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# Infinite Dimensional Multipoint Methods and the Solution of Two Point Boundary Value Problems

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## Abstract

Consider the problem of determining the roots of an equation of the form $F(x) = 0$ where $F$ maps the Banach space $X$ into itself. Convergence theorems for the iterative solution of $F(x) = 0$ are proved for multipoint algorithms of the form $x_{n+1} = x_n - \phi_\alpha(x_n)$, $\alpha \geqq 1$, where $\phi_\alpha(x) = \sum_{j=1}^{\alpha}(F'_x)^{-1}F(x-\phi_{j-1}(x))$ and $\phi_0(x) = 0$. The theorems are applied to the solution of two point boundary value problems of the form $\dot{y} = f(y,t)$, $g(y(0))+h(y(1)) = c$. A set $\{A(t),B,C\}$ of matrices is called boundary compatible if the linear two point boundary value problem $\dot{y} = A(t)y+k(t)$, $By(0)+Cy(1) = d$ has a unique solution for all $k(t)$ and $d$. Then, under certain conditions, there are boundary compatible sets such that the problem $\dot{y} = f(y,t)$, $g(y(0))+h(y(1))$ has the equivalent integral representation

$$y(t) = \Lambda(t)\{c-g(y(0))-h(y(1))+By(0)+Cy(1)\} + \int_0^1\Gamma(t,s)\{f(y(s),s)-A(s)y(s)\}ds \qquad \text{(i)}$$

where $\Lambda$ and $\Gamma$ are Green's matrices for the linear problem $\dot{y} = A(t)y+k(t)$, $By(0)+Cy(1) = d$. Equation (i) is viewed as an operator equation of the form $F(x) = (I-T)(x) = 0$ and convergence conditions for the iterative solution of (i) are deduced from the general theorems. Explicit interpretations of the convergence results are given in terms of $f,g,h$ and some illustrative numerical examples are presented.

## 1. Introduction

Considerable effort has been devoted to the study of higher order methods for the iterative solution of equations of the form $F(x) = 0$ where $F$ maps the Banach space $X$ into itself (see, for example, [3], [5], [7], [8], [9]). Most of these methods require commensurately high order derivatives of $F$ and so, are often of limited practical utility. Here, we consider a class of multipoint methods whose order of convergence does not explicitly depend upon higher order derivatives. More precisely, we examine a family of methods of order $\alpha$ which require $(\alpha-1)$ evaluations of $F$, a single inversion of $F'$, and no explicit evaluations of higher derivatives of $F$ except in the convergence analysis where a uniform bound on $F''$ is used.

We deal with the class of multipoint methods given by

$$x_{n+1} = x_n - \phi_\alpha(x_n) = \psi_\alpha(x_n) \tag{1.1}$$

for integers $\alpha \geqq 1$ where

$$\phi_\alpha(x) = \sum_{j=1}^{\alpha} (F'_x)^{-1} F(x - \phi_{j-1}(x)) \tag{1.2}$$

and $\phi_0(x) = 0$. We prove a number of convergence theorems for the entire class. The first theorem involves the order of convergence of the algorithms. In the second theorem, we present conditions under which the iterations (1.1) converge to a unique zero of $F$. This result is a simple application of the contraction mapping theorem. The final theorem consists of practical convergence conditions analogous to Kantorovich's theorem on the convergence of Newton's method ([8]). We then apply the algorithms (1.1) to the iterative solution of two point boundary value problems of the form

$$\dot{y} = f(y,t), \quad g(y(0))+h(y(1)) = c \tag{1.3}$$

on [0,1]. Convergence conditions are deduced from the general theorems.

We note that Traub considers the class (1.1) in [9] for the case of nonlinear equations on the real line. Here, we consider the class in an infinite dimensional setting. We also note that if $\alpha = 1$, then the algorithm is simply Newton's method.

We observe that fixed point problems are also covered in our development. More precisely, if $T$ is a map of $X$ into itself and if we let $F = I-T$, then the equations

$$F(x) = 0 \tag{1.4}$$

and

$$x = T(x) \tag{1.5}$$

are equivalent. The formulations (1.4) and (1.5) will be used interchangeably throughout the sequel. For example, in the case of (1.5), we have

$$(1.6)\qquad \psi_\alpha(x) = [(I-T'_x)^{-1}(T-T'_x)]^\alpha(x)$$

for $\alpha \geqq 1$ as is easily proved by induction on $\alpha$. If we define a mapping $Q(\cdot,\cdot)$ of $X \times X$ into $X$ by

$$(1.7)\qquad Q(x,y) = [(I-T'_x)^{-1}(T-T'_x)](y)$$

then (1.6) may be written in the form

$$(1.8)\qquad \psi_\alpha(x) = Q(x,\psi_{\alpha-1}(x))$$

for $\alpha \geqq 1$ and (1.1) becomes

$$(1.9)\qquad x_{n+1} = \psi_\alpha(x_n) = Q(x_n,\psi_{\alpha-1}(x_n))$$

for $\alpha \geqq 1$. Thus, for example, the Newton's method iteration is given by

$$(1.10)\qquad x_{n+1} = \psi_1(x_n) = Q(x_n,x_n)$$

and the modified Newton's method iteration is given by

$$(1.11)\qquad x_{n+1} = Q(x_o,x_n)$$

where $x_o$ is a fixed initial guess. Convergence proofs for the method based on $\psi_1(x)$ can be found in [5], [6], [7] and [8] and convergence proofs for the method based on $\psi_2(x)$ can be found in [3].

## 2. Convergence Analysis

We now prove a number of convergence theorems. We begin with

DEFINITION 2.1 *Let* $\psi(\cdot)$ *map* $X$ *into itself and suppose that the algorithm* $x_{n+1} = \psi(x_n)$ *converges to a fixed point* $x^*$ *of* $T$. *Then the algorithm converges with order* $p \geqq 1$ *if*

$$(2.2) \qquad \lim_{n \to \infty} \frac{\|\psi(x_n)-x^*\|}{\|\psi(x_{n-1})-x^*\|^p} = C > 0$$

*where* $C$ *is a constant.*

We then have

THEOREM 2.3 *Let* $T$ *map* $X$ *into itself. Suppose that* (i) $T$ *is twice continuously differentiable on the closed sphere* $\bar{S} = \bar{S}(x^*,r)$ *where* $x^*$ *is a fixed point of* $T$; (ii) $[I-T'_x]^{-1}$ *exists and is uniformly bounded on* $\bar{S}$ *with*

$$(2.4) \qquad \sup_{x \in \bar{S}} \{\|(I-T'_x)^{-1}\|\} \leqq B;$$

(iii) $T''_x$ *is uniformly bounded on* $\bar{S}$ *with*

$$(2.5) \qquad \sup_{x \in \bar{S}} \{\|T''_x\|\} \leqq K;$$

*and* (iv) *the constants* $r, B$ *and* $K$ *satisfy the inequalities*

$$(2.6) \qquad r < 1 \quad \text{and} \quad BK \leqq 2/3.$$

*Then, for an initial guess* $x_o$ *in* $\overline{S}$, *the sequence* $\{\psi_\alpha(x_n)\}$ (*with* $\psi_\alpha$ *given by* (1.6)) *lies in* $\overline{S}$ *and converges to* $x^*$ *with order at least* $\alpha+1$. *Moreover, the rate of convergence is given by*

$$(2.7) \qquad \|\psi_\alpha(x_n)-x^*\| \leq c_\alpha\|\psi_\alpha(x_{n-1})-x^*\|^{\alpha+1} \leq c_\alpha^{n+1}\|x_o-x^*\|^{(\alpha+1)^{n+1}}$$

*where the constants* $c_\alpha$ *are given by*

$$(2.8) \qquad \begin{aligned} c_1 &= BK/2 \\ c_\alpha &= \left(1+\frac{c_{\alpha-1}}{2}r^{\alpha-1}\right)BKc_{\alpha-1} \end{aligned}$$

*for* $\alpha \geqq 2$.

*Proof:* The proof is by induction on $\alpha$. We first consider the case $\alpha = 1$. Since $x_o \in \overline{S}$, we have

$$(2.9) \qquad \begin{aligned} x^*-\psi_1(x_o) &= x^*-(I-T'_{x_o})^{-1}(T-T'_{x_o})x_o \\ &= (I-T'_{x_o})^{-1}[(I-T)x^*-(I-T)x_o-(I-T'_{x_o})(x^*-x_o)] \end{aligned}$$

and, hence, in view of (i), (ii) and (iii),

$$(2.10) \qquad \|x^*-\psi_1(x_o)\| \leq \tfrac{1}{2}BK\|x^*-x_o\|^2 = c_1\|x^*-x_o\|^2 \leq \frac{r^2}{3}$$

Since $r < 1$, $\psi_1(x_o) \in \overline{S}$. Now assume that $x_n = \psi_1(x_{n-1}) \in \overline{S}$. By an identical argument, we deduce that

$$(2.11)\qquad \|x^*-x_{n+1}\| = \|x^*-\psi_1(x_n)\| \leq c_1\|x^*-x_n\|^2 \leq r^2/3$$

and hence, that $x_{n+1} = \psi_1(x_n) \in \bar{S}$. It follows that $\psi_1(x_n) \in \bar{S}$ for all $n \geq 0$. Repeated application of (2.11) shows that $\|x^*-x_{n+1}\| \leq c_1^{n+1}\|x^*-x_o\|^{2^{n+1}} \leq r^{2^{n+1}}$ and so, $\lim_{n\to\infty}\|x^*-x_n\| = 0$. Thus the theorem is true for $\alpha = 1$.

We now suppose that the theorem holds for all $m \leq \alpha$ and we shall show that it then holds for $\alpha+1$. Since $x_o \in \bar{S}$ and $\psi_\alpha(x_o) \in \bar{S}$, we have

$$(2.12)\quad \psi_{\alpha+1}(x_o)-x^* = (I-T'_{x_o})^{-1}(T-T'_{x_o})\psi_\alpha(x_o)-x^*$$
$$= (I-T'_{x_o})^{-1}[(T'_{x^*}-T'_{x_o})(\psi_\alpha(x_o)-x^*)-(I-T)\psi_\alpha(x_o)+(I-T)x^*+(I-T'_{x^*})(\psi_\alpha(x_o)-x^*)]$$

and

$$(2.13)\qquad \|\psi_{\alpha+1}(x_o)-x^*\| \leq B[K\|x^*-x_o\|\|\psi_\alpha(x_o)-x^*\| + \tfrac{1}{2}K\|\psi_\alpha(x_o)-x^*\|^2]$$
$$\leq B[Kc_\alpha\|x^*-x_o\|^{\alpha+2} + \tfrac{1}{2}Kc_\alpha^2\|x^*-x_o\|^{2(\alpha+1)}]$$
$$\leq BKc_\alpha\|x^*-x_o\|^{\alpha+2}[1 + \frac{c_\alpha}{2}r^\alpha]$$
$$\leq c_{\alpha+1}\|x^*-x_o\|^{\alpha+2}$$

Since $c_{\alpha+1} \leq 1$[+], $x_1 = \psi_{\alpha+1}(x_o) \in \bar{S}$. Now suppose that $x_o,\ldots,x_n$ are in $\bar{S}$. Then $\psi_\alpha(x_n)$ is in $\bar{S}$ and we have

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[+]This follows by induction on $\alpha$ for if $c_{\alpha-1} \leq 1$, then $c_\alpha \leq (1+1/2)BKc_{\alpha-1} \leq (3/2)(2/3)c_{\alpha-1} \leq 1$.

$$(2.14)\quad \psi_{\alpha+1}(x_n)-x^* = (I-T'_{x_n})^{-1}[(T'_{x^*}-T'_{x_o})(\psi_\alpha(x_n)-x^*)-(I-T)\psi_\alpha(x_n)+(I-T)x^* + (I-T'_{x^*})(\psi_\alpha(x_n)-x^*)]$$

$$(2.15)\quad \begin{aligned}\|\psi_{\alpha+1}(x_n)-x^*\| &\leqq B[K\|x^*-x_n\|\|\psi_\alpha(x_n)-x^*\| + \tfrac{K}{2}\|\psi_\alpha(x_n)-x^*\|^2] \\ &\leqq B[Kc_\alpha\|x^*-x_n\|^{\alpha+2} + \tfrac{K}{2}c_\alpha^2\|x^*-x_n\|^{2(\alpha+1)}] \\ &\leqq c_{\alpha+1}\|x^*-x_n\|^{\alpha+2}\end{aligned}$$

Thus, $x_{n+1} = \psi_{\alpha+1}(x_n) \in \overline{S}$ and so, by induction on $n$, $\psi_{\alpha+1}(x_n) \in \overline{S}$ for all $n$. Repeated application of (2.15) allows us to conclue that $\{\psi_{\alpha+1}(x_n)\}$ converges to $x^*$ and that the rate of convergence is given by (2.7). Thus, the theorem holds for $\alpha+1$. The proof is now complete.

We now turn our attention to a convergence theorem for the algorithms $x_{n+1} = \psi_\alpha(x_n)$, $\alpha \geqq 1$. We begin with some simple lemmas.

LEMMA 2.16 *Suppose that $T$ is differentiable and that $(I-T'_x)^{-1}$ exists for all $x$ in the domain, $\mathscr{D}(T)$, of $T$. Then $x^* \in \mathscr{D}(T)$ is a fixed point of $T$ if and only if $x^*$ is a fixed point of $\psi_\alpha$ for all $\alpha \geqq 1$.*

*Proof:* If $x^* = T(x^*)$, then

$$(2.17)\quad x^*-T'_{x^*}(x^*) = (I-T'_{x^*})x^* = T(x^*)-T'_{x^*}(x^*)$$

and so,

$$(2.18)\quad x^* = [(I-T'_{x^*})^{-1}(T-T'_{x^*})](x^*) = \psi_1(x^*)$$

since $(I-T'_{x*})^{-1}$ exists. If we assume that $\psi_\alpha(x*) = x*$, then $\psi_{\alpha+1}(x*) = Q(x*,\psi_\alpha(x*)) = Q(x*,x*) = Q(x*,\psi_o(x*)) = \psi_1(x*) = x*$ be virtue of (1.8) and the convention $\psi_o(x) = x$. Thus, $\psi_\alpha(x*) = x*$ for all $\alpha \geqq 1$ by induction. Conversely, if $\psi_1(x*) = x*$, then $(I-T'_{x*})x* = (I-T'_{x*})\psi_1(x*) = T(x*)-T'_{x*}(x*)$ and so, $x*$ is a fixed point of $T$.

LEMMA 2.19 *Suppose that* (i) $T$ *is twice continuously differentiable on the closed sphere* $\bar{S} = \bar{S}(x_o,r)$; (ii) $[I-T'_x]^{-1}$ *exists and is uniformly bounded on* $\bar{S}$ *with*

(2.20) $$\sup_{x \in \bar{S}} \{\|(I-T'_x)^{-1}\|\} \leqq D;$$

*and,* (iii) $T''_x$ *is uniformly bounded on* $\bar{S}$ *with*

(2.21) $$\sup_{x \in \bar{S}} \{\|T''_x\|\} \leqq M.$$

*Then the mapping* $Q(x,y) = [(I-T'_x)^{-1}(T-T'_x)]y$ *has partial derivatives* $Q_1(x,y)(\cdot)$ *and* $Q_2(x,y)(\cdot)$ *with respect to* $x$ *and* $y$, *respectively, and*

(2.22) $$Q_1(x,y)(\cdot) = (I-T'_x)^{-1}T''_x(\cdot)(I-T'_x)^{-1}(T-I)y$$

(2.23) $$Q_2(x,y)(\cdot) = (I-T'_x)^{-1}(T'_y-T'_x)(\cdot)$$

*for all* $x,y$ *in* $\bar{S}$.

Proof: Let $x, y$ be elements of $\bar{S}$ and let $h, k$ be increments in $x$ and $y$, respectively, with $x+h$ and $y+k$ in $\bar{S}$. Then

$$(2.24) \qquad Q(x+h,y)-Q(x,y) = (I-T'_{x+h})^{-1}(T-T'_{x+h})y-(I-T'_x)^{-1}(T-T'_x)y$$
$$= [(I-T'_x)-(T'_{x+h}-T'_x)]^{-1}[(T-T'_x)-(T'_{x+h}-T'_x)]y$$
$$- (I-T'_x)^{-1}(T-T'_x)y$$

However, $[(I-T'_x)-(T'_{x+h}-T'_x)]^{-1} = [I-(I-T'_x)^{-1}(T'_{x+h}-T'_x)]^{-1}(I-T'_x)^{-1} = [I-U]^{-1}(I-T'_x)^{-1}$ where

$$(2.25) \qquad U = (I-T'_x)^{-1}(T'_{x+h}-T'_x)$$

provided that $(I-U)^{-1}$ exists. Now (i), (ii) and (iii) together imply that $\|U\| \leqq DM\|h\|$ and so, $(I-U)^{-1}$ will exist for all $h$ with $\|h\| \leqq \delta/DM$ where $\delta < 1$. Moreover, we then have $(I-U)^{-1} = I+U+U^2+\cdots$ . We also note that

$$(2.26) \qquad U = (I-T'_x)^{-1}T''_x(h) + \mathcal{O}(\|h\|^2)$$

as $\|(I-T'_x)^{-1}\| \leqq D$. It follows that

$$(2.27) \qquad Q(x+h,y)-Q(x,y) = (I-U)^{-1}(I-T'_x)^{-1}[(T-T'_x)-(T'_{x+h}-T'_x)]y$$
$$-(I-T'_x)^{-1}(T-T'_x)y$$
$$= (I-U)^{-1}[(I-T'_x)^{-1}(T-T'_x)-U-(I-U)(I-T'_x)^{-1}(T-T'_x)]y$$

$$= (I+U+U^2+\cdots)[U\{(I-T'_x)^{-1}(T-T'_x)-I\}]y$$
$$= (U+U^2+\cdots)(I-T'_x)^{-1}(T-I)y$$
$$= (I-T'_x)^{-1}T''_x(h)(I-T'_x)^{-1}(T-I)y + \mathcal{O}(\|h\|^2)$$

and hence, that the partial derivative of $Q$ with respect to $x$ exists and is given by (2.22). As for the partial derivative with respect to $y$, we have

$$(2.28) \qquad Q(x,y+k)-Q(x,y) = (I-T'_x)^{-1}[T(y+k)-T'_x(y+k)-T(y)-T'_xy]$$

and

$$(2.29) \qquad \|Q(x,y+k)-Q(x,y)-(I-T'_x)^{-1}(T'_y-T'_x)k\| = \|(I-T'_x)^{-1}[T(y+k)-T(y)-T'_yk]\| \leq \frac{DM}{2}\|k\|^2$$

It follows that the partial derivative of $Q$ with respect to $y$ exists and is given by (2.23).

COROLLARY 2.30 <u>Suppose that conditions (i), (ii) and (iii) of the lemma are satisfied. If $x$ is an element of $\overline{S}$ such that $\psi_{\alpha-1}(\cdot)$ is differentiable at $x$ and $\psi_{\alpha-1}(x)$ is in $\overline{S}$, then $\psi_\alpha(\cdot)$ is differentiable at $x$ and</u>

$$(2.31) \qquad \psi'_{\alpha,x}(\cdot) = Q_1(x,\psi_{\alpha-1}(x))(\cdot) + Q_2(x,\psi_{\alpha-1}(x))\psi'_{\alpha-1,x}(\cdot) .$$

<u>Proof:</u> Simply apply the lemma and the chain rule for Frechet derivatives

(note that $\psi_\alpha(x) = Q(x,\psi_{\alpha-1}(x)))$.

LEMMA 2.32 (*c.f.* [8]) *Let* $V$ *be a map of* $X$ *into itself. Suppose that* (i) $V$ *is continuously differentiable on* $\bar{S} = \bar{S}(x_o,r)$ *with*

(2.33) $$\sup_{x \in \bar{S}} \{\|V'_x\|\} \leqq \delta < 1$$

*for some* $\delta$; *and* (ii) *there is an* $\eta > 0$ *such that*

(2.34) $$\|V(x_o)-x_o\| \leqq \eta$$

*and* $(\eta/1-\delta) \leqq r$. *Then the sequence* $x_n = V(x_{n-1})$ *converges to the unique fixed point* $x^*$ *of* $V$ *in* $\bar{S}$ *and the rate of convergence is given by*

(2.35) $$\|x^*-x_n\| \leqq \frac{\delta}{1-\delta}\|x_n-x_{n-1}\| \leqq \frac{\delta^n}{1-\delta}\|x_1-x_o\|$$

*for* $n = 1,2,\ldots$ .

We now have

THEOREM 2.36 *Suppose that* (i) $T$ *is twice continuously differentiable on* $\bar{S} = \bar{S}(x_o,r)$; (ii) $(I-T'_x)^{-1}$ *exists and is uniformly bounded on* $\bar{S}$ *with*

(2.37) $$\sup_{x \in \bar{S}} \{\|(I-T'_x)^{-1}\|\} \leqq B;$$

(iii) $T''_x$ *is uniformly bounded on* $\bar{S}$ *with*

(2.38) $$\sup_{x \in \overline{S}} \{\|T''_x\|\} \leq K;$$

(iv) T-I *is uniformly bounded on* $\overline{S}$ *with*

(2.39) $$\sup_{x \in \overline{S}} \{\|(T-I)x\|\} \leq M;$$

*and,* (iv) *there is an* $\eta_\alpha > 0$ *such that*

(2.40) $$\|\psi_\alpha(x_o)-x_o\| \leq \eta_\alpha$$

(2.41) $$\eta_\alpha \leq (1-h_\alpha)r$$

*where* $h_\alpha = \alpha(B^2KM) < 1$ *for each* $\alpha \geq 1$. *Then, for each* $\alpha \geq 1$, *the multipoint sequence* $\{\psi_\alpha(x_n)\}$ *based on the initial guess* $x_o$ *converges to the unique fixed point* $x^*$ *of* T *in* $\overline{S}$ *and the rate of convergence is given by*

(2.42) $$\|x^*-\psi_\alpha(x_n)\| \leq \frac{h_\alpha}{1-h_\alpha}\|\psi_\alpha(x_n)-\psi_\alpha(x_{n-1})\| \leq \frac{h_\alpha^{n+1}}{1-h_\alpha}\|x_1-x_o\|$$

*for each* $\alpha \geq 1$.

*Proof:* We first show that $\psi_\alpha(\cdot)$ is differentiable on $\overline{S}$ and that

(2.43) $$\sup_{x \in \overline{S}} \{\|\psi'_{\alpha,x}\|\} \leq h_\alpha < 1$$

for each $\alpha \geq 1$. Suppose that $\alpha = 1$. Then $\psi_1(\cdot)$ is differentiable on

$\overline{S}$ (as $\psi_o(x) = Ix$) and

$$(2.44)\quad \sup_{x\in\overline{S}}\{\|\psi'_{1,x}\|\} \leq \sup_{x\in\overline{S}}\{\|Q_1(x,x)\|\} + \sup_{x\in\overline{S}}\{\|Q_2(x,x)\|\cdot 1\}$$

$$\leq \sup_{x\in\overline{S}}\{\|(I-T'_x)^{-1}T''_x(I-T'_x)^{-1}(T-I)x\|\}$$

$$\leq B^2KM = h_1 < 1$$

by virtue of corollary 2.30 and the hypotheses of the theorem. Since $\|\psi_1(x)-x_o\| \leq \|\psi_1(x)-\psi_1(x_o)\| + \|\psi_1(x_o)-x_o\|$, it follows from the mean value theorem that $\|\psi_1(x)-x_o\| \leq h_1r + (1-h_1)r = r$. Thus, $\psi_1(x) \in \overline{S}$ for all $x$ in $\overline{S}$. Suppose now that $\psi_\beta(\cdot)$ is differentiable on $\overline{S}$, that $\psi_\beta(x) \in \overline{S}$ for all $x$ in $\overline{S}$, and that (2.43) holds if $\beta \leq \alpha$. Then $\psi_{\alpha+1}$ is differentiable on $\overline{S}$ by virtue of corollary 2.30 and

$$(2.45)\quad \sup_{x\in\overline{S}}\{\|\psi'_{\alpha+1,x}\|\} \leq \sup_{x\in\overline{S}}\{\|Q_1(x,\psi_\alpha(x))\|\} + \sup_{x\in\overline{S}}\{\|Q_2(x,\psi_\alpha(x))\|\|\psi'_{\alpha,x}\|\}$$

$$\leq \sup_{x\in\overline{S}}\{\|(I-T'_x)^{-1}T''_x(I-T'_x)^{-1}(T-I)\psi_\alpha(x)\|\}$$

$$+ \sup_{x\in\overline{S}}\{\|(I-T'_x)^{-1}(T'_{\psi_\alpha(x)}-T'_x)\|\}h_\alpha$$

$$\leq B^2KM+BK \sup_{x\in\overline{S}}\{\|\psi_\alpha(x)-x\|\}h_\alpha .$$

But $\|\psi_\alpha(x)-x\| \leq \|\psi_\alpha(x)-\psi_{\alpha-1}(x)\|+\cdots+\|\psi_1(x)-x\|$ and $\|\psi_\beta(x)-\psi_{\beta-1}(x)\| = \|(I-T'_x)^{-1}(T-I)\psi_{\beta-1}(x)\|$ so that

$$(2.46)\quad \sup_{x\in\overline{S}}\{\|\psi'_{\alpha+1,x}\|\} \leq B^2KM+B^2KM\alpha h_\alpha$$

$$\leq h_1(1+\alpha h_\alpha) \leq (\alpha+1)h_1 = h_{\alpha+1} < 1$$

since $h_\alpha < 1$. Thus, (2.43) holds for each $\alpha \geq 1$ by induction. Moreover, $\psi_\alpha(x) \in \overline{S}$ for all $x$ in $\overline{S}$.

It now follows from lemma 2.32 that the sequence $\{\psi_\alpha(x_n)\}$ converges to the unique fixed point $x^*_\alpha$ of $\psi_\alpha(\cdot)$ in $\overline{S}$. However, the proof of lemma 2.16 shows that $x^*_1 = x^*$ is a fixed point of $T$ and hence, by lemma 2.16, $x^* = x^*_\alpha$ for each $\alpha \geq 1$. The rate of convergence inequality (2.42) follows from lemma 2.32 and so the theorem is established.

We now state and prove a basic convergence theorem for the multipoint algorithm (1.1). The results are analogous to Kantorovich's theorem on the convergence of Newton's method ([8]). We deal with a map $F$ of $X$ into itself and with the algorithms

$$x_{n+1} = \psi_\alpha(x_n) = x_n - \sum_{j=1}^{\alpha} (F'_{x_n})^{-1} F(\psi_{\alpha-1}(x_n)) \tag{2.47}$$

for $\alpha = 1,2,\ldots$ . Moreover, we write $x_{n,\alpha}$ to indicate that we are considering a particular element of the class of algorithms (2.47). We then have

THEOREM 2.48 <u>Suppose that</u> (i) $F$ <u>is twice continuously differentiable on</u> $\overline{S}_\alpha = \overline{S}(x_{o,\alpha}, r_\alpha)$; (ii) $(F'_{x_{o,\alpha}})^{-1}$ <u>exists and</u> $\|(F'_{x_{o,\alpha}})^{-1}\| \leq B_{o,\alpha}$; (iii) $\|(F'_{x_{o,\alpha}})^{-1}\| \|F(x_o,\alpha)\| \leq d_{o,\alpha}$; (iv) $F''_x$ <u>is uniformly bounded on</u> $\overline{S}_\alpha$ <u>with</u>

$$\sup_{x \in \overline{S}_\alpha} \{\|F''_x\|\} \leq K_\alpha; \tag{2.49}$$

(iv) $\eta_{o,\alpha} = \gamma^{(\alpha)}_{o,\alpha} d_{o,\alpha}$ <u>where</u> $\gamma^{(1)}_{o,\alpha} = 1$ <u>and</u> $\gamma^{(j)}_{o,\alpha}$ <u>is given by</u>

(2.50) $$\gamma_{o,\alpha}^{(j)} = 1 + \frac{1}{2}\sum_{i=1}^{j-1}\{(K_\alpha B_{o,\alpha} d_{o,\alpha})^i (\prod_{k=1}^{i} \gamma_{o,\alpha}^{(k)})\}$$

*for* $2 \leqq j \leqq \alpha$; *and* (v) *the following relations are satisfied*

(2.51) $$h_{o,\alpha} = K_\alpha B_{o,\alpha}\ \eta_{o,\alpha} \leqq 1/2$$

(2.52) $$r_\alpha = \frac{16}{11}\ \eta_{o,\alpha}$$

(2.53) $$\gamma_{o,\alpha}^{(\alpha)} \leqq 2$$

*for* $\alpha = 1,2,\ldots$ .[+] *Then the multipoint sequence* $\{\psi_\alpha(x_{n,\alpha})\}$ *converges to a zero* $x_\alpha^*$ *of* F *in* $\bar{S}_\alpha$ *with order at least* $\alpha+1$ *and the rate of convergence is given by*

(2.54) $$\|x_\alpha^* - x_{n,\alpha}\| \leqq \frac{16}{11}\left(\frac{5}{16}\right)^n (2h_{o,\alpha})^{(\alpha+1)^n - 1}\ \eta_{o,\alpha}$$

*for* $\alpha = 1,2,\ldots$ .

*Proof:* We give the proof in three steps. We first show that all the inverses $(F'_{x_{n,\alpha}})^{-1}$ exist and that all the $x_{n,\alpha}$ are in $\bar{S}_\alpha$. Next we prove that $\{x_{n,\alpha}\}$ is a Cauchy sequence in $\bar{S}_\alpha$ and hence has a limit $x_\alpha^*$ in $\bar{S}_\alpha$. Finally, we show that $F(x_\alpha^*) = 0$ and that (2.54) is valid.

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[+] We note that (2.53) is automatically satisfied for $1 \leqq \alpha \leqq 10$ and that, for large $\alpha$, we can choose $h_{o,\alpha}$ smaller than 1/2 to insure the validity of (2.53). However, it is unlikely that large values of $\alpha$ would be used practically. We also note that the particular choice 16/11 is not quite optimum but is adequate for our purposes.

For each $\alpha \geqq 1$, we define $B_{n,\alpha}$, $d_{n,\alpha}$, $\eta_{n,\alpha}$ and $h_{n,\alpha}$, recursively, by setting

(2.55) $$B_{n,\alpha} = B_{n-1,\alpha}/1-h_{n-1,\alpha}$$

(2.56) $$d_{n,\alpha} = 2^{\alpha-2} h^{\alpha}_{n-1,\alpha}\ \eta_{n-1,\alpha}$$

(2.57) $$\eta_{n,\alpha} = \frac{5}{16}\ 2^{\alpha} h^{\alpha}_{n-1,\alpha}\ \eta_{n-1,\alpha}$$

(2.58) $$h_{n,\alpha} = K_{\alpha} B_{n,\alpha}\ \eta_{n,\alpha}$$

for $n \geqq 1$. We now prove by induction that

(2.59) $$(F'_{x_{n,\alpha}})^{-1} \text{ exists and is linear;}$$

(2.60) $$\|(F'_{x_{n,\alpha}})^{-1}\| \leqq B_{n,\alpha};$$

(2.61) $$\|(F'_{x_{n,\alpha}})^{-1}\| \|F(x_{n,\alpha})\| \leqq d_{n,\alpha};$$

(2.62) $$\|x_{n+1,\alpha} - x_{n,\alpha}\| \leqq \eta_{n,\alpha};$$

(2.63) $$h_{n,\alpha} \leqq 1/2;$$

and

(2.64) $$\|x_{n+1,\alpha} - x_{o,\alpha}\| \leqq r_{\alpha}$$

for $n = 0,1,2,\ldots$ . For $n = 0$, (2.59)-(2.61) and (2.63) are simply hypotheses of the theorem and (2.64) will follow from (2.52) and (2.62). Thus, we need only show that $\|x_{1,\alpha}-x_{o,\alpha}\| \leqq \eta_{o,\alpha}$.

We begin by showing that $\psi_j(x_{o,\alpha})$ is an element of $\overline{S}_\alpha$ for $1 \leqq j \leqq \alpha-1$. This is done by induction on $\alpha$. For $\alpha = 2$, we have

$$\|\psi_1(x_{o,\alpha})-x_{o,\alpha}\| = \|(F'_{x_{o,\alpha}})^{-1}F(x_{o,\alpha})\| \leqq d_{o,\alpha} \leqq \eta_{o,\alpha} \leqq r_\alpha$$

so that $\psi_1(x_{o,\alpha}) \in \overline{S}_\alpha$. By expanding $F(\psi_1(x_{o,\alpha}))$ about $x_{o,\alpha}$, we find that

$$\|F(\psi_1(x_{o,\alpha}))\| \leqq \frac{K_\alpha}{2} d_{o,\alpha}^2 \tag{2.65}$$

and hence, that

$$\|\psi_2(x_{o,\alpha})-x_{o,\alpha}\| \leqq d_{o,\alpha}\left(1 + \frac{K_\alpha B_{o,\alpha} d_{o,\alpha}}{2}\right) = d_{o,\alpha}\gamma_{o,\alpha}^{(2)} \leqq r_\alpha \tag{2.66}$$

so that $\psi_2(x_{o,\alpha}) \in \overline{S}_\alpha$. This argument can be repeated to show that $\|\psi_j(x_{o,\alpha})-x_{o,\alpha}\| \leqq d_{o,\alpha}\gamma_{o,\alpha}^{(j)} \leqq r_\alpha$ and hence that $\psi_j(x_{o,\alpha}) \in \overline{S}_\alpha$ for $1 \leqq j \leqq \alpha-1$. Moreover, by expanding $F(\psi_j(x_{o,\alpha}))$ in a Taylor series about $x_{o,\alpha}$, we have

$$\|F(\psi_j(x_{o,\alpha}))\| \leqq K_\alpha \frac{\|F(\psi_{j-1}(x_{o,\alpha}))\|}{\|F'_{x_{o,\alpha}}\|} \|\psi_{j-1}(x_{o,\alpha})-x_{o,\alpha}\| + \frac{K_\alpha}{2} \frac{\|F(\psi_{j-1}(x_{o,\alpha}))\|^2}{\|F'_{x_{o,\alpha}}\|^2} \tag{2.67}$$

for $1 \leqq j \leqq \alpha-1$. It follows that

$$\|F(\psi_j(x_{o,\alpha}))\| \leq K_\alpha^j B_{o,\alpha}^{j-1} d_{o,\alpha}^{j+1} \gamma_{o,\alpha}^{(1)} \cdots \gamma_{o,\alpha}^{(j)} \tag{2.68}$$

for $1 \leq j \leq \alpha-1$. Since $x_{1,\alpha}-x_{o,\alpha} = -\sum_{j=0}^{\alpha-1}(F'_{x_{o,\alpha}})^{-1}F(\psi_j(x_{o,\alpha}))$, we deduce that $\|x_{1,\alpha}-x_{o,\alpha}\| \leq \eta_{o,\alpha}$ or, in other words, that (2.62) holds for $n = 0$.

We now examine the transition from $n = 0$ to $n = 1$. Since $x_{1,\alpha}$ and $x_{o,\alpha}$ are in the convex set $\overline{S}_\alpha$, we have $\|F'_{x_{o,\alpha}}-F'_{x_{1,\alpha}}\| \leq K_\alpha\|x_{1,\alpha}-x_{o,\alpha}\|$. It follows that $\|F'_{x_{o,\alpha}}\|-\|F'_{x_{1,\alpha}}\| \leq K_\alpha \eta_{o,\alpha}$ and hence, that

$$\|F'_{x_{1,\alpha}}\| \geq \left(1 - \frac{K_\alpha \eta_{o,\alpha}}{\|F'_{x_{o,\alpha}}\|}\right)\|F'_{x_{o,\alpha}}\| \geq (1-h_{o,\alpha})\|F'_{x_{o,\alpha}}\| > 0 \tag{2.69}$$

But (2.69) implies that $(F'_{x_{1,\alpha}})^{-1}$ exists, is linear, and satisfies

$$\|(F'_{x_{1,\alpha}})^{-1}\| \leq \frac{B_{o,\alpha}}{1-h_{o,\alpha}} = B_{1,\alpha} \tag{2.70}$$

so that (2.59) **and** (2.60) hold for $n = 1$. To verify (2.61) for $n = 1$, we expand $F(x_{1,\alpha})$ in a Taylor series about $\psi_{\alpha-1}(x_{o,\alpha})$ to obtain

$$\begin{aligned} &\|F(x_{1,\alpha})-F(\psi_{\alpha-1}(x_{o,\alpha})) + F'_{\psi_{\alpha-1}(x_{o,\alpha})}(F'_{x_{o,\alpha}})^{-1}F(\psi_{\alpha-1}(x_{o,\alpha}))\| \\ &\quad \leq \frac{K_\alpha}{2}\,\frac{\|F(\psi_{\alpha-1}(x_{o,\alpha}))\|^2}{\|F'_{x_{o,\alpha}}\|^2} \end{aligned} \tag{2.71}$$

and

$$\|F(x_{1,\alpha})\| \leq \frac{K_\alpha^\alpha B_{o,\alpha}^{\alpha-1} d_{o,\alpha}^{\alpha+1}}{2} \prod_{j=1}^{\alpha} \gamma_{o,\alpha}^{(j)} \tag{2.72}$$

Since $\eta_{o,\alpha} = d_{o,\alpha}\gamma_{o,\alpha}^{(\alpha)}$, $\gamma_{o,\alpha}^{(\alpha)} \leq 2$, and $\gamma_{o,\alpha}^{(\alpha)} \geq \gamma_{o,\alpha}^{(j)}$, we have

$$(2.73)\qquad \|(F'_{x_{1,\alpha}})^{-1}\|\|F(x_{1,\alpha})\| \leq \frac{K_\alpha^\alpha B_{o,\alpha}^\alpha d_{o,\alpha}^\alpha}{2(1-h_{o,\alpha})} \left(\prod_{j=1}^{\alpha-1} \gamma_{o,\alpha}^{(j)}\right) \eta_{o,\alpha}$$

$$\leq 2^{\alpha-2} h_{o,\alpha}^\alpha \eta_{o,\alpha} = d_{1,\alpha}$$

so that (2.61) holds for $n = 1$.

Now to verify that (2.62) holds for $n = 1$, we shall show that

$$(2.74)\qquad \|x_{2,\alpha}-x_{1,\alpha}\| = \|\psi_\alpha(x_{1,\alpha})-x_{1,\alpha}\| \leq d_{1,\alpha}\gamma_{1,\alpha}^{(\alpha)}$$

where $\gamma_{1,\alpha}^{(j)}$ is given by

$$(2.75)\qquad \gamma_{i,\alpha}^{(j)} = 1 + \frac{1}{2}\sum_{i=1}^{j-1}\{(K_\alpha B_{1,\alpha} d_{1,\alpha})^i \prod_{k=1}^{i} \gamma_{1,\alpha}^{(k)}\}$$

for $2 \leq j \leq \alpha$ and $\gamma_{1,\alpha}^{(1)} = 1$. We note that $\gamma_{1,\alpha}^{(j)} \leq \gamma_{o,\alpha}^{(j)} \leq 2$ for all $j \leq \alpha$ (by a simple recursive calculation). Assuming for the moment that (2.74) is valid, we have

$$(2.76)\qquad \|x_{2,\alpha}-x_{1,\alpha}\| \leq d_{1,\alpha}[1 + \frac{1}{2}\sum_{j=1}^{\alpha-1}\{(K_\alpha B_{1,\alpha} d_{1,\alpha})^j (\prod_{k=1}^{j} \gamma_{1,\alpha}^{(k)})\}]$$

$$\leq d_{1,\alpha}[1 + \frac{1}{2}\sum_{j=1}^{\alpha-1}\{(\frac{K_\alpha B_{o,\alpha} 2^{\alpha-2} h_{o,\alpha}^\alpha \mathcal{N}_{o,\alpha}}{(1-h_{o,\alpha})})^j 2^j\}$$

$$\leq d_{1,\alpha}[1 + \frac{1}{2}\sum_{j=1}^{\alpha-1}\{(2^\alpha h_{o,\alpha}^{\alpha+1})^j\}$$

$$\leq d_{1,\alpha}[1 + \frac{1}{2}\sum_{j=1}^{\alpha-1} 2^{-j}] \leq \frac{5}{4}\, d_{1,\alpha}$$

Since $d_{1,\alpha} = 2^{\alpha-2} h_{o,\alpha}^{\alpha} \eta_{o,\alpha}$, we conclude that

$$(2.77) \qquad \|x_{2,\alpha}-x_{1,\alpha}\| \leq \frac{5}{16} 2^{\alpha} h_{o,\alpha}^{\alpha} \eta_{o,\alpha} = \eta_{1,\alpha}$$

or, in other words, that (2.62) holds for $n = 1$.

We now establish (2.74). We observe that

$$(2.78) \qquad \psi_{\alpha}(x_{1,\alpha})-x_{1,\alpha} = -(F'_{x_{1,\alpha}})^{-1}[F(x_{1,\alpha})+\dots+ F(\psi_{\alpha-1}(x_{1,\alpha}))]$$

and hence, that (2.74) will hold if

$$(2.79) \qquad \|F(\psi_j(x_{1,\alpha}))\| \leq \frac{K_{\alpha}^{j} B_{1,\alpha}^{j-1} d_{1,\alpha}^{j+1}}{2} \left( \prod_{i=1}^{j} \gamma_{1,\alpha}^{(i)} \right)$$

for $1 \leq j \leq \alpha-1$. Now (2.79) can be established in exactly the same way as (2.68) once we have shown that $\psi_j(x_{1,\alpha}) \in \overline{S}_{\alpha}$ for $1 \leq j \leq \alpha-1$. But this can be done by induction on $\alpha$. For $\alpha = 2$, we have $\|\psi_1(x_{1,\alpha})-x_{o,\alpha}\| \leq \|\psi_1(x_{1,\alpha})-x_{1,\alpha}\| + \|x_{1,\alpha}-x_{o,\alpha}\| \leq d_{1,\alpha}+ \eta_{o,\alpha} \leq (1 + 2^{\alpha-2} h_{o,\alpha}^{\alpha}) \eta_{o,\alpha} \leq \frac{5}{4} \eta_{o,\alpha} \leq r_{\alpha}$ so that $\psi_1(x_{1,\alpha}) \in \overline{S}_{\alpha}$. By expanding $F(\psi_1(x_{1,\alpha}))$ about $x_{1,\alpha}$, we find that $\|F(\psi_1(x_{1,\alpha}))\| \leq \frac{K_{\alpha}}{2} d_{1,\alpha}^2$ and hence, that $\|\psi_2(x_{1,\alpha})-x_{o,\alpha}\| \leq \|\psi_2(x_{1,\alpha})-\psi_1(x_{1,\alpha})\| + \|\psi_1(x_{1,\alpha})-x_{1,\alpha}\| + \|x_{1,\alpha}-x_{o,\alpha}\| \leq \frac{K_{\alpha}}{2} B_{1,\alpha} d_{1,\alpha}^2 + d_{1,\alpha} + \eta_{o,\alpha} \leq d_{1,\alpha}\gamma_{1,\alpha}^{(2)} + \eta_{o,\alpha} \leq \frac{5}{4} d_{1,\alpha} + \eta_{o,\alpha} \leq (1 + \frac{5}{16}) \eta_{o,\alpha} = r_{\alpha}$. Thus, $\psi_2(x_{1,\alpha}) \in \overline{S}_{\alpha}$. The argument can be repeated to show that $\psi_j(x_{1,\alpha}) \in \overline{S}_{\alpha}$ for $j \leq \alpha-1$ since $\gamma_{1,\alpha}^{(j)} \leq 5/4$ (cf. (2.76)). Thus (2.62) holds for $n = 1$.

As for (2.63), we have

$$(2.80)\qquad h_{1,\alpha} = K_\alpha B_{1,\alpha}\,\eta_{1,\alpha} = \frac{K_\alpha B_{o,\alpha}}{(1-h_{o,\alpha})}\left(\frac{5}{16}\right)2^\alpha h^\alpha_{o,\alpha}\,\eta_{o,\alpha}$$

$$\leq \frac{5}{8}\,2^\alpha h^{\alpha+1}_{o,\alpha} \leq \frac{5}{8}\,h_{o,\alpha} \leq \frac{1}{2}$$

Now, $\|x_{2,\alpha}-x_{o,\alpha}\| \leq \|x_{2,\alpha}-x_{1,\alpha}\| + \|x_{1,\alpha}-x_{o,\alpha}\| \leq \left(\frac{5}{16}\,2^\alpha h^\alpha_{o,\alpha}+1\right)\eta_{o,\alpha} \leq \frac{21}{16}\,\eta_{o,\alpha} \leq r_\alpha$ and so, (2.64) holds for $n = 1$.

If we now assume that (2.59)-(2.64) hold for $m \leq n-1$, then we can show by exactly the same arguments used in going from $n = 0$ to $n = 1$ that (2.59)-(2.64) are satisfied for $n$. Thus, by induction, the relations (2.59)-(2.64) hold for all $n \geq 0$.

Now, it follows from (2.57) and (2.63) that $\eta_{n,\alpha} \leq \left(\frac{5}{16}\right)^n \eta_{o,\alpha}$ and hence that the series $\sum_{n=0}^{\infty} \eta_{n,\alpha}$ is convergent. Since

$$(2.81)\qquad \|x_{n+m,\alpha}-x_{n,\alpha}\| \leq \sum_{j=0}^{m-1} \eta_{n+j,\alpha}$$

we conclude that $x_{n,\alpha}$ is a Cauchy sequence in $\overline{S}_\alpha$ and so converges to an element $x^*_\alpha$ of $\overline{S}_\alpha$.

We claim that $x^*_\alpha$ is a zero of $F$. In view of the analog of (2.72) for arbitrary $n$, we have

$$(2.82)\qquad \|F(x_{n+1,\alpha})\| \leq K^\alpha_\alpha B^{\alpha-1}_{n,\alpha} d^{\alpha+1}_{n,\alpha}\,2^{\alpha-2}\gamma^{(\alpha)}_{n,\alpha}$$

$$\leq \frac{h^\alpha_{n,\alpha}2^{\alpha-2}d_{n,\alpha}\gamma^{(\alpha)}_{n,\alpha}}{B_{n,\alpha}}$$

$$\leq \frac{h^\alpha_{n,\alpha}2^{\alpha-2}}{B_{o,\alpha}}\,\eta_{n,\alpha} \leq \frac{1}{4B_{o,\alpha}}\left(\frac{5}{16}\right)^n \eta_{o,\alpha}$$

(using the analog of (2.74)). It follows that $\lim_{n\to\infty} \|F(x_{n,\alpha})\| = \|F(x^*_\alpha)\| = 0$ as $F$ is continuous.

All that remains is the establishment of the rate of convergence inequality (2.54). But, since $\|x^*_\alpha - x_{n,\alpha}\| \leq \sum_{j=0}^{\infty} \eta_{n+j,\alpha}$, (2.54) will follow from the estimate

$$\eta_{n,\alpha} \leq \left(\tfrac{5}{16}\right)^n (2h_{o,\alpha})^{(\alpha+1)^n - 1}\, \eta_{o,\alpha} \tag{2.83}$$

and the fact that $h_{o,\alpha} \leq 1/2$. But (2.83) is a direct consequence of the relations

$$\eta_{n,\alpha} = \tfrac{5}{16}\,(2h_{n-1,\alpha})^{\alpha}\, \eta_{n-1,\alpha} \tag{2.84}$$

$$h_{n,\alpha} \leq \tfrac{1}{2}(2h_{o,\alpha})^{(\alpha+1)^n} \tag{2.85}$$

which follow from the definitions of $\eta_{n,\alpha}$ and $h_{n,\alpha}$.+ Thus, the proof of the theorem is complete.

## 3. Two Point Boundary Value Problems

We consider the (normalized) two point boundary value problem (TPBVP)

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+The argument is as follows:

$$\begin{aligned}
\eta_{n,\alpha} &= \left(\tfrac{5}{16}\right)(2h_{n-1,\alpha})^{\alpha}\, \eta_{n-1,\alpha} \\
&\leq \left(\tfrac{5}{16}\right)^n (2^{\alpha})^n \left[(2h_{o,\alpha})^{(\alpha+1)^{n-1} + \cdots + (\alpha+1)}\right]^{\alpha} (2^{-\alpha})^{n-1} h^{\alpha}_{o,\alpha}\eta_{o,\alpha} \\
&\leq \left(\tfrac{5}{16}\right)^n [(2h_{o,\alpha})]^{\frac{[(\alpha+1)^n - (\alpha+1)]\alpha}{\alpha}} (2h_{o,\alpha})^{\alpha}\, \eta_{o,\alpha} \\
&\leq \left(\tfrac{5}{16}\right)^n (2h_{o,\alpha})^{(\alpha+1)^n - 1}\, \eta_{o,\alpha}
\end{aligned}$$

$$(3.1) \qquad \dot{y} = f(y,t) \ , \quad g(y(0))+h(y(1)) = c$$

where $f,g,h$ are vector valued functions and $c$ is an element of $R_p$. We first review some results relating to the development of equivalent integral equation representations of the TPBVP (3.1) (see, for example, [7]). Since linear TPBVP's will play an important role in the integral equation representations, we begin our discussion with a consideration of linear TPBVP's.

Consider the linear TPBVP

$$(3.2) \qquad \dot{y} = A(t)y+k(t) \ , \quad By(0)+Cy(1) = d$$

where $A(t),B,C$ are $p \times p$ matrices and $k(t)$, $d$ are p-vectors. We recall the following

PROPOSITION 3.3 Suppose that (i) the functions $A(t)$ and $k(t)$ are integrable on $[0,1]$; (ii) there is an integrable function $m(t)$ on $[0,1]$ with $\|A(t)\| \leq m(t)$, $|k(t)| \leq m(t)$ and $\int_0^1 |m(t)|\,dt < \infty$; and, (iii) $\det(B+C\Phi^A(1,0)) \neq 0$ where $\Phi^A(t,s)$ is the fundamental matrix of $\dot{y} = A(t)y$. Then (3.2) has a unique solution $\psi(t)$ on $[0,1]$ which can be written in the form

$$(3.4) \qquad \psi(t) = \Lambda(t)d + \int_0^1 \Gamma(t,s)k(s)\,ds$$

where the Green's matrices $\Lambda$ and $\Gamma$ are given by

$$(3.5) \qquad \Lambda(t) = \Phi^A(t,0)[B+C\Phi^A(1,0)]^{-1}$$

and

$$(3.6)\qquad \Gamma(t,s) = \begin{cases} \Phi^A(t,0)[B+C\Phi^A(1,0)]^{-1}B\Phi^A(0,s) & 0 \leq s < t \\ -\Phi^A(t,0)[B+C\Phi^A(1,0)]^{-1}C\Phi^A(1,s) & t < s \leq 1 \end{cases}$$

for all $t,s$ in $[0,1]$.

Proof: (see [4] or [7]).

DEFINITION 3.7 Let $A(t)$, $B,C$ be $p \times p$ matrices. Then $\{A(t),B,C\}$ is called a boundary compatible set if (i) $A(t)$ is measurable on $[0,1]$; (ii) there is an integrable function $m(t)$ on $[0,1]$ with $\|A(t)\| \leq m(t)$ and $\int_0^1 m(t)dt < \infty$; and, (iii) $\det(B+C\Phi^A(1,0)) \neq 0$ where $\Phi^A(t,s)$ is the fundamental matrix of $\dot{y} = A(t)y$.

PROPOSITION 3.8 Let $B$ and $C$ be $p \times p$ matrices. Then there is a matrix $A(t)$ such that $\{A(t),B,C\}$ is a boundary compatible set if and only if the matrix $[B,C]$ has full rank.

Proof: (see [7]).

Propositions (3.3) and (3.8) form the basis for the integral equation representation of (3.2). In particular, we have

THEOREM 3.9 Let $D$ be a domain in $R_p$ and let $I$ be an open interval containing $[0,1]$. Suppose that (i) $f(y,t)$ is measurable in $t$ for each fixed $y$ and continuous in $y$ for each fixed $t$ on $D \times I$; (ii) there is a measurable function $m(t)$ with $\|f(y,t)\| \leq m(t)$ on $D \times I$ and $\int_I m(t)dt < \infty$; (iii) $g$ and $h$ map $D$ into itself; and, (iv) $\{A(t),B,C\}$ is a boundary compatible set. Then the TPBVP (3.1) has the equivalent[+]

[+]This means that an absolutely continuous function $\psi(t)$ is a solution of (3.1) if and only if it is a solution of the integral equation.

*integral representation*

$$(3.10) \qquad y(t) = \Lambda^{ABC}(t)[c-g(y(0))-h(y(1))+By(0)+Cy(1)] + \int_0^1 \Gamma^{ABC}(t,s)\{f(y(s),s)-A(s)y(s)\}ds$$

*where* $\Lambda^{ABC}(t)$ *and* $\Gamma^{ABC}(t,s)$ *are the Green's matrices of the linear problem determined by* $\{A(t),B,C\}$.

*Proof:* (see [7]).

We are now ready to apply the multipoint algorithms developed in section 2 to the solution of (3.1). Assuming that the conditions of theorem (3.9) are satisfied, we can define a mapping $T^{ABC}$ of the Banach space $X = \mathscr{C}([0,1],R_p)$ into itself by setting

$$(3.11) \qquad T^{ABC}(x) = \Lambda^{ABC}(t)\{c-g(x(0))-h(x(1))+Bx(0)+Cx(1)\} + \int_0^1 \Gamma^{ABC}(t,s)\{f(x(s),s)-A(s)x(s)\}ds$$

Then, (3.10) is equivalent to the fixed point problem

$$(3.12) \qquad x = T^{ABC}(x)$$

on $\mathscr{C}([0,1],R_p)$ and we can apply the multipoint algorithms to (3.12).

Now, in order to interpret theorem 2.48 explicitly in terms of f,g and h, we require the Frechet derivatives of the operator $T^{ABC}$. These derivatives are given by

$$(3.13)\qquad (T_x^{ABC})'(w) = \Lambda^{ABC}(t)\{(B-\frac{\partial g}{\partial x}(x(0)))w(0) + (C-\frac{\partial h}{\partial x}(x(1)))w(1)\} + \int_0^1 \Gamma^{ABC}(t,s)\{(\frac{\partial f}{\partial x}(x(s),s)-A(s))w(s)\}ds$$

and

$$(3.14)\qquad (T_x^{ABC})''(u,v) = \Lambda^{ABC}(t)\{[(\frac{\partial}{\partial x}(-\frac{\partial g}{\partial x}(x(0)))),u(0)]v(0) + [(\frac{\partial}{\partial x}(-\frac{\partial h}{\partial x}(x(1)))),u(1)]v(1)\} + \int_0^1 \Gamma^{ABC}(t,s)\{[(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}(x(s),s))),u(s)]v(s)\}ds$$

where, for example,

$$(3.15)\qquad [(\frac{\partial}{\partial x}(-\frac{\partial g}{\partial x}(x(0)))),u(0)] = \sum_{i=1}^{p}(\frac{\partial}{\partial x_i}(-\frac{\partial g}{\partial x}(x(0))))u_i(0)$$

and it is assumed that the indicated partial derivatives exist. We now have

THEOREM 3.16 Let $J = \{A(t),B,C\}$ be a boundary compatible set and let $x_{o,\alpha}(t)$ be an element of $\mathscr{L}([0,1],R_p)$. Suppose that (i) $f(x,t)$, $\frac{\partial f}{\partial x}(x(t),t)$ and $\frac{\partial^2 f}{\partial x^2}(x(t),t)$ are defined, continuous in $x$ and essentially bounded in $t$ for all elements $(x,t)$ of the graphs of the functions $x$ in $\bar{S}_\alpha = \bar{S}(x_{o,\alpha},r_\alpha)$; (ii) $g(x)$, $\frac{\partial g}{\partial x}(x(0))$ and $\frac{\partial^2 g}{\partial x^2}(x(0))$ are defined and continuous for all values $x(0)$ of the functions $x$ in $\bar{S}_\alpha$; (iii) $h(x)$, $\frac{\partial h}{\partial x}(x(1))$, and $\frac{\partial^2 h}{\partial x^2}(x(1))$ are defined and continuous for all values $x(1)$ of the functions $x$ in $\bar{S}_\alpha$; and, (iv) there are positive real numbers $\delta_\alpha$, $B_{o,\alpha}$, $d_{o,\alpha}$, $K_{1,\alpha}$, $K_{2,\alpha}$, $K_\alpha$ and $\eta_{o,\alpha}$ such that

$$(3.17)\quad \sup_{i}\ \sup_{t\in[0,1]}\ \{\sum_{k=1}^{p}\{|\sum_{j=1}^{p}\lambda_{ij}^{J}(t)(b_{jk}-\frac{\partial g_j}{\partial x_k}(x_{o,\alpha}(0)))|$$

$$+|\sum_{j=1}^{p}\lambda_{ij}^{J}(t)(c_{jk}-\frac{\partial h_j}{\partial x_k}(x_{o,\alpha}(1)))| + \int_0^1|\sum_{j=1}^{p}\gamma_{ij}^{J}(t,s)(\frac{\partial f_j}{\partial x_k}(x_{o,\alpha}(s),s)-a_{jk}(s))|ds\}\}$$

$$\leq \delta_\alpha < 1;$$

$$(3.18)\qquad \|x_{o,\alpha}-T^J(x_{o,\alpha})\| \leq (1-\delta_\alpha)d_{o,\alpha};$$

$$(3.19)\quad \sup_{x\in\overline{S}_\alpha}\ \sup_{i}\ \sup_{t\in[0,1]}\ \{\sum_{k=1}^{p}\sum_{\ell=1}^{p}\int_0^1|\sum_{j=1}^{p}\gamma_{ij}^{J}(t,s)\frac{\partial^2 f_j}{\partial x_k\partial x_\ell}(x(s),s)|ds\} \leq K_{1,\alpha};$$

$$(3.20)\quad \sup_{x\in\overline{S}_\alpha}\ \sup_{i}\ \sup_{t\in[0,1]}\ \{\sum_{k=1}^{p}\sum_{\ell=1}^{p}\{|\sum_{j=1}^{p}\lambda_{ij}^{J}(t)\frac{\partial^2 q_j}{\partial x_k\partial x_\ell}(x(0))|+|\sum_{j=1}^{p}\lambda_{ij}^{J}(t)\frac{\partial^2 h_j}{\partial x_k\partial x_\ell}(x(1))|\}\}$$

$$\leq K_{2,\alpha};$$

$$(3.21)\qquad \eta_{o,\alpha} = \gamma_{o,\alpha}^{(\alpha)}d_{o,\alpha}$$

(<u>where</u> $\gamma_{o,\alpha}^{(j)}$ <u>is given by</u> (2.50)) ;

$$(3.22)\qquad K_\alpha = K_{1,\alpha}+K_{2,\alpha};$$

$$(3.23)\qquad B_{o,\alpha} = 1/1-\delta_\alpha;$$

$$(3.24)\qquad h_{o,\alpha} = K_\alpha B_{o,\alpha}\ \eta_{o,\alpha} \leq 1/2;$$

$$(3.25)\qquad r_\alpha = \frac{16}{11}\ \eta_{o,\alpha};$$ <u>and</u>

(3.26) $$\gamma_{o,\alpha}^{(\alpha)} \leq 2$$

<u>where</u> $\lambda_{ij}^{J}(t)$ <u>and</u> $\gamma_{ij}^{J}(t,s)$ <u>are the elements of the Green's matrices</u> $\Lambda^{J}(t)$ <u>and</u> $\Gamma^{J}(t,s)$, <u>respectively. Then the multipoint sequence</u> $\{\psi_{\alpha}^{J}(x_{n,\alpha})\}$ (<u>with</u> $\psi_{\alpha}^{J}(x_{n,\alpha})$ <u>given by</u> (2.47) <u>where</u> $F = F^{J} = I-T^{J}$) <u>converges to a solution</u> $x_{\alpha}^{*}(t)$ <u>of</u> (3.1) <u>in</u> $\overline{S}_{\alpha}$ <u>and the rate of convergence is given by</u> (2.54) <u>for</u> $\alpha = 1,2,\ldots$ .

<u>Proof</u>: We simply verify that the hypotheses of theorem 2.48 are satisfied by the mapping $F^{J} = I-T^{J}$. In view of (3.13) and (3.14), we see that (i), (ii) and (iii) imply that $F^{J}$ is twice continuously differentiable on $\overline{S}_{\alpha}$ (see, for example, [8]). Moreover, from (3.13) and (3.17), we deduce that $(F^{J})'^{-1}_{x_{o,\alpha}} = [I-(T^{J})'_{x_{o,\alpha}}]^{-1}$ exists and that $\|(F^{J})'^{-1}_{x_{o,\alpha}}\| \leq 1/1-\delta_{\alpha}$. Combining this with (3.18), we find that $\|(F^{J})'^{-1}_{x_{o,\alpha}}\|\|F^{J}(x_{o,\alpha})\| \leq d_{o,\alpha}$. From (3.14), (3.19), (3.20), and (3.22), we have $\sup_{x \in \overline{S}_{\alpha}} \|(F^{J})''_{x}\| \leq K_{1,\alpha}+K_{2,\alpha} = K_{\alpha}$. In view of our other assumptions, we immediately see that the hypotheses of theorem 2.48 hold. Hence, $F^{J}(x_{\alpha}^{*}) = (I-T^{J})(x_{\alpha}^{*}) = 0$ and so, $x_{\alpha}^{*}(t)$ is a solution of (3.12) (a fortiori, a solution of (3.1)). Thus, the theorem is established.

The basic strength of theorem 3.16 lies in the possibility of replacing the sequence of operator iterations $x_{n+1,\alpha} = \psi_{\alpha}(x_{n,\alpha})$ by an equivalent sequence of linear TPBVP's. To illustrate what is involved, let us consider the third order method generated by $\psi_2$. Beginning with an initial guess $x_o$ and proceeding formally, we have

(3.27) $$x_{n+1} = \{[I-(T^{J})'_{x_n}]^{-1}[T-(T^{J})'_{x_n}]\}^{2}(x_n)$$

where $J = \{A(t),B,C\}$ is a boundary compatible set. But (3.27) is equivalent to the pair of equations

(3.28a) $$z_n = (T^J)'_{x_n} z_n + [T-(T^J)'_{x_n}](x_n)$$

(3.28b) $$x_{n+1} = (T^J)'_{x_n} x_{n+1} + [T-(T^J)'_{x_n}](z_n)$$

But these equations are both linear and of exactly the same form. Now, let $A_n(s) = \frac{\partial f}{\partial x}(x_n(s),s)$, $B_n = \frac{\partial g}{\partial x}(x_n(0))$ and $C_n = \frac{\partial h}{\partial x}(x_n(1))$. Then it can easily be shown using (3.11) and (3.13) that (3.28a) and (3.28b) are equivalent to the pair of integral equations

(3.29a) $$\begin{aligned} z_n(t) = {} & \Lambda^J(t)\{c-g(x_n(0))-h(x_n(1))+B_n x_n(0)+C_n x_n(1) \\ & - B_n z_n(0)-C_n z_n(1)+Bz_n(0)+Cz_n(1)\} \\ & + \int_0^1 T^J(t,s)\{[A_n(s)z_n(s)+f(x_n(s),s)-A_n(s)x_n(s)]-A(s)z_n(s)\}ds \end{aligned}$$

(3.29b) $$\begin{aligned} x_{n+1}(t) = {} & \Lambda^J(t)\{c-g(z_n(0))-h(z_n(1))+B_n z_n(0)+C_n z_n(1) \\ & - B_n x_{n+1}(0)-C_n x_{n+1}(1)+Bx_{n+1}(0)+Cx_{n+1}(1)\} \\ & + \int_0^1 T^J(t,s)\{[A_n(s)x_{n+1}(s)+f(z_n(s),s)-A_n(s)z_n(s)]-A(s)x_{n+1}(s)\}ds \end{aligned}$$

However, these integral equations are equivalent to the linear TPBVP's

(3.30a) $$\dot{z}_n = A_n(s)z_n + [f(x_n(s),s)-A_n(s)x_n(s)]$$
$$c_n = B_n z_n(0) + C_n z_n(1)$$

(3.30b) $$\dot{x}_{n+1} = A_n(s)x_{n+1}(s) + [f(z_n(s),s)-A_n(s)z_n(s)]$$
$$d_n = B_n x_{n+1}(0) + C_n x_{n+1}(1)$$

where $c_n = c-g(x_n(0))-h(x_n(1))+B_n x_n(0)+C_n x_n(1)$ and $d_n = c-g(z_n(0))-$ $-h(z_n(1))+B_n z_n(0)+C_n z_n(1)$. Now, assuming that the conditions of theorem 3.16, are satisfied, we deduce that (3.27) has a solution and hence that all the pairs (3.28)-(3.30) have solutions. Thus, under the assumptions of the theorem, the multipoint algorithm $x_{n+1} = \psi_2(x_n)$ is equivalent to the successive solution of the pairs of linear TPBVP's (3.30). Since the Jacobian $A_n(s)$ is the same in (3.30a) and (3.30b), we are actually only required to solve the same linear TPBVP at each stage for different forcing functions and so, only one integration of the homogeneous equation is required at each step. Thus, the extra computation required to obtain higher order convergence is small. This represents the major advantage of the multipoint methods. In the general case $x_{n+1,\alpha} = \psi_\alpha(x_{n,\alpha})$, the iteration is equivalent to the solution of $\alpha$ linear TPBVP's with the same homogeneous part.

## 4. Example 1: Temperature Distribution in a Homogeneous Rod+

Consider the nonlinear TPBVP

(4.1) $$\ddot{\theta}(t) = \beta f(\theta(t)), \quad \theta(0) = \theta_o, \quad \theta(\ell) = \theta_\ell$$

+See [6].

which describes the steady state temperature distribution $\theta(\cdot)$ in a homogeneous rod of length $\ell$ where $f(\cdot)$ is the rate of heat generation. We shall suppose that $f(\cdot)$ is given by $f(\theta(t)) = \exp(\theta(t))$, that the units are normalized so that $\ell = 1$, and that $\theta(0) = \theta(1) = 0$. Thus, we wish to solve the TPBVP

$$(4.2) \qquad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \beta \exp(x_1(t)) \end{bmatrix} ; \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$. We now have

THEOREM 4.3 <u>Suppose that</u> $0 < \beta \leq .9$ <u>and that</u> $r = \frac{16}{11}\eta_0 \cong .75$ <u>where</u> $\eta_0 = d_0(1 + \frac{\beta e^r}{4} d_0)$, $d_0 = B\frac{\beta}{2}$, <u>and</u> $B = 1$. <u>Then the multipoint sequence</u> $\underset{\sim}{x}_{n+1} = \psi_2(\underset{\sim}{x}_n)$ <u>with</u> $\underset{\sim}{x}_0 = \underset{\sim}{0}(\cdot)$ <u>converges to a solution</u> $\underset{\sim}{x}^*$ <u>of</u> (4.2) <u>in</u> $\overline{S} = \overline{S}(\underset{\sim}{x}_0, r)$ <u>and the rate of convergence is given by</u>

$$(4.4) \qquad \|\underset{\sim}{x}^* - \underset{\sim}{x}_n\| \leq \frac{16}{11}\left(\frac{5}{16}\right)^n (2h_0)^{3^n - 1}\, \eta_0$$

<u>where</u> $h_0 = \frac{\beta e^r \eta_0}{2} \leq .5$.

<u>Proof:</u> We simply verify the hypotheses of theorem 3.16. We first observe that

$$(4.5) \qquad \frac{\partial f}{\partial \underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ \beta \exp(x_1) & 0 \end{bmatrix}, \quad \frac{\partial g}{\partial \underset{\sim}{x}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \frac{\partial h}{\partial \underset{\sim}{x}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and that

$$(4.6)\qquad \frac{\partial^2 f}{\partial \underset{\sim}{x}^2} = \left[\frac{\partial}{\partial x_j}\left[\frac{\partial f_i}{\partial x_k}\right]\right]_{j=1,2} = \begin{cases} \begin{bmatrix} 0 & 0 \\ \beta\ \exp(x_1) & 0 \end{bmatrix} & j=1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & j=2 \end{cases}$$

$$(4.7)\qquad \frac{\partial^2 g}{\partial \underset{\sim}{x}^2} = \underset{\sim}{0}\ ,\quad \frac{\partial^2 h}{\partial \underset{\sim}{x}^2} = \underset{\sim}{0}\ .$$

Thus, the hypotheses (i), (ii) and (iii) of theorem 3.16 hold. Moreover, if we let $\underset{\sim}{A}(t) = \frac{\partial f}{\partial \underset{\sim}{x}}(\underset{\sim}{x}_o(t))$, $\underset{\sim}{B} = \frac{\partial g}{\partial \underset{\sim}{x}}$ and $\underset{\sim}{C} = \frac{\partial h}{\partial \underset{\sim}{x}}$, then the set $J = \{\underset{\sim}{A}(t),\underset{\sim}{B},\underset{\sim}{C}\}$ is a boundary compatible set (as is easily checked).

Now let $\delta = 0$ and $K_2 = 0$. Then the inequalities (3.17) and (3.20) hold in our case in view of the definition of $J$ and (4.7). Moreover, the operator $T^J$ is given by

$$(4.8)\qquad T^J(\underset{\sim}{x}) = \beta\int_0^1 \Gamma^J(t,s)\begin{bmatrix} 0 \\ \exp(x_1(s)) - x_1(s) \end{bmatrix} ds$$

where $\Gamma^J(t,s)$ is the Green's matrix which corresponds to $J$. Writing (4.8) in component form, we have

$$(4.9)\qquad \begin{aligned} T^J(\underset{\sim}{x})_1(t) &= \beta\int_0^1 \gamma_{12}^J(t,s)[\exp(x_1(s)) - x_1(s)]ds \\ T^J(\underset{\sim}{x})_2(t) &= \beta\int_0^1 \gamma_{22}^J(t,s)[\exp(x_1(s)) - x_1(s)]ds \end{aligned}$$

where

$$(4.10) \qquad \gamma_{12}^{J}(t,s) = \begin{cases} t(s-1) & t < s \\ s(t-1) & s < t \end{cases}$$

$$(4.11) \qquad \gamma_{22}^{J}(t,s) = \begin{cases} s-1 & t < s \\ s & s < t \end{cases}$$

for $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$. Setting $K_1 = K = \frac{\beta e^r}{2}$, we deduce that

$$(4.12) \qquad \sup_{\underset{\sim}{x} \in \overline{S}} \sup_{i=1,2} \sup_{t\in[0,1]} \{ \sum_{k=1}^{2} \sum_{\ell=1}^{2} \int_0^1 | \sum_{j=1}^{2} \gamma_{ij}^{J}(t,s) \frac{\partial^2 f_j}{\partial x_k \partial x_\ell}(\underset{\sim}{x}(s)) | ds \}$$

$$\leqq \sup_{\underset{\sim}{x} \in \overline{S}} \sup_{i=1,2} \sup_{t\in[0,1]} \{ \beta \int_0^1 | \gamma_{i2}^{J}(t,s) \exp(x_1(s)) | ds \}$$

$$\leqq \beta e^r \sup_{i=1,2} \{ \sup_{t\in[0,1]} \int_0^1 | \gamma_{i2}^{J}(t,s) | ds \}$$

But $\int_0^1 | \gamma_{12}^{J}(t,s) | ds = \int_0^t | s(t-1) | ds + \int_t^1 | t(s-1) | ds = (t-t^2)/2$ and $\int_0^1 | \gamma_{22}^{J}(t,s) | ds = \int_0^t s ds + \int_t^1 (1-s) ds = t^2-t+1/2$. It follows that

$$(4.13) \qquad \beta e^r \sup_{i=1,2} \{ \sup_{t\in[0,1]} \int_0^1 | \gamma_{i2}^{J}(t,s) | ds \} \leqq \frac{\beta e^r}{2} = K$$

and hence that (3.19) holds in our case. Moreover, in view of the definition of $T^J$, we can easily see that (3.18) will hold with $d_o = \frac{\beta}{2}$. All that remains is to check that (3.21), (3.24) and (3.26) hold.

Now (3.21) holds by the definition of $\eta_o$. As regards (3.24), we have

$$(4.14) \qquad h_o = K \cdot 1 \eta_o = \frac{\beta e^r}{2} \eta_o = \frac{11}{32} \beta e^r r$$

since $r = \frac{16}{11}\eta_o$. But $r \simeq .75$ and $11(.9)e^{.75}(.75) \leqq 16$ as $e^{.75} \leqq 2.12$. Thus, (3.24) holds. Moreover, since $\gamma_{0,2}^{(2)} = 1 + \frac{K}{2}Bd_o = 1 + \frac{1}{2}\frac{\beta^2}{4}e^r$,

$$\gamma_{0,2}^{(2)} \leqq 1 + \frac{(.9)^2}{8}e^{.75} \leqq 1 + \frac{(.81)(2.12)}{8} \leqq 1.215 \leqq \frac{5}{4} < 2 \tag{4.15}$$

so that (3.26) is satisfied. Thus, the theorem is established.

Of course, an analogous theorem could be proved for any of the multipoint algorithms.

The pair of linear TPBVP's (3.30a) and (3.30b) here take the form

$$\begin{aligned} &\dot z_{n,1} = z_{n,2} \\ &\dot z_{n,2} = \beta\ \exp(x_{n,1}(s))z_{n,1} + \beta\ \exp(x_{n,1}(s))[1-x_{n,1}(s)] \\ &z_{n,1}(0) = z_{n,1}(1) = 0 \end{aligned} \tag{4.16a}$$

$$\begin{aligned} &\dot x_{n+1,1} = x_{n+1,2} \\ &\dot x_{n+1,2} = \beta\ \exp(x_{n,1}(s))x_{n+1,1} - \beta\ \exp(x_{n,1}(s))z_{n,1}(s) + \beta\ \exp(z_{n,1}(s)) \\ &x_{n+1,1}(0) = x_{n+1,1}(1) = 0 \end{aligned} \tag{4.16b}$$

and theorem 4.3 insures the convergence of the sequence $\underset{\sim}{x}_{n+1}(\cdot)$ to the solution of (4.2). The equations (4.18a) and (4.18b) were integrated numerically using a modified fourth order Runge-Kutta method and the results of the computations are indicated in Tables I and II. Table I contains the number of iterations required for "convergence"[+] for various

[+]Convergence is here construed to mean that

$$\|\underset{\sim}{x}_{n+1} - \underset{\sim}{x}_n\| = \sum_{i=1}^{2}\{\max_k |x_{n+1,i}(t_k) - x_{n,i}(t_k)|\} \leqq 10^{-6}$$

where the $t_k$ are the points in the integration routine.

values of $\beta$, while Table II contains the actual solutions. The results in Table II for $\beta = 1$ compare quite favorably with those presented by Bellman in [2]. We also observe that, although theorem 4.3 guaranteed convergence only for $0 < \beta \leqq .9$, the actual computations converged for values of $\beta > .9$.

TABLE I

| $\beta$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| Iterations Required | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 | 4 |

5. Example 2: An Oscillation Problem

Consider the nonlinear differential equation

$$(5.1)\qquad \ddot{y}(t) + 6y(t) + \beta y^2(t) + \cos t = 0$$

which describes an oscillator with a nonlinear restoring force. We wish to determine periodic solutions of (5.1) with period $2\pi$ and so, we impose the boundary conditions

$$(5.2)\qquad y(0)-y(2\pi) = 0, \quad \dot{y}(0)-\dot{y}(2\pi) = 0$$

The boundary value problem (5.1), (5.2) can be written in vector form as

$$(5.3)\quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -6x_1(t)-\beta x_1^2(t)-\cos t \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1(2\pi) \\ x_2(2\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$. We now have

THEOREM 5.4 Suppose that $0 < \beta \leq .5$ and that $r = \frac{16}{11}\eta_o$ where $\eta_o = d_o(1 + \frac{4.16\,\beta d_o}{2})$ and $d_o = .2$. Then the multipoint sequence $\underset{\sim}{x}_{n+1} = \psi_2(\underset{\sim}{x}_n)$ with $\underset{\sim}{x}_o(\cdot) = \underset{\sim}{0}(\cdot)$ converges to a solution $\underset{\sim}{x}^*$ of (5.3) in $\bar{S} = \bar{S}(\underset{\sim}{x}_o, r)$ and the rate of convergence is given by

$$\|\underset{\sim}{x}^* - \underset{\sim}{x}_n\| \leq \frac{16}{11}\left(\frac{5}{16}\right)^n (2h_o)^{3^n - 1}\eta_o \tag{5.5}$$

where $h_o = 4.16\beta\eta_o \leq .5$.

Proof: We simply verify the hypotheses of theorem 3.16. We first observe that

$$\frac{\partial f}{\partial \underset{\sim}{x}} = \begin{bmatrix} 0 & 1 \\ -6-2\beta x_1 & 0 \end{bmatrix}, \quad \frac{\partial g}{\partial \underset{\sim}{x}} = \underset{\sim}{I}, \quad \frac{\partial h}{\partial \underset{\sim}{x}} = -\underset{\sim}{I} \tag{5.6}$$

and that

$$\frac{\partial^2 f}{\partial \underset{\sim}{x}^2} = \left[\frac{\partial}{\partial x_j}\left[\frac{\partial f_i}{\partial x_k}\right]\right]_{j=1,2} = \begin{cases} \begin{bmatrix} 0 & 0 \\ -2\beta & 0 \end{bmatrix} & j = 1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & j = 2 \end{cases} \tag{5.7}$$

$$\frac{\partial^2 g}{\partial \underset{\sim}{x}^2} = \underset{\sim}{0}, \quad \frac{\partial^2 h}{\partial \underset{\sim}{x}^2} = \underset{\sim}{0} \tag{5.8}$$

Thus, the hypotheses (i), (ii) and (iii) of theorem 3.16 are satisfied.

Moreover, if we let $\underset{\sim}{A}(t) = \frac{\partial f}{\partial \underset{\sim}{x}}(\underset{\sim}{x}_o(t))$, $\underset{\sim}{B} = \frac{\partial g}{\partial \underset{\sim}{x}}$ and $\underset{\sim}{C} = \frac{\partial h}{\partial \underset{\sim}{x}}$, then the set $J = \{\underset{\sim}{A}(t), \underset{\sim}{B}, \underset{\sim}{C}\}$ is a boundary compatible set (as is easily checked).

Now let $\delta = 0$ and $K_2 = 0$. Then the inequalities (3.17) and (3.20) hold in our case in view of the definition of $J$ and (5.8). Moreover, the operator $T^J$ is given by

$$(5.9) \qquad T^J(\underset{\sim}{x}) = \int_0^{2\pi} \Gamma^J(t,s) \begin{bmatrix} 0 \\ -\beta x_1^2(s) - \cos s \end{bmatrix} ds$$

where $\Gamma^J(t,s)$ is the Green's matrix which corresponds to $J$. Writing (5.9) in component form, we have

$$(5.10) \qquad \begin{aligned} T^J(\underset{\sim}{x})_1(t) &= \int_0^{2\pi} \gamma_{12}^J(t,s)[-\beta x_1^2(s) - \cos s]ds \\ T^J(\underset{\sim}{x})_2(t) &= \int_0^{2\pi} \gamma_{22}^J(t,s)[-\beta x_1^2(s) - \cos s]ds \end{aligned}$$

where

$$(5.11) \qquad \gamma_{12}^J(t,s) = \begin{cases} -\frac{1}{c}\cos a(\pi - s + t) & t \leq s \\ -\frac{1}{c}\cos a(\pi - t + s) & s < t \end{cases}$$

$$(5.12) \qquad \gamma_{22}^J(t,s) = \begin{cases} \frac{a}{c}\sin a(\pi - s + t) & t \leq s \\ -\frac{a}{c}\cos a(\pi - t + s) & s < t \end{cases}$$

for $0 \leq s \leq 2\pi$, $0 \leq t \leq 2\pi$ and where $a = \sqrt{6}$ and $c = 2\sqrt{6}\sin(\pi\sqrt{6})$. Using (5.7) and the estimates $\int_0^{2\pi} |\gamma_{12}^J(t,s)| ds \leq \frac{2}{c}[4 + \frac{\sin(\pi a)}{a}]$, $\int_0^{2\pi} |\gamma_{22}^J(t,s)| ds \leq \frac{2a}{c}[5 - \cos(\pi a)]$, we deduce that (3.19) holds in our case for $K = 4.16\beta$. Moreover, in view of the definition of $T^J$, we can readily check that (3.18) will be satisfied with $d_o = .2$ since $T^J(\underset{\sim}{Q})_1(t) = \frac{\cos t}{5}$ and $T^J(\underset{\sim}{Q})_2(t) = \frac{\sin t}{5}$. As (3.21) holds by the definition of $\eta_o$, all that remains is to verify (3.24) and (3.26).

Regarding (3.24), we have

$$h_o = Kl\eta_o = (4.16)\beta(.2)(1 + \frac{4.16\beta(.2)}{2}) \leq \frac{1}{2} \tag{5.13}$$

since $\beta \leqq .5$. Moreover, since $\gamma_{o,2}^{(2)} = 1 + \frac{Kd_o}{2}$, (3.26) is clearly satisfied here. Thus, the theorem is established.

Again an analogous theorem could be proved for any of the multipoint algorithms.

The pair of linear TPBVP's (3.30a) and (3.30b) here take the form

$$\begin{aligned} \dot{z}_{n,1} &= z_{n,2} \\ \dot{z}_{n,2} &= -(6+2\beta x_{n,1}(t))z_{n,1} + \beta x_{n,1}^2(t) - \cos t \\ z_{n,1}(0) &= z_{n,1}(2\pi),\ z_{n,2}(0) = z_{n,2}(2\pi) \end{aligned} \tag{5.13a}$$

$$\begin{aligned} \dot{x}_{n+1,1} &= x_{n+1,2} \\ \dot{x}_{n+2,2} &= -(6+2\beta x_{n,1}(t))x_{n+1,1} + \beta z_{n,1}(2x_{n,1}(t) - z_{n,1}) - \cos t \\ x_{n+1,1}(0) &= x_{n+1,1}(2\pi),\ x_{n+1,2}(0) = x_{n+1,2}(2\pi) \end{aligned} \tag{5.13b}$$

and theorem 5.4 insures the convergence of the sequence $\underset{\sim}{x}_{n+1}(\cdot)$ to the solution of (5.3). The equations (5.13a) and (5.13b) were integrated numerically using a modified Runge-Kutta method and the results of the computations are indicated in Tables III and IV. Table III contains the number of iterations required for "convergence"+ for various values of $\beta$, while Table IV contains the actual solutions. We note that the actual computations again converged for larger values of $\beta$ than .5.

TABLE III

| $\beta$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|
| Iterations Required | 2 | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 4 |

+Convergence is here construed to mean that

$$\|\underset{\sim}{x}_{n+1}-\underset{\sim}{x}_n\| = \sum_{i=1}^{2} \{\max_k |x_{n+1,i}(t_k)-x_{n,i}(t_k)|\} \leq 10^{-6}$$

where the $t_k$ are the points in the integration routine.

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