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OPTIMUM DESIGN OF COMMUNICATION LINKS WITH NOISY PHASE REFERENCE

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Houston, Texas

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ABSTRACT

A binary communication link is analyzed, and an expression is obtained for bit error probability when the phase reference is noisy. Graphical results are presented which allow selection of modulation index to minimize the error probability.

OPTIMUM DESIGN OF COMMUNICATION LINKS WITH NOISY PHASE REFERENCE*

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SUMMARY

The receiver phase reference of a binary communications link is used for coherent demodulation and bit detection. A noisy phase reference increases the probability of bit error. The probability of bit error has been obtained as a function of modulation index for a phase-shift-keyed link in which the phase reference is obtained from a phase-locked loop preceded by a band-pass limiter.

INTRODUCTION

In binary phase-shift-keyed (PSK) communication links, the receiver phase reference is commonly obtained by a phase-locked loop. The phase reference then is used for coherent demodulation and bit detection. If the reference is noisy, the probability of bit error will increase relative to the noiseless reference case. If a limiter precedes the phase-locked loop, the probability of bit error will not be the same as it would be without a limiter.

One of the parameters that determines the probability of error is the peak phase deviation (modulation index). The results presented in this document allow the designer to determine the modulation index which yields the minimum probability of error. This can be considered to be an optimum modulation index. A comparison is made of the results obtained when using an "exact" and an "approximate" expression for the probability of error.

The effect of a noisy phase reference on probability of error has been considered previously (refs. 1 to 3). However, the effect of both the limiter and the modulation index has received very little attention. References 4 to 7 present discussions on phase-locked loop noise when a limiter is used; however, the effect of the modulation index on probability of error is not considered. Therefore, the results presented in this paper take into consideration the effect of both the limiter and the modulation index.

* Parts of this paper were presented at the IEEE National Telemetering Conference April 8-11, 1968, Houston, Texas.

SYMBOLS

A	signal-to-noise ratio in the phase-locked loop bandwidth B_L
a	amplitude
B_i	limiter bandwidth
B_L	phase-locked loop bandwidth
B_{LO}	one-sided loop bandwidth at the design point
$E\left[P_E(\phi)\right]$	expected value of probability of bit error
$e(t)$	modulated signal
$\mathrm{erfc}(x)$	complementary error function
exp	exponential
$f(T)$	fraction of data power passed by the band-pass filter
g	fraction of the data spectrum passed by the limiter filter
$I_O(A)$	modified Bessel function, first kind, zero order
N_O	noise spectral density
P_c	carrier or synchronization power in the PSK signal
P_d	data power in the PSK signal
P_E	expected probability of a data bit error
$P_E(\phi)$	probability of a data bit error for phase error ϕ
P_T	total power in the PSK signal
$p(\phi)$	probability density function for phase error ϕ
T	bit period
t	time

x, y	variables of integration
α	limiter suppression factor
α_O	limiter suppression factor at the loop design point
β	peak phase shift, peak phase deviation, or modulation index
Γ	limiter performance factor affecting the signal-to-noise ratio A
$\theta(t)$	modulating data signal
ρ_H	signal-to-noise ratio in the limiter bandwidth B_i
σ_ϕ^2	variance of phase noise
ϕ	phase error
ω_c	carrier frequency

PROBABILITY OF BIT ERROR

The model under investigation is shown in figure 1. The phase-locked loop tracks the input signal and generates a phase reference signal which is used for coherent demodulation and bit detection. The case in which the PSK signal contains both the data signal and the synchronization information (position 1 of the switch) is of primary interest. The results can be extended to the case in which a separate synchronization subcarrier is used (position 2 of the switch) if the total power divides between the PSK signal and the synchronization signal according to $\sin^2\beta$ and $\cos^2\beta$, respectively (as explained in the appendix).

For the model shown, the probability of a data bit error for a phase error ϕ that is constant[1] during the data bit period in a coherent binary PSK link is given in

[1]Lindsey (ref. 4) shows that the probability of error during a bit period when the phase error is not constant is lower than when the phase error is constant. Thus, the results presented here can be considered as an upper bound on the probability of bit error.

reference 8 as

$$P_E(\phi) = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2TP_d/N_O}\cos\phi}^{\infty} \exp\left(-\frac{x^2}{2}\right)dx \tag{1}$$

where T is the bit period, P_d is the data power, and N_O is the noise spectral density. It is assumed that the modulation method is such that the total power P_T consists of data power P_d and carrier or synchronization power P_c (as explained in the appendix). The word "carrier" is used here to include any PSK modulated sinusoid (such as a PSK modulated subcarrier).

The probability density function for the phase error ϕ of the phase reference obtained from the phase-locked loop is given in reference 1 as

$$p(\phi) = \frac{\exp(A \cos \phi)}{2\pi I_O(A)} \tag{2}$$

where A is the signal-to-noise ratio in the phase-locked loop bandwidth B_L and $I_O(A)$ is the modified Bessel function of the first kind, zero order. The regions of validity of equation (2) are discussed in reference 1. For a second order phase-locked loop, the regions of validity depend on the loop constants and the loop signal-to-noise ratio.

In equation (1), $P_E(\phi)$ is defined with respect to a random variable. To obtain numerical results, the expected value of the probability of bit error is used. The expected value of the probability of bit error is obtained by applying the definition for the expected value.

$$E\left[P_E(\phi)\right] = P_E = \int_{\phi} P_E(\phi)p(\phi)d\phi \tag{3}$$

By combining equations (1), (2), and (3), the result is

$$P_E = \frac{1}{\pi}\int_0^{\pi} \frac{\exp(A \cos \phi)}{I_O(A)} \operatorname{erfc}\left(\sqrt{\frac{2TP_d}{N_O}} \cos \phi\right) d\phi \tag{4}$$

where erfc(x) is the complementary error function

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(- \frac{y^2}{2}\right) dy \tag{5}$$

From the appendix

$$P_d = P_T f(T) \sin^2 \beta \tag{6a}$$

and

$$P_c = P_T \cos^2 \beta \tag{6b}$$

where P_d, P_T, and P_c are data power, total power, and carrier power, respectively, and β is the peak phase shift, also known as the peak phase deviation or the modulation index. The factor $f(T)$ is the fraction of the data power passed by the bandpass filter in figure 1.

The "signal" to the phase-locked loop is the carrier component of the total signal power. Thus, the loop signal-to-noise ratio is obtained from references 4 and 7 as

$$A = \frac{P_T \cos^2 \beta}{\frac{N_O B_{LO}}{3} \Gamma \left(1 + 2 \frac{\alpha \cos \beta}{\alpha_O}\right)} \tag{7}$$

where B_{LO} is the one-sided loop bandwidth at the design point. The factor Γ is

$$\Gamma = \frac{1 + 0.345\rho_H}{0.862 + 0.690\rho_H} \tag{8}$$

The limiter suppression factor α is approximated by

$$\alpha = \sqrt{\frac{0.7854\rho_H + 0.4768\rho_H^2}{1 + 1.024\rho_H + 0.4768\rho_H^2}} \tag{9}$$

In equation (7), the limiter suppression factor α is multiplied by cos ß since only the carrier voltage amplitude affects the loop bandwidth. In the preceding equations, ρ_H is the signal-to-noise ratio in the limiter bandwidth B_i and is given by

$$\rho_H = \frac{P_T \cos^2 \beta + gP_T \sin^2 \beta}{N_O B_i} \tag{10}$$

The first term in the numerator is the carrier power. The second term is the data power passed by the limiter filter. The factor g is found by calculating the fraction of the total data spectrum passed by the limiter filter.

The limiter suppression factor at the loop design point is α_O and is the value of α evaluated at the specified design point signal-to-noise ratio.

Equations (4), (7), (8), (9), and (10) now can be combined, and the probability of bit error given by equation (4) can be calculated. The integration can be performed analytically (ref. 2); however, the result contains an infinite series of Bessel functions. Alternatively, equation (4) can be evaluated on a digital computer. This evaluation has been made, and the results are presented in figures 2 to 9 which comprise two groups of four figures. The first group of figures is for a band-pass limiter bandwidth B_i of 7 kHz. The second group of figures is for a B_i of 20 kHz. Within each group, the first three figures are arranged according to increasing values of TP_T/N_O. The fourth figure in each group shows the probability of bit error versus TP_T/N_O when the peak phase shift is chosen to be optimum. Also shown in each figure is a "noiseless reference" curve which is a plot of equation (1) with $\phi = 0$ and which shows the probability of bit error when the phase error is zero.

The value of f(T) in equation (6a) was set equal to unity for all calculations. The value of g (eq. (10)) was calculated for each B_i and bit rate with the assumption that the data frequency spectrum was that of a split-phase encoded random data bit stream. Table I gives the resulting values of g that were used.

To evaluate the effect of g, the calculations also were made for $g = 1$ at $T = 0.0001$. The results were essentially the same as for $g = 0.640$, which indicated very little dependence of probability of error on values of g. Thus, the curves in figures 2 to 9 can be used for all binary data waveforms including the split-phase encoded waveform.

The value of α_O was obtained by calculating the value of α for a loop signal-to-noise ratio of unity in $2B_{LO}$. The values of α_O and B_{LO} are shown in the figures.

By referring to the figures, it can be seen that an optimum value of modulation index ß minimizes the probability of bit error. The optimum value of ß increases with an increasing bit rate (decreasing T). By comparison of figures 2, 3, and 4, it can be seen that the optimum ß is weakly dependent on TP_T/N_O. The effect of the limiter bandwidth also can be seen by comparison of figures 2, 3, and 4 with figures 6, 7, and 8.

COMPARISON WITH AN APPROXIMATE RESULT

An equation for the approximation of the probability of bit error has been obtained in terms of the variance of the phase noise (ref. 3). The equation for the probability of bit error is

$$P_E = \text{erfc}\sqrt{\frac{2TP_d}{N_O}}\left\{1 + \frac{TP_d}{N_O}\left[1 + \left(\frac{2TP_d}{N_O} - 1\right)\cot^2\beta\right]\sigma_\phi^{\,2}\right\} \tag{11}$$

where $P_d = P_T f(T)\sin^2\beta$. The variance of the phase noise when the reference is obtained from a phase-locked loop is

$$\sigma_\phi^{\,2} = \frac{1}{\pi}\int_0^{\pi}\frac{\phi^2\exp(A\cos\phi)}{I_O(A)}\,d\phi \tag{12}$$

Equation (11) also has been evaluated on a digital computer. It was found that the results differed significantly from the more exact expression. Figure 10 shows the difference for a particular case. Similar differences were obtained for other values of T, B_i, and TP_T/N_O. It is concluded that the approximation is not satisfactory for calculating the probability of bit error (particularly for values of β which approach 90°).

CONCLUDING REMARKS

The probability of bit error has been obtained as a function of modulation index for a phase-shift-keyed link in which the phase reference is obtained from a phase-locked loop preceded by a band-pass limiter. The results apply whenever the power divides between data power and carrier power (synchronization power) according to $\sin^2\beta$ and $\cos^2\beta$, respectively, where β is the modulation index. The resulting curves allow selection of an optimum modulation index. An approximate expression for the probability of bit error has been evaluated, and the results have been found to differ significantly from the more exact results.

Manned Spacecraft Center
National Aeronautics and Space Administration
Houston, Texas, May 17, 1968
914-50-17-08-72

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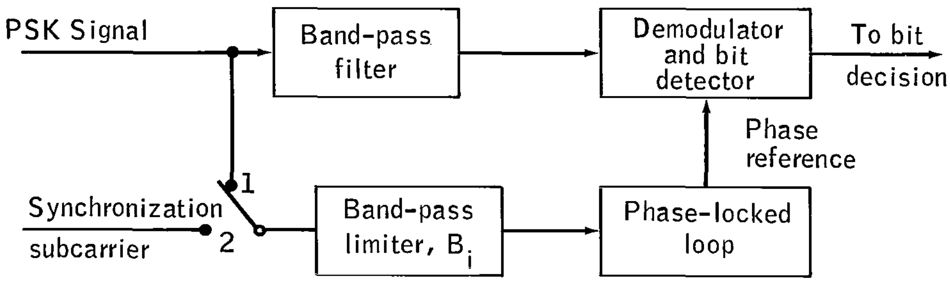
TABLE I. - VALUES OF g

T, sec	g, B_i = 7 kHz	g, B_i = 20 kHz
0.004	1.000	1.000
.002	.956	1.000
.001	.923	1.000
.0005	.856	.940
.0002	.350	.856
.0001	.060	.640



Switch position 1:

Power in PSK signal = total power = P_T

Data power in the PSK signal = $P_d = P_T \sin^2\beta$

Carrier or synchronization power in the PSK signal = $P_c = P_T \cos^2\beta$

Switch position 2:

Power in PSK signal = data power = $P_d = P_T \sin^2\beta$

Carrier or synchronization power = $P_c = P_T \cos^2\beta$

Total power = $P_T = P_d + P_c$

Figure 1.- Analysis model.

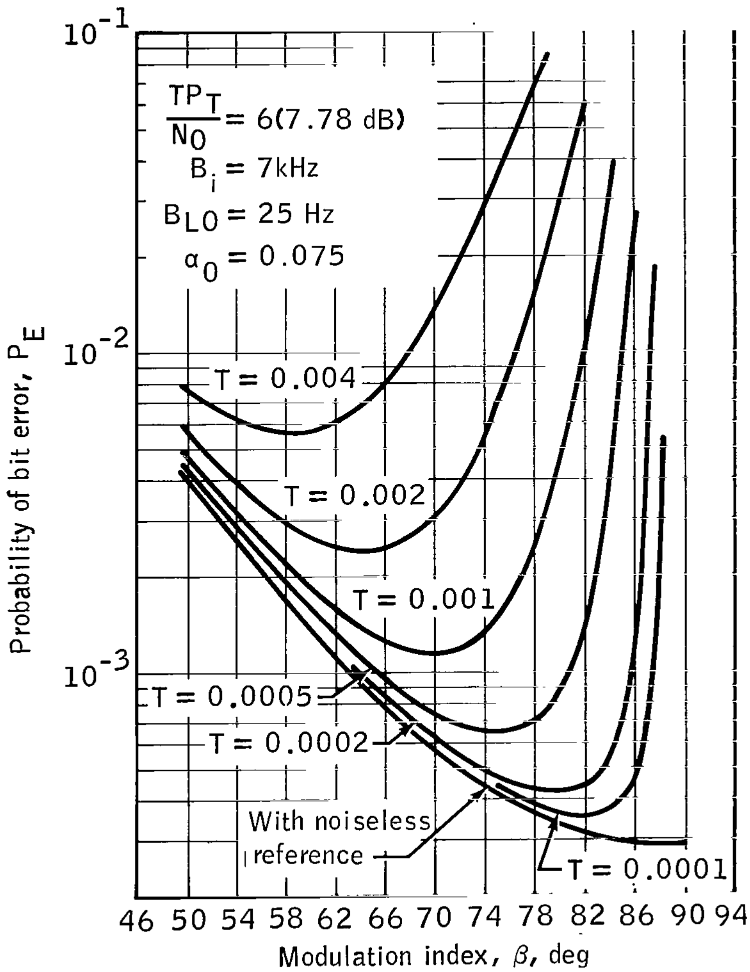


Figure 2. - Probability of bit error curves, set 1.

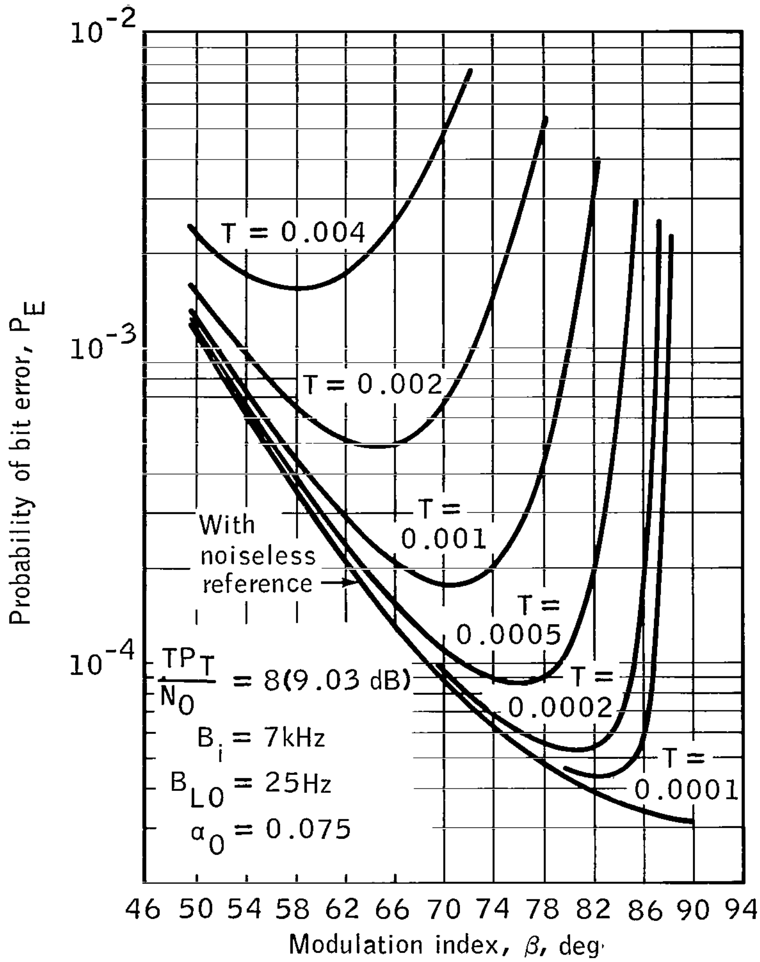


Figure 3.- Probability of bit error curves, set 2.

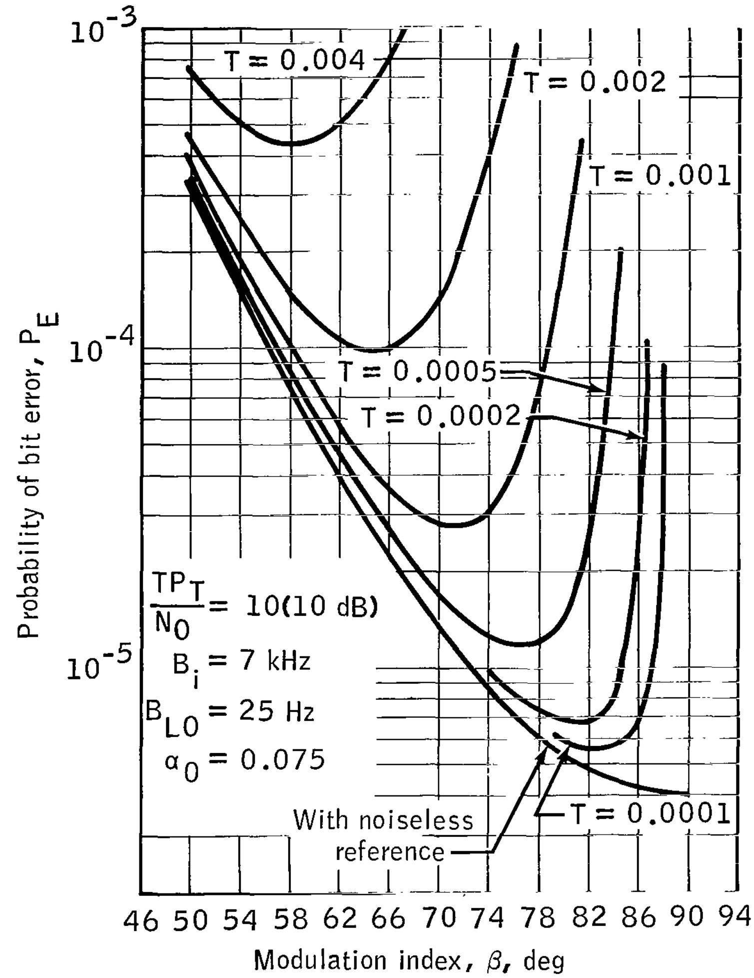


Figure 4. - Probability of bit error curves, set 3.

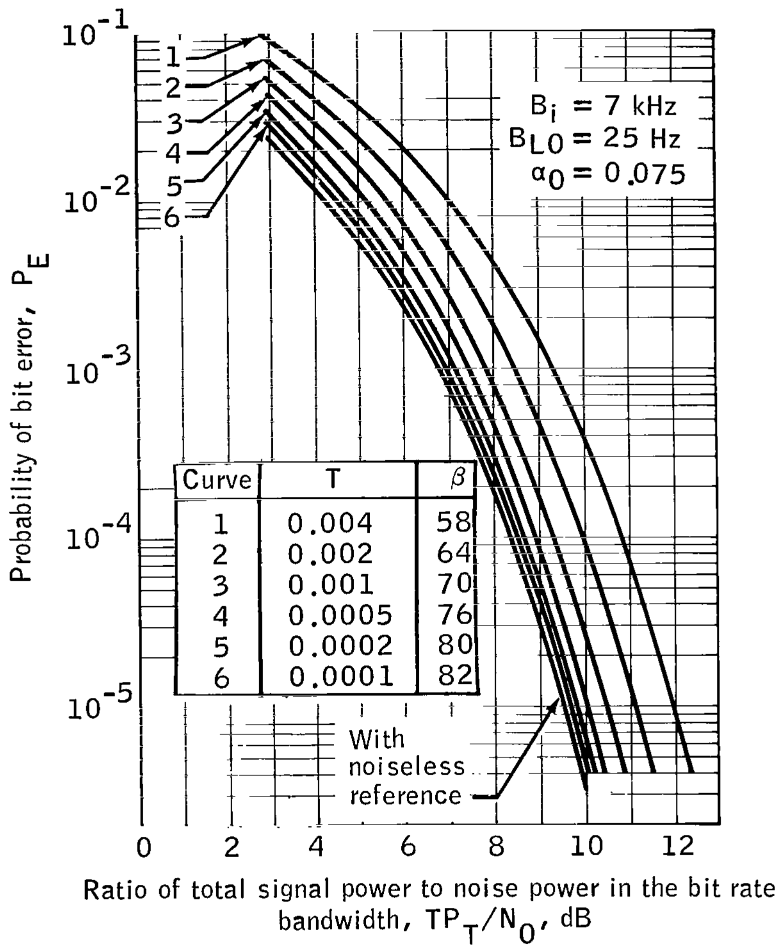


Figure 5. - Probability of bit error curves, set 4.

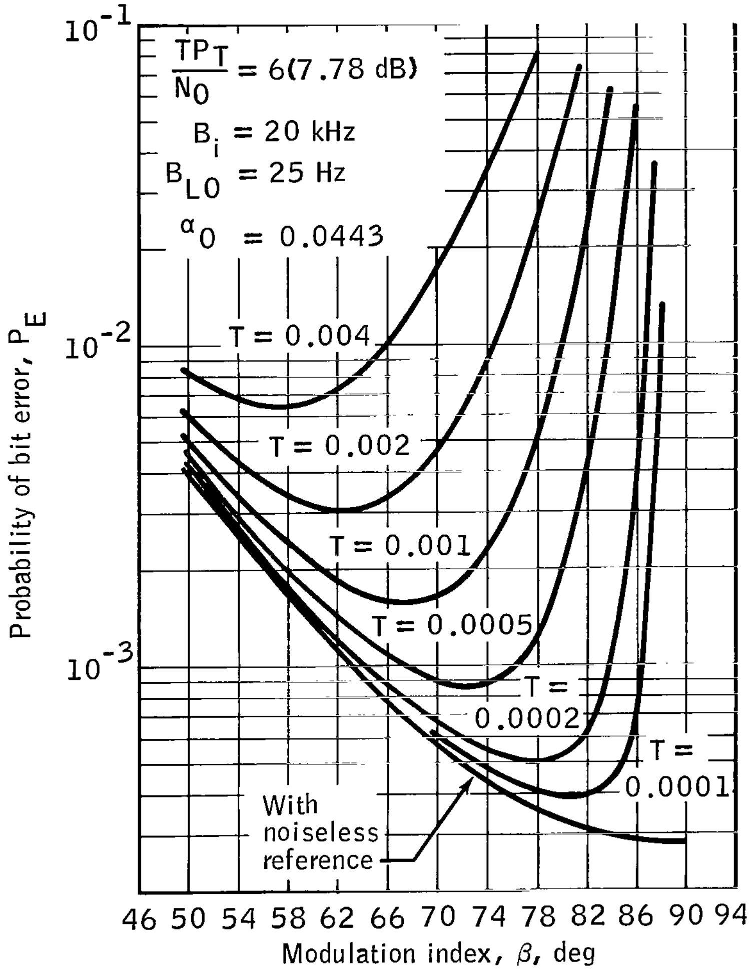


Figure 6. - Probability of bit error curves, set 5.

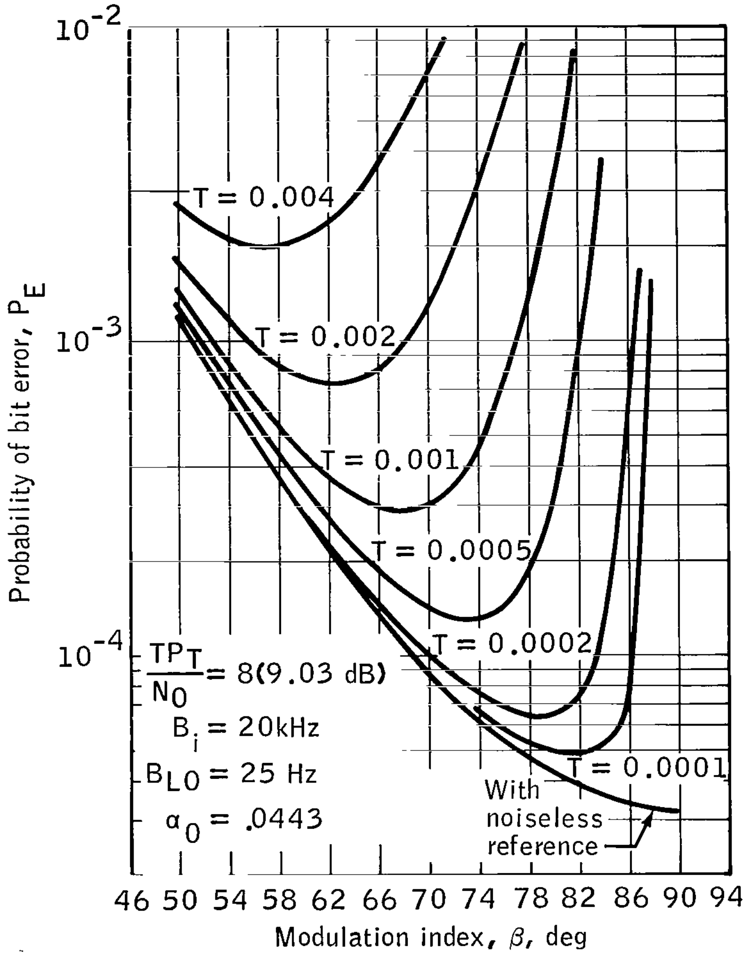


Figure 7. - Probability of bit error curves, set 6.

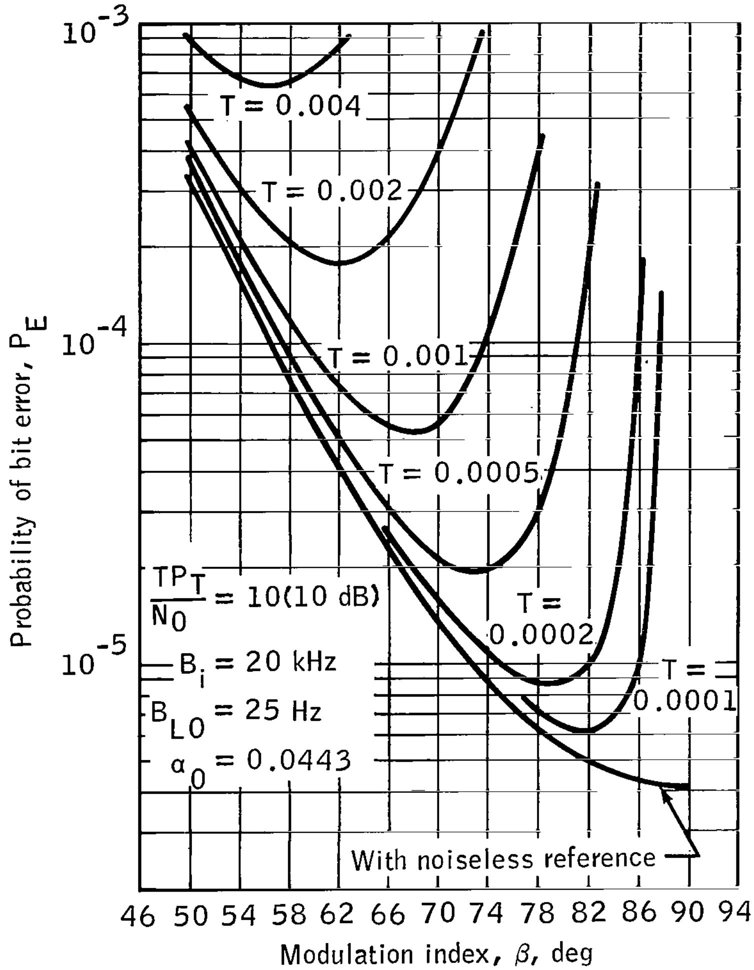


Figure 8. - Probability of bit error curves, set 7.

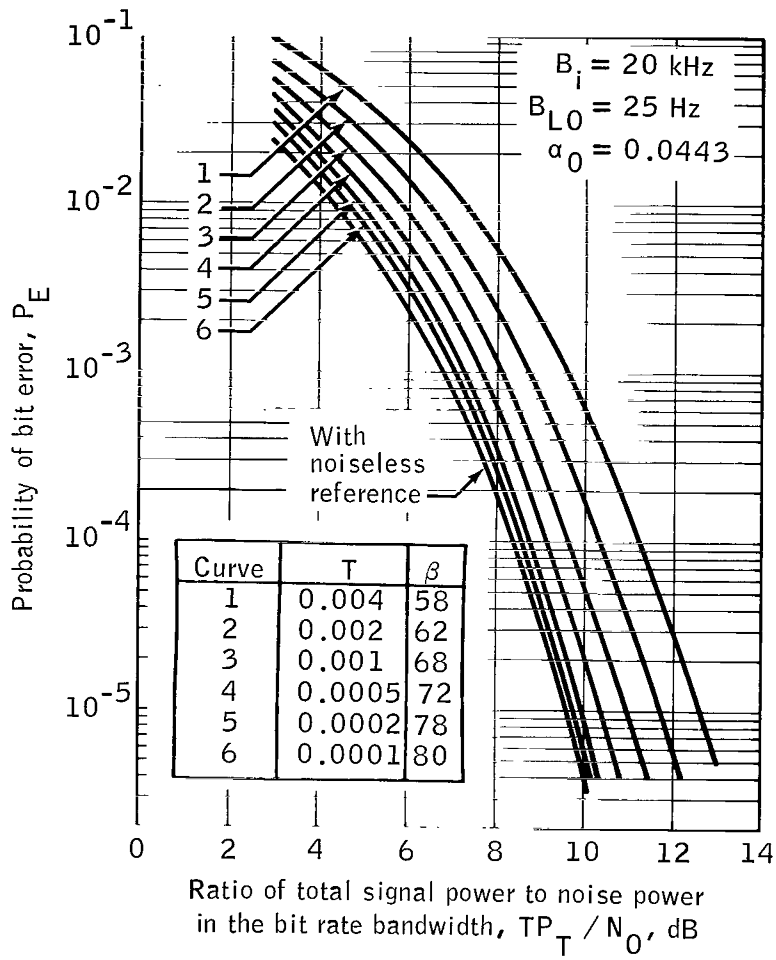


Figure 9. - Probability of bit error curves, set 8.

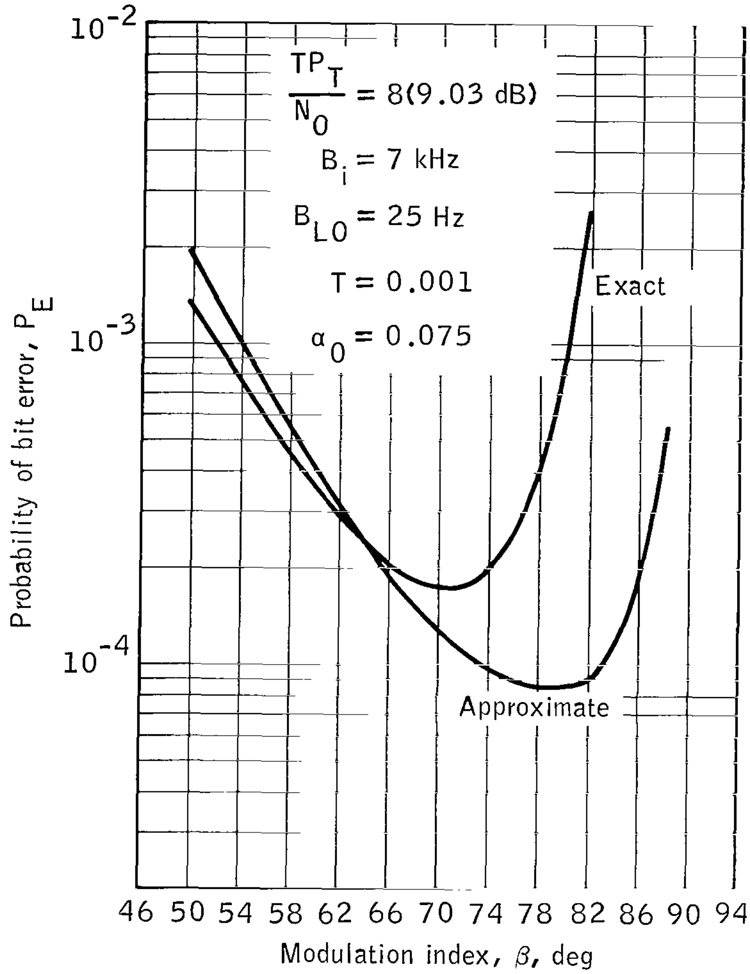


Figure 10.- Probability of bit error comparison.

APPENDIX

DIVISION OF POWER BETWEEN CARRIER AND DATA SIGNAL

The process of binary phase modulation (phase shift keying) of the carrier signal causes a division of the total power. The power divides so that part of the power is contained in the data sidebands, and the remainder is at the carrier frequency.

The equation for the modulated signal can be written as

$$e(t) = a \sin \left[\omega_c t + \beta\theta(t)\right] = a\left[\sin \omega_c t \cos \beta\theta(t) + \cos \omega_c t \sin \beta\theta(t)\right] \tag{13}$$

where a is the amplitude, ω_c is the carrier frequency, β is the modulation index (peak phase shift or deviation), and $\theta(t)$ is the modulating data signal.

The function $\theta(t)$ represents a binary waveform that takes on the values ± 1. Equation (13) then can be rewritten as

$$e(t) = a\left[\cos \beta \sin \omega_c t + \sin \beta\theta(t) \cos \omega_c t\right] \tag{14}$$

The first term in equation (14) is an unmodulated carrier with a power of

$$P_c = \frac{a^2 \cos^2 \beta}{2} \tag{15}$$

The second term in equation (14) is a modulated carrier with a power of

$$P_d = \frac{a^2 \sin^2 \beta}{2} \tag{16}$$

The total power in the signal is

$$P_T = \frac{a^2}{2} \tag{17}$$

Therefore, the carrier power and the data power are given by

$$P_c = P_T \cos^2 \beta \tag{18}$$

$$P_d = P_T \sin^2 \beta \tag{19}$$

Equations (18) and (19) state that the total power divides between the carrier signal and the data signal (when the modulation is binary) according to $\cos^2 \beta$ and $\sin^2 \beta$, respectively.

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