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ON ELECTRODYNAMICS OF A GYROTROPIC MEDIUM

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SUMMARY

Formulas are obtained for the radiating energy and the radiation field of charges moving in a gyrotropic medium. These formulas are applied to the case of oscillator rotation and also to the case of radiation of an electron moving in a given medium with constant velocity (Čerenkov effect).

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I. INTRODUCTION

Ginzburg considered in its time [1] the question of the field and of emitted energy by charges moving in an anisotropic medium (see also [2]). However, the possibility of optical activity (gyrotropy) was not taken into account in Guinzburg's works (*). Meanwhile, the emission of electrons, moving in gyrotropic media, has certain specific singularities, thus representing a known interest.

We shall resolve the stated problem by the method constituting a further generalization of the Hamilton method (see [3]), which was applied by Ginzburg in the works previously indicated [1].

2. EQUATIONS FOR POTENTIALS IN A GYROTROPIC MEDIUM

We shall start from field equations in a medium where there is a charge e moving with velocity $\vec{v}$, and also from relations between $\vec{D}$ and $\vec{E}$

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(*) A medium is called gyrotropic, whenever electromagnetic waves, propagating in it with a specific velocity (normal waves), have, generally speaking, an elliptical polarization.

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c} e\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_e) + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}; \tag{2.1}$$

$$\mathrm{div}\,\mathbf{D} = 4\pi e\delta(\mathbf{r}-\mathbf{r}_e); \tag{2.2}$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}; \tag{2.3}$$

$$\mathrm{div}\,\mathbf{H} = 0; \tag{2.4}$$

$$\mathbf{D} = (\varepsilon_{\alpha\beta})\,\mathbf{E}. \tag{2.5}$$

In the case of gyrotropic medium considered here, the dielectric constant $(\varepsilon_{\alpha\beta})$ $(\alpha,\ \beta = x, y, z)$ represents a complex tensor of second rank [4]. In the absence of absorption this tensor is Hermitian [4]:

$$\varepsilon_{\alpha\beta} = \overset{*}{\varepsilon}_{\beta\alpha}. \tag{2.6}$$

Let us introduce, as usual, instead of field intensities the potentials

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\,\varphi,\quad \mathbf{H} = \mathrm{rot}\,\mathbf{A}. \tag{2.7}$$

Note that because of gyrotropy property and (2.6) it is practical to consider during intermediate operations the field vectors as complex quantities. For example, the real value of an electric field is $\vec{E} + \overset{*}{\vec{E}}$, where $\vec{E}$ is a quantity standing in (2.7) etc.

By substituting (2.7), (2.3), (2.5) into Eqs.(2.1), (2.2), we shall obtain for the potentials, the following equations:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}\frac{\partial^2 A_\beta}{\partial t^2}\,\mathbf{e}_\alpha - \mathrm{grad}\,\mathrm{div}\,\mathbf{A} -$$
$$-\frac{1}{c}\frac{\partial}{\partial t}\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}\frac{\partial\varphi}{\partial x_\beta}\,\mathbf{e}_\alpha + \mathrm{k.\ c.} = -4\pi e\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_e), \tag{2.8}$$

$$\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}\left(\frac{1}{c}\frac{\partial}{\partial t}\frac{\partial A_\beta}{\partial x_\alpha} + \frac{\partial^2\varphi}{\partial x_\beta\,\partial x_\alpha}\right) + \mathrm{k.\ c.} = -4\pi e\delta(\mathbf{r}-\mathbf{r}_e), \tag{2.9}$$

where δ is a delta-function; $\vec{r}_e$ is the radius-vector of the electron, $\vec{e}_\alpha$ are unitary vectors of coordinate axes, k. c. means a complex-conjugate expression.

For the solution of these equations we shall make use of an additional condition, constituting a generalization of the well known condition div $\vec{A} = 0$.

$$\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}\frac{\partial A_\beta}{\partial x_\alpha} + \mathrm{k.\ c.} = 0. \tag{2.10}$$

As a result of this condition's application to (2.9), the equation for the scalar potential is separated and, upon simplification with the aid of (2.6) it assumes the form:

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$$\sum_{\gamma} \varepsilon_{\gamma\gamma} \frac{\partial^2 \operatorname{Re} \varphi}{\partial x_\gamma^2} + 2 \sum_{\substack{\alpha, \beta \\ \alpha > \beta}} \operatorname{Re} \varepsilon_{\alpha\beta} \frac{\partial^2 \operatorname{Re} \varphi}{\partial x_\alpha \partial x_\beta} = -2\pi e \delta (\mathbf{r} - \mathbf{r}_e), \tag{2.11}$$

where Re means the real part of z.

3. RADIATION ENERGY

When computing the field energy of a moving electron, we start from the expression

$$\mathscr{H} = \frac{1}{4\pi} \int \mathbf{E}\mathbf{D}^* d\tau + \frac{1}{4\pi} \int \mathbf{H}\mathbf{H}^* d\tau. \tag{3.1}$$

In the presence of dispersion, i.c., when $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}(\omega)$, we must make use of the general expression

$$\mathscr{H} = \frac{1}{2\pi} \iint_{\tau\, t} \mathbf{E} \frac{\partial \mathbf{D}^*}{\partial t} d\tau\, dt + \frac{1}{4\pi} \int \mathbf{H}\mathbf{H}^* d\tau. \tag{3.2}$$

If we introduce expressions (2.5), (2.7) into (3.1) or (3.2), and if we utilize conditions (2.10), we shall obtain that the energy of the field is divided in two parts:

- the energy linked with the longitudinal part of D,

$$\mathscr{H}_l = \frac{1}{4\pi} \int_{\tau} \sum_{\alpha, \beta} \varepsilon_{\alpha\beta} \frac{\partial \varphi}{\partial x_\beta} \frac{\partial \varphi^*}{\partial x_\alpha} d\tau, \tag{3.3}$$

and

- the energy $\mathscr{H}_{tr}$, linked with the transverse part of D and corresponding to the radiation. The expression for $\mathscr{H}_{tr}$ may be written in the form:

$$\mathscr{H}_{tr} = \frac{1}{4\pi} \int_{\tau} \mathbf{E}^{tr} \mathbf{D}^{tr*} d\tau + \frac{1}{4\pi} \int_{\tau} \mathbf{H}\mathbf{H}^* d\tau, \tag{3.4}$$

or, in the presence of dispersion

$$\mathscr{H}_{tr} = \frac{1}{2\pi} \iint_{\tau\, t} \mathbf{E}^{tr} \frac{\partial \mathbf{D}^{tr*}}{\partial t} d\tau\, dt + \frac{1}{4\pi} \int_{\tau} \mathbf{H}\mathbf{H}^* d\tau, \tag{3.5}$$

where

$$\mathbf{E}^{tr} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad D_\alpha^{tr} = -\frac{1}{c} \sum_{\beta} \varepsilon_{\alpha\beta} \frac{\partial A_\beta}{\partial t}. \tag{3.6}$$

Hence it may be seen that the above chosen basic condition (2.10) has a simple physical sense: it expresses the fact that the radiation field is transverse

$$\operatorname{div} \mathbf{D}^{tr} = 0. \tag{3.7}$$

In order to compute the energy emitted by the electron, we shall apply the Hamilton method, used by Guinzburg [1] in the problem of anisotropic medium.

Let us expand the vectorial potential in Fourier series within the limits of a cube with edge L = 1, considering the field at cube boundaries as periodic and let us substitute Eq.(2.8) by a system of equations for the coefficients of this series.

We shall write the series in question in the form:

$$\bar{A} = \sqrt{4\pi}\, c \sum_{l,\lambda} a_{l\lambda} q_{l\lambda} e^{ik_\lambda \bar{r}}, \tag{3.8}$$

where

$$a_{l\lambda} = a_{l\lambda 1} + i a_{l\lambda 2}; \tag{3.9}$$

$q_{l\lambda}(t)$, say the "field coordinates" are the functions of time sought for. Relation (3.9) expresses the case when the polarization of the emitted waves in the considered gyrotropic medium is elliptical. Vectors $a_{l\lambda 1}$ and $a_{l\lambda 2}$ define the plane of the ellipse which describes the end of the electric vector. The directions and the relative values of these vectors are determined from Eqs. (III), (IV) of the Appendix. The index i = 1, 2 corresponds to different types of polarization (the so called ordinary and extraordinary waves; see Appendix).

Let us substitute expression (3.8) for A into (2.8) and multiply the so obtained equation by $\sqrt{4\pi c}\,(a_{l\mu 1} - i a_{l\mu 2})\, e^{-ik_\mu r}$ and let us integrate it over the volume in which the expansion is performed. Having utilized conditions (2.6), (2.10) and the periodicity conditionm we obtain a differential equation with respect to $q_{l\mu}(t)$

$$Q_{l\mu 1} \ddot{q}_{l\mu} + Q_{l\mu 2} q_{l\mu} = \sqrt{4\pi}\, e\mathbf{v}\, (a_{l\mu 1} - i a_{l\mu 2})\, e^{-ik_\mu r_e}, \tag{3.10}$$

where

$$Q_{l\mu 1} = \sum_{\alpha,\beta} \varepsilon_{\alpha\beta}\, a_{l\mu\beta}\, a^*_{l\mu\alpha}, \tag{3.11}$$

$$Q_{l\mu 2} = c^2 [k_\mu^2 a_{l\mu} a^*_{l\mu} - (k_\mu a_{l\mu})(k_\mu a^*_{l\mu})]. \tag{3.12}$$

Eq.(3.10) is substantially simplified and coincides in form with the equation corresponding to the case of vacuum [3], provided we subordinate the coefficients $Q_{l\mu 1}$ and $Q_{l\mu 2}$ to conditions:

$$Q_{l\mu 1} = 1, \tag{3.13}$$

$$Q_{l\mu 2} = \omega_{l\mu}^2 = k_\mu^2 c^2 / n_{l\mu}^2. \tag{3.14}$$

These conditions, constituting according to (3.11) and (3.12) the normalization conditions of $\vec{a}$, are evidently not independent; indeed, substituting into (3.11)-(3.14) the expressions for n^2, $\vec{E}$ (see Appendix) and performing a series of simplifications, we become convinced that the fulfillment of one of the indicat d conditions converts the other one into an identity.

Let us now transform the expression (3.4) for the radiation energy. To that effect we shall substitute in it (3.6), (3.8), utilize the normalization condition, transforming the first addend with the help of (3.11), (3.13), the second addend with the aid of (3.12), (3.14), and we shall apply the condition of periodicity. As a result we obtain

$$\mathcal{H}_{tr} = \sum_{l,\mu} (\dot{q}_{l\mu} \dot{q}^*_{l\mu} + \omega^2_{l\mu} q_{l\mu} q^*_{l\mu}). \tag{3.15}$$

Inasmuch as the radiation field's energy is separated from the energy of the carried field, the obtained expression remains valid also in the case of the presence of dispersion, when one must start from formula (3.5) (for more details on this, see [1]).

4. RADIATION OF AN ELECTRON MOVING UNIFORMLY IN A GYROTROPIC MEDIUM

We shall consider as a first application of the above expounded theory the radiation of an electron, uniformly moving in a gyrotropic medium with a velocity higher than the phase velocity of light in that medium (Cerenkov effect). The radius-vector of the electron varies in this case according to the law:

$$\mathbf{r}_e = \mathbf{v}t. \tag{4.1}$$

The functions $q_\mu(t)$ entering into (3.15) are determined as the solution of Eqs.(3.10) at conditions (3 13), (3.14) and (4.1). These solutions, satisfying the initial conditions $q_\mu = q_\mu = 0$ at $t = 0$, are the following:

$$q_{l\mu} = \frac{\sqrt{4\pi} e\, [(\mathbf{v}\mathbf{a}_{l\mu 1}) - (\mathbf{v}\mathbf{a}_{l\mu 2})]}{\omega^2_{l\mu} - \omega^2_c} \left[e^{-i\omega_c t} - \frac{1}{2}\left(1 + \frac{\omega_c}{\omega_{l\mu}}\right) e^{-i\omega_{l\mu} t} - \right.$$
$$\left. - \frac{1}{2}\left(1 - \frac{\omega_c}{\omega_{l\mu}}\right) e^{i\omega_{l\mu} t} \right], \tag{4.2}$$

where

$$\omega_c = (\mathbf{k}_\mu \mathbf{v}). \tag{4.3}$$

As may be seen from Eq.(3.10), and also directly from (4.2), the solutions, accruing with time and corresponding to the radiation, will take place only at resonance condition, i. e. at $\omega_{l\mu} = \omega_c$. Therefore, taking into account (3.14), we obtain the well known condition for Cerenkov radiation:

$$\cos \vartheta_{l\mu} = 1 / n_{l\mu}\beta, \tag{4.4}$$

where $\vartheta_{l\mu}$ is the angle between $\vec{v}$ and $\vec{k}_\mu$, $\beta = v/c$. This condition follows also from simple interferential considerations and, for that reason, it cannot depend on the properties of the medium. The essential peculiarity of a gyrotropic (and generally anisotropic) medium is the dependence of the refractive index n_i on the propagation direction $n_{i\mu} = n_{i\mu}(\vartheta, \phi)$, where ϕ is the azimuthal angle in the spherical system of coordinates, in which the direction of $\vec{v}$ is taken for the polar axis.

The wave normals of monochromatic waves of Čerenkov radiation form, generally speaking, two families of conical surfaces, corresponding to ordinary and extraordinary waves.

Let us substitute the solution (4.2) into (3.15) and leave in the obtained sum only the terms corresponding to radiation. Further we shall pass from the sum to the integral by means of the well known Hamiltonian method [1, 3], utilizing the expression for the number of waves with frequencies in the range ω, $\omega + d\omega$ and wave vectors lying in the solid angle equal to unity [3].

Performing integration over the angle ϑ and taking into account that in the general case the quantity $n_{i\mu}$ depends on ϑ, ϕ, we obtain (*)

$$(\mathscr{H}_{1r})_i = \frac{e^2 t}{2\pi c^2 v} \int\limits_{\omega} \int\limits_{\varphi=0}^{2\pi} \frac{[(v a_{i1})^2 + (v a_{i2})^2]\, n_i^2\, \omega\, d\omega\, d\varphi}{1 - (1/n_i \sqrt{n_i^2\beta^2 - 1})\,(dn_i/d\vartheta)}, \tag{4.5}$$

whereupon condition (4,4) must be observed. Let us recall that i = 1, 2 is related respectively to ordinary and extraordinary waves. Integration over is performed in the interval, where the condition $v \geqslant c/n_i(\omega)$ is fulfilled.

Formula (4.5) differs in its form from the corresponding formula for an inactive anisotropic medium (see [1, 2]) by the presence of not one but two terms of the type $(\vec{v}\vec{a})^2$. This is linked with the elliptical polarization of norma; waves, which is characterized by two vectors, $\vec{a}_1$, and $\vec{a}_2$. The expressions $\vec{a}$, n, cos $(\widehat{av})$ and others, entering into (4.5) have in the given case another, much more complex form than for the inactive medium (see Appendix).

In case of isotropic medium formula (4.5) is substantially simplified not only on account of polarization nonlinearity as this takes also place in the preceding case, but also on the strength of the fact that now n does not depend on the direction, that is, $dn/a\vartheta = 0$. As follows from (3.11)-(3.14), in an isotropic medium $(\vec{v}\vec{a})^2 = (v^2/\varepsilon)\sin^2\vartheta$, whence, taking into account (4.4) and the independence of parameters of ϕ, we obtain from (4.5) the well known Frank and Tamm formula [5] for Čerenkov losses

$$\mathscr{H}_{1r} = \frac{e^2 v t}{c^2} \int\limits_{\varepsilon\beta^2>1} \left(1 - \frac{1}{\varepsilon\beta^2}\right) \omega\, d\omega. \tag{4.5'}$$

Calculation of Radiation for the Simplest Case

We shall limit ourselves to application of formula (4.5) to the simplest case of a medium characterized by the unique "gyrotropy parameter" ε_g.

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} \varepsilon & -i\varepsilon_g & 0 \\ i\varepsilon_g & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \tag{4.6}$$

characterizing a certain gyrotropic crystal. As may be seen, at $\varepsilon_g = 0$, we obtain a standard isotropic dielectric with dielectric constant ε. For $\varepsilon_g \neq 0$ the properties of the medium in the given direction depend on the angle θ ($\cos\theta = \gamma$) , which forms this direction with physically outlined axis OZ.

The medium is then found to be birefringent and gyrotropic. In case of tensor (4.6), the index of refraction is determined by formula (V) and the ratio a_2/a_1 by formula (VI). The directions of vectors $\vec{a}_1$ and $\vec{a}_2$ is taken along the major axes of the ellipse described by the end of vector $\vec{E}$ (see Appendix).

In order to find the values of a_1 according to (3.11) - (3.14), we shall make use of relations (VI) - (VII). As a result, we shall obtain

$$a_1^2 = \frac{\left[(n^2\gamma^2-\varepsilon)(n^2-\varepsilon)-\varepsilon_g^2\right]^2 + n^4\gamma^2(1-\gamma^2)(n^2-\varepsilon)^2}{\varepsilon\left[(n^2\gamma^2-\varepsilon)(n^2-\varepsilon)-\varepsilon_g^2\right]^2 + \left[\varepsilon(n^2-\varepsilon)^2+\varepsilon_g^2(2n^2-\varepsilon)\right]n^4\gamma^2(1-\gamma^2)} \tag{4.7}$$

The value of a_2 is determined by formulas (4.7) and (VI), taking into account the proportionality of the quantities a_1, a_2 and E_1, E_2.

We shall consider two physically defined cases of electron motion in the considered crystal: the motion along the axiz OZ and the motion perpendicularly to this axis (*).

a) <u>The Electron Moves along the Axis OZ</u>. According to the properties of vector $\vec{E}$ (see Appendix), in the given case, we have

$$(\mathbf{v}\mathbf{a}_2) = 0; \quad \vartheta = 0 \tag{4.8}$$

From considerations of symmetry it follows that both mentioned cone families will be in the given case circular with axis directed along OZ and with uniform distribution of radiation intensity along the cones' cross-section.

Condition (4.4) for Čerenkov radiation yields, upon substitution in it of the expression for $n_{1,2}^2$ [see (V)], an equation relative to $\gamma^2 = \cos^2\theta$, whose solution will be

$$\gamma_{1,2}^2 = \frac{\beta(2\varepsilon^2-\varepsilon_g) \mp \varepsilon_g\sqrt{4\varepsilon+\varepsilon_g^2\beta^2}}{2\beta\left[\beta^2\varepsilon(\varepsilon^2-\varepsilon_g^2)-\varepsilon_g^2\right]}. \tag{4.9}$$

The conditions

$$0 \leqslant \gamma_{1,2}^2 \leqslant 1 \tag{4.10}$$

determine the regions of parameters ε_1, ε_g, β, in which radiation takes place

$$\text{a}')\ \varepsilon \geqslant \frac{1}{\beta^2},\ \varepsilon_g^2 \leqslant \frac{(\varepsilon\beta^2-1)^2}{\beta^4};\ \text{a}'')\ -\frac{1}{\beta^2} \leqslant \varepsilon \leqslant 0,\ -\frac{4\varepsilon}{\beta^2} \leqslant \varepsilon_g^2 \leqslant \frac{(\varepsilon\beta^2-1)^2}{\beta^4};$$

$$\text{b}')\ \varepsilon \geqslant \frac{1}{\beta^2},\ \varepsilon_g \geqslant \frac{\varepsilon\beta^2-1}{\beta^2};\ \text{b}'')\ 0 \leqslant \varepsilon \leqslant \frac{1}{\beta^2},\ \varepsilon_g \geqslant \frac{1-\varepsilon\beta^2}{\beta^2};\ \text{b}''')\ \varepsilon \leqslant 0,\ \varepsilon_g \leqslant \frac{\varepsilon\beta^2-1}{\beta^2}; \tag{4.11}$$

$$\text{c}')\ \varepsilon > \frac{1}{\beta^2},\ \varepsilon_g \leqslant \frac{1-\varepsilon\beta^2}{\beta^2};\ \text{c}'')\ 0 \leqslant \varepsilon \leqslant \frac{1}{\beta^2},\ \varepsilon_g \leqslant \frac{\varepsilon\beta^2-1}{\beta^2};\ \text{c}''')\ \varepsilon \leqslant 0,\ \varepsilon_g \geqslant \frac{1-\varepsilon\beta^2}{\beta^2}.$$

In regions a) both types of waves are emitted, - in regions b) only type-1 (ordinary waves) and in regions c) only type-2 (extraordinary waves).

Let us now make use of formulas (4.7), (V), (VII), and also of conditions (4.4), (4,8) and the symmetry condition. Upon simplifications we obtain from the basic formula (4.5) the following expression for energy losses by the electrom to emission

$$(\mathcal{H}_{tr})_{1,2} = \frac{e^2 v_l}{2c^2} \int \left(1 - \frac{1}{\varepsilon\beta^2}\right)\left[1 \pm \frac{\varepsilon_g \beta (1 + \varepsilon\beta^2)}{(1 - \varepsilon\beta^2)\sqrt{4\varepsilon + \varepsilon_g^2 \beta^2}}\right] \omega \, d\omega \tag{4.12}$$

where integration spreads over the region (4.11).

b) The Electron Moves Perpendicularly to Axis OZ. In this case the radiation pattern no longer has a circular symmetry relative to axis OZ as in the preceding case. The conical surfaces of both families now have a complex shape, whereupon the radiation intensity on the different generatrices is not identical (it is dependent upon the azimuthal angle ϕ). Let us denote by ψ_1 and ψ_2 the angles forming vectors $\vec{a}_1$ and $\vec{a}_2$ with electron velocity direction $\vec{v}$ (let the latter be directed along OY)

As follows from the properties of the considered gyrotropic crystal indicated in the Appendix, following are the relations between the basic angles characterizing the emitted waves:

$$\delta^2 = 1 - \frac{\gamma^2}{\sin^2\varphi}, \tag{4.13}$$

$$\cos^2\psi_1 = \frac{\delta^2 \sin^2 (\widehat{E_1, OZ})}{\cos^2\varphi + \delta^2 \sin^2\varphi}, \tag{4.14}$$

$$\cos^2\psi_2 = \frac{(1 - \delta^2)\cos^2\varphi}{\cos^2\varphi + \delta^2 \sin^2\varphi}, \tag{4.15}$$

where

$$\delta = \cos\vartheta. \tag{4.16}$$

The expression for $\sin (\widehat{E , OZ})$ is determined from (VII). If in (V) we introduce for n^2 the expression for γ^2 from (4.13) requiring the fulfillment of the radiation condition (4.4), we obtain with respect to δ^2 a quadratic equation with coefficients dependent on ε, ε_g, β, ϕ. The solution of this equation is as follows:

$$\delta_{1,2}^2 = \frac{\beta\,[2\varepsilon^2 - \varepsilon_g \cos^2\varphi] \mp \varepsilon_g \sqrt{\varepsilon_g^2 \beta^2 \cos^4\varphi + 4\varepsilon(\varepsilon\beta^2 - 1)\sin^2\varphi}}{2\beta\left[\varepsilon\left(\varepsilon^2 - \varepsilon_g^2\right)\beta^2 + \varepsilon_g^2 \sin^2\varphi\right]}. \tag{4.17}$$

The inequalities

$$0 \leqslant \delta^2 \leqslant 1 \tag{4.18}$$

determine the regions of parameters ε, ε_g, β, in which tadiation takes place

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$$\text{a)}\quad \varepsilon \geqslant \frac{1}{\beta^2},\quad \varepsilon_g \leqslant \frac{\sqrt{\varepsilon(\varepsilon\beta^2-1)}}{\beta};$$

$$\text{b')}\ \varepsilon \geqslant \frac{1}{\beta^2},\quad \varepsilon_g \geqslant \frac{\sqrt{\varepsilon(\varepsilon\beta^2-1)}}{\beta};\ \text{b'')}\ \varepsilon \leqslant 0,\ \varepsilon_g \leqslant -\frac{\sqrt{\varepsilon(\varepsilon\beta^2-1)}}{\beta}; \qquad (4.1\ )$$

$$\text{c')}\ \varepsilon \geqslant \frac{1}{\beta^2},\quad \varepsilon_g \leqslant -\frac{\sqrt{\varepsilon(\varepsilon\beta^2-1)}}{\beta};\qquad \text{c'')}\ \varepsilon \leqslant 0,\ \varepsilon_g \geqslant \frac{\sqrt{\varepsilon(\varepsilon\beta^2-1)}}{\beta}.$$

Both types of waves are emitted in regions a), only type-1 (ordinary waves) and type-2 (extraordinary waves) are emitted respectively in regions b) and c).

In order to obtain a formula of losses to emission, it is necessary to substitute into (4.5) the expression for a_1 and a_2 [see (4.7) and (VI)], the expression for ψ_1 and ψ_2 [see (4.14) and (4.15)] and utilize the expression (V) and conditions (4.4) and (4.11). As a result of the indicated actions, and upon simplification, we obtain for $(\mathscr{H}_{tr})_{1,2}$ the expression:

$$(\mathscr{H}_{tr})_{1,2} = \frac{e^2 vt}{4\pi c^2}\int_{\omega}\int_{\varphi=0}^{2\pi}\left(1-\frac{1}{\varepsilon\beta^2}\right)\left[1 \pm \frac{\varepsilon_g \beta\cos^2\varphi}{\sqrt{4\varepsilon(\varepsilon\beta^2-1)\sin^2\varphi+\beta^2\varepsilon_g^2\cos^4\varphi}}\right]\omega\, d\omega, \qquad (4.20)$$

where integration over ω is spread to regions, whose boundaries are defined by (4.19).

Obviously, integration in formulas (4.13) and (4.20) may carried out to the end at least, in principle, under the condition that the components of the dielectric tensor (4.6) ε and ε_g are given as a function of frequency.

As may be seen from (4.20), in regions, where both types of waves are e emitted, the expression for aggregate losses to radiation coincides in form with expressions (4.5'), giving losses in an isotropic medium with dielectric constant ε. Generally speaking, there will be no quantitative coincidence, since the integration regions over ω will be different.

In the limiting case $\varepsilon_g \to 0$, that is, at transition to isotropic medium, it follows from (4.13) and (4.20) that the losses, constituting in the sum (4.5'), are divided equally between ordinary and extraordinary waves. Meanwhile, it is obvious that no birefringence can take place in an isotropic medium and that there is only a unique value $n^2_{1,2} = \varepsilon$. There is however no contradiction. For $\varepsilon_g \neq 0$, the "normal" waves are ellitpically polarized ones, and the same polarization character remains as when $\varepsilon_g \to 0$, mthat is, in an isotropic medium. But the superimposition of these types of waves yields in that medium a single linearly-polarized wave, which is easy to verify.

Note that in both cases considered integrals (4.13) and (4.20) express sing losses to Čerenkov radiation, mau diverge at one of the integration limits. This case, impossible for an isotropic medium, was already noted in the work of author [2], devoted to electron losses in a nonactive anisotropic medium. In the limiting case of transition to isotropic medium, the region, where there is divergence, just as in the preceding case, degenerates at the point $\varepsilon = 0$. n an isotropic medium this point
to Cerenkov, but to polarization losses.

The question of calculation of polarization losses in an isotropic medium with the help of the Hamiltonian method, and also the calculation of total, as well as of losses to radiation in a gyrotropic medium by other methods will be considered by the present author in a subsequent work.

5. OSCILLATOR EMISSION

Let us now find by means of the basic formula (3.15) the radiation energy of an oscillator in a gyrotropic medium. The radius-vector of the radiating electron $\vec{r}_e$ varies according to the law:

$$\mathbf{r}_e = \mathbf{r}_0 e^{i\omega_0 t}. \tag{5.1}$$

If we assume, as usual, that oscillator dimensions $(\sim \mathbf{r}_0)$ are small by comparison with the wavelength, we shall be in a position to consider that $e^{-i\mathbf{k}_\mu \mathbf{r}_e} \sim 1$. Taking this into account, and also (5.1), Ew.(3.10) takes the form:

$$\ddot{q}_{l\mu} + \omega_{l\mu}^2 q_{l\mu} = \sqrt{4\pi}\, e\,[(\mathbf{r}_0 \mathbf{a}_{l\mu 2}) + i\,(\mathbf{r}_0 \mathbf{a}_{l\mu 1})]\,\omega_0 e^{i\omega_0 t}. \tag{5.2}$$

The solution of such an equation for initial conditions $q_i = \dot{q}_i = 0$ at $t = 0$ is as follows:

$$q_{l\mu} = \frac{\sqrt{4\pi}\, e\, \omega_0 [(\mathbf{r}_0 \mathbf{a}_{l\mu 2}) + i\,(\mathbf{r}_0 \mathbf{a}_{l\mu 1})]}{\omega_{l\mu}^2 - \omega_0^2} \Big[e^{-i\omega_0 t} - \frac{1}{2}\Big(1 + \frac{\omega_0}{\omega_{l\mu}}\Big) e^{-i\omega_{l\mu} t} - \\ - \frac{1}{2}\Big(1 - \frac{\omega_0}{\omega_{l\mu}}\Big) e^{i\omega_{l\mu} t}\Big]. \tag{5.3}$$

Let us substitute (5.3) into (3.15) and take into account, as is usually done in the Hamiltonian method [1, 3], the number of oscillations with frequency in the range ω, $\omega + d\omega$. As a result, we obtain the following expression for the energy of waves with the i-th type of polarization (that is, of ordinary and extraordinary waves), whose normals lie in the solid angle $d\Omega$

$$\mathcal{H}_{l\mu} = \frac{e^2 \omega_0^4 n_{l\mu}^3 t}{4\pi c^3} \,[(\mathbf{r}_0 \mathbf{a}_{l\mu 1})^2 + (\mathbf{r}_0 \mathbf{a}_{l\mu 2})^2]\, d\Omega. \tag{5.4}$$

Here $\mathbf{a}_{l\mu 1}$, $\mathbf{a}_{l\mu 2}$ are determined according to (3.11)-(3.14) and (III); as to the determination of $n_{i\mu}$, see the Appendix. This expression differs from the case of inactive anisotropic medium [1] by the presence of two and not one term $(\vec{r}_0\vec{a})^2$, which is linked with the elliptical polarization of emitted waves, and by another form of $\vec{a}$ and $\underline{n}$.

*** T H E E N D *

Appendix and References follow..

A P P E N D I X

Let us bring forth certain indispensable data on the propagation of plane waves in a gyrotropic nonabsorbing medium. The relationship between vector $\vec{D}$ and $\vec{E}$ is given in the case in question by complex Hermitian tensor $(\varepsilon_{\alpha\beta})$.

$$D_\alpha = \sum_{\beta=1} \varepsilon_{\alpha\beta} E_\beta. \tag{I}$$

On the other hand, in case of plane waves one may obtain from Maxwellian equations the well known expression

$$D_\alpha = -n^2 \sum_{\beta=1}^{3} (\varkappa_\alpha \varkappa_\beta - \delta_{\alpha\beta}) E_\beta \quad (\alpha = 1, 2, 3), \tag{II}$$

where n is the index of refraction, $\vec{k}$ is the unitary vector of wave normal, $\delta_{\alpha\beta}$ is the Kronecker symbol.

According to (I) and (II), we obtain a homogenous system of equations with respect to E_x, E_y, E_z:

$$\sum_{\beta=1}^{3} [n^2 (\varkappa_\alpha \varkappa_\beta - \delta_{\alpha\beta}) + \varepsilon_{\alpha\beta}] E_\beta = 0 \qquad (\alpha = 1, 2, 3). \tag{III}$$

Equating to zero this system's determinant, and taking advantage of the Hermitian state of tensor $\varepsilon_{\alpha\beta}$, we obtain relative to n^2 a biquadratic equation, which is not appropriate to be written here. From this equation two different expressions are obtained for n^2, of which one corresponds to the ordinary wave (n_1^2), and the other - to the extraordinary wave (n_2^2).

The system (III) and the expression for n^2 provide the possibility of finding with a precision to the constant multiplier of vector $\vec{E}$ component, having the form

$$\mathbf{E} \sim \mathbf{E}_1 + i\mathbf{E}_2 \tag{IV}$$

where $\vec{E}_1$ and $\vec{E}_2$ define the plane of the ellipse which describes the end of the electric vector. From (III) one may obtain an unlimited number of pairs of vectors $\vec{E}_1$ and $\vec{E}_2$. In the general case these vectors are not mutually perpendicular, nor are they perpendicular to $\vec{k}$.

In the particular case of a simplest gyrotropic crystal, described by tensor (4.6), the expressions for n, $\vec{E}$, etc. are substantially simplified In this case the solution of the equation for n^2 yields

$$n_{1,2}^2 = \frac{1}{2\varepsilon} \{[2\varepsilon^2 - \varepsilon_g^2 (1 - \gamma^2)] \pm \varepsilon_g \sqrt{\varepsilon_g^2 (1 - \gamma^2)^2 + 4\varepsilon^2\gamma^2}\}, \tag{V}$$

where $\gamma = \cos\theta$, θ being the angle between the wave normal and the axis OZ of the crystal.

Now it is practical to take for vectors $\vec{E}_1$ and $\vec{E}_2$ the main axes of the ellipse that describes the end of vector $\vec{E}$; vector $\vec{E}_1$ lies in a plane formed by the axis of the crystal and the wave normal $\vec{k}$, while E_2 is perpendicular to this plane. The ratio of the quantities $\vec{E}_1^2$ and $\vec{E}_2^2$ is

$$\frac{E_2^2}{E_1^2} = \frac{n^4\gamma^2(1-\gamma_*)\varepsilon_g^2}{n^4\gamma^2(1-\gamma^2)(n^2-\varepsilon)^2 + \left[(n^2\gamma^2-\varepsilon)(n^2-\varepsilon)-\varepsilon_g^2\right]^2} \qquad \text{(VI)}$$

The angle between vector $\vec{E}_1$ and the axis OZ of the crystal is determined by the relation

$$\mathrm{tg}\,(\widehat{\vec{E}_1, OZ}) = \frac{n^2\gamma\sqrt{1-\gamma^2}\,(n^2-\varepsilon)}{(n^2-\varepsilon)(n^2\gamma^2-\varepsilon)-\varepsilon_g^2}. \qquad \text{(VII)}$$

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*** END OF APPENDIX ***

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