ON THE STABILITY OF RANDOMLY SAMPLED SYSTEMS

by

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Abstract

Randomly sampled linear systems with linear or non-linear feedback loops are studied by a stochastic Liapunov function method. The input, in this paper, is assumed zero (driven systems will be treated in a companion paper). Improved criteria for stability (with probability one, on s^{th} moment $s > 1$, or in mean square) are given, when the sequence of holding times are independent. The method is relatively straightforward to apply (especially in comparison with the direct methods), and allows the study with non-linear feedback, or non-stationary holding times. A randomly sampled Luré problem is studied. Numerical results, describing some interesting phenomena (such as jitter stabilized systems) are presented.

1. Introduction

The paper is concerned with systems of the type of Fig. 1, where c is a row vector, $\dot{x} = Ax$ is assumed to be asymptotically stable, and $f(\cdot)$ may be either a linear or non-linear element. The case of scalar valued f and u is of main interest, although the method is obviously usable when f and u are vector valued. The sampler samples at a sequence of random times $\tau_1, \ldots, \tau_n, \ldots$ and the holding intervals Δ_n, defined by $\Delta_n = \tau_{n+1} - \tau_n$ are assumed to be mutually independent. The input u may be random. In this paper, however, u will be assumed to be zero. Results for the 'driven' system - concerning ergodicity of the outputs (for ergodic u), recurrence properties, and estimates of the moments of the output, will appear in a companion paper. Conditions on A, B, f and $\{\Delta_n\}$ under which various stability properties hold, are obtained. Numerical results, illustrating some properties of interest are also presented. There has been a fair amount of work on such sampling systems [1] - [4]. The problem occurs in several physical situations - perhaps especially in sampling for digital-analog conversion where some random jitter is involved.

Past work, e.g. [2], has generally dealt with specific, and relatively simple (often scalar with linear f) cases, and has involved (even for scalar cases with linear f) quite direct and very tedious calculations. In this paper, an approach based on the idea of a stochastic Liapunov function [5] - [7] is taken. All results on the system of Fig. 1 that were available (as known to the authors) can be obtained much more easily, and a number of easy extensions to the conditions for stability are given. Furthermore, non-linear f (see, e.g., the randomly sampled Luré problem of Section 4) can sometimes be treated, and path excursion probabilities and moment estimates obtained and probability are convergence as well as convergence in certain moments can be studied. In addition, it is not necessary that the Δ_n be identically distributed. Also, as common to Liapunov-function-like methods, perturbation results are also possible. E.g., if a system has a certain type of stability, then so will a slightly perturbed system. Furthermore (as will appear) the estimates (except for ergodicity) for driven systems usually do not even require stationarity of the inputs.

2. Stability Theorems

Let $u \equiv 0$. From Fig. 1,

$$\dot{x} = Ax - KBf(cx(\tau_n)) \quad \text{for} \quad \tau_{n+1} > t \geq \tau_n. \tag{1}$$

Let x_n, denote $x(\tau_n)$. Then

$$\begin{aligned} x_{n+1} &= e^{A\Delta_n} x_n - e^{A\Delta_n} \int_o^{\Delta_n} e^{-As} ds KBf(cx_n) \\ &= e^{A\Delta_n} x_n + A^{-1}(1-e^{A\Delta_n}) KBf(cx_n) \\ &\equiv F_n X_n + G_n f(cx_n), \end{aligned} \tag{2}$$

where the (F_n, G_n) are independent in n. If x_o is independent of $\{\Delta_n\}$, then the sequence $\{x_n\}$ is easily verified to be a (discrete time) Markov process. The following stability lemma will be central to the development of the sequel.

<u>Lemma</u> 1. (For proof see [8] or [7].) *Let* $\{Y_n\}$ *be a Markov process, and* $V(Y)$ *a non-negative function. Suppose that*

$$E_Y V(Y_{n+1}) - V(Y) = -k(Y) \leq 0 \tag{3}$$

(where E_Y *is the expectation given that* $Y_n = Y$*). Then the sequence* $\{V(Y_n)\}$ *is a non-negative super martingale and there is a* $V \geq 0$ *so that* $V(Y_n) \to V$ w.p.1. *(with probability one), and*

$$P_Y\{\sup_{\infty > n \geq 0} V(Y_n) \geq \epsilon\} \leq V(Y)/\epsilon \tag{4}$$

(where P_Y *is the probability given that* $Y_o = Y$*). Also* $E_Y \sum_o^\infty k(Y_n) \leq V(Y)$ *and* $k(Y_n) \to 0$ w.p.1. *(The last statement* $(k(Y_n) \to 0)$ *is merely the Borel-Cantelli Lemma. See* [7, p. 71].)

<u>Lemma</u> 2. *Suppose the conditions of Lemma 1, except that*

$$E_Y V(Y_{n+1}) - V(Y) \leq -\gamma V(Y), \quad \gamma > 0. \tag{5}$$

Then[†]

$$P_Y\{\sup_{\infty > n \geq 0} V(Y_n)(1-\gamma)^{-n} \geq \epsilon\} \leq V(Y)/\epsilon$$

[†]Note that $V(Y) \geq 0$ implies that $\gamma \leq 1$.

and

$$E_Y V(Y_n)/(1-\gamma_1)^n \to 0 \quad \text{where} \quad 0 < \gamma_1 < \gamma.$$

Proof. The proof is similar to that of [7], Theorem 3, p. 86. It is readily verified that $\tilde{V}(x,n) = V(Y)/(1-\gamma_1)^n$ and $\hat{V}(x,n) = V(Y)/(1-\gamma)^n$ are (time-dependent) non-negative super-martingales. The lemma then follows from Lemma 1; in particular

$$P_Y\{\sup_{\infty > n \geq 0} V(Y)(1-\gamma)^{-n} \geq \epsilon\} = P_Y\{\sup_{\infty > n \geq 0} \hat{V}(Y,n) \geq \epsilon\} \leq \hat{V}(Y,0)/\epsilon = V(Y)/\epsilon$$

Q.E.D.

Next, the case of linear f is treated. Let $f(\alpha) = \alpha$ (without loss of generality). Then

$$x_{n+1} = A_n x_n, \quad A_n = e^{A\Delta_n} + A^{-1}(1-e^{A\Delta_n})KBC. \tag{6}$$

Theorem 1. Suppose that there is a non-negative function $V(x)$ such that $E_x V(x_{n+1}) - V(x) = -k(x) \leq 0$. Then the conclusions of Lemmas 1 and 2 hold for $Y_n = x_n$.

3. Linear Feedback

Example 1. Consider the scalar linear system, where

$\dot{x} = -ax + K\epsilon$, $B = c = 1$. Then $A_n = [(1+K/a)e^{-a\Delta_n} - K/a]$, $a > 0$. Let $V(x) = |x|^s$ for some $s > 0$. Then

$$E_x|x_{n+1}|^s - |x|^s = E\{|(1+K/a)e^{-a\Delta_n} - K/a|^s - 1\}|x|^s. \quad (7)$$

By Theorem 1, a sufficient condition for $|x_n| \to 0$ w.p.1. and $E|x_n|^s \to 0$ is that

$$1 - E|(1+K/a)e^{-a\Delta_n} - K/a|^s \equiv \gamma > 0. \quad (8)$$

Via a more tedious direct method, Leneman [2] succeeded only in showing that (8) is sufficient for $\mathcal{E}|x_n|^2 \to 0$ for $s = 2$. (In fact, it is worth repeating that, if one is interested in w.p.1. convergence, the conditions for mean square stability are not necessary.) (8) is an improvement over the available result, since, as $s \to 0$, the range of K for which (8) holds increases. Note also that it is <u>not</u> required that either K or the value of a be constant, nor that the Δ_n be identically distributed. This illustrates another advantage over the direct method. Also by Lemma 2, (where $0 < \gamma < 1$ is defined by (8))

$$P_x\{\sup_{\infty > n \geq 0} |x_n|^s (1-\gamma)^{-n} \geq \epsilon^s\} \leq |x|^s/\epsilon \quad (9)$$

Intersample Behavior for Example 1. Between samples, for random t_o satisfying $\tau_{n+1} \geq t \geq \tau_n$,

$$x_t = [(1+K/a)e^{-a(t-\tau_n)} - K/a]x_{\tau_n}.$$

Thus, there is some real M so that for any non-random t, $|x_t| \leq M|x_{n(t)}|$, where $n(t)$ is the τ_n immediately preceeding the fixed t. It will be shown that for any $\epsilon > 0$ there is a set of paths of probability $\geq 1 - \epsilon$ so that $|x_{n(t)}| < \epsilon$ for sufficiently large t, for this set; this will imply that $|x_t| \to 0$ w.p.1. To prove the assertion, note that, since $|x_n| \to 0$ w.p.1., for any $\epsilon > 0$ there is an N and a set of paths of probability $\geq 1 - \epsilon/2$ so that for these paths $|x_n| < \epsilon$ for $n \geq N$. To complete the demonstration note that, since, for any finite N,

$$P\{\Delta_1+\cdots+\Delta_N < T\} \to 1 \quad \text{as} \quad T \to \infty,$$

there is a $T < \infty$ so that

$$P\{\text{at least } N \text{ occurrences of sampling by time } T\} \geq 1 - \epsilon/2.$$

Observe that the arguement does not require that the Δ_n be identically distributed.

n Dimensional Linear Systems. Useful Liapunov functions other than quadratic forms or functions of quadratic forms have not been found for the n-dimensional problem. Nevertheless, some useful results can be obtained - especially by comparison with direct methods of calculation.

Theorem 2. Suppose that there are positive definite symmetric matrices P and C so that $EA_n'PA_n - P = -C$. Then $x_n \to 0$ w.p.1., and $x_t \to 0$ w.p.1. There is some $1 > \gamma > 0$ so that

$$P_x\{\sup_{\infty > n \geq 0} x_n'Px_n(1-\gamma)^{-n} \geq \epsilon\} \leq x'Px/\epsilon. \tag{10}$$

If the Δ_n are identically distributed, then the existence of such a P and C are necessary as well as sufficient for mean square stability (i.e., for $Ex_n^2 \to 0$). Let the Δ_n be identically distributed. Let the n^2 dimensional vectors $\mathscr{P}$ and $\mathscr{C}$ denote the vectors containing the elements of P and C and write the linear operation (on P) $EA_n'PA_n - P = -C$ as $\mathscr{A}\mathscr{P} - \mathscr{P} = -\mathscr{C}$. Then a necessary and sufficient condition for mean square stability (and sufficient for w.p.1. stability) is that the eigenvalues of $\mathscr{A}$ are inside the unit circle.

(Note that, by symmetry of P, C, the vectors $\mathscr{P}, \mathscr{C}$

need only be $n(n+1)/2$ dimensional. Also, the last statement also follows from the 'Kronecker Product' method, see [3], [9].)

Proof. Let $V(x) = x'Px$. Then $E_x V(x_{n+1}) - V(x) = x'[EA_n'PA_n - P]x = -x'Cx$, and the first assertion follows from Theorem 1. Equation (10) follows from Lemma 2 since, for some $\gamma > 0$, $C > \gamma P$ in the sense that $C - \gamma P$ is positive definite. Now let Δ_n be identically distributed. Suppose $\{x_n\}$ is mean square stable; thus $E|x_n|^2 \to 0$ exponentially [9]. Now given C positive definite, the matrix P defined by

$$x'Px = E_x \sum_{0}^{\infty} x_n'Cx_n < \infty$$

$$P = C + EA_1'CA_1 + EA_1'A_2'CA_2A_1 + \cdots$$

yields a suitable Liapunov function - hence the first necessary condition.

From what has been said, it is clear that mean square stability implies that the iteration: $P_{n+1} = EA_n'P_nA_n + C$ (with $P_0 = C$) is convergent for any matrix C. Since this is equivalent to an iteration of the type $\mathscr{P}_{n+1} = \mathscr{A}\mathscr{P}_n + \mathscr{C}$, the $\mathscr{P}_n$ must converge for any vector $\mathscr{C}$. Hence the eigenvalues of $\mathscr{A}$ must be inside the unit circle. In any case, the convergence of x_t to zero w.p.1. is proved exactly as for the scalar case, and the details will not be repeated. Q.E.D.

<u>Numerical Data.</u> To use Theorem 2, one must choose P, then test the function $x'Px$. Suppose the system (6) is asymptotically stable with no jitter, i.e., when $\Delta_n = \Delta$, a real number. Then write $A_n = A_\Delta$. There is a quadratic Liapunov function $x'Px$ for the system $x_{n+1} = A_\Delta x_n$, and the use of this Liapunov function, in the presence of jitter, yields some useful bounds on the jitter with which stability is guaranteed. In fact, the following procedure was used. Fix the gain K and delay Δ, and choose C positive definite, then compute P so that $A_\Delta' P A_\Delta - P = -C$. Then add jitter until $EA_n' P A_n - P$ is no longer positive definite. The run of Table 1 is for the system of Fig. 2 with the jitter uniformly distributed with mean $\Delta = 1$, $K = 10$, $r_1 = 1$, $r_2 = 2$. The jitter model was used in order to simulate a system with nominal sampling time Δ and symmetric errors. The holding time error then can be no bigger than Δ. So $2\Delta = \delta$ is the maximum jitter allowed. The matrices C varied over the family

$$C = \begin{bmatrix} 1 & 0 \\ 0 & c^2 \end{bmatrix}.$$

For each c^2, a P (for $\Delta = 1$, and no jitter) was computed. Then the corresponding maximum jitter J (the supremum of the values of

δ for which $EA_n'PA_n - P$ is negative definite) was computed.

c^2	$J = \max \delta$ for stability
0	.40
.05	.35
.1	.31
.2	.26
.5	.18
1.0	.13

Table 1.

For this problem, the maximum jitter was computed as about 0.42. Hence, the Liapunov derived method yields a satisfactory estimate here. It is worth noting that the bounds improved as $c^2 \to 0$. This suggests that there is a method of selecting a 'best' Liapunov function, but it is not yet understood.

The data concerned with the method involving computation of the eigenvalues of $\mathscr{A}$ is plotted in Figs. 3 - 6. The gain is plotted as the maximum % jitter (Jitter/ (2 average jitter) = $J/2\Delta$) for which the system is asymptotically stable (m.sq - and, of course w.p.1.). The system is that of Fig. 2. There are several noteworthy aspects of the data. Let $r_1 = 1$, $r_2 = 2$, $K_1 = 6$. For the deterministic problem where $\Delta_n \equiv \Delta$, the system is unstable for holding times larger than 1.42. In Fig. 3, for $K = 6$, $\Delta = 1$, the stochastic system is stable for Jitter $\leqq 83\%$ or $\delta \leqq 1.66$. Thus when the random hold is in the interval [1.42, 1.66], the system (at that particular hold) is operating in a deterministically unstable region, yet it is stable.

It is interesting to observe that, with the jitter model used, the maximum gain is increasingly insensitive to the jitter (or, conversely, the maximum allowed jitter is increasingly sensitive to the gain) as Δ increases. Apparently for large $\Delta = E\Delta_n$, a slight decrease in gain allows for a sizeable increase in the allowable jitter. However, it is surprising that (for $\Delta = 2$) for only a slight decrease in the gain for which the deterministic system ($\Delta_n = \Delta$) is marginally stable, the random system is stable with 100% jitter. Furthermore, for large average holds (e.g., $\Delta = 2$ in Fig. 4), jitter may have the effect of stabilizing the system. This is illustrated in Fig. 5, where the maximum modulus of the eigenvalues of $\mathscr{A}$ are plotted vs. the absolute jitter δ, for $E\Delta_n = \Delta = 2$, $K = 5.86$, $r_1 = 1$, $r_2 = 4$. Note that the maximum modulus is near unity for $\delta = 0$. As δ increases, the maximum modulus first decreases, then increases and at δ 1.85 the maximum modulus is again unity.

Further elaboration of this point appears in Fig. 6, where the maximum modulus of the eigenvalues of $\mathscr{A}$ are plotted vs. the jitter δ. Note that the jitter initially does reduce the eigenvalues — and actually does slightly stabilize the (deterministically unstable) system. Stabilization, via the use of 'white noise' coefficients does occur and is understood in certain very simple continuous time problems-see examples 1, 2, Chapter 2 of [5]. However, in the case here, the reasons for the stabilization, however slight it is, are not, as yet, satisfactorily understood. It is fairly clear that the jitter does allow the random holding time to take smaller values than the nominal a certain part of the time, and that the stabilizing effect of these smaller holding times outweighs the unstabilizing effect of the longer holding times - until the jitter becomes too large, but a more detailed explanation is not available.

4. Nonlinear Systems

Although the non-linear problem cannot be treated to the same degree as the linear problem owing to the lack of suitable Liapunov functions for the deterministic problem (giving necessary as well as sufficient conditions) with non-linear feedback, some quite useful results can still be easily obtained. First, there is an obvious generalization of the scalar case of Example 1 as follows:

Example 2. The system is

$$x_{n+1} = e^{-a\Delta_n} x_n - \frac{1}{a}(1-e^{-a\Delta_n})Kf(x_n)$$

$$= G_n x_n + F_n Kf(x_n).$$

Let $V(x) = |x|^s$, and $0 \leqq f(x_n)/x_n = u_n \leqq u < \infty$. Then

$$E_x V(x_{n+1}) - V(x) = E_x |G_n x + F_n K f(x_n)|^s - |x|^s$$

$$= E_x |G_n x + F_n K u_n x|^s - |x|^s = |x|^s E_x \{|G_n + F_n K u_n|^s - 1\}.$$

Thus, as long as $Ku_n \leqq Ku$ is less than the least K which causes instability in Example 1, there is asymptotic stability here. The scalar nature of the problem made it easy to treat.

A Sampled Lure Problem. A generic class of important problems for which the stochastic Liapunov method (although not the direct method, as developed to date) yields some results is depicted in the (Luré type problem) of Fig. 7. The system equations are[†]

$$\begin{aligned} \dot{x} &= Ax - mf(\sigma_n) \qquad & t_{n+1} \geqq t \geqq t_n \\ \dot{\sigma} &= cx & \sigma_n = \sigma(t_n) \end{aligned} \tag{11}$$

where A is asymptotically stable, $f(0) > 0$, $0 \leqq \int_0^\sigma f(\alpha)d\alpha \to \infty$ as $\sigma \to \pm\infty$, and

$$|df(\alpha)/d\alpha| \leqq u. \tag{12}$$

[†]Note that C, b_n (as well as d to follow) are row vectors.

Later, the estimate

$$q \int_{\sigma_1}^{\sigma_2} f(\alpha)d\alpha \leq qf(\sigma_1)(\sigma_2-\sigma_1) + \frac{\mu|q|}{2}(\sigma_2-\sigma_1)^2 \tag{13}$$

following [10], Equation 5, will be used. Note that (13) is exact if f is linear. A straightforward integration of (11) yields

$$\begin{aligned} x_{n+1} &= A_n x_n + a_n f(\sigma_n) \\ \sigma_{n+1} &= \sigma_n - b_n x_n + r_n f(\sigma_n) \end{aligned} \tag{14}$$

$$\begin{aligned} A_n &= e^{A\Delta_n}, \quad a_n = A^{-1}(I-e^{A\Delta_n})m \\ b_n &= cA^{-1}(I-e^{A\Delta_n}) \\ r_n &= cA^{-1}(\Delta_n I-A^{-1}(I-e^{A\Delta_n})]m. \end{aligned} \tag{15}$$

For the rest of the development, the Liapunov function (16) used in [10] for a discrete time Luré problem will be used,

$$V(x,\sigma) = x'Hx + q\int_0^{\sigma} f(\alpha)d\alpha, \tag{16}$$

where $q > 0$ and H is positive definite symmetric. Now, the coefficients (A_n, a_n, b_n, r_n) are still independent in n, and computing with (16) gives

$$E_{x,\sigma}V(x_{n+1},\sigma_{n+1}) - V(x,\sigma)$$

$$= x'E[A_n'HA_n-H]x + 2E[a_n'HA_n]xf(\sigma) + E(a_n'Ha_n)f^2(\sigma) + \tag{17}$$

$$+ qE_{x,\sigma}\int_{\sigma}^{\sigma_{n+1}} f(\alpha)d\alpha].$$

where $x_n = x$ and $\sigma_n = \sigma$ is used. Using the estimate (13) in (17) gives

$$E_{x,\sigma}V(x_{n+1},\sigma_{n+1}) - V(x,\sigma) \leqq x'Cx + 2dxf(\sigma) + \rho f^2(\sigma) \tag{18}$$

$$\begin{aligned} C &= E[A_n'HA_n - H + \tfrac{\mu}{2}qb_n'b_n] \\ d &= E[a_n'HA_n - \tfrac{\mu}{2}qr_nb_n - \tfrac{q}{2}b_n] \\ \rho &= E[a_n'Ha_n + qr_n + \tfrac{\mu}{2}r_n^2]. \end{aligned} \tag{19}$$

Under the condition

$$\begin{aligned} &C \text{ negative definite} \\ &\rho > dC^{-1}d' \end{aligned} \tag{20}$$

the matrix

$$\begin{bmatrix} C & d' \\ d & \rho \end{bmatrix}$$

is negative definite and, hence, by Lemmas 1 and 2, (18) and (20) imply that x_n and $\sigma_n \to 0$ w.p.1. Thus the system is asymptotically stable w.p.1. Also, by Lemmas 1 and 2, $E(x_n'x_n+f^2(\sigma_n)) \to 0$. The proof that $x_t \to 0$, $\sigma_t \to 0$ w.p.1. as $t \to \infty$ is done exactly as for the scalar linear case. Note, again, that it is not really necessary for the Δ_n to be identically distributed.

Example 3. In order to show that the condition (20) is not vacuous, a simple example will be given. In general, following the example of the linear case for Table 1, one may do some mild experimentation with H and q to improve the Liapunov function. For the simple example, let $G(s) = 1/(1+s)$, $E\Delta_n = \Delta \sim .7$, so that $e^{-\Delta} = 1/2$, and let the jitter be distributed as in Fig. 2b with $\delta = \Delta/2 = .35$.

Then ρ, d and C, q and H are scalars and the relation (equality in (20)) $\rho - d^2C^{-1} = 0$ (with constraint $C \leq 0$) may be solved for μ in terms of q and H. The q and H maximizing the allowable range of μ (μ is equivalent to the gain K) may then be obtained. In the present case

$$-C = .745H - .127\mu q$$

$$-d = .245H - .307\mu q - .25q$$

$$-\rho = -.255H + 1.445q - .782\mu q$$

or, by (20)

$$C = -.745H + .127\mu q < 0 \tag{21a}$$

$$100H^2 + 25q^2 - 484qH + (160qH+135q^2) + 37.6\mu^2 q^2 < 0. \tag{21b}$$

Solving (21b) for μ (in terms of H/q), and maximizing the μ over H/q yields that the system of Fig. 7 is asymptotically stable w.p.1. (and also in the mean square sense) if

$$\mu < .785, \tag{22}$$

with the maximizing μ and H/q, C is negative.

Repeating the same procedure for no jitter gives asymptotic stability for

$$\mu < 1.31.$$

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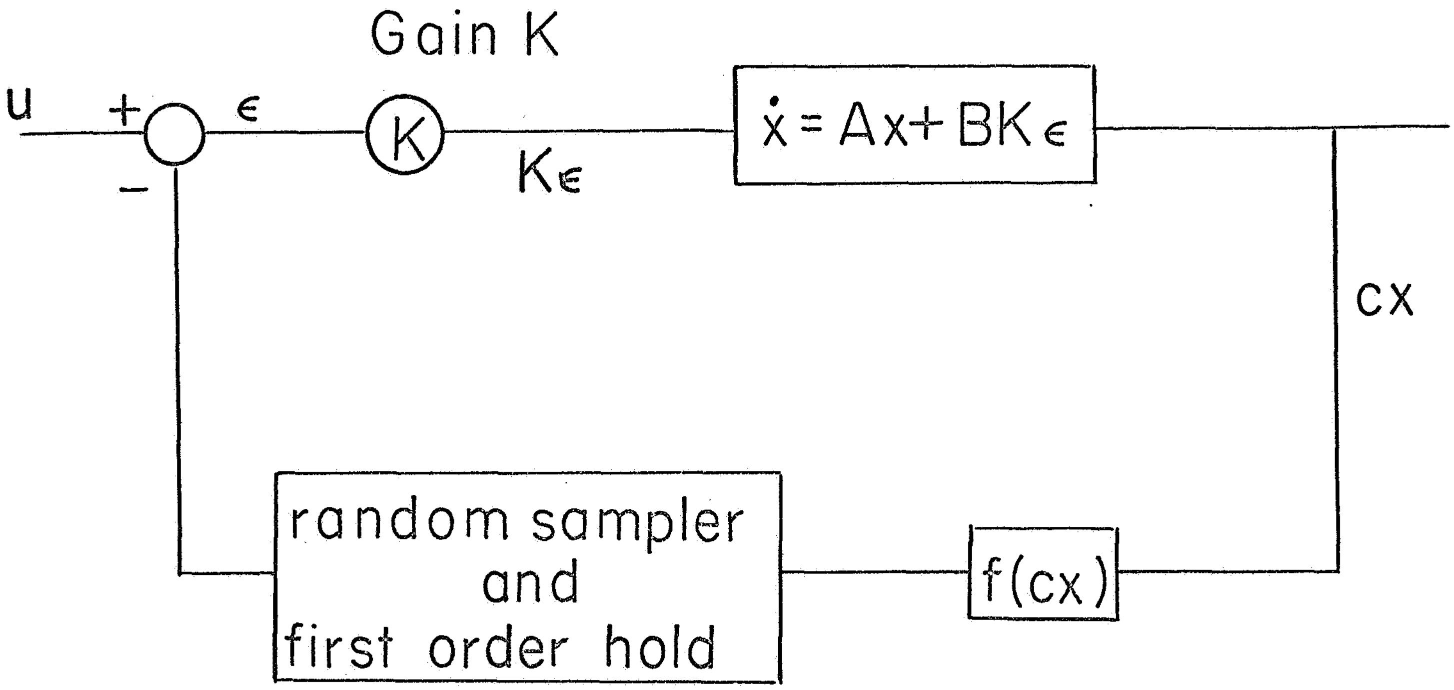

FIG. 1 BASIC SYSTEM

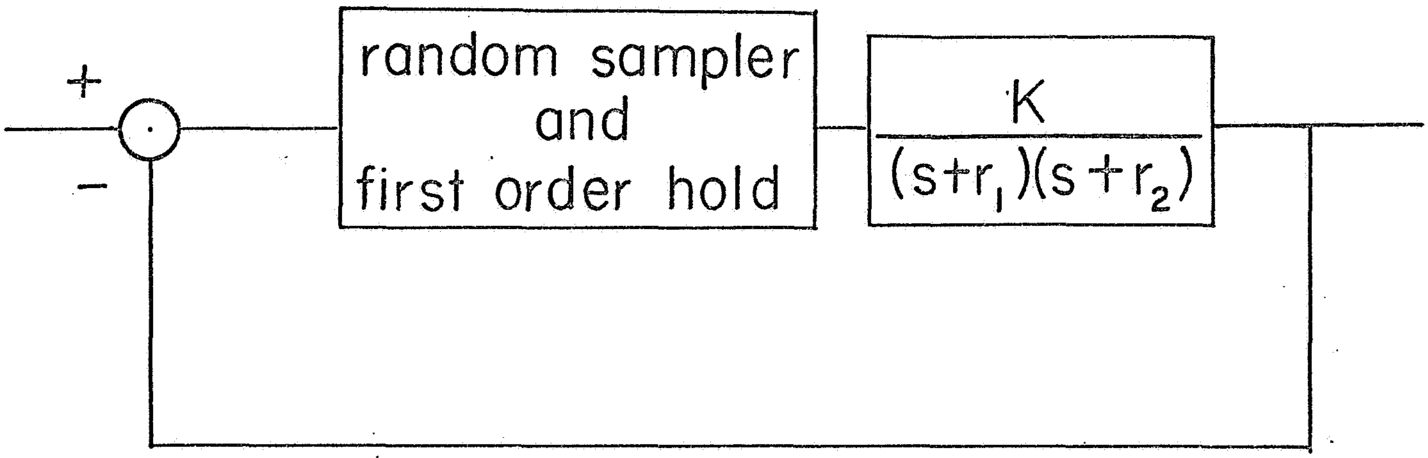

FIG. 2a

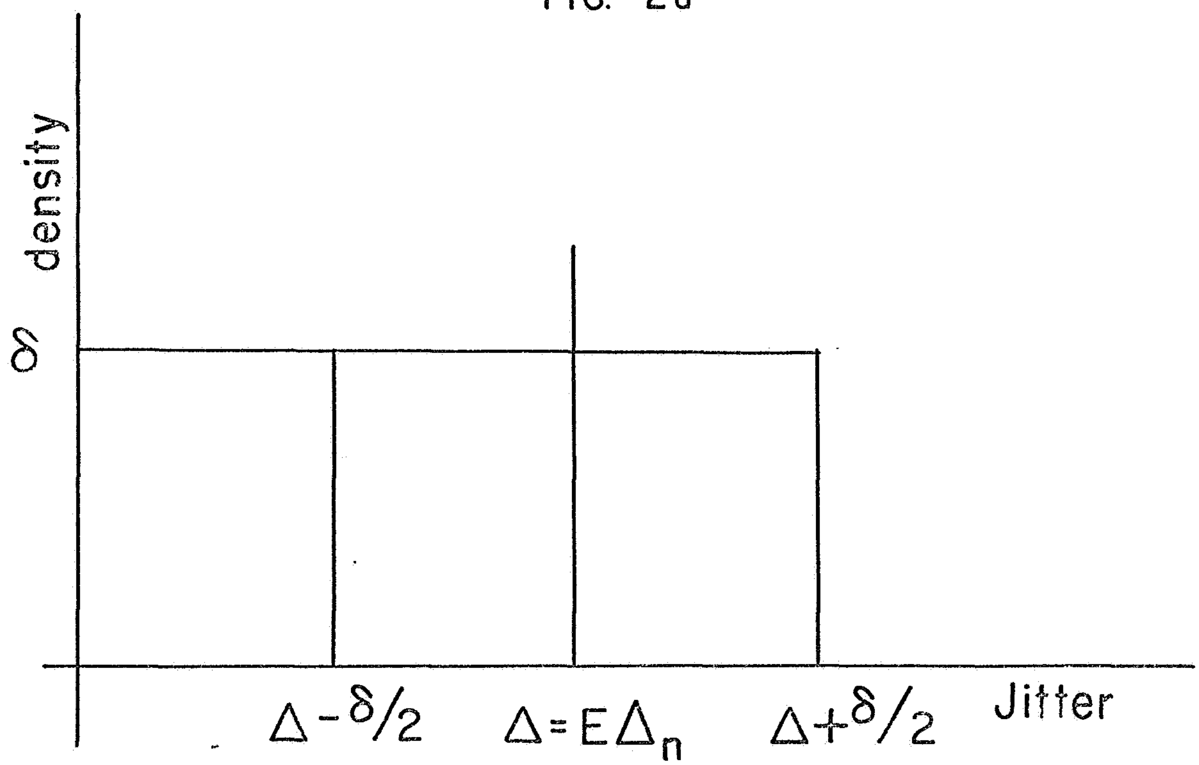

FIG. 2b THE JITTER DENSITY

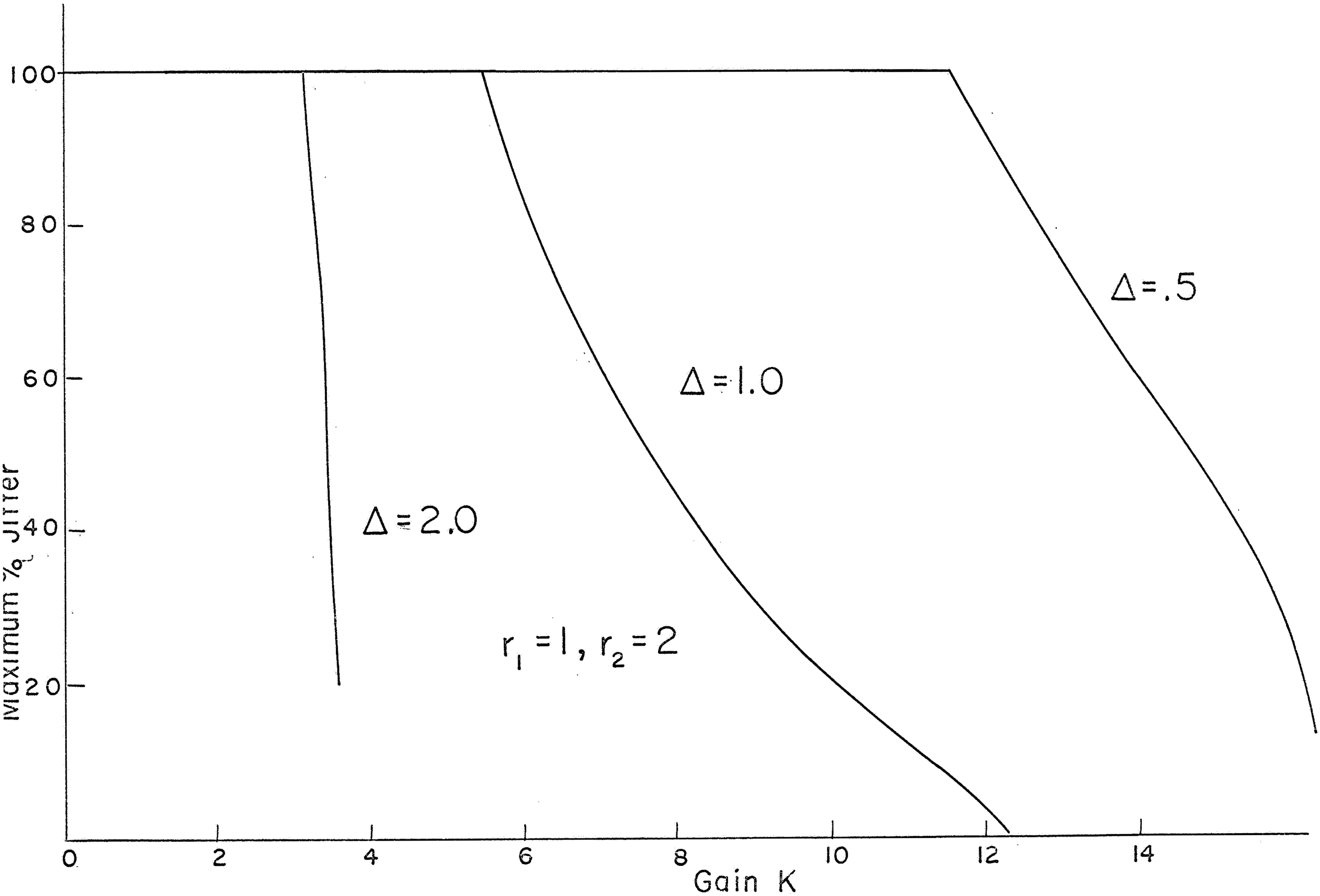

FIG. 3 MAXIMUM % JITTER VS GAIN K (% JITTER $\equiv \delta/2\Delta$)

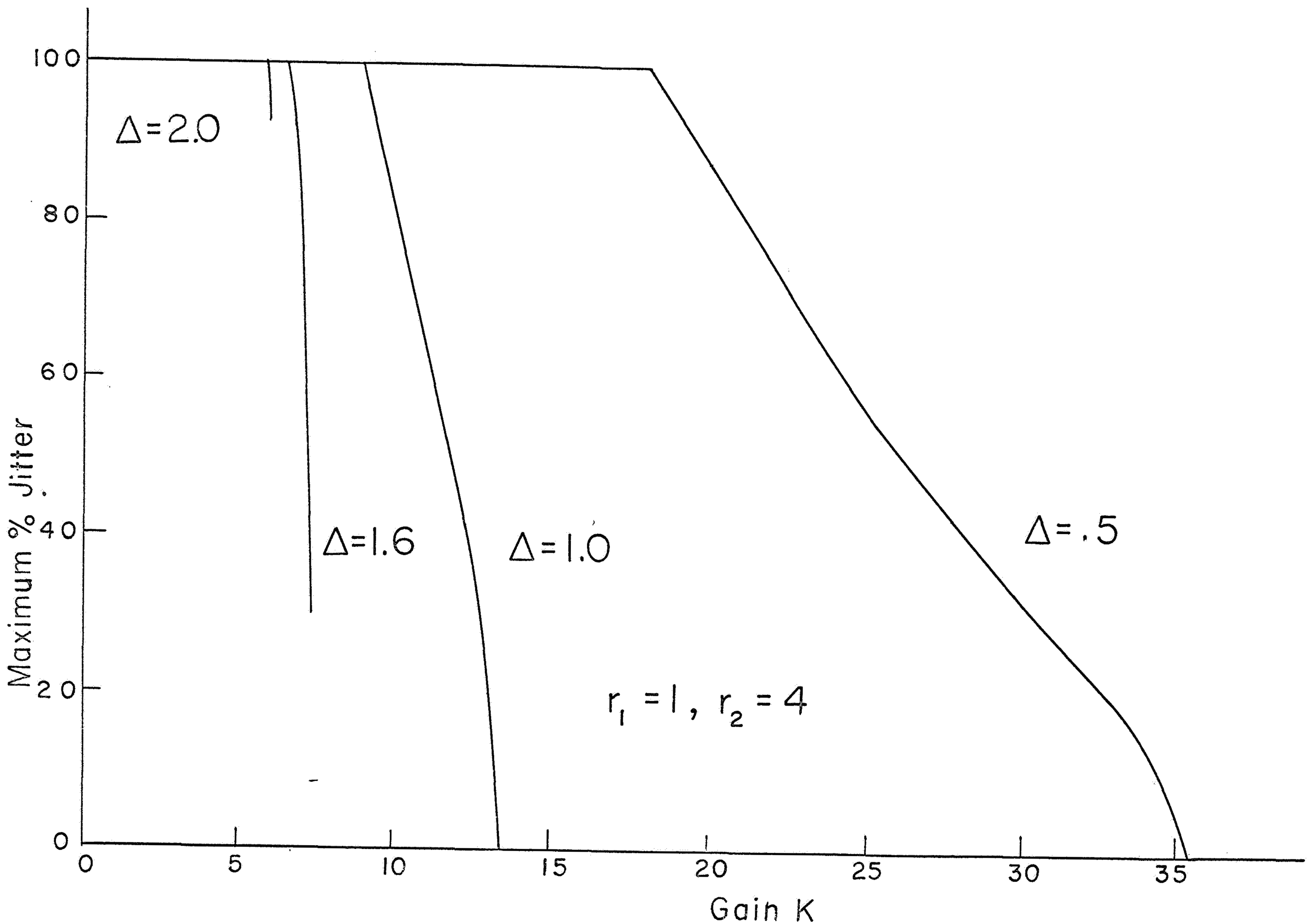

FIG. 4 MAXIMUM % JITTER VS GAIN K

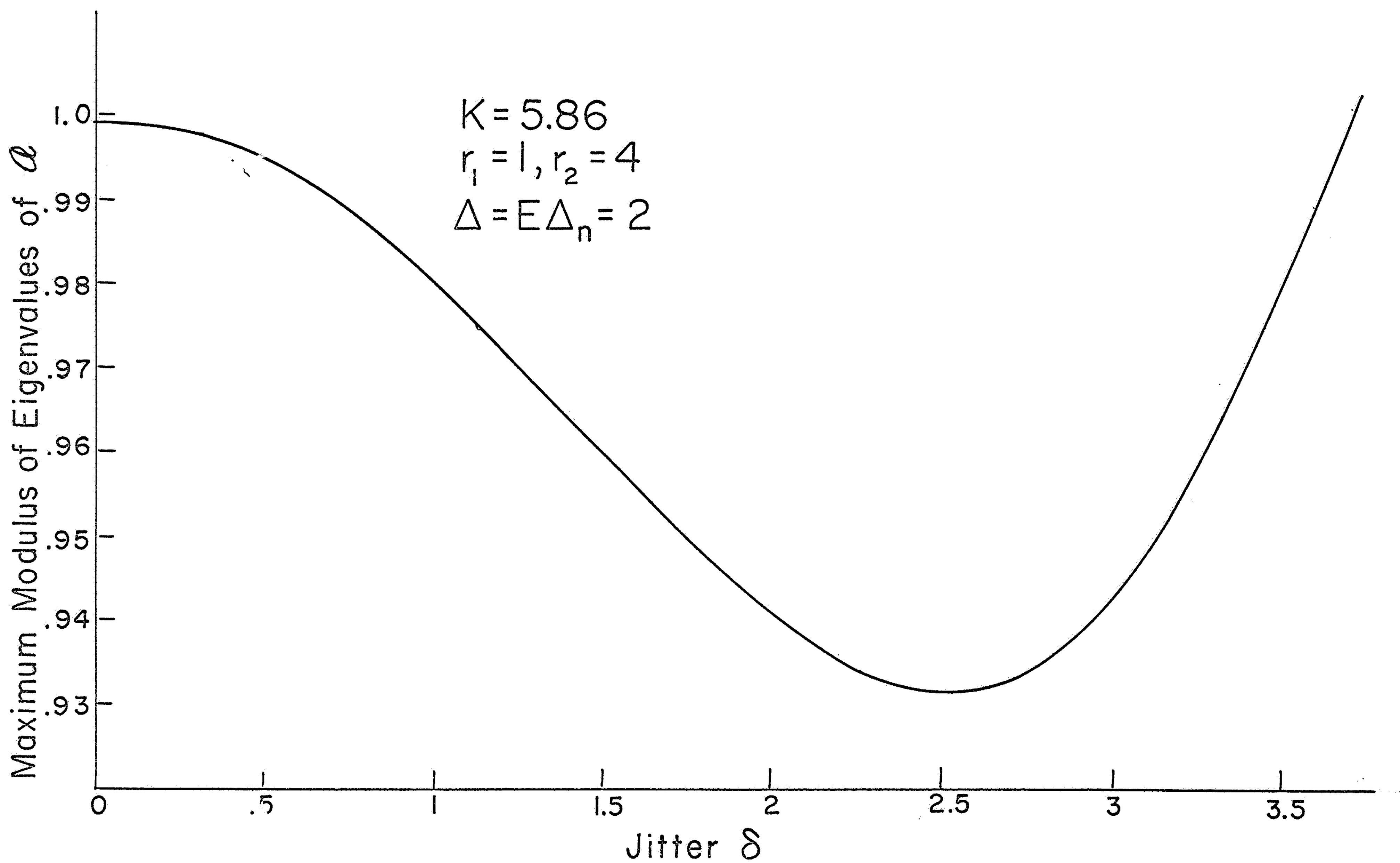

FIG. 5 MAXIMUM MODULUS VS JITTER

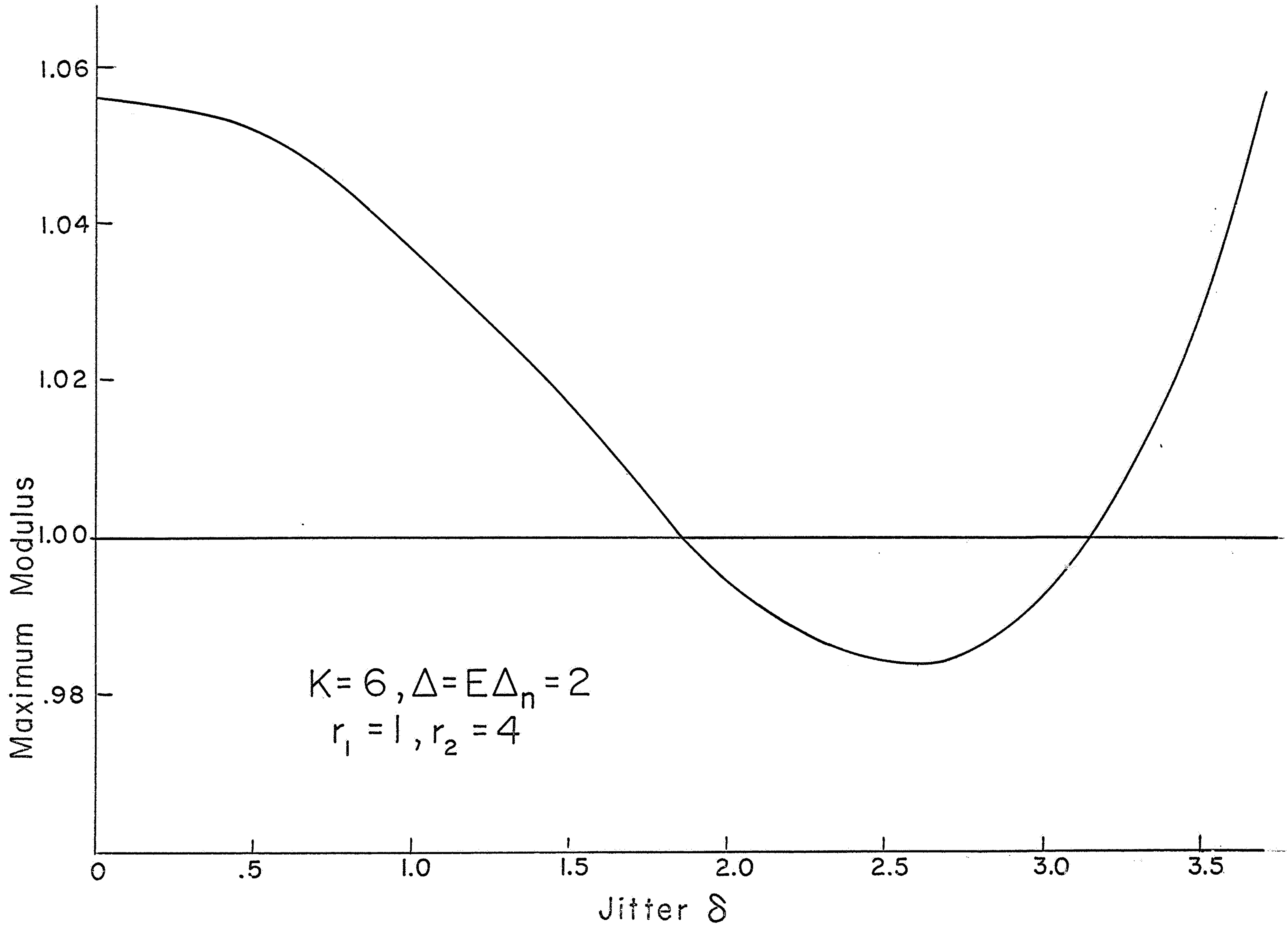

FIG. 6 MAXIMUM MODULUS OF EIGENVALUES OF $\mathcal{A}$ VS. JITTER δ

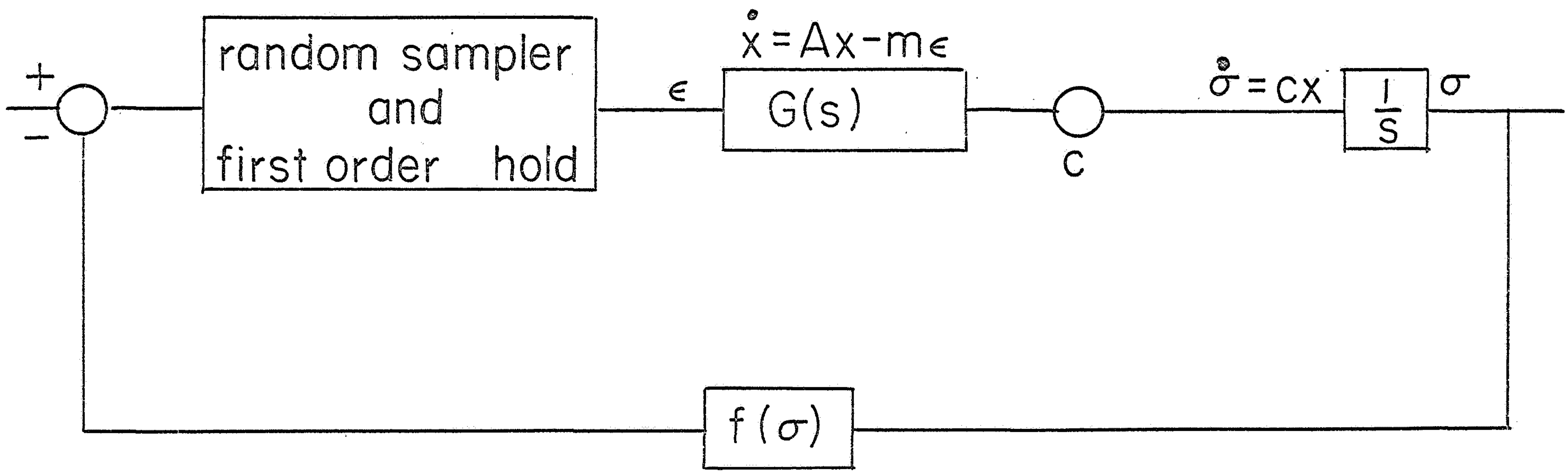

FIG. 7 THE RANDOMLY SAMPLED LURE PROBLEM