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COMPUTER MECHANIZATION OF REENTRY EQUATIONS WITH FOUR DEGREES OF FREEDOM

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Prepared by
THE UNIVERSITY OF MICHIGAN
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1. Introduction

An effective method for controlling the flight path of a vehicle during reentry into the atmosphere is to utilize aerodynamic lift forces. By controlling the bank angle of the vehicle the lift force vector can be directed upwards or downwards to effect the down-range impact point, or sidewards to effect the cross-range impact point. In computing trajectories for such vehicles there are fundamentally four degrees of freedom which must be represented. These are the three translational degrees of freedom and one rotational, namely the roll motion.

In solving the translational equations for reentry there are sizable computing advantages in using a modified flight-path axis system called the H-frame [1]. The purpose of this report is to add the fourth degree of freedom to these equations in order to allow the trajectory computation for lifting reentry vehicles with controlled bank angle.

2. The H-Frame Translational Equations

The vehicle translational equations of motion will be referred to the H-frame. The H-frame axes x_h, y_h, z_h have their origin at the center of gravity of the vehicle. The z_h axis is directed radially inward toward the center of mass of the earth and the x_h axis is horizontal and points forward in the plane of the motion. The y_h axis is therefore horizontal and perpendicular to the plane of the motion. (See Figure 2.1). The vehicle velocity V_p with respect to an inertial reference frame with origin at the center of the earth has the horizontal component U_h along x_h and the vertical component W_h along z_h (positive downward). By definition the y_h component of velocity

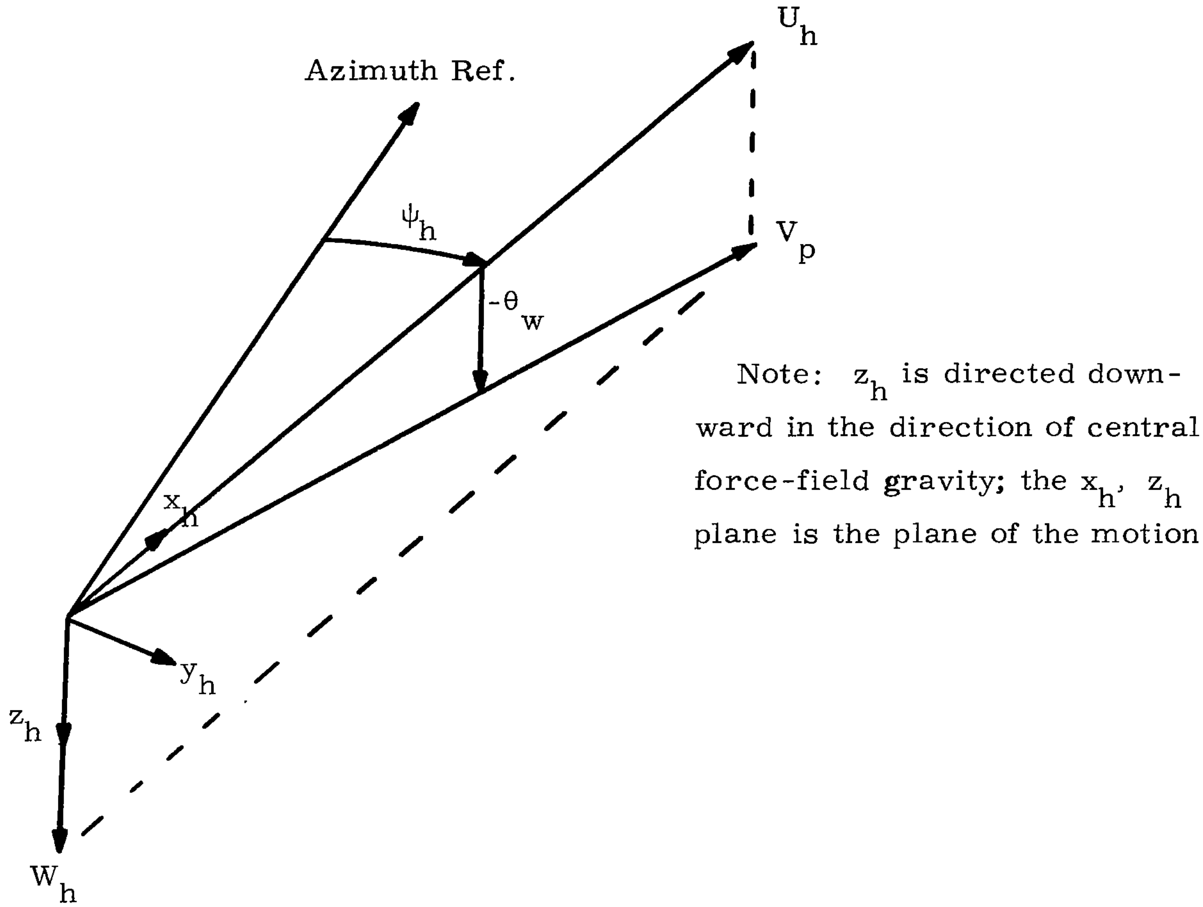

Figure 2.1. H-Frame Axis System.

is zero. Summing forces along the H-frame axes, we obtain the following three translational equations of motion [1]:

$$\dot{U}_h - \frac{U_h W_h}{r} = \frac{X_h}{m} \tag{2.1}$$

$$r_h U_h = \frac{Y_h}{m} \tag{2.2}$$

$$\dot{W}_h + \frac{U_h^2}{r} = \frac{Z_h}{m} + \frac{g_0 r_0^2}{r^2} \tag{2.3}$$

where m is the vehicle mass, X_h, Y_h, and Z_h are the external forces not including inverse-square gravity along x_h, y_h, and z_h, respectively, r is the radial distance from vehicle to the center of the earth, g_0 is the gravity acceleration at reference radius r_0, and r_h is the H-frame angular velocity component along z_h. If we measure the flight-path heading angle ψ_h with respect to an azimuthal reference direction in the horizontal plane (e.g., north), as shown in Figure 2.1, then the following relation holds [1]:

$$\dot{\psi}_h = r_h + \frac{U_h \sin\psi_h}{r} \tan L$$

or from Eq. (2.2)

$$\dot{\psi}_h = \frac{Y_h}{mU_h} + \frac{U_h \sin\psi_h}{r} \tan L \tag{2.4}$$

where L is the latitude. Finally the latitude L and longitude λ are given by the equations [1]

$$\dot{L} = \frac{U_h \cos\psi_h}{r} \tag{2.5}$$

$$\dot{\lambda} = \frac{U_h \sin\psi_h}{r} - r_n \tag{2.6}$$

where r_n is the earth spin rate. Equation (2.1) can be integrated and solved for U_h to obtain the following:

$$U_h = \frac{1}{r}\left[\int_0^t \frac{rX_h}{m}\,dt' + (rU_h)\Big|_{t=0}\right] \tag{2.7}$$

This is simply the angular momentum integral. Eqs. (2.3) through (2.7) represent the translational equations for the H-frame system, where we note that the time rate of change of altitude h is given by

$$\dot{h} = \dot{r} = -W_h \tag{2.8}$$

Often for simplicity one neglects the earth-rotation and assumes that the azimuthal reference direction lies in the plane of the motion.

3. Rotational Equations

In the discussion that follows we will assume that the vehicle during reentry exhibits a prescribed angle of attack α (in fact, we can consider this a control-input) but zero angle of sideslip. This means that the body axes x, y, z will differ from the flight-path axes x_w, y_w, and z_w only by the angle of attack α, as shown in Figure 3.1. We will now proceed to develop the equations for the H-frame force components X_h, Y_h, and Z_h in terms of the flight-path axis force components X_w, Y_w, and Z_w, along with the relationships that include the body-axis roll rate P.

Let us denote the flight-path angular velocity components along the body x, y, and z axes by p, q, and r, respectively. Similarly, we denote the body-axis angular velocity components along the body axes by P, Q, and R (i.e., conventional roll, pitch, and yaw rates). Noting that the only difference between the flight-path and body-axis angular velocity vectors is the angle of attack rate $\dot{\alpha}$, which is directed along the y axis, we can write

$$p = P, \quad q = Q - \dot{\alpha}, \quad r = R \tag{3.1}$$

The flight-path axis angular rate components p, q, and r can then be transformed to components p_w, q_w, and r_w along the flight-path axes, which, in

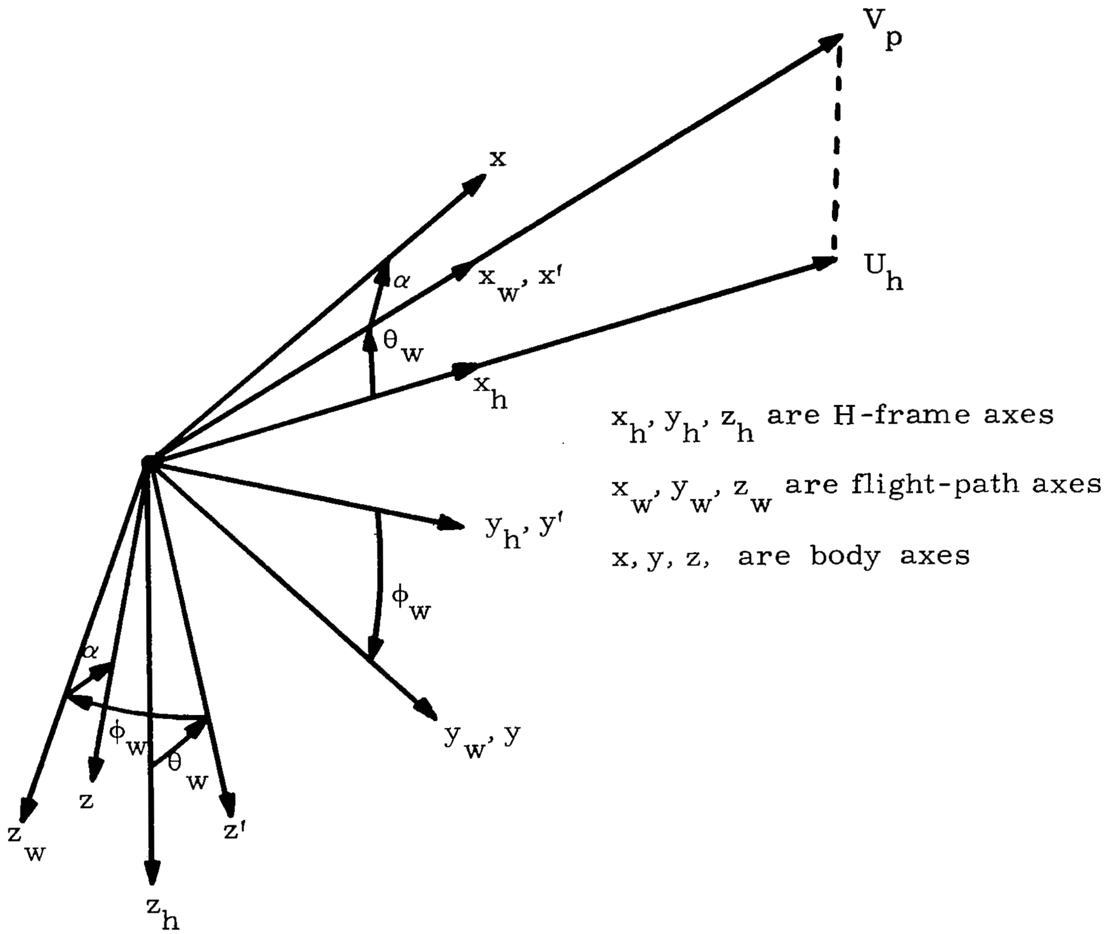

Figure 3.1. H-Frame, Flight-Path, and Body Axes.

terms of P, Q, and R from Eq. (3.1) yields

$$p_w = P\cos\alpha + R\sin\alpha \tag{3.2}$$

$$q_w = Q - \dot{\alpha} \tag{3.3}$$

$$r_w = -P\sin\alpha + R\cos\alpha \tag{3.4}$$

But the flight-path axis yaw rate r_w along z_w is given by $r_w = Y_w/mV_p$ and it is clear from Figure 3.1 that $Y_w = m g_0 r_0^2 \cos\theta_w \sin\phi_w / r^2$ in the absence of any side force due to thrusting or aerodynamic forces. Thus $r_w = g_0 r_0 \cos\theta_w \sin\phi_w / r^2 V_p$ and from Eq. (3.4)

$$R = \left(\frac{g_0 r_0^2}{r^2 V_p}\cos\theta_w \sin\phi_w + P\sin\alpha\right)\frac{1}{\cos\alpha} \tag{3.5}$$

Substituting this into Eq. (3.2) we obtain

$$p_w = P(\cos\alpha + \sin\alpha\tan\alpha) + \frac{g_0 r_0^2}{r^2 V_p}\cos\theta_w \sin\phi_w \tan\alpha \tag{3.6}$$

which is the required relationship between body-axis roll rate P and flight-path axis roll-rate p_w.

The flight-path axis bank angle rate $\dot{\phi}_w$ is related to the flight-path axis angular velocities by the formula

$$\dot{\phi}_w = p_w + \dot{\psi}_w \sin\theta_w \tag{3.7}$$

In writing this equation we have neglected the angular rate of the E frame (local North, East, and downward) with respect to the I(inertial) frame. But $\dot{\psi}_w = \dot{\psi}_h$ in Figure 2.1. From Eqs. (3.6) and (3.7) we then have

$$\dot{\phi}_w = P(\cos\alpha+\sin\alpha\tan\alpha) + \frac{g_0 r_0^2}{r^2 V_p}\cos\theta_w \sin\phi_w \tan\alpha + \dot{\psi}_h \sin\theta_w \tag{3.8}$$

where from Figure 2.1 we see that

$$\sin\theta_w = -\frac{W_h}{V_p} \tag{3.9}$$

and

$$V_p = \frac{U_h}{\cos\theta_w} \tag{3.10}$$

Finally, reference to Figure (3.1) shows that the H-frame force components X_h, Y_h, and Z_h are given by

$$X_h = X_w \cos\theta_w + Z_w \cos\phi_w \sin\theta_w \tag{3.11}$$

$$Y_h = -Z_w \sin\phi_w$$

$$Z_h = -X_w \sin\theta_w + Z_w \cos\phi_w \cos\theta_w \tag{3.12}$$

The flight-path axis forces X_w and Z_w are in turn given by

$$X_w = -\tfrac{1}{2}\rho V_p^2 A C_D \tag{3.13}$$

and

$$Z_w = -\tfrac{1}{2}\rho V_p^2 A C_L \tag{3.14}$$

where C_D is the drag coefficient, C_L is the lift coefficient, A is the characteristic area, and ρ is the atmospheric density. Note that we have neglected the effect of earth rotation in using the velocity V_p with respect to the inertial reference frame for the dynamic pressure calculation. For an equatorial plane this can be corrected by merely adding the eastward earth velocity to U_h in Eq. (3.10). For low-inclination orbits such a correction would be approximately correct.

For instrumentation purposes it may be necessary to compute the body-axis bank angle ϕ from the flight-path bank angle ϕ_w. In general this relationship can be shown to be [3]

$$\tan\phi = \frac{-\sin\theta_w \sin\beta + \cos\theta_w \sin\phi_w \cos\beta}{-\sin\theta_w \sin\alpha\cos\beta - \cos\theta_w \sin\phi_w \sin\alpha \sin\beta + \cos\theta_w \cos\phi_w \cos\alpha} \tag{3.15}$$

where β is the sideslip angle. Since we have assumed in the development here that $\beta = 0$, Eq. (3.15) can be simplified to the expression

$$\tan \phi = \frac{\cos \theta_w \sin \phi_w}{\cos \theta_w \cos \phi_w \cos \alpha - \sin \theta_w \sin \alpha} \tag{3.16}$$

Note that for $\alpha << 1$ and $\theta_w << 1$ Eq. (3.16) reduces to $\tan \phi \cong \tan \phi_w$. Thus by assuming $\phi = \phi_w$ we are only making errors of order α^2 and $\theta_w \alpha$. Similarly, reference to Eq. (3.8) shows that for $\alpha << 1$ and $r \cong r_0$, we can write

$$\dot{\phi}_w \cong P + \frac{g_0 \alpha}{V_p} \cos \theta_w \sin \phi_w + \dot{\psi}_h \sin \theta_w \tag{3.17}$$

where, again, the error is of order α^2.

4. Overall Mechanization

Figure 4.1 shows a block diagram of the equations as presented in Sections 2 and 3. Although it is probably not worthwhile for a reentry computation such as has been considered here, one can add an energy constraint to improve overall computational accuracy [2] in addition to the momentum integral already incorporated into the equation for horizontal velocity U_h. It should be noted that the equations in Figure 4.1 are exact in every respect except that the effect of eastward component of atmospheric velocity due to earth rotation has not been included as part of the velocity in the dynamic-pressure calculation, and the expression for $\dot{\phi}_w$ has neglected the rotation of the E frame with respect to the I frame [see Eq. (3.7)].

Note that thrusting forces can be added to the mechanization in Figure 4.1. Since these are normally referred to body axes, they would need to be resolved through α to the flight-path axes and then added to X_w and Z_w, respectively.

5. Use of Dimensionless Variables and Perturbation Quantities

In order to simplify the equations presented in the previous sections, particularly with respect to computer scaling, it is worthwhile to introduce a number of dimensionless variables. First let us define a dimensionless

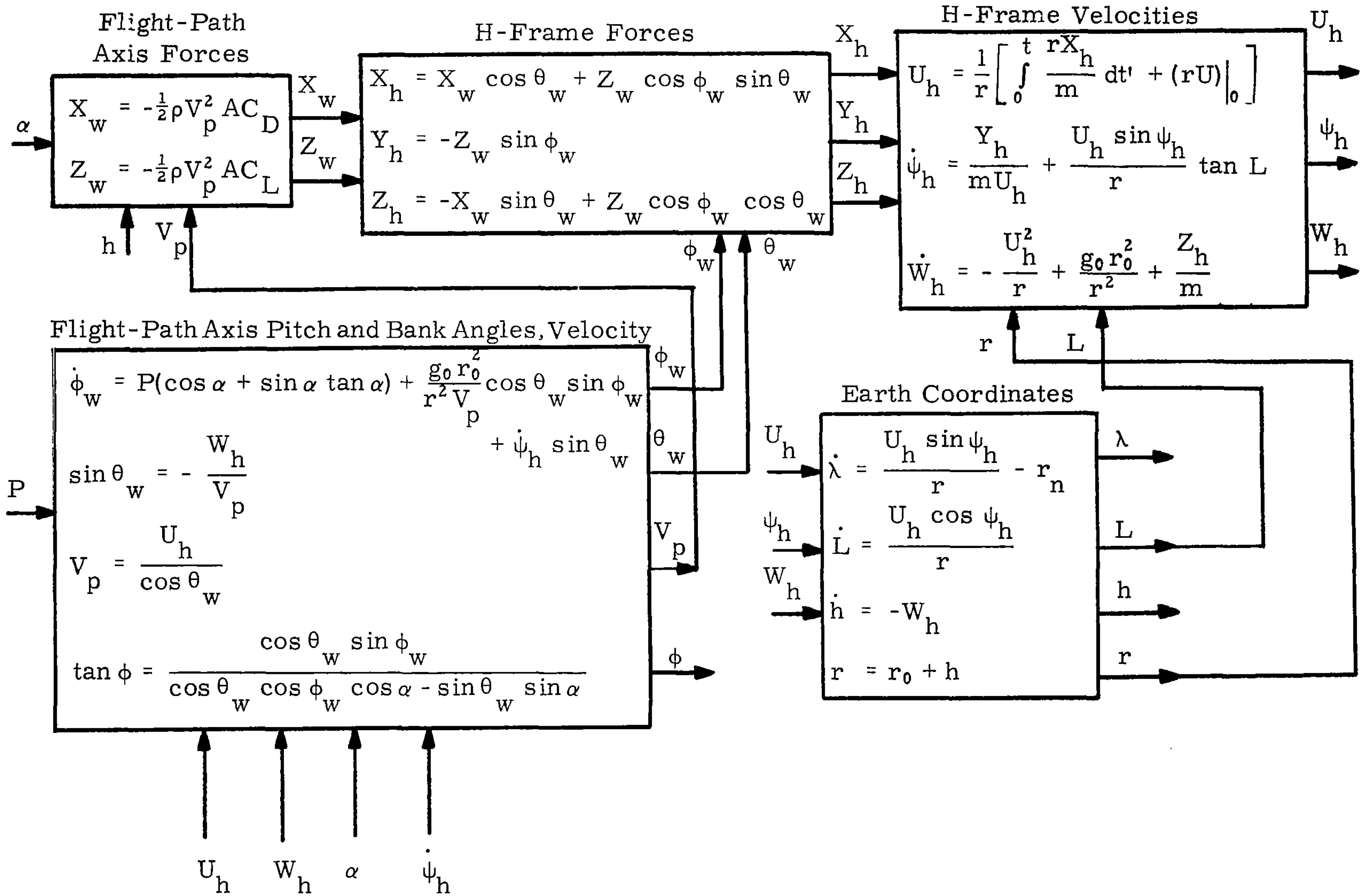

Figure 4.1. Block Diagram of Four-Degree-of Freedom Reentry Equations.

perturbation radius

$$\delta\rho = \frac{r}{r_0} - 1 \tag{5.1}$$

Next we note that the circular-orbit velocity at radius r_0 is given by $\sqrt{g_0 r_0}$. This permits the definition of the following dimensionless velocity variables:

$$u_h = \frac{U_h}{\sqrt{g_0 r_0}} \tag{5.2}$$

$$w_h = \frac{W_h}{\sqrt{g_0 r_0}} \tag{5.3}$$

and

$$v_p = \frac{V_p}{\sqrt{g_0 r_0}} \tag{5.4}$$

Finally, we introduce dimensionless time τ given by

$$\tau = \sqrt{\frac{g_0}{r_0}}\ t \tag{5.5}$$

In terms of these new variables the translational equations of motion (2.3) through (2.8) become

$$u_h = \frac{1}{1 + \delta\rho}\left[\int_0^\tau (1 + \delta\rho)\frac{X_h}{m g_0}\, d\tau' + (1 + \delta\rho)u_h\Big|_{\tau'=0} \right] \tag{5.6}$$

$$\frac{d\psi_h}{d\tau} = \frac{Y_h}{m g_0 u_h} + \frac{u_h \sin\psi_h}{1 + \delta\rho}\tan L$$

$$\frac{dw_h}{d\tau} = \frac{1}{1 + \delta\rho}\left[\frac{1}{1 + \delta\rho} - u_h^2\right] + \frac{Z_h}{m g_0} \tag{5.7}$$

$$\frac{d\lambda}{d\tau} = \frac{u_h \sin\psi_h}{1 + \delta\rho} - r_n\sqrt{\frac{r_0}{g_0}} \tag{5.8}$$

$$\frac{dL}{d\tau} = \frac{u_h \cos\psi_h}{1 + \delta\rho} \tag{5.9}$$

$$\frac{d\delta\rho}{d\tau} = -w_h \qquad (5.10)$$

$$\frac{h}{r_0} = \frac{r_0 - r_{earth}}{r_0} + \delta\rho \qquad (5.11)$$

Similarly, Eqs. (3.8), (3.9), and (3.10) become

$$\frac{d\phi_w}{d\tau} = \sqrt{\frac{r_0}{g_0}}\, P(\cos\alpha + \sin\alpha\tan\alpha) + \frac{1}{(1+\delta\rho)^2 v_p}\cos\theta_w \sin\phi_w + \frac{d\psi_h}{d\tau}\sin\theta_w \qquad (5.12)$$

$$\sin\theta_w = -\frac{w_h}{v_p} \qquad (5.13)$$

$$v_p = \frac{u_h}{\cos\theta_w} \qquad (5.14)$$

If the vehicle computation involves one or more orbits before reentry, then there is a scaling advantage in representing the horizontal velocity in terms of a variation, $\delta u_h = u_h - 1$ [1].

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