements must be faced. It was suggested in par. 8.4 that the energy decay parameter F be considered as one way of specifying dynamic stability requirements for situations involving nonlinear behavior. A considerable amount of research is needed before this can be done.

8.6.2.4 Lyapunov stability, Zubov's theorem: Owing to the "roughness" of Zubov's theorem, as discussed in Sec. 9, and because of its similarity to the characteristic equation approach, some refinement of this approach may lead to more efficient stability assessment for nonlinear and/or nonautonomous equations than the time history approach currently offers. Whether or not this refinement can be accomplished requires additional study.

9. LYAPUNOV STABILITY

9.1 Introduction

The purpose of this section is to explain the second or direct method of Lyapunov. This method and its derivatives allow the determination of the stability behavior of nonlinear and nonautonomous ordinary differential equations oi perturbed motion without solving these equations. It will be shown that when applied to ordinary, linear, autonomous differential equations of perturbed motion, the direct method of Lyapunov yields the same results as the familiar characteristic equation method described in par. 8.2. In that sense, the characteristic equation method is essentially a special case of the direct method of Lyapunov. A dynamic stability criterion for nonlinear airplane equations of motion is developed on the basis of Zubov's theorem. This criterion has practical significance since it allows the establishment of stability when the equations of motion have a general form — namely, the large perturbation form. However, as will be shown later, stability obtained in this manner has its limitations. In addition, a method is presented by which it is possible to determine the conditions under which the linear small perturbation equations of motion give the correct answer of stability.

An appreciation of the scope and potential of the direct method of Lyapunov in solving problems of stability determination can be gained from reading one or more of refs. 25, 52, 53 and 54. An example of the potential of the method is shown in Hahn's book, "Theory and Application of Lyapunov's Direct Method" (ref. 25), where more than 20 different kinds of stability and instability are discussed. References 25 and 54 combined contain more than 30 pages of bibliography and references, an indication of the scope involved.

Before stating the two main theorems of the direct method of Lyapunov, it is necessary to define the mathematical meaning of stability used in conjunction with the method. Reference 25 gives the definitions for stability (stability of the equilibrium) of the perturbed equations of motion as follows:

The perturbed equations of motion may be written in the general form

$$\{\dot{X}\} = \{R(\{X\}, t)\} \tag{9.1}$$

where $\{R(\{x\}, t)\}$ is a column matrix of functional relationships between the motion variables x_i and, if the equations are nonautonomous, t. It can be demonstrated that this form can be obtained for the equations of motion considered herein. Setting $\{\dot{\delta}\} = \{0\}$ and carrying out the matrix multiplication $[F(\{x\}, t)]\{x\}$ results in:

$$\{\dot{x}\} = [F(\{x\},t)]\{x\} = \{R(\{x\},t)\}$$

The particular solutions for a set of initial disturbances, $\{x_0\}$, introduced at t_0 are given by:

$$\{x\} = \{P(t, \{x_0\}, t_0)\} \tag{9.2}$$

The equilibrium is the state before initial disturbance, given by:

$$\{P(t < t_0, \{x_0\}, t_0)\} = \{0\} \tag{9.3}$$

9.2 Definitions of Stability According to Lyapunov

By Lyapunov's definitions, the equilibrium is:

a) Stable if there exists an $\epsilon > 0$ and a $\delta > 0$ such that:

$$|\{x_0\}| < \delta \tag{9.4}$$

implies

$$|\{P(t, \{x_0\}, t_0)\}| < \epsilon \tag{9.5}$$

The significance of a) is that for small enough initial disturbances, $\{x_0\}$, the motion $\{x\} = \{P\}$ remains close to the equilibrium (undisturbed motion), i.e. the solutions (motions) are not divergent.

b) Quasi-asymptotically stable if there exists a $\delta_0 > 0$ such that when:

$$|\{x_0\}| < \delta_0 \tag{9.6}$$

then:

$$\lim_{t \to \infty} \{P(t, \{x_0\}, t_0)\} = \{0\} \tag{9.7}$$

The significance of b) is that for small enough initial disturbances $\{x_0\}$ and no additional disturbances for $t \geq t_0$ the perturbation variables eventually return to zero (condition for $t < t_0$), which is exactly the original equilibrium.

c) Asymptotically stable if it is both stable and quasi-asymptotically stable.

It follows that for a), a new equilibrium may be achieved or a condition of neutral stability, in the usual sense, may exist. Conditions for unstable equilibrium as well as some enlightening remarks concerning both stability and instability may be found in refs. 25, 52, and 53.

It is observed that satisfying b), and hence c), with respect to the limiting process $(t \to \infty)$ can only be done in the analytical sense, i.e. if the solutions are expressed as explicit functions of time. Also, in order that physically realizable limits are not violated, it may be unrealistic to apply this limit even in the analytical sense. When it is necessary to use numerical integration techniques to solve the equations, it is obviously not possible to satisfy b). Thus it may be deduced that only the most ideal problems can be treated in this manner and that most real, physical problems require the use of a more sophisticated approach — for example, "stability in a finite (time) interval." (Reference 25 presents a discussion of this approach.) For practical purposes of stability behavior determination, if an airplane appears to behave in a finite time interval in such a manner that equation (9.7) would be satisfied if $(t \to \infty)$, it may be considered to have asymptotic stability. This is, of course, a liberty being applied without mathematical proof.

Lyapunov has derived a technique that does not require knowledge of the solution or its behavior for $t \to \infty$. This technique, known as the "direct method," is discussed in par. 9.3.

9.3 The Direct Method of Lyapunov

Lyapunov studied the relationship between an arbitrary function and the differential equation of perturbed motion (equation 9.1), and deduced the following stability criteria (theorems).

Theorem 1

$\{\dot{x}\} = \{R(\{x\}, t)\}$ has a stable equilibrium if there exists a positive definite function $V = V(\{x\}, t)$ whose total derivative:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=0}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \frac{\partial V}{\partial t} + \left\{\frac{\partial V}{\partial x_i}\right\}^T\{\dot{X}\} = \frac{\partial V}{\partial t} + \left\{\frac{\partial V}{\partial x_i}\right\}\{R(\{X\}, t)\}$$

for the differential equation is nonpositive.

Theorem 2

$\{\dot{x}\} = \{R(\{x\}, t)\}$ has an asymptotically stable equilibrium if, in addition to theorem 1,

$$\lim_{t \to \infty} \{P(t, \{X_o\}, t_o)\} = \{0\}$$

for all $t \geq t_o$ and dV/dt is negative definite.

Proofs of these theorems may be found in ref. 25. However, further clarification of the theorems is important for a better understanding of the direct method. For that reason, the following interpretation is presented.

The total derivative of V is given by:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial t} \tag{9.8}$$

since $\partial x_i/\partial t = \dot{x}_i$,

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i \tag{9.9}$$

or

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \left\{\frac{\partial V}{\partial x_i}\right\}^T\{\dot{x}\} \tag{9.10}$$

As already defined in equation (9.1), and in the theorems above,

$$\{\dot{x}\} = \{R(\{x\}, t)\}$$

therefore:

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \left\{\frac{\partial V}{\partial X_i}\right\}^T \left\{R\left(\{X\}, t\right)\right\} \tag{9.11}$$

This establishes the relationship between V and the equations of motion. Now consider asymptotic stability for a two-dimensional problem. Regardless of the behavior of

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} P_1(t, \{x_0\}, t_0) \\ P_2(t, \{x_0\}, t_0) \end{Bmatrix}$$

and because of theorem 2, the geometric relation shown in fig. 17 holds between V and {x}.

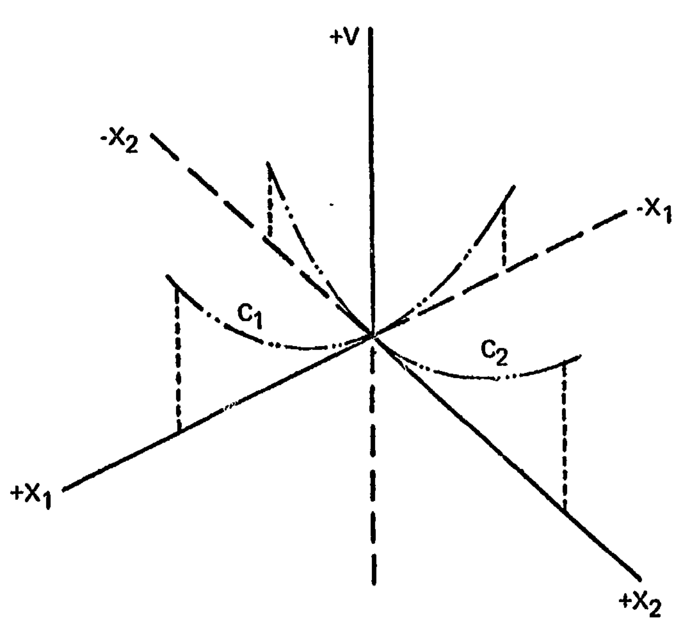


FIGURE 17. – GEMETRIC RELATIONSHIP BETWEEN THE LYAPUNOV FUNCTION V AND THE MOTION VARIABLES X_1 AND X_2

In fig. 17, the curves C1 and C2 are the projections of $V(\{P\}, t)$ on the $(V, -x_i)$ planes. This makes sense because $V > 0$ for all $(\{x\}, t)$, since it is positive definite. Furthermore, theorem 2 implies that:

$$\left.\begin{array}{ll} \text{a)} & \text{If } X_i < 0 \text{ and increasing, then: } V \to 0 \\ \text{b)} & \text{If } X_i > 0 \text{ and decreasing, then: } V \to 0 \end{array}\right\} i = 1, 2$$

due to $|\{x\}| \to 0$. This gives the following relations (from fig. 17):

$$\frac{\partial V}{\partial x_i} > 0 \quad (x_i > 0) \tag{9.12}$$

$$\frac{\partial V}{\partial x_i} < 0 \quad (x_i < 0) \tag{9.13}$$

Returning to the general case, where $i = 1, 2, \ldots, n$, consider the isolated situation where:

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x_i} = 0$$

for all i except k. Then, for $dV/dt < 0$, from equation (9.10):

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_k} \dot{x}_k < 0 \tag{9.14}$$

If $x_k > 0$, then from inequality (9.12) $\partial V/\partial x_k > 0$.

If $\partial V/\partial x_k > 0$, then $\dot{x}_k < 0$ to satisfy inequality (9.14).

Therefore, the relationship:

$$\dot{x}_k < 0 \quad (x_k > 0) \tag{9.15}$$

holds and automatically implies that $x_k = P_k(t, \{x_0\}, t_0)$ is converging toward $x_k = 0$, $\dot{x}_k = 0$. A similar argument for inequality (9.13) yields a similar conclusion (stationary or convergent) for $x_k < 0$:

$$\dot{x}_k > 0 \quad (x_k < 0) \tag{9.16}$$

Since theorem 2 requires dV/dt to be negative definite, x_k is converging toward zero. Thus the motion is returning to the equilibrium X_{k_1} because $x_k \to 0$ implies that $X_k = X_{k_1} + x_k$ approaches X_{k_1}.

This approach may be extended to the case $\partial V/\partial t \neq 0$, $\partial V/\partial x_i \neq 0$ for any i. Since relations (9.12) and (9.13) hold for all i, assume that in

$$\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i$$

not all $(\partial V/\partial x_i)\,\dot{x}_i$ are negative, but $dV/dt < 0$. For $(\partial V/\partial x_i)\dot{x}_i > 0$, either (1) $\partial V/\partial x_i$, $\dot{x}_i < 0$ or (2) $\partial V/\ x_i$, $\dot{x}_i > 0$. Consider (1) as depicted in fig. 18. Let x_i be positive and diverging and $\dot{V} < 0$ and observe $V = V(x_i(t), t)$.

The projections of $V(x_i(t), t)$ onto the (V, x), (V, t), and (x_i, t) planes give the curves $V(x_i)$, $V(t)$, and $x_i(t)$, respectively. It is obvious that for $\partial V/\partial x_i > 0$ (case (1)) and $\partial V/\partial t < 0$, $\dot{x}_i > 0$, so case (1) is not acceptable; that is, the projected curve $x'_i(t)$ for $x'_i < 0$ would not intersect $V(x_i(t), t)$. A similar discussion of case (2) would lead to its elimination also. Therefore, all terms $\partial V/\partial x_i\ \dot{x}_i$ must satisfy:

$$\frac{\partial V}{\partial x_i}\,\dot{x}_i < 0 \qquad (i = 1, 2, \cdots, n) \tag{9.17}$$

and further must satisfy one or the other of inequalities (9.15), (9.16). Similar arguments and rationalizations can be used to consider all the possible variations of dV/dt and theorems 1 and 2. The major point to be established is the relationship of dV/dt to the differential equations of motion, as in equation (9.11), and the resulting implications, namely inequalities (9.15) and (9.16). Theorems on instability and more elaborate discussions concerning phase space (n dimensions) may be found in refs. 25, 52, and 53.

9.4 Connection of the Direct Method With the Characteristic Equation Method

9.4.1 Derivation of Criterion. - Lyapunov functions are rather arbitrary, as mentioned previously, but there are some convenient forms that are easier to work with than others. Because of their definiteness properties, quadratic forms are particularly convenient. For example, the familiar characteristic equation method can be proven using the direct method of Lyapunov with the Lyapunov function:

$$V = \frac{1}{2}\{X\}^T\{X\} \tag{9.18}$$

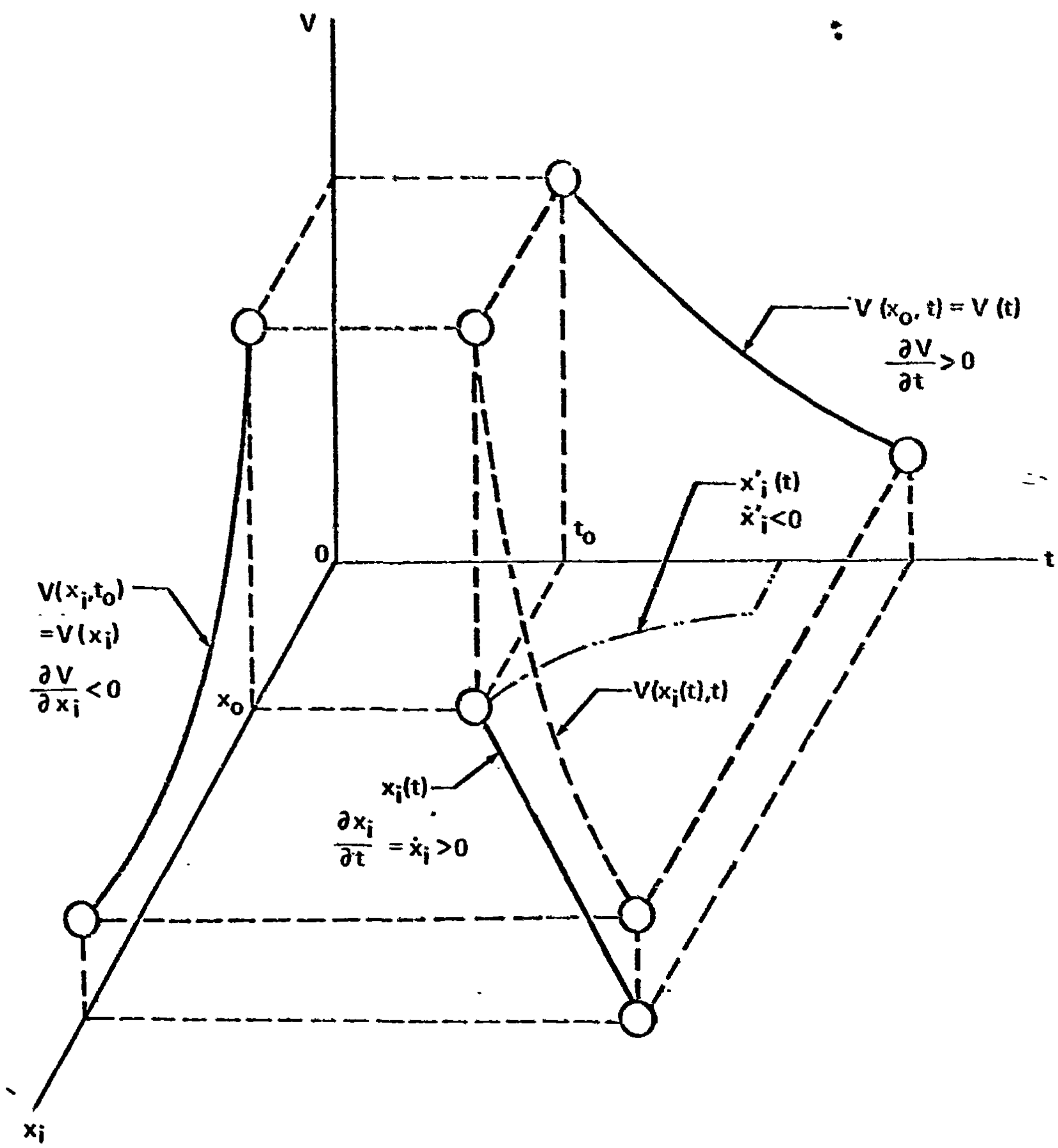


FIGURE 18. – GEOMETRIC RELATION BETWEEN V, x_i, AND t

The familiar rigid-airplane, autonomous, linear, small perturbation equations of motion may be written:

$$\{\dot{x}\} = [A]\{x\} \tag{9.19}$$

where $\left|A\right| = \left|A_{ij}\right|$ = constant n x n matrix. The usual procedure in solving these equations is to take the Laplace transform of (9.19), giving:

$$s\,x(s) - x(t_o) = [A]\,x(s) \tag{9.20}$$

This reduces to:

$$\left[s[1] - [A]\right]\{x(s)\} = \{x(t_o)\} \tag{9.21}$$

the solution of which is:

$$\{x(s)\} = \left[s[1] - [A]\right]^{-1}\{x(t_o)\} \tag{9.22}$$

The denominator of the right-hand side of equation (9.22) is the characteristic form:

$$\left\| [s] - [A] \right\| \tag{9.23}$$

The roots S_i are found by solving the n^{th}-degree polynomial in s:

$$\left\| [s] - [A] \right\| = 0 \tag{9.24}$$

From dynamic stability criterion 8.1 of Sec. 8, it is already known that if the real parts σ_i of the roots $S_i = \sigma_i + j\omega_i$ are negative, the airplane is called stable. The reason for this statement of stability is that solutions to equation (9.22) can be written in the familiar form:

$$x_j = \sum_{i=1}^{n} C_{ij}\, e^{(\sigma_i + jw)t}$$

Now consider the direct method of Lyapunov, using Lyapunov function (9. 18). It is a positive definite quadratic form. The total derivative of V is given by:

$$\frac{dV}{dt} = \frac{1}{2}\frac{d}{dt}\left(\{X\}^T\{X\}\right) = \frac{1}{2}\{\dot{X}\}^T\{X\} + \frac{1}{2}\{X\}^T\{\dot{X}\} \qquad (9.25)$$

From equation (9. 19):

$$\{\dot{X}\}^T = \{X\}^T[A]^T \qquad (9.26)$$

Substituting equations (9. 19) and (9. 26) into equation (9. 25) yields:

$$\frac{dV}{dt} = \frac{1}{2}\{X\}^T\left[[A]^T + [A]\right]\{X\} \qquad (9.27)$$

or

$$\frac{dV}{dt} = \{X\}^T\frac{1}{2}\left[[A]^T + [A]\right]\{X\} \qquad (9.28)$$

Equation (9. 28) is a real symmetric quadratic form and has definiteness properties determined by the eigenvalues of:

$$\frac{1}{2}\left[[A]^T + [A]\right] \qquad (9.29)$$

Let the necessarily real eigenvalues λ of expression (9. 29) be determined by solving the determinant:

$$\left\| \frac{1}{2}\left[[A]^T + [A]\right] - \lambda[1] \right\| = 0 \qquad (9.30)$$

The eigenvalues λ_A of $[A]$ have real parts $Re(\lambda_A)$ such that:

$$\lambda_{min.} \leq Re(\lambda_A) \leq \lambda_{max.} \qquad (9.31)$$

which can be concluded from ref. 25, page 37.

From theorem 1, equation (9.19) will have a stable equilibrium if dV/dt (equation (9.28) is negative (nonpositive). Since the right-hand side of equation (9.28) is a quadratic form, then for stability the eigenvalues λ must satisfy:

$$\lambda \leq 0 \tag{9.32}$$

However, to rule out neutral stability, it is necessary to assume asymptotic stability. From equation (9.18), it is obvious that

$$\lim_{|\{x\}| \to 0} V(\{x\}, t) = 0$$

and if, instead of expression (9.32), the inequality

$$\lambda < 0 \tag{9.33}$$

is satisfied, then theorem 2 will be satisfied. Using inequality (9.33), inequalities (9.31) may be rewritten:

$$\lambda_{min.} \leq Re(\lambda_A) \leq \lambda_{max.} < 0 \tag{9.34}$$

From this it immediately follows that:

$$Re(\lambda_A) < 0 \tag{9.35}$$

This corresponds exactly to the stability criterion of characteristic equations as expressed by 8.1. This demonstrates that the characteristic equation method is a consequence of the more general direct method of Lyapunov.

For more complex problems than those represented by the conventional equations (9.19), the application of the direct method becomes very complicated and the matrix manipulations unwieldy. Consider as an example the case where the perturbed equations of motion are nonautonomous but linear:

$$\{\dot{x}\} = [A(t)]\{x\} \tag{9.36}$$

A Lyapunov function that allows a statement of a sufficient condition for stability is.

$$V = \varphi(t)\{x\}^T [B(t)]\{x\} \tag{9.37}$$

as shown in ref. 25. A quick assessment of the complicated form dV/dt will assume should show that a more practical (even though perhaps less accurate) approach is desirable.

One choice that has proven fruitful is a Lyapunov function representing energy. For example, if the motion velocities are given by $\{z\}$, a Lyapunov function equal to the kinetic energy will be:

$$V = \frac{1}{2} \{z\}^T [M] \{z\} \tag{9.38}$$

where $[M]$ is the diagonal matrix of inertial characteristics. Another example is the total energy:

$$V = \frac{1}{2} \{z\}^T [M] \{z\} + \int_0^w P(w)\,dw \tag{9.39}$$

where

$$\int_0^w P(w)\,dw$$

represents the potential energy.

In the general case of equation (9.36), however, this can be as difficult to deal with as solving the equations of motion by numerical integration.

The choice of Lyapunov functions is unlimited. For a particular Lyapunov function it will usually be found that for disturbances within some bound, say $|\{x_0\}| \leq h_i$, stability will be guaranteed and for some $h_2 > h_1$, if $|\{x_0\}| \geq h_2$, instability may be guaranteed. If $h_1 \rightarrow \infty$, there is no instability, i.e. the equilibrium is stable for all disturbances and h_2 does not exist. On the other hand, if $h_2 \rightarrow 0$, there is no h_1 and the equilibrium is unstable for all disturbances, no matter how small. This is because h_1 and h_2 are always positive and h_2 cannot be less than h_1. If h_1 and h_2 are equal, then h_1 (or h_2) is a boundary of initial disturbances between stability and instability, i.e. a "stability houndary." However, in general, $h_1 \neq h_2$, and for $h_1 < |\{x_0\}| < h_2$ neither stability nor instability can be deduced. Choosing a different Lyapunov function can shift h_1 and h_2, providing $h_1 \neq h_2$. This does not change the stability behavior, but it is possible that the size of the "gray" area $h_1 < \|\{x_0\}\| < h_2$ can be reduced. If several Lyapunov functions are tried, the greatest h_1 and the least h_2 define upper and lower bounds for stability and instability respectively. Thus it is observed that there should be a "best" Lyapunov function which will give the most accurate values of h_1 and h_2. However, it may be impossible or at least impractical to try finding a "best" Lyapunov function which gives the most accurate stability boundary. The definition of an approximate stability boundary is better than no boundary at all, so with a view toward practicality, a straightforward approach using a theorem attributed to Zubov is now presented.

The equations of motion for an elastic (or rigid) airplane may be written in the general matrix form:

$$\{\dot{X}\} = [F]\{X\} \tag{9.40}$$

where $[F]$ may be any form factorable from $\{R(\{x\}, t)\}$ of equation (9.1). Zubov, according to Hahn in ref. 25, proved the following theorem for $[F] = [F(\{x\}, t)]$.

<u>Theorem 3</u>

The equilibrium of equation (9.40) is stable if all eigenvalues (depending on $\{x\}$ and t) of the matrix

$$\frac{1}{2}\left[[F]^T + [F]\right]$$

are nonpositive in a certain domain R_h, t_0; that is, if the roots λ_i of the equation

$$\left\| \frac{1}{2}\left[[F]^T + [F]\right] - \lambda [\,I\,] \right\| = 0 \tag{9.41}$$

satisfy the following conditions:

$$\lambda_i \leq 0 \qquad \begin{array}{l} i = 1, 2, \cdots, n \\ t \geq t_0;\ |\{x\}| \leq h \end{array} \tag{9.42}$$

This theorem can be proven using the Lyapunov function $V = 1/2\{x\}^T\{x\}$, in a manner similar to the one used in showing equivalence of Lyapunov's direct method and the characteristic equation method. Fortunately, most airplanes have practical limits for admissible x_i values.* This allows determination of an h where:

$$h = \sum_{i=1}^{n} (x_{i\,max})^2 \tag{9.43}$$

Also, it is not possible to test all $\{x\}$ and t satisfying $|\{x\}| \leq h$ and $t \geq t_0$, since there exists an infinite number of each. However, a representative set** of $\{x\}$ and t can be substituted into [F] and the eigenvalues λ_i found for the specific values of $\{x\}$ and t.

Stability as determined by Zubov's theorem requires $\lambda_i < 0$ as a criterion. For convenience, equation (9.41) will be called the quasi-characteristic equation when $\{x_R\}$ and t_R are chosen from a representative set of $\{x\}$ and t and substituted into the equation. Thus the quasi-characteristic equation is:

$$\frac{1}{2}\left[[F(\{x_A\}, t_A)]^T + [F(\{x_R\}, t_R)]\right] - \lambda [I] = 0 \tag{9.44}$$

where:

$$\{x_A\} \in \{x\};\ |\{x_R\}| \leq h = \sum_{i=1}^{n} (x_{i\,max})^2 \qquad t_R \geq t_o$$

With this definition and on the basis of theorem 3, the following dynamic stability criterion will be defined for equations of the type of (9.40).

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*Already discussed in par. 8.5.

**A discussion of what is meant by "representative set" is given in par. 8.5.

Dynamic Stability Criterion

An airplane whose equations of motion are given by

$$\{\dot{x}\} = [F]\{x\}$$

will be called stable if the roots of the quasi-characteristic equation (9.44) are all negative (nonpositive) for a representative set of the variables in R_h, t_o.

Stability Boundary

An approximate stability boundary for an airplane whose equations of motion are given by

$$\{\dot{x}\} = [F]\{x\}$$

will be those values of $\{x_o\}$ for which at least one of the roots of the quasi-characteristic equation vanishes and for which values $|\{x_R\}| > |\{x_o\}|$ yield one or more positive roots.

It is believed that this criterion, once properly computerized, could be a significant breakthrough in the analysis of airplane stability in nonlinear and/or nonautonomous situations. There are some limitations, however, and these are discussed next.

9.4.2 Limitations. - In using Zubov's theorem 3, it is recognized that using a particular Lyapunov function ($V = 1/2\{x\}^T\{x\}$) does not necessarily give a "best" stability boundary. In fact, the existence of positive eigenvalues of the quasi-characteristic equation does not necessarily imply instability.

This is a serious limitation that must be recognized when applying Zubov's theorem. A simple example will illustrate it. For a very simple form of perturbed equations of motion (linear and autonomous), assume the following:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{Bmatrix} = \begin{bmatrix} -1 & 1 \\ -7 & -3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_1 \end{Bmatrix} = [A]\{X\} \tag{9.45}$$

Using [A] from equation (9.45):

$$\frac{1}{2}\left[[A]^T + [A]\right] = \begin{bmatrix} -1 & -3 \\ -3 & -3 \end{bmatrix} \tag{9.46}$$

and the eigenvalues are given by the roots λ of:

$$\left\| \begin{bmatrix} -1 & -3 \\ -3 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right\| = 0 \tag{9.47}$$

Expanding equation (9.45), the following is obtained:

$$\lambda^2 + 4\lambda + 3 - 9 = 0$$

which may be factored into

$$(\lambda + 2 - \sqrt{10})(\lambda + 2 + \sqrt{10}) = 0$$

giving the roots:

$$\begin{aligned} \lambda_1 &= +1.16 \\ \lambda_2 &= -5.16 \end{aligned} \tag{9.48}$$

Therefore, Zubov's theorem is not satisfied because of $\lambda_1 > 0$.

Now consider the characteristic equation approach. As shown previously, the following determines the stability:

$$\left\| \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ -7 & -3 \end{bmatrix} \right\| = 0 \tag{9.49}$$

This reduces to:

$$\left\| \begin{bmatrix} S+1 & -1 \\ 7 & S+3 \end{bmatrix} \right\| = 0$$

and further to:

$$s^2 + 4s + 3 + 7 = 0 \tag{9.50}$$

This may be factored into

$$(s + 2 - j\sqrt{6})(s + 2 + j\sqrt{6}) = 0$$

giving the roots:

$$s_{1,2} = -2 \pm j\sqrt{6} \tag{9.51}$$

These indicate a stable solution.

9.4.3 <u>Conclusions.</u> - If the equations of motion of an airplane satisfy Zubov's theorem, stability is guaranteed. However, the simple example above shows that positive eigenvalues do not necessarily imply instability and, in fact, a stable system was shown to violate the condition of Zubov's theorem. Because of the "roughness" of this approach, a more accurate approach may be required. Research into this area seems to be needed.

9.5 Stability According to the First Approximation*

The development up to here has shown how Lyapunov stability theory applies to nonlinear and/or nonautonomous equations of motion. What is yet to be cleared up is the question of what conditions must exist for the linearized equations to be used as an approximation to the nonlinear equations. This question has two aspects: response and stability. It is entirely possible that linearized equations yield the correct answer with regard to the stability of an airplane but yield unacceptable approximations to the response behavior.

The answer to the question of response behavior can be obtained through generation and judgment of response time histories. Without the benefit of such time histories in a particular situation, engineering judgment plays the dominant role. Reference 23 formulates such conditions that must be met by the linearized equations to adequately predict airplane response.

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*That is, according to the linearized equations

The answer to the question of stability can be obtained by using the direct method of Lyapunov, as will be discussed in the remainder of this section.

Consider the large-perturbation equations of motion in the form:

$$\{\dot{x}\} = \{R(\{x\}, t)\} = [A(t)]\{x\} + \{K(\{x\}, t)\} \tag{9.52}$$

where $\{K(\{x\}, t)\}$ is a column matrix of higher order terms; i. e., it contains products of motion variables, etc. This can be demonstrated for the large perturbation equations of motion.

Setting $\{\delta\} = \{0\}$ and all $x_i = 0$ in $[F(\{x\}, t)]$, then using $\text{sgn}(x_i) = 0$; $(x_i = 0)$:

$$[F(\{x\}, t)]_{\{x\}=0} = [A(t)]$$

Further:

$$\{K(\{x\}, t)\} = \Big[[F(\{x\}, t)] - [A(t)]\Big]\{x\}$$

For the autonomous case:

$$\{\dot{x}\} = [A]\{x\} + \{K(\{x\})\} \tag{9.53}$$

In the following it is implied that the nonlinear terms $\{K(\{x\}, t)\}$ are "sufficiently" small. The meaning of "sufficient" has not been clearly established. The following theorem can be proven:

Theorem 4 (theorem 26.2 in ref. 25)

If the motions of the linearized differential equation given below have intensive behavior*, then the complete and the linearized differential equations have the same stability behavior where:

a) The complete equation is given by equation (9.52) or (9.53).

b) The reduced (linearized) equation is given by:

$$\{\dot{x}\} = [A(t)]\{x\} \tag{9.54}$$

c) For the autonomous case, the reduced equation becomes:

$$\{\dot{x}\} = [A]\{x\} \tag{9.55}$$

Equations (9.54) and (9.55) are called "first approximations."

Note that there are two conditions to be satisfied here:

a) The nonlinear terms must be sufficiently small.

b) The behavior of the first approximation must be exponentially stable or unstable.

Also observe that both are necessary conditions. Condition b) can be relatively easy to satisfy in the autonomous case. The follov ·.g discussion of items a) and b) for the autonomous case has been extracted from ref. 23, par. 2.3, pp. 15-19, with appropriate changes to fit the discussion here.

9.6 The Validity of Linear Small Perturbation Equations of Motion in Predicting Stability Behavior

For sufficiently small perturbations the linear approximation theorem of Lyapunov (theorem 4, herein) is applicable. For such sufficiently small disturbances, this theorem merely restates the well-known fact that the stability characteristics of the uncontrolled airplane can be obtained from the roots of the characteristic equation of the linear approximation to the equations of

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*Intensive behavior means that every motion admits, along at least one of its branches, exponential stability or instability for all $\{x_0\}$ or $t > t_0$. See ref. 25 for further discussion of intensive behavior.

motion. In other words, if all the roots of equation (9.55) have negative real parts, the airplane is stable. If one the roots has a positive real part, the airplane is called unstable. If one of the roots is zero, the set of equations is critical according to Lyapunov* and the stability characteristics must be obtained from the nonlinear equations.

It was stated above that the theorem of linear approximation is valid for sufficiently small perturbations. It is intuitively acceptable that for infinitesimally small disturbances (initial conditions leading to a motion deviating slightly from the flight considered) a linear approximation will yield correct results. This knowledge is of little value to the airplane designer, since stability must be ensured in an environment of finite (and sometimes quite large) disturbances. The problem is therefore to define a domain of initial disturbances within which the small-disturbance theory will correctly predict the stability behavior of the airplane.

Texts on airplane stability and control, such as refs. 13 through 16, either fail to bring this problem up or to solve it in a useful manner. This is surprising because the key to its solution can be found in applying Lyapunov's direct method. The following development will clarify this point.

For the linear, autonomous equations of motion (equation 9.55) it is always possible to construct a Lyapunov function V (ref. 53, p. 57). Construction of this function V can be carried out as follows. Assume:

$$V = \frac{1}{2}\sum_{i,j} B_{i,j}\, x_i\, x_j = \frac{1}{2}\{x\}^T[B]\{x\}$$
$$[B]^T = [B]$$

If V is to be a Lyapunov function for the linear part of equation (9.53), it must satisfy:

$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_i \frac{\partial V}{\partial x_i}\left(a_{i1}x_1 + \cdots + a_{in}x_n\right) = C$$

*For further discussion of the meaning of cr[illegible] havior, see ref. 25.

where C is any negative definite form. It is convenient to select:

$$C = -\left(x_1^2 + \cdots + x_n^2\right) \tag{9.58}$$

The coefficients a_{i_j} are the elements of [A] in equation (9.55). It turns out that [B] will be positive definite if and only if the eigenvalues of [A] have negative real parts. A unique solution for [B] can always be found (ref. 53, p. 57).

The function V obtained in this manner is a Lyapunov function for the linear equations. It is also a Lyapunov function for the nonlinear equations in some small neighborhood of the origin $\{x(0)\} = \{0\}$ if it satisfies theorem 1 in the sense that for the complete equation (9.53):

$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i} \dot{x}_i \leq 0 \tag{9.59}$$

It is emphasized that even though V has been derived for the linear equations, x_i in inequality (9.59) must be computed for the nonlinear set. Thus, in (9.59):

$$\{\dot{x}_i\} = [A]\{x_i\} + \{K(\{x\})\}$$

By checking inequality (9.59) systematically for combinations of values of initial disturbances, a domain of initial disturbances is found within which the linear approximation is valid.

The domain of small disturbances found in this manner guarantees the validity of small-disturbance theory for disturbances inside the domain.

Outside the domain there still exists a possibility that small-disturbance theory applies. Because this method of constructing the domain will at least verify whether or not the domain is large enough to be practical, this last fact is not considered a serious disadvantage. Chetayev and Malkin have discussed the problem of enlarging the domain of initial disturbances in ref. 52 and in ref. 53 (pp. 71-73) respectively.

A useful observation is the following. It is possible to include in the nonlinear part $\{K(\{x\})\}$ of equation (9.52) expressions representing nonlinearities in the aerodynamic forces and moments. In this manner the effect of aerodynamic nonlinearities on the size of the domain of initial disturbances can be determined.

Having computed the domain of validity of the linear approximation, attention can be focused on methods to determine the stability behavior. It is assumed that the domain so found has a practical size, meaning that it is not infinitesimal in nature.

An additional possibility is observed in the following: If in equation (9.56) $[B] = [1]$ is arbitrarily chosen, and if at the same time $\{K(\{x\})\}$ is factorable into $[K'(\{x\})]\{x\}$, equation (9.53) can be written:

$$\{\dot{X}\} = \Big[[A] + [K'(\{X\})]\Big]\{X\} \tag{9.60}$$

or:

$$\{\dot{X}\} = [F(\{X\})]\{X\} \tag{9.61}$$

Since, now,

$$V = \frac{1}{2}\{X\}^T\{X\} \tag{9.62}$$

it is observed that Zubov's theorem (theorem 3), is a consequence of the previous discussion. Note also that although Zubov's theorem may yield less accurate answers, it eliminates the need to solve for $[B]$.

If the nonautonomous case is treated, the approach is the same, but $[B(t)]$, and sometimes an arbitrary positive function $\varphi(t)$, is required as indicated in equation (9.35):

$$V = \varphi(t)\{X\}^T[B(t)]\{X\}$$

If $[B(t)] = [1]$ and $\varphi(t) = 1$ are chosen arbitrarily, then the most general case of Zubov's theorem,

$$\dot{V} = \{x\}^T \frac{1}{2}\left[\left[F(\{x\},t)\right] + \left[F(\{x\},t)\right]^T\right]\{x\}$$

follows immediately. Here again the need to solve for $[B] = [B(t)]$ is eliminated along with choosing $\varphi(t)$.

It should be evident that even in the limited presentation given here there are apparent practical applications of Lyapunov stability theory in airplane stability analysis. More research in this area may uncover even more practical applications or better ways to approach current problems.

10. CONCLUSIONS AND RECOMMENDATIONS

A careful derivation of the equations of motion for an elastic airplane has been presented in Secs. 4, 5, and 6. The aerodynamic and structural operators needed to solve the equations of motion are discussed in Sec. 6 and in app. B.

The development of equations of motion for the completely elastic airplane relies on the influence coefficient concept for both structural and aerodynamic representations. The advantage of this concept is that, in itself, it does not require a commitment to any particular aerodynamic or structural model except that superposition must be valid. This restricts the aerodynamic theory (whatever it may be) to small angles of incidence, and restricts the structural theory (whatever it may be) to small strain and displacement and Hooke's law.

Several assumptions made in the derivation have important consequences with regard to the restrictions imposed on the analysis. By carrying out additional research it would be possible to remove several assumptions from the analysis. The most restrictive assumptions are listed below, together with recommendations for additional research.

G1 Constant mass and mass distribution. — It was stated that this assumption implies that no fuel slosh is accounted for. This assumption can be removed realistically only by assuming a model for the fuel tanks, the baffling arrangement, etc., and then including equations accounting for the dynamic behavior of the fuel and its effect on the entire airplane. This has not been done in this report and would require additional research.

S2 Small strains and displacements. — For elastic airplanes with very long, slender bodies or wings, it is possible that the linear force-deflection relation is violated. If this is ever felt to be important, a careful investigation must be made of the static and dynamic structural representations used in this report. This will require additional theoretical and experimental research.

A2 Aerodynamic influence coefficients for zero sideslip. — An important consequence of the restriction to zero sideslip of aerodynamic influence coefficient theory is that at present no matrix expressions can be generated for sideslip forces and moments on total airplane configurations. The restriction to zero sideslip in aerodynamic influence coefficient theory also makes it impossible to develop matrix equations describing the steady-state equilibrium of elastic airplanes under sideslip conditions. To remove this restriction will require additional theoretical and experimental research. Another important limitation of aerodynamic influence coefficient theory is that it is valid only for small angles of incidence.

The static and dynamic stability criteria have been derived for an elastic airplane. It has been shown that the basic form of these criteria is the same for rigid, equivalent elastic, and completely elastic airplanes.

Static and dynamic stability criteria have been summarized in tables 8 and 9 respectively.

Physical interpretations have been presented for those derivatives that appear in the stability criteria. Where practical, the stability criteria have been related to known flight experience. Also, the relationship between stability parameters, stability criteria, and handling qualities has been discussed.

Specific conclusions that were reached can be summarized as follows:

a) The mathematical formulation of stability criteria is the same for the rigid, equivalent elastic, and completely elastic airplanes.

b) Static longitudinal stability is, in general, a prerequisite for dynamic longitudinal stability.

c) Static stability is not necessarily required for good handling qualities.

The following areas are recommended for additional research:

a) A study should be made of the effect of airplane elasticity on the behavior of phugoid and short period with all speed derivatives properly accounted for.

b) It will be necessary to develop a capability for calculating time histories, including elastic degrees of freedom and unsteady aerodynamic effects.

c) The energy decay method for judging stability behavior and its relation to flying qualities needs to be further explored.

d) More research is needed to establish the practical application of Lyapunov theory to airplane stability analysis.

e) Further research is needed before a generally acceptable procedure can be defined with which a decision can be made as to the minimum number of elastic degrees of freedom that are needed in stability and response studies.

f) A discrepancy was found with regard to the interpretation of Routh's test functions. Reference 49 was found to disagree with ref. 48. This discrepancy has not been further investigated.

g) Not considered under the scope of this contract is the case of stability under continuously acting disturbances (for example, gust). It is felt that a thorough understanding of this type of stability is essential in studying the upset and recovery behavior of airplanes. It is highly recommended that study of this type of stability be initiated.

h) This report does not deal with controlled airplanes. Research is needed to establish the stability criteria (qualitative and quantitative) for an elastic airpl e when controlled by an automatic system.

It has been shown that the conventional notion of associating static longitudinal stability C_{m_α} with stability of the flight path is generally correct only if the speed derivatives C_{L_u}, and C_{m_u} can be neglected. In addition, it has been shown that this holds true for steady climbs and dives provided dynamic pressure remains reasonably constant. The additional assumption that the thrust derivatives be negligible is also required. Whether or not this assumption is justified depends on many factors, particularly on the location of the engines.

Directional stability C_{n_β} has been shown to affect stability of the flight path (spiral stability). Positive C_{n_β} actually hurts spiral stability. It has been shown that positive dihedral effect $C_{\ell_\beta} < 0$ is necessary for the spiral mode to be stable, but it was reasoned that some degree of spiral instability must be tolerated in view of the detrimental effect of C_{ℓ_β} on Dutch roll.

For the completely elastic airplane, it is concluded that a numerical evaluation is needed to determine the effect of normal modes on stability of the flight path.

11. REFERENCES

This section includes all of the references for the Summary Report and the three appendixes.

1. Bisplinghoff, R. L.: and Ashley, H.: Principles of Aeroelasticity. John Wiley and Sons, Inc., 1962.

2. Milne, R. D.: Dynamics of the Deformable Airplane, Parts I and II. Her Majesty's Stationery Office, London, 1964.

3. Frazer, R. A.; Duncan, W. J.; and Collar, A. R.: Elementary Matrices. Cambridge University Press, 1952.

4. Etkin, Bernard: Dynamics of Flight. John Wiley and Sons, Inc., 1962.

5. Schwendler, R. G.; and MacNeal, H.: Optimum Structural Representation in Aeroelastic Analysis. ASD-TR-61-680, Computer Engineering Associates, March 1962.

6. Anon.: USAF Stability and Control Handbook. AF33(616)-6460, Douglas Aircraft Company, 1960.

7. Lamb, H.: Hydrodynamics. Second ed., Dover Publications, 1945.

8. Sokolnikoff, I. S.: Mathematical Theory of Elasticity. McGraw-Hill Book Company, Inc., 1956.

9. Hildebrand, F. B.: Advanced Calculus for Engineers. Prentice-Hall, Inc., 1957.

10. Anon.: Military Specification Flying Qualities of Piloted Airplanes. MIL-F-8785 (ASG), April 17, 1959.

11. Anon.: British Civil Airworthiness Requirements. Section D Aeroplanes, Air Registration Board, Issue 8, February 1, 1966.

12. Anon.: Proposal for a Revised Military Specification, Flying Qualities of Piloted Airplanes. MIL-F-8785 (ASG), with Substantiating Text, Bureau of Naval Weapons, Washington, D.C., Report No. NADC-ED-6282, January 18, 1963.

13. Seckel, E.: Stability and Control of Airplanes and Helicopters. Academic Press, Inc., 1964.

14. Babister, A. W.: Aircraft Stability and Control. Pergamon Press, Inc., 1961.

15. Perkins, C. D.; and Hage, R. E.: Airplane Performance, Stability and Control. John Wiley and Sons, Inc., 1957.

16. Kolk, W. R.: Modern Flight Dynamics. Prentice-Hall, Inc., 1961.

17. Abramson, H. N.: The Dynamics of Airplanes. The Ronald Press Company, 1958.

18. Chestnut, H.; and Mayer, R. W.: Servomechanisms and Regulating System Design. Second ed., John Wiley and Sons, Inc., 1963.

19. Graham, D.; and McRuer, D.: Analysis of Nonlinear Feedback Control Systems. John Wiley and Sons, Inc., 1961.

20. Pastel, M. P.; and Thaler, G. J.: Analysis and Design of Nonlinear Feedback Control Systems. McGraw-Hill Book Company, Inc., 1962.

21. Levinson, Emanuel: Nonlinear Feedback Control Systems. Electro-Technology, July through December 1962.

22. Davis, Harold T.: Introduction to Nonlinear Differential and Integral Equations. Government Printing Office, September 1960.

23. Roskam, J.: On Some Linear and Nonlinear Stability and Response Characteristics of Rigid Airplanes and a New Method to Integrate Nonlinear Ordinary Differential Equations. PhD Dissertation, University of Washington, July 1965.

24. Jaffe, Peter: A Generalized Approach to Dynamic-Stability Flight Analysis. JPL Technical Report No. 32-757, July 1965.

25. Hahn Wolfgang, ed. and trans.: Theory and Application of Liapunov's Direct Method. Prentice-Hall, Inc., 1963.

26. Bisplinghoff; Ashley; and Hoffman: Aeroelasticity. Addison-Wesley Publishing Company, Inc., 1955.

27. Ashley; and Landahl: Aerodynamics of Wings and Bodies. Addison-Wesley Publishing Company, Inc., 1965.

28. Miles, J. W.: The Potential Theory of Unsteady Supersonic Flow. Cambridge University Press, 1959.

29. Chester, W.: Supersonic Flow Past Wing-Body Combinations. The Aeronautical Quarterly, Vol. IV, August 1953, pp. 287-314.

30. Ward, G. N.: Linearized Theory of Steady High-Speed Flow. Cambridge University Press, 1955.

31. Van Dyke, Milton D.: Supersonic Flow Past Oscillating Airfoils Including Nonlinear Thickness Effects. NACA Report 1183, 1954.

32. Bryson, Arthur E.: Stability Derivatives for a Slender Missile with Application to a Wing-Body-Vertical Tail Configuration. J. Aeron. Sci., Vol. 20, May 1953.

33. Landau; and Lifshitz (J. B. Sykes and J. S. Ball, trans.): Mechanics. Addison-Wesley Publishing Company, Inc., 1960.

34. Hayes, W. D.: and Probstein, R. F.: Hypersonic Flow Theory. Academic Press, Inc., 1959.

35. Woodward, F.; LaRowe, E.; and Love, J. E.: Analysis and Design of Supersonic Wing-Body Combinations, Including Flow Properties in Near Field. Part I and II, NASA Report CR-73107, 1967.

36. Anon.: Flight Control and Fire Control System Manual. Vol II, AE-61-4 II, Bureau of Aeronautics (prepared by Northrop Corporation), September 1952.

37. Francis, J. G. F.: The QR Transformation. Part I, The Computer Journal, Vol. 4, No. 3, October 1961, pp. 263-271. Part II, IBIO, January 1962, pp. 332-345.

38. Wilkinson, J. H.: The Algebraic Eigenvalue Problem. Clarendon Press, Oxford, 1965.

39. Pearce, B. F.; Johnson, W. A.; and Siskind, R. K.: Analytical Study of Approximate Longitudinal Transfer Functions for a Flexible Airframe. ASD-TDR-62-279, June 1962.

40. Samson, F. J.; and Petersen, Harry E.: MIMIC Programming Manual, SEG-TR-67-31, July 1967.

41. Van Dyke, M.: Perturbation Methods in Fluid Mechanics. Academic Press, Inc., 1964.

42. Fung, Y. C.: An Introduction to the Theory of Aeroelasticity. John Wiley and Sons, Inc., 1955.

43. Milne, R. D.: Some Remarks on the Dynamics of Deformable Bodies. AIAA Journal, Vol. 6, March 1968.

44. Anon.: Air Worthiness Standards, Transport Category Airplanes. FAR, Part 25 Federal Aviation Administration.

45. Kuo, B. C.: Automatic Control Systems. Prentice-Hall, Inc., 1962.

46. Korn, G. A.; and Korn, T. M.: Mathematical Handbook for Scientists and Engineers. McGraw-Hill Book Company, Inc., 1961.

47. Hamming, R. W.: Numerical Methods for Scientists and Engineers. McGraw-Hill Book Company, Inc., 1962.

48. Anon.: International Dictionary of Applied Mathematics. D. van Nostrand Company, Inc., 1960.

49. Duncan, W. J.: The Principles of the Control and Stability of Aircraft. Cambridge University Press, 1956.

50. Rheinfurth, Mario H.; and Swift, Frederick W.: A New Approach to the Explanation of the Flutter Mechanism. NASA TN D-3125, 1966.

51. Haus, F. C.; Czinczenheim, J.; and Moulin, L.: The Use of Analog Computers in Solving Problems of Flight Mechanics. Agardograph 44, June 1960.

52. Chetayev, N. G.: The Stability of Motion. Pergamon Press, Inc., 1961.

53. Malkin, I. G.: Theory of Stability of Motion. AEC-TR-3352, translated from the publication of the State Publishing House of Technological-Theoretical Literature, Moscow-Leningrad, 1952.

54. Lefferts, Eugene J.: A Guide to the Application of the Lyapunov Direct Method to Flight Control Systems. NASA CR-209, 1966.

55. Bryan, G. H.: Stability in Aviation. The Macmillan Company, 1911.

56. Milne-Thomson, L. M.: Theoretical Aerodynamics. The Macmillan Company, 1948.

57. Green, A. E.; and Zerna, W.: Theoretical Elasticity. Clarendon Press, Oxford, 1960.

58. Pearce, B. F.: Topics on Flexible Airplane Dynamics, Part I. ASD-TDR-63-334, Systems Technology, Inc., Inglewood, California, 1963.

59. Heaslet, Max A.; Lanar, Harvard; and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to 3-Dimensional Supersonic Airfoil Problems. NASA Report No. 889, Ames Aeronautical Laboratory, Moffett Field, California, April 14, 1947.

60. Pope, A : Basic Wing and Airfoil Theory. McGraw-Hill Book Company, Inc., 1952.

61. Nelson, H. C.; and Berman, J. H.: Calculations on the Forces and Moments for an Oscillating Wing-Aileron Combination in Two-Dimensional Potential Flow at Sonic Speeds. NACA TN 2590, 1952.

62. Rubbert, P. E.; et al.: A General Method for Determining the Aerodynamic Characteristics of Fan-in-Wing Configurations. Vol. I - Theory and Application. USAAVLABS Technical Report 67-61A, 1967.

63. Campbell, J. P.; Johnson, J. L., Jr.; and Hewes, D. E.: Low-Speed Study of the Effect of Frequency on the Stability Derivatives of Wings Oscillating in Yaw with Particular Reference to High Angle-of-Attack Conditions. NACA RM L55H05, 1955.

64. Weissinger, J.: The Lift Distribution of Swept Back Wings. NACA TM 1120, 1947.

65. Tekhonov; and Samarskii (A.R.M. Robson and P. Basu, trans.): Equations of Mathematical Physics. Pergamon Press, Inc., 1963.

66. Royal Aeronautical Society Data Sheets, 1955.

67. Anon.: Aerodynamic Characteristics of Non-Straight-Taper Wings. AFFDL-TR-66-73, General Dynamics, Fort Worth Division, October 1966.

68. Gray, W. L.; and Schenk, K. M.: Method for Calculating the Subsonic Steady-State Loading on an Airplane with a Wing of Arbitrary Planform and Stiffness. NACA TN 3030, 1953.

69. Johnson, J. L., Jr.: Low-Speed Measurements of Static Stability, Damping in Yaw, and Damping in Roll of a Delta, a Swept and an Unswept Wing for Angles-of-Attack from 0° to 90°. NACA RM L56B01, 1956.

70. Wiley, H. G.: The Significance of Nonlinear Damping Trends Determined for Current Aircraft Configurations. NASA U-59-15, September 1966.

71. Flack, Nelson D.: AFFTC Stability and Control Techniques. USAF Flight Test Center, Edwards Air Force Base, California, AFFTC-TN-59-21, 1959.

72. Campbell, J. P.; and McKinney, M. O.: Summary of Methods for Calculating Dynamic Lateral Stability and Response and for Estimating Lateral Stability Derivatives. NACA TR 1098, 1952.

73. Kohlman, D. L.; and Drake, L. R.: Handbook for Estimating C_{ℓ_β} for Rigid and Elastic Airplanes at Subsonic and Supersonic Speeds. Center for Research Inc., Engineering Science Division, The University of Kansas, Lawrence, Kansas, 1966.

74. Pearce, B. F.; and Siskind, R. K.: Topics on Flexible Airplane Dynamics, Part II. The Application of Flexible Airframe Transfer Function Approximations and the Sensitivity of Airframe Transfer Functions to Elastic Mode Shapes. ASD-TDR-63-334, Part II, July 1963.

75. Pass, H. R.; Pearce, B. F.; and Wolkovitch, J.: Topics on Flexible Airplane Dynamics, Part III. Coupling of the Rigid and Elastic Degrees of Freedom on an Airframe. ASD-TDR-63-334, Part III, July 1963.

76. Pass, H. R.; and Pearce, B. F.: Topics of Flexible Airplane Dynamics, Part IV. Coupling of the Rigid and Elastic Degrees of Freedom of an Airframe-Autopilot System. ASD-TDR-63-334, Part IV, July 1963.