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RAREFIED GAS FLOW OVER A SPHERE

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by

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FOREWORD

This report is based upon the thesis submitted by the author in partial fulfillment of the requirements for the Ph.D. degree in Engineering at UCLA. The research described here was supported by the National Aeronautics and Space Administration (under Grant Nsg 237-62) and North American Aviation, Inc., (through a predoctoral fellowship) and by the UCLA Department of Engineering.

ABSTRACT

The problem of flow over a sphere is investigated within the framework of kinetic theory of gases. Solutions are sought which describe the flow field for a large range of fluid densities. The governing equation for all density levels is the Boltzmann equation, which is a nonlinear integro differential equation. Instead of attempting to solve the Boltzmann equation exactly, one may be concerned primarily with certain mean quantities such as velocity, density, pressure, etc., but not the distribution function itself. One is then led to consider the moment equations of the Boltzmann equation or the Maxwell equation of transfer.

The essence of the moment method consists in finding the unknown parametric functions introduced in the velocity distribution function f. In practice, the moment method can best be initiated with full knowledge of f in the free molecule flow limit. For the problem of flow over a closed body this information is often not available. It is found that in the free molecule flow limit the distribution function for the flow over a sphere can be represented by

$$f = f_1 = f_1^{(M)}(\vec{x},\vec{\xi}) + G(\vec{x},\vec{\xi})$$

for all molecules which have velocity vectors lying in a cone subtended by the sphere, and

$$f = f_2 = f_2^{(M)}(\vec{x},\vec{\xi}) + G(\vec{x},\vec{\xi})$$

for all molecules whose velocity vector is directed into the region external to this conical region. The functions $f_1^{(M)}$, $f_2^{(M)}$ are Maxwellian distribution functions evaluated by the conditions at the sphere and the free stream respectively. The function $G(\vec{x},\vec{\xi})$ is determined by satisfying the moment equations from the homogeneous Maxwell equation of transfer and the boundary conditions on the solid surface.

The formulation for studies in the transition regime is accomplished by introducing parametric functions into $f_1^{(M)}, f_2^{(M)}$ and $G(\vec{x}, \vec{\xi})$. Closure of the moment method is accomplished by taking the number of moments equal to the number of parametric functions.

The present investigation for flow over a sphere is restricted to the two limiting cases; low speed flow, where the Mach number is very small, and the high speed approximation for very large Mach numbers. In the low speed approximation the solution is found by solving the six moment equations corresponding to; continuity, radial momentum, tangential momentum, energy, shear stress, and radial heat flux. Analytical solutions are obtained for the six equations and the computed drag and heat transfer compare favorably with existing measurements.

For the high speed case four moments are taken but the governing partial differential equations are nonlinear and a simple separation of variables cannot be found for the general case. However, if an expansion for small angles $(\theta \ll 1)$ is assumed, the resulting ordinary differential equations can be integrated numerically. Furthermore, if one also makes the assumption of large mean free path, an analytical solution can be obtained. The results from the high speed analysis are found to show acceptable agreement with the drag and density measurements.

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A	Area
$A_2(4)$	= 1.3682, Equation (2.5)
$A_n^{(\pm)}(x)$	Separation variable for low speed flow, Equation (4.25)
a	Constant in intermolecular potential
$a_i = (a_1, a_2, a_3)$	Generalized coordinates, Appendix (A)
$B_n^{(\pm)}(x)$	Separation variable for low speed flow, Equation (4.25)
b_i	Integration constants
$C_n^{(\pm)}(x)$	Separation variable for low speed flow, Equation (4.25)
C_D	Drag coefficient
c_p	Specific heat of constant pressure
$c_i = \xi_i - q_i$	Relative particle velocity
$\bar{c}$	Average particle speed
D	Drag
$D_n^{(\pm)}(x)$	Separation constant for low speed flow, Equation (4.25)
$f(\vec{x}, \vec{\xi}, t)$	Distribution function
$f^{(0)} \equiv \frac{n}{(2\pi kT)^{3/2}} \exp\left\{-\frac{\lvert\vec{\xi}-\vec{q}\rvert^2}{2kT}\right\}$	Local Maxwellian distribution function
$f_i^{(M)} \equiv \frac{n_i}{(2\pi kT_i)^{3/2}} \exp\left\{-\frac{\lvert\vec{\xi}-\vec{u}_i\rvert^2}{2kT_i}\right\}$	Generalized cartesian Maxwellian distribution function
$F(\theta; \lambda)$	Equation (4.14)
$G(\vec{x}, \vec{\xi})$	Equation (3.1)
$g_i^{(n)}(\vec{x})$	Spatial variation of $G(\vec{x}, \vec{\xi})$
$H_i(\vec{x})$	Generalized parametric functions
$h(\vec{x}, \vec{\xi}) = f - f^{(0)}$	Perturbation distribution function

LIST OF SYMBOLS (Continued)

h	Enthalpy
$h_i = (h_1, h_2, h_3)$	Stretching factor, Appendix (A)
k	Boltzmann constant
$K_n = \frac{\lambda}{L}$	Knudsen number
$K_i^{(j)}$	Constants in free molecule flow solution
L	Characteristic length
$M \equiv \xi_\infty / \sqrt{\gamma R T_\infty}$	Free stream Mach number
m	Particle mass
$\bar{n} \equiv \int f d\vec{\xi}$	Number density
$n_i(r, \theta)$	Parametric density function in $f_i^{(M)}$
$N_i(r, \theta) \equiv n_i(r, \theta) - 1$	
$N^{(\pm)}(r, \theta) = N_1(r, \theta) \pm N_2(r, \theta)$	
$\bar{p} = \bar{n} k T$	Pressure
$\bar{p}_{ij} = \bar{p} \delta_{ij} - \bar{\tau}_{ij}$	Pressure tensor
$\bar{P}_{ij} \equiv m \int \xi_i \xi_j f d\vec{\xi}$	
$\bar{P}_{ijk} \equiv m \int \xi_i \xi_j \xi_k f d\vec{\xi}$	
$\bar{P}_{ijkl} \equiv m \int \xi_i \xi_j \xi_k \xi_l f d\vec{\xi}$	
$\bar{Q}_j$	Heat flux in j^{th} direction
$\bar{Q}_{AVE}$	Heat transfer averaged over the area
$\vec{q}$	Mean velocity vector
$q_i \equiv \frac{1}{\bar{n}} \int \xi_i f d\vec{\xi}$	Mean velocity component
$R \equiv \frac{k}{m}$	Gas constant
r	Radial coordinate
r_0	Sphere radius

LIST OF SYMBOLS (Continued)

$(R_e)_L \equiv \frac{\rho q L}{\mu}$	Reynolds number based on length L
S_T	Stanton number
T	Temperature
$T_i(r,\theta)$	Parametric temperature function in $f_i^{(M)}$
$t_i(r,\theta) \equiv T_i(r,\theta) - 1$	
$t^{(\pm)}(r,\theta) \equiv t_1(r,\theta) \pm t_2(r,\theta)$	
t	Time
$\vec{u}_i(r,\theta) = (u_{r_i}, u_{\theta_i}, u_{\phi_i})$	Parametric velocity function for spherical coordinates in $f_i^{(M)}$
$u_i(r,\theta) \equiv u_{r_i}(r,\theta)$	
$u^{(\pm)}(r,\theta) \equiv u_1(r,\theta) \pm u_2(r,\theta)$	
$U^{(\pm)}(x) \equiv A_1^{(\pm)}(x)$	
$v_i(r,\theta) \equiv u_{\theta_i}(r,\theta)$	
$v^{(\pm)}(r,\theta) \equiv v_1(r,\theta) \pm v_2(r,\theta)$	
$V^{(\pm)}(x) \equiv B_1^{(\pm)}(x)$	
$W_1(x)$	Defined in Equation (5. 1)
$\vec{x} = (x_1, x_2, x_3)$,	cartesian coordinates
(r, θ, ϕ),	spherical coordinates
$x \equiv \sqrt{1 - \left(\frac{r_0}{r}\right)^2}$	Transformed coordinate
z	Number of degrees of freedom

LIST OF SYMBOLS (Continued)

$\alpha \equiv \cos^{-1}\left(\frac{r_0}{r}\right)$	Figure 1
α_i	Integration constants from low speed four moment solution
$\overline{\alpha_i}$	Integration constants from low speed six moment solution
β_0	Integration constant from heat transfer solution
γ	Ratio of specific heats
$\delta(y)$	Delta function at y=0
δ_{ij}	Kronecker delta
$\epsilon_0 \equiv \sqrt{\frac{T_b}{T_\infty}} / \gamma M$	
$\epsilon \equiv \frac{T_b}{T_\infty} - 1$	
η	Angle in velocity space for spherical coordinates
θ	Angle in physical space for spherical coordinates
λ	Mean free path
$\tilde{\lambda} \equiv \frac{\lambda_\infty}{r_0}$	Knudsen number based on free stream mean free path and body radius
μ	Viscosity coefficient
ν	Collision frequency appearing in the BGK equation
$\vec{\xi}$	Particle velocity vector
$\xi \equiv \sqrt{\xi_r^2 + \xi_\theta^2 + \xi_\phi^2}$	Particle velocity
$d\vec{\xi} \equiv d\xi_i \, d\xi_j \, d\xi_k$	Differential volume in velocity space
$\bar{\rho} \equiv m\bar{n}$	Density
$\sigma(\vec{\xi})$	Source term in Grad's formulation

$\tau_{ij} \equiv m \int c_i c_j f \, d\vec{\xi}$ $= \overline{P}_{ij} - m \overline{q}_i \overline{q}_j$	Stress
$\Phi(\vec{\xi})$	Arbitrary function of particle velocity
$\Delta\Phi$	Change in Φ produced by collisions
$\hat{\phi}(r,\theta)$	Velocity potential
ϕ	Azimuthal angle in physical space for spherical coordinates
$\vec{\Omega}$	Instantaneous local angular velocity associated with coordinate curvature
$d\Omega$	Collision cross-section
ϖ	Angle in velocity space for spherical coordinates

Subscripts

$(\ )_\infty$	Free stream quantities
$(\ )_i$	i^{th} component of a vector or used to denote Lees' two stream Maxwellian distribution function
$(\ )_b$	Quantities evaluated at the body surface
$(\ )_{AVE}$	Averaged over the area

Superscripts

$(\ )'$	Collision partner; used in collision integral for the Boltzmann equation
$(\ )^*$	Value of the quantity after a collision
$\tilde{(\ )}$	Nondimensional variable
$(\ )^{FMF}$	Free molecule flow value
$\overline{(\ )}$	Mean value
$(\ )^{(B)}$	Basset's solution

CHAPTER I – INTRODUCTION

The motion of a finite body in a fluid of infinite extent for all density levels is in general characterized by two parameters; the Mach number, M , and the mean free path, λ (henceforth abbreviated MFP). The two extremes of the MFP, $\lambda = 0$ and $\lambda = \infty$, correspond respectively to the continuum limit and the free molecule flow limit (henceforth abbreviated FMF limit). In the continuum limit the particle density is high enough so that collisions between particles dominate whereas in the FMF limit the collisions between particles are of secondary importance in comparison to collisions between particles and solid boundaries.

The governing equations in the continuum limit are the Navier Stokes equations whose solutions in general differ greatly between subsonic and supersonic flow. In the FMF limit the mean quantities such as density, pressure, and temperature at the solid surface are found directly by making mass, momentum, and energy balances. The solution in this limit was investigated for simple convex geometries by Ashley[1] and Heinemann[2] for all values of the Mach number. For any value of the MFP between the two extremes there exists the transition region. When the MFP is not small the transport relations which are normally adopted in the continuum regime will no longer be valid.

The Boltzmann equation for the distribution function, $f(\vec{x},\vec{\xi},t)$, is generally accepted as the fundamental equation for the entire range of MFP. The Boltzmann equation may be written

$$\frac{D}{Dt}f(x_i,\xi_i,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} + \frac{\partial f}{\partial \xi_i}\frac{\partial \xi_i}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \qquad (1.1)$$

where

$f(x_i, \xi_i, t)\, d\vec{x}\, d\vec{\xi} \equiv$ number of particles in the combined volumes $d\vec{x}$ and $d\vec{\xi}$ at the same time

$\frac{\partial x_i}{\partial t} \equiv \xi_i \equiv$ particle velocity

$$\frac{\partial \xi_i}{\partial t} \equiv \frac{F_i}{m} - (\vec{\Omega} \times \vec{\xi})_i$$

$\vec{\Omega} \equiv$ instantaneous local angular velocity associated with coordinate curvature

$(\vec{\Omega} \times \vec{\xi}) = 0$ for Cartesian coordinates

The collision integral is given by

$$\left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} = \int |\vec{\xi}' - \vec{\xi}|\, d\Omega \iiint (f^* f^{*\prime} - f f')\, d\vec{\xi}' \tag{1.2}$$

Equation (1. 1) in orthogonal curvilinear coordinates is given in Appendix (A).

If the Boltzmann equation is properly nondimensionalized and all the normalized quantities are designated by a tilde, $(\tilde{\ })$, we find

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} + \tilde{\xi}_i \frac{\partial \tilde{f}}{\partial \tilde{x}_i} + \left[\tilde{F}_i - (\vec{\Omega} \times \vec{\xi})_i\right] \frac{\partial \tilde{f}}{\partial \tilde{\xi}_i} = \frac{1}{K_n}\left(\frac{\partial \tilde{f}}{\partial \tilde{t}}\right)_{\text{collisions}} \tag{1.3}$$

where

$K_n \equiv \frac{\lambda}{L} =$ Knudsen Number

In the continuum limit ($K_n = 0$) the collision integral vanishes. This leads to the well-known Maxwellian distribution function which describes a situation in which the collisions between particles are so numerous that local equilibrium is always maintained. For the FMF limit ($K_n = \infty$) Equation (1.3) becomes the homogeneous Boltzmann equation

$$\frac{\partial \tilde{f}}{\partial t} + \tilde{\xi}_i \frac{\partial \tilde{f}}{\partial \tilde{x}_i} + \left[\tilde{F}_i - (\vec{\Omega} \times \vec{\xi})_i\right] \frac{\partial \tilde{f}}{\partial \tilde{\xi}_i} = 0 \quad . \qquad (1.4)$$

which may be solved by the method of characteristics.

The difficulties in solving the Boltzmann equation for a finite value of K_n are obvious. The Boltzmann equation is in general a nonlinear integro differential equation to which an exact solution for a realistic boundary value problem has not been found. In view of these difficulties, various methods of approximations have been applied to the Boltzmann equation as summarized by Lees.[3]

The most well-known approximation is the Chapman Enskog[4] method which assumes an expansion in the Knudsen number

$$f = f^{(0)} + K_n f^{(1)} + \cdots$$

where

$$f^{(0)} \equiv \text{local Maxwellian distribution function}$$

The convergence of this expansion has never been shown and one would conjecture that the expansion is valid only for $K_n \ll 1$.

An analogous expansion for the other extreme of the MFP is the Knudsen iteration

$$f = f^{FMF} + \frac{1}{K_n} f^{(1)} + \cdots$$

where $\quad f^{FMF} \equiv$ FMF distribution function

Applications of this method to flow problems between parallel plates[3] have shown that the Knudsen iteration is invalid for this choice of geometry. Although no general statement of validity can be made about this scheme, it appears to be more questionable than the Chapman-Enskog expansion.

Another familiar approximation is to replace the collision integral by a simple relaxation equation

$$\left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \equiv \nu\left(f^{(0)} - f\right) \tag{1.5}$$

$\nu \equiv$ characteristic frequency

This simple kinetic model is most often referred to as the BGK[5] model and the resulting simplified Boltzmann equation is usually called the Krook equation. Although this "linearization" (Equation 1.5 is linear if $f^{(0)}$ is a constant Maxwellian) greatly simplifies the Boltzmann equation, the relationship between this linear model and the full collision integral has never been fully established.

Most of the previous attempts to solve the problem of a sphere moving in an infinite fluid have employed one or a combination of the preceeding three assumptions. For example, in attempting to extend the continuum solution into the transition regime, one can utilize the Chapman-Enskog expansion to make a correction to the continuum result. This is usually accomplished by modifying the "no slip" boundary condition at the solid surface. From a simple kinetic theory model the slip velocity is found to be proportional to the MFP.

For large MFP, many investigators have made use of the Knudsen iteration to obtain the near FMF solutions. These results are characterized by complicated computational procedures and the validity of extending these solutions into the transition regime remains questionable. The great effort expended to obtain approximate solutions to the Boltzmann equation is further indication of the difficulty in finding solutions for all values of the MFP.

Instead of attempting to solve the Boltzmann equation exactly, one may be concerned primarily with certain mean quantities such as velocity, density, pressure, etc., but not the distribution function itself. One is then led to consider the moment equations of the Boltzmann equation or the Maxwell equation of transfer.

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CHAPTER II – MOMENT METHOD SOLUTION OF THE BOLTZMANN EQUATION

II. A Maxwell's Equation of Transfer

If $\Phi(\xi_i)$ is any function of particle velocity, the mean value of Φ is defined by

$$\overline{n\Phi} \equiv \int f \Phi \, d\vec{\xi} \tag{2.1}$$

where for $\Phi = 1$ the number density is given by

$$\bar{n} = \int f d\vec{\xi}$$

Multiplying the Boltzmann equation by $\Phi(\xi_i)$ and integrating over velocity space, $d\vec{\xi}$, one obtains the Maxwell equation of transfer

$$\frac{\partial}{\partial t}(\overline{n\Phi}) + \frac{\partial}{\partial x_i}(\overline{n\Phi\xi_i}) - \frac{\partial}{\partial \xi_i}\left[\,\overline{\left(\frac{F_i}{m} - (\vec{J}' \times \vec{\xi})_i\right) n\Phi}\,\right] = \Delta\Phi \tag{2.2}$$

where

$$\Delta\Phi \equiv \int |\vec{\xi}' - \xi| \, d\Omega \int (\Phi^* - \Phi) f f' \, d\vec{\xi}' d\vec{\xi} \tag{2.3}$$

$(\Phi^* - \Phi) \quad \equiv \quad$ change in Φ due to collision

If an infinite number of moments are taken (an infinite number of Φ's) and the resulting equations are satisfied, the solution is equivalent to solving the Boltzmann equation exactly. But to solve an infinite set of equations would be at least as difficult to solve as the original Boltzmann equation.

The problem of evaluating the collision integral remains, but in this case the integration is extended to the six dimensional velocity space $d\vec{\xi}' d\vec{\xi}$ which simplifies the computation. Furthermore, if $\Phi(\xi_i)$ is taken to be any of the five scalar invariants for perfectly elastic collisions; mass $(\Phi = m)$, momentum $(\Phi = m\xi_i,\ i=1,2,3)$ and energy $(\Phi = m\xi^2/2)$, the integral $\Delta\Phi$ vanishes since $\Phi = \Phi^*$ for these moments. For moments which correspond to nonzero $\Delta\Phi$ a great simplification is achieved by adopting Maxwell's inverse fifth power law for the force field between colliding particles. This force field corresponds to an intermolecular potential given by

$$\psi = \frac{a}{r^4} \tag{2.4}$$

where

$$a = \text{constant}$$

For this potential, the collision integral is independent of the relative velocity of colliding particles $|\vec{\xi}' - \vec{\xi}|$ and can be readily integrated.[35] It was shown by Maxwell that the collision integral, $\Delta\Phi$, for the nonvanishing moments corresponding to the stress $(\Phi = m\xi_j\xi_k)$ and to the heat flux $(\Phi = m\xi_j\xi^2/2)$ are given by[35]

$$\Delta\Phi(m\xi_j\xi_k) = 3\pi\, A_2(4)\sqrt{\frac{2a}{m}}\; \bar{n}\, \bar{p}_{jk} \tag{2.5}$$

$$A_2(4) = \text{pure number}$$

and

$$\Delta\Phi(m\xi_j\xi^2/2) = 3\pi\, A_2(4)\sqrt{\frac{2a}{m}}\, \bar{n}\left[-\frac{2}{3}\bar{Q}_j + \sum_{i=1}^{3}\bar{q}_i\bar{p}_{ij}\right] \tag{2.6}$$

The viscosity coefficient for the Maxwellian molecule is found to be[35]

$$\mu = \frac{\sqrt{\frac{m}{2a}}}{3\pi A_2(4)}\, kT = \frac{\sqrt{\frac{m}{2a}}}{3\pi A_2(4)}\left(\frac{\bar{p}}{\bar{n}}\right) \tag{2.7}$$

Assuming Maxwell's relationship between the viscosity and the MFP at the free stream conditions one obtains

$$\mu_\infty = \frac{1}{2} m n_\infty \lambda_\infty \bar{c} \tag{2.8}$$

where
$$\bar{c} \equiv \sqrt{8kT_\infty/\pi m}$$

and substituting Equations (2. 7) and (2. 8) into Equations (2. 5) and (2. 6), one obtains

$$\Delta\Phi(m\xi_j\xi_k) = \frac{\sqrt{\frac{\pi k T_\infty}{2m}}}{\lambda_\infty}\left(\frac{\bar{n}}{n_\infty}\right)\bar{p}_{jk} \tag{2.9}$$

and

$$\Delta\Phi(m\xi_j\xi^2/2) = \frac{\sqrt{\frac{\pi k T_\infty}{2m}}}{\lambda_\infty}\left(\frac{\bar{n}}{n_\infty}\right)\left[-\frac{2}{3}\bar{Q}_j + \sum_{i=1}^{3} \bar{q}_i\bar{p}_{ij}\right] \tag{2.10}$$

Although this choice for the intermolecular potential is highly idealized it affords the greatest simplification while preserving the nonlinear character of the collision integral. If a more accurate description of a real gas is found to be necessary more realistic intermolecular potentials can be used. However from previous investigations[3,16,17,23] in which the Maxwell inverse fifth power law was utilized gross aerodynamics quantities compared favorably with the experimental results.

II. B Criticisms of the Moment Method

Two major criticisms of the moment method are

(1) Truncation of the Equations

(2) Closure

(1) Truncation of the Equations

A finite number (N) of moments is normally taken in all schemes involving the moment method. This necessary truncation is one of the main criticisms, nevertheless it allows the introduction of some physical insight at a very early stage. For example, by taking the moments corresponding to the five scalar collision invariants, the integral $\Delta\Phi$ vanishes and a set of equations is obtained which reduces to the familiar conservation in continuum fluid mechanics. For a given problem some moments are more important than others. As a rule of thumb, knowledge of the lower moments which have obvious physical meaning are preferred. The error made in truncating the equations cannot be determined since no convergence of the moment method for the Boltzmann equation has been studied systematically.

(2) Closure problem

A second difficulty in the moment technique is that the N^{th} moment equation contains the $(N+1)^{ST}$ moment. A closure problem exists since a complete formulation requires that the number of dependent variables be equal to the number of equations. The general procedure to effect the closure is to assume that the distribution function may be expressed in terms of "N" parametric functions of the spatial variables,

$$f = f(\vec{\xi}, H_1(\vec{x}), H_2(\vec{x}), \ldots, H_N(\vec{x}))$$

where

$$H_i(\vec{x}) \equiv \text{ parametric functions}$$

The "N" moment equations now result in "N" equations in the "N" unknown parametric functions $(H_1, H_2, \ldots, H_N)$. The shortcoming of this procedure is that there is no unique method to choose the parametric functions. This is a common failing of most integral techniques; e.g., the Raleigh-Ritz Method or the Karman-Pohlhausen Method in boundary layer theory, but the validity should be judged by the results produced.

In spite of these valid criticisms, the moment method appears to offer the most promise to obtain results for the complete range of fluid densities while retaining the essential nonlinear features of the Boltzmann equation.

II.C Moment Method Solutions

(1) Grad's Thirteen Moment Method

Grad assumed a distribution function which was a perturbation over the local Maxwellian by the local stresses and heat fluxes

$$f = f^{(0)} \left\{ 1 + \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\overline{\tau}_{ij} c_i c_j}{2 \overline{p} RT} - \sum_{i=1}^{3} \frac{\overline{Q}_i c_i}{\overline{p} RT} \left(1 - \frac{c^2}{5RT} \right) \right\} \tag{2.11}$$

When the moment equations corresponding to the five scalar invariants are computed, thirteen dependent variables appear in the five conservation equations. These thirteen functions of the spacial coordinates are

$$\overline{n}, \overline{q}_i, T, \overline{\tau}_{ij}, \overline{Q}_i$$

The stress $\overline{\tau}_{ij}$ is a symmetric tensor and represents only six unknowns.

As pointed out by Lees,[3] Grad's method gives qualitatively good results for relatively simple problems such as the low speed Couette flow, but introduces undesirable couplings between stresses and heat fluxes for more difficult problems. The chief criticism, however, is that in using polynomials in the particle velocity the distribution function is continuous in velocity space and thus cannot exhibit the discontinuity in velocity which is essential in the FMF limit for flows with solid boundaries

(2) Mott-Smith Bimodel Method[7]

Mott-Smith employed a bimodel distribution function to study the structure of a strong normal shock wave.

He assumed that the distribution function was given by the sum of two full range Maxwellian type distributions

$$f = f_\alpha^{(M)} + f_\beta^{(M)}$$

where $f_\alpha^{(M)}$ represents the supersonic upstream particles and $f_\beta^{(M)}$ represents the subsonic downstream particles. The Maxwellian distribution for $f_\alpha^{(M)}$ was assumed to be

$$f_\alpha^{(M)} = \frac{n_\alpha(\vec{x})}{(2\pi R T_\alpha(\vec{x}))^{3/2}} \exp\left\{-\frac{[\vec{\xi}-\vec{u}_\alpha(\vec{x})]^2}{2RT_\alpha(\vec{x})}\right\} \qquad (2.12)$$

and for $f_\beta^{(M)}$, α is replaced by β. In general each of the distribution functions contains five parametric functions $(n_\alpha, T_\alpha, \vec{u}_\alpha)$ which must be determined from the moment equations. If this model is applied to problems with solid boundaries the bimodel distribution would be incorrect in the FMF limit.

(3) Lees Two Stream Maxwellian[3]

Lees generalized Mott-Smith's formulation so that the following requirements were satisfied by the distribution function. First, the distribution function must have a discontinuity in the velocity space which is essential in the FMF limit; second, the distribution function must be capable of providing a smooth transition from the FMF limit to the continuum regime, and; third, that it lead to the simplest set of differential equations and boundary conditions consistent with the first two requirements.

The first condition of discontinuity in velocity or two-sidedness of the distribution function is fulfilled if one could generalize the FMF solution. The time independent Boltzmann equation with no external forces in cartesian coordinates for the FMF limit is given by

$$\xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \xi_3 \frac{\partial f}{\partial x_3} = 0 \tag{2.13}$$

The characteristics of Equation (2. 13) are given by

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \frac{dx_3}{\xi_3} = \frac{d\xi_1}{0} = \frac{d\xi_2}{0} = \frac{d\xi_3}{0} \tag{2.14}$$

which shows that the distribution function is constant along the particle trajectories

$$\xi_1 = b_1 \qquad \xi_2 = b_2 \qquad \xi_3 = b_3$$

$$x_1 = b_4 + \frac{b_1}{b_4} x_2 \qquad x_2 = b_5 + \frac{b_2}{b_3} x_3 \qquad x_1 - \frac{b_1}{b_2} x_3 = b_6$$

where $b_i =$ constant

These characteristics are straight lines in the physical space (x_1, x_2, x_3) moving with unchanged velocity ($\xi_1 = b_1, \xi_2 = b_2, \xi_3 = b_3$) in both directions from solid surfaces. As stated by Lees[3] the distribution function at a given point in physical space is governed by the "line of sight" principle of geometrical optics. Lees separates the space into the two regions shown in Figure 1.

According to the "line of sight" principle the effect of the body at a point $P(x_1, x_2, x_3)$ (Figure 1) is confined to the conical surface generated by tangent lines from $P(x_1, x_2, x_3)$ to the surface. For the simple boundary conditions of diffuse reemission (see Section III. B) at the solid surfaces the distribution function for all the particles with velocity vectors in the cone, region (1), directed away from the surface is the Maxwellian characterized by the velocity and temperature of the solid surface. The remaining velocity vectors, region (2), are characterized by the free stream Maxwellian.

Based on these observations Lees generalized the FMF solution by the following representation:

For $\vec{\xi}$ lying in region (1) the distribution function is given by

$$f = f_1^{(M)}(\vec{\xi}, \vec{x}) = \frac{n_1(\vec{x})}{(2\pi R T_1(\vec{x}))^{3/2}} \exp\left\{ - \frac{[\vec{\xi} - \vec{u}_1(\vec{x})]^2}{2RT_1(\vec{x})} \right\} \tag{2.15}$$

and a similar expression for region (2)

The quantities, $n_i(\vec{x})$, $T_i(\vec{x})$, $\vec{u}_i(\vec{x})$; $i = 1, 2$, represent ten parametric functions of $\vec{x}$ which can be determined from ten partial differential equations resulting from taking ten moment equations. Since the distribution function is completely determined once the parametric functions are found all microscopic quantities such as velocity, density, and pressure can be computed.

The particular choice of the ten moment equations is not unique. But in all problems the goal is to satisfy the conservation equations and at least one moment resulting in a nonvanishing collision integral.

Although the method proposed by Lees has obvious shortcomings, it has been sucessfully applied to problems concerned with a fluid of finite extent and/or nonlinear problems with plane boundaries.

II. D Free Molecule Flow Solution

Any method which proposes to be true in the FMF limit, such as Lees' Method, must begin with the correct FMF distribution function. The correct FMF distribution function results from solving the Boltzmann equation in the limit $K_n \to \infty$, where the collision integral vanishes. The time independent Boltzmann equation with no external forces is given for spherical coordinates (Appendix (A))

$$\xi_r \frac{\partial f}{\partial r} + \frac{\xi_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{1}{r}\Big[(\xi_\theta^2+\xi_\phi^2)\frac{\partial f}{\partial \xi_r} + (\cot\theta\,\xi_\phi^2 - \xi_r\xi_\theta)\frac{\partial f}{\partial \xi_\theta}$$

$$- (\xi_r\xi_\phi + \xi_\theta\xi_\phi\cot\theta)\frac{\partial f}{\partial \xi_\theta}\Big] = 0 \tag{2.16}$$

where symmetry with respect to the angle ϕ has been assumed.

The characteristics of Equation (2. 16) are

$$\frac{dr}{\xi_r} = \frac{d\theta}{\frac{\xi_\theta}{r}} = \frac{d\phi}{0} = \frac{d\xi_r}{\frac{(\xi_\theta^2+\xi_\phi^2)}{r}} = \frac{d\xi_\theta}{\frac{1}{r}(\cot\theta\,\xi_\phi^2 - \xi_r\xi_\theta)} = \frac{-d\xi_\phi}{\frac{1}{r}(\cot\theta\,\xi_\theta\xi_\phi + \xi_r\xi_\theta)} \tag{2.17}$$

In cartesian coordinates the characteristics for the homogeneous Boltzmann equation were found from a simple integration (Section II. C. 3). But in the spherical case the equations given by Equation (2. 17) are much more difficult to integrate.

The following exposition illustrates this difficulty. Lees[3] assumed that the distribution function could be generalized from a cartesian type Maxwellian (Equation 2. 15)

$$f_i^{(M)}(\vec{\xi},\vec{x}) = \frac{n_i(\vec{x})}{(2\pi R T_i(\vec{x}))^{3/2}} \exp\left\{-\frac{[\vec{\xi}-\vec{u}_i(\vec{x})]^2}{2RT_i(\vec{x})}\right\} \tag{2.15}$$

where

$$\vec{x} = (r, \theta, \phi)$$

$$\vec{u}_i = (u_{ri}, u_{\theta i}, u_{\phi i})$$

When Equation (2. 15) is substituted into Equation (2. 16) the solution requires that

$$\begin{aligned}
\left[u_{ri}(r,\theta)\right]^{FMF} &= K_i^{(1)} \cos\theta \\
\left[u_{\theta i}(r,\theta)\right]^{FMF} &= K_i^{(2)} \sin\theta \\
\left[n_i(r,\theta)\right]^{FMF} &= K_i^{(3)} \\
\left[T_i(r,\theta)\right]^{FMF} &= K_i^{(4)}
\end{aligned} \tag{2. 18}$$

where

$K_i^{(j)}$ = constant

$u_{\phi i} \equiv 0$ from symmetry

The uniform flow conditions in the free stream $(r = \infty)$ requires that the following boundary conditions be satisfied for all values of the MFP

$$\begin{aligned}
f &= f_2 \\
u_{r_2}(\infty,\theta) &= -q_\infty \cos\theta \\
u_{\theta 2}(\infty,\theta) &= -q_\infty \sin\theta \\
n_2(\infty,\theta) &= n_\infty \\
T_2(\infty,\theta) &= T_\infty
\end{aligned} \tag{2. 19}$$

These conditions are consistent with the requirements of Equation (2. 18)

The boundary conditions for diffuse reemission at the surface of a stationary sphere $(r = r_0)$ requires (see Section (III. B)) that for all values of the MFP

$$
\begin{aligned}
f &= f_1 \\
u_{r_1}(r_0,\theta) &= 0 \\
u_{\theta_1}(r_0,\theta) &= 0 \\
T_1(r_0,\theta) &= T_b
\end{aligned}
\tag{2.20}
$$

In order to satisfy the condition of zero mass flux at the surface in the FMF limit the number density of the reflected particles is given in Appendix E to be

$$
n_b\sqrt{T_b} = n_\infty\sqrt{T_\infty}\left\{e^{-\frac{\gamma}{2}M^2\cos^2\theta} + \sqrt{\frac{\pi\gamma}{2}}M\cos\theta\left[1+\operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}M\cos\theta\right)\right]\right\} \tag{2.21}
$$

Since the reflected particles are identified as group (1), n_1 at the surface is identified with n_b and must be a function of the angle θ to insure the condition of zero mass transfer at the surface. Therefore, only three of the four conditions given by Equation (2. 18) are satisfied at the sphere surface.

Equation (2. 21) shows that a generalization of the cartesian type Maxwellian distribution function cannot satisfy the boundary conditions on a body with finite curvatures and the Boltzmann equation in the FMF limit. If the homogeneous Boltzmann equation is not satisfied the resulting moment equations cannot be correct in the FMF limit.

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CHAPTER III – PRESENT FORMULATION

III.A Definition of the Distribution Function

It was shown in the preceding section that the solution for the FMF distribution function in spherical coordinates requires a solution of five complicated first order differential equations. Even if a solution could be found it would most likely be too complicated to generalize for use in the moment method. An approximate method to satisfy the homogeneous Boltzmann equation is to modify Lees' formulation by defining the following distribution function (Figure 1)

$$f = f_1 = f_1^{(M)}(\vec{x}, \vec{\xi}) + G(\vec{x}, \vec{\xi}) \quad \text{in region (1)} \tag{3.1}$$

and

$$f = f_2 = f_2^{(M)}(\vec{x}, \vec{\xi}) + G(\vec{x}, \vec{\xi}) \quad \text{in region (2)} \tag{3.2}$$

where $f_i^{(M)}(\vec{x}, \vec{\xi})$ is the generalized cartesian Maxwellian given by Lees.

The function $G(\vec{x}, \vec{\xi})$ is determined from the following conditions

(1) The boundary conditions are satisfied.

(2) The moment equations are satisfied in the FMF limit.

(3) $G(\vec{x}, \vec{\xi})$ takes the simplest form consistent with (1) and (2).

The average value of any function of particle velocity becomes

$$\overline{n\Phi} \equiv \int f \Phi \, d\vec{\xi} = \int_{\text{region}(1)} f_1^{(M)} \Phi \, d\vec{\xi} + \int_{\text{region}(2)} f_2^{(M)} \Phi \, d\vec{\xi} + \int_{\text{region}(1)\text{ and }(2)} G \Phi \, d\vec{\xi} \tag{3.3}$$

The advantage of defining f_1 and f_2 so that they both contain the same function $G(\vec{x}, \vec{\xi})$ is indicated in the last term where the integration extends over the complete velocity space. This will lead to a great simplification in computing the moments.

The explicit form of the function $G(\vec{x},\vec{\xi})$ is determined by the following considerations.

The cartesian type Maxwellian $f_i^{(M)}$ satisfies the Boltzmann equation in the FMF limit but not the boundary conditions on the curved surface. If $G(\vec{x},\vec{\xi})$ is taken to be zero except at a finite number of points in the velocity space, f_1 and f_2 would satisfy the Boltzmann equation except at these finite points in velocity space. Since the moments of the distribution function are integrals defined in velocity space, $G(\vec{x},\vec{\xi})$ must be integrable over that space. A quantity having these two properties is the Dirac delta function. The function $G(\vec{x},\vec{\xi})$ will be assumed to take the following general form in terms of the delta function and its derivatives

$$G(\vec{x},\vec{\xi}) \equiv \sum_{n=0}^{\infty} \sum_{i=1}^{3} g_i^{(n)}(\vec{x})\, \delta(\xi_j)\, \delta(\xi_k)\, \frac{d^n}{d\xi_i^n}\left[\delta(\xi_i)\right] \tag{3.4}$$

where the cyclic order of permutation of the indices (i, j, k) must be followed and

$$\delta(\xi_i) \equiv \frac{d^0}{d\xi_i^0}\left[\delta(\xi_i)\right]$$

This choice of representation for $G(\vec{x},\vec{\xi})$ is certainly arbitrary, just as the functional form of the distribution function f. However, the consideration of the unique representation of f has never been a major issue in any integral method. The simplification of integrating the function $G(\vec{x},\vec{\xi})$ over the complete velocity space is apparent since the integral of the delta function over a finite interval depends on whether the argument of the delta function lies within the interval. A physical interpretation of $G(\vec{x},\vec{\xi})$ is given in Appendix M.

From the definition of $G(\vec{x},\vec{\xi})$ the contribution of G to the average value of Φ is found to be

$$\int G(\vec{x},\vec{\xi})\,\Phi(\vec{\xi})\,d\vec{\xi} = \tag{3.5}$$

region (1) and (2)

$$\iiint_{-\infty}^{\infty} G(\vec{x},\vec{\xi})\,\Phi(\vec{\xi})\,d\vec{\xi} = \sum_{n=0}^{\infty}\sum_{i=1}^{3}(-1)^n g_i^{(n)}(\vec{x})\left[\frac{d^n\Phi(\xi_i,0,0)}{d\xi_i^{\,n}}\right]_{\xi_i=0}$$

In arriving at Equation (3.5) the following integral property of the delta function has been used

$$\int_{-\infty}^{\infty} F(x)\,\frac{d^n}{dx^n}\left[\delta(x-x_0)\right]dx = (-1)^n\left[\frac{d^nF(x)}{dx^n}\right]_{x=x_0}$$

To conserve the total number of particles the requirement that

$$\iiint_{-\infty}^{\infty} G(\vec{x},\vec{\xi})\,d\vec{\xi} = 0 \tag{3.6}$$

is imposed. This condition is satisfied if

$$g_i^{(0)}(\vec{x}) = 0 \qquad i=1,2,3 \tag{3.7}$$

It is found that the simplest form for $G(\vec{x},\vec{\xi})$ for the flow over a sphere is an expansion in terms of derivatives with respect to the radial particle velocity, ξ_r. The particle velocity vector is

$$\vec{\xi} = (\xi_r, \xi_\theta, \xi_\phi)$$

and the spatial coordinates are

$$\vec{x} = (r, \theta, \phi)$$

Therefore, for spherical symmetry

$$G(\vec{x}, \vec{\xi}) = G(r, \theta; \vec{\xi})$$

and the integral is found to be

$$\int G(r,\theta;\vec{\xi})\,\Phi(\xi_r, \xi_\theta, \xi_\phi)\, d\vec{\xi} = \sum_{n=1}^{\infty} (-1)^n g^n(r,\theta) \left[\frac{d^n \Phi(\xi_r, 0, 0)}{d\xi_r^{\,n}} \right]_{\xi_r = 0} \tag{3.8}$$

The function $G(r,\theta;\vec{\xi})$ is determined in terms of the parametric functions appearing in $f_i^{(M)}(r,\theta;\vec{\xi})$ and thus introduces no additional variables into the distribution function. For example, in the problem of the flow over a sphere with spherical symmetry the number of parametric functions becomes eight:

$$n_i(r,\theta)$$

$$T_i(r,\theta)$$

$$\vec{u}_i = (u_{ri}(r,\theta),\ u_{\theta i}(r,\theta),\ 0)$$

$$u_{\phi i} = 0 \qquad \text{from symmetry}$$

$$i = 1, 2$$

These eight functions are determined by satisfying eight moment equations. Once these functions are determined, all mean quantities such as velocity, density, and pressure can be computed.

In spherical coordinates Maxwell's equation of transfer is given by (see Appendix A).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\int \xi_r \Phi f\, d\vec{\xi}\right\} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\int \xi_\theta \Phi f\, d\vec{\xi}\right\}$$

$$-\frac{1}{r}\int f\left\{\left[\xi_\theta^2+\xi_\phi^2\right]\frac{\partial\Phi}{\partial\xi_r} + \left[\cot\theta\, \xi_\phi^2 - \xi_r\xi_\theta\right]\frac{\partial\Phi}{\partial\xi_\theta}\right.$$

$$\left. -\left[\cot\theta\, \xi_\theta\xi_\phi + \xi_r\xi_\phi\right]\frac{\partial\Phi}{\partial\xi_\phi}\right\} d\vec{\xi} = \Delta\Phi$$

(3.9)

The eight moment equations necessary to determine the eight parametric functions will be chosen to correspond to the equations of continuity $(\Phi_1 = m,\ \Delta\Phi_1 = 0)$, radial momentum $(\Phi_2 = m\xi_r,\ \Delta\Phi_2 = 0)$ tangential momentum $(\Phi_3 = m\xi_\theta,\ \Delta\Phi_3 = 0)$, energy $(\Phi_4 = \frac{1}{2}m\xi^2,\ \Delta\Phi_4 = 0)$, shear stress $(\Phi_5 = m\xi_r\xi_\theta,\ \Delta\Phi_5 = (\frac{p}{\mu})\,\overline{p}_{r\theta})$, radial stress $(\Phi_6 = m\xi_r\xi_r,\ \Delta\Phi_6 = (\frac{p}{\mu})\,\overline{p}_{rr})$, tangential stress $(\Phi_7 = m\xi_\theta\xi_\theta,\ \Delta\Phi_7 = (\frac{p}{\mu})\,\overline{p}_{\theta\theta})$, and the radial heat flux $(\Phi_8 = \frac{m}{2}\xi_r\xi^2,\ \Delta\Phi_8 = (\frac{p}{\mu})[-\frac{2}{3}\overline{Q}_r + \overline{q}_r\overline{p}_{rr} + \overline{q}_\theta\overline{p}_{r\theta} + \overline{q}_\phi\overline{p}_{r\phi})$
The eight moment equations are given in Appendix B.

As stated by Lees[3] no integral method can be expected to predict phenomena such as flow separation or the details of wake formation behind bluff bodies in the continuum limit. In addition, the investigation of flow over a sphere will be restricted to the two limiting cases, $M \ll 1$ (low speed) and $M \gg 1$ (hypersonic). In the low speed case the solution can be investigated from the FMF limit to the Stokes flow regime which spans a large range of Knudsen numbers. For the continuum hypersonic limit the flow in front of the body is relatively unaffected by the wake on the rear portion of the body and again a wide range of Knudsen numbers may be investigated. These two limiting cases are discussed in Chapter IV and Chapter V.

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III. B Boundary Conditions

The determination of the boundary conditions at a solid surface is in itself one of the most fundamental and difficult problems in rarefied gas dynamics. For simplicity, diffuse reemission from solid boundaries will be assumed. Referring to Figure 1, the incident particles at a convex solid surface belong to group (2) and the reemitted particles belong to group (1). From the definition of the two stream distribution function all particles with velocity vectors directed away from the surface belong to group (1) and all others to group (2). For diffuse reemission the emitted particles have a Maxwellian velocity distribution corresponding to the wall temperature and the local surface velocity.

Therefore, at the body surface

$\vec{x} = \vec{x}_b$:

$$\vec{u}_1(\vec{x}_b) = \vec{u}_b$$
$$T_1(\vec{x}_b) = T_b$$

The condition of zero mass transfer at the wall is satisfied by

$$(n\vec{q})_{NORMAL} \equiv \int (\vec{\xi}) f d\vec{\xi}_{NORMAL} = 0 \quad \text{at} \quad \vec{x} = \vec{x}_b$$

The free stream is assumed to be in equilibrium and the distribution function is given by the local Maxwellian $f = f_2^{(M)}$ with

$$\vec{u}_2(\infty) = \vec{q}_\infty$$
$$T_2(\infty) = T_\infty$$
$$n_2(\infty) = n_\infty$$

Also since

$$f = f_2^{(M)} + G(r,\theta;\vec{\xi}) \qquad \text{in region (2)}$$

$G(r,\theta;\vec{\xi})$ must vanish as $r \to \infty$.

For the flow over a sphere with symmetry with respect to "ϕ" the boundary conditions at $r = \infty$ becomes

$$\begin{aligned} &(i)\ u_{r_2}(\infty,\theta) = -q_\infty \cos\theta \\ &(ii)\ u_{\theta_2}(\infty,\theta) = q_\infty \sin\theta \\ &(iii)\ T_2(\infty,\theta) = T_\infty \\ &(iv)\ n_2(\infty,\theta) = n_\infty \end{aligned} \qquad (3.10.a)$$

If the sphere is stationary, $\vec{u}_b = 0$, the boundary conditions at the surface $r = r_0$ becomes

$$\begin{aligned} &(v)\ u_{r_1}(r_0,\theta) = 0 \\ &(vi)\ u_{\theta_1}(r_0,\theta) = 0 \\ &(vii)\ T_1(r_0,\theta) = T_b \end{aligned} \qquad (3.10.b)$$

(T_b is constant if the sphere is assumed to have infinite conductivity)

$$(viii)\ (\overline{nq_r})_{r=r_0} = 0$$

To satisfy boundary condition (viii), the radial velocity is computed from Equation (3.3) by taking $\Phi = \xi_r$.

$$\overline{nq_r} = \int_{(1)} f_1^{(M)} \xi_r d\vec{\xi} + \int_{(2)} f_2^{(M)} \xi_r d\vec{\xi} + \int_{(1)\text{ and }(2)} G\, \xi_r d\vec{\xi} \qquad (3.11)$$

The last term is found from Equation (3.8) to be

$$\int G(r,\theta;\vec{\xi})\xi_r d\vec{\xi} = -\ q^{(1)}(r,\theta) \qquad (3.12)$$

It will be found later that $q^{(1)}(r,\theta)$ has the following form

$$q^{(1)} \sim \frac{1}{r^2}$$

This satisfies the condition that $G(r,\theta;\vec{\xi})$ vanish at $r=\infty$. Furthermore, this behavior of $q^{(1)}$ is analogous to the effect of a doublet (or dipole) at the center of the sphere.

For invicid, irrotational, and incompressible flow over a sphere, the velocity potential is given by

$$\hat{\phi}(r,\theta) = q_\infty r\cos\theta + q_\infty \frac{r_0^3}{2r^2} \cos\theta$$

The first term is the effect of the free stream and the second term represents a doublet at the center of the sphere. The doublet guarantees the vanishing of the normal velocity at the surface of the sphere

$$(q_r)_{r=r_0} = \left(\frac{\partial\hat{\phi}}{\partial r}\right)_{r=r_0} = 0$$

Analogous to the situation in potential flow theory, the appearance of $q^{(1)}$ effectively reduces the velocity of the impinging free stream particles by placing a particle source at the center of the sphere in such a way that the boundary condition of zero mass transfer at the body is satisfied.

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III. C Grad's Asymptotic Solution

H. Grad[8] presented a survey of the flow regimes for the problem of flow over an object. In discussing some of the more interesting limiting cases he observed that even for large MFP the elementary FMF theory is not uniformly valid at large distances from the body. The FMF theory is valid for a distance r from the body which is small in comparison to the MFP, λ. At a distance r comparable to λ the incident and reflected streams interact and both will be altered. In the limit as r becomes very large the Knudsen number based on r ; $K_n = \frac{\lambda}{r}$, becomes zero for any finite value of the MFP and an equilibrium situation or continuum flow conditions exist far from the body. This nonuniformity in the flow field is not confined to the case of very large MFP but exists for all values of λ and depends only on the length scale of interest. For example, if the MFP is small the region of FMF conditions lies in a very thin layer of order λ from the body. Thus in every case FMF conditions exist near the body, continuum flow conditions far from the body, and a complex transition zone in between.

Grad's interest was with the correct limiting solution of the Krook equation far from a small object. In cartesian coordinates for steady flow and no external forces the Krook equation becomes

$$\xi_i \frac{\partial f}{\partial x_i} = \nu \left(f^{(0)} - f \right) \tag{3.13}$$

Far from the small body located at the origin the disturbance produced by the body appears as a point singularity. This observation led Grad to modify the Krook equation by the addition of a point source at the origin.

$$\xi_i \frac{\partial f}{\partial x_i} = \nu \left(f^{(0)} - f \right) + \sigma(\vec{\xi}) \delta(\vec{x}) \tag{3.14}$$

The source $\nu(\vec{\xi})$ is a function of the particle velocity and $\delta(\vec{x})$ is the delta function.

Sirovich[9] showed that the source term is given by

$$\nu(\vec{\xi}) = - \int_A h(\vec{x}, \vec{\xi}) \vec{\xi} \cdot d\vec{A} + O\left(\frac{L}{r}\right)^3 \tag{3.15}$$

where $h(\vec{x}, \vec{\xi})$ is the perturbation distribution function when f is linearized about the free stream Maxwellian and

A ≡ surface of the body
L ≡ characteristic body length

The integral (3.15) represents the perturbation mass flux from the body and $\nu(\vec{\xi})$ may be interpreted as a particle source which adsorbs and emits particles such that the total number of particles is conserved. To order $\left(\frac{L}{r}\right)^3$ the source is completely determined by the boundary conditions on the surface and the body geometry.

M. H. Rose[10] applied Grad's formulation to the computation of the drag on a sphere for the near FMF limit in the high speed limit. She found that as the Knudsen number becomes infinite the source, $\nu(\vec{\xi})$, becomes the net mass flow from the source of the FMF perturbation solution. This solution represents the difference at any point in space between the stream of particles reflected by the body and those unable to reach the point due to the presence of the body.

The source term introduced into the Krook equation can be compared to the assumed form of the distribution function in the present study

$$f_i = f_i^{(M)}(r, \theta; \vec{\xi}) + G(r, \theta; \vec{\xi})$$

The function $G(r,\theta;\vec{\xi})$ was introduced to satisfy the boundary conditions at the surface of the body and the FMF limit. This is analogous to the function of Grad's source term, $\sigma(\vec{\xi})$.

A further comparison can be made after taking moments of the modified Krook equation corresponding to ; $\Phi = m, m\xi_i, \frac{m\xi^2}{2}$, which results in the five scaler conservation equations

Continuity ($\Phi = m$)

$$\frac{\partial}{\partial x_i}(\overline{mnq_{bi}}) = m\,\delta(\vec{x}) \int \sigma \, d\vec{\xi} \tag{3.16.a}$$

Momentum ($\Phi = m\xi_i$)

$$\overline{mq_{bj}}\,\frac{\partial \overline{q}_{bi}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{p}_{ij}}{\partial x_j} - \delta(\vec{x})\,\overline{mq_{bi}} \int \sigma \, d\vec{\xi}$$

$$+ m\,\delta(\vec{x}) \int \sigma \xi_i \, d\vec{\xi} \tag{3.16.b}$$

Energy ($\Phi = m\xi^2/2$)

$$\overline{mnq_{bi}}\,\frac{\partial}{\partial x_i}\left(h + \frac{q^2}{2}\right) = \frac{\partial}{\partial x_i}\left(\overline{p}_{ij}\overline{q}_{bj} - \overline{Q}_i\right)$$

$$- m\,\delta(\vec{x})\left(h + \frac{q^2}{2}\right)\int \sigma \, d\vec{\xi} + \frac{m\,\delta(\vec{x})}{2} \int \sigma \xi^2 \, d\vec{\xi} \tag{3.16.c}$$

Recall that these equations are asymptotic forms far from the body and the terms involving $\sigma(\vec{\xi})$ represent the disturbance due to the body at the origin. This disturbance can manifest itself in many ways but it must behave as a momentum sink if there is to be drag and an energy sink if transfer of energy between the free stream and the body occurs. From Equation (3.16.a) conservation of mass requires that

$$\int \sigma(\vec{\xi})\, d\vec{\xi} = 0 \tag{3.17}$$

which guarantees that the total number of particles is conserved. If there is no exchange of energy between body and stream Equation (3.16.c) requires that

$$\int \sigma(\vec{\xi})\, \xi^2\, d\vec{\xi} = 0 \tag{3.18}$$

These restrictions on the integrals involving the source $\sigma(\vec{\xi})$ will be shown to be analogous to the integral properties associated with the function $G(r, \theta, \vec{\xi})$ in the present formulation.

Although the similarities between Grad's formulation and the present one are numerous, Grad considers the asymptotic solution for the Krook equation far from the body whereas in the present study no approximation is made for the collision integral and the entire flow field is considered for all values of the MFP.

CHAPTER IV – LOW SPEED FLOW

IV. A Introduction

The drag and heat transfer for a sphere at low mean speeds will be investigated within the framework of the general formulation. Near the continuum regime for small MFP Basset[11] made a slip flow correction to the Stokes' drag formula. Goldberg[12] presented results according to Grad's thirteen moment equations for small Knudsen numbers. For large MFP the technique of Knudsen iteration has been employed by Liu, Pang, and Jew[13] for the Boltzmann equation and by Willis[14] using the Krook equation. Recently Lees and Brinker (private communication) have obtained numerical solutions for the drag based on the moment method, but the assumed form of the distribution function and the number of moments are all quite different from the present investigation. In their studies detailed analysis of the singularity in the governing equations is made and an elaborate numerical scheme is devised for the integration.

In the present study the distribution function was defined to be

$$f = f_1 = f_1^{(M)}(r,\theta;\vec{\xi}) + G(r,\theta;\vec{\xi}) \qquad \text{in region (1)}$$

and (4.1)

$$f = f_2 = f_2^{(M)}(r,\theta;\vec{\xi}) + G(r,\theta;\vec{\xi}) \qquad \text{in region (2)}$$

where $f_i^{(M)}$ is the generalized cartesian Maxwellian distribution function given in spherical coordinates with angular symmetry with respect to ϕ (Figure 2)

$$f_i^{(M)}(r,\theta;\vec{\xi}) = \tag{4.2}$$

$$\frac{n_i(r,\theta)}{(2\pi R T_i(r,\theta))^{3/2}} \exp\left\{-\left[\frac{(\xi_r - U_i(r,\theta))^2 + (\xi_\theta - V_i(r,\theta))^2 + \xi_\phi^2}{2RT_i(r,\theta)}\right]\right\}$$

where

$$u_i(r,\theta) \equiv u_{ri}(r,\theta)$$

$$v_i(r,\theta) \equiv u_{\theta i}(r,\theta)$$

To compute the moments it is convenient to transform the rectangular velocity space $(\xi_r, \xi_\theta, \xi_\phi)$ into spherical coordinates (ξ, ω, η). From Figures 3 and 4

$$\xi \equiv \sqrt{\xi_r^2 + \xi_\theta^2 + \xi_\phi^2}$$

$$\omega \equiv \cos^{-1}\left(\frac{\xi_r}{\sqrt{\xi_r^2+\xi_\theta^2+\xi_\phi^2}}\right) \qquad (4.3)$$

$$\eta \equiv \tan^{-1}\left(\frac{\xi_\phi}{\xi_\theta}\right)$$

The angle ω is the conical angle which separates the velocity space into two zones

$$f = f_1 \quad \text{for} \quad 0 < \omega < \frac{\pi}{2} - \alpha \qquad (4.4)$$

$$f = f_2 \quad \text{for} \quad \frac{\pi}{2} - \alpha < \omega < \pi$$

where

$$\alpha \equiv \cos^{-1}\left(\frac{r_0}{r}\right)$$

The integration limits for ξ and η are

$$0 \leqslant \xi \leqslant \infty$$

$$0 \leqslant \eta \leqslant 2\pi$$

The differential volume in spherical velocity space is

$$d\vec{\xi} = \xi^2 \sin\omega \; d\xi \, d\omega \, d\eta$$

and the average value of $\Phi(\xi_i)$ becomes

$$\overline{n\Phi} = \int_0^{\frac{\pi}{2}-\alpha}\int_0^{2\pi}\int_0^{\infty} \Phi f_1^{(M)} \xi^2 \sin\omega \, d\xi \, d\eta \, d\omega$$

$$+\int_{\frac{\pi}{2}-\alpha}^{\pi}\int_0^{2\pi}\int_0^{\infty} \Phi f_2^{(M)} \xi^2 \sin\omega \, d\xi \, d\eta \, d\omega \qquad (4.5)$$

$$+ \sum_{n=1}^{\infty} (-1)^n g^{(n)}(r,\theta) \left[\frac{d^n \Phi(\xi_r, 0, \beta)}{d\xi_r{}^n} \right]_{\xi_r = 0}$$

All mean quantities can be evaluated in terms of the eight parametric functions but the integrals are complicated by the form of $f_i^{(M)}$. However, a great simplification is achieved at this stage by introducing the low speed approximation into $f_i^{(M)}$. Making the approximation of low mean speed; i.e.,

$$u_i(r,\theta),\ v_i(r,\theta) \ll \sqrt{kT_\infty}$$

the squares of the mean velocities can be neglected and $f_i^{(M)}$ becomes

$$f_i^{(M)} \cong \frac{n_i}{(2\pi RT_i)^{3/2}} \exp\left\{-\frac{\xi^2}{2RT_i} + \frac{\xi u_i \cos\omega + \xi v_i \sin\omega \cos\eta}{RT_i}\right\}$$

$$\cong \frac{n_i}{(2\pi RT_i)^{3/2}} \left[1 + \frac{u_i}{RT_i}\xi\cos\omega + \frac{v_i}{RT_i}\xi\sin\omega\cos\eta\right] \exp\left(\frac{-\xi^2}{2RT_i}\right)$$

(4.6)

The moments that appear in the eight moment equations are given in Appendix B. They are computed from Equations (4.5) and (4.6) and are listed in Appendix C. The function $G(r,\theta;\vec{\xi})$ contributes to only three of the moments; $\overline{nq_r}$, $\overline{P}_{rr}$, and $\overline{P}_{rrr}$, which contain; $g^{(1)}$, $g^{(2)}$, and $g^{(3)}$, respectively.

The function $g^{(1)}(r,\theta)$ is determined by satisfying the boundary condition of vanishing radial velocity at the sphere surface; $g^{(2)}(r,\theta)$ is found by requiring that the normal pressure be correct in the FMF limit, and $g^{(3)}(r,\theta)$ insures the correct heat flux in the FMF limit. All three functions must satisfy the moment equations in the FMF limit. In short, the functional forms of $g^{(i)}$ in the FMF limit can be determined uniquely. Details of the determination of the $g^{(i)}$'s are presented in the next section.

IV.B Determination of $G(r,\theta;\vec{\xi})$

The function $g^{(1)}$ appears in only the moment corresponding to the radial velocity. From Appendix C

$$
\begin{aligned}
\widetilde{nq_r} &\equiv \frac{\overline{nq_r}}{n_\infty q_\infty} \\
&= \frac{(1-x^2)}{\sqrt{2\pi\gamma}\,M}\left(\tilde{n}_1\sqrt{\tilde{T}_1}-\tilde{n}_2\sqrt{\tilde{T}_2}\right) + \frac{\tilde{n}_1\tilde{u}_1+\tilde{n}_2\tilde{u}_2}{2} \qquad (4.7)\\
&\quad - \frac{x^3}{2}\left(\tilde{n}_1\tilde{u}_1-\tilde{n}_2\tilde{u}_2\right) - g^{(1)}(x,\theta)/n_\infty q_\infty
\end{aligned}
$$

At the surface of the sphere, $r=r_0$ $(x=0)$, the condition of zero mass flux requires that

$$
(\widetilde{nq_r})_{x=0} = \frac{\left(\tilde{n}_1\sqrt{\tilde{T}_1}-\tilde{n}_2\sqrt{\tilde{T}_2}\right)_{x=0}}{\sqrt{2\pi\gamma}\,M} + \left(\frac{\tilde{n}_2\tilde{u}_2}{2}\right)_{x=0} - \frac{g^{(1)}(0,\theta)}{n_\infty q_\infty} = 0 \qquad (4.8)
$$

The boundary condition $u_1(0,\theta)=0$ has been used in obtaining Equation (4.8)

The solution of the Boltzmann equation in the FMF limit in Section II.D required that the following conditions be satisfied for the generalized cartesian Maxwellian distribution function, $f_i^{(M)}$,

$$
\begin{aligned}
\tilde{n}_2^{FMF} &= 1 \\
\tilde{T}_2^{FMF} &= 1 \\
\tilde{u}_2^{FMF} &= -\cos\theta \\
\tilde{T}_1^{FMF} &= T_0/T_\infty \qquad (4.9)\\
\tilde{n}_1^{FMF} &\equiv \left(\frac{n_1}{n_\infty}\right)^{FMF} = K_1^{(3)} = \text{constant}
\end{aligned}
$$

Therefore, Equation (4. 8) in the FMF limit becomes

$$\frac{K_1^{(3)}\sqrt{\frac{T_b}{T_\infty}}-1}{\sqrt{2\pi\gamma}\;M} - \frac{\cos\theta}{2} - \frac{\left[g^{(1)}(0,\theta)\right]^{FMF}}{n_\infty g_\infty} = 0 \qquad (4.10)$$

Although the angle dependence must be incorporated in $g^{(1)}$, Equation (4. 10) contains two unknowns $K_1^{(3)}$ and $g^{(1)}$. In order to determine each uniquely, a second condition must be imposed. One can, without loss of generality, specify that $g^{(1)}$ vanishes at the stagnation point $(\theta = 0)$, i. e.,

$$g^{(1)}(0,0) = 0 \qquad (4.11)$$

Therefore $K_1^{(3)}$ is found to be

$$K_1^{(3)} \equiv \left(\frac{\tilde{n}_1}{n_\infty}\right)^{FMF} = \sqrt{\frac{T_\infty}{T_b}}\left(1 + \sqrt{\frac{\pi\gamma}{2}}\,M\right) \qquad (4.12)$$

The value of $\tilde{n}_1^{FMF}$ given by Equation (4. 12) corresponds to the value of the body number density, n_b, in the FMF solution Appendix E evaluated at the stagnation point. The function $\left[g^{(1)}(0,\theta)\right]^{FMF}$ is found from Equation (4. 10)

$$\left[g^{(1)}(0,\theta)\right]^{FMF} = \frac{n_\infty g_\infty}{2}\left(1-\cos\theta\right) \qquad (4.13)$$

Following the procedure of Lees' moment formulation, the result from Equation (4. 12) can be generalized for finite MFP to give

$$\left(\tilde{n}_1\sqrt{\tilde{T}_1}\right)_{x=0} = \left(\tilde{n}_2\sqrt{\tilde{T}_2}\right)_{x=0} - \sqrt{\frac{\pi\gamma}{2}}\,M\,\tilde{n}_2(0,0)\,\tilde{u}_2(0,0) + F(\theta,\lambda) \qquad (4.14)$$

where $F(\theta,\lambda)$ is an arbitrary function of θ and λ which must vanish in the FMF limit, i. e., $F(\theta;\infty) \equiv 0$

The boundary condition given by Equation (4.8) becomes

$$\frac{q^{(1)}(0,\theta)}{n_\infty q_\infty} = \frac{1}{2}\left[\tilde{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \tilde{n}_2(0,0)\tilde{u}_2(0,0)\right] + \frac{F(\theta;\lambda)}{n_\infty q_\infty} \tag{4.15}$$

It is shown in Appendix D that in order to satisfy the continuity equation in the FMF limit the x-dependence of $\left[q^{(1)}(x,\theta)\right]^{FMF}$ is

$$\left[q^{(1)}(x,\theta)\right]^{FMF} \sim (1-x^2) = \frac{1}{r^2}$$

Prescribing the same dependence on x for all values of the MFP completes the determination of $q^{(1)}(x,\theta)$

$$\frac{q^{(1)}(x,\theta)}{n_\infty q_\infty} = \frac{(1-x^2)}{2}\left[\tilde{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \tilde{n}_2(0,0)\tilde{u}_2(0,0)\right] + \frac{(1-x^2)}{2}\frac{F(\theta;\lambda)}{n_\infty q_\infty} \tag{4.16}$$

The determination of $q^{(2)}(x,\theta)$ and $q^{(3)}(x,\theta)$ is accomplished in similar fashion. Since $q^{(2)}$ appears only in the radial stress, $\overline{P}_{rr}$, it is determined uniquely by requiring that $\overline{P}_{rr}$ be equal to the normal pressure in the FMF limit. As given in Appendix E.

$$p^{FMF} = 1 + \frac{\epsilon}{4} + \sqrt{\frac{2\gamma}{\pi}}\, M\cos\theta\left(1+\frac{\pi}{4}\right) \tag{4.17}$$

Again generalizing the FMF result for finite MFP one finds that at $x=0$

$$\frac{2mq^{(2)}(0,\theta)}{kn_\infty T_\infty} = -\sqrt{\frac{\pi\gamma}{8}}\, M\left[\tilde{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \tilde{n}_2(0,0)\tilde{u}_2(0,0)\right] \tag{4.18}$$

In order to satisfy the radial momentum equation in the FMF limit $q^{(2)}$ is found in Appendix D t. be proportional to $\frac{1}{r^2} = (1-x^2)$ so that the final form becomes

$$\frac{2mq^{(2)}(x,\theta)}{kn_\infty T_\infty} = -(1-x^2)\sqrt{\frac{\pi\gamma}{8}}\, M\left[\tilde{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \tilde{n}_2(0,0)\tilde{u}_2(0,0)\right] \tag{4.19}$$

The third function $q^{(3)}(r,\theta)$ which appears only in the triple moment, $\overline{P}_{rrr}$, is determined in a completely analogous way as $q^{(2)}(x,\theta)$. Instead of matching the normal pressure and satisfying the radial momentum equation, the radial heat flux in the FMF limit is matched and the energy equation is satisfied. Finally

$$\begin{aligned}\frac{3mq^{(3)}(x,\theta)}{kn_\infty T_\infty q_\infty} = {} & (1-x^2)\left(\frac{T_b}{T_\infty}\right)\left[\tilde{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \tilde{n}_2(0,0)\tilde{u}_2(0,0)\right] \\ & + (1-x^2)\sqrt{\frac{\gamma}{2\pi}}\, M\left[1-n_2(0,\theta)\left(V_2^2(0,\theta)+U_2^2(0,\theta)\right)\right]\end{aligned} \tag{4.20}$$

The determination of the $q^{(i)}$'s result in the modified boundary conditions

$x=1$ $(r=\infty)$:

(i) $\tilde{u}_2(1,\theta) = -\cos\theta$

(ii) $\tilde{v}_2(1,\theta) = \sin\theta$

(iii) $\tilde{T}_2(1,\theta) = 1$

(iv) $\tilde{n}_2(1,\theta) = 1$ (4.21)

$x=0$ $(r=r_0)$:

(v) $\tilde{u}_1(0,\theta)=0$

(vi) $\tilde{v}_1(0,\theta)=0$ (4.22)

(vii) $\tilde{T}_1(0,\theta)=T_b$

(viii) $\left(\tilde{n}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\sqrt{\tilde{T}_2}\right)_{x=0} = -\sqrt{\frac{\pi\gamma}{2}}\, M\tilde{n}_2(0,0)\tilde{u}_2(0,0)$

$+ F(\theta;\lambda))$ (4.23)

where $F(\theta;\infty) = 0$

IV. C Linearized Moment Equations

The eight moment equations corresponding to continuity, radial momentum, tangential momentum, energy, shear stress, radial stress, tangential stress, and radial heat flux are generated from the Maxwell Equation of Transfer Appendix B. These moment equations are given in terms of the parametric functions in Appendix C. Introducing the following linearization

$$\tilde{n}_i = 1 + \tilde{N}_i$$
$$\tilde{T}_i = 1 + \tilde{t}_i$$

and the following dependent variables

$$N^{(\pm)} \equiv \tilde{N}_1 \pm \tilde{N}_2$$
$$t^{(\pm)} \equiv \tilde{t}_1 \pm \tilde{t}_2$$
$$u^{(\pm)} \equiv \tilde{u}_1 \pm \tilde{u}_2$$
$$V^{(\pm)} \equiv \tilde{V}_1 \pm \tilde{V}_2$$

one obtains the following set of governing equations accurate to $O(\mu^2)$

Continuity

$$\frac{(1-x^2)^2}{x}\frac{\partial}{\partial x}\left(N^{(-)}+\frac{1}{2}t^{(-)}\right) + \sqrt{2\pi\sigma}\, M\Big\{\frac{(1-x^2)}{2x}\frac{\partial u^{(+)}}{\partial x} - \frac{x^2(1-x^2)}{2}\frac{\partial u^{(-)}}{\partial x} + u^{(+)} + \frac{1}{2}\left(\cot\theta\, V^{(+)} + \frac{\partial V^{(+)}}{\partial \theta}\right) + \frac{(x^3-3x)}{2}\left(u^{(-)} + \frac{\cot\theta}{2} V^{(-)} + \frac{1}{2}\frac{\partial V^{(-)}}{\partial \theta}\right)\Big\} = 0 \tag{4.23.a}$$

Radial Momentum

$$\frac{1}{2x}\frac{\partial}{\partial x}\left(N^{(+)}+t^{(+)}\right) - \frac{x^2}{2}\frac{\partial}{\partial x}\left(N^{(-)}+t^{(-)}\right) + \sqrt{\frac{2\sigma}{\pi}}\, M\Big\{\frac{(1-x^4)}{x}\frac{\partial u^{(-)}}{\partial x} + (1-x^2)\left(u^{(-)} + \cot\theta\frac{V^{(-)}}{2} + \frac{1}{2}\frac{\partial V^{(-)}}{\partial \theta}\right)\Big\} = 0 \tag{4.23.b}$$

Tangential Momentum

$$\frac{1}{2}\frac{\partial}{\partial\theta}\left(N^{(+)}+t^{(+)}\right)+\frac{(x^3-3x)}{4}\frac{\partial}{\partial\theta}\left(N^{(-)}+t^{(-)}\right)$$
$$+\sqrt{\frac{\gamma}{2\pi}}M(1-x^2)^2\left[\frac{(1-x^2)}{x}\frac{\partial V^{(-)}}{\partial x}-V^{(-)}+\frac{\partial u^{(-)}}{\partial\theta}\right]=0 \qquad (4.23.c)$$

Energy (with conintuity)

$$\frac{\partial}{\partial x}\left(N^{(-)}-\frac{7}{2}t^{(-)}\right)=0 \qquad (4.23.d)$$

Shear Stress

$$(1-x^2)^2\frac{\partial}{\partial\theta}\left(N^{(-)}+\frac{3}{2}t^{(-)}\right)+\sqrt{2\pi\gamma}\,M\left\{\frac{(1-x^2)}{2x}\frac{\partial V^{(+)}}{\partial x}\right.$$
$$+\frac{3x^2}{4}(1-x^2)\left(x^2-\frac{5}{3}\right)\frac{\partial V^{(-)}}{\partial x}+\frac{x^3}{4}(5-3x^2)\left(V^{(-)}-\frac{\partial u^{(-)}}{\partial\theta}\right)$$
$$\left.+\frac{1}{2}\left(\frac{\partial u^{(+)}}{\partial\theta}-V^{(+)}\right)\right\}=-\sqrt{\frac{\pi\gamma}{2}}\frac{M}{\tilde{\lambda}}(1-x^2)^{3/2}V^{(-)} \qquad (4.23.e)$$

Radial Stress

$$\frac{2(1-x^2)(1-x^4)}{x}\frac{\partial}{\partial x}\left(N^{(-)}+\frac{3}{2}t^{(-)}\right)+\sqrt{2\pi\gamma}\,M\left\{\frac{3(1-x^2)}{2x}\frac{\partial u^{(+)}}{\partial x}\right.$$
$$-\frac{3x^4}{2}(1-x^2)\frac{\partial u^{(-)}}{\partial x}+u^{(+)}+\frac{1}{2}\left(\cot\theta\, V^{(+)}+\frac{\partial V^{(+)}}{\partial\theta}\right)+$$
$$\left.\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)\left(u^{(-)}+\cot\theta\, V^{(-)}+\frac{1}{2}\frac{\partial V^{(-)}}{\partial\theta}\right)\right\}$$
$$=\frac{-\pi}{\tilde{\lambda}}\left\{\frac{x\sqrt{1-x^2}}{2}\left(N^{(-)}+t^{(-)}\right)+\sqrt{\frac{2\gamma}{\pi}}\frac{M}{3}\sqrt{1-x^2}\,(1+3x^2)\,u^{(-)}\right.$$
$$\left.+\frac{4}{3\sqrt{1-x^2}}\frac{m\,g^{(2)}(x,\theta)}{k n_\infty T_\infty}\right\} \qquad (4.23.f)$$

Tangential Stress (with continuity, Energy, and Radial Stress Equations)

$$\left(\cot\theta - \frac{\partial}{\partial\theta}\right)\left[-V^{(+)} + \frac{1}{4}\left(\frac{3x^5}{2} - 5x^3 + \frac{15x}{2}\right)V^{(-)}\right] = 0 \tag{4.23.g}$$

Radial Heat Flux (with Radial Momentum)

$$x^2(1-x^2)\left[\frac{1}{5}\frac{\partial}{\partial x}\left(N^{(-)}+t^{(-)}\right) - \frac{\partial t^{(-)}}{\partial x}\right] - \frac{(1-x^2)}{x}\left[\frac{1}{5}\frac{\partial}{\partial x}\left(N^{(+)}+t^{(+)}\right) - \frac{\partial t^{(+)}}{\partial x}\right]$$
$$= -\frac{4\sqrt{1-x^2}}{15\tilde{\lambda}}\left[-\frac{1}{2}\left(N^{(-)} - \frac{7}{2}t^{(-)}\right) + \sqrt{\frac{\pi\gamma}{8}}\left(u_2(0,\theta) - u_2(0,0)\right)\right] \tag{4.23.h}$$

The boundary conditions in the linearized case become

$x = 1\ (r = \infty)$:

$$\begin{aligned}
&(i)\quad \tilde{u}_2(1,\theta) = -\cos\theta\\
&(ii)\quad \tilde{v}_2(1,\theta) = \sin\theta\\
&(iii)\quad \tilde{t}_2(1,\theta) = 0\\
&(iv)\quad \tilde{N}_2(1,\theta) = 0
\end{aligned} \tag{4.24.a}$$

$x = 0\ (r = r_0)$:

$$\begin{aligned}
&(v)\quad \tilde{u}_1(0,\theta) = 0\\
&(vi)\quad \tilde{v}_1(0,\theta) = 0\\
&(vii)\quad \tilde{t}_1(0,\theta) = \epsilon\\
&\qquad\qquad \frac{T_b}{T_\infty} \equiv 1+\epsilon\\
&(viii)\quad N^{(-)}(0,\theta) + \frac{1}{2}t^{(-)}(0,\theta)\\
&\qquad = -\sqrt{\frac{\pi\gamma}{2}}\,M\,\tilde{u}_2(0,0) + F(\theta,\lambda)
\end{aligned} \tag{4.24.b}$$

where $F(\theta;\infty) \equiv 0$

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IV.D General Separation of Variables

If the flow is symmetric with respect to the X_1 axis $(\theta = 0)$; see Figure 5, then the radial velocity is an even function of θ; $u(r,\theta) = u(r,-\theta)$, and the tangential velocity is an odd function of θ, $v(r,\theta) = -v(r,-\theta)$. The assumption of symmetry with respect to $\theta = 0$ is true in the FMF limit and in the continuum limit if flow separation does not occur.

The scalar density and temperature are also even functions of θ. The free stream boundary conditions naturally reflect the same dependence on θ.

$$\tilde{u}_2(\infty,\theta) = -\cos\theta$$

$$\tilde{v}_2(\infty,\theta) = \sin\theta$$

$$\tilde{n}_2(\infty,\theta) = 1$$

$$\tilde{T}_2(\infty,\theta) = 1$$

Using the well known result from Fourier analysis that an even function of θ in the interval $(-\pi,\pi)$ can be represented by a cosine series and an odd function in the same interval can be expressed in a sine series, the following separation of variables is assumed

$$u^{(\pm)}(x,\theta) = \sum_{n=0}^{\infty} A_n^{(\pm)}(x)\cos(n\theta)$$

$$v^{(\pm)}(x,\theta) = \sum_{n=0}^{\infty} B_n^{(\pm)}(x)\sin(n\theta)$$

$$N^{(\pm)}(x,\theta) = \sum_{n=0}^{\infty} C_n^{(\pm)}(x) \cos(n\theta)$$

$$t^{(\pm)}(x,\theta) = \sum_{n=0}^{\infty} D_n^{(\pm)}(x) \cos(n\theta) \tag{4.25}$$

Substituting into the eight moment equations in Section IV. C yields the ordinary differential equations given in Appendix F. If the series defined by (4.25) are truncated after $n=N$ terms; e.g.,

$$u^{(\pm)}(x,\theta) = \sum_{n=0}^{N} A_n^{(\pm)}(x) \cos(n\theta)$$

the number of equations becomes $(7N+5)$ in the $(6N+8)$ dependent variables; $\left(A_n^{(+)}, A_n^{(-)}, B_1^{(+)}, B_1^{(-)}, C_n^{(+)}, C_n^{(-)}, D_n^{(+)}, D_n^{(-)}\right)$, $n=0,1,\ldots,N$

Equating the number of equations to the number of unknowns yields $N=3$. In other words, truncation of the series by setting $N=3$ provides the closure to the problem. This corresponds to a system of 26 equations in 26 dependent variables. Although the set of 26 ordinary differential equations can be solved numerically, an alternate procedure is persued which eventually leads to analytic solutions.

IV. E Six Moment Formulation

The problem of interest is the computation of the drag and heat transfer for all values of the MFP when the mean speed of the free stream is low. Before one solves the complete problem, separate studies on two simple cases are illustrative. First, consider the heat conduction problem with no convection where all the "dynamic" variables (u_1, u_2, v_1, v_2) may be assumed constant. Second, investigate the drag problem with no heat transfer where the "thermodynamic" variables (n_1, n_2, T_1, T_2) may be taken as constants. From the results of the two separate problems the coupled heat transfer and drag computation can be formulated in terms of six moment equations.

IV. E. 1 Conductive Heat Transfer (Four Moment Solution)

The pure conductive heat transfer from a sphere was investigated by Lees.[15] He used the four moment equations corresponding to continuity, radial momentum, energy, and radial heat flux to determine the four "thermodynamic" variables; n_1, n_2, T_1, T_2 . The remaining four "dynamic" variables; u_1, u_2, v_1, v_2 were assumed constant, but in the absence of an external stream the boundary conditions require that

$$u_1 = u_2 = v_1 = v_2 = 0$$

The four linearized equations from Section IV. C become,

<u>Continuity</u>

$$\frac{\partial}{\partial x}\left(N^{(-)} + \frac{1}{2} t^{(-)}\right) = 0 \tag{4.26.a}$$

<u>Radial Momentum</u>

$$\frac{\partial}{\partial x}\left(N^{(+)} + t^{(+)}\right) - x^3 \frac{\partial}{\partial x}\left(N^{(-)} + t^{(-)}\right) = 0 \tag{4.26.b}$$

Energy (and continuity)

$$\frac{\partial}{\partial x}\left(N^{(-)} - \frac{7}{2}t^{(-)}\right) = 0 \tag{4.26.c}$$

Radial Heat Flux

$$\begin{aligned} & x^2(1-x^2)\left[\frac{1}{5}\frac{\partial}{\partial x}\left(N^{(-)}+t^{(-)}\right) - \frac{\partial t^{(-)}}{\partial x}\right] \\ & - \left(\frac{1-x^2}{x}\right)\left[\frac{1}{5}\frac{\partial}{\partial x}\left(N^{(+)}+t^{(+)}\right) - \frac{\partial t^{(+)}}{\partial x}\right] \\ & = \frac{2\sqrt{1-x^2}}{15\,\bar{\lambda}}\left(N^{(-)} - \frac{7}{2}t^{(-)}\right) \end{aligned} \tag{4.26.d}$$

Boundary Conditions:

$x=1$ $(r=\infty)$: (iii) $t^{(+)}(1) - t^{(-)}(1) = 0$

(iv) $N^{(+)}(1) - N^{(-)}(1) = 0$

$x=0$ $(r=r_0)$: (vii) $t^{(+)}(0) + t^{(-)}(0) = 2\epsilon$

(viii) $N^{(-)}(0) + \frac{1}{2}t^{(-)}(0) = 0$

Since the equations are independent of Θ the Equations (4.26.a) through (4.26.d) are ordinary differential equations. The solution of this set of equations is obtained readily

$$\begin{aligned} t^{(+)}(x) &= \beta_0\left(1 + \frac{8}{15\bar{\lambda}}\sqrt{1-x^2}\right) \\ t^{(-)}(x) &= \beta_0 \\ N^{(+)}(x) &= -\beta_0\left(\frac{1}{2} + \frac{8}{15\bar{\lambda}}\sqrt{1-x^2}\right) \\ N^{(-)}(x) &= -\frac{1}{2}\beta_0 \end{aligned} \tag{4.27}$$

where $\beta_0 \equiv \dfrac{\epsilon}{1+\dfrac{4}{15\tilde{\lambda}}}$

The radial heat flux at the sphere surface is given by Appendix C

$$\bar{Q}_r(0,\theta) = n_\infty \sqrt{\frac{2}{\pi m}}\,(kT_\infty)^{3/2}\, t^{(-)}(0)$$

$$= n_\infty \sqrt{\frac{2}{\pi m}}\,(kT_\infty)^{3/2} \frac{\epsilon}{1+\dfrac{4}{15\tilde{\lambda}}} \tag{4.28}$$

The average heat transfer becomes Appendix H.

$$\bar{Q}_{AVE} = n_\infty \sqrt{\frac{2}{\pi m}}\,(kT_\infty)^{3/2} \frac{\epsilon}{1+\dfrac{4}{15\tilde{\lambda}}} \tag{4.29}$$

Normalizing the heat transfer by the FMF limit gives the result obtained by Lees[15]

$$\frac{\bar{Q}_{AVE}}{\left(\bar{Q}_{AVE}\right)^{FMF}} = \frac{1}{1+\dfrac{4}{15\tilde{\lambda}}} \tag{4.30}$$

IV. E. 2 Drag (Four Moment Solution)

The drag problem for the low speed flow without heat transfer can be formulated in terms of four moments. A four moment solution for the low speed flow over a cylinder was presented by Liu and

Passamaneck.[16] In the four moment formulation the "thermodynamic" variables are assumed constant and the "dynamic" variables are governed by the four moment equations; continuity, radial momentum, tangential momentum, and shear stress.

The "thermodynamic" variables are determined by the boundary conditions to be

$$\begin{aligned}
\tilde{N}_1 &= -\frac{\epsilon}{2} - \sqrt{\frac{\pi}{2}}\, M\, u^{(+)}(0,0) \\
\tilde{N}_2 &= 0 \\
\tilde{t}_1 &= \epsilon \\
\tilde{t}_2 &= 0
\end{aligned} \tag{4.31}$$

The four equations from Section IV. C become:

Continuity

$$\frac{(1-x^2)}{2x}\frac{\partial u^{(+)}}{\partial x} - \frac{x^2(1-x^2)}{2}\frac{\partial u^{(-)}}{\partial x} + u^{(+)} + \frac{\cot\theta}{2}v^{(+)} + \frac{1}{2}\frac{\partial v^{(+)}}{\partial\theta}$$

$$+\frac{(x^3-3x)}{2}\left[u^{(-)} + \frac{\cot\theta}{2}v^{(-)} + \frac{1}{2}\frac{\partial v^{(-)}}{\partial\theta}\right] = 0$$

(4. 32. a)

Radial Momentum

$$\frac{(1+x^2)}{x}\frac{\partial u^{(-)}}{\partial x} + u^{(-)} + \frac{\cot\theta}{2}v^{(-)} + \frac{1}{2}\frac{\partial v^{(-)}}{\partial\theta} = 0 \tag{4. 32. b}$$

Tangential Momentum

$$\frac{(1-x^2)}{x}\frac{\partial v^{(-)}}{\partial x} - v^{(-)} + \frac{\partial u^{(-)}}{\partial x} = 0 \qquad (4.32.c)$$

Shear Stress

$$\frac{(1-x^2)}{2x}\frac{\partial v^{(+)}}{\partial x} + \frac{3x^2}{4}(1-x^2)\left(x^2-\frac{5}{3}\right)\frac{\partial v^{(-)}}{\partial x} + \frac{x^3}{4}(5-3x^2)\left(v^{(-)} - \frac{\partial u^{(-)}}{\partial \theta}\right)$$

$$+\frac{1}{2}\left(\frac{\partial u^{(+)}}{\partial \theta} - v^{(+)}\right) = -\frac{(1-x^2)^{3/2}\, v^{(-)}}{2\tilde{\lambda}} \qquad (4.32.d)$$

Introduction of the following separation of variables

$$u^{(\pm)}(x,\theta) = U^{(\pm)}(x)\cos\theta$$
$$v^{(\pm)}(x,\theta) = V^{(\pm)}(x)\sin\theta \qquad (4.33)$$

($U^{(\pm)}$ and $V^{(\pm)}$ replace $A_1^{(\pm)}$ and $B_1^{(\pm)}$ respectively in the general separation of variables) leads to the following set of differential equations:

Continuity

$$\frac{(1-x^2)}{2x}\frac{dU^{(+)}}{dx} - \frac{x^2(1-x^2)}{2}\frac{dU^{(-)}}{dx} + U^{(+)} + V^{(+)}$$
$$+\frac{(x^3-3x)}{2}\left(U^{(-)} + V^{(-)}\right) = 0 \qquad (4.34.a)$$

Radial Momentum

$$\frac{(1+x^2)}{x}\frac{dU^{(-)}}{dx} + U^{(-)} + V^{(-)} = 0 \qquad (4.34.b)$$

Tangential Momentum

$$\frac{(1-x^2)}{x}\frac{dV^{(-)}}{dx} - \left(U^{(-)} + V^{(-)}\right) = 0 \qquad (4.34.c)$$

Shear Stress

$$\frac{(1-x^2)}{2x}\frac{dV^{(+)}}{dx} + \frac{3x^2}{4}(1-x^2)\left(x^2-\frac{5}{3}\right)\frac{dV^{(-)}}{dx} + \frac{x^2}{4}(5-3x^2)\left(U^{(-)}+V^{(-)}\right)$$

$$-\frac{1}{2}\left(U^{(+)}+V^{(+)}\right) = -\frac{(1-x^2)^{3/2}\,V^{(-)}}{2\tilde{\lambda}} \tag{4.34.d}$$

The associated boundary conditions are

$$x=1\ (r=\infty):\quad \begin{array}{ll} (i) & U^{(+)}(1)-U^{(-)}(1)=-2 \\ (ii) & V^{(+)}(1)-V^{(-)}(1)=2 \end{array}$$

$$x=0\ (r=r_0):\quad \begin{array}{ll} (v) & U^{(+)}(0)+U^{(-)}(0)=0 \\ (vi) & V^{(+)}(0)+V^{(-)}(0)=0 \end{array} \tag{4.35}$$

The two momentum equations, which involve only $U^{(-)}$ and $V^{(-)}$ can be solved to yield

$$U^{(-)}(x) = \alpha_0 + C_1\sqrt{\frac{1-x^2}{1+x^2}} \tag{4.36.a}$$

$$V^{(-)}(x) = -\alpha_0 + C_1\sqrt{\frac{1+x^2}{1-x^2}} \tag{4.36.b}$$

Since the boundary condition at $x=1$ requires that $V^{(-)}(1)$ be finite, the constant C_1 must be zero. Therefore,

$$U^{(-)}(x) = -V^{(-)}(x) = \alpha_0 \tag{4.37}$$

The remaining equations can now be integrated to give

$$U^{(1)}(x) = \frac{\alpha_0}{3\tilde{\lambda}} x^2\sqrt{1-x^2} - 2\alpha_1 + \alpha_2\sqrt{1-x^2} \tag{4.38.a}$$

$$V^{(1)}(x) = \frac{\alpha_0}{6\tilde{\lambda}} (x^2-2)\sqrt{1-x^2} - \frac{\alpha_2}{2}\sqrt{1-x^2} + 2\alpha_1 \tag{4.38.b}$$

where

$$\begin{aligned} \alpha_0 &\equiv \frac{1}{1+\frac{1}{3\tilde{\lambda}}} \\ \alpha_1 &\equiv \frac{1}{2}\left(\frac{2+3\tilde{\lambda}}{1+3\tilde{\lambda}}\right) \\ \alpha_2 &\equiv \frac{2}{1+3\tilde{\lambda}} \end{aligned} \tag{4.39}$$

The drag coefficient C_D is computed in Appendix G

$$C_D = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\left(2+\frac{\pi}{4}\right)}{M}\,\alpha_0 = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\left(2+\frac{\pi}{4}\right)}{M}\,\frac{3\tilde{\lambda}}{1+3\tilde{\lambda}} \tag{4.40}$$

It is found that if the shear stress equation is replaced by either the tangential stress or radial stress equations the constant; $\frac{1}{3}$, appearing in α_0 is increased slightly to $\frac{5}{12}$. This can be interpreted as an indication of convergence for the moment method.

IV. E. 3 Coupled Heat Transfer and Drag (Six Moment Solution)

The simplest formulation which accounts for both heat transfer and drag can be obtained by recognizing that for the two separate problems, a total of six different moment equations was used; continuity (both), radial momentum (both), tangential momentum (drag), energy (heat transfer), shear stress (drag), and radial heat flux (heat transfer). It is obvious that the two separate solutions do not satisfy all of the six moment equations and the boundary conditions. Moreover, the six moment equations contain eight dependent variables; the "thermodynamic" set (n_1, n_2, T_1, T_2) and the "dynamic" set (u_1, u_2, v_1, v_2). In order to complete the formulation two additional moment equations must be added or two of the dependent variables must be discarded. The former choice, as was seen in Section IV. D, leads to 26 ordinary differential equations which can only be solved numerically. The latter choice leads to a set of equations which can be integrated directly.

Let us assume that the separation of variables for the dynamic variables is the same as for the four moment equations

$$\begin{aligned} u^{(\pm)}(x,\theta) &= U^{(\pm)}(x)\cos\theta \\ v^{(\pm)}(x,\theta) &= V^{(\pm)}(x)\sin\theta \end{aligned} \tag{4.41}$$

Following the results for the drag problem, we shall require at the onset that

$$U^{(-)}(x) = -V^{(-)}(x) = \bar{\alpha}_0 = \text{constant} \tag{4.42}$$

Thus, the remaining six dependent variables are:

$$u^{(+)}(x,\theta),\ v^{(+)}(x,\theta),\ N^{(+)}(x,\theta),\ N^{(-)}(x,\theta),\ t^{(+)}(x,\theta),\ t^{(-)}(x,\theta)$$

The separation of variables for the "thermodynamic" quantities is taken to be the first two terms of the general separation of variables from Appendix F

$$\begin{aligned} N^{(\pm)}(x,\theta) &= C_0^{(\pm)}(x) + C_1^{(\pm)}(x)\cos\theta \\ t^{(\pm)}(x,\theta) &= D_0^{(\pm)}(x) + D_1^{(\pm)}(x)\cos\theta \end{aligned} \tag{4.43}$$

Substituting the assumed separation of variables into the six equations gives ten ordinary differential equations for the ten dependent variables, $U^{(+)}$, $V^{(+)}$, $C_0^{(+)}$, $C_0^{(-)}$, $C_1^{(+)}$, $C_1^{(-)}$, $D_0^{(+)}$, $D_0^{(-)}$, $D_1^{(+)}$, $D_1^{(-)}$

<u>Continuity</u>

$$\frac{d}{dx}\left(C_0^{(-)}+\frac{1}{2}D_0^{(-)}\right)=0 \tag{4.44.a}$$

$$\frac{(1-x^2)^2}{x}\frac{d}{dx}\left(C_1^{(-)}+\frac{1}{2}D_1^{(-)}\right)+\sqrt{2\pi\gamma}\,M\left\{\frac{(1-x^2)}{2x}\frac{dU^{(+)}}{dx}+U^{(+)}+V^{(+)}\right\}=0 \tag{4.45.a}$$

<u>Radial Momentum</u>

$$\frac{d}{dx}\left(C_0^{(+)}+D_0^{(+)}\right)-x^3\frac{d}{dx}\left(C_0^{(-)}+D_0^{(-)}\right)=0 \tag{4.44.b}$$

$$\frac{d}{dx}\left(C_1^{(+)}+D_1^{(+)}\right)-x^3\frac{d}{dx}\left(C_1^{(-)}+D_1^{(-)}\right)=0 \tag{4.45.b}$$

<u>Tangential Momentum</u>

$$C_1^{(+)}+D_1^{(+)}+\frac{(x^3-3x)}{2}\left(C_1^{(-)}+D_1^{(-)}\right)=0 \tag{4.45.c}$$

<u>Energy</u>

$$\frac{d}{dx}\left(C_0^{(-)}-\frac{7}{2}D_0^{(-)}\right)=0 \tag{4.44.c}$$

$$\frac{d}{dx}\left(C_1^{(-)}-\frac{7}{2}D_1^{(-)}\right)=0 \tag{4.45.d}$$

Shear Stress

$$-(1-x^2)^2\left(C_1^{(-)}+\frac{3}{2}D_1^{(-)}\right)+\sqrt{2\pi\gamma}\,M\left\{\frac{(1-x^2)}{2x}\frac{dV^{(+)}}{dx}\right.$$
$$\left.-\frac{1}{2}\left(U^{(+)}+V^{(+)}\right)\right\}=\sqrt{\frac{\pi\gamma}{2}}\,\frac{M\bar{\alpha}_0}{\bar{\lambda}}\,(1-x^2)^{3/2}$$

(4. 45. e)

where $$\bar{\alpha}_0 \equiv U^{(-)}(x)=-V^{(-)}(x)$$

Radial Heat Flux

$$x^2(1-x^2)\left\{\frac{1}{5}\frac{d}{dx}\left(C_0^{(-)}+D_0^{(-)}\right)-\frac{dD_0^{(-)}}{dx}\right\}-\frac{(1-x^2)}{x}\left\{\frac{1}{5}\frac{d}{dx}\left(C_0^{(+)}+D_0^{(+)}\right)\right.$$
$$\left.-\frac{dD_0^{(+)}}{dx}\right\}=\frac{4\sqrt{1-x^2}}{15\bar{\lambda}}\left[\frac{1}{2}\left(C_0^{(-)}-\frac{7}{2}D_0^{(-)}\right)+\sqrt{\frac{\pi\gamma}{2}}\,M\,U^{(+)}(0)\right]$$

(4. 44. d)

$$x^2(1-x^2)\left\{\frac{1}{5}\frac{d}{dx}\left(C_1^{(-)}+D_1^{(-)}\right)-\frac{dD_1^{(-)}}{dx}\right\}-\frac{(1-x^2)}{x}\left\{\frac{1}{5}\frac{d}{dx}\left(C_1^{(+)}+D_1^{(+)}\right)\right.$$
$$\left.-\frac{dD_1^{(+)}}{dx}\right\}=\frac{4\sqrt{1-x^2}}{15\bar{\lambda}}\left[\frac{1}{2}\left(C_1^{(-)}-\frac{7}{2}D_1^{(-)}\right)-\sqrt{\frac{\pi\gamma}{2}}\,M\,U^{(+)}(0)\right]$$

(4. 45. f)

It is easily seen that within these ten equations only four equations (4. 44. a, b, c, d) contain the four variables $\left(C_0^{(+)}, C_0^{(-)}, D_0^{(+)}, D_0^{(-)}\right)$ and the remaining six equations (4. 45. a, b, c, d, e, f) contain the other six variables $\left(U^{(+)}, V^{(+)}, C_1^{(+)}, C_1^{(-)}, D_1^{(+)}, D_1^{(-)}\right)$. The first set of variables will be labeled the "thermodynamic" set and the second set the "dynamic" variables for the six moment solution. The equations for each set can be solved independently of the other, the only coupling is through the boundary conditions.

The boundary conditions become

$X=1$ ($r=\infty$):

$$\text{(i)}\quad U^{(+)}(1) = \bar{\alpha}_0 - 2$$

$$\text{(ii)}\quad V^{(+)}(1) = 2 - \bar{\alpha}_0$$

$$\text{(iii)}\quad D_0^{(+)}(1) - D_0^{(-)}(1) = 0$$

$$D_1^{(+)}(1) - D_1^{(-)}(1) = 0 \qquad (4.46.\text{a})$$

$$\text{(iv)}\quad C_0^{(+)}(1) - C_0^{(-)}(1) = 0$$

$$C_1^{(+)}(1) - C_1^{(-)}(1) = 0$$

$X=0$ ($r=r_0$)

$$\text{(v)}\quad U^{(+)}(0) = -\bar{\alpha}_0$$

$$\text{(vi)}\quad V^{(+)}(0) = \bar{\alpha}_0$$

$$\text{(vii)}\quad D_0^{(+)}(0) + D_0^{(-)}(0) = 2\epsilon \qquad (4.46.\text{b})$$

$$D_1^{(+)}(0) + D_1^{(-)}(0) = 0$$

$$\text{(viii)}\quad C_0^{(-)}(0) + \frac{1}{2} D_0^{(-)}(0) = \sqrt{\frac{\pi\gamma}{2}}\, M \bar{\alpha}_0$$

$$C_1^{(-)}(0) + \frac{1}{2} D_1^{(-)}(0) = F_1(\lambda)$$

$$F_1(\infty) = 0$$

Integration of Equations (4. 44. a) through (4. 44. d) yields

$$D_0^{(+)}(x) = \beta_0\left[1 + \frac{8}{15\tilde{\lambda}}\sqrt{1-x^2}\,\right]$$

$$D_0^{(-)}(x) = \beta_0$$

$$C_0^{(+)}(x) = \sqrt{\frac{\pi\gamma}{2}}\, M \bar{\alpha}_0 - \beta_0\left[\frac{1}{2} + \frac{8}{15\tilde{\lambda}}\sqrt{1-x^2}\right]$$

$$C_0^{(-)}(x) = \sqrt{\frac{\pi\gamma}{2}}\, M \bar{\alpha}_0 - \frac{\beta_0}{2} \qquad (4.47)$$

where

$$\beta_0 \equiv \frac{\epsilon}{1 + \frac{4}{15\tilde{\lambda}}}$$

This reduces to Lees' heat conduction solution if $\bar{\alpha}_0$ is zero. The constant $\bar{\alpha}_0$ is determined from the solution of the "dynamic" variables. The solution for the "dynamic" variables is found to be

$$U^{(-)}(x) = -V^{(-)}(x) = \bar{\alpha}_0$$

$$U^{(+)}(x) = -2\bar{\alpha}_1 + \frac{\bar{\alpha}_0}{3\tilde{\lambda}}\left[x^2\sqrt{1-x^2} - \frac{(2 - x^2 + x^4/2)}{30}\right] + \bar{\alpha}_2\sqrt{1-x^2}$$

$$V^{(+)}(x) = 2\bar{\alpha}_1 + \frac{\bar{\alpha}_0}{3\tilde{\lambda}}\left[\frac{(1+x^2)}{30} - \frac{x^4}{60} - \left(1 - \frac{x^2}{2}\right)\sqrt{1-x^2}\right] - \frac{\bar{\alpha}_2}{2}\sqrt{1-x^2}$$

$$C_1^{(+)}(x) = -D_1^{(+)}(x) = \frac{M}{15\tilde{\lambda}}\sqrt{\frac{\pi\gamma}{2}}\,\bar{\alpha}_0\left(2\sqrt{1-x^2} - 1\right) \qquad (4.48)$$

$$C_1^{(-)}(x) = -D_1^{(-)}(x) = \frac{-M}{15\tilde{\lambda}}\sqrt{\frac{\pi\gamma}{2}}\,\bar{\alpha}_0 = 2F_1(\lambda)$$

where

$$\bar{\alpha}_0 = \frac{1}{1 + \frac{41}{120\tilde{\lambda}}}$$

$$\bar{\alpha}_1 = \frac{40 + 60\tilde{\lambda}}{41 + 120\tilde{\lambda}} \tag{4.49}$$

$$\bar{\alpha}_2 = \frac{248}{3(41 + 120\tilde{\lambda})}$$

It should be noted that in this solution the quantities $C_1^{(\pm)}$ and $D_1^{(\pm)}$ associated with $N^{(\pm)}$ and $t^{(\pm)}$ are of order M, but the velocity components are of order unity. Secondly, this solution is very similar to the four moment solution for the "dynamic" variables. In the four moment formulation the solution was found to be

$$U^{(-)}(x) = -V^{(+)}(x) = \alpha_0 = \frac{1}{1 + \frac{1}{3\tilde{\lambda}}}$$

whereas in the six moment the corresponding constant was found to be

$$\bar{\alpha}_0 = \frac{1}{1 + \frac{41}{120\tilde{\lambda}}}$$

which exhibits a difference of $\frac{1}{120}$ in the coefficient of $\tilde{\lambda}^{-1}$. Similar results can be shown for $\bar{\alpha}_1$ and $\bar{\alpha}_2$. This finding shows heuristically the possibility of convergence. However, no systematical study for proof of convergence in the moment method has been attempted.

Computation of the Drag

The radial stress and the shear stress at the sphere surface are given by

$$(\tilde{P}_{rr})_{\chi=0} = 1 + \frac{\epsilon}{4\left(1+\frac{4}{15\tilde{\lambda}}\right)} + \sqrt{\frac{2\gamma}{\pi}}\, M\bar{\alpha}_0 \left(1+\frac{\pi}{4}\right)\cos\theta \tag{4.50.a}$$

and

$$(P_{r\theta})_{\chi=0} = -\sqrt{\frac{\gamma}{2\pi}}\, M\bar{\alpha}_0 \sin\theta \tag{4.50.b}$$

From Appendix G the drag coefficient is found to be

$$C_D = (C_D)_{P_{rr}} + (C_D)_{P_{r\theta}} = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\bar{\alpha}_0}{M}\left(2+\frac{\pi}{4}\right) \tag{4.51}$$

where $$\bar{\alpha}_0 = \frac{120\tilde{\lambda}}{41+120\tilde{\lambda}}$$

The first term is the contribution of radial stress given by

$$(C_D)_{P_{rr}} = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\bar{\alpha}_0}{M}\left(1+\frac{\pi}{4}\right)$$

and the second term the contribution due to the shear stress

$$(C_D)_{P_{r\theta}} = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\bar{\alpha}_0}{M}$$

Again it is seen that C_D for the six moment solution differs from that for the four moment by only the difference between α_0 and $\bar{\alpha}_0$.

Computation of the Heat Transfer

The radial heat flux at the surface is given by

$$\tilde{Q}_r(g\theta) = \frac{\sqrt{\frac{2}{\pi\gamma}}}{M}\,\frac{\epsilon}{\left(1+\frac{4}{15\tilde{\lambda}}\right)} + \bar{\alpha}_0\cos\theta\left[\frac{\epsilon}{1+\frac{4}{15\tilde{\lambda}}} - \frac{1}{4} + \frac{1}{15\tilde{\lambda}}\right]$$

$$+ \frac{\bar{\alpha}_0^2\cos^2\theta}{15\tilde{\lambda}}\sqrt{\frac{\pi\gamma}{2}}\, M - \frac{1}{2}\sqrt{\frac{\gamma}{2\pi}}\, M\left(1+\bar{\alpha}_0^2\cos^2\theta\right) \tag{4.52}$$

From Appendix H the average heat transfer is found to be

$$\tilde{Q}_{AVE} = \frac{\sqrt{\frac{2}{\pi\gamma}}\frac{\epsilon}{M}}{1+\frac{4}{15\tilde{\lambda}}} - \frac{M}{2}\sqrt{\frac{\gamma}{2\pi}}\left[1+\frac{\bar{\lambda}_o^2}{2}\left(1-\frac{2\pi}{15\tilde{\lambda}}\right)\right] \tag{4.53}$$

The adiabatic temperature ratio, ϵ_A , is computed by requiring that $\tilde{Q}_{AVE} = 0$, thus

$$\epsilon_A = \frac{\gamma M^2}{4}\left(1+\frac{4}{15\tilde{\lambda}}\right)\left[1+\frac{\bar{\lambda}_o^2}{2}\left(1-\frac{2\pi}{15\tilde{\lambda}}\right)\right] \tag{4.54}$$

The average heat transfer in terms of ϵ_A is given by

$$\tilde{Q}_{AVE} = \frac{\sqrt{\frac{2}{\pi\gamma}}\left(\epsilon-\epsilon_A\right)}{M\left(1+\frac{4}{15\tilde{\lambda}}\right)} \tag{4.55}$$

The Stanton number normalized by the FMF limit as given in Appendix G is

$$\frac{S_T}{(S_T)^{FMF}} = \frac{1}{1+\frac{4}{15\tilde{\lambda}}}$$

which is equal to $\frac{\tilde{Q}_{AVE}}{(\tilde{Q}_{AVE})^{FMF}}$ in the pure conduction problem. However, the ratio of the average heat transfer is not equal to the ratio of the Stanton number since the quantity $(\epsilon-\epsilon_A)$ is not equal to $(\epsilon-\epsilon_A)^{FMF}$.

IV. E. 4 Comparison with Experimental Data and Other Theories

Drag

The drag coefficient was found to be

$$C_D = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\tilde{\alpha}_0}{M}\left(2+\frac{\pi}{4}\right) \tag{4.56}$$

where

$$\tilde{\alpha}_0 \equiv \begin{cases} \dfrac{1}{1+\dfrac{41}{120\tilde{\lambda}}} & \text{for the 6-moment solution} \\[2ex] \dfrac{1}{1+\dfrac{1}{3\tilde{\lambda}}} & \text{for the 4-moment solution} \end{cases}$$

In the FMF limit $\tilde{\alpha}_0^{FMF} = 1$ and the correct FMF solution is obtained

$$C_D^{FMF} = \frac{8}{3}\sqrt{\frac{2}{\pi\gamma}}\,\frac{\left(2+\frac{\pi}{4}\right)}{M} \tag{4.57}$$

For the continuum limit the drag coefficient becomes

$$C_D \cong 8\sqrt{\frac{2}{\pi\gamma}}\,\frac{\left(2+\frac{\pi}{4}\right)}{M}\,\tilde{\lambda} \tag{4.58}$$

This result can be compared to the Stoke's drag formula by recalling that the viscosity was assumed to be related to the MFP through

$$\mu_\infty = \frac{1}{2} m n_\infty \lambda_\infty \bar{c}$$

which gives

$$\tilde{\lambda} \equiv \frac{\lambda_\infty}{r_0} = \sqrt{\frac{\pi\gamma}{2}}\,\frac{M}{(Re)_{r_0}} \tag{4.59}$$

where the Reynolds number is defined by

$$(Re)_{r_0} \equiv \frac{m n_\infty q_\infty r_0}{\mu_\infty}$$

Equation (4.58) may then be rewritten as

$$C_D \cong \frac{22}{(Re)_{r_0}} \tag{4.60}$$

Equation (4. 60) is seen to be qualitatively similar to Stokes solution given by

$$C_D = \frac{12}{(Re)_{r_0}} \tag{4.61}$$

but differs significiently in the constant. Thus, the present estimation of sphere drag incurs a quantitative difference in the Stokes flow regime. This is caused, not by the formulation of the problem, but by the method of separation of variables. One can readily see the difference by comparing Equation (4. 41) with the well known Stokes solution.

The present solution when compared with Millikan's oil drop experiments[17] in Figure 6 shows acceptable correlation for the complete experimental range of MFP $\left(1 \leq \tilde{\lambda} \leq \infty\right)$. This solution is also compared with the near FMF solutions by Willis[14] and Liu, Pang, and Jew[13] for large values of the MFP.

For small values of the MFP Basset's[11] slip flow correction to Stoke's formula is given by

$$C_D = \frac{12}{(Re)_{r_0}} \left(\frac{1+\tilde{\lambda}}{1+2\tilde{\lambda}}\right) \tag{4.62}$$

Goldberg's[12] solution using Grad's Thirteen Moment Method predicts

$$C_D = \frac{12}{(Re)_{r_0}} \left[\frac{\left(1+\frac{15\tilde{\lambda}}{2}\right)\left(1+2\tilde{\lambda}\right) + \frac{6\tilde{\lambda}^2}{\pi}}{\left(1+\frac{15}{2}\tilde{\lambda}\right)\left(1+3\tilde{\lambda}\right) + \frac{9}{15\tilde{\lambda}}\left(4+9\tilde{\lambda}\right)\tilde{\lambda}^2}\right]$$

(4. 63)

Both of these solutions are compared with the present theory in Figure 6. It is easily seen that both Basset's and Goldberg's solution are not correct in the FMF limit.since they predict an infinite drag coefficient as $\tilde{\lambda} \rightarrow \infty$.

Heat Transfer

The average heat transfer was found to be correct in the FMF limit. Comparison of the Stanton number with the experimental data of Kavanau and Drake[18] is given in Figure 7. The comparison is again seen to be acceptable.

Flow Field

The only analytic solution available which can be compared with the present solution is Basset's[11] slip flow correction to Stoke's solution. The velocity components for the six moment solution are found to be Appendix C and Section IV. E. 3)

$$\tilde{q}_r = \bar{\alpha}_0 \cos\theta \left\{ \frac{(1-x^2)}{2} - \frac{x^3}{2} + \frac{31}{45\tilde{\lambda}} \frac{\sqrt{1-x^2}}{2} - \frac{2+3\tilde{\lambda}}{6\tilde{\lambda}} + \frac{1}{6\tilde{\lambda}} \left[x^2\sqrt{1-x^2} - \frac{(2-x^2+x^4/2)}{30} \right] \right\} \tag{4.64}$$

$$\tilde{q}_\theta = \bar{\alpha}_0 \sin\theta \left\{ \frac{x(3-x^2)}{4} + \frac{2+3\tilde{\lambda}}{6\tilde{\lambda}} - \frac{31}{45\tilde{\lambda}} \frac{\sqrt{1-x^2}}{4} + \frac{1}{6\tilde{\lambda}} \left[\frac{(1+x^2)}{40} - \frac{x^4}{60} - \left(1-\frac{x^2}{2}\right)\sqrt{1-x^2} \right] \right\} \tag{4.65}$$

where

$$\bar{\alpha}_0 = \frac{120\tilde{\lambda}}{41+120\tilde{\lambda}}$$

and the shear stress and normal stress are given by

$$\tilde{P}_{r\theta} = -\sqrt{\frac{\gamma}{2\pi}}\, M\,(1-\chi^2)^2\,\bar{\alpha}_0 \sin\theta \tag{4.66}$$

$$\tilde{P}_{rr} = 1 + \sqrt{\frac{2\gamma}{\pi}}\, M(1+\chi^2)\cos\theta\,\left(1+\frac{\pi}{4}+\chi^2\right)\bar{\alpha}_0 + \sqrt{\frac{\pi\gamma}{8}}\, M\chi^2(1+\chi)\,\bar{\alpha}_0 + \frac{(1-2\chi^2)\epsilon}{4\left(1+\frac{4}{15\tilde{\lambda}}\right)} \tag{4.67}$$

The corresponding quantities from Basset's solution which are valid for small values of $\tilde{\lambda}$ are denoted by the superscript (B)

$$\tilde{q}_{br}^{(B)} = -\cos\theta\left\{1-\frac{3}{2}\sqrt{1-\chi^2}\left(\frac{1+\tilde{\lambda}}{1+2\tilde{\lambda}}\right)+\frac{(1-\chi^2)^{3/2}}{2}\,\frac{(1-\tilde{\lambda})}{(1+2\tilde{\lambda})}\right\} \tag{4.68}$$

$$\tilde{q}_{b\theta}^{(B)} = \sin\theta\left\{1-\frac{3}{4}\sqrt{1-\chi^2}\left(\frac{1+\tilde{\lambda}}{1+2\tilde{\lambda}}\right)-\frac{(1-\chi^2)^{3/2}}{4}\,\frac{(1-\tilde{\lambda})}{(1+2\tilde{\lambda})}\right\} \tag{4.69}$$

$$\tilde{P}_{r\theta}^{(B)} = -\sqrt{\frac{\gamma}{2\pi}}\, M\,(1-\chi^2)^2 \sin\theta\;\frac{3\tilde{\lambda}(1-\tilde{\lambda})}{1+2\tilde{\lambda}} \tag{4.70}$$

$$\tilde{P}_{rr}^{(B)} = 1+\sqrt{\frac{2\gamma}{\pi}}\, M(1-\chi^2)\cos\theta\left\{\frac{3\tilde{\lambda}(1+5\tilde{\lambda})}{2(1+2\tilde{\lambda})}+\frac{3\chi^2\tilde{\lambda}(1-\tilde{\lambda})}{(1+2\tilde{\lambda})}\right\} \tag{4.71}$$

Each of the corresponding pairs are compared separately in Figures 8 through 11.

Radial Velocity (Figure 8)

Both solutions $\tilde{q}_r$ and $\tilde{q}_r^{(B)}$ satisfy the boundary conditions $\tilde{q}_r(1,\theta) = -\cos\theta$ and $\tilde{q}_r(0,\theta) = 0$. It is seen in Figure 8 that both solutions are almost identical in the continuum regime, but differ considerably in the FMF limit. Since the slip flow assumption is obviously incorrect in the FMF limit, one expects that the present solution for the radial velocity gives a better description for a large range of MFP.

Tangential Velocity (Figure 9)

The boundary condition at $x=1$, $\tilde{q}_\theta(1,\theta) = \sin\theta$, is satisfied by both solutions, but their values at the body, $x=0$, are not identical

$$\tilde{q}_\theta(0,\theta) = \frac{60\tilde{\lambda}}{41+120\tilde{\lambda}} \sin\theta \tag{4.72}$$

$$\tilde{q}_\theta^{(B)}(0,\theta) = \frac{3\tilde{\lambda}}{2+4\tilde{\lambda}} \sin\theta \tag{4.73}$$

Equation (4.72) is correct in the FMF limit but also compares favorably with Basset's solution for small MFP. For small $\tilde{\lambda}$

$$\tilde{q}_\theta(0,\theta) \cong \frac{60\tilde{\lambda}}{41} \sin\theta \cong \frac{3}{2}\left(1-\frac{1}{40}\right)\tilde{\lambda}\sin\theta \tag{4.74}$$

and

$$\tilde{q}_\theta^{(B)}(0,\theta) \cong \frac{3}{2}\tilde{\lambda}\sin\theta \tag{4.75}$$

Thus in the continuum limit the present solution is in error by 2.5% whereas in the FMF limit Basset's result is in error by 25%. The results presented in Figure 9 show that the present analysis correctly predicts the tangential velocity over a wide range of Knudsen numbers.

Shear Stress (Figure 10)

The ratio of shear stresses for the two solutions is given by

$$\frac{\tilde{P}_{r\theta}}{\tilde{P}_{r\theta}^{(B)}} = \left(\frac{120\tilde{\lambda}}{41+120\tilde{\lambda}}\right)\left(\frac{1+2\tilde{\lambda}}{3\tilde{\lambda}-3\tilde{\lambda}^2}\right) \tag{4.76}$$

For small $\tilde{\lambda}$ Equation (4.76) becomes

$$\frac{\tilde{P}_{r\theta}}{\tilde{P}_{r\theta}^{(B)}} = \frac{\frac{120}{41}\tilde{\lambda}}{1+\frac{120}{41}\tilde{\lambda}}\left[\frac{1+3\tilde{\lambda}+0(\tilde{\lambda}^2)}{3\tilde{\lambda}}\right]$$

$$\cong 1+0(\tilde{\lambda}^2) \tag{4.77}$$

where the following expansion has been used

$$\frac{1}{1+\tilde{\lambda}} \cong 1-\tilde{\lambda}+0(\tilde{\lambda}^2)$$

This ratio is seen to be unity to order $(\tilde{\lambda}^2)$ for small $\tilde{\lambda}$. For $\tilde{\lambda}$ equal to unity, Basset's solution is zero and changes sign for $\tilde{\lambda}$ greater than one. Basset's result is therefore incorrect for $\tilde{\lambda}$ greater than unity. Again the comparison between the two solutions illustrates that the present model is applicable over a wide wide range of Knudsen numbers.

Radial Stress (Figure 11)

Since the shear stress from the present study compares favorably with Basset's solution, the discrepancy in the drag for small MFP (Figure 6) is mainly due to the difference in the radial stress for the two solutions. This difference is easily observed if the radial stress is compared at the body.

Taking $\epsilon = 0$ one finds

$$\frac{(\tilde{P}_{rr})_{\lambda=0} - 1}{(\tilde{P}_{rr}^{(B)})_{\lambda=0} - 1} = \frac{80}{41}\left(1+\frac{\pi}{4}\right) \frac{1+2\tilde{\lambda}}{(1+5\tilde{\lambda})\left(1+\frac{120}{41}\tilde{\lambda}\right)} \tag{4.78}$$

Since this ratio is not close to unity for $\tilde{\lambda}$ very small, the radial stress from the present study errs in the continuum limit. This conclusion regarding the radial stress $\bar{P}_{rr}$ is not surprising since the six moment formulation herein excludes the moment equation corresponding to the radial stress. The discrepancies in the radial stress are shown in Figure 11.

CHAPTER V – HIGH SPEED FLOW

V.A Introduction

Most analytical efforts on high speed flow over a sphere have been confined to near FMF theories, valid approximately for $K_n > 10$ (e.g., Baker and Charwat,[19] Rose,[10] Willis,[20] or to small departures from the continuum limit (e.g., Van Dyke).[21] In the transition regime, theoretical works on nonlinear flows were initiated by Mott-Smith[7] on the shock structure problem. Subsequently, boundary value problems such as Couette flow (Liu and Lees,[22] Lubonski)[23] and Raleigh's problem (Chu)[24] were treated. Mathematical models were also suggested by Rott and Whittenburg[25] and by Hamel.[26]

The general formulation presented in Chapter III separates the distribution function f into two parts: $f = f_1$ for all molecules whose velocity vector lies in a cone subtended by the body, and $f = f_2$ for molecules whose velocity vector lies outside of the conical region. At high supersonic or hypersonic speeds the following simplifications are introduced

(1) The oncoming stream can be compared to a high speed molecular beam, thus the moment contribution due to f_2 may be evaluated over the entire velocity space including the vacant conical region.

(2) For the distribution function f_1, which may be intutively related to the reemitted particles, the approximation of small mean speed can be made and the distribution function f_1 can be linearized as in the low speed case. This is analogous to the "cold wall" approximation in high-speed gasdynamics.

These simplifications are very similar to the two fluid model of Rott and Whittenburg.[25] In fact, all the multi-fluid models such as that of Baker and Charwat[19] (sphere), Lubonski[23] (Couette flow), and Hamel[26] (piston problem), bear resemblences. Such methods might be judged by how well they provide the capability to solve actual boundary value problems without a prohibitive amount of labor.

In spherical coordinates f_1 and f_2 are represented by

$$f_1 = f_1^{(M)}(\vec{x},\vec{\xi}) = \frac{n_1(r,\theta)}{(2\pi R T_b)^{3/2}} \exp\left\{-\frac{(\xi_r - W_1\cos\theta)^2 + (\xi_\theta + W_1\sin\theta)^2 + \xi_\phi^2}{2RT_b}\right\} \tag{5.1}$$

where the "velocity components" of the body distribution function are assumed to be

$$u_1 \equiv W_1(x)\cos\theta$$
$$v_1 \equiv -W_1(x)\sin\theta$$

and

$$f_2 = f_2^{(M)}(\vec{x},\vec{\xi}) + G(\vec{x},\vec{\xi}) \tag{5.2}$$

where

$$f_2^{(M)}(\vec{x},\vec{\xi}) = \frac{n_2(r,\theta)}{(2\pi R T_\infty)^{3/2}} \exp\left\{-\frac{(\xi_r - u_2(r,\theta))^2 + (\xi_\theta - v_2(r,\theta))^2 + \xi_\phi^2}{2RT_\infty}\right\} \tag{5.3}$$

The function $G(\vec{x},\vec{\xi})$ does not appear in f_1 because the contribution to the moment from f_2 will be found by integrating over the complete velocity space. (See previous section.) The temperture parameters $T_1 = T_b$ and $T_2 = T_\infty$ are assumed constant and equal respectively to the wall temperature and the free stream temperature. Three parametric functions; n_2, u_2, v_2 appear in f_1 and two; n_1, W_1 in. f_1.

To compute the moments f_1 is linearized at this stage and its contribution to the moment is the same as in the low speed case.

The average value of $\Phi(\xi_i)$ becomes

$$\overline{n\Phi} = \int_0^{\frac{\pi}{2}-\alpha}\int_0^{2\pi}\int_0^{\infty} \Phi f_1^{(M)} \xi^2 \sin\omega \, d\xi \, d\eta \, d\omega$$

$$+ \iiint_{-\infty}^{\infty} \Phi f_2^{(M)} \, d\xi_r \, d\xi_\theta \, d\xi_\phi \tag{5.4}$$

$$+ \sum_{n=1}^{\infty} (-1)^n g^{(n)}(r,\theta) \left[\frac{d^n \Phi(\xi_r,0,0)}{d\xi_r^n} \right]_{\xi_r=0}$$

All average quantities can be evaluated in terms of the five parametric functions $(n_1, W_1, n_2, u_2, V_2)$.

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V.B Four Moment Formulation

The computation of sphere drag without heat transfer for the low speed case was formulated in terms of four moments. In the same way the high speed flow problem can be formulated in terms of the following four moment equations; continuity, radial momentum, tangential momentum, and radial stress. The radial stress is taken instead of the shear stress or tangential stress because the normal stress is considered to be more significant in the hypersonic approximation. The four moment equations are given in Appendix I in terms of the five parametric functions; n_1, w_1, n_2, u_2, v_2 .

The associated boundary conditions are:

at $x=1\ (r=\infty)$:

$$\begin{aligned} \tilde{n}_2(1) &= 1 \\ \tilde{u}_2(1,\theta) &= -\cos\theta \\ \tilde{v}_2(1,\theta) &= \sin\theta \end{aligned} \qquad (5.5)$$

at $x=0\ (r=r_0)$:

$$\tilde{w}_1(0) = 0$$

$$(\widetilde{nq_r})_{x=0} = 0 \qquad (5.6)$$

where the density has been normalized by n_∞ and the velocities by q_∞. Again if four moments are utilized, one of the five parametric functions must be prescribed at the outset. The selection of this one parametric function depends upon the desired information. The two alternatives will be discussed from an intuitive viewpoint.

The free stream velocity functions $\tilde{u}_2$ and $\tilde{v}_2$ cannot be fixed since they must always satisfy the boundary conditions at $x=1$ but must tend to zero at the body as the MFP decreases. The number density of the emitted particles, $\tilde{n}_1$, cannot be constant if the effect of collisions are properly taken into account. Therefore, the consideration is focused on $\tilde{n}_2$ and $\tilde{w}_1$.

If $\tilde{n}_2(x)$ is assumed constant (equal to unity from the boundary condition) the number density of the free stream particles is not affected by the collisions. However, since $\tilde{w}_1(x)$ is not fixed, momentum exchange between the high speed free stream and the low speed body stream can still occur. This assumption is expected to lead to a correct estimation of the momentum flux (drag) but it may yield incorrect predictions of the density field. The description of the flow field by this model is analogous to the first collision models suggested by Baker and Charwat,[19] Kinslow and Potter,[27] and Wainwright.[28]

On the other hand, if one assumes that $\tilde{w}_1(x)$ be constant (equal to zero from the boundary conditions) and allows $\tilde{n}_2(x)$ to vary, the density near the body would increase as the MFP decreases so that it forms a shockwave in the continuum limit. But if $\tilde{w}_1(x)$ is taken to be zero, the momentum exchange between the high speed stream and the low speed stream is incompletely described, hence, drag estimates may be in error.

V.C Determination of $G(\vec{r},\vec{\xi})$

The moments which appear in the four moment equations given in Appendix B are computed from Equations (5.1) and (5.2) and listed in Appendix I. The function $G(r,\theta;\vec{\xi})$ contributes to only two of the moments, $\widetilde{nq}_r$ and $\widetilde{\Sigma}_{rr}$, which contain $g^{(1)}$ and $g^{(2)}$ respectively. Just as in the low speed case $g^{(1)}$ is determined by satisfying the boundary condition of vanishing radial velocity at the sphere surface and $g^{(2)}$ is found by requiring that the normal pressure be correct in the FMF limit.

The radial velocity is given by Appendix I

$$\widetilde{nq}_r = \widetilde{n}_2\widetilde{u}_2 + (1-x^2)\frac{\widetilde{n}_1\sqrt{\widetilde{T}_1}}{\sqrt{2\pi\gamma}M} + \frac{(1-x^3)}{2}\widetilde{n}_1\widetilde{w}_1 - \frac{g^{(1)}(x,\theta)}{n_\infty q_\infty} \tag{5.7}$$

At the sphere surface $r = r_0\ (x=0)$ the condition of zero mass flux requires that

$$(\widetilde{nq}_r)_{x=0} = \left(\frac{\widetilde{n}_1\sqrt{\widetilde{T}_1}}{\sqrt{2\pi\gamma}M}\right)_{x=0} + (\widetilde{n}_2\widetilde{u}_2)_{x=0} - \frac{g^{(1)}(0,\theta)}{n_\infty q_\infty} = 0 \tag{5.8}$$

The boundary condition $\widetilde{w}_1(0)=0$ has been used in Equation (5.8)

Following the same procedure as for the low speed case, we find that $g^{(1)}(x,\theta)$ is given by

$$\frac{g^{(1)}(x,\theta)}{n_\infty q_\infty} = (1-x^2)\left[\widetilde{n}_2(0,\theta)\,\widetilde{u}_2(0,\theta) - \widetilde{n}_2(0,0)\,\widetilde{u}_2(0,0)\right] \tag{5.9}$$

The variation with respect to x is necessary to satisfy the continuity equation in the FMF limit just as in the low speed case. Appendix D. The determination of $g^{(2)}$ is again completely analogous to the low speed problem (see Appendix E). For large Mach numbers

$$p^{FMF} \cong \gamma M^2 \cos^2\theta + \sqrt{\frac{\pi\gamma}{2}}\, M \sqrt{\frac{T_b}{T_\infty}} \cos\theta + 1 \tag{5.10}$$

and $g^{(2)}(x,\theta)$ is found to be

$$\frac{2m g^{(2)}(x,\theta)}{n_\infty k T_\infty} = -(1-x^2)\sqrt{\frac{\pi\gamma}{2}}\, M \sqrt{\frac{\gamma}{T_1}} \left[\hat{n}_2(0,\theta)\tilde{u}_2(0,\theta) - \hat{n}_2(0,0)\hat{u}_2(0,0)\right] \tag{5.11}$$

A comparison between $g^{(1)}$ and $g^{(2)}$ for the high speed case and low speed case shows that $g^{(1)}$ and $g^{(2)}$ for the present case differ by a factor of two from the low speed case.

V.D Separation of Variables — Near Stagnation Point

The boundary conditions require that

at $X=1\ (r=\infty)$:

$$\text{(i)}\quad \tilde{u}_2(1,\theta) = -\cos\theta$$

$$\text{(ii)}\quad \tilde{V}_2(1,\theta) = \sin\theta \qquad (5.12.a)$$

$$\text{(iii)}\quad \tilde{n}_2(1) = 1$$

at $X=0\ (r=r_0)$: (5.12.b)

$$\text{(iv)}\quad \tilde{W}_1(0) = 0$$

$$\text{(v)}\quad \left(\tilde{n}_1\sqrt{T_1}\right)_{X=0} = -\sqrt{2\pi\gamma}\, M\,\tilde{n}_2(0)\,\tilde{u}_2(0,0)$$

Since the FMF solution indicates that all of the parametric functions are independent of X and equal to their boundary values, the following separation of variables can be assumed

$$\tilde{u}_2(X,\theta) = U_2(X)\cos\theta$$

$$\tilde{V}_2(X,\theta) = V_2(X)\sin\theta$$

$$\tilde{n}_2(X) = N_2(X) \qquad (5.13)$$

$$\tilde{n}_1\tilde{W}_1(X) = U_1(X)$$

$$\tilde{n}_1(X) = \frac{M}{\sqrt{T_1}}\, N_1(X)$$

The definition of $\tilde{n}_1$ eliminates the Mach number from the boundary condition at the body. The boundary conditions become

$X=1\ (r\to\infty)$:

$$\text{(i)}\quad U_2(1)=-1$$

$$\text{(ii)}\quad V_2(1)=1$$

$$\text{(iii)}\quad N_2(1)=1 \tag{5.14.a}$$

$X=0\ (r=r_0)$:

$$\text{(iv)}\quad U_1(0)=0$$

$$\text{(v)}\quad N_1(0)=-\sqrt{2\pi\gamma}\,N_2(0)\,U_2(0) \tag{5.14.b}$$

The moment equations corresponding to the assumed separation of variables are given in Appendix K. It is clear that the equations are not separable because of their nonlinearity. However, if the trigonometric functions are expanded in power series, the equations can be separated in terms of the powers of θ . For small values of the angle θ only the lowest powers need be considered. This expansion for small θ is truly a stagnation point expansion. The equations for small θ become

Continuity

$$\frac{(1-X^2)}{X}\frac{d}{dX}N_2U_2+\frac{(1-X^2)(1-X^3)}{2X}\frac{dU_1}{dX}+2N_2(U_2+V_2)$$

$$+\frac{(1-X^2)^2}{X\sqrt{2\pi\gamma}}\frac{dN_1}{dX}+O(\theta^2)=0 \tag{5.15.a}$$

Radial Momentum

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2^2 + 2N_2U_2(U_2+V_2) + \frac{(1-x^2)}{x\gamma M^2}\frac{dN_2}{dx}$$

$$+\frac{(1-x^2)(1-x^3)}{2x}\epsilon_0\frac{dN_1}{dx} + \sqrt{\frac{2\gamma}{\pi}}\epsilon_0\frac{(1-x^2)(1-x^4)}{x}\frac{dU_1}{dx} + O(\theta^2) = 0 \qquad (5.15.b)$$

Tangential Momentum

$$\theta\left\{\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2V_2 + 3N_2V_2\,(U_2+V_2) - \sqrt{\frac{\gamma}{2\pi}}\epsilon_0\frac{(1-x^2)^3}{x}\frac{dU_1}{dx}\right\} \qquad (5.15.c)$$

$$+\,O(\theta^3) = 0$$

Radial Stress

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2^3 + 2N_2U_2^2(U_2+V_2) + \frac{3(1-x^2)}{x\gamma M^2}\frac{d}{dx}N_2U_2$$

$$+\frac{2N_2(U_2+V_2)}{\gamma M^2} + \frac{3}{2}\frac{(1-x^2)(1-x^5)}{x}\gamma\epsilon_0^2\frac{dU_1}{dx}$$

$$+\frac{(1-x^2)(1-x^4)}{x}\sqrt{\frac{2\gamma}{\pi}}\epsilon_0^2\frac{dN_1}{dx} + O(\theta^2) \qquad (5.15.d)$$

$$= \frac{\sqrt{\frac{\pi}{2\gamma}}}{\gamma M^3\tilde{\lambda}}\;\frac{\tilde{n}\,\tilde{p}_{rr}}{\sqrt{1-x^2}}$$

$$= \frac{-\sqrt{\frac{\pi}{2\gamma}}}{3\tilde{\lambda}\sqrt{\tilde{T}_1}}\sqrt{\frac{1-x}{1+x}}\left[N_1N_2U_2^2 + \sqrt{\frac{2\gamma}{\pi}}(1+x)\,U_1N_2U_2^2\right]$$

where $\epsilon_0 \equiv \dfrac{\sqrt{\tilde{T}_1}}{\gamma M}$

The continuity, radial momentum, and radial stress equations have leading terms which are of order unity but the leading term in the tangential momentum equation is proportional to θ . The tangential momentum equation is therefore of lower order than the other three equations. This is further indication that the tangential component of the velocity is of less importance than the normal component in the stagnation point region.

The next term in θ appearing in the four moment equations is θ^2 . Therefore, the omission of the other terms contributes an error of the order θ^2 for the system.

In the stagnation point expansion the role of the function $G(r,\theta;\vec{\xi})$ is greatly reduced. Since the dependence of $g^{(1)}$ and $g^{(2)}$ on the angle θ was found to be

$$g^{(i)} \cong (1-\chi^2)\left[\cos\theta - 1\right] \qquad i=1,2$$

$G(r,\theta;\vec{\xi})$ vanishes at the stagnation point $\theta=0$. The stagnation point expansion gives

$$g^{(i)} \cong (1-\chi^2)\left[\theta^2 + O(\theta^4) \cdots\right]$$

which means that $G(r,\theta;\vec{\xi})$ is always negligible if the solution is determined to an error of order θ^2 .

V.E Solutions

Since there are five parametric functions $(N_1, U_1, N_2, U_2, V_2)$ and only four moment equations one of the five must be prescribed in advance. From Section V. B the alternatives are

$$(1)\quad N_2(x) = 1$$

or

$$(2)\quad U_1(x) = 0$$

One may also elect to adopt both simplifications, $N_2(x) = 1$, and $U_1(x) = 0$, and neglect the relatively unimportant tangential momentum equation (Equation (5. 15. c)). Then the system consists of three moment equations and three undetermined parametric functions. Introducing the hypersonic approximation, $M \gg 1$, one finds that this simple "three moment" formulation gives some fairly interesting analytical results.

All equations to be considered will neglect terms of order $\frac{1}{M^2}$ which leaves only two parameters

$$\epsilon_0 \equiv \frac{\sqrt{\frac{T_b}{T_\infty}}}{\gamma M} \qquad \text{and} \qquad \lambda\sqrt{T_1} \equiv \frac{\lambda_\infty}{r_0}\sqrt{\frac{T_b}{T_\infty}}$$

in the system of equations. The three cases; $N_2(x) = 1$, $U_1(x) = 0$, and the "three moment" solution will be discussed separately in the following sections.

V. E. 1 Four Moment Solution, $N_2(x) = 1$

This assumption implies that the flux of high speed free stream particles to the body is not influenced by collisions with emitted molecules. The molecular density is reduced only by reducing the number of emitted molecules that can reach a given point in space. In the usual first collision methods the molecular density is further reduced by reducing the number of free stream particles incident on the surface which in turn also reduces the number of emitted particles. This reduction occurs if it is assumed that all particles which experience a collision (scattered particles) are not incident on the body. Both the present model and the first collision theories indicate that the effect of collisions near the body is a partial shielding of the body from direct momentum transfer from the free stream. This shielding results in a reduction of the drag from the FMF limit.

Although this assumption will be shown to give a good estimate of the momentum flux (drag) the density variation with the MFP may be in error. The first collision model will always predict a decrease in density as the number of collisions increases. The increase in the number of collisions or collision frequency corresponds to a decrease in the MFP. However, in the continuum limit, $\lambda \to 0$, the density must increase in such a way that a gasdynamic shock is formed in front of a blunt body. A more direct way to illustrate this result is to examine the equation for the density in terms of these assumptions.

From Appendix I the density at the body $X = 0$ is given by

$$(\tilde{n})_{X=0} = N_2(0) + \frac{M}{\sqrt{T_1}} \frac{N_1(0)}{2} \tag{5.16}$$

The boundary condition of vanishing radial velocity requires that

$$N_1(0) = -\sqrt{2\pi\gamma}\, N_2(0)\, U_2(0) \tag{5.17}$$

Since $U_2(0)$ is always negative substitution of Equation (5. 17) into (5. 16) gives

$$(\tilde{n}_2)_{X=0} = N_2(0) \left(1 + \sqrt{\frac{\pi\gamma}{2}} \frac{M}{\sqrt{T_1}} \| U_2(0) \| \right) \tag{5.18}$$

where $\|U_2(0)\| \equiv$ absolute value of $U_2(0)$

and $$0 \leqslant \|U_2(0)\| \leqslant 1$$

If $N_2(x)$ is assumed to be unity Equation (5. 18) predicts that the density at the body decreases as the MFP decreases. In the limit as $\lambda \to 0$ Equation (5. 18) gives

$$(\tilde{n})_{\lambda=0} = \left(\frac{n}{n_\infty}\right)_{x=0} = 1$$

which is clearly incorrect for the high speed limit.

It is found however that the density distribution will be very close to the FMF distribution for values of the parameter, $\tilde{\lambda}\sqrt{\tilde{T}_1} \equiv \frac{\lambda_\infty}{r_0}\sqrt{\frac{T_b}{T_\infty}}$, as small as order (1). This result is consistant with Probstein's[29] conjecture which will be discussed later.

The equations for $N_2(x) = 1$ become

<u>Continuity</u>

$$\frac{(1-x^2)}{x}\frac{dU_2}{dx} + \frac{(1-x^2)(1-x^3)}{2x}\frac{dU_1}{dx} + 2(U_2+V_2) + \frac{(1-x^2)^2}{x\sqrt{2\pi\gamma}}\frac{dN_1}{dx} = 0$$

(5. 19. a)

<u>Radial Momentum</u>

$$2U_2\left[\frac{(1-x^2)}{x}\frac{dU_2}{dx} + U_2+V_2\right] + \frac{(1-x^2)}{x}\epsilon_0\left[\frac{(1-x^3)}{2}\frac{dN_1}{dx} + \sqrt{\frac{2\gamma}{\pi}}(1-x^4)\frac{dU_1}{dx}\right] = 0$$

(5. 19. b)

<u>Tangential Momentum</u>

$$\frac{(1-x^2)}{x}\frac{d}{dx}U_2V_2 + 3V_2(U_2+V_2) - \sqrt{\frac{\gamma}{2\pi}}\,\epsilon_0\frac{(1-x^2)^3}{x}\frac{dU_1}{dx} = 0$$

(5. 19. c)

Radial Stress

$$\frac{(1-x^2)}{x}\frac{dU_2}{dx} + \frac{2}{3}(U_2+V_2) = \frac{-\sqrt{\frac{\pi}{2\gamma}}}{9\tilde{\lambda}\sqrt{T_1}}\sqrt{\frac{1-x}{1+x}}\left[N_1 + \sqrt{\frac{2\gamma}{\pi}}(1+x)U_1\right]$$

(5. 19. d)

Boundary Conditions

$$\begin{aligned} U_2(1) &= -1 \\ V_2(1) &= 1 \\ U_1(0) &= 0 \\ N_1(0) &= -\sqrt{2\pi\gamma}\,U_2(0) \end{aligned} \qquad (5.20)$$

This system of equations and boundary conditions is integrated numerically. The results for the drag computed from Appendix L are presented in Figure 13. The density distribution is compared with the experimental data and the solution for $U_1(x) = 0$ in Figures 15 and 16.

V. E. 2 Four Moment Solution, $U_1(x) = 0$

Independent of the assumptions, $U_1(x)$ is always zero on the sphere surface for diffuse reemission. Therefore the assumption of $U_1(x)$ being identically zero is quite correct in a small region close to the body. It also allows $N_2(x)$ to increase near the body which is essential to predict the density distribution for any finite MFP. The density of the high speed free stream particles is increased by collisions with the emitted stream. This effect can be described qualitatively by imagining that the emitted particle is converted through a collision into a member of the high speed stream.

This model results in a very good estimate of the mass flux (density) but overestimates the drag. The drag or momentum flux is overestimated because the assumption of $U_1(x)$ equal to zero reverses the shielding effect of the previous assumption, $N_2(x) = 1$, and the first collision methods. That is, instead of shielding the body from direct momentum transfer from the free stream, the emitted particles through collisions increase the population of the high speed particles thereby increasing the momentum transfer to the body.

The equations for $U_1(x) = 0$ become

Continuity

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2 + 2N_2(U_2+V_2) + \frac{(1-x^2)^2}{x\sqrt{2\pi\delta}}\frac{dN_1}{dx} = 0 \tag{5.21.a}$$

Radial Momentum

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2^2 + 2N_2U_2(U_2+V_2) + \frac{(1-x^2)(1-x^3)}{2x}\epsilon_0\frac{dN_1}{dx} = 0 \tag{5.21.b}$$

Tangential Momentum

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2V_2 + 3N_2V_2(U_2+V_2) = 0 \tag{5.21.c}$$

Radial Stress

$$\frac{(1-x^2)}{x}\frac{d}{dx}N_2U_2^3 + 2N_2U_2^2(U_2+V_2) = -\frac{\sqrt{\frac{\pi}{2\delta}}}{3\tilde{\lambda}\sqrt{T_1}}\sqrt{\frac{1-x}{1+x}}\,N_1N_2U_2^2 \tag{5.21.d}$$

with the boundary conditions

$$
\begin{aligned}
N_2(1) &= 1 \\
\mho_2(1) &= -1 \\
\mathcal{V}_2(1) &= 1 \\
N_1(0) &= -\sqrt{2\pi\gamma}\, N_2(0)\,\mho_2(0)
\end{aligned}
\tag{5.22}
$$

It can be seen from the momentum equations that the assumption of $\mho_1(x)$ equal to zero reduces the momentum exchange between the two streams. In fact when the parameter $\epsilon_0 \equiv \frac{\sqrt{\tilde{T}_1}}{\gamma M}$ in the radial momentum equation is taken to be zero (hypersonic limit) there is no momentum exchange between the two streams. In this case the two momentum equations can be integrated to yield

$$N_2\mho_2^4/\mathcal{V}_2^2 = C_1 \tag{5.23.a}$$

and

$$\mathcal{V}_2 = \frac{-\mho_2}{1+\frac{C_2}{\sqrt{1-x^2}}} \tag{5.23.b}$$

Since $\mho_2(x)$ is not zero the constant C_2 in Equation (5.23.b) is zero if $\mathcal{V}_2(x)$ is finite at $x=1$. Equations (5.23.a) and (5.23.b) thus combine to give

$$N_2\mho_2^2 = -N_2\mho_2\mathcal{V}_2 = C_1 \tag{5.24}$$

For $C_2=0$ the continuity and radial stress equations can be solved for $N_2(x)$ and $N_1(x)$ to give

$$N_1(x) = C_3 \exp\left\{ \frac{-\pi}{3\tilde{\lambda}\sqrt{T_1}} \int^X \frac{x(1-x)}{(1-x^2)^{5/2}} N_2(x)\,dx \right\} \tag{5.25.a}$$

and

$$\sqrt{\frac{C_1}{N_2(x)}} = \frac{\sqrt{\frac{\pi}{2\gamma}}}{3\tilde{\lambda}\sqrt{T_1}} \int^X \frac{x(1-x)}{(1-x^2)^{3/2}} N_1(x)\,dx + C_4 \tag{5.25.b}$$

with the boundary conditions

$$N_2(1) = 1 \quad \text{and} \quad N_1(0) = \sqrt{2\pi\gamma C_1 N_2(0)} \tag{5.25.c}$$

Equations (5.25.a) and (5.25.b) are coupled nonlinear integral equations can be solved numerically or by an iteration procedure. The iteration can be effected by assuming that N_2 and N_1 appearing in the integrands of Equations (5.25.a) and (5.25.b) respectively are constant for the first iteration.

The two quantities $N_2 \overline{u_2^2}$ and $N_2 \overline{u_2 v_2}$ which appear respectively in the normal stress and shear stress are the leading terms in the drag formula Appendix L. Therefore if the constant C_1 is evaluated at $X = 1$ the drag can only be the FMF value to order $\left(\frac{1}{M}\right)$. The numerical results presented in Figure 13 verify the last statement. The drag for the case of $\epsilon_0 = .101$ deviates only slightly from the FMF value. The density distribution is compared with the experimental data from Reference 28 in Figure 16.

V.E.3 Three Moment Solution

If the equations in Section V.D are taken to order Θ the tangential momentum equation can be neglected since the leading term in this equation is order Θ^2. The adoption of both assumptions; $N_2(x) = 1$ and $U_1(x) = 0$, then completes the formulation by three moments. Since analytical solutions are possible in this case this investigation will hopefully shed some light into the entire study of the sphere drag problem for high speed flow.

The three equations are

Continuity

$$\frac{(1-x^2)}{x}\frac{dU_2}{dx} + 2(U_2+V_2) + \frac{(1-x^2)^2}{x\sqrt{2\pi\gamma}}\frac{dN_1}{dx} = 0 \tag{5.26.a}$$

Radial Momentum

$$\frac{(1-x^2)}{x}\frac{dU_2^2}{dx} + 2U_2(U_2+V_2) + \frac{(1-x^2)(1-x^3)}{2x}\epsilon_0\frac{dN_1}{dx} = 0 \tag{5.26.b}$$

Radial Stress

$$\frac{(1-x^2)}{x}\frac{d}{dx}U_2^3 + 2U_2^2(U_2+V_2) = -\frac{\sqrt{\frac{\pi}{2\gamma}}}{3\tilde{\lambda}\sqrt{\hat{T}_1}}N_1U_2^2 \tag{5.26.c}$$

with the boundary conditions

$$\begin{aligned} U_2(1) &= -1 \\ V_2(1) &= 1 \\ N_1(0) &= -\sqrt{2\pi\gamma}\,U_2(0) \end{aligned} \tag{5.27}$$

For ϵ_0 equal to zero the equations can be integrated to yield

$$N_1(x) = C_1 \exp\left\{\frac{-\pi}{9\tilde{\lambda}\sqrt{\hat{T}_1}} \frac{(1-x^3)}{(1-x^2)^{3/2}}\right\} \tag{5.28.a}$$

$$U_2(x) = \frac{-\sqrt{\frac{\pi}{2\gamma}}}{3\tilde{\lambda}\sqrt{\hat{T}_1}} C_1 \, I(\tilde{\lambda}\sqrt{\hat{T}_1}, x) + C_2 \tag{5.28.b}$$

$$V_2(x) = -U_2(x) - \frac{(1-x^2)}{2x}\left[\frac{dU_2}{dx} + \frac{(1-x^2)}{\sqrt{2\pi\gamma}}\frac{dN_1}{dx}\right] \tag{5.28.c}$$

where

$$I(\tilde{\lambda}\sqrt{\hat{T}_1}, x) \equiv \int^x \frac{x(1-x)}{(1-x^2)^{3/2}} \exp\left\{\frac{-\pi}{9\tilde{\lambda}\sqrt{\hat{T}_1}} \frac{(1-x^3)}{(1-x^2)^{3/2}}\right\} dx$$

(5.29)

$$C_1 \equiv \sqrt{2\pi\gamma} \Big/ \left\{\exp\left(\frac{-\pi}{9\tilde{\lambda}\sqrt{\hat{T}_1}}\right) + \frac{\pi}{3\tilde{\lambda}\sqrt{\hat{T}_1}} I(\tilde{\lambda}\sqrt{\hat{T}_1}, 1)\right\}$$

$$C_2 \equiv \frac{-C_1}{\sqrt{2\pi\gamma}} \exp\left(\frac{-\pi}{9\tilde{\lambda}\sqrt{\hat{T}_1}}\right)$$

In the near FMF limit the constants C_1 and C_2 become

$$C_1 \cong \sqrt{2\pi\gamma}$$

$$C_2 \cong -\exp\left(\frac{-\pi}{9\tilde{\lambda}\sqrt{\hat{T}_1}}\right) \tag{5.30}$$

For $N_2(x) = 1$ the drag coefficient given in Appendix L is determined by U_2 and V_2 at the body. From Equations (5.28.a) through (5.28.b) V_2 at the body is given by

$$V_2(0) = -U_2(0)\left\{ 1 + \frac{\pi}{3\lambda\sqrt{T_1}} \frac{\left(1 - \frac{\sqrt{2\pi\gamma}\,\epsilon_0}{4U_2(0)}\right)}{\left(1 - \frac{\sqrt{2\pi\gamma}\,\epsilon_0}{U_2(0)}\right)} \right\}$$

Expanding the second term for small values of ϵ_0 one obtains

$$V_2(0) \cong -U_2(0)\left(1 + \frac{\pi}{3\lambda\sqrt{T_1}}\right) - \frac{\pi}{4\lambda\sqrt{T_1}}\sqrt{2\pi\gamma}\,\epsilon_0 + O(\epsilon_0^2) \tag{5.30}$$

The drag coefficient is given by

$$C_D = U_2^2(0)\left[2 + \frac{\pi}{3\lambda\sqrt{T_1}}\right] - \sqrt{2\pi\gamma}\,\epsilon_0 U_2(0)\left[\frac{2}{3} - \frac{\pi}{4\lambda\sqrt{T_1}}\right] \tag{5.31}$$

But for the near FMF limit $U_2(0)$ may be written as

$$U_2(0) \cong -\exp\left(\frac{-\pi}{9\lambda\sqrt{T_1}}\right) \tag{5.32}$$

and the drag coefficient becomes

$$C_D = 2 + \frac{2}{3}\sqrt{2\pi\gamma}\,\epsilon_0 - \frac{\pi}{\lambda\sqrt{T_1}}\left[\frac{1}{9} + \frac{2}{3}\sqrt{2\pi\gamma}\,\epsilon_0\left(\frac{1}{9} + \frac{3}{8}\right)\right] \tag{5.33}$$

In the FMF limit Equation (5.33) is correct to $O\left(\frac{1}{M^2}\right)$

$$C_D^{FMF} = 2 + \frac{2}{3}\sqrt{2\pi\gamma}\,\epsilon_0 \tag{5.34}$$

The drag computation for the near FMF solution is presented in Figure 14 in comparison with the near FMF solution of Willis[20] and Rose[10] and the experimental data. The density distribution given by this solution (Figure (15)) is acceptable for $\lambda\sqrt{T_1} = O(1)$, but as in the first case, $N_2(x) = 1$, the variation with the MFP is incorrect.

V. F Nonuniformity of the High Speed Flow Field

Probstein[29] suggested that for the high speed flow past a blunt body the significant MFP is that defined for collisions between the free stream and emitted particles. This MFP will be called the body MFP and denoted by λ_b. A simple kinetic theory calculation[29] gives the following relation between λ_b and the free stream MFP λ_∞

$$\frac{\lambda_b}{r_0} = \frac{4}{\sqrt{\pi\gamma}} \frac{\sqrt{\frac{T_b}{T_\infty}}}{M} \frac{\lambda_\infty}{r_0} = \frac{4}{\sqrt{\pi\gamma}} \frac{\tilde{\lambda}\sqrt{\tilde{T}_1}}{M} \tag{5.35}$$

Probstein[29] conjectures that for

$$\frac{\lambda_b}{r_0} > 1 \tag{5.36}$$

the first collision solutions can be used to define the flow field and aerodynamic properties. Since the first collision methods cannot predict a sharp density gradient no shock-like structure can exist. But when

$$\frac{\lambda_b}{r_0} \cong \frac{1}{3} \tag{5.37}$$

a shock-like structure will begin to form and the first collision methods are no longer valid. This highly nonlinear "cascading" effect in the flow field is caused by the increase in the collision frequency between the incident and emitted particles.

Probstein's hypothesis is qualitatively verified from the results of the present theory. It is seen in Figure 15 that the density distribution for $\tilde{\lambda}\sqrt{\tilde{T}_1} \gg 1$ is essentially the same as the FMF value and is almost identical for either of the two assumptions $N_2(x) = 1$ or $U_1(x) = 0$. The drag however is predicted accurately by the assumption $N_2(x) = 1$.

For $\tilde{\lambda}\sqrt{T_1} < 1$ a large departure from the FMF density distribution is observed experimentally which is predicted only by assuming $U_1(x) = 0$. The drag in this regime is not predicted accurately by either model.

The relation between λ_b and λ_∞ given in Equation (5.35) shows that even if λ_∞ is very large λ_b may be small for the high speed limit $(M \to \infty)$. This implies that even if the free stream MFP indicates a near FMF situation, the region near the body may be in the continuum regime. The flow field description is therefore not uniform and suggests the following flow model within the framework of the present theory. The characteristic lengths are λ_b near the body and the sphere radius r_0 for the region distant from the sphere. The existence of two length scales suggest an "inner" and "outer" expansion similar to the boundary layer concept in ordinary fluid mechanics. In the present case the inner solution can be found from the assumption $U_1(x) = 0$ in a region of order (λ_b). This solution must be matched at the interface of the two zones with the outer solution obtained for $N_2(x) = 1$. The assumption of $U_1(x) = 0$ leads to a constant value of the drag which can be evaluated at the outer boundary of the inner region. Since the assumption of $N_2(x) = 1$ results in a correct estimation of the drag, then this combination of an outer region $(N_2 = 1)$ and an inner region $(U_1 = 0)$ could give a good estimate for both the drag and the density. The details of the correct expansion and the matching conditions are not obvious and they are beyond the scope of the present study.

V.G Comparison with Experimental Data and Other Theories

(1) Drag

Many investigators have stated that for the low density high speed flow over a blunt body the free stream MFP, λ_∞, is not the most meaningful parameter for correlation of experimental results. This point is verified by the scatter of experimental data shown on Figure 12. In the present study, the parameter

$$\tilde{\lambda}\sqrt{\tilde{T}_1} \equiv \frac{\lambda_\infty}{r_0}\sqrt{\frac{T_b}{T_\infty}}$$

always appears in the analysis as a group while the free stream MFP, λ_∞, never appears by itself. This group of parameters is related to the body MFP (Equation (5.35))

$$\tilde{\lambda}\sqrt{\tilde{T}_1} = \frac{\sqrt{\pi\gamma}}{4} M\left(\frac{\lambda_b}{r_0}\right) \tag{5.35}$$

It is seen from Figure 13 that if the abscissa is changed from $\frac{\lambda_\infty}{r_0}$ to $\tilde{\lambda}\sqrt{\tilde{T}_1}$ the same experimental data from Figure 12 exhibit considerably better correlation. It should be noted that although the parameter $\epsilon_0 \equiv \frac{\sqrt{\frac{T_b}{T_\infty}}}{\gamma M}$ is the same for both sets of data the Mach numbers for the Masson, Morris, and Bloxsom[32] data is at least three times as great as those for Kinslow and Potter's[27] results.

Also shown on Figure 13 are the numerical solutions from Sections V.E.1, V.E.2, and V.E.3. Figure 14 compares the same experimental data with the analytic near FMF solution from Section V.E.3 and the near FMF theories of Willis[20] and Rose.[10]

The four moment solution, assuming $N_2(x) = 1$, from Section V. E. 1 appears to correlate better with the data from Reference 32. The numerical solution for $U_1(x) = 0$ from Section V. E. 2 shows that the drag changes only slightly from the FMF limit. The numerical results of the "three moment" solution is seen to be very similar to the solution resulting from $N_2 = 1$ for $4 \leq \tilde{\lambda}\sqrt{\tilde{T}_1} \leq \infty$, but diverges significantly for $\tilde{\lambda}\sqrt{\tilde{T}_1} < 1$. This may be qualitatively explained by the fact that the entire tangential momentum equation in this case has been neglected. However, the near FMF computation for the "three moment" solution is shown to compare very favorably with the experimental data of Reference 27 and the other near FMF theories.

(2) Density

The results of the density computation verify Probstein's[29] conjecture as stated in Section V. F. The density distribution (Figure 15) is very close to the FMF value for $1 \leq \tilde{\lambda}\sqrt{\tilde{T}_1} \leq \infty$ and is accurately predicted by both the results of Section V. E. 1 and Section V. E. 2. However, the results from Section V. E. 1 predict incorrectly that the density decreases as the MFP decreases (this was discussed in detail in Sections V. E. 1 and V. E. 2. For small values of the MFP the solution from Section V. E. 1 appears to be incorrect, while the assumption that $U_1(x) = 0$ results in a very steep density gradient which appears in Figure 16 to give a good description near the body.

CHAPTER VI – CONCLUSION AND FUTURE WORK

The investigation of the flow over a sphere was accomplished by employing a modification of Lees' two stream Maxwellian distribution function in Maxwell's moment equations. Although the Mach number was restricted to the two limiting cases; low speed flow $(M \ll 1)$ and high speed flow $(M \gg 1)$, the Knudsen number or MFP was unrestricted.

In the low speed case an analytic solution was obtained for the coupled drag and heat transfer problem in terms of six moment equations. This solution for the drag compared favorably with Millikan's[17] oil drop experiments for the complete range of experimental Knudsen numbers. The predicted heat transfer also showed good agreement with the experimental results of Kavanau and Drake.[18] Although no systematic study of convergence for the moment method used in the present study was made, a heuristic proof was attempted by choosing different moments of the Boltzmann equation which gave essentially the same result.

Unlike the situation for the low speed flow, the high speed flow problem depends critically on which moments are taken. The most probable cause is the fact that the high speed moment equations are nonlinear in comparison to the linear low speed equations. Because of this uncertainty, physical intuition must be applied to choose the most important moments for a particular problem. The predicted sphere drag and density field using the following four moments; continuity, radial momentum, tangential momentum, and radial stress, were found to be adequate for a wide range of Knudsen number. In particular the results indicate that for a wide range of the parameter $\tilde{\lambda}\sqrt{\overline{T}_1}$; $1 \leq \tilde{\lambda}\sqrt{\overline{T}_1} \leq \infty$, the density distribution varies only slightly from the FMF result. For small values of this parameter the first indication of the formation of a gas dynamic shock was observed. It was also found that an analytic solution could be found by making the assumption of large MFP. This near FMF solution was found to be in complete agreement with the near FMF theories of Willis[20] and Rose.[10]

All the results indicate that the moment method formulated in the present study gives considerable insight into the nature of the transition from highly rarefied flows to the continuum regime for the problem of flow over a sphere. No other method of studying the complete range of Knudsen number has been successful. Because this method is applicable over a wide range of Knudsen numbers, the details of the flow field may be in error. For example the choice of the distribution function is obviously oversimplified and one cannot expect it to be correct; especially for the nonlinear high speed flow. However, the results from the present study confirm the conjecture that the gross flow quantities such as drag and heat transfer are adequately predicted.

Future efforts to extend the results of the present study can be directed in the following areas:

(1) Carry out the numerical solution of the eight moment equations for the low speed approximation to verify that the six moment solution was adequate.

(2) Complete the rigorous "inner" and "outer" matching scheme for the high speed approximation.

(3) Carry out the numerical solution of the high speed flow equations without making the near stagnation point approximation.

(4) Apply the general formulation to other geometrical shapes.

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APPENDIX A

BOLTZMANN EQUATION AND MAXWELL'S EQUATION OF TRANSFER IN CURVALINEAR COORDINATES

The Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + \left[F_i - (\vec{\Omega} \times \vec{\xi})_i\right] \frac{\partial f}{\partial \xi_i} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} \tag{A. 1}$$

In orthogonal curvilinear coordinates the spatial gradient is given by

$$\sum_{i=1}^{3} \frac{\partial f}{\partial x_i} = \sum_{i=1}^{3} \frac{\vec{e}_i}{h_i} \frac{\partial f}{\partial a_i}$$

where

$$(a_1, a_2, a_3) = \begin{cases} (x_1, x_2, x_3) & \text{cartesian} \\ (r, \theta, z) & \text{cylindrical} \\ (r, \theta, \phi) & \text{spherical} \end{cases}$$

$$(h_1, h_2, h_3) = \begin{cases} (1, 1, 1) & \text{cartesian} \\ (1, r, 1) & \text{cylindrical} \\ (1, r, r\sin\theta) & \text{spherical} \end{cases}$$

and

$$\vec{e}_i \equiv \text{unit vector in } i^{\text{th}} \text{ coordinate}$$

Lees[3] shows that the curvature term is given by

$$\sum_{i=1}^{3} (\vec{\Omega} \times \vec{\xi})_i \frac{\partial f}{\partial \xi_i} = \sum_{i=1}^{3} \left\{ \frac{1}{h_i h_j} \left(\xi_j^2 \frac{\partial h_j}{\partial a_i} - \xi_i \xi_j \frac{\partial h_i}{\partial a_j} \right) + \frac{1}{h_i h_k} \left(\xi_k^2 \frac{\partial h_k}{\partial a_i} - \xi_i \xi_k \frac{\partial h_i}{\partial a_k} \right) \right\} \frac{\partial f}{\partial \xi_i}$$

where the cyclical order of permutation of the indices i, j, k is required.

Therefore, the Boltzmann equation in curvilinear coordinates becomes

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \frac{\xi_i}{h_i} \frac{\partial f}{\partial a_i} + \sum_{i=1}^{3} F_i \frac{\partial f}{\partial \xi_i} \\
&+ \sum_{i=1}^{3} \left\{ \frac{1}{h_i h_j} \left(\xi_j^2 \frac{\partial h_j}{\partial a_i} - \xi_i \xi_j \frac{\partial h_i}{\partial a_j} \right) + \frac{1}{h_i h_k} \left(\xi_k^2 \frac{\partial h_k}{\partial a_i} - \xi_i \xi_k \frac{\partial h_i}{\partial a_k} \right) \right\} \frac{\partial f}{\partial \xi_i} \\
&= \left(\frac{\partial f}{\partial t} \right)_{\text{collisions}}
\end{aligned}
\tag{A. 2}
$$

Maxwell's equation of transfer is obtained by multiplying Equation (A. 2) by any function of particle velocity and integrating over the velocity space, $d\vec{\xi}$. The resulting equation is obtained by an alternate method by Lees. [3]

$$
\begin{aligned}
&\frac{\partial}{\partial t} \left\{ \int f \Phi \, d\vec{\xi} \right\} + \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial a_i} \left\{ h_j h_k \int f \xi_i \Phi \, d\vec{\xi} \right\} \\
&- \int f \left\{ \sum_{i=1}^{3} \frac{F_i}{m} \frac{\partial \Phi}{\partial \xi_i} \right\} d\vec{\xi} - \int f \left\{ \sum_{i=1}^{3} \left[\frac{1}{h_i h_j} \left(\xi_j^2 \frac{\partial h_j}{\partial a_i} - \xi_i \xi_j \frac{\partial h_i}{\partial a_j} \right) \right. \right. \\
&\left. \left. + \frac{1}{h_i h_k} \left(\xi_k^2 \frac{\partial h_k}{\partial a_i} - \xi_i \xi_k \frac{\partial h_i}{\partial a_k} \right) \right] \frac{\partial \Phi}{\partial \xi_i} \right\} d\vec{\xi} \\
&= \Delta \Phi
\end{aligned}
\tag{A. 3}
$$

APPENDIX B

EIGHT MOMENT EQUATIONS

Maxwell's equation of transfer in spherical coordinates for symmetry with respect to ϕ (see Figure 2). is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\int f\xi_r\Phi\, d\vec{\xi}\right\} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\int f\xi_\theta\Phi\, d\vec{\xi}\right\}$$

$$-\frac{1}{r}\int f\left\{(\xi_\theta^2+\xi_\phi^2)\frac{\partial\Phi}{\partial\xi_r} + (\cot\theta\,\xi_\phi^2-\xi_r\xi_\theta)\frac{\partial\Phi}{\partial\xi_\theta}\right.$$

$$\left.-(\xi_r\xi_\phi+\cot\theta\,\xi_\theta\xi_\phi)\frac{\partial\Phi}{\partial\xi_\phi}\right\}d\vec{\xi} = \Delta\Phi$$

1. Continuity Equation ($\Phi_1 = m$, $\Delta\Phi_1 = 0$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\overline{nq}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\overline{nq}_\theta\right) = 0$$

$$\overline{nq}_i \equiv \int f\xi_i\, d\vec{\xi} \qquad \text{(B. 1)}$$

2. Radial Momentum Equation ($\Phi_2 = m\xi_r$, $\Delta\Phi_2 = 0$)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\overline{P}_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\overline{P}_{r\theta}\right) - \frac{1}{r}\left(\overline{P}_{\theta\theta}+\overline{P}_{\phi\phi}\right) = 0$$

$$\overline{P}_{ij} \equiv m\int f\xi_i\xi_j\, d\vec{\xi} \qquad \text{(B. 2)}$$

3. <u>Tangential Momentum Equation</u> $(\Phi_3 = m\xi_\theta,\ \Delta\Phi_3 = 0)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{P}_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\overline{P}_{\theta\theta}) - \left(\frac{\cot\theta\,\overline{P}_{\phi\phi} - \overline{P}_{r\theta}}{r}\right) = 0 \quad \text{(B. 3)}$$

4. <u>Energy Equation</u> $(\Phi_4 = m\xi^2/2,\ \Delta\Phi_4 = 0)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\overline{P}_{rrr} + \overline{P}_{r\theta\theta} + \overline{P}_{r\phi\phi})\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\,(\overline{P}_{rr\theta} + \overline{P}_{\theta\theta\theta} + \overline{P}_{\theta\phi\phi})\right] = 0 \quad \text{(B. 4)}$$

$$\overline{P}_{ijk} \equiv m\int f\,\xi_i\xi_j\xi_k\,d\vec{\xi}$$

5. <u>Shear Stress Equation</u> $(\Phi_5 = m\xi_r\xi_\theta,\ \Delta\Phi_5 = (\frac{P}{\mu})\overline{P}_{r\theta})$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{P}_{rr\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\overline{P}_{r\theta\theta}) - \frac{1}{r}(\overline{P}_{\theta\theta\theta} + \overline{P}_{\theta\phi\phi} + \cot\theta\,\overline{P}_{r\phi\phi} - \overline{P}_{rr\theta}) = \left(\frac{P}{\mu}\right)\overline{P}_{r\theta} \quad \text{(B. 5)}$$

6. <u>Radial Stress Equation</u> $(\Phi_6 = m\xi_r^2,\ \Delta\Phi_6 = (\frac{P}{\mu})\overline{P}_{rr})$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{P}_{rrr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\overline{P}_{rr\theta}) - \frac{2}{r}(\overline{P}_{r\theta\theta} + \overline{P}_{r\phi\phi}) = \left(\frac{P}{\mu}\right)\overline{P}_{rr} \quad \text{(B. 6)}$$

7. <u>Tangential Stress Equation</u> $\left(\Phi_7 = mq_\theta^2,\quad \Delta\Phi_7 = \left(\frac{\bar{p}}{\mu}\right)\bar{P}_{\theta\theta}\right)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\bar{P}_{r\theta\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\bar{P}_{\theta\theta\theta}\right)$$

$$-\frac{2}{r}\left(\cot\theta\,\bar{P}_{\theta\phi\phi} - \bar{P}_{r\theta\theta}\right) = \left(\frac{\bar{p}}{\mu}\right)\bar{P}_{\theta\theta} \qquad \text{(B. 7)}$$

8. <u>Radial Heat Flux Equation</u> $\left(\Phi_8 = \frac{m\xi_r\xi^2}{2},\right.$

$$\left.\Delta\Phi_8 = \left(\frac{\bar{p}}{\mu}\right)\left[-\frac{2}{3}\bar{Q}_r + \bar{q}_r\bar{P}_{rr} + \bar{q}_\theta\bar{P}_{r\theta}\right]\right)$$

$$\frac{1}{2r^2}\frac{\partial}{\partial r}\left[r^2\left(\bar{P}_{rrrr} + \bar{P}_{rr\theta\theta} + \bar{P}_{rr\phi\phi}\right)\right]$$

$$+\frac{1}{2r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\bar{P}_{rrr\theta} + \bar{P}_{r\theta\theta\theta} + \bar{P}_{r\theta\phi\phi}\right)\right]$$

$$-\frac{1}{2r}\left[\bar{P}_{rr\theta\theta} + \bar{P}_{rr\phi\phi} + \bar{P}_{\theta\theta\theta\theta} + \bar{P}_{\phi\phi\phi\phi} + 2\bar{P}_{\theta\theta\phi\phi}\right] \qquad \text{(B. 8)}$$

$$= \left(\frac{\bar{p}}{\mu}\right)\left[-\frac{2}{3}\bar{Q}_r + \bar{q}_r\bar{P}_{rr} + \bar{q}_\theta\bar{P}_{r\theta}\right]$$

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APPENDIX C

COMPUTATION OF THE MOMENTS – LOW SPEED FLOW

The moments appearing in the eight moment equations in Appendix B are computed with the linearized distribution function given in Section (IV. A). The parametric functions appearing in the distribution are normalized by the free stream quantities $(n_\infty, T_\infty, q_\infty)$ and results in the following non-dimensional variables.

$$\tilde{n}_i(r,\theta) \equiv \frac{n_i(r,\theta)}{n_\infty}$$

$$\tilde{T}_i(r,\theta) \equiv \frac{T_i(r,\theta)}{T_\infty}$$

$$\tilde{u}_i(r,\theta) \equiv \frac{u_i(r,\theta)}{q_\infty}$$

$$\tilde{v}_i(r,\theta) \equiv \frac{v_i(r,\theta)}{q_\infty}$$

The coordinate r defined in the semi-infinite interval

$$r_0 \leqslant r \leqslant \infty$$

is transformed to the independent variable x defined in the finite interval

$$0 \leqslant x \leqslant 1$$

through the transformation

$$x \equiv \sqrt{1 - \left(\frac{r_0}{r}\right)^2}$$

The moments become

$$\tilde{n} \equiv \frac{\bar{n}}{n_\infty} = \frac{\tilde{n}_1+\tilde{n}_2}{2} - \frac{x}{2}(\tilde{n}_1-\tilde{n}_2) + \sqrt{\frac{\gamma}{2\pi}}\, M(1-x^2)\left(\frac{\tilde{n}_1\tilde{u}_1}{\sqrt{\tilde{T}_1}} - \frac{\tilde{n}_2\tilde{u}_2}{\sqrt{\tilde{T}_2}}\right)$$

$$\tilde{nq}_r \equiv \frac{\overline{nq_r}}{n_\infty q_\infty} = \frac{(1-x^2)}{\sqrt{2\pi\gamma}\,M}\left(\tilde{n}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\sqrt{\tilde{T}_2}\right) + \frac{\tilde{n}_1\tilde{u}_1+\tilde{n}_2\tilde{u}_2}{2}$$

$$- \frac{x^3}{2}(\tilde{n}_1\tilde{u}_1 - \tilde{n}_2\tilde{u}_2) - \frac{g^{(1)}(r,\theta)}{n_\infty q_\infty}$$

$$\tilde{nq}_\theta = \frac{\tilde{n}_1\tilde{v}_1+\tilde{n}_2\tilde{v}_2}{2} + \frac{(x^3-3x)}{4}(\tilde{n}_1\tilde{v}_1 - \tilde{n}_2\tilde{v}_2)$$

$$\tilde{P}_{rr} \equiv \frac{\bar{P}_{rr}}{kn_\infty T_\infty} = \frac{\tilde{n}_1\tilde{T}_1+\tilde{n}_2\tilde{T}_2}{2} - \frac{x^3}{2}(\tilde{n}_1\tilde{T}_1 - \tilde{n}_2\tilde{T}_2)$$

$$+ \sqrt{\frac{2\gamma}{\pi}}\, M(1-x^4)\left(\tilde{n}_1\tilde{u}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\tilde{u}_2\sqrt{\tilde{T}_2}\right) + \frac{2m\,g^{(2)}(r,\theta)}{kn_\infty T_\infty}$$

$$\tilde{P}_{\theta\theta} = \frac{\tilde{n}_1\tilde{T}_1+\tilde{n}_2\tilde{T}_2}{2} + \frac{(x^3-3x)}{4}(\tilde{n}_1\tilde{T}_1 - \tilde{n}_2\tilde{T}_2)$$

$$+ \sqrt{\frac{\gamma}{2\pi}}\, M(1-x^2)^2\left(\tilde{n}_1\tilde{u}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\tilde{u}_2\sqrt{\tilde{T}_2}\right)$$

$$\tilde{P}_{\phi\phi} = \tilde{P}_{\theta\theta}$$

$$\tilde{P}_{rrr} \equiv \frac{\bar{P}_{rrr}}{kn_\infty T_\infty q_\infty} = \frac{\sqrt{\frac{2\gamma}{\pi}}}{M}(1-x^4)\left(\tilde{n}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{T}_2^{3/2}\right)$$

$$+ \frac{3}{2}(\tilde{n}_1\tilde{u}_1\tilde{T}_1 + \tilde{n}_2\tilde{u}_2\tilde{T}_2) - \frac{3x^5}{2}(\tilde{n}_1\tilde{u}_1\tilde{T}_1 - \tilde{n}_2\tilde{u}_2\tilde{T}_2) - \frac{6m\,g^{(3)}(r,\theta)}{kn_\infty T_\infty q_\infty}$$

$$\tilde{P}_{\theta\theta\theta} = \frac{3}{2}(\tilde{n}_1\tilde{v}_1\tilde{T}_1 + \tilde{n}_2\tilde{v}_2\tilde{T}_2) + \frac{3}{16}(-15x+10x^3-3x^5)(\tilde{n}_1\tilde{v}_1\tilde{T}_1 - \tilde{n}_2\tilde{v}_2\tilde{T}_2)$$

$$\tilde{P}_{\phi\phi\phi} = 0$$

$$\tilde{P}_{r\theta\theta} = \frac{\tilde{n}_1\tilde{u}_1\tilde{T}_1+\tilde{n}_2\tilde{u}_2\tilde{T}_2}{2} + \frac{(1-x^2)^2}{\sqrt{2\pi\gamma}\,M}\left(\tilde{n}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{T}_2^{3/2}\right)$$

$$+ \frac{3}{4}\left(x^5 - \frac{2x^3}{3}\right)(\tilde{n}_1\tilde{u}_1\tilde{T}_1 - \tilde{n}_2\tilde{u}_2\tilde{T}_2)$$

$$\tilde{P}_{rr\theta} = \frac{\tilde{n}_1\tilde{v}_1\tilde{T}_1 + \tilde{n}_2\tilde{v}_2\tilde{T}_2}{2} + \frac{3}{4}\left(x^5 - \frac{5x^3}{3}\right)\left(\tilde{n}_1\tilde{v}_1\tilde{T}_1 - \tilde{n}_2\tilde{v}_2\tilde{T}_2\right)$$

$$\tilde{P}_{r\phi\phi} = \tilde{P}_{r\theta\theta}$$

$$\tilde{P}_{rr\phi} = 0$$

$$\tilde{P}_{\theta\phi\phi} = \frac{1}{3}\tilde{P}_{\theta\theta\theta}$$

$$\tilde{P}_{rrrr} \equiv \frac{\bar{P}_{rrrr}}{k n_\infty T_\infty q_\infty^2} = \frac{3}{2\gamma M^2}\left[\left(\tilde{n}_1\tilde{T}_1^2 + \tilde{n}_2\tilde{T}_2^2\right) - x^5\left(\tilde{n}_1\tilde{T}_1^2 - \tilde{n}_2\tilde{T}_2^2\right)\right]$$

$$+ \frac{8}{\sqrt{2\pi\gamma}\,M}\left(1 - x^6\right)\left(\tilde{n}_1\tilde{u}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{u}_2\tilde{T}_2^{3/2}\right)$$

$$\tilde{P}_{rr\theta\theta} = \frac{1}{2\gamma M^2}\left[\left(\tilde{n}_1\tilde{T}_1^2 + \tilde{n}_2\tilde{T}_2^2\right) + \frac{3}{2}\left(x^5 - \frac{5x^3}{3}\right)\left(\tilde{n}_1\tilde{T}_1^2 - \tilde{n}_2\tilde{T}_2^2\right)\right]$$

$$+ \frac{\sqrt{\frac{2}{\pi\gamma}}}{M}\left(2x^6 - 3x^4 + 1\right)\left(\tilde{n}_1\tilde{u}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{u}_2\tilde{T}_2^{3/2}\right)$$

$$\tilde{P}_{rr\phi\phi} = \tilde{P}_{rr\theta\theta}$$

$$\tilde{P}_{rrr\theta} = \frac{\sqrt{\frac{2}{\pi\gamma}}}{M}\left(2x^6 - 3x + 1\right)\left(\tilde{n}_1\tilde{v}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{v}_2\tilde{T}_2^{3/2}\right)$$

$$\tilde{P}_{r\theta\theta\theta} = \frac{3\left(1 - x^2\right)^3}{\sqrt{2\pi\gamma}\,M}\left(\tilde{n}_1\tilde{v}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{v}_2\tilde{T}_2^{3/2}\right)$$

$$\tilde{P}_{r\theta\phi\phi} = \frac{1}{3}\tilde{P}_{r\theta\theta\theta}$$

$$\tilde{P}_{\theta\theta\theta\theta} = \frac{3}{2\gamma M^2}\left[\left(\tilde{n}_1\tilde{T}_1^2 + \tilde{n}_2\tilde{T}_2^2\right) + \frac{15}{8}\left(-x + \frac{2x^3}{3} - \frac{x^5}{5}\right)\left(\tilde{n}_1\tilde{T}_1^2 - \tilde{n}_2\tilde{T}_2^2\right)\right]$$

$$+ \frac{3\left(1 - x^2\right)^3}{\sqrt{2\pi\gamma}\,M}\left(\tilde{n}_1\tilde{u}_1\tilde{T}_1^{3/2} - \tilde{n}_2\tilde{u}_2\tilde{T}_2^{3/2}\right)$$

$$\tilde{P}_{\theta\theta\phi\phi} = \frac{1}{3}\tilde{P}_{\theta\theta\theta\theta}$$

$$\tilde{P}_{\phi\phi\phi\phi} = \tilde{P}_{\theta\theta\theta\theta}$$

From the definition of $\overline{P}_{ij}$ the pressure and the stress components become

pressure;

$$\tilde{p} \equiv \frac{\overline{p}}{kn_\infty T_\infty} = \frac{1}{3}\left[\tilde{P}_{rr} + \tilde{P}_{\theta\theta} + \tilde{P}_{\phi\phi}\right] - \frac{\gamma M^2}{3}\left[\tilde{n}\tilde{q}_r^2 + \tilde{n}\tilde{q}_\theta^2\right]$$

stress;

$$\tilde{\tau}_{ij} \equiv \frac{\overline{\tau}_{ij}}{kn_\infty T_\infty} = \tilde{P}_{ij} - \gamma M^2\, \tilde{n}\tilde{q}_i\tilde{q}_j$$

and the stress tensor;

$$\tilde{p}_{ij} \equiv \tilde{p}\,\delta_{ij} - \tilde{\tau}_{ij} = -\tilde{P}_{ij} + \tilde{p}\,\delta_{ij} + \gamma M^2 \tilde{n}\tilde{q}_i\tilde{q}_j$$

In the low speed approximation $(M \ll 1)$ these quantities are

$$\tilde{p} \cong \frac{1}{3}\left[\tilde{P}_{rr} + \tilde{P}_{\theta\theta} + \tilde{P}_{\phi\phi}\right] = \frac{\tilde{n}_1\tilde{T}_1 + \tilde{n}_2\tilde{T}_2}{2} - \frac{x}{2}\left(\tilde{n}_1\tilde{T}_1 - \tilde{n}_2\tilde{T}_2\right)$$
$$+ \frac{2}{3}\sqrt{\frac{2\gamma}{\pi}}\, M(1-x^2)\left(\tilde{n}_1\tilde{u}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\tilde{u}_2\sqrt{\tilde{T}_2}\right) + \frac{2}{3}\,\frac{m g^{(2)}(r,\theta)}{kn_\infty T_\infty}$$

$$\tilde{p}_{rr} \cong \tilde{p} - \tilde{P}_{rr} = -\frac{x}{2}(1-x^2)\left(\tilde{n}_1\tilde{T}_1 - \tilde{n}_2\tilde{T}_2\right)$$
$$- \sqrt{\frac{2\gamma}{\pi}}\, M(1-x^2)\left(\frac{1+3x^2}{3}\right)\left(\tilde{n}_1\tilde{u}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\tilde{u}_2\sqrt{\tilde{T}_2}\right) - \frac{4}{3}\,\frac{m g^{(2)}(r,\theta)}{kn_\infty T_\infty}$$

$$\tilde{p}_{\theta\theta} \cong \tilde{p} - \tilde{P}_{\theta\theta} = -\frac{1}{2}\,\tilde{p}_{rr}$$

$$\tilde{p}_{r\theta} \cong -\tilde{P}_{r\theta} = -\sqrt{\frac{\gamma}{2\pi}}\, M(1-x^2)^2\left(\tilde{n}_1\tilde{v}_1\sqrt{\tilde{T}_1} - \tilde{n}_2\tilde{v}_2\sqrt{\tilde{T}_2}\right)$$

The radial heat flux defined by

$$\overline{Q}_r \equiv m \int f \frac{c_r c^2}{2} d\vec{\xi}$$

is found to be

$$\tilde{Q}_r \equiv \frac{\overline{Q}_r}{k n_\infty T_\infty q_\infty} = \frac{1}{2}\left[\tilde{P}_{rrr} + \tilde{P}_{r\theta\theta} + \tilde{P}_{r\phi\phi}\right]$$

$$+ \frac{\tilde{q}_r}{2}\left(3\tilde{P}_{rr} + \tilde{P}_{\theta\theta} + \tilde{P}_{\phi\phi}\right) - \tilde{q}_\theta \tilde{P}_{r\theta} - \tilde{q}_\phi \tilde{P}_{r\phi}$$

$$+ \gamma M^2 \tilde{n} \tilde{q}_r \left(\tilde{q}_r^2 + \tilde{q}_\theta^2 + \tilde{q}_\phi^2\right)$$

Neglecting squares of velocities or assuming that $(M \ll 1)$

$$\tilde{Q}_r \cong \frac{1}{2}\left[\tilde{P}_{rrr} + \tilde{P}_{r\theta\theta} + \tilde{P}_{r\phi\phi}\right]$$

$$- \frac{\tilde{q}_r}{2}\left(3\tilde{P}_{rr} + \tilde{P}_{\theta\theta} + \tilde{P}_{\phi\phi}\right) - \tilde{q}_\theta \tilde{P}_{r\theta} - q_\phi \tilde{P}_{r\phi}$$

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APPENDIX D

MOMENT EQUATIONS – LOW SPEED FLOW

The tilde over the non-dimensional variables have been suppressed.

1. Continuity Equation

$$\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left(\frac{n_1\sqrt{T_1}-n_2\sqrt{T_2}}{\sqrt{2\pi\gamma}\,M}\right)+\frac{(1-x^2)}{2x}\frac{\partial}{\partial x}(n_1u_1+n_2u_2)$$

$$-\frac{x^2}{2}(1-x^2)\frac{\partial}{\partial x}(n_1u_1-n_2u_2)+\frac{(x^3-3x)}{2}(n_1u_1-n_2u_2)$$

$$+\frac{(x^3-3x)}{4}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)(n_1v_1-n_2v_2)+(n_1u_1+n_2u_2) \tag{D. 1}$$

$$+\frac{1}{2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)(n_1v_1+n_2v_2)+\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left[\frac{g^{(1)}(x,\theta)}{n_0 q_{\infty}(1-x^2)}\right]=0$$

In the FMF limit the parametric functions were found to be

$$u_1=v_1=0$$
$$u_2=-\cos\theta$$
$$v_2=\sin\theta$$
$$n_1, T_1, T_2 = \text{CONSTANT}$$

Therefore conservation of mass requires that

$$\left[g^{(1)}(x,\theta)\right]^{FMF}\cong(1-x^2)=\frac{1}{r^2}$$

If this result is generalized for a finite MFP by

$$g^{(1)}(x,\theta)=(1-x^2)\,g^{(1)}(\theta)$$

$g^{(1)}(x,\theta)$ does not appear in the continuity equation.

2. Radial Momentum Equation

$$\frac{(1-x^2)}{2x}\frac{\partial}{\partial x}(n_1T_1+n_2T_2) - \frac{x^2}{2}(1-x^2)\frac{\partial}{\partial x}(n_1T_1-n_2T_2)$$

$$+\frac{(1-x^2)(1-x^4)}{x}\sqrt{\frac{2\gamma}{\pi}}\,M\frac{\partial}{\partial x}(n_1u_1\sqrt{T_1}-n_2u_2\sqrt{T_2})$$

$$+\sqrt{\frac{2\gamma}{\pi}}\,M(1-x^2)^2\left\{\frac{1}{2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)(n_1v_1\sqrt{T_1}-n_2v_2\sqrt{T_2})+(n_1u_1\sqrt{T_1}-n_2u_2\sqrt{T_2})\right]$$

$$+\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left[\frac{2m\,g^{(2)}(x,\theta)}{kn_0T_0(1-x^2)}\right]=0 \qquad \text{(D. 2)}$$

Again in the FMF limit

$$\left[g^{(2)}(x,\theta)\right]^{FMF}\cong(1-x^2)=\frac{1}{r^2}$$

therefore, generalizing for the case of finite MFP

$$g^{(2)}(x,\theta)=(1-x^2)\,g^{(2)}(\theta)$$

and $g^{(2)}(x,\theta)$ does not appear in the radial momentum equation.

3. Tangential Momentum Equation

$$\frac{(1-x^2)^3}{x}\sqrt{\frac{\gamma}{2\pi}}\,M\frac{\partial}{\partial x}(n_1v_1\sqrt{T_1}-n_2v_2\sqrt{T_2})$$

$$+\frac{1}{2}\frac{\partial}{\partial\theta}\left[n_1T_1+n_2T_2+\frac{(x^3-3x)}{2}(n_1T_1-n_2T_2)\right] \qquad \text{(D. 3)}$$

$$+\sqrt{\frac{\gamma}{2\pi}}\,M(1-x^2)^2\left[\frac{\partial}{\partial\theta}(n_1u_1\sqrt{T_1}-n_2u_2\sqrt{T_2})-(n_1v_1\sqrt{T_1}-n_2v_2\sqrt{T_2})\right]=0$$

4. Energy Equation

$$-\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left[\frac{6mg^{(3)}(x,\theta)}{kn_\infty T_\infty q_\infty(1-x^2)}\right]+\frac{2\sqrt{\frac{2}{\pi\gamma}}}{M}\frac{(1-x^2)^2}{x}\frac{\partial}{\partial x}\left(n_1T_1^{3/2}-n_2T_2^{3/2}\right)$$

$$+\frac{5}{2}\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left(n_1u_1T_1+n_2u_2T_2\right)-\frac{5x^2}{2}(1-x^2)\frac{\partial}{\partial x}\left(n_1u_1T_1-n_2u_2T_2\right)$$

$$+5\left[n_1u_1T_1+n_2u_2T_2+\frac{1}{2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left(n_1v_1T_1+n_2v_2T_2\right)\right] \qquad \text{(D. 4)}$$

$$+\frac{5x}{2}(x^2-3)\left[\left(n_1u_1T_1-n_2u_2T_2\right)+\frac{1}{2}\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left(n_1v_1T_1-n_2v_2T_2\right)\right]=0$$

Again as for $g^{(1)}$ and $g^{(2)}$, $g^{(3)}(x,\theta)$ is taken to be

$$g^{(3)}(x,\theta)\cong(1-x^2)=\frac{1}{r^2}$$

which satisfies the MFM limit and does not appear in the energy equation.

5. Shear Stress Equation

$$\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left(\frac{n_1v_1T_1+n_2v_2T_2}{2}\right)+\frac{3}{4}(1-x^2)\left(x^4-\frac{5x^2}{3}\right)\frac{\partial}{\partial x}\left(n_1v_1T_1-n_2v_2T_2\right)$$

$$+(1-x^2)^2\frac{\partial}{\partial\theta}\left(\frac{n_1T_1^{3/2}-n_2T_2^{3/2}}{\sqrt{2\pi\gamma}\,M}\right)+\frac{\partial}{\partial\theta}\left[\frac{n_1u_1T_1+n_2u_2T_2}{2}\right.$$

$$\left.+\frac{3}{4}\left(x^5-\frac{5x^3}{3}\right)\left(n_1u_1T_1-n_2u_2T_2\right)\right] \qquad \text{(D. 5)}$$

$$-\frac{1}{2}\left[n_1v_1T_1+n_2v_2T_2+\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)\left(n_1v_1T_1-n_2v_2T_2\right)\right]$$

$$=\frac{r_0\left(\frac{\tilde{\rho}}{\mu}\right)\tilde{p}_{r\theta}}{q_\infty\sqrt{1-x^2}}=\frac{\sqrt{\frac{2}{\pi\gamma}}}{\tilde{\lambda}M}\frac{\tilde{n}\,\tilde{p}_{r\theta}}{\sqrt{1-x^2}}$$

where $\left(\frac{\tilde{\rho}}{\mu}\right)=\sqrt{\frac{\pi kT_\infty}{2m}}\left(\frac{\hat{n}}{\lambda_\infty}\right)$ and $\tilde{\lambda}\equiv\frac{\lambda_\infty}{r_0}$

6. Radial Stress Equation

$$\frac{(1-x^2)(1-x^4)}{x}\,\frac{\sqrt{\frac{2}{\pi\gamma}}}{M}\,\frac{\partial}{\partial x}\left(n_1T_1^{3/2}-n_2T_2^{3/2}\right)+\frac{3}{2}\,\frac{(1-x^2)}{x}\,\frac{\partial}{\partial x}\left(n_1u_1T_1+n_2u_2T_2\right)$$

$$-\frac{3x^4}{2}(1-x^2)\frac{\partial}{\partial x}\left(n_1u_1T_1-n_2u_2T_2\right)+\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left[\frac{n_1v_1T_1+n_2v_2T_2}{2}\right.$$

$$\left.+\frac{3}{4}\left(x^5-\frac{5x^3}{3}\right)\left(n_1v_1T_1-n_2v_2T_2\right)\right]+\left(n_1u_1T_1+n_2u_2T_2\right) \qquad \text{(D. 6)}$$

$$+\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)\left(n_1u_1T_1-n_2u_2T_2\right)$$

$$=\frac{r_0\left(\frac{\tilde{\rho}}{\mu}\right)\tilde{p}_{rr}}{q_\infty\sqrt{1-x^2}}=\frac{\sqrt{\frac{\pi}{2\gamma}}}{\tilde{\lambda}M}\,\frac{\tilde{n}\,\tilde{p}_{rr}}{\sqrt{1-x^2}}$$

7. Tangential Stress Equation

$$\frac{(1-x^2)}{2x}\frac{\partial}{\partial x}\left(n_1u_1T_1+n_2u_2T_2\right)+\frac{(1-x^2)^3}{x\sqrt{2\pi\gamma}\,M}\frac{\partial}{\partial x}\left(n_1T_1^{3/2}-n_2T_2^{3/2}\right)$$

$$+\frac{3}{4}(1-x^2)\left(x^4-\frac{5x^2}{3}\right)\frac{\partial}{\partial x}\left(n_1u_1T_1-n_2u_2T_2\right)$$

$$+\left(\cot\theta+3\frac{\partial}{\partial\theta}\right)\left[\frac{n_1v_1T_1+n_2v_2T_2}{2}+\left(\frac{-15x+10x^3-3x^5}{16}\right)\left(n_1v_1T_1-n_2v_2T_2\right)\right]$$

$$+2\left(n_1u_1T_1+n_2u_2T_2\right)+\frac{1}{4}\left(-15x+10x^3-3x^5\right)\left(n_1u_1T_1-n_2u_2T_2\right)$$

$$=\frac{r_0\left(\frac{\tilde{\rho}}{\mu}\right)\tilde{p}_{\theta\theta}}{q_\infty\sqrt{1-x^2}} \qquad \text{(D. 7)}$$

$$=\frac{\sqrt{\frac{\pi}{2\gamma}}}{\tilde{\lambda}M}\,\frac{\tilde{n}\,\tilde{p}_{\theta\theta}}{\sqrt{1-x^2}}$$

8. Radial Heat Flux Equation

$$\frac{5}{4\gamma M^2}\frac{(1-x^2)}{x}\frac{\partial}{\partial x}\left(n_1T_1^2+n_2T_2^2\right)-\frac{5}{4\gamma M^2}x^2(1-x^2)\frac{\partial}{\partial x}\left(n_1T_1^2-n_2T_2^2\right)$$

$$+\frac{6(1-x^2)^2}{\sqrt{2\pi\gamma}\,M}\left\{\frac{(1+x^2)}{x}\frac{\partial}{\partial x}\left(n_1u_1T_1^{3/2}-n_2u_2T_2^{3/2}\right)+\left(n_1u_1T_1^{3/2}-n_2u_2T_2^{3/2}\right)\right\}$$

$$+\frac{3(1-x^2)^2}{\sqrt{2\pi\gamma}\,M}\left\{\left(\cot\theta+\frac{\partial}{\partial\theta}\right)\left(n_1v_1T_1^{3/2}-n_2v_2T_2^{3/2}\right)\right\}$$

$$=\frac{\sqrt{\frac{\pi}{2\gamma}}}{\tilde{\lambda}M}\left[-\frac{2}{3}\tilde{n}\tilde{Q}_r+\tilde{n}\tilde{q}_{\theta r}\tilde{P}_{rr}+\tilde{n}\tilde{q}_{\theta\theta}\tilde{P}_{r\theta}\right] \qquad (D.8)$$

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APPENDIX E

FREE MOLECULE FLOW LIMIT

1. Number density[1]

$$\frac{n_b\sqrt{T_b}}{n_\infty\sqrt{T_\infty}} = \exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right) + \sqrt{\frac{\pi\gamma}{2}}M\cos\theta\left[1+\operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}M\cos\theta\right)\right]$$

where

$$\operatorname{erf}(x) \equiv$$

at the stagnation point $\theta = 0$

$$\left.\frac{n_b\sqrt{T_b}}{n_\infty\sqrt{T_\infty}}\right|_{\theta=0} = \exp\left(-\frac{\gamma}{2}M^2\right) + \sqrt{\frac{\pi\gamma}{2}}M\left[1+\operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}M\right)\right]$$

low speed approximation $(M \to 0)$

$$\left.\frac{n_b\sqrt{T_b}}{n_\infty\sqrt{T_\infty}}\right|_{\theta=0} \cong 1 + \sqrt{\frac{\pi\gamma}{2}}M + O(M^2)$$

high speed approximation $(M \to \infty)$

$$\left.\frac{n_b\sqrt{T_b}}{n_\infty\sqrt{T_\infty}}\right|_{\theta=0} \cong \sqrt{2\pi\gamma}\,M + O(1)$$

Normal pressure[1]

The incident pressure $\tilde{p}_\infty$ is given by

$$\tilde{p}_\infty \equiv \frac{p_\infty}{n_\infty k T_\infty} = \sqrt{\frac{\gamma}{2\pi}}\, M\cos\theta \, \exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right)$$

$$+ \frac{1}{2}\left(1+\gamma M^2\cos^2\theta\right)\left[1+ \mathrm{erf}\left(\sqrt{\frac{\gamma}{2}}\, M\cos\theta\right)\right]$$

and the pressure due to the molecules reflected at the surface is

$$\tilde{p}_b \equiv \frac{p_b}{n_b k T_\infty} = \frac{1}{2}\sqrt{\frac{T_b}{T_\infty}}\, \exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right)$$

$$+ \frac{1}{2}\sqrt{\frac{\pi\gamma}{2}}\, M\cos\theta \sqrt{\frac{T_b}{T_\infty}}\left[1+ \mathrm{erf}\left(\sqrt{\frac{\gamma}{2}}\, M\cos\theta\right)\right]$$

for diffuse reemission from the surface

$$\tilde{p}^{FMF} = \tilde{p}_\infty + \tilde{p}_b$$

$$= \exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right)\left[\sqrt{\frac{\gamma}{2\pi}}\, M\cos\theta + \frac{1}{2}\sqrt{\frac{T_b}{T_\infty}}\right]$$

$$+ \left[1+\mathrm{erf}\left(\sqrt{\frac{\gamma}{2}}\, M\cos\theta\right)\right]\left[\frac{1}{2} + \frac{\gamma M^2}{2}\cos^2\theta + \frac{1}{2}\sqrt{\frac{T_b}{T_\infty}}\sqrt{\frac{\pi\gamma}{2}}\, M\cos\theta\right]$$

where

$$\mathrm{erf}(x) \equiv \frac{2x}{\sqrt{\pi}}\left(1 - \frac{x^2}{1!\,3} + \frac{x^4}{2!\,5} \cdots\cdots \right) ;\quad x^2 < \infty$$

$$\mathrm{erf}(\infty) = 1$$

low speed approximation $(M \to 0)$

$$\tilde{p}^{FMF} \cong \sqrt{\frac{\gamma}{2\pi}}\, M\cos\theta + \frac{1}{2}\sqrt{\frac{T_b}{T_\infty}}\left(\frac{1}{2} + \sqrt{\frac{\pi\gamma}{2}}\,\frac{M\cos\theta}{2}\sqrt{\frac{T_b}{T_\infty}}\right)\left(1 + \frac{2\sqrt{\frac{\gamma}{2}}\, M\cos\theta}{\sqrt{\pi}}\right)$$

for $\quad \epsilon \equiv \frac{T_b}{T_\infty} - 1 \quad$ where $\quad \epsilon = O(M)$

$$\tilde{p}^{FMF} \cong 1 + \frac{\epsilon}{4} + \sqrt{\frac{2\gamma}{\pi}}\, M\cos\theta\left(1 + \frac{\pi}{4}\right) + O(M^2)$$

high speed approximation $(M \to \infty)$

$$\tilde{p}^{FMF} \cong \gamma M^2 \cos^2\theta + \sqrt{\frac{\pi\gamma}{2}}\, M\cos\theta\sqrt{\frac{T_b}{T_\infty}} + 1$$

3. Shear stress[1]

For diffuse reemission the shear stress becomes

$$\tilde{p}_{r\theta}^{FMF} = -\sqrt{\frac{\gamma}{2\pi}}\, M\sin\theta\left\{\exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right) + \sqrt{\frac{\pi\gamma}{2}}\, M\cos\theta\left[1 + \operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}\, M\cos\theta\right)\right]\right\}$$

low speed approximation $(M \to 0)$

$$\tilde{p}_{r\theta}^{FMF} \cong -\sqrt{\frac{\gamma}{2\pi}}\, M\sin\theta + O(M^2)$$

high speed approximation $(M \to \infty)$

$$\tilde{p}_{r\theta}^{FMF} \cong -\gamma M^2 \sin\theta\cos\theta$$

4. Radial Heat Flux[32]

For diffuse reemission

$$\tilde{Q}_r^{FMF} \equiv \frac{\bar{Q}_r^{FMF}}{k n_\infty T_\infty q_\infty} = \frac{1}{\sqrt{2\pi\gamma}\,M}\Big\{\frac{1}{2}\exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right)$$

$$+\left[2\frac{T_b}{T_\infty} - \frac{\gamma M^2}{2} - \frac{5}{2}\right]\Big[\exp\left(-\frac{\gamma}{2}M^2\cos^2\theta\right)$$

$$+\sqrt{\frac{\pi\gamma}{2}}\,M\cos\theta\left(1+\operatorname{erf}\left(\sqrt{\frac{\gamma}{2}}\,M\cos\theta\right)\right)\Big]\Big\}$$

for the low speed approximation $(M \to 0)$

and $\epsilon \equiv \frac{T_b}{T_\infty} - 1$ where $\epsilon = O(M)$

$$\tilde{Q}^{FMF} \cong \sqrt{\frac{2}{\pi\gamma}}\,\frac{\epsilon}{M} + \left(\epsilon - \frac{1}{4}\right)\cos\theta + \sqrt{\frac{\gamma}{2\pi}}\,M\left[\left(\epsilon-\frac{1}{2}\right)\cos^2\theta - \frac{1}{2}\right]$$

$$-\frac{\gamma}{4}M^2\cos\theta + O(M^3)$$

APPENDIX F

SEPARATION OF VARIABLES – LOW SPEED FLOW

Using the well known result from Fourier analysis that an even function of θ in the interval $(-\pi, \pi)$ can be expanded in a cosine series and an odd function in the same interval can be expanded in a sine series, the following separation of variables is assumed

$$
\begin{aligned}
U^{(\pm)}(x,\theta) &= \sum_{n=0}^{\infty} A_n^{(\pm)}(x)\cos(n\theta) \\
V^{(\pm)}(x,\theta) &= \sum_{n=0}^{\infty} B_n^{(\pm)}(x)\sin(n\theta) \\
N^{(\pm)}(x,\theta) &= \sum_{n=0}^{\infty} C_n^{(\pm)}(x)\cos(n\theta) \\
t^{(\pm)}(x,\theta) &= \sum_{n=0}^{\infty} D_n^{(\pm)}(x)\cos(n\theta)
\end{aligned}
\tag{F.1}
$$

Substituting into the eight mo... ...nt e...uations yields the following ordinary differential equations.

Continuity

$$
\begin{aligned}
n=1:\quad & \frac{(1-x^2)}{x}\frac{d}{dx}\left(C_1^{(-)}+\tfrac{1}{2}D_1^{(-)}\right) + \sqrt{2\pi\gamma}\,M\left\{\frac{(1-x^2)}{2x}\frac{dA_1^{(+)}}{dx}\right. \\
& \left. -\frac{x^2(1-x^2)}{2}\frac{dA_1^{(-)}}{dx} + A_1^{(+)} + B_1^{(+)} + \frac{(x^3-3x)}{2}\left(A_1^{(-)}+B_1^{(-)}\right)\right\} = 0
\end{aligned}
\tag{F.2.a}
$$

$$
\begin{aligned}
n\neq 1:\quad & \frac{(1-x^2)}{x}\frac{d}{dx}\left(C_n^{(-)}+\tfrac{1}{2}D_n^{(-)}\right) + \sqrt{2\pi\gamma}\,M\left\{\frac{(1-x^2)}{2x}\frac{dA_n^{(+)}}{dx}\right. \\
& \left. -\frac{x^2(1-x^2)}{2}\frac{dA_n^{(-)}}{dx} + A_n^{(+)} + \frac{(x^3-3x)}{2}A_n^{(-)}\right\} = 0
\end{aligned}
\tag{F.2.b}
$$

Radial Momentum

$n=1$:
$$-\frac{1}{2x}\frac{d}{dx}\left(C_1^{(+)}+D_1^{(+)}\right) - \frac{x^2}{2}\frac{d}{dx}\left(C_1^{(-)}+D_1^{(-)}\right)$$
$$+\sqrt{\frac{2\gamma}{\pi}}\,M(1-x^2)\left\{\frac{(1+x^2)}{x}\frac{dA_1^{(-)}}{dx} + A_1^{(-)} + B_1^{(-)}\right\} = 0 \qquad \text{(F. 3. a)}$$

$n \neq 1$:
$$\frac{1}{2x}\frac{d}{dx}\left(C_n^{(+)}+D_n^{(+)}\right) - \frac{x^2}{2}\frac{d}{dx}\left(C_n^{(-)}+D_n^{(-)}\right)$$
$$+\sqrt{\frac{2\gamma}{\pi}}\,M(1-x^2)\left\{\frac{(1+x^2)}{x}\frac{dA_n^{(-)}}{dx} + A_n^{(-)}\right\} = 0 \qquad \text{(F. 3. b)}$$

Tangential Momentum

$n=1$:
$$-\frac{1}{2}\left(C_1^{(+)}+D_1^{(+)}\right) - \frac{(x^3-3x)}{4}\left(C_1^{(-)}+D_1^{(-)}\right)$$
$$+\sqrt{\frac{\gamma}{2\pi}}\,M(1-x^2)^2\left\{\frac{(1-x^2)}{x}\frac{dB_1^{(-)}}{dx} - A_1^{(-)} - B_1^{(-)}\right\} = 0 \qquad \text{(F. 4. a)}$$

$n \neq 1$:
$$n\left\{\frac{1}{2}\left(C_n^{(+)}+D_n^{(+)}\right) + \frac{(x^3-3x)}{4}\left(C_n^{(-)}+D_n^{(-)}\right)\right.$$
$$\left.+\sqrt{\frac{\gamma}{2\pi}}\,M(1-x^2)^2 A_n^{(-)}\right\} = 0 \qquad \text{(F. 4. b)}$$

Energy

$$\frac{d}{dx}\left(C_n^{(-)} - \frac{7}{2}D_n^{(-)}\right) = 0 \qquad \text{(F. 5)}$$

or

$$C_n^{(-)} - \frac{7}{2}D_n^{(-)} = \delta_n$$

Shear Stress

$$n=1:\quad -(1-x^2)^2\left(C_1^{(-)}+\frac{3}{2}D_1^{(-)}\right)+\sqrt{2\pi\delta}\,M\left\{\frac{(1-x^2)}{2x}\frac{dB_1^{(+)}}{dx}\right.$$

$$+\frac{3}{4}x^2(1-x^2)\left(x^2-\frac{5}{3}\right)\frac{dB_1^{(-)}}{dx}+\frac{x^3}{4}(5-3x^2)\left(A_1^{(-)}+B_1^{(-)}\right)$$

$$\left.-\frac{1}{2}\left(A_1^{(+)}+B_1^{(+)}\right)\right\}=-\sqrt{\frac{\pi\delta}{2}}\,\frac{M}{\lambda}(1-x^2)^{3/2}B_1^{(-)} \qquad \text{(F. 6. a)}$$

$$n\neq 1:\quad n\left\{-(1-x^2)^2\left(C_n^{(-)}+\frac{3}{2}D_n^{(-)}\right)\right.$$

$$\left.+\sqrt{2\pi\delta}\,M\left(\frac{x^3}{4}(5-3x^2)A_n^{(-)}-\frac{1}{2}A_n^{(+)}\right)\right\}=0$$

(F. 6. b)

Radial Stress

$$n=0:\quad 2\frac{(1-x^2)(1-x^4)}{x}\frac{d}{dx}\left(C_0^{(-)}+\frac{3}{2}D_0^{(-)}\right)+\sqrt{2\pi\delta}\,M\left\{\frac{3}{2}\frac{(1-x^2)}{x}\frac{dA_0^{(+)}}{dx}\right.$$

$$\left.-\frac{3x^4}{2}(1-x^2)\frac{dA_0^{(-)}}{dx}+A_0^{(+)}+\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)A_0^{(-)}\right\}$$

$$=\frac{-\pi}{\lambda}\left\{\frac{x\sqrt{1-x^2}}{2}\left(C_0^{(-)}+D_0^{(-)}\right)+\sqrt{\frac{2\delta}{\pi}}\,\frac{M}{3}\sqrt{1-x^2}\,(1+3x^2)A_0^{(-)}\right. \qquad \text{(F. 7. a)}$$

$$\left.-\frac{\sqrt{1-x^2}}{3}\sqrt{\frac{\pi\delta}{8}}\,M\left[A_0^{(+)}(0)-A_0^{(-)}(0)-2\tilde{u}_2(0,0)\right]\right\}$$

$$n=1:\quad 2\frac{(1-x^2)(1-x^4)}{x}\frac{d}{dx}\left(C_1^{(-)}+\frac{3}{2}D_1^{(-)}\right)+\sqrt{2\pi\delta}\,M\left\{\frac{3}{2}\frac{(1-x^2)}{x}\frac{dA_1^{(+)}}{dx}\right.$$

$$\left.-\frac{3x^4}{2}(1-x^2)\frac{dA_1^{(-)}}{dx}+A_1^{(+)}+B_1^{(+)}+\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)\left(A_1^{(-)}+B_1^{(-)}\right)\right\} \qquad \text{(F. 7. b)}$$

$$=\frac{-\pi}{\lambda}\left\{\frac{x\sqrt{1-x^2}}{2}\left(C_1^{(-)}+D_1^{(-)}\right)+\frac{M}{3}\sqrt{\frac{2\delta}{\pi}}\sqrt{1-x^2}\,(1+3x^2)A_1^{(-)}\right.$$

$$\left.-\frac{\sqrt{1-x^2}}{3}\sqrt{\frac{\pi\delta}{8}}\,M\left(A_1^{(+)}(0)-A_1^{(-)}(0)\right)\right\}$$

$$n>1: \quad \frac{2(1-x^2)(1-x^4)}{x}\frac{d}{dx}\left(C_n^{(-)}+\frac{3}{2}D_n^{(-)}\right)+\sqrt{2\pi\delta}\,M\left\{\frac{3}{2}\frac{(1-x^2)}{x}\frac{dA_n^{(+)}}{dx}\right.$$

$$-\frac{3x^4}{2}(1-x^2)\frac{dA_n^{(-)}}{dx}+A_n^{(+)}+\frac{3}{2}\left(x^5-\frac{5x^3}{3}\right)A_n^{(-)}\Big\}$$

$$=\frac{-\pi}{\tilde\lambda}\left\{\frac{x\sqrt{1-x^2}}{2}\left(C_n^{(-)}+D_n^{(-)}\right)+\sqrt{\frac{2\delta}{\pi}}\frac{M}{3}\sqrt{1-x^2}\,(1+3x^2)A_n^{(-)}\right. \tag{F. 7. c}$$

$$\left.-\frac{\sqrt{1-x^2}}{3}\sqrt{\frac{\pi\delta}{8}}\,M\left(A_n^{(+)}(0)-A_n^{(-)}(0)\right)\right\}$$

Tangential Stress

$$n\neq 1: \qquad B_n^{(\pm)}(x)=0 \tag{F. 8}$$

($B_1^{(\pm)}(x)$ satisfies the equation identically)

Radial Heat Flux

$$n=0: \quad x^2(1-x^2)\left[\frac{1}{5}\frac{d}{dx}\left(C_0^{(-)}+D_0^{(-)}\right)-\frac{dD_0^{(-)}}{dx}\right]$$

$$-\frac{(1-x^2)}{x}\left[\frac{1}{5}\frac{d}{dx}\left(C_0^{(+)}+D_0^{(+)}\right)-\frac{dD_0^{(+)}}{dx}\right] \tag{F. 9. a}$$

$$=\frac{-4\sqrt{1-x^2}}{15\,\tilde\lambda}\left\{-\frac{\delta_0}{2}+\sqrt{\frac{\pi\delta}{8}}\frac{M}{2}\left(A_0^{(+)}(0)-A_0^{(-)}(0)\right)-\sqrt{\frac{\pi\delta}{8}}\,M\,\tilde u_z(0,0)\right\}$$

$$n>0: \quad x^2(1-x^2)\left[\frac{1}{5}\frac{d}{dx}\left(C_n^{(-)}+D_n^{(-)}\right)-\frac{dD_n^{(-)}}{dx}\right]$$

$$-\frac{(1-x^2)}{x}\left[\frac{1}{5}\frac{d}{dx}\left(C_n^{(+)}+D_n^{(+)}\right)-\frac{dD_n^{(+)}}{dx}\right] \tag{F. 9. b}$$

$$=-\frac{4\sqrt{1-x^2}}{15\,\tilde\lambda}\left\{-\frac{\delta_n}{2}+\sqrt{\frac{\pi\delta}{8}}\frac{M}{2}\left(A_n^{(+)}(0)-A_n^{(-)}(0)\right)\right\}$$

If the series are truncated after $n = N$ terms, e.g.,

$$U^{(\pm)}(x,\theta) = \sum_{n=0}^{N} A_n^{(\pm)}(x) \cos(n\theta)$$

the number of equations becomes $(7N+5)$ in the $(6N+8)$ dependent variables;

$$\left(A_n^{(+)}, A_n^{(-)}, B_1^{(+)}, B_1^{(-)}, C_n^{(+)}, C_n^{(-)}, D_n^{(+)}, D_n^{(-)} \right), \quad n = 0, 1, \ldots, N$$

The boundary conditions become;

$X = 1$ $(r = \infty)$:

(i) $\tilde{u}_2(1,\theta) = -\cos\theta$ or $\begin{cases} A_n^{(+)}(1) - A_n^{(-)}(1) = 0 \,; & n \neq 1 \\ A_1^{(+)}(1) - A_1^{(-)}(1) = -2 \end{cases}$

(ii) $\tilde{V}_2(1,\theta) = \sin\theta$ or $B_1^{(+)}(1) - B_1^{(-)}(1) = 2$

(iii) $t_2(1,\theta) = 0$ or $D_n^{(+)}(1) - D_n^{(-)}(1) = 0$

(iv) $N_2(1,\theta) = 0$ or $C_n^{(+)}(1) - C_n^{(-)}(1) = 0$

$X = 0$ $(r = r_0)$:

(v) $\tilde{u}_1(0,\theta) = 0$ or $A_n^{(+)}(0) + A_n^{(-)}(0) = 0$

(vi) $\tilde{V}_1(0,\theta) = 0$ or $B_1^{(+)}(0) + B_1^{(-)}(0) = 0$

(vii) $t_1(0,\theta) = \epsilon$ or $\begin{cases} D_0^{(+)}(0) + D_0^{(-)}(0) = 2\epsilon \\ D_n^{(+)}(0) + D_n^{(-)}(0) = 0\,, \quad n>0 \end{cases}$

(viii) $$\left(N^{(-)} + \tfrac{1}{2} t^{(-)}\right)_{x=0} = -\sqrt{\frac{\pi r}{2}}\, M\, \tilde{u}_2(0,0) + F(\theta,\lambda)$$

or
$$C_0^{(-)}(0) + \tfrac{1}{2} D_0^{(-)}(0) = -\sqrt{\frac{\pi r}{2}}\, M\, \tilde{u}_2(0,0)$$

$$C_n^{(-)}(0) + \tfrac{1}{2} D_n^{(-)}(0) = F_n(\lambda) \quad ; \quad n>0$$

where
$$F(\theta,\lambda) \equiv \sum_{n=1}^{\infty} F_n(\lambda) \cos(n\theta)$$

APPENDIX G

COMPUTATION OF THE DRAG-LOW SPEED FLOW

The drag on the sphere is computed from the normal stress and the shear stress on the body. Since the boundary conditions require that the radial velocity vanish on the body the normal stress and the shear stress are given by $\bar{P}_{rr}$ and $\bar{P}_{r\theta}$. The drag becomes

$$D \equiv \int_A (\bar{P}_{rr})_{r=r_0} \cos\theta \, dA - \int_A (\bar{P}_{r\theta})_{r=r_0} \sin\theta \, dA \qquad \text{(G. 1)}$$

where $dA = r_0^2 \sin\theta \, d\theta \, d\phi$ for a sphere

Taking into account the symmetry with respect to ϕ , Equation (G. 1) can be integrated

$$D = 2\pi r_0^2 \left\{ \int_0^{\pi} (\bar{P}_{rr})_{r=r_0} \cos\theta \sin\theta \, d\theta - \int_0^{\pi} (\bar{P}_{r\theta})_{r=r_0} \sin^2\theta \, d\theta \right\} \qquad \text{(G. 2)}$$

The drag coefficient is defined by

$$C_D \equiv \frac{D}{\frac{1}{2} m n_\infty q_\infty^2 \pi r_0^2} \qquad \text{(G. 3)}$$

$$= \frac{4}{\gamma M^2} \left\{ \int_0^{\pi} (\tilde{P}_{rr})_{r=r_0} \cos\theta \sin\theta \, d\theta - \int_0^{\pi} (\tilde{P}_{r\theta})_{r=r_0} \sin^2\theta \, d\theta \right\}$$

where

$$\tilde{P}_{ij} \equiv \frac{\bar{P}_{ij}}{n_\infty k T_\infty}$$

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APPENDIX H

COMPUTATION OF THE AVERAGE HEAT TRANSFER

The average heat transfer for a sphere is given by

$$\overline{Q}_{AVE} \equiv \frac{1}{A}\int_A (\overline{Q}_r)_{r=r_0} dA \tag{H. 1}$$

where

$dA \equiv r_0^2 \sin\theta \, d\theta \, d\phi$ for a sphere

$(\overline{Q}_r)_{r=0} \equiv$ radial heat flux at the sphere surface

For spherical symmetry and in terms of the non-dimensional quantities

$$\tilde{Q}_{AVE} \equiv \frac{\overline{Q}_{AVE}}{n_0 k T_\infty q_\infty} = \frac{1}{2}\int_0^\pi (\tilde{Q}_r)_{r=r_0} \sin\theta \, d\theta \tag{H. 2}$$

where where $\tilde{Q}_r \equiv \dfrac{\overline{Q}_r}{n_0 k T_\infty q_\infty}$

The Stanton number is defined by

$$S_T \equiv \frac{\overline{Q}_{AVE}}{(T_b - T_A)\rho_\infty c_p q_\infty} = \frac{\tilde{Q}_{AVE}}{\left(\frac{T_b}{T_\infty} - \frac{T_A}{T_\infty}\right)\left(\frac{m c_p}{k}\right)} \tag{H. 3}$$

where $T_A \equiv$ adiabatic wall temperature

But for a perfect gas the specific heat coefficient at constant pressure is given by

$$c_p = \left(\frac{\gamma}{\gamma-1}\right)\frac{k}{m} \qquad \text{or} \qquad \frac{m c_p}{k} = \left(\frac{\gamma}{\gamma-1}\right)$$

This means that the definition of the Stanton number depends on the molecular model, e.g.,

$$\gamma \equiv \frac{z+2}{z} \qquad z \equiv \text{number of degrees of freedom}$$

In the derivation of the collision integral for the Boltzmann equation only the translational energy was taken into account. Therefore, all the results are rigorously valid only for a monatomic gas ($z = 3$). If the results are to be compared with experiments for polyatomic gases the Stanton number must be defined in a way which is independent of γ .

APPENDIX I

MOMENTS FOR HIGH SPEED FLOW

The definitions of the moments and the non-dimensional variables are the same as for the low speed approximation.

$$\tilde{n} \equiv \frac{n}{n_\infty} = \tilde{n}_2 + (1-x)\frac{\tilde{n}_1}{2} + \sqrt{\frac{\gamma}{2\pi}}\, M (1-x^2) \frac{\tilde{n}_1 \tilde{w}_1}{\sqrt{\tilde{T}_1}} \cos\theta$$

$$\tilde{n}\tilde{q}_r \equiv \frac{\overline{nq_r}}{n_\infty q_\infty} = \tilde{n}_2\tilde{u}_2 + (1-x^2)\frac{\tilde{n}_1\sqrt{\tilde{T}_1}}{\sqrt{2\pi\gamma}\, M} + \frac{(1-x^3)}{2}\tilde{n}_1\tilde{w}_1\cos\theta - \frac{g^{(1)}(x,\theta)}{n_\infty q_\infty}$$

$$\tilde{n}\tilde{q}_\theta = \tilde{n}_2\tilde{v}_2 - \frac{\tilde{n}_1\tilde{w}_1}{2}\left(1+\frac{x^3-3x}{2}\right)\sin\theta$$

$$\tilde{P}_{rr} \equiv \frac{\overline{P_{rr}}}{n_\infty k T_\infty} = \gamma M^2 \tilde{n}_2\tilde{u}_2^2 + \tilde{n}_2\tilde{T}_2 + (1-x^3)\frac{\tilde{n}_1\tilde{T}_1}{2}$$
$$+ \sqrt{\frac{2\gamma}{\pi}}\, M(1-x^4)\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1}\cos\theta + \frac{2m\, g^{(2)}(x,\theta)}{n_\infty k T_\infty}$$

$$\tilde{P}_{\theta\theta} = \gamma M^2 \tilde{n}_2\tilde{v}_2^2 + \tilde{n}_2\tilde{T}_2 + \frac{1}{2}\left(1+\frac{x^3-3x}{2}\right)\tilde{n}_1\tilde{T}_1$$
$$+ \sqrt{\frac{\gamma}{2\pi}}\, M(1-x^2)^2\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1}\cos\theta$$

$$\tilde{P}_{\phi\phi} = \tilde{n}_2\tilde{T}_2 + \left(1+\frac{x^3-3x}{2}\right)\frac{\tilde{n}_1\tilde{T}_1}{2} + \sqrt{\frac{\gamma}{2\pi}}\, M(1-x^2)^2\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1}\cos\theta$$

$$\tilde{P}_{r\theta} = \gamma M^2 \tilde{n}_2\tilde{u}_2\tilde{v}_2 - (1-x^2)^2\sqrt{\frac{\gamma}{2\pi}}\, M\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1}\sin\theta$$

$$\tilde{P}_{rrr} \equiv \frac{\overline{P_{rrr}}}{n_\infty k T_\infty q_\infty} = \gamma M^2 \tilde{n}_2\tilde{u}_2^3 + 3\tilde{n}_2\tilde{u}_2\tilde{T}_2 + \frac{\sqrt{\frac{2}{\pi\gamma}}}{M}(1-x^4)\tilde{n}_1\tilde{T}_1^{3/2}$$
$$+ \frac{3}{2}(1-x^5)\tilde{n}_1\tilde{w}_1\tilde{T}_1\cos\theta$$

$$\tilde{P}_{\theta\theta\theta} = \gamma M^2 \tilde{n}_2\tilde{v}_2^3 + 3\tilde{n}_2\tilde{v}_2\tilde{T}_2 - \frac{3}{2}\left[1+\frac{1}{8}\left(-15x+10x^3-3x^5\right)\right]\tilde{n}_1\tilde{w}_1\tilde{T}_1\sin\theta$$

$$\tilde{P}_{\phi\phi\phi} = 0$$

$$\tilde{P}_{r\theta\theta} = \gamma M^2\, \tilde{n}_2 \tilde{u}_2 \tilde{v}_2^2 + \tilde{n}_2 \tilde{u}_2 \tilde{T}_2 + \frac{(1-x^2)^2}{\sqrt{2\pi\gamma}\,M}\, \tilde{n}_1 \tilde{T}_1^{3/2}$$

$$+\frac{1}{2}\left(1+\frac{3x^5}{2}-\frac{5x^3}{2}\right)\tilde{n}_1\tilde{w}_1\tilde{T}_1\cos\theta$$

$$\tilde{P}_{r\phi\phi} = \tilde{n}_2 \tilde{u}_2 \tilde{T}_2 + \frac{1}{2}\left(1+\frac{3x^5}{2}-\frac{5x^3}{2}\right)\tilde{n}_1\tilde{w}_1\tilde{T}_1\cos\theta + \frac{(1-x^2)^2}{\sqrt{2\pi\gamma}\,M}\,\tilde{n}_1\tilde{T}_1^{3/2}$$

$$\tilde{P}_{rr\phi} = 0$$

$$\tilde{P}_{\theta\phi\phi} = \tilde{n}_2 \tilde{v}_2 \tilde{T}_2 - \frac{1}{2}\left[1+\frac{1}{8}\left(-15x+10x^3-3x^5\right)\right]\tilde{n}_1\tilde{w}_1\tilde{T}_1\sin\theta$$

$$\tilde{n}\tilde{p} \equiv \frac{1}{3}\sum_{i=1}^{3}\tilde{n}\,\tilde{P}_{ii} - \frac{\gamma M^2}{3}\sum_{i=1}^{3}(\tilde{n}\tilde{q}_i)^2$$

$$\tilde{n}\tilde{p}_{ij} = \tilde{n}\tilde{p}\,\delta_{ij} - \tilde{n}\,\tilde{P}_{ij} + \gamma M^2\left(\tilde{n}\tilde{q}_i\,\tilde{n}\tilde{q}_j\right)$$

APPENDIX J

MOMENT EQUATIONS-HIGH SPEED FLOW

Continuity

$$\frac{(1-X^2)}{X}\frac{\partial}{\partial X}(\tilde{n}_2\tilde{u}_2) + 2\tilde{n}_2\tilde{u}_2 + \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\tilde{n}_2\tilde{v}_2$$

$$+ \frac{(1-X^2)(1-X^3)}{2X}\frac{\partial}{\partial X}(\tilde{n}_1\tilde{w}_1)\cos\theta + \frac{(1-X^2)^2}{X\sqrt{2\pi\gamma}}\frac{\partial}{\partial X}\left(\frac{\tilde{n}_1\sqrt{\tilde{T}_1}}{M}\right)$$

$$- \frac{1}{2}\left(1 + \frac{X^3-3X}{2}\right)\sin\theta\frac{\partial}{\partial\theta}(\tilde{n}_1\tilde{w}_1) = 0 \qquad \text{(J. 1)}$$

where $g^{(1)}(x_i, \xi)$ is assumed proportional to $(1-X^2)$ to satisfy the free molecule flow limit as in the low speed case (Appendix D).

Radial Momentum

$$\gamma M^2\left\{\frac{(1-X^2)}{X}\frac{\partial}{\partial X}(\tilde{n}_2\tilde{u}_2^2) + 2\tilde{n}_2\tilde{u}_2^2 + \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\tilde{n}_2\tilde{u}_2\tilde{v}_2\right.$$

$$\left.-\tilde{n}_2\tilde{v}_2^2\right\} + \frac{(1-X^2)}{X}\frac{\partial}{\partial X}(\tilde{n}_2\tilde{T}_2) + \frac{(1-X^2)(1-X^4)}{2X}\frac{\partial}{\partial X}(\tilde{n}_1\tilde{T}_1)$$

$$+ \sqrt{\frac{2\gamma}{\pi}}M\left\{\frac{(1-X^2)(1-X^3)}{2X}\frac{\partial}{\partial X}(\tilde{n}_1\tilde{T}_1) + \frac{(1-X^2)^2(1+X^2)}{X}\frac{\partial}{\partial X}(\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1})\cos\theta\right.$$

$$\left.- \frac{(1-X^2)^2}{2}\sin\theta\frac{\partial}{\partial\theta}(\tilde{n}_1\tilde{w}_1\sqrt{\tilde{T}_1})\right\} = 0 \qquad \text{(J. 2)}$$

where $g^{(2)}(x, \theta)$ is assumed proportional to $(1-X^2)$ to satisfy the free molecule flow limit as in the low speed case (Appendix D).

Tangential Momentum

$$\gamma M^2 \left\{ \frac{(1-x^2)}{x} \frac{\partial}{\partial x}\left(\tilde{n}_2 \tilde{u}_2 \tilde{v}_2\right) + 3\tilde{n}_2 \tilde{u}_2 \tilde{v}_2 + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) \tilde{n}_2 \tilde{v}_2^2 \right\}$$

$$+ \sqrt{\frac{\gamma}{2\pi}}\, M \left\{ -\frac{(1-x^2)^3}{x} \frac{\partial}{\partial x}\left(\tilde{n}_1 \tilde{w}_1 \sqrt{\tilde{T}_1}\right) \sin\theta + (1-x^2)\cos\theta \frac{\partial}{\partial\theta}\left(\tilde{n}_1 \tilde{w}_1 \sqrt{\tilde{T}_1}\right) \right.$$

$$\left. + \frac{\partial}{\partial\theta}\left(\tilde{n}_2 \tilde{T}_2\right) + \frac{1}{2}\left(1 + \frac{x^3 - 3x}{2}\right) \frac{\partial}{\partial\theta}\left(\tilde{n}_1 \tilde{T}_1\right) = 0 \right.$$

(J. 3)

Radial Stress

$$\gamma M^2 \left\{ \frac{(1-x^2)}{x} \frac{\partial}{\partial x}\left(\tilde{n}_2 \tilde{u}_2^3\right) + 2\tilde{n}_2 \tilde{u}_2^3 + \left(\cot\theta + \frac{\partial}{\partial\theta}\right) \tilde{n}_2 \tilde{v}_2 \tilde{u}_2^2 \right.$$

$$\left. - 2\tilde{n}_2 \tilde{u}_2 \tilde{v}_2^2 \right\} + 3\frac{(1-x^2)}{x} \frac{\partial}{\partial x}\left(\tilde{n}_2 \tilde{u}_2 \tilde{T}_2\right)$$

$$+ \frac{(1-x^2)(1-x^4)}{x} \frac{\sqrt{\frac{2}{\pi\gamma}}}{M} \frac{\partial}{\partial x}\left(\tilde{n}_1 \tilde{T}_1^{3/2}\right) + 2\tilde{n}_2 \tilde{u}_2 \tilde{T}_2$$

(J.4)

$$+ \frac{3}{2} \frac{(1-x^2)(1-x^5)}{x} \cos\theta \frac{\partial}{\partial x}\left(\tilde{n}_1 \tilde{w}_1 \tilde{T}_1\right)$$

$$+ \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\left(\tilde{n}_2 \tilde{v}_2 \tilde{T}_2\right) - \left(1 + \frac{3x^5 - 5x^3}{2}\right) \frac{\sin\theta}{2} \frac{\partial}{\partial\theta}\left(\tilde{n}_1 \tilde{w}_1 \tilde{T}_1\right)$$

$$= \frac{\sqrt{\frac{\pi}{2\gamma}}}{\tilde{\lambda} M} \frac{\tilde{n} \tilde{\rho}_{rr}}{\sqrt{1-x^2}}$$

APPENDIX K

SEPARATION OF VARIABLES – HIGH SPEED FLOW

Substituting the separation of variables assumed in Section V.D, the four moment equations from Appendix J become;

Continuity

$$\cos\theta \left\{ \frac{(1-x^2)}{x} \frac{d}{dx} N_2 U_2 + \frac{(1-x^2)(1-x^2)}{2x} \frac{dU_1}{dx} + 2N_2(U_2+V_2) \right\} \qquad \text{(K. 1)}$$

$$+ \frac{(1-x^2)^2}{x\sqrt{2\pi\gamma}} \frac{dN_1}{dx} = 0$$

Radial Momentum

$$\gamma M^2 \left\{ \cos^2\theta \left[\frac{(1-x^2)}{x} \frac{d}{dx} N_2 U_2^2 + 2N_2U_2(U_2+V_2) \right] \right.$$

$$\left. - \sin^2\theta \, N_2 V_2 (U_2+V_2) \right\} + \frac{(1-x^2)}{x} \frac{dN_2}{dx} \qquad \text{(K. 2)}$$

$$+ \frac{(1-x^2)(1-x^2)}{2x} M\sqrt{T_1} \frac{dN_1}{dx} + \cos\theta \sqrt{\frac{2\gamma}{\pi}} M\sqrt{T_1} \frac{(1-x^2)(1-x^2)}{x} \frac{dU_1}{dx} = 0$$

Tangential Momentum

$$\sin\theta \cos\theta \, \gamma M^2 \left\{ \frac{(1-x^2)}{x} \frac{d}{dx} N_2U_2V_2 + 3N_2V_2(U_2+V_2) \right\}$$

$$-\sin\theta \sqrt{\frac{\gamma}{2\pi}} M\sqrt{T_1} \frac{(1-x^2)^3}{x} \frac{dU_1}{dx} = 0 \qquad \text{(K. 3)}$$

Radial Stress

$$\gamma M^2 \left\{ \cos^3\theta \left[\frac{(1-x^2)}{x} \frac{d}{dx} N_2 U_2^3 + 2 N_2 U_2^2 (U_2 + V_2) \right] \right.$$

$$\left. - 2 \sin^2\theta \cos\theta \, N_2 U_2 V_2 \, (U_2 + V_2) \right\}$$

$$+ \cos\theta \left\{ \frac{3(1-x^2)}{x} \frac{d}{dx} N_2 U_2 + \frac{3}{2} \frac{(1-x^2)(1-x^5)}{x} \tilde{T}_1 \frac{dU_1}{dx} + 2 N_2 (U_2 + V_2) \right\}$$

$$+ \frac{(1-x^2)(1-x^4)}{x} \sqrt{\frac{2}{\pi\gamma}} \, \tilde{T}_1 \frac{dN_1}{dx} = \frac{\sqrt{\frac{\pi}{2\gamma}}}{\tilde{\lambda} M} \frac{\tilde{n} \tilde{\rho}_{rr}}{\sqrt{1-x^2}} \tag{K. 4}$$

APPENDIX L

COMPUTATION OF THE DRAG — HIGH SPEED FLOW

In the hypersonic limit the drag of a sphere in FMF is determined by integrating over the "front" half of the sphere from $\theta = 0$ to $\theta = \frac{\pi}{2}$. For example if the drag coefficient is written as the sum

$$C_D = C_D^{(1)} + C_D^{(2)} \tag{L. 1}$$

where

$C_D^{(1)} \equiv$ contribution to the drag by integrating over $0 \leq \theta \leq \frac{\pi}{2}$

$C_D^{(2)} \equiv$ contribution to the drag by integrating over $\frac{\pi}{2} \leq \theta \leq \pi$

It can easily be shown that in the FMF limit

$$\operatorname*{limit}_{\lambda \to \infty} C_D^{(2)} = 0$$

This result is analogous to the Newtonian impact model in hypersonic continuum flow where the drag is determined by the surfaces on which the free stream impinges.

The normal stress on the body is given by $(-\bar{P}_{rr})_{x=0}$ and and the shear stress is given by $(\bar{P}_{r\theta})_{x=0}$. The drag is then given by

$$D = 2\pi r_0^2 \left\{ \int_0^{\frac{\pi}{2}} (\bar{P}_{rr})_{x=0} \cos\theta \sin\theta \, d\theta - \int_0^{\frac{\pi}{2}} (\bar{P}_{r\theta})_{x=0} \sin^2\theta \, d\theta \right\} \tag{L. 2}$$

Although the present theory is, strictly speaking, valid only near the stagnation point the parametric functions appearing in the normal stress and shear stress will be evaluated at the stagnation point then integrated over the surface to obtain the drag. This procedure is similar to Lees correction of the Newtonian flow pressure coefficient (Reference 34). The normal stress on the body is

$$
\begin{aligned}
(\tilde{P}_{rr})_{x=0} &\equiv \frac{(P_{rr})_{x=0}}{n_\infty k T_\infty} \\
&= \gamma M^2 N_2(0)\, U_2^2(0) \cos^2\theta + N_2(0) \\
&\quad + \frac{1}{2} M \sqrt{\tilde{T}_1}\, N_1(0) + \frac{2m g^{(2)}(0,\theta)}{n_\infty k T_\infty}
\end{aligned}
\qquad \text{(L. 3)}
$$

and the shear stress is given by

$$
(\tilde{P}_{r\theta})_{x=0} \equiv \frac{(P_{r\theta})_{x=0}}{n_\infty k T_\infty} = \gamma M^2 N_2(0)\, U_2(0)\, V_2(0) \sin\theta \cos\theta \qquad \text{(L. 4)}
$$

where

$$
N_1(0) = -\sqrt{2\pi\gamma}\; N_2(0)\, U_2(0)
$$

and

$$
\frac{2m g^{(2)}(0,\theta)}{n_\infty k T_\infty} = -\sqrt{\frac{\pi\gamma}{2}}\, M \sqrt{\tilde{T}_1} \left[U_2(0)\cos\theta - U_2(0) \right] N_2(0)
$$

Substituting these equations into Equation (L. 2) gives for the drag coefficient

$$
\begin{aligned}
C_D &\equiv \frac{D}{\frac{1}{2} m n_\infty q_\infty^2 \pi r_0^2} \\
&= N_2(0)\, U_2(0) \left[U_2(0) - V_2(0) \right] - \frac{2}{3} \sqrt{\frac{2\pi}{\gamma}}\, \frac{\sqrt{\tilde{T}_1}}{M}\, N_2(0)\, U_2(0) \\
&\quad + \frac{2 N_2(0)}{\gamma M^2}
\end{aligned}
\qquad \text{(L. 5)}
$$

In the FMF limit

$$C_D^{FMF} = 2 + \frac{2}{3}\sqrt{\frac{2\pi}{\gamma}}\frac{\sqrt{\tilde{T}_1}}{M} + \frac{2}{\gamma M^2} \tag{L. 6}$$

which is the correct value given in Reference 1. In the continuum limit the drag coefficient obtained from the Newtonian impact theory gives

$$C_D = 1$$

which is reduced to

$$C_D = \frac{3}{4}$$

if a correction is made for the surface curvature.

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APPENDIX M

PHYSICAL INTERPRETATION OF $G(\vec{x},\vec{\xi})$

Chimieleski and Ferzinger[36] show that the local Maxwellian distribution function

$$f_i^{(M)} = \frac{n_i}{(2\pi RT_i)^{3/2}} \exp\left\{-\frac{|\vec{\xi}-\vec{u}_i|^2}{2RT_i}\right\} \tag{M. 1}$$

can be represented in the sense of generalized functions by an expansion in terms of the delta function and its derivatives and a power series in $T_i^{1/2}$.

$$f_i^{(M)} = n_i \sum_{n=0}^{\infty} (-1)^n \frac{\delta^n(\vec{C}_i)(2RT_i)^{\frac{n}{2}}(n+1)}{2^n \Gamma(\frac{n}{2}+1)} \tag{M. 2}$$

where $\vec{C}_i = \vec{\xi} - \vec{u}_i$

In the present study the distribution function is assumed to be composed of two parts; the first a generalized Maxwellian distribution function and the second an expansion in terms of derivatives of the delta function. The representation of the distribution function in the present study is given by Equations (3. 1) and (3. 2) and Figure M. 1

$$f = f_1 = f_1^{(M)}(r,\theta;\vec{\xi}) + G(r,\theta;\vec{\xi}) \quad \text{in region (1)} \tag{M. 3}$$

and

$$f = f_2 = f_2^{(M)}(r,\theta;\vec{\xi}) + G(r,\theta;\vec{\xi}) \quad \text{in region (2)} \tag{M. 4}$$

Both of the functions $f_1^{(M)}$ and $f_2^{(M)}$ are given by

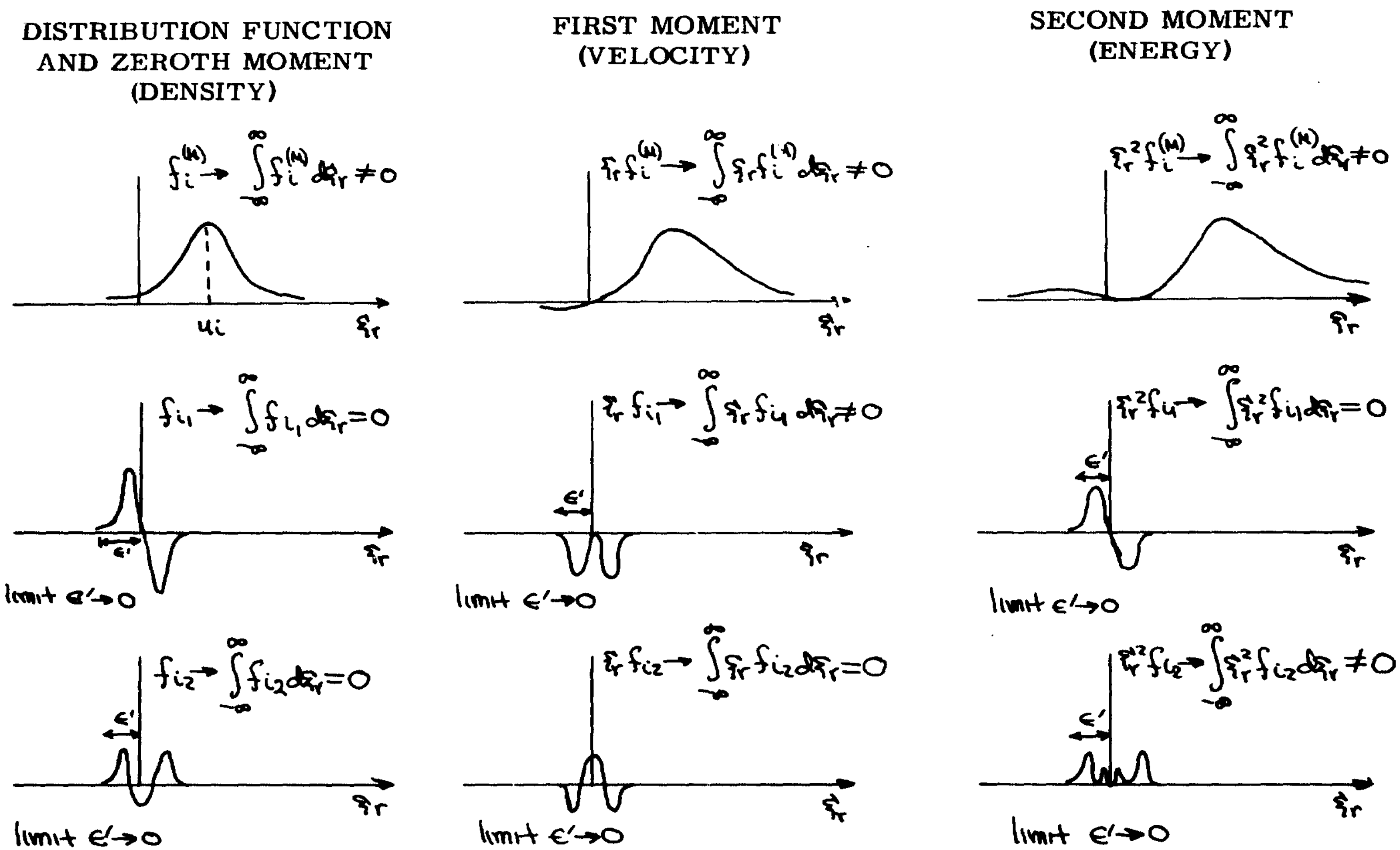

DISTRIBUTION FUNCTION AND ZEROTH MOMENT (DENSITY)
FIRST MOMENT (VELOCITY)
SECOND MOMENT (ENERGY)
limit ε'→0
limit ε'→0
limit ε'→0
limit ε'→0
limit ε'→0
limit ε'→0

FIGURE M. 1

$$f_i^{(M)}(r,\theta;\vec{\xi}) = \frac{n_i(r,\theta)}{(2\pi R T_i(r,\theta))^{3/2}} \exp\left\{-\frac{\left[(\xi_r - U_i(r,\theta))^2 + (\xi_\theta - V_i(r,\theta))^2 + \xi_\phi^2\right]}{2RT_i(r,\theta)}\right\} \quad \text{(M. 5)}$$

$i = 1, 2$

The function $G(r,\theta;\vec{\xi})$ appearing in both f_1 and f_2 are assumed to take the form

$$G(r,\theta;\vec{\xi}) = \sum_{n=0}^{\infty} \sum_{i=1}^{3} g_i^{(n)}(r,\theta)\, \delta(\xi_j)\, \delta(\xi_k) \frac{d^n}{d\xi_i^n}[\delta(\xi_i)] \quad \text{(M. 6)}$$

subject to the following restriction

$$\iiint_{-\infty}^{\infty} G(r,\theta;\vec{\xi})\, d\vec{\xi} = 0 \quad \text{(M. 7)}$$

This last condition guarentees that $G(r,\theta;\vec{\xi})$ does not contribute to the total number of particles in physical space; i. e., no matter how $G(r,\theta;\vec{\xi})$ affects the flow field, it does not act as a mass source. Equation (M. 7) is always satisfied if the first term $g_i^{(0)}(r,\theta)$ in Equation (M. 6) is taken to be zero. The summation for $G(r,\theta;\vec{\xi})$ in Equation (M. 6) then extends from n=1 to infinity.

Since it is difficult to sketch the distribution function in all three velocity coordinates, the following one dimensional velocity distribution will be utilized to illustrate the physical meaning of the assumed form of the distribution function. Taking the radical particle velocity, ξ_r , to be the single independent variable in velocity space, one may write the one dimensional distribution function in the form

$$f_i(r,\theta;\xi_r) = S_i(r,\theta) \exp\left\{-\frac{[\xi_r - U_i(r,\theta)]^2}{2RT_i(r,\theta)}\right\} + G(r,\theta;\xi_r) \quad \text{(M. 8)}$$

where

$$S_i(r,\theta) \equiv \text{ function of } T_i(r,\theta) \text{ and } n_i(r,\theta)$$

$$G(r,\theta,\xi_r) = g^{(0)}(r,\theta)\,\delta(\xi_r) + g^{(1)}(r,\theta)\,\frac{d}{d\xi_r}\delta(\xi_r) + g^{(2)}(r,\theta)\,\frac{d^2}{d\xi_r^2}\delta(\xi_r) + \cdots \tag{M. 9}$$

From Equation (M. 7) the first term of $G(r,\theta;\xi_r)$ is equal to zero and f_i becomes

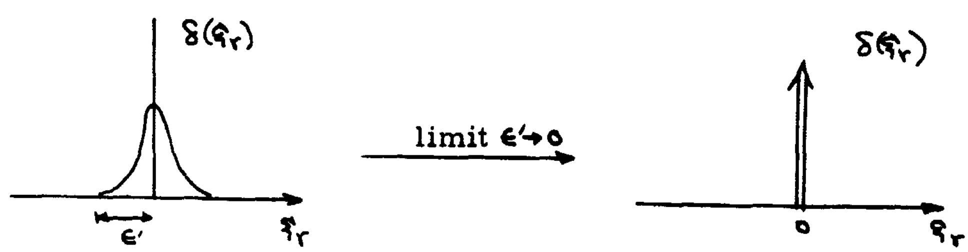
$$f_i(r,\theta,\xi_r) = S_i(r,\theta)\exp\left\{-\frac{[\xi_r - u_i(r,\theta)]^2}{2RT_i(r,\theta)}\right\} + g^{(1)}(r,\theta)\,\frac{d}{d\xi_r}\delta(\xi_r) + g^{(2)}(r,\theta)\,\frac{d^2}{d\xi_r^2}\delta(\xi_r) + \cdots$$

$$\equiv f_i^{(N)} + f_{i_1} + f_{i_2} + \cdots \tag{M. 10}$$

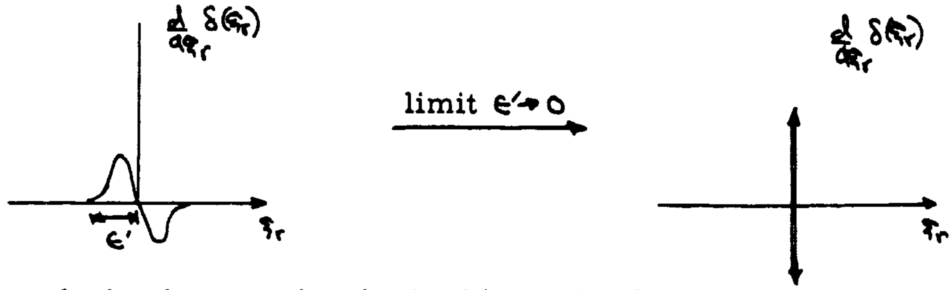
Before any interpretation of the distribution function is attempted, a graphical representation of the delta function and its derivatives is instructive. Although the delta function (or for that matter, any symbolic function) is defined only by its integral property

$$\int_{-\infty}^{\infty} F(x)\,\delta(x-x_0)\,dx = F(x_0) \tag{M. 11}$$

it can be interpreted as the limit of the following schematic representation as ϵ' tends to zero.

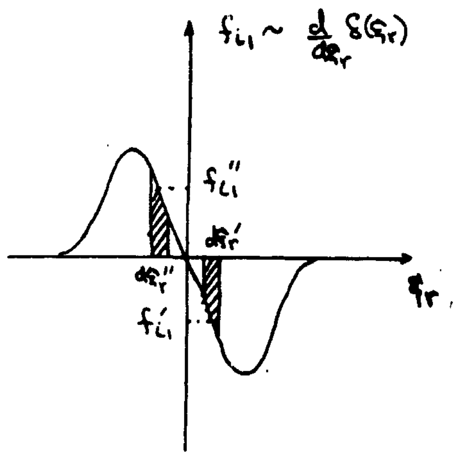


The first derivative is the dipole given by



Higher derivatives can be obtained by a simple extension of this procedure. The first three terms of the distribution function $f_i^{(w)}$, f_{i_1} and f_{i_2} are given schematically in the first column in the Table 1 at the end of this appendix.

One can interpret the contribution from $G(r,\theta;\xi_r)$ to the distribution function f_i from a physical viewpoint in the following way. The representation of the dipole is given by



The incremental velocity element $d\xi_r''$ is chosen in the following way

$$
\begin{aligned}
d\xi_r'' &= d\xi_r' \\
f_{i_1}'' &= -f_{i_1}'
\end{aligned}
\qquad \text{(M. 12)}
$$

The number of particles described by f_{i_1} at a position in physical space is given by

$$N = \int_{-\infty}^{\infty} f_{i_1}(\bar{x}, \xi_r)\, d\xi_r \tag{M. 13}$$

Therefore, in the incremental element $d\xi_r'$

$$dN' \cong f_{i_1}'(\bar{x}, \xi_r')\, d\xi_r' \tag{M. 14}$$

and in the element $d\xi_r''$

$$dN'' \cong f_{i_1}''(\bar{x}, \xi_r'')\, d\xi_r'' \tag{M. 15}$$

However the conditions specified by Equation (M. 13) require that the same $dN' + dN''$ is zero, i. e.,

$$\begin{aligned} dN' + dN'' &\cong f_{i_1}'\, d\xi_r' + f_{i_1}''\, d\xi_r'' \\ &\cong -f_{i_1}''\, d\xi_r'' + f_{i_1}''\, d\xi_r'' = 0 \end{aligned}$$

Therefore, one can interpret the dipole as contributing an equal number of particles dN' and dN'' with small velocities whose sum is always zero. In this way the dipole does not contribute to the density in physical space. Although the contribution from the dipole to the total number of particles is zero it does make a net contribution to the mean velocity since the integral

$$\int_{-\infty}^{\infty} \xi_r f_{i_1}\, d\xi_r$$

is not zero. A similar interpretation can be given to the second derivative of the delta function except that its contribution to the energy is nonzero; i. e.,

$$\int_{-\infty}^{\infty} \xi_r^2 f_{i_2}\, d\xi_r \neq 0$$

In terms of the moments of the distribution function, the physical interpretation is clear. From the first column in Table 1 the function $G(r,t;\xi_r)$ does not contribute to the total number of particles, in the second column only the Maxwellian, $f_i^{(M)}$, and the second term, f_{i1}, will contribute to the mean velocity. Therefore, the dipole term acts as a momentum source. In the last column the dipole does not contribute to the moment corresponding to the translational energy but the third term, f_{i2}, does contribute and can be interpreted as an energy source.

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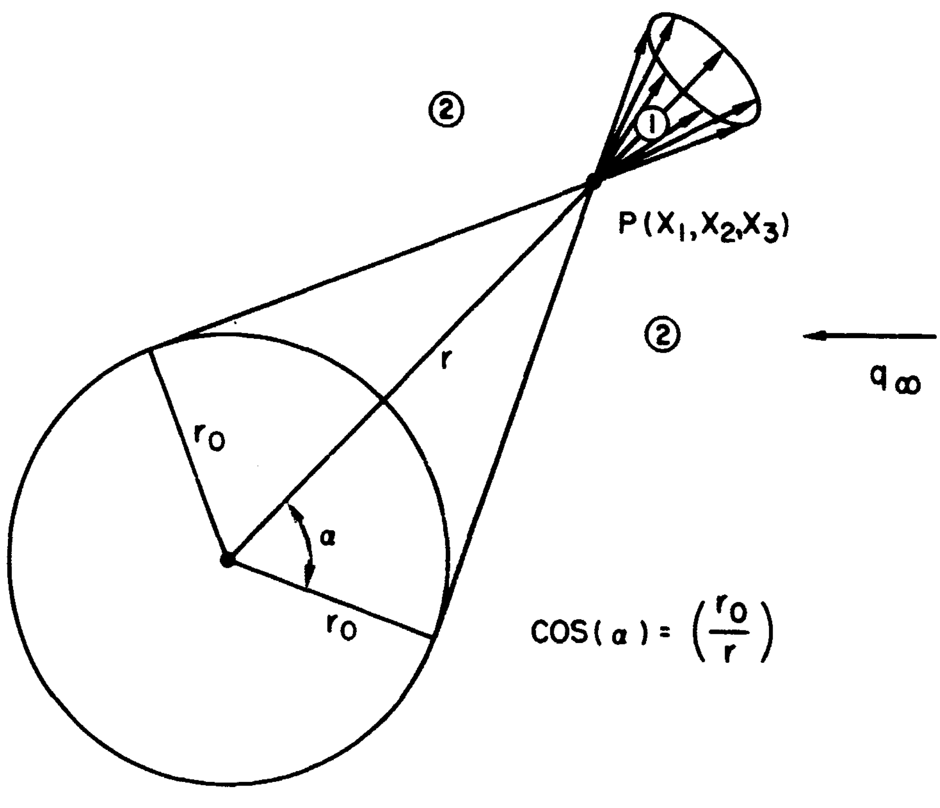
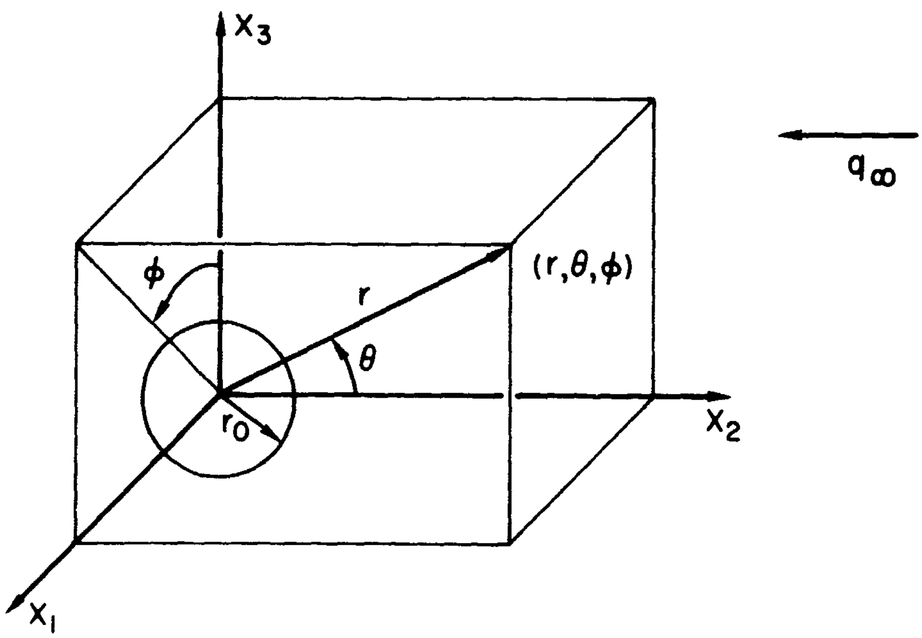


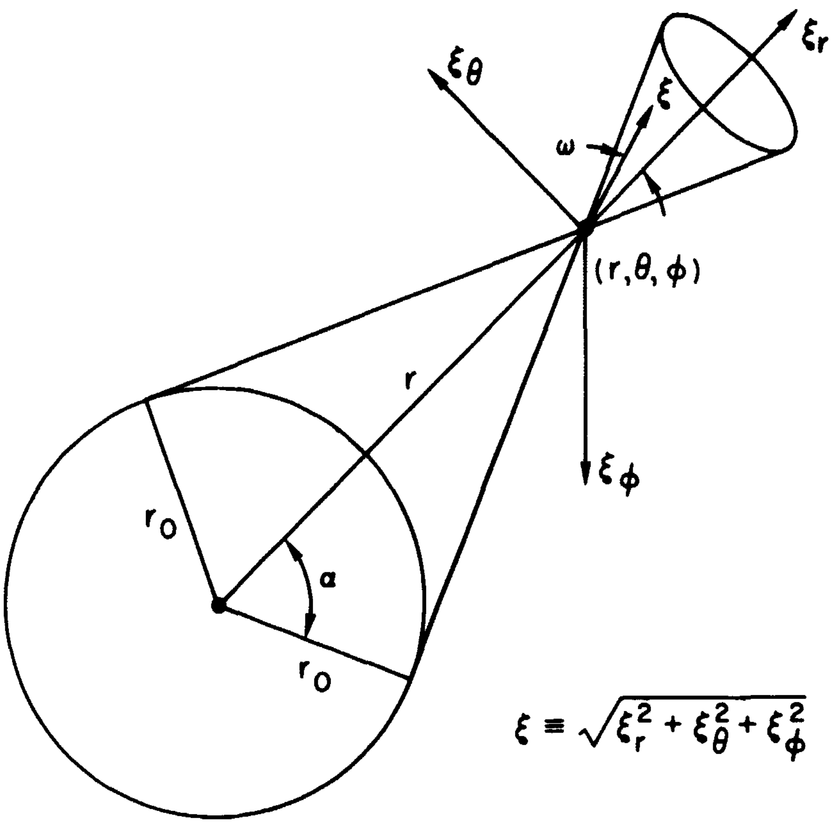
DIVISION OF VELOCITY SPACE

FIGURE 1



PHYSICAL SPACE

FIGURE 2

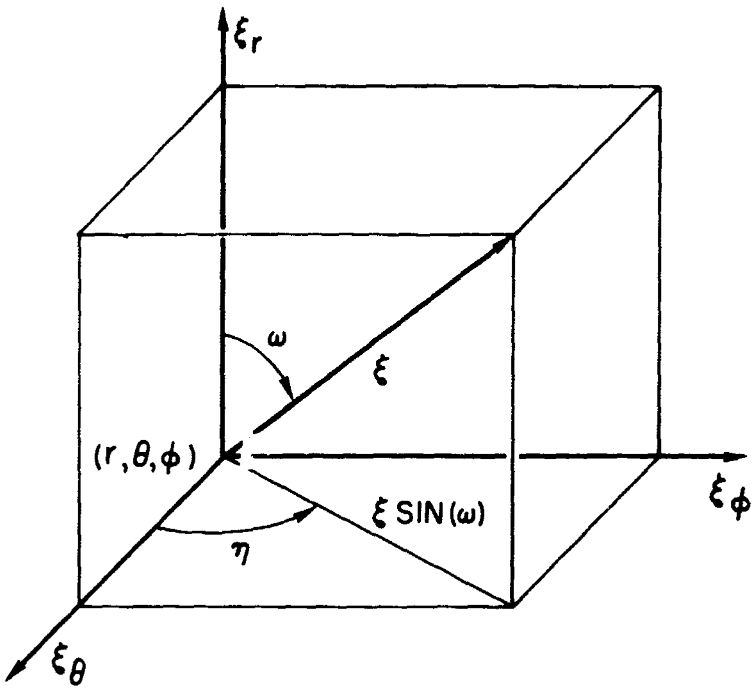


*ω IS THE CONICAL ANGLE WEIGHTING FUNCTION FOR THE DISTRIBUTION FUNCTION

$$f = \begin{cases} f_1 \text{ FOR } 0 < \omega < \pi/2 - \alpha \\ f_2 \text{ FOR } \pi/2 - \alpha < \omega < \pi \end{cases}$$

VELOCITY SPACE $(\xi_r, \xi_\theta, \xi_\phi)$

FIGURE 3



$$\xi \equiv \sqrt{\xi_r^2+\xi_\theta^2+\xi_\phi^2}$$

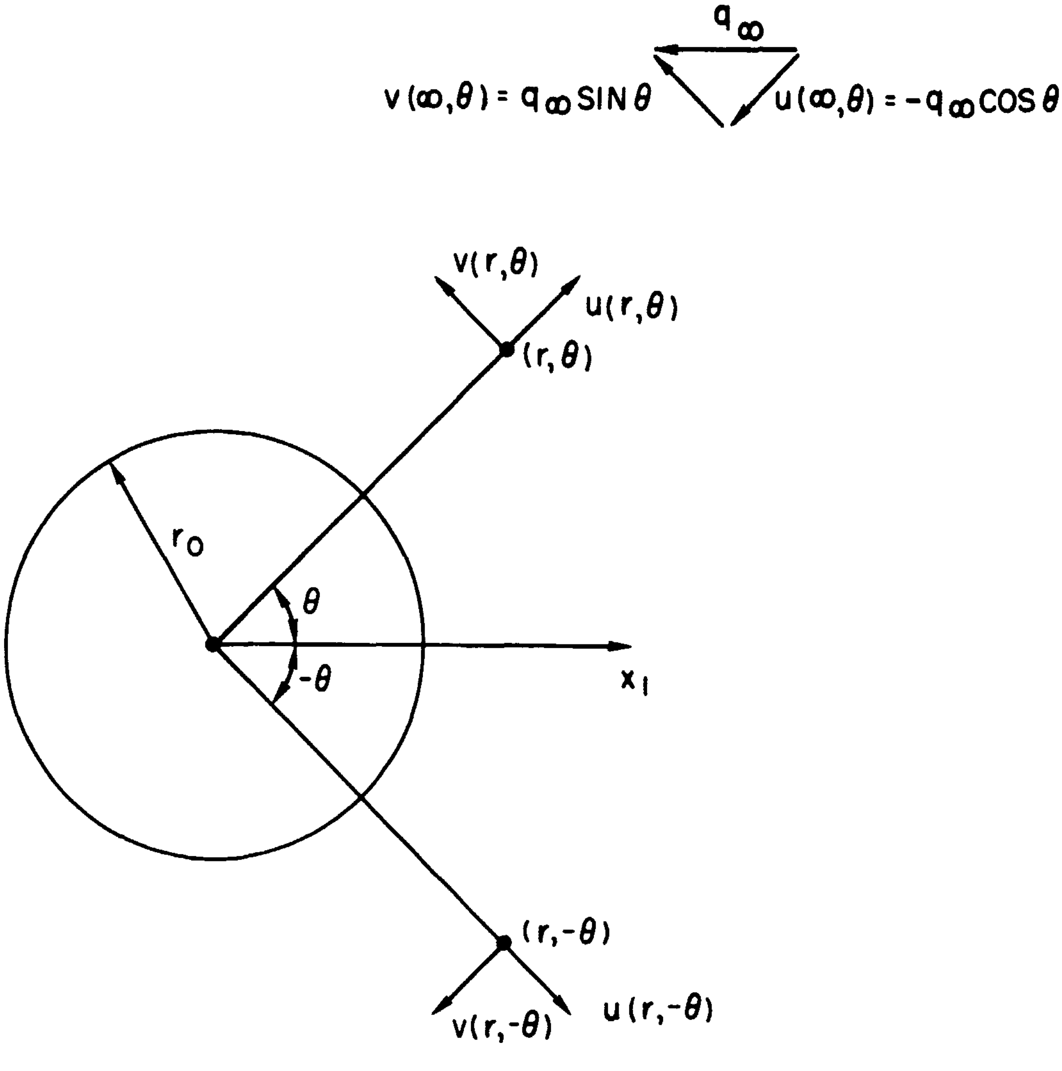
$$\omega \equiv \mathrm{COS}^{-1}\left[\xi_r \Big/ \sqrt{\xi_r^2+\xi_\theta^2+\xi_\phi^2}\,\right]$$

$$\eta \equiv \mathrm{TAN}^{-1}\left[\xi_\phi/\xi_\theta\right]$$

$$d\xi \equiv d\xi_r\,d\xi_\theta\,d\xi_\phi = \xi^2\,\mathrm{SIN}(\omega)\,d\xi\,d\omega\,d\eta$$

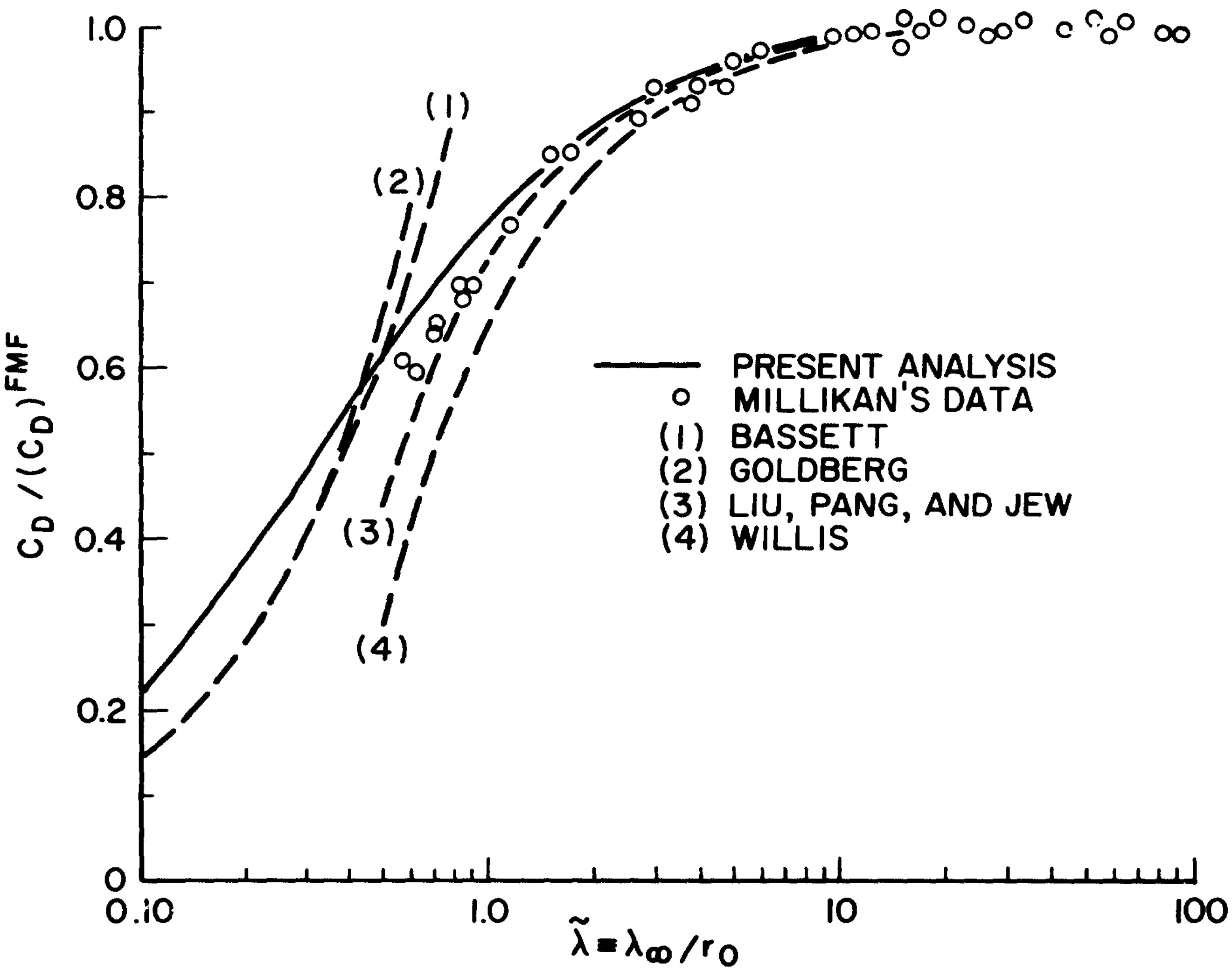
VELOCITY SPACE IN SPHERICAL COORDINATES (ξ, ω, η)

FIGURE 4



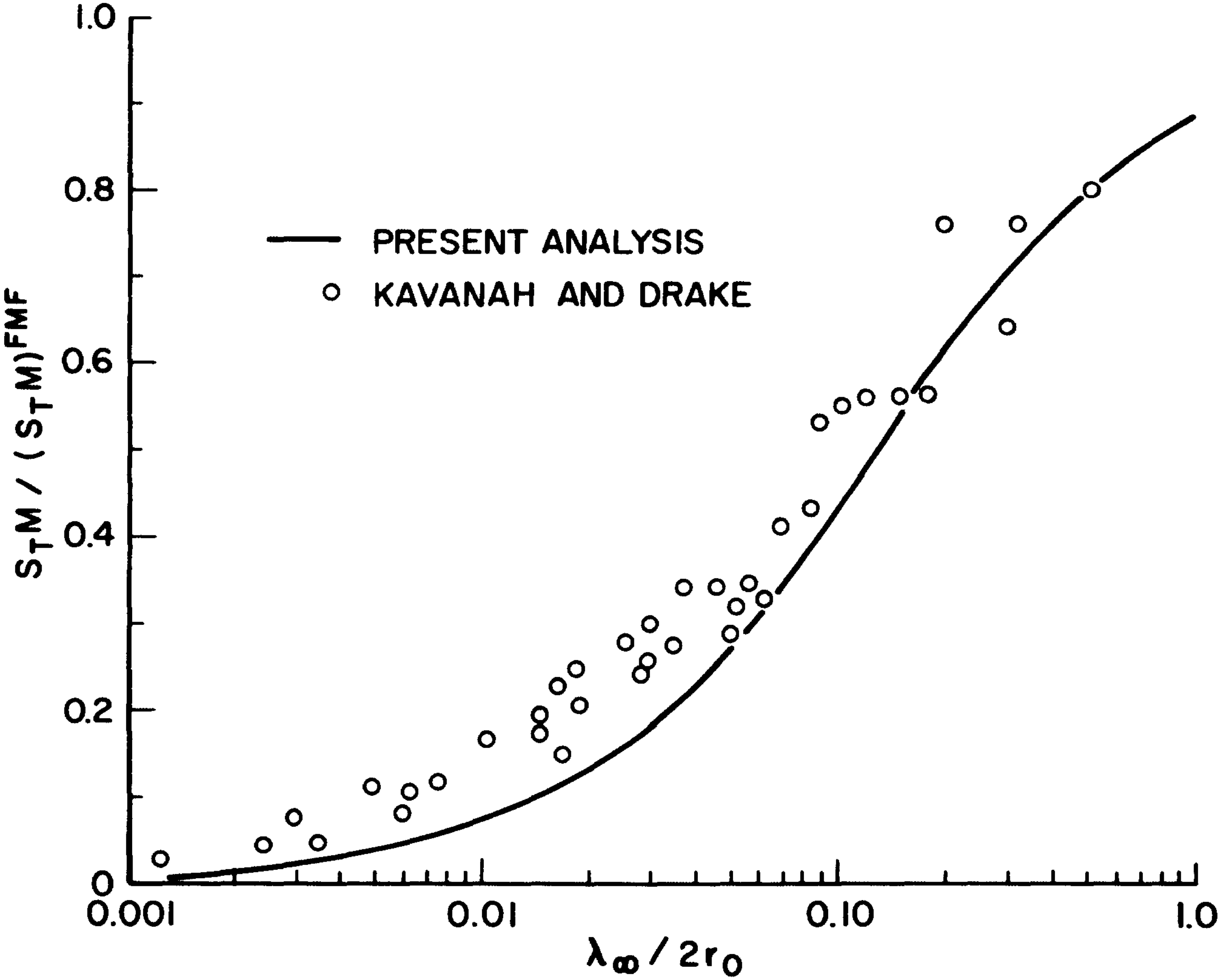
SEPARATION OF VARIABLES – LOW SPEED FLOW

FIGURE 5

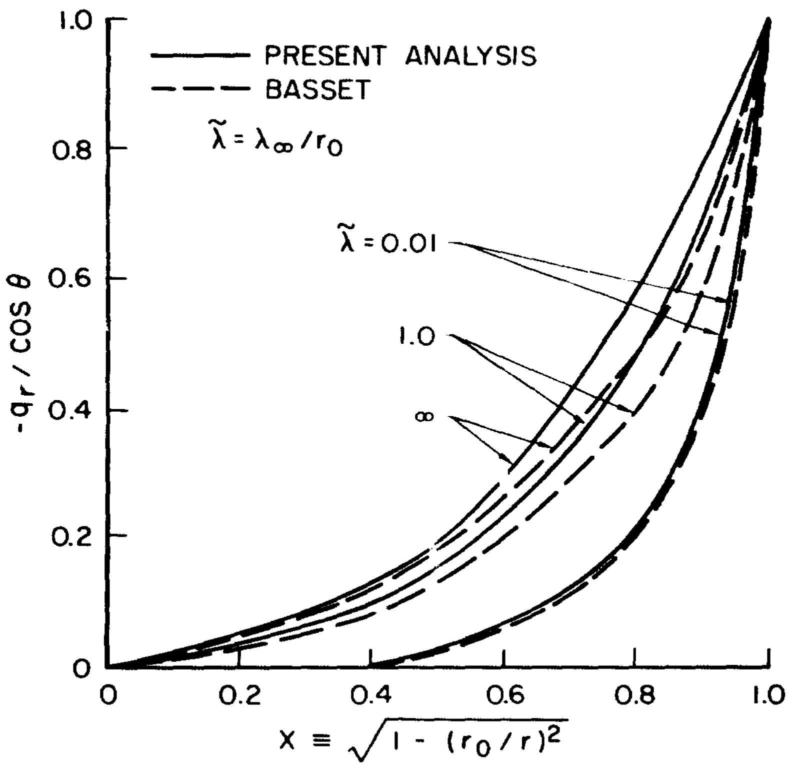


SPHERE DRAG – LOW SPEED FLOW

FIGURE 6

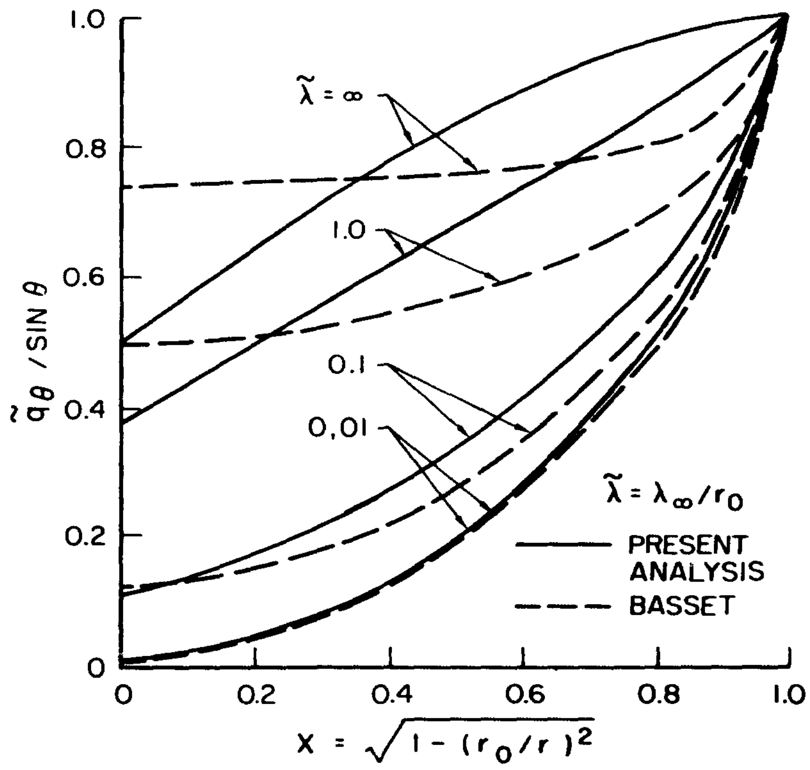


HEAT TRANSFER TO A SPHERE – LOW SPEED FLOW
FIGURE 7



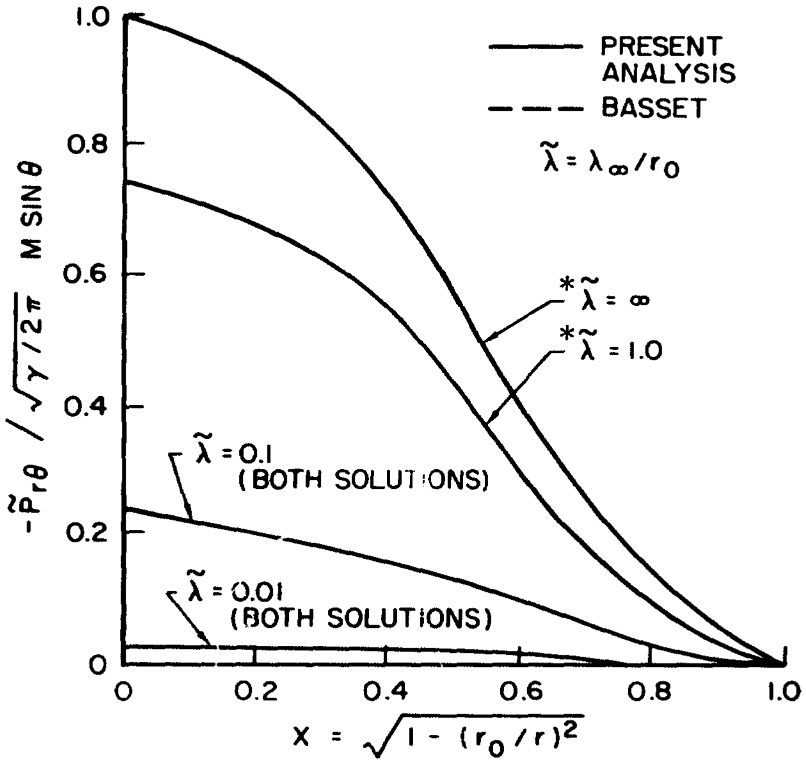
RADIAL VELOCITY – LOW SPEED FLOW

FIGURE 8



TANGENTIAL VELOCITY – LOW SPEED FLOW

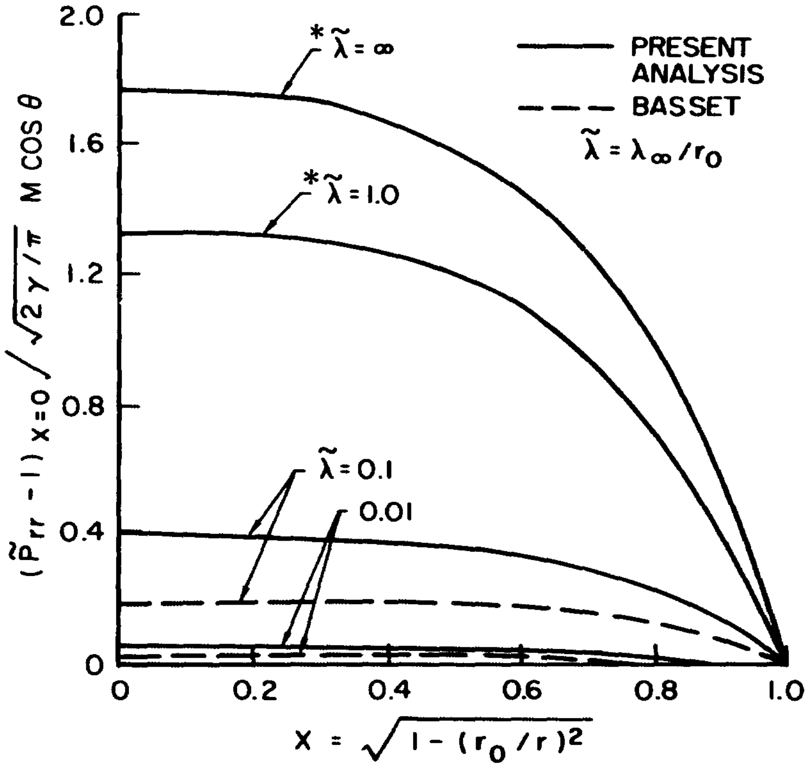
FIGURE 9



*BASSET'S SOLUTION IS ZERO FOR $\tilde{\lambda}=1$ AND CHANGES SIGN FOR $\tilde{\lambda}>1$

SHEAR STRESS – LOW SPEED FLOW

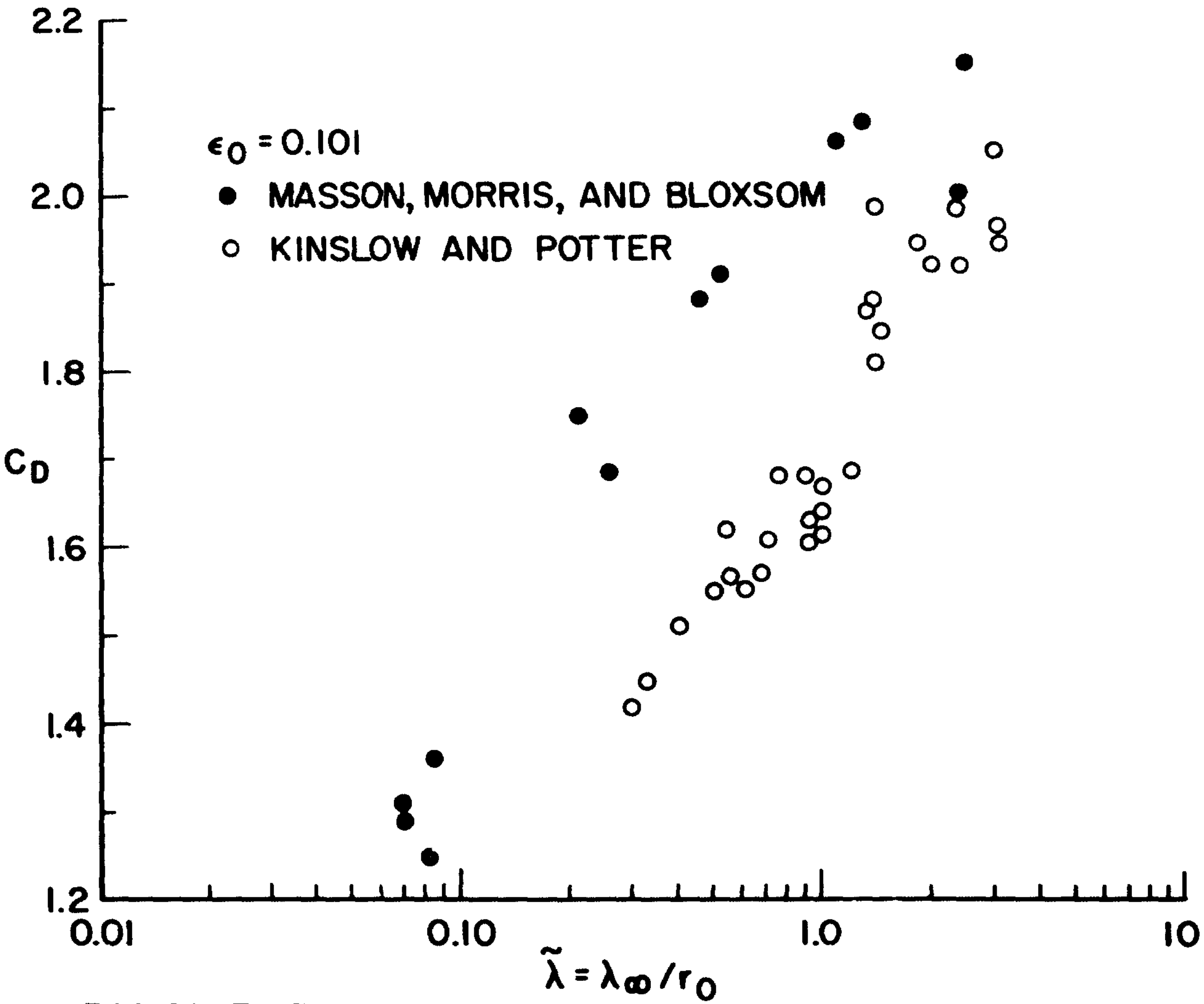
FIGURE 10



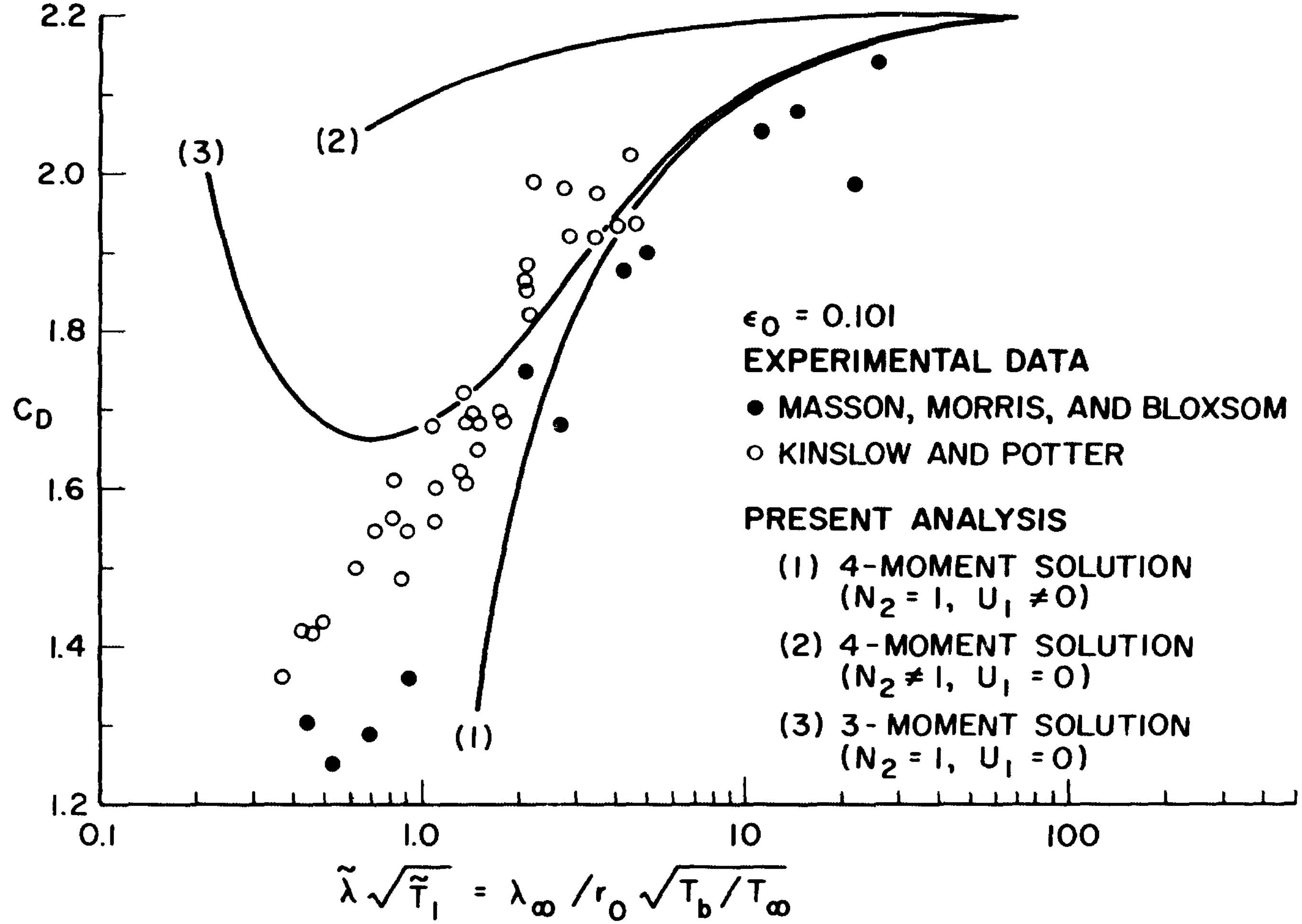
*BASSET'S SOLUTION IS GREATER THAN THE FREE MOLECULE FLOW LIMIT FOR $\tilde{\lambda} \geq 1$

RADIAL STRESS – LOW SPEED FLOW

FIGURE 11

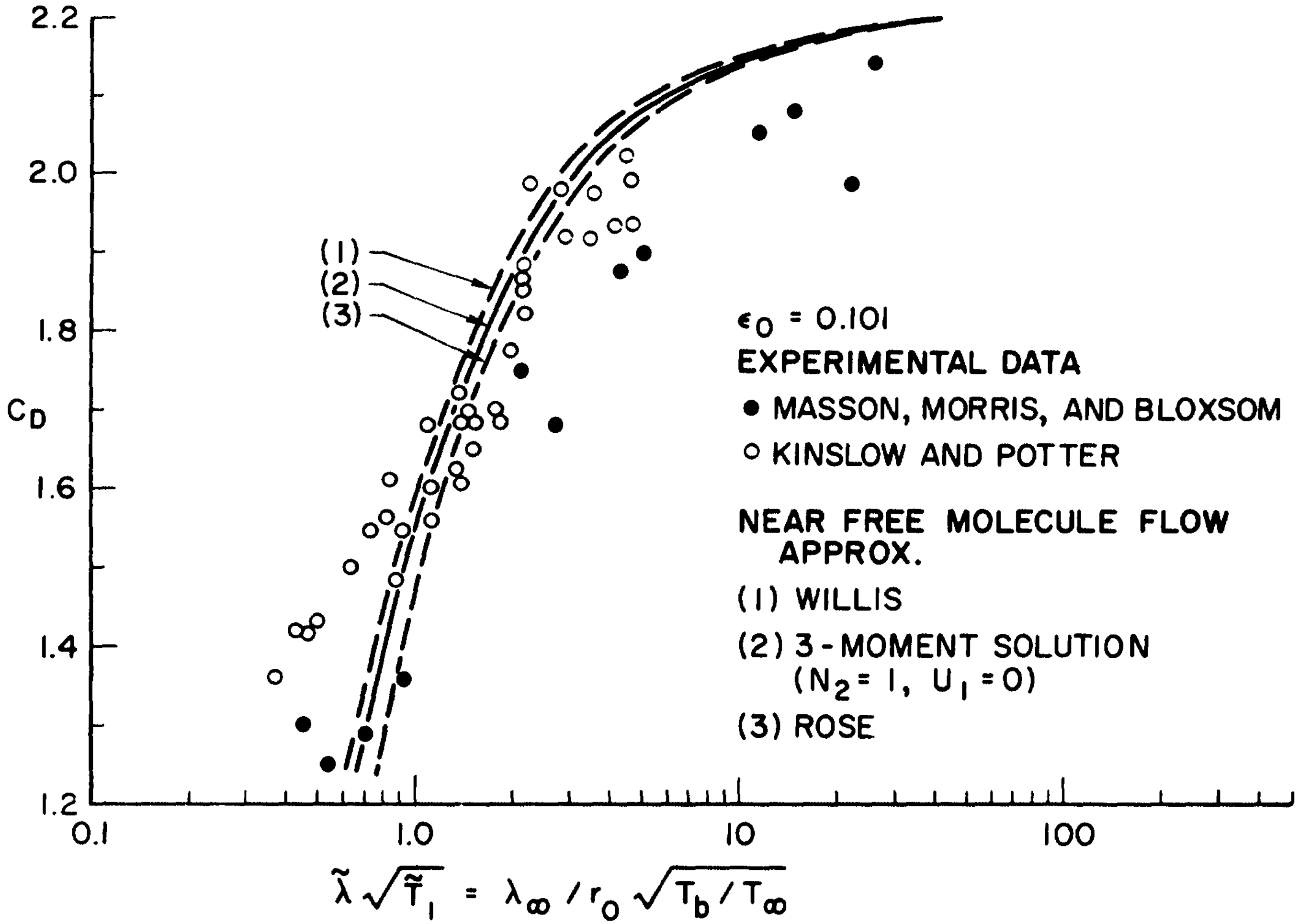


DRAG COEFFICIENT – HIGH SPEED FLOW – EXPERIMENTAL DATA
FIGURE 12



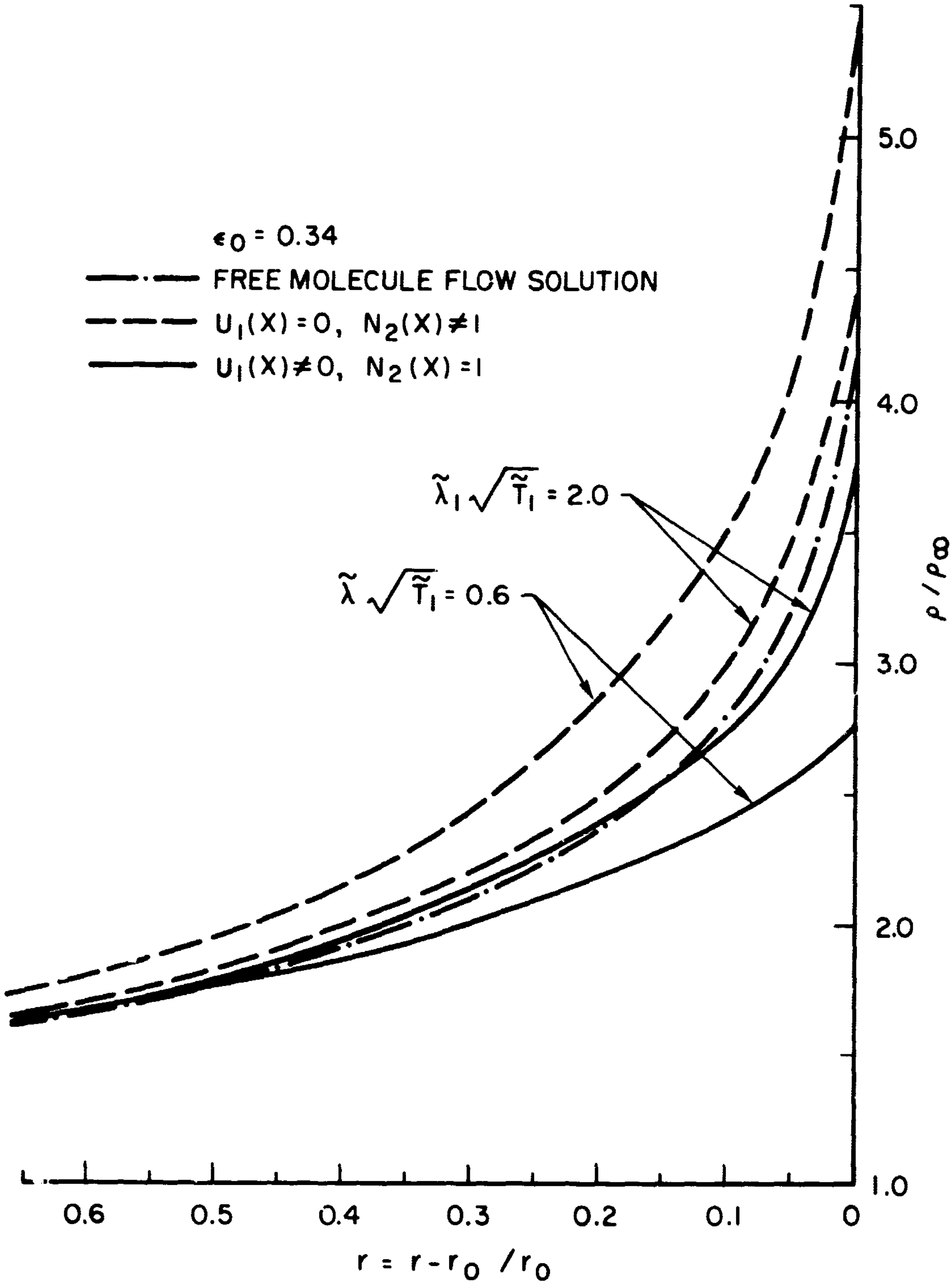
DRAG COEFFICIENT – HIGH SPEED FLOW
NUMERICAL SOLUTIONS

FIGURE 13



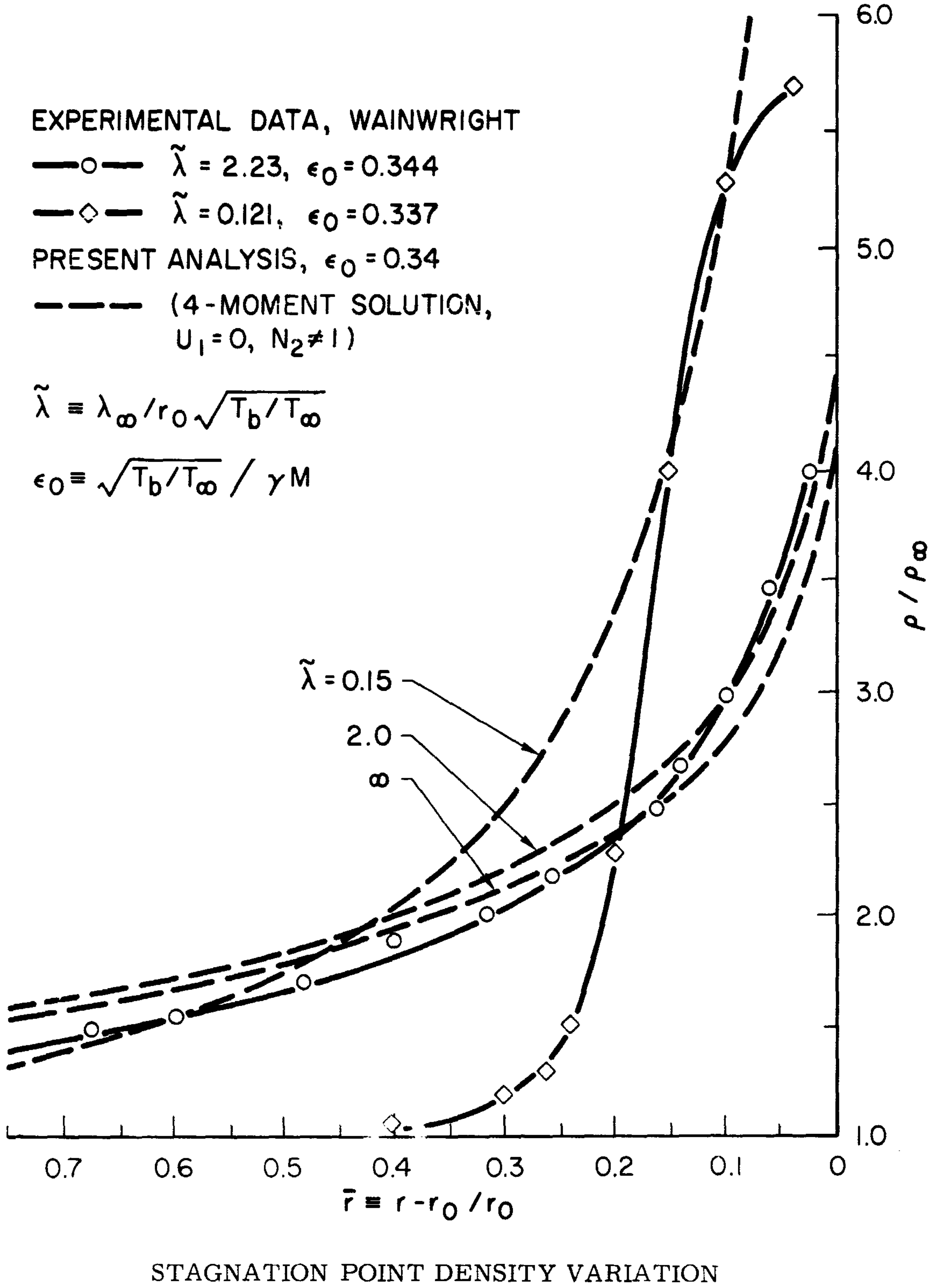
DRAG COEFFICIENT – HIGH SPEED FLOW
NEAR FREE MOLECULE FLOW SOLUTIONS

FIGURE 14



STAGNATION POINT DENSITY VARIATION
(HIGH SPEED FLOW)

FIGURE 15



STAGNATION POINT DENSITY VARIATION
(HIGH SPEED FLOW); COMPARISON WITH
EXPERIMENTAL DATA

FIGURE 16