NASr-21(05)

N 69 14260

NASA CR 98601

CASE FILE
COPY

MEMORANDUM
RM-5615-NASA
DECEMBER 1968

ON THE DETERMINATION OF SURFACE MOTIONS OF THE EARTH'S CORE

R. H. Ball, Anne B. Kahle and E. H. Vestine

PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND Corporation
SANTA MONICA • CALIFORNIA

NASr-21(05)

MEMORANDUM
RM-5615-NASA
DECEMBER 1968

ON THE DETERMINATION OF SURFACE MOTIONS OF THE EARTH'S CORE

R. H. Ball, Anne B. Kahle and E. H. Vestine

This research is supported by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

This study is presented as a competent treatment of the subject, worthy of publication. The Rand Corporation vouches for the quality of the research, without necessarily endorsing the opinions and conclusions of the authors.

Published by The RAND Corporation

PREFACE

From a knowledge of the geomagnetic field at the earth's surface and its changes with time, we have been able, in recent work, to determine some aspects of the surface motion of the fluid core of the earth, where the field originates. In this paper we have further developed our methods to include some of the fluid mechanics of a rotating core. As well as improving our fluid velocity determinations, these methods make possible an estimate of the hitherto unknown electric current patterns.

This study is one of a series intended to add to our understanding of the magnetic field, including its origin, maintainence, and long-term changes. The results will make possible improved predictions of the strength and patterns of the earth's magnetic field as it affects the radiation belts, and will aid in estimating the magnetic fields likely to be found on other planets. Other recent publications in this series include RM-5091-NASA, Estimated Surface Fluid Motions of the Earth's Core; RM-5192-NASA, Nature of Surface Flow in the Earth's Central Core; and RM-5193-NASA, Comparison of Estimates of Surface Fluid Motions of the Earth's Core for Various Epochs.

ABSTRACT

In earlier papers, the authors derived fluid motions near the surface of the earth's central core from geomagnetic data, using the frozen-flux assumption. Another estimate of the poloidal part of the motion was derived by Rikitake from geomagnetic data using a different method. In the present paper, the general problem of inferring fluid velocities in the core from magnetic data is discussed and previous results are compared with one another. The earlier analysis is extended by allowing for small contributions to secular change from magnetic diffusion, while constraining the velocity to satisfy a quasi-geostrophic condition. The latter dynamic condition is derived from first principles, and allows for electromagnetic forces in addition to the Coriolis force. The under-determined system of equations is solved by applying a variational principle which requires nonsingular solutions corresponding to a given magnetic Reynolds number. Solutions are shown for several values of the Reynolds number and for the first time include estimates of surface electrical-current patterns.

ACKNOWLEDGMENTS

The authors are indebted to Dr. John Kern for valuable discussions, and to Art Richmond for suggesting ideas mentioned in Section 2 and for furnishing his preliminary results on westward drift.

We also wish to thank Dr. Mary Romig for reading and commenting upon an earlier version of the manuscript, Dr. Conway Leovy for his comments on Section 3, and E. S. Batten for reviewing the entire manuscript.

CONTENTS

1. INTRODUCTION

In four previous articles (Vestine and Kahle, 1966; Kahle, Vestine and Ball, 1967; Kahle, Ball, and Vestine, 1967; and Vestine, Ball, and Kahle, 1967) the authors have derived approximate descriptions of fluid motions supposed to take place near the surface of the earth's core. The basic concept underlying this work was that the lines of magnetic flux are moved about by the fluid as though frozen into it, and that the resulting changes in magnetic induction observed at the earth's surface can be used to infer the fluid motions.

Rikitake (1967) has also derived core motions from magnetic data by a different method, wherein one infers the poloidal motions required to produce, over a long period of time, the observed nondipole field from an assumed toroidal magnetic field. This method is different from, and in a sense complementary to, our previous procedure. In our procedure, both the toroidal and poloidal components of the velocity were inferred from the (essentially) instantaneous measured values of the magnetic field and the secular change, with diffusion neglected. Rikitake utilizes a two-step process in which a poloidal magnetic field is first generated by the interaction of a toroidal field (assumed to be of the T_2^o type) with an arbitrary poloidal velocity field, and then diffuses into the mantle to become the nondipole field. We may note that Rikitake, when calculating the equilibrium configuration, neglected transport of the poloidal field, whereas in our complementary approach we neglected diffusion.

It is of considerable interest to see that, despite the differences between our theory and Rikitake's, his poloidal velocity field is similar in its main features to our results, as Rikitake has pointed out (1967). It is conceivable that both theories may be useful, in the sense that whereas the instantaneous rate of change of magnetic field at the core surface may be dominated by transport, the time-averaged or equilibrium properties of the surface field could still be strongly influenced by diffusion of the field from inside the core. However, we disagree with Rikitake's contention that transport of the poloidal field is negligible relative to diffusion and transport of the toroidal field. As we shall show in Section 2, the toroidal magnetic field cannot contribute directly to the secular change field (defined as the rate of change of the poloidal magnetic field) at the surface of or outside the core. This was implicit in our previous work and also that of Roberts and Scott (1965).

In this paper we generalize our previous theory to include diffusion, and add dynamical information. In doing so, we discuss in more detail some of the problems encountered in the determination of the core velocity on the basis of surface measurements.

One aspect of the problem involves the determination of the values of the electric and magnetic fields at the interface between the core and the mantle. This is complicated by inaccuracies in the surface observations of magnetic fields, the absence of data on electric fields, and the difficulty of extrapolation through a mantle of uncertain properties. Except for a few remarks, we shall generally ignore these difficulties by regarding the mantle as nonconductive, by truncating the spherical-harmonic series for the magnetic field, and by regarding the electric field as completely unknown.

The other aspect of the problem, to which this paper is mainly devoted, is the determination of the fluid motion near the core surface from whatever information we have about field values at the core/mantle interface. Roberts and Scott (1965) derived a number of results concerning this problem, and we have also discussed it in our previous papers. We first extend this discussion to include diffusion in Section 2.

Roberts and Scott showed that the important field quantities should be continuous across the thin fluid boundary layer at the surface of the core. The determination of the surface velocity then rests upon finding dynamical constraints equal to the number of significant unknown variables, and the latter number depends upon what assumptions are made about core conditions. For example, if the electric and magnetic fields were both known at the core and magnetic diffusion were assumed negligible, the velocity would be uniquely determined by the induction equation of hydromagnetism.

Without the electric-field data, the velocity is not uniquely determined (Roberts and Scott, 1965; Backus, 1968). This was the case for our earlier work, although we have argued (Vestine, Ball, and Kahle, 1967) that the form of the unobservable part of the motion is limited in such a way that if the magnetic field is sufficiently complex, one might still be able to determine the broad-scale features of the velocity pattern. (Unique numerical answers were obtained by representing the velocity fields in terms of finite series, but this proves nothing.) Our new results tend to support this argument for the poloidal component of flow, but not so well for the toroidal flow (Section 5).

Roberts and Scott also showed that in the absence of diffusion, an arbitrary secular-change field could not be obtained from a nonsingular velocity field through induction. They proposed to use this fact as a means of improving secular-change data (removing uncertainties) by forcing the data into consistency with a finite velocity field. However, our results in this and previous papers indicate that the required discrepancy in secular change is quite large (at least for the finite series used to represent the velocity). Therefore we are moved to consider diffusion to explain part of the observed secular change, especially with regard to the dipole terms.

When diffusion cannot be neglected, additional constraints are needed to determine the velocity, and in Section 3 we derive a plausible constraint from the Navier-Stokes equation. However, when both toroidal and poloidal currents contribute significantly to diffusion, this still leaves too few conditions for the number of unknowns.

In Section 4 we attempt to resolve the dilemma between the singularity or inconsistency in the equations that occur without diffusion and the indeterminacy that occurs with diffusion by formulating a variational principle in which diffusion is minimized. The variational principle contains the magnetic Reynolds number as an arbitrary parameter, for different values of which we proceed to solve the equations. We thus adopt a heuristic approach in which the behavior of the system (the velocity and current patterns and the fit to secular change) is observed for a range of Reynolds number, and allow the results to suggest the most reasonable value.

2. IMPLICATIONS OF THE INDUCTION EQUATION

In our previous studies, we deduced the velocity at the core surface from the hydromagnetic induction equation alone, assuming that diffusion of the magnetic field could be neglected. However, that procedure had the unsatisfactory feature of solving a formally indeterminate equation.

Furthermore, there is reason to doubt whether diffusion is as negligible as previously assumed. We shall therefore reexamine more carefully what can be deduced from the induction equation, taking account of the possibility of diffusion.

The electromagnetic theory of the core is based on the standard equations of hydromagnetism (Elsasser, 1950, 1956), wherein the core fluid is assumed to be a good conductor obeying an isotropic Ohm's law and both displacement and charge-convection currents are neglected. For convenience we begin by recalling the following well-known equations

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \tag{2.1}$$

$$\text{curl } \vec{B} = 4\pi\vec{J} \tag{2.2}$$

$$\vec{B} = \text{curl } \vec{A} \tag{2.3}$$

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} \tag{2.4}$$

where $\vec{J}$ is the current, $\vec{E}$ the electric field, $\vec{B}$ the magnetic induction and ϕ and $\vec{A}$ are scalar and vector potentials (all in e.m.u.). The conductivity, σ, will be assumed uniform in the core. Electric charge need not be considered explicitly.

Equations (2.1) through (2.4) may be combined to obtain the "induction" equation

$$\vec{v} \times \vec{B} = -\vec{E} + \vec{J}/\sigma = \nabla\phi + \frac{\partial\vec{A}}{\partial t} + \frac{\text{curl } \vec{B}}{4\pi\sigma} \tag{2.5}$$

The more familiar form of this equation is obtained by taking the curl, which yields

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}(\vec{v} \times \vec{B}) + \frac{\nabla^2 \vec{B}}{4\pi\sigma} \tag{2.6}$$

The two terms on the right side of Eq. (2.6) represent the effects of induction and diffusion, respectively. The order of magnitude of the ratio of these two terms is usually characterized -- using a scale analysis -- by the magnetic Reynolds number

$$R_m = 4\pi\sigma L|\vec{v}| \tag{2.7}$$

where L is the characteristic length scale. In the core it is usually assumed that $\sigma \sim 3 \times 10^{-6}$ e.m.u., $|v| \sim 10$ km/year = .03 cm/sec, and $L \sim 1000$ km, so $R_m \sim 100$. This implies that diffusion can be neglected for many purposes, and one thereby obtains the "frozen-flux" approximation

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}\ (\vec{v} \times \vec{B}) \tag{2.8}$$

which is equivalent to neglecting $\vec{J}/\sigma$ in Eq. (2.5). This is the equation used in our previous papers.

Returning to the general case, we examine what Eq. (2.5) implies about the velocity of the fluid. By taking the cross-product of the radial unit vector $\hat{i}_r$ with Eq. (2.5), one obtains

$$\vec{v} = \frac{1}{B_r}\left[\vec{B}\, v_r + \hat{i}_r \times \left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right) + \frac{\hat{i}_r \times \vec{J}}{\sigma}\right] \tag{2.9}$$

At the surface of the core, the normal (radial) component of the velocity, v_r, must vanish.* The normal component of $\vec{B}$ and the tangential components of $\nabla\phi$ and $\partial\vec{A}/\partial t$ must be continuous across the core/mantle interface (assumed infinitesimally thin), but the tangential components

*We assume, for simplicity, that the core/mantle surface is spherical, but the argument does not depend on this. Only the toroidal-poloidal representation requires this geometry.

of the current, $\vec{J}_T$, may be discontinuous. To determine $\vec{v}$ from Eq. (2.9), it is sufficient to know ϕ and $\partial\vec{A}/\partial t$ just outside the core and $\vec{J}_T$ just inside.

One might, in principle, determine $\nabla_T\phi$ and $\partial\vec{A}/\partial t$ by making surface measurements and extrapolating them through the mantle -- except for the higher-frequency components, which are shielded by the mantle. However, in practice one lacks sufficient information about either the values of the electric field at the earth's surface or the precise properties of the mantle with which to determine ϕ. On the other hand, one has no means of measuring the core surface current $\vec{J}_T$, so one can only guess at its order of magnitude through assumptions about the structure of the magnetic field in the core, using $\vec{J} = \text{curl}\ \vec{B}/(4\pi)$. In our previous studies, we assumed that $\vec{J}$ is negligible just inside the core (the frozen-field approximation), whereby the velocity at the core surface is given by

$$\vec{v} = (B_r)^{-1}\ \hat{i}_r \times [\nabla\phi + \partial\vec{A}/\partial t] \qquad (2.10)$$

which is just the integrated form of Eq. (2.8).

It should be noted that one expects the tangential velocity immediately adjacent to the boundary to be zero and to increase to a finite value inward through a thin fluid boundary layer, as pointed out by Roberts and Scott (1965). For convenience we will refer to the interface of this boundary layer with the mantle as the "top," and also identify an imaginary surface in the region where the velocity achieves its mainstream value as the "bottom" of the boundary layer, as illustrated schematically in Fig. 1. Now consider the values of the velocity in the mainstream of the fluid near the bottom of this boundary layer, denoted by $\vec{v}_o$. Roberts and Scott argue that $\vec{B}$ will be unchanged across such a layer (i.e., continuous in the infinitesimal-layer sense) and that v_r will still be negligible at the bottom. It follows that $\nabla_T\phi$ and $\partial\vec{A}/\partial t$ are also continuous, so we may write the tangential currents at the top and bottom of the layer, respectively, as

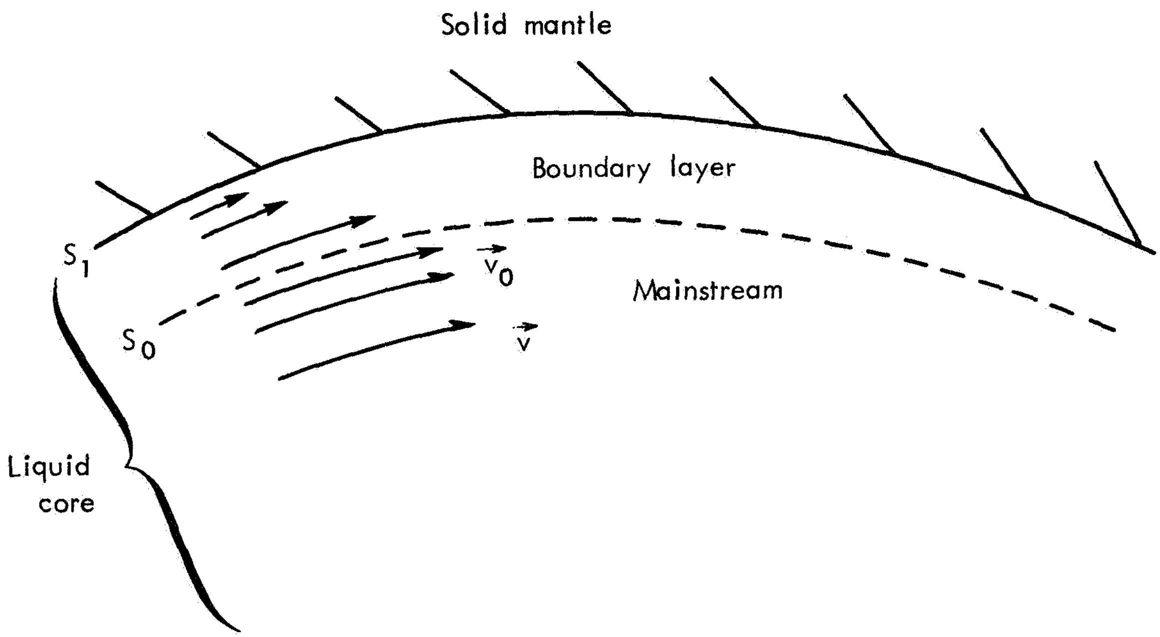

Fig. 1 -- Schematic diagram of Boundary Layer Surfaces. S_o denotes an imaginary surface where velocity achieves mainstream values, denoted as "bottom" of boundary layer. S_1 is the core-mantle interface or "top" of boundary layer.

$$\vec{J}_{T_1} = \sigma\left[-\nabla_T\phi - (\partial\vec{A}/\partial t)_T\right] \qquad (2.11)$$

$$\vec{J}_{T_o} = \sigma\left[-\nabla_T\phi - (\partial\vec{A}/\partial t)_T - B_r\hat{i}_r \times \vec{v}_o\right] \qquad (2.12)$$

which results from taking the cross-product of $\hat{i}_r$ with Eq. 2.9. Hence the excess current at the top of the boundary layer is

$$\vec{J}_{T_1} - \vec{J}_{T_o} = +\sigma B_r\hat{i}_r \times \vec{v}_o \qquad (2.13)$$

One might say that this excess current is induced by the field lines which are dragged through the slower-moving fluid in the boundary layer. It should be emphasized that when we speak of "surface current" in Section 4, we shall mean $\vec{J}_{T_o}$, the current at the outer part of the mainstream, rather than the strict surface value $\vec{J}_{T_1}$.

Since the frozen-flux approximation refers to the mainstream of the fluid, one therefore assumes that $\vec{J}_{T_o}$ is negligible in this case. Although $\vec{J}_{T_1}$ is not necessarily small, its effect will be negligible for a thin boundary layer. Equation (2.10) is still appropriate if one understands the velocity to mean $\vec{v}_o$. The consequence of this argument is that the boundary layer may be simply ignored if it is as thin as Roberts and Scott have argued. One simply allows the fluid to slip as if frictionless. We shall discuss the thickness further in the next section.

Equation (2.10) contains a singularity on those curves for which B_r vanishes, unless the numerator also vanishes thereon. Roberts and Scott discussed this singularity (in other terms) and showed that it implies a constraint on the secular-change field, $\partial\vec{A}/\partial t$. They proposed to use this constraint to improve the data on secular change. However, we shall show below that the diffusion or current term is not so obviously negligible, a result which implies that the errors in secular-change data that these workers propose to correct might instead be attributed to a toroidal surface current.

To clarify the field relationships, we can express the solenoidal velocity and magnetic vectors in terms of toroidal and poloidal fields (Elsasser, 1956). Let

$$\vec{v} \equiv - \text{curl}\ (\hat{i}_r \chi) - \text{curl curl}\ (\hat{i}_r \mu) = \hat{i}_r \times \nabla\chi + \hat{i}_r \nabla_T^2 \mu - \nabla_T \left(\frac{\partial\mu}{\partial r}\right) \quad (2.14)$$

$$\vec{A} \equiv \hat{i}_r T - \hat{i}_r \times \nabla_T S + \nabla\xi \quad (2.15)$$

$$\begin{aligned}\vec{B} &= \text{curl}\ (\hat{i}_r T) + \text{curl curl}\ (\hat{i}_r S) \\ &= - \hat{i}_r \times \nabla_T T - \hat{i}_r \nabla_T^2 S + \nabla_T \left(\frac{\partial S}{\partial r}\right)\end{aligned} \quad (2.16)$$

$$\vec{J} = (4\pi)^{-1} \left[-\hat{i}_r \nabla_T^2 T + \nabla_T \left(\frac{\partial T}{\partial r}\right) + \hat{i}_r \times \nabla_T \left(\nabla_T^2 S + \frac{\partial^2 S}{\partial r^2} \right) \right] \quad (2.17)$$

where we have used tangential operators defined by*

$$\nabla_T \equiv \frac{\hat{i}\theta}{r} \frac{\partial}{\partial\theta} + \frac{\hat{i}\lambda}{r \sin\theta} \frac{\partial}{\partial\lambda} \quad (2.18)$$

$$\nabla_T^2 \equiv \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\lambda^2} \right] \quad (2.19)$$

(The scalars T and χ determine the toroidal components or stream functions of the magnetic and velocity fields, while S and μ determine the poloidal components. These roles are reversed for the $\vec{A}$ and $\vec{J}$ fields, wherein

---

*Some useful identities for these operators are: $\nabla \cdot \nabla_T = \nabla_T \cdot \nabla_T = \nabla_T^2$; $\nabla_T \cdot \nabla = \nabla_T^2 + 2/r\ \partial/\partial r$; curl $(\hat{i}_r F) = - \hat{i}_r \times \nabla_T F$; curl curl $(\hat{i}_r F)$ = - curl $(\hat{i}_r \times \nabla_T F) = -\hat{i}_r \nabla_T^2 F + \nabla_T(\partial F/\partial r) = - \hat{i}_r(\nabla_T^2 F + \partial^2 F/\partial r^2) + \nabla(\partial F/\partial r)$; curl $(\nabla_T F)$ = - curl $(\hat{i}_r\ \partial F/\partial r) = \hat{i}_r \times \nabla_T\ (\partial F/\partial r)$; $\nabla_T \cdot \hat{i}_r = 2/r$.

T determines the poloidal and S the toroidal components.) The scalar ξ is of no physical significance and is determined by the choice of gauge; for example, the condition $\nabla.\vec{A} = 0$ implies that

$$\nabla^2\xi = -\nabla.(\hat{i}_r T) = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T) \tag{2.20}$$

In terms of the poloidal-toroidal representation, and using the fact that $v_{ro} = 0$, we can write the velocity at the surface S_o as

$$\begin{aligned}\vec{v}_o &= (B_r)^{-1}\,\hat{i}_r \times \left(\nabla\phi + \frac{\partial\vec{A}}{\partial t} + \vec{J}_o/\sigma\right) \\ &= -\frac{1}{\nabla_T^2 S}\left\{\hat{i}_r \times \nabla_T\left(\phi + \frac{\partial\xi}{\partial t}\right) + \nabla_T\left(\frac{\partial S}{\partial t}\right)\right. \\ &\quad \left. + \frac{1}{4\pi\sigma}\left[\hat{i}_r \times \nabla_T\left(\frac{\partial T}{\partial r}\right) - \nabla_T\left(\nabla_T^2 S + \frac{\partial^2 S}{\partial r^2}\right)\right]\right\}\end{aligned} \tag{2.21}$$

It is obvious that we can always eliminate the singularity (at $B_r \equiv -\nabla_T^2 S = 0$) by appropriate choices of the (unknown) functions $\partial T/\partial r$ and $\partial^2 S/\partial r^2$. In particular, diffusion of poloidal lines of force, characterized by $(\nabla_T^2 S + \partial^2 S/\partial r^2)$, is equivalent to an adjustment in the secular change, $\partial S/\partial t$, as we stated above.

One can also show that the secular change is independent of the toroidal magnetic field. Consider the radial component of the differential form of the induction equation, which can be derived by multiplying Eq. (2.21) by $\nabla_T^2 S$ and taking the tangential divergence:

$$\frac{\partial B_r}{\partial t} \equiv -\nabla_T^2\left(\frac{\partial S}{\partial t}\right) = \nabla_T\cdot\left(\vec{v}_o \nabla_T^2 S\right) - \frac{1}{4\pi\sigma}\nabla_T^2\left(\nabla_T^2 S + \frac{\partial^2 S}{\partial r^2}\right) \tag{2.22}$$

Obviously the toroidal field has no direct effect on secular change through either transport or diffusion. What occurs is a two-step process, like that used by Rikitake (1967), in which the toroidal field deeper in the core is distorted by inductive motions into poloidal field, which can then penetrate the surface only by diffusion. In this context no explicit assumption need be made about the toroidal field, although we expect from the low ratio of mantle-to-core conductivity (estimated at about 10^{-3}) that J_r will be negligible at the boundary. This implies that $T_o \sim 0$, although $\partial T/\partial r$ may be substantial.

It was pointed out to the authors by Mr. Arthur Richmond that diffusion might reasonably be expected to have a substantial effect on secular change, in contrast to the conclusion based upon the magnetic-Reynolds-number argument. The explanation for this paradox is that at the surface of the core, only the weak poloidal field can contribute to the induction term, whereas the diffusion term is affected by the much stronger poloidal magnetic field which may exist somewhat deeper in the core (see below). To state it another way, one sees from Eq. (2.22) that the length scale which characterizes the induction term is the horizontal scale, whereas the vertical length scale -- which might be much shorter -- is involved in the diffusion term. This situation requires that the actual or "effective" Reynolds number at the core surface be defined by a more careful scale analysis.

We can define a specific number $\bar{R}_m$ in terms of root-mean-square values (denoted by $\langle\ \rangle_{rms}$) of the terms in Eq. (2.22) as

$$\bar{R}_m \equiv \frac{4\pi\sigma \left\langle \nabla_T \cdot (\vec{v}_o \nabla_T^2 S) \right\rangle_{rms}}{\left\langle \nabla_T^2 \left(\nabla_T^2 S + \partial^2 S/\partial r^2 \right) \right\rangle_{rms}} \sim \frac{4\pi\sigma\ V\ L_R^2}{L_H} \sim \left(L_R/L_H \right)^2\ R_m \qquad (2.23)$$

where we assume that the standard R_m is defined by a length scale of the order L_H (L_H and L_R denote horizontal and radial length scales). Thus $\bar{R}_m$ could be of order one if $L_R \sim L_H/10$. This situation is plausible if one considers that coupling of fields by fluid motions deeper in the core could result in poloidal field derivatives comparable to those of the toroidal field; i.e (see Eq. 2.16),**

*At a reasonably high Reynolds number, R_m, the motion can create very large poloidal fields from toroidal fields, or vice versa. The ultimate ratio depends upon the dynamical situation, and therefore takes one into dynamo theory.

$$\partial^2 S/\partial r^2 \sim \partial T/\partial r \tag{2.24}$$

Since the toroidal field deeper in the core (B^T) may be larger than the surface poloidal field (B_o^P),* Eq. (2.24) implies that the poloidal field has a small vertical length scale viz.

$$\frac{L_H^2 B_o^P}{L_R^2} \approx \frac{\partial^2 S}{\partial r^2} \sim \frac{\partial T}{\partial r} \approx B^T \; L_H/L_R$$

or

$$(L_R/L_H) \sim \frac{B_o^P}{B^T} \tag{2.25}$$

To summarize the situation, we suggest that the vertical length scale of the magnetic field in the outer part of the core may be smaller than the horizontal scale or the typical length scale deeper in the core. The latter stipulation is necessary because the typical magnetic Reynolds number for most of the core must be large to have an efficient dynamo.

The possibility of a small $\bar{R}_m$ at the surface is also suggested by our previous results for the core velocity [e.g., Table 4 (Vestine et al., 1967)], wherein we always found a large discrepancy in the fit to the dipole terms; i.e., between the measured values of the spherical harmonic coefficients $\dot{g}_1^o$ and $\dot{h}_1^1$, and the values calculated from our derived velocity field, $\dot{B}_r = - \nabla_T \cdot (B_r \vec{v}_o)$. A typical result was (for epoch 1960): $\dot{g}_1^o$ (measured) = 20.3 γ/year and $\dot{g}_1^o$ (calculated) = 5.3 γ/year. This discrepancy could be explained by a toroidal surface current of the form

$$\vec{J} = \sigma(15\gamma/\text{yr.})a^3/b \; \hat{i}_r \times \nabla_T(\cos\theta) \tag{2.26}$$

---

*Strong toroidal magnetic fields are required in most dynamo theories (Bullard and Gellman, 1954; Elsasser, 1956; Braginskiy, 1967).

(a and b are the radii of earth's outer surface and core, respectively). But this current implies, according to Eq. (2.17), a decrease of the dipole component with depth given by

$$4\pi\sigma(15\gamma/\text{yr.}) = \frac{d^2 g_1^o}{dr^2} \sim \frac{g_1^o}{L_R^2} \qquad (2.27)$$

The implied vertical length scale is therefore about $L_R \simeq 400$ km for $\sigma = 3 \times 10^{-6}$ e.m.u. From another point of view, the fraction of change in $\dot{g}_1^o$ associated with diffusion implies a characteristic time of decay for the dipole field of $\tau \simeq g_1^o/\dot{g}_1^o = 2000$ years, which is close to the historically observed decay rate. This result is consistent with a picture in which the main dipole field of the earth is, at present, generated by freely decaying currents in the outer 400 km or so of the core (depending on the conductivity) superimposed on smaller motion-induced currents that make no long-term contribution to the dipole.

Although it is conceivable that this discrepancy (the inability to represent the observed secular change by induction) is due to the finite velocity representation (only terms up to n = m = 4 in the spherical harmonic coefficients can be handled), numerical experiments indicate that this is not the primary problem. It appears that the fundamental constraint [the singularity of Eq. (2.10) at $B_r = 0$] cited by Roberts and Scott is the real problem--an arbitrary secular change cannot be obtained from a given main field through induction by a nonsingular velocity field. Furthermore, since the discrepancy is large for the well-determined dipole terms, it seems unlikely that it could be attributed to errors in the data--unless there is a strong coupling, via the motion, to higher-order, poorly determined terms in the magnetic field.

To summarize, we see that the induction equation permits one to infer the core surface velocity if both electric and magnetic field data are available and if the current can be neglected. However, since electric-field data are not available, and there is evidence that the current is not completely negligible, additional information is needed.

3. FLUID-DYNAMICAL CONSTRAINT

We have so far used the magnetic induction equation, which describes the effect of motions on the magnetic field. To obtain further information, we turn to the equation for the dynamics of the fluid, the Navier-Stokes equation. In the rotating frame of reference of the earth, this equation can be written*

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} + 2\vec{\Omega} \times \vec{v} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})\right]$$

$$= -\nabla p + \vec{J} \times \vec{B} - \rho\ \nabla\Phi_g + \rho\nu\nabla^2\vec{v} \tag{3.1}$$

where $\vec{\Omega}$ is the angular velocity of the earth, ρ the fluid density, p the pressure, Φ_g the gravitational potential, and ν the kinematic viscosity.

The important feature of this equation for large-scale core motions, as in meteorology, is the very small magnitude of the inertial terms relative to that of the forces, except for the Coriolis term (Elsasser, 1956); i.e., the a priori assumption of small accelerations requires a balancing of forces.** This balance gives us an important constraint on the unknown variables.

In addition to neglecting the inertial terms, we may further simplify Eq. (3.1) by taking account of the presumably small compressibility of the core fluid. Let

$$\rho(\vec{r},t) = \rho_o[1 + \varepsilon(\vec{r},t)] \tag{3.2}$$

---

*We shall not take account of the precessional motion or torques discussed by Malkus (1963), since the average value (over one daily rotation) of these forces will not contribute to the final result we derive in this section.

**The inertial terms $\partial\vec{v}/\partial\vec{t}$ and $(\vec{v}.\nabla)\vec{v}$ are about 10^{-5} times the Coriolis terms for the scales assumed in Section 2.

where ρ_o is constant and $\varepsilon << 1$. Reasonable estimates of ε are sufficiently small that we can assume

$$\mathrm{div}\ \vec{v} = 0 \qquad (3.3)$$

and further neglect ε in all terms of Eq. (3.1) except the large gravitational term (Boussinesq approximation). One can also rewrite $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ as $-1/2\ \nabla|\vec{\Omega} \times \vec{r}|^2$, and include it in an effective gravitational potential

$$\Phi \equiv \Phi_g - 1/2|\vec{\Omega} \times \vec{r}|^2 \qquad (3.4)$$

With these simplifications, the approximate form of the Navier-Stokes equation for the core can be written (Elsasser, 1956; Hide and Roberts, 1961)

$$2\vec{\Omega} \times \vec{v} = -\nabla P + \vec{J} \times \vec{B}/\rho_o - \varepsilon\nabla\Phi + \nu\nabla^2\vec{v} \qquad (3.5)$$

where $P \equiv p/\rho_o + \Phi$. The viscous drag term, $\nu\nabla^2\vec{v}$, is important only in the boundary layer. The Lorentz force, $\vec{J} \times \vec{B}$, is generally about $0.1/\bar{R}_m$ times the Coriolis term. Thus one expects that it may be significant in limited areas of the core surface -- where J and B are larger than average and v small -- if $\bar{R}_m$ is near one; i.e., if diffusion is important in the induction equation.

It should be noted that Eq. (3.5) has been derived on the assumption of quasi-steady motions of the fluid. It is possible that there exist turbulent or wave-like motions which have much shorter time or length scales (Malkus, 1963; Hide, 1966). Such motions would presumably not be observable at the surface of the earth, since high-frequency components of secular change are shielded by the mantle. A possible difficulty is that these motions might be coupled to the observable (large-scale) motions by the nonlinear terms in the equation of motion and in the induction equation. However, to couple effectively, the turbulent motions must have wave numbers and frequencies whose sums or differences are comparable with the motions under study, and the interaction must proceed unidirectionally over a long period of time.

The effect might then require an alteration of the equations of motion for the observable motions in such a way that they would differ from Eq. (3.1). Coupling might lead to frictional effects such as those found in atmospheric problems in the "eddy viscosity" or Reynolds stress. It should therefore be understood that Eq. (3.5) neglects turbulent effects, except as they may be included in the viscosity coefficient ν.

The significance of a fluid boundary layer at the core/mantle interface has been discussed by Roberts and Scott (1965) (as mentioned in Section 2), who based their conclusions upon work of Stewartson (1957, 1960a,b). Although we are in general agreement with them about the significance of the boundary, we believe that their conclusions on the detailed structure of the boundary are susceptible of improvement. In particular, the result of Stewartson's which appears to have been used (1960a) did not include the Coriolis effect, and further assumed an equilibrium between induction and diffusion in the boundary layer for which $\partial\vec{B}/\partial t = 0$. It therefore appears desirable to examine the structure of the boundary layer under more appropriate assumptions.

In the Appendix, we derive an approximate solution of Eq. (3.5) which gives the radial dependence of the variables in the boundary layer. The main assumption is that the thickness of the layer is much less than the horizontal scale length. The general expression found for the fluid velocity in the core, Eq. (A.28), is

$$\begin{aligned}\vec{v}(r,\theta,\lambda) &= \hat{i}_r\, v_r(r,\theta,\lambda)\\ &+ \vec{V}_{MT}(r,\theta,\lambda)\left\{1 - \exp[(r-b)/\delta]\cos[k_o(b-r)]\right\} \qquad (3.6)\\ &+ \hat{i}_r \times \vec{V}_{MT}(r,\theta,\lambda)\exp[(r-b)/\delta]\sin[k_o(b-r)]\left(\frac{\cos\theta}{|\cos\theta|}\right)\end{aligned}$$

where

$$k_o \equiv \sqrt{\Omega/\nu}\; f(\theta,\lambda)$$

$$\delta \equiv \sqrt{\nu/\Omega}\; f(\theta,\lambda)/|\cos\theta|$$

$$f(\theta,\lambda) \equiv +\sqrt{-\gamma_o + \sqrt{\gamma_o^2 + \cos^2\theta}} \tag{3.7}$$

$$\gamma_o \equiv \sigma B_r^2/(2\Omega\rho_o)$$

and where v_r must vanish at $r = b$ and $\vec{V}_{MT}$ is arbitrary, except that these quantities must satisfy $\nabla\cdot\vec{v} = 0$ and are assumed to vary only slowly across the boundary layer (i.e., v_r is zero at S_o as well as at S_1). The velocity at the surface S_o, as discussed in Section 2, is understood to be

$$\vec{v}_o(\theta,\lambda) \equiv \vec{v}(b - \bar{\varepsilon},\theta,\lambda) \cong \vec{V}_{MT}(b,\theta,\lambda) \tag{3.8}$$

where $\bar{\varepsilon} > \delta$ is some depth greater than the boundary-layer thickness, but small relative to mainstream scale lengths.

The nominal width of the boundary layer, δ, is given approximately, for the case of $\gamma_o << |\cos\theta|$ and $\nu = 10^3\ \mathrm{cm^2/sec}$, by

$$\delta \simeq \sqrt{\frac{\nu}{\Omega|\cos\theta|}} \sim \frac{3.7 \times 10^3}{\sqrt{|\cos\theta|}} \tag{3.9}$$

which is the usual Ekman boundary layer, well known in ordinary fluid mechanics. However, sufficiently close to the equator, Eq. (3.7) approximates to

$$\delta \approx \sqrt{\frac{\nu\rho_o}{\sigma B_r^2}} \sim \frac{10^5 \text{ cm}}{|B_r|} \tag{3.10}$$

The last result is precisely that given by Roberts and Scott, and is due to the fact that the Coriolis force term they neglected as a matter of convenience vanishes at the equator. Hence we see that except at the equator the boundary layer is generally thinner than that found by Roberts and Scott, a result which strengthens their conclusion that the magnetic field is continuous across this layer. [This conclusion is examined in the Appendix, Eqs. (A.26) and (A.27).]

Focusing attention on the variables at the surface of the mainstream of the core (S_o in Fig. 1), we see that the fluid equations, consisting of Eqs. (A.3) and (A.12), may be summarized as

$$2\vec{\Omega} \times \vec{v}_o = -\nabla P_o + \frac{\vec{J}_o \times \vec{B}}{4\pi\rho_o} - \varepsilon\nabla\Phi \tag{3.11}$$

$$v_{or} = 0 \tag{3.12}$$

We also found [see Appendix, Eq. (A.13)] that the assumption of negligible mantle conductivity implies that (at S_o)

$$J_{or} = 0 \tag{3.13}$$

If the surface of the core is an equipotential of the effective gravitational potential Φ, then at this surface

$$\nabla\Phi = \hat{i}_n g \tag{3.14}$$

where $\hat{i}_n$ is the normal to the core surface.* Hence we can eliminate the gravitational term from Eq. (3.5) by taking its cross-product with $\hat{i}_n$. If the

*Small deviations of the normal vector from the gravity plumb line might result in large tangential forces. By excluding such deviations, the present theory ignores the possibility of "bumps" on the lower mantle surface, such as those discussed by Kern [1965].

core surface is a smooth spheroid, it is sufficiently accurate for the geometrical analysis to take $\hat{i}_n = \hat{i}_r$; i.e., to neglect the small aspherical effects. (This was already assumed in the electromagnetic theory.) Thus

$$\begin{aligned} \hat{i}_r \times [2\vec{\Omega} \times \vec{v}_o] &\equiv -2\Omega \cos\theta\, \vec{v}_o \\ &= -\hat{i}_r \times \nabla_T P + (B_r/\rho_o)\vec{J}_o \end{aligned} \tag{3.15}$$

4. SURFACE FLOW ANALYSIS

The complete equations for the surface flow consist of Eqs. (2.12) and (3.15), together with Eqs. (3.12) and (3.13), which can be summarized as

$$\vec{J}_o = \sigma\left[-\nabla_T\phi + \hat{i}_r \times \nabla_T\dot{S} - B_r\hat{i}_r \times \vec{v}_o\right] \tag{4.1}$$

and

$$\vec{v}_o \cos\theta = \hat{i}_r \times \nabla_T P/(2\Omega) - B_r\,\vec{J}_o/(2\Omega\rho_o) \tag{4.2}$$

We can eliminate $\vec{J}_o$ between these equations (and also eliminate $\hat{i}_r \times \vec{v}_o$ in terms of $\vec{v}_o$) to obtain

$$\vec{v}_o = \frac{1}{\left(\cos^2\theta + \gamma_o^2\right)}\left[\left(\cos\theta\ \hat{i}_r \times \nabla_T P - \gamma_o\nabla_T P\right)/(2\Omega) + (\cos\theta\ \gamma_o/B_r)(\nabla_T\phi - \hat{i}_r \times \nabla_T\dot{S}) + (\gamma_o^2/B_r)(\hat{i}_r \times \nabla_T\phi + \nabla_T\dot{S})\right] \tag{4.3}$$

where

$$\gamma_o \equiv \sigma B_r^2/(2\Omega\rho_o) \tag{4.4}$$

Resubstitution of $\vec{v}_o$ into Eq. (4.1) gives

$$\vec{J}_o = \frac{\sigma}{\left(\cos^2\theta + \gamma_o^2\right)}\left[(-\nabla_T\phi + \hat{i}_r \times \nabla_T\dot{S})\cos^2\theta - \gamma_o\cos\theta(\hat{i}_r \times \nabla_T\phi + \nabla_T\dot{S}) + \left(\frac{B_r}{2\Omega}\right)(\cos\theta\ \nabla_T P + \gamma_o\hat{i}_r \times \nabla_T P)\right] \tag{4.5}$$

It is noteworthy that $\vec{v}_o$ and $\vec{J}_o$ have finite limits when either $\cos\theta = 0$ or $B_r = 0$ (but not both); for example,

$$\lim_{\theta \to \pi/2} \vec{v}_o = \frac{\hat{i}_r \times \nabla_T \phi + \nabla_T \dot{S}}{B_r} - \frac{\rho_o \nabla_T P}{\sigma B_r^2} \tag{4.6}$$

and

$$\lim_{B_r \to 0} \vec{v}_o = \frac{\hat{i}_r \times \nabla_T P}{2\Omega \cos\theta} \tag{4.7}$$

Hence the expressions (4.3) and (4.5) have only a finite number of isolated singular points -- the intersections of the geographic equator with the curves $B_r = 0$. It should be possible to remove these singularities by proper choices of the functions ϕ and P. By contrast, the frozen-flux equation, (2.10), has a singularity on the entire curve $B_r = 0$.

Equations (4.3) and (4.5) involve the two unknown functions ϕ and P, so $\vec{v}_o$ is still essentially undermined. However, if the effective magnetic Reynolds number $\bar{R}_m$ defined in Eq. (2.23) is large, then $\bar{J}_o$ must be small, and in the limit $\vec{J}_o = 0$, Eq. (4.7) implies that

$$-\nabla_T \phi + \hat{i}_r \times \nabla_T \dot{S} + \frac{B_r \nabla_T P}{2\Omega \cos\theta} = 0 \tag{4.8}$$

which implies from Eq. (4.5) that

$$\vec{v}_o = \frac{\hat{i}_r \times \nabla_T P}{2\Omega \cos\theta} \tag{4.9}$$

or

$$\vec{v}_o = \frac{\hat{i}_r \times \nabla_T \phi + \nabla_T \dot{S}}{B_r} \qquad (4.10)$$

Equations (4.9) and (4.10) are respectively the geostrophic condition and the frozen-flux approximation. Taken together [i.e., Eq. (4.8)], they provide sufficient information to determine all quantities, but the form of the equations is such that they do not always possess a simultaneous solution -- even if Eq. (4.10) were nonsingular by itself. This follows from the fact that the quantities $\hat{i}_r \times \nabla_T P$ and $\hat{i}_r \times \nabla_T \phi$ are toroidal vectors, whereas the given quantity $\nabla_T \dot{S}$ is a poloidal vector.* For example, if B_r were proportional to cos θ (dipole field), $\nabla_T \dot{S}$ could not be different from zero. Hence the simultaneous satisfaction of Eqs. (4.11) and (4.12) for arbitrary B_r and $\dot{B}_r$ is a much more severe constraint than the singularity discussed by Roberts and Scott (1965).

We have found that the data are apparently not perfectly consistent with this constraint; thus either one must have a nonzero current or else the fluid dynamical approximations we have made in deriving Eq. (4.2) are incorrect. (The third possibility, errors in the data, can also be represented as nonzero current, as discussed in Section 2.)

The point of view we should like to adopt is to regard the fluid-dynamic constraint, Eq. (4.2), as correct and the electric current as small but not entirely negligible. This would be the case if $\bar{R}_m$ were moderately large, say of the order of 10. We shall seek an approximate solution for the system for which the velocity is nonsingular and which corresponds to the least value of $\vec{J}_o$. This condition may be expressed mathematically by specifying that the integral

$$I \equiv \int\int |\vec{J}_o|^2 dS \qquad (4.11)$$

over the surface of the core be a minimum, where ϕ and P are treated as independent variables and $\vec{v}_o$ is required to be nonsingular (in some

*Poloidal and toroidal vectors on a spherical surface are linearly independent. They are orthogonal in the sense that the surface integral of their inner product vanishes identically.

sense) as an auxiliary condition. To make this condition explicit, we shall specify that*

$$\iint |\vec{v}_o|^2 dS = \text{constant} < \infty \qquad (4.12)$$

Assuming the constant in Eq. (4.12) is not varied, we can combine Eqs. (4.11) and (4.12) by Lagrange's method of multipliers to obtain the variational principle

$$\delta I = \delta \iint \left[|\vec{J}_o|^2 + \Lambda \; 2\Omega\rho_o\sigma |\vec{v}_o|^2) \right] dS \qquad (4.13)$$

$$= 0$$

where the factor $2\Omega\rho_o\sigma$ has been used to render the multiplier Λ dimensionless. Since the constant in Eq. (4.12) has not been specified, the multiplier Λ cannot be determined from the variational principle. It is, at the outset, a free parameter related to the effective Reynolds number $\bar{R}_m$.

The Euler-Lagrange equations which are obtained from Eq. (4.13) by varying ϕ and P are

$$\nabla_T \cdot \vec{X} = 0 \qquad (4.14)$$

and

$$\nabla_T \cdot \vec{Y} = 0 \qquad (4.15)$$

---

*Although the specification of a bounded integral is not mathematically equivalent to a nonsingular velocity field, it is a more convenient analytical formulation and will suffice to provide the desired result in terms of the numerical methods used here.

where

$$\vec{X} \equiv \left[(\cos^2\theta + \Lambda\gamma_o)(\nabla_T\phi - \hat{i}_r \times \nabla_T\dot{S}) + \frac{B_r}{2\Omega}(\Lambda\hat{i}_r \times \nabla_T P - \cos\theta\ \nabla_T P)\right] \times (\gamma_o^2 + \cos^2\theta)^{-1} \quad (4.16)$$

and

$$\vec{Y} \equiv \left[B_r \cos\theta\ (\hat{i}_r \times \nabla_T\dot{S} - \nabla_T\phi) - \Lambda B_r(\nabla_T\dot{S} + \hat{i}_r \times \nabla_T\phi) + (\rho_o/\sigma)(\Lambda + \gamma_o)\nabla_T P\right] \times (\gamma_o^2 + \cos^2\theta)^{-1} \quad (4.17)$$

Integration of these equations gives the solutions

$$\vec{X} = (2\Omega\rho_o\Lambda/\sigma)\ \text{curl}\ (\hat{i}_r U) \equiv -(2\Omega\rho_o\Lambda/\sigma)\hat{i}_r \times \nabla_T U \quad (4.18)$$

$$\vec{Y} = (2\Omega\rho_o\Lambda/\sigma)\ \text{curl}\ (\hat{i}_r V) \equiv -(2\Omega\rho_o\Lambda/\sigma)\hat{i}_r \times \nabla_T V \quad (4.19)$$

where U and V are arbitrary functions on the core surface (a constant factor $2\Omega\rho_o\Lambda/\sigma$ has been inserted for later convenience).

We can solve Eqs. (4.16) and (4.17) to obtain P and ϕ in terms of U, V, S and B_r:

$$\hat{i}_r \times \nabla_T P = 2\Omega B_r\left(\cos\theta\ \nabla_T U + \Lambda\hat{i}_r \times \nabla_T U\right) + 2\Omega(\cos^2\theta + \gamma_o\Lambda)\ \nabla_T V \quad (4.20)$$

$$\hat{i}_r \times \nabla_T\phi + \nabla_T\dot{S} = \left(B_r^2 + 2\Omega\rho_o\Lambda/\sigma\right)\nabla_T U + B_r\left(\cos\theta\ \nabla_T V - \Lambda\hat{i}_r \times \nabla_T V\right) \quad (4.21)$$

Using Eqs. (4.20), (4.21), (4.3), and (4.5), we obtain expressions for $\vec{v}_o$ and $\vec{J}_o$:

$$\vec{v}_o = B_r \nabla_T U + \cos\theta\ \nabla_T V \tag{4.22}$$

$$\vec{J}_o = \Lambda \left(\sigma B_r \nabla_T V + 2\Omega\rho_o\ \hat{i}_r \times \nabla_T U \right) \tag{4.23}$$

We can also obtain differential equations for U and V by eliminating P and ϕ from Eqs. (4.20) and (4.21) by operating with the tangential divergence:

$$\begin{aligned} \frac{\nabla_T \cdot (\hat{i}_r \times \nabla_T P)}{2\Omega} \equiv 0 = {} & B_r \cos\theta\ \nabla_T^2 U + \nabla_T B_r \cdot (\cos\theta\ \nabla_T U + \Lambda \hat{i}_r \times \nabla_T U) \\ & - \frac{B_r \sin\theta}{r} \nabla_\theta U + (\cos^2\theta + \gamma_o \Lambda) \nabla_T^2 V - \frac{2 \sin\theta \cos\theta}{r} \nabla_\theta V \\ & + \frac{\sigma B_r \Lambda}{\Omega\rho_o} \nabla_r B_r \cdot \nabla_T V \end{aligned} \tag{4.24}$$

$$\begin{aligned} \nabla_T \cdot \left(\hat{i}_r \times \nabla_T \phi + \nabla_T \dot{S} \right) \equiv \nabla_T^2 \dot{S} \equiv -\dot{B}_r \\ = (B_r^2 + 2\Omega\rho_o \Lambda/\sigma) \nabla_T^2 U + 2B_r\ \nabla_T B_r \cdot \nabla_T U \\ + \nabla_T B_r \cdot \left(\cos\theta\ \nabla_T V - \Lambda \hat{i}_r \times \nabla_T V \right) \\ - \frac{B_r \sin\theta}{r} \nabla_\theta V + B_r \cos\theta\ \nabla_T^2 V \end{aligned} \tag{4.25}$$

Equations (4.24) and (4.25) are restatements of the fluid-dynamical constraint, (4.2), and the induction equation, (4.1), respectively, in terms of the functions U and V and the parameter Λ. The variational principle has therefore eliminated the indeterminacy (by effectively adding two new conditions) and has led to expressions which are inherently nonsingular. The remaining task is to choose Λ and to solve Eqs. (4.24) and (4.25) in terms of known values of B_r and $\dot{B}_r$.

We have pursued a heuristic approach by solving the equations for several different values of Λ and examining how the solutions vary with Λ. Equation (4.23) indicates that $\vec{J}_o$ is explicitly proportional to Λ (ignoring the implicit dependence of U and V on Λ), so one expects $\bar{R}_m$ to be approximately inversely proportional to Λ. We have solved Eqs. (4.24) and (4.25) simultaneously by least squares numerical approximation for several values of Λ, corresponding to Reynolds numbers ($\bar{R}_m$) between 20 and about one.

The method of solution of the equations is similar to that used in our earlier papers (Kahle, Vestine, and Ball, 1967), involving extrapolation of main geomagnetic field and secular change data for epoch 1960 (Cain et al., 1967) to the core. Spherical harmonic expansions of the magnetic field are terminated at order and degree four, and the functions U and V are expanded in spherical harmonics to the same order. The 48 coefficients of U and V are obtained by evaluating Eqs. (4.29) and (4.25) at 612 grid points and applying a least-squares and matrix-inversion technique.

Values of $\vec{J}_o$ and $\vec{v}_o$ are computed from Eqs. (4.22) and (4.23). It is convenient to represent these quantities by poloidal and toroidal potentials, wherein*

$$\vec{v}_o = \hat{i}_r \times \nabla_T \chi - \nabla_T \psi \qquad (4.26)$$

and

$$\vec{J}_o = \hat{i}_r \times \nabla_T \kappa - \nabla_T \tau \qquad (4.27)$$

The additional numerical step required to obtain the functions ψ, χ, κ, and τ from U and V is a cumbersome but essential part of the process, since the introduction of the intermediate functions U and V is precisely the means by which singularities have been eliminated.

---

*These new definitions are related to Eqs. (2.14) through (2.17) by: $\chi \equiv \chi$, $\psi \equiv \partial\mu/\partial r$ ($\nabla_T^2\mu = 0$ at core surface), $\kappa = (4\pi)^{-1}(\nabla_T^2 S + \partial^2 S/\partial r^2)$, and $\tau = -(4\pi)^{-1}\, \partial T/\partial r$ ($\nabla_T^2 T = 0$ at core surface).

5. DISCUSSION OF RESULTS

Initial results obtained from the variational principle, Eq. (4.13), gave rather small values of the zonal flow velocity (the longitude-independent part of χ), including a westward drift term of about .005°/yr. Also, the fit to the secular change field was poor for values of $\bar{R}_m$ greater than about one.

This result was apparently due to the unnecessarily severe restriction imposed on the zonal flow velocity by the form of the variational principle. One can show that zonal flow terms, given by

$$\vec{v}^z = \hat{i}_r \times \nabla_T \left[r \sum_{n=1}^{\infty} A_n^o P_n^o(\theta)\right] \qquad (5.1)$$

have no effect on the fluid constraint, Eq. (4.2). Therefore one can obtain greater freedom in satisfying the induction equation (fitting secular-change data) by altering the side condition of the variational principle, Eq. (4.12), so that the integral of $|\vec{v}_o - \vec{v}^z|^2$ is constrained rather than $|\vec{v}_o|^2$. This change leaves Eqs. (4.23) and (4.24) unaltered, but adds $\vec{v}^z$ to the right side of Eq. (4.22) so that

$$\vec{v}_o = \vec{v}^z + B_r \nabla_T U + \cos\theta \, \nabla_T V \qquad (5.2)$$

It also adds terms with the additional (not entirely independent) unknowns A_n° to Eq. (4.25). The results in this paper were obtained with these modified equations.

With the modified equations we obtained a better fit to the secular-change data and reasonable values of zonal velocity. Table 1 shows the derived secular-change coefficients for several values of $\bar{R}_m$, together with original data of Cain (1967) for epoch 1960 and also the comparable results obtained by our previous method (Kahle, Vestine, and Ball, 1967). The fit is acceptable at $\bar{R}_m = 1.2$, with all coefficients having the correct sign and an r.m.s. error of only 13 percent. As $\bar{R}_m$ is increased, the fit declines only gradually for most coefficients, although a few -- notably $\dot{h}_1^1$ -- deteriorate much more rapidly. The tendency for

Table 1

COMPARISON OF DERIVED SECULAR CHANGE COEFFICIENTS

n	m	Coefficients Derived from Variational Principle				Previous Method	
		$\Lambda = .0175$, $\bar{R}_m = 1.2$	$\Lambda = .008$, $\bar{R}_m = 2.2$	$\Lambda = .004$, $\bar{R}_m = 4.3$	$\Lambda = .001$, $\bar{R}_m = 20$	$R = \infty$	Original Data (Cain 1960)
		$\dot{g}_n^m$					
1	0	+ 12.8	+ 13.2	+ 12.1	+ 7.2	+ 5.4	+ 14.0
1	1	+ 7.5	+ 6.7	+ 7.9	+ 10.4	+ 16.8	+ 8.8
2	0	- 20.1	- 18.1	- 16.8	- 15.3	- 17.1	- 23.3
2	1	- 1.3	- 2.4	- 2.8	- 3.8	- 4.4	- 0.1
2	2	- 3.0	- 1.7	- 1.3	- 1.2	- 1.1	- 4.6
3	0	- 1.2	- 1.7	- 0.9	- 1.9	- 1.4	- 0.9
3	1	- 8.6	- 6.8	- 6.4	- 6.4	- 6.4	- 10.6
3	2	+ 2.5	+ 2.1	+ 1.3	+ 0.9	+ 0.9	+ 2.3
3	3	- 5.4	- 4.5	- 3.2	- 1.9	- 4.0	- 5.9
4	0	+ 1.4	+ 1.0	+ 0.6	+ 0.5	+ 0.8	+ 1.5
4	1	+ 0.5	+ 0.04	+ 0.1	- 0.4	- 0.7	+ 0.9
4	2	- 1.6	- 1.4	- 1.0	- 0.8	- 0.9	- 1.8
4	3	+ 0.9	+ 0.8	+ 0.5	+ 0.2	+ 0.6	+ 0.7
4	4	- 3.0	- 2.9	- 2.9	- 2.8	- 2.8	- 3.0
		$\dot{h}_n^m$					
1	1	- 2.3	+ 0.9	+ 4.5	+ 8.2	+ 1.7	- 3.7
2	1	- 13.2	- 11.4	- 10.5	- 10.4	- 12.8	- 14.3
2	2	- 16.1	- 13.7	- 10.7	- 8.2	- 13.7	- 16.6
3	1	+ 5.8	+ 5.5	+ 5.1	+ 4.2	+ 5.3	+ 5.2
3	2	+ 1.7	+ 0.6	+ 0.2	- 0.4	+ 1.5	+ 2.5
3	3	- 6.7	- 7.1	- 7.8	- 8.3	- 7.5	- 7.0
4	1	- 2.2	- 2.5	- 2.5	- 2.6	- 2.1	- 2.2
4	2	- 0.2	- 0.6	- 1.0	- 1.5	- 0.3	- 0.1
4	3	+ 1.9	+ 2.3	+ 2.6	+ 3.0	+ 2.1	+ 1.9
4	4	- 6.1	- 5.5	- 4.9	- 4.7	- 6.2	- 6.5
r.m.s. error							
	core surface	12.6%	21.9%	33.0%	42.6%	30.0%	
	earth surface	13.4%	25.5%	35.9%	48.7%	38.1%	

coefficients of higher order to be represented somewhat better than those of lower order is due, at least in part, to the fact that the numerical fit was made at the core surface, where higher-order terms are considerably amplified.

The values of westward drift velocity (A_1^o) obtained with the modified equations all lie in the narrow range .12 to .15°/yr. (.13°/yr equals .025 cm/sec or 7.9 km/yr linear velocity at the equator on the core surface.) This is closer to the values normally found; in particular, it agrees well with the estimated range .13 ± .3°/yr calculated by Richmond [1968] by a very different method. Furthermore, in contrast to our previous method, the value obtained by our new method appears to be relatively insensitive to slight changes in the data, such as order and degree of truncation.

Contour maps of the functions ψ, χ, κ, and τ, which characterize $\vec{v}_o$ and $\vec{J}_o$, are shown in Figs. 2 through 4 for several values of $\bar{R}_m$. Coefficients of the spherical harmonic expansions for these functions, defined by the general form

$$\psi = b\sum_{n=1}^{4} \sum_{m=1}^{n} P_n^m(\theta) \left(\alpha_n^m \cos m\lambda + \beta_n^m \sin m\lambda\right)$$

are given in Tables 2 and 3. Figure 2 shows ψ (the poloidal part of the velocity) for $\bar{R}_m = 1.2$, 4.3, and 20, plus the corresponding result from our previous method, denoted by $\bar{R}_m = \infty$. The patterns show a small change in form and a gradual increase in intensity as $\bar{R}_m$ increases. The $\bar{R}_m = 20$ case clearly resembles the old $\bar{R}_m = \infty$ case, which is interesting because the latter was obtained with no fluid-dynamic constraint. Contours of the toroidal velocity represented by the stream function χ -- minus the uniform westward drift term $A_1^o \cos\theta$ -- are shown in Fig. 3 for the same four cases. Again the form and intensity change only gradually as $\bar{R}_m$ is increased, but here the patterns do not tend to converge toward the old $\bar{R}_m = \infty$ result -- at least up to $\bar{R}_m = 20$. The $\bar{R}_m = \infty$ pattern shows a much greater intensity (note the different contour interval) and a different pattern of flow, although in some of the more intense

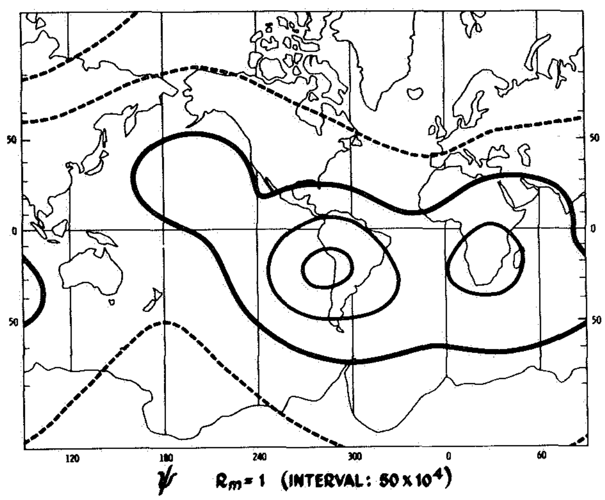

Fig. 2a

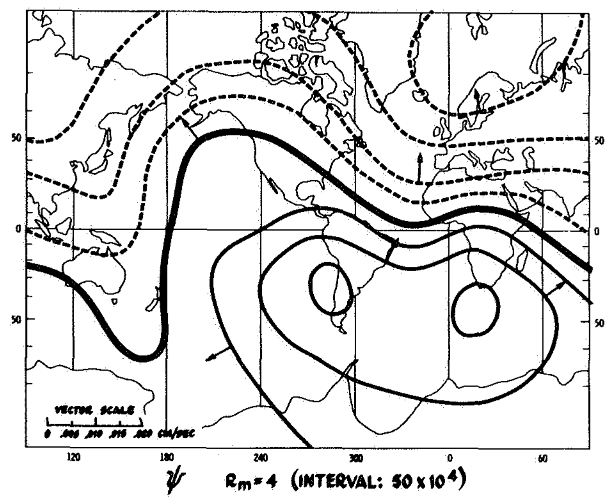

Fig. 2b

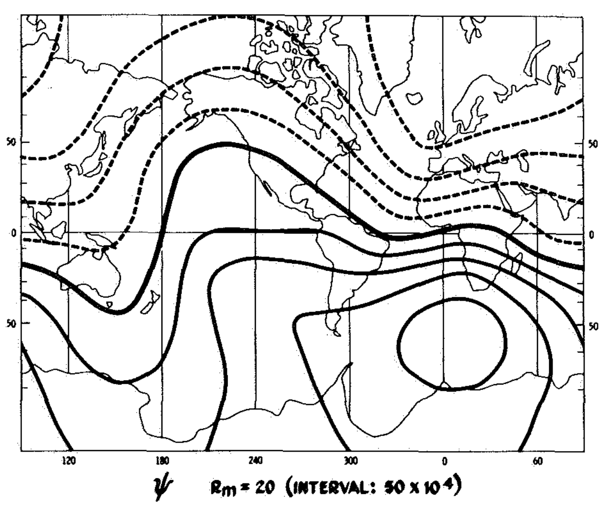

Fig. 2c

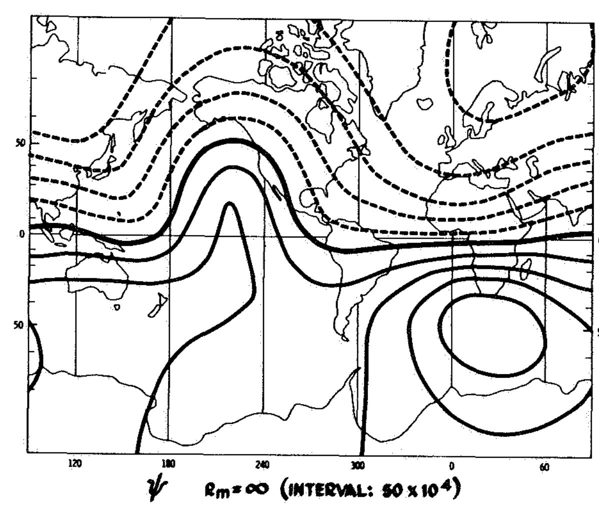

Fig. 2d

Contours of velocity potential ψ ($\vec{v} = -\nabla_T \psi$) for poloidal component of flow, magnetic Reynolds numbers, $\overline{Rm}$ = 1.2, 4.3, 20, and ∞; case of $\overline{Rm} = \infty$ derived by previous method (Kahle, Vestine, and Ball, 1967).

Contour intervals are 50 x 10^4 cm^2/sec. Positive values are shown as solid lines (—), negative values by dashed lines (---), and zero contours by heavy solid lines (━). Velocity vectors are added to $\overline{Rm}$ = 4 case for clarity.

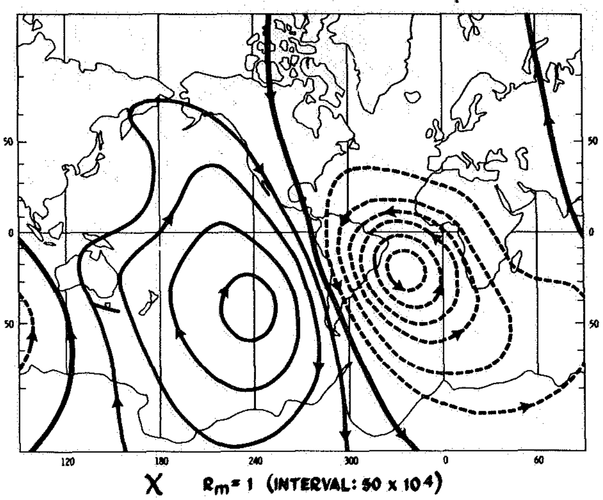

Fig. 3a

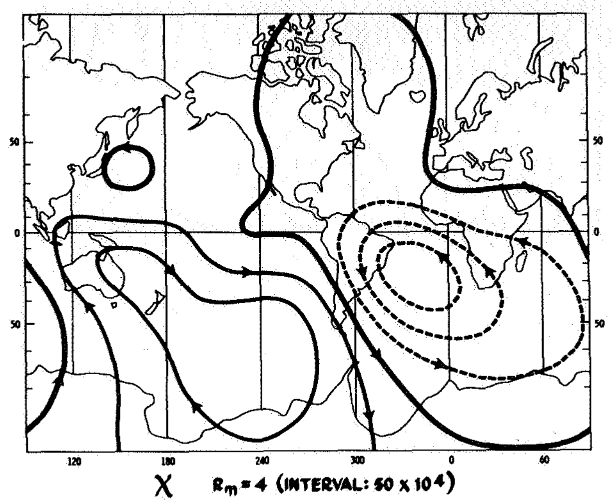

Fig. 3b

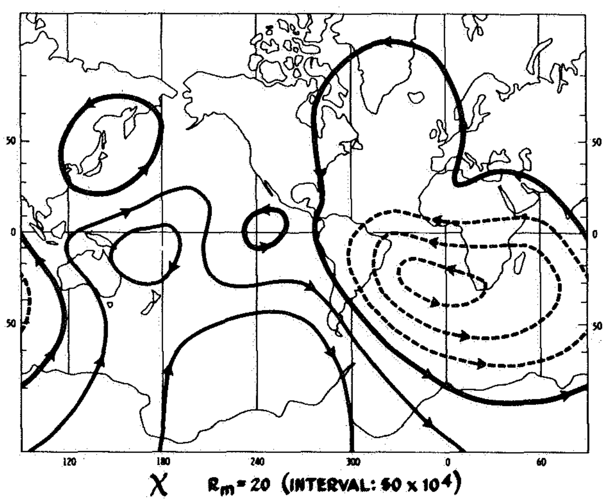

Fig. 3c

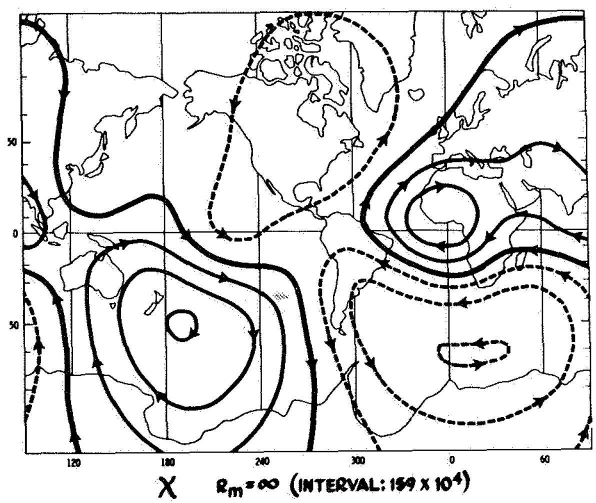

Fig. 3d

Contours of stream function χ $\left(\vec{v} = \hat{i}_r \times \nabla\chi\right)$ for toroidal component of flow, for magnetic Reynolds numbers, $\overline{Rm}$ = 1.2, 4.3, 20, and ∞. Westward drift term is not included; i.e., contours shown are $\chi' \equiv \chi - A_1^o \cos\theta$. Contour intervals are in cm^2/sec. Arrows are added to denote direction of flow.

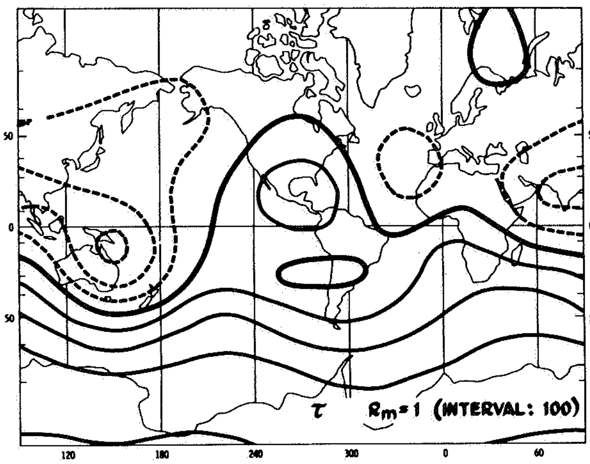

Fig. 4a

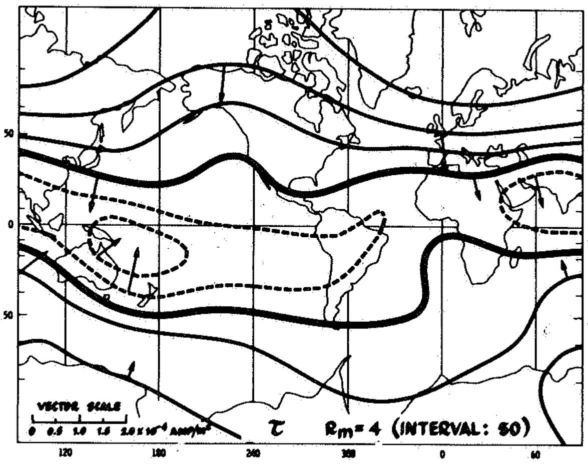

Fig. 4b

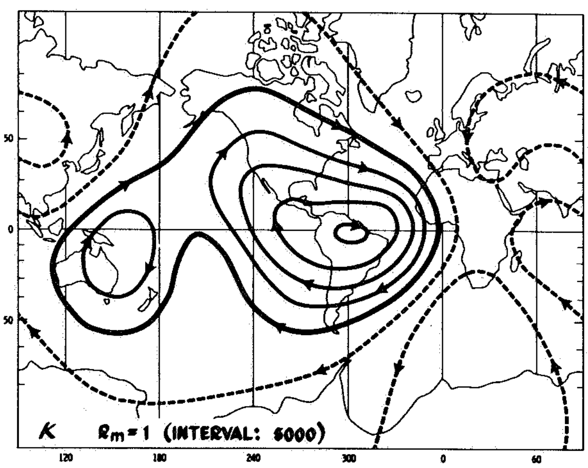

Fig. 4c

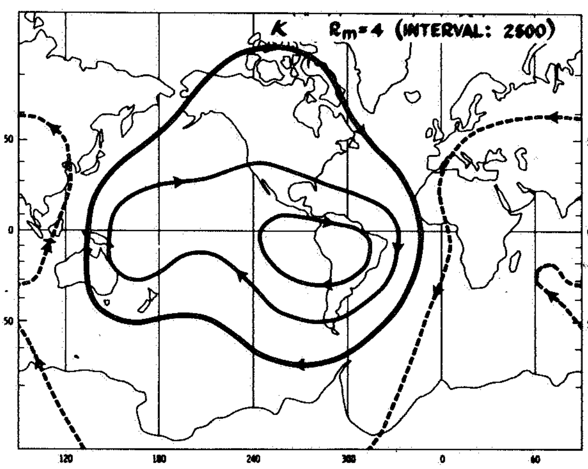

Fig. 4d

Contours of current functions τ (poloidal current) and κ (toroidal current) $\left(\vec{J} = i_r \times \nabla\kappa - \nabla_T\tau\right)$. Contour intervals differ for each case (given in units of amp/m or 10^{-3} abamp/cm). κ contours do not include uniform zonal flow; i.e., only $\kappa - \alpha_1^0 \cos\theta$ is shown.

Table 2

COEFFICIENTS OF VELOCITY POTENTIALS

		χ Field (10^{-3} cm/sec)					ψ Field (10^{-3} cm/sec)				
		Variational Principle				Previous	Variational Principle				Previous
n	m	$\Lambda = .0175$ $\bar{R}_m = 1.2$	$\Lambda = .008$ $\bar{R}_m = 2.2$	$\Lambda = .004$ $\bar{R}_m = 4.3$	$\Lambda = .001$ $\bar{R}_m = 20$	Method $\bar{R}_m = \infty$	$\Lambda = .0175$ $\bar{R}_m = 1.2$	$\Lambda = .008$ $\bar{R}_m = 2.2$	$\Lambda = .004$ $\bar{R}_m = 4.3$	$\Lambda = .001$ $\bar{R}_m = 20$	Method $\bar{R}_m = \infty$
		α_n^m					α_n^m				
1	0	+ 23.24	+ 25.81	+ 27.58	+ 29.56	+ 19.87	− 0.717	− 2.357	− 3.527	− 4.488	− 5.372
1	1	− 2.16	− 2.21	− 2.37	− 2.38	− 0.81	+ 0.585	+ 0.702	+ 0.586	+ 0.177	− 1.001
2	0	− 2.41	− 1.96	− 2.12	− 3.44	− 3.98	− 1.490	− 1.741	− 1.196	− 0.330	− 1.019
2	1	+ 1.08	+ 1.86	+ 2.12	+ 1.86	+ 9.52	− 1.075	− 1.739	− 2.100	− 2.388	− 2.013
2	2	− 0.44	− 0.57	− 0.37	− 0.04	+ 4.48	− 0.376	− 0.399	− 0.260	+ 0.026	+ 0.092
3	0	− 0.24	− 0.32	− 0.52	− 1.07	+ 0.86	+ 0.310	+ 0.125	+ 0.027	− 0.216	+ 0.453
3	1	− 0.24	− 0.32	− 0.25	+ 0.06	− 9.18	− 0.275	− 0.313	− 0.178	+ 0.130	+ 0.275
3	2	+ 0.47	+ 0.48	+ 0.37	+ 0.28	+ 0.59	+ 0.487	+ 0.315	+ 0.069	− 0.302	− 0.884
3	3	+ 0.23	+ 0.36	+ 0.20	− 0.14	+ 2.09	− 0.303	− 0.372	− 0.240	+ 0.011	+ 0.346
4	0	+ 0.03	+ 0.01	+ 0.17	+ 0.70	+ 3.62	− 0.658	− 0.843	− 0.848	− 0.714	− 0.847
4	1	− 0.33	− 0.46	− 0.49	− 0.49	+ 2.85	+ 0.037	− 0.275	− 0.474	− 0.600	− 0.114
4	2	− 0.10	− 0.13	− 0.23	− 0.52	+ 1.80	− 0.163	− 0.290	− 0.305	− 0.162	+ 0.342
4	3	+ 0.12	+ 0.17	+ 0.19	+ 0.09	− 2.11	− 0.023	− 0.081	− 0.210	− 0.523	− 0.265
4	4	+ 0.04	+ 0.15	+ 0.35	+ 0.62	+ 3.37	+ 0.080	+ 0.198	+ 0.208	− 0.096	− 0.347
		β_n^m					β_n^m				
1	1	− 0.23	+ 0.22	+ 0.38	− 0.32	+ 3.85	− 0.532	− 1.343	− 1.790	− 1.709	− 0.289
2	1	+ 0.52	+ 0.78	+ 0.90	+ 0.84	+ 2.93	+ 0.421	− 0.045	− 0.255	− 0.352	− 1.642
2	2	+ 1.17	+ 0.75	+ 0.22	− 0.21	− 0.34	+ 0.296	+ 0.826	+ 1.133	+ 1.178	+ 1.477
3	1	− 0.45	− 0.72	− 0.86	− 0.95	− 1.37	+ 0.062	− 0.064	− 0.160	− 0.255	+ 0.031
3	2	− 0.50	− 0.31	+ 0.02	+ 0.59	+ 0.74	+ 0.186	+ 0.371	+ 0.275	− 0.042	+ 0.637
3	3	+ 0.34	+ 0.62	+ 0.67	+ 0.35	− 0.93	+ 0.455	+ 0.512	+ 0.380	+ 0.157	− 0.768
4	1	+ 0.46	+ 0.77	+ 0.91	+ 0.61	+ 1.95	− 0.399	− 0.461	− 0.374	− 0.232	− 0.471
4	2	− 0.04	+ 0.21	+ 0.37	− 0.38	+ 1.49	+ 0.014	+ 0.219	+ 0.381	+ 0.521	+ 0.470
4	3	− 0.68	− 0.88	− 0.85	− 0.66	− 1.34	− 0.251	− 0.416	− 0.267	+ 0.115	− 0.728
4	4	+ 0.23	+ 0.28	+ 0.29	+ 0.36	− 1.82	+ 0.424	+ 0.637	+ 0.661	+ 0.485	+ 0.247

Table 3

COEFFICIENTS OF CURRENT POTENTIALS

		τ (Poloidal Current Function) (10^{-11} abamp/cm^2)				κ (Toroidal Current Function) (10^{-11} abamp/cm^2)				Equivalent κ Derived From Error in Previous Method (10^{-11} abamp/cm^2)
n	m	Λ = .0175 R_m = 1.2	Λ = .008 R_m = 2.2	Λ = .004 R_m = 4.3	Λ = .001 R_m = 20	Λ = .0175 R_m = 1.2	Λ = .008 R_m = 2.2	Λ = .004 R_m = 4.3	Λ = .001 R_m = 20	
		α_n^m				α_n^m				
1	0	- 43.8	- 2.7	- 4.68	- 5.79	+ 2714	+ 2708	1823	491	+ 1750
1	1	+ 21.0	+ 13.8	+ 15.03	+ 0.68	- 628	- 830	- 590	- 68	- 1630
2	0	+ 41.8	+ 54.0	+ 22.06	+ 8.33	- 2720	- 1331	- 574	- 64	- 786
2	1	- 13.4	- 7.2	+ 8.00	+ 0.88	+ 273	+ 36	- 36	- 18	+ 815
2	2	- 8.6	- 3.8	+ 1.61	+ 0.07	- 1033	- 396	- 80	- 12	- 646
3	0	- 19.7	- 0.7	+ 0.94	+ 2.34	+ 347	+ 481	+ 302	+ 65	+ 100
3	1	- 3.5	- 3.6	- 1.48	- 0.20	- 988	- 166	+ 67	+ 33	- 960
3	2	- 5.0	+ 5.7	- 3.20	+ 1.32	+ 370	+ 318	+ 201	+ 41	+ 318
3	3	+ 6.4	+ 5.4	- 2.16	+ 1.00	- 830	- 620	- 341	- 33	- 423
4	0	+ 15.1	+ 10.3	+ 2.98	+ 0.88	+ 408	+ 241	+ 84	+ 2	+ 200
4	1	+ 17.4	+ 21.1	+ 10.85	+ 2.79	+ 251	+ 215	+ 129	+ 24	+ 497
4	2	- 2.0	- 2.5	- 0.30	- 0.28	- 172	- 166	- 114	- 16	- 258
4	3	- 0.5	- 2.0	- 2.10	- 0.36	+ 276	+ 250	+ 114	+ 16	+ 25
4	4	+ 8.4	+ 6.0	+ 1.01	+ 0.27	- 261	- 123	- 61	- 29	- 57
		β_n^m				β_n^m				
1	1	- 25.5	- 6.6	+ 5.72	+ 1.62	- 2639	- 1792	- 1162	- 440	- 1100
2	1	- 19.3	- 10.5	- 1.46	- 3.03	- 875	- 277	+ 6	+ 38	- 290
2	2	+ 25.2	+ 11.6	+ 7.09	- 1.36	- 1754	- 1130	- 402	+ 39	- 540
3	1	+ 18.3	+ 13.9	+ 10.73	+ 2.41	+ 216	+ 269	+ 174	+ 44	- 250
3	2	- 13.3	- 11.7	- 7.47	+ 1.15	+ 1025	+ 549	+ 197	+ 6	+ 240
3	3	+ 0.7	- 3.5	- 3.39	+ 0.77	+ 213	+ 325	+ 125	- 24	+ 12
4	1	+ 11.2	+ 10.0	+ 4.38	+ 2.61	+ 186	+ 73	- 5	- 12	- 20
4	2	+ 9.5	- 0.2	+ 3.09	- 2.14	+ 360	+ 101	+ 60	- 8	+ 66
4	3	+ 21.0	+ 15.8	+ 1.06	+ 0.36	- 355	- 290	- 129	- 10	- 72
4	4	+ 11.6	+ 6.9	+ 2.98	+ 0.40	- 1080	- 689	- 316	- 38	- 100

regions (South Africa and South America) the flows are still similarly directed. Consideration of the coefficients in Table 2 indicates that higher-order harmonics are much more dominant in χ for the $\bar{R}_m = \infty$ solution, a circumstance which is probably related to the singularity discussed in Section 2.

Figure 4 shows the poloidal and toroidal parts of the current, τ and κ, respectively, for the two cases $\bar{R}_m = 1.2$ and 4.3. (Of course there are no currents in the $\bar{R}_m = \infty$ case.) For these quantities, the magnitudes vary nearly inversely with $\bar{R}_m$, but the patterns again tend to retain a similar form, especially κ. Coefficients of the spherical harmonic expansion of τ and κ are given in Table 3. τ begins to change more rapidly as $\bar{R}_m$ decreases to a value of one and less. Above a value of one, the pattern is predominantly one of current upflow at both poles and downflow in a belt around the equator.

To the extent that they have physical validity, the surface current patterns give one some information about the magnetic fields deeper in the core -- specifically, about the rate of change with depth. The τ field relates to diffusion of the toroidal magnetic field $[\tau \equiv -(4\pi)^{-1} \partial T/\partial r]$. (T is assumed to be zero at the surface.) The τ patterns shown in Fig. 4 indicate a toroidal magnetic field that increases with depth and is predominantly of the T_2^o type [i.e., varying like the spherical harmonic $P_2^o(\theta)$] -- directed eastward in the northern hemisphere and westward in the southern hemisphere. Such a field is postulated in several dynamo theories (Bullard 1949; Bullard and Gellman, 1954; Parker, 1955; Elsasser, 1956).

The κ function is related to diffusion of the poloidal magnetic field, which results from both radial and horizontal gradients $[\kappa \equiv (4\pi)^{-1}(\nabla_T^2 S + \partial^2 S/\partial r^2)]$. The horizontal gradients are presumed known from surface data, but since we have included spherical harmonics only to order four, such derivatives can produce only diffusion corresponding to $\bar{R}_m$ of the order 10^2 or more. Hence the value of κ is mostly determined by the unknown vertical derivatives for the low values of $\bar{R}_m$ under consideration. The calculated coefficients of χ are roughly inversely proportional to $\bar{R}_m$ for larger values, but some coefficients tend to level off to a constant value as $\bar{R}_m$ approaches one. For example, the axial dipole term approaches 2.7×10^{-8} abamp/cm^2 near $\bar{R}_m = 1$ (see Table 3). This value implies that the dipole term decreases with depth with a scale length of 430 km, which is comparable with the value inferred in Section 2 from our previous theory. It accounts for a diffusive decay of 13.3 γ/yr, or 95 percent of the observed secular change of the axial dipole component of the magnetic field at the earth's surface. Contour maps of κ for two cases are shown in Fig. 4. In Table 3 we also show the equivalent κ corresponding to the error in secular change found by our previous method. This κ field resembles the presently derived values to a striking degree, especially for $\bar{R}_m = 4.3$, which tends to reinforce the idea that this diffusion is necessary to explain the observed secular change.

In summary, we have obtained a family of particular solutions for the velocity and current fields near the surface of the core. These solutions -- in terms of finite spherical-harmonic expansions -- tend to satisfy the original equations more closely as the magnetic Reynolds number is reduced. The r.m.s. fit to the secular change field is within 13 percent at $\bar{R}_m = 1.2$, and is still within 33 percent at $\bar{R}_m = 4.3$. No particular value of $\bar{R}_m$ can be singled out as a clear turning point to indicate a preferred choice within this range. Fortunately, the velocity field does not change much over this range of $\bar{R}_m$, so that it is fairly well determined. The corresponding surface currents are also roughly determined in form, but much less well in magnitude. Both the form and magnitude of the currents are physically reasonable for values of $\bar{R}_m$ of the order of one or larger.

The particular family of solutions was obtained by adding a plausible fluid-dynamic constraint to the hydromagnetic induction equation and imposing an additional variational condition. The variational condition guarantees, essentially, that the velocity and current fields will (1) be nonsingular and (2) will have a fixed ratio of r.m.s. velocity and current (i.e. $\bar{R}_m$). The condition does not guarantee that such a solution exists, and indeed the results indicate that only for sufficiently small $\bar{R}_m$ can one satisfy the equations to a desired degree of approximation. (The ability to satisfy the equations may also depend on the completeness of the finite spherical-harmonic representations, but numerical experiments indicated that this was not the primary source of error in fitting secular change.)

While this variational principle is rather arbitrary, it does appear to be the simplest such condition which achieves the desired result of nonsingularity. The principle, Eq. (4.13), as well as its modified form, contains both $\vec{J}_o$ and $\vec{v}_o$ quadratically, with constant coefficients, and without their derivatives. If positive powers of B_r are included, the resulting Euler-Lagrange equations are not manifestly nonsingular. For example, if one were to minimize the integral of the work done by the fluid against electromagnetic forces near the surfaces of the core, $\vec{v}_o \cdot (\vec{J}_o \times \vec{B}) = B_r \hat{i}_r \cdot (\vec{v}_o \times \vec{J}_o)$, then the resulting equations for $\vec{v}_o$ and $\vec{J}_o$ would have the forms (analogous to Eqs. 4.22 and 4.23)

$$\vec{v}_o = \nabla_T K/B_r - \hat{i}_r \times \nabla_T L$$

$$\vec{J}_o = \sigma \hat{i}_r \times \nabla_T K + (2\Omega\rho_o \cos\theta)\hat{i}_r \times \nabla_T L/B_r$$

which are singular at $B_r = 0$.

It would be interesting to investigate solutions obtained by imposing alternative side conditions based on other physically reasonable assumptions. For example, one such physical condition might be based on the fact that the electric potential at the core surface is directly related to the currents which flow in the mantle, and therefore to the core/mantle torque coupling. Another possibility might be to alter the fluid dynamic constraint to take account of gravitational stresses resulting from deformations of the core/mantle interface as inferred from other geophysical considerations (heat flow, gravity anomalies, etc.) Such alternative approaches would cf course have to also lead to nonsingular velocity fields and reasonable representations of the observable data.

We have found a plausible solution which satisfactorily reproduces the secular change. While it has not been proven to be unique, it is satisfying to note the close similarity with the ψ part of the previous $\bar{R}_m = \infty$ solution, which was derived with fewer conditions. The chief difference appears to be that the addition of diffusion has made it possible to satisfy the induction equation with a less complex χ field.

It should be noted that the present numerical method does not allow us to use the data to best advantage, since evaluation of the fields at the core overemphasizes the poorly known higher-order components of B_r and $\dot{B}_r$. A matrix form of the induction equation, such as that given by Roberts and Scott, would enable one to utilize better statistical weighting in performing the least-squares fit. The additional complexity this would have necessitated did not seem warranted for the present purpose, which was to study the effect of a particular constraint on the system.

APPENDIX

ANALYSIS OF BOUNDARY LAYER

We seek an approximate solution of Eq. (3.5) in the boundary layer with the assumption that the scale length for horizontal variations is much larger than the vertical scale length, or boundary-layer thickness.

It is convenient to separate each of the dynamical variables into a mainstream component (denoted by subscript M), which is assumed to have negligible variation in the boundary layer, and a boundary-layer component (denoted by subscript B), which is assumed to vanish outside the boundary layer.

$$\vec{B} = \vec{B}_M + \vec{B}_B$$

$$\vec{v} = \vec{V}_M + \vec{V}_B \qquad (A.1)$$

$$\vec{J} = \vec{J}_M + \vec{J}_B \text{ , etc.}$$

The theory may be pursued in a formal way by expanding the equations in powers of the boundary-layer thickness, δ. We shall discuss only the lowest order, in which horizontal gradients are neglected relative to vertical gradients. Furthermore, we shall assume a priori that the absolute magnitude of the thickness is sufficiently small that the finite boundary current, $\vec{J}_B$, produces a negligible change in magnetic field; i.e., $\vec{B}_B$ is assumed to be zero since

$$|\vec{B}_B| \sim \delta \; |\text{curl } \vec{B}_B| = \delta \; |\vec{J}_B|/(4\pi) \qquad (A.2)$$

This approximation will be verified a posteriori.

Since the B variables vanish outside the boundary layer, the M variables must satisfy the equations individually in the main stream. Furthermore, the assumed slow vertical variation implies that the viscous term is negligible. Hence in the mainstream Eq. (3.5) becomes

$$2\vec{\Omega} \times \vec{V}_M = -\nabla P_M + \vec{J}_M \times \vec{B}_M/\rho_o - \varepsilon\nabla\Phi \qquad \textbf{(A.3)}$$

By continuation, this must also hold in the boundary layer, so subtracting it from the original form of Eq. (3.5) gives

$$2\vec{\Omega} \times \vec{V}_B = -\nabla P_B + \vec{J}_B \times \vec{B}_M/\rho_o + \upsilon\nabla^2\vec{V}_B \qquad \textbf{(A.4)}$$

By a similar process, one separates Eq. (2.1) into

$$\vec{J}_M = \sigma\ [-\nabla\varphi_M - \partial\vec{A}_M/\partial t + \vec{V}_M \times \vec{B}_M] \qquad \textbf{(A.5)}$$

and

$$\vec{J}_B = \sigma\ [-\nabla\varphi_B + \vec{V}_B \times \vec{B}_M] \qquad \textbf{(A.6)}$$

where the continuity of $\vec{B}$ has been invoked to neglect $\partial\vec{A}_B/\partial t$.

At the core/mantle interface (surface S_1 in Fig. 1), one has the boundary condition

$$\vec{v}_1 \equiv \vec{V}_{M1} + \vec{V}_{B1} = 0 \qquad \textbf{(A.7)}$$

and if mantle currents are neglected,

$$J_{r1} \equiv J_{Mr1} + J_{Br1} = 0 \qquad \textbf{(A.8)}$$

The boundary-layer components of velocity and current must individually satisfy the solenoidal condition

$$\nabla\cdot\vec{V}_B = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_{Br}) + \nabla_T\cdot\vec{V}_{BT} = 0 \qquad \textbf{(A.9)}$$

or

$$\frac{\partial V_{Br}}{\partial r} = -\frac{2}{r} V_{Br} - \nabla_T \cdot \vec{V}_{BT} \tag{A.10}$$

and similarly,

$$\frac{\partial J_{Br}}{\partial r} = -\frac{2\ J_{Br}}{r} - \nabla_T \cdot \vec{J}_{BT} \tag{A.11}$$

Since these conditions must be satisfied throughout the boundary layer, it is clear that the vertical (radial) components of $\vec{V}_B$ and $\vec{J}_B$ must be of the order

$$|V_{Br}| \sim (\delta/L_H)|\vec{V}_{BT}| \tag{A.12}$$

$$|J_{Br}| \sim (\delta/L_H)|\vec{J}_{BT}| \tag{A.13}$$

and we therefore neglect these vertical components. Eqs. (A.7) and (A.8) then imply that the mainstream vertical components are also negligible at $r = b$ and hence also at the lower boundary $r \sim b - \delta$ (surface S_o in Fig. 1). These are the only boundary conditions which constrain the mainstream solutions.

Using the approximations just derived, one can write the vertical components of Eqs. (A.4) and (A.6) as

$$-2\Omega \sin\theta\ V_{B\lambda} = -\frac{\partial P_B}{\partial r} + (J_{B\theta}B_{M\lambda} - J_{B\lambda}B_{M\theta})/\rho_o + \nu \frac{\partial^2 V_{Br}}{\partial r^2} \tag{A.14}$$

and

$$\frac{\partial \varphi_B}{\partial r} = V_{B\theta}B_{M\lambda} - V_{B\lambda}B_{M\theta} \tag{A.15}$$

Eq. (A.15) implies that the horizontal gradient of ϕ_B should be negligible, and Eq. (A.14) implies the same about P_B, provided that we assume the kinematic viscosity ν is of order $\Omega\delta^2$. The horizontal components of Eqs. (A.4) and (A.6) may then be written

$$2(\vec{\Omega} \times \vec{V}_B)_T \equiv 2\Omega \cos \theta(\hat{i}_\lambda V_{B\theta} - \hat{i}_\theta V_{B\lambda}) = (B_{Mr}/\rho_o)(\hat{i}_\theta J_{B\lambda} - \hat{i}_\lambda J_{B\theta}) + \nu \frac{\partial^2 \vec{V}_{BT}}{\partial r^2} \tag{A.16}$$

and

$$\vec{J}_{BT} = \sigma(\vec{V}_B \times \vec{B}_M)_T = \sigma B_{Mr}(\hat{i}_\theta V_{B\lambda} - \hat{i}_\lambda V_{B\theta}) \tag{A.17}$$

Elimination of $\vec{J}_{BT}$ gives

$$V_{B\lambda} = \frac{\gamma_o V_{B\theta}}{\cos \theta} - \frac{\nu}{2\Omega \cos \theta} \frac{\partial^2 V_{B\theta}}{\partial r^2} \tag{A.18}$$

and

$$V_{B\theta} = - \frac{\gamma_o V_{B\lambda}}{\cos \theta} + \frac{\nu}{2\Omega \cos \theta} \frac{\partial^2 V_{B\lambda}}{\partial r^2} \tag{A.19}$$

where

$$\gamma_o \equiv \frac{\sigma B_{Mr}^2}{2\Omega\rho_o} \tag{A.20}$$

Ignoring the horizontal dependence (θ and λ), we may obtain the fundamental modes by considering the radial mode

$$\vec{V}_B = \vec{u}(\theta,\lambda) \exp (ikr) \tag{A.21}$$

Eq. (A.18) yields the dispersion relation

$$\cos^2\theta + \left(\gamma_o + \frac{k^2\nu}{2\Omega}\right)^2 = 0 \qquad \text{(A.22)}$$

The allowed solution of this is

$$k = k_o - i/\delta \qquad \text{(A.23)}$$

where

$$k_o \equiv \sqrt{\Omega/\nu}\ \ f(\theta,\lambda)$$

$$\delta \equiv \sqrt{\nu/\Omega}\ \ f(\theta,\lambda)/|\cos\ \theta|$$

$$f(\theta,\lambda) \equiv \sqrt{-\ \gamma_o + \sqrt{\gamma_o^2 + \cos^2\theta}}$$

Using this eigenvalue, one obtains the solution of Eqs. (A.18) and (A.19) as

$$\vec{V}_B = -\exp[(r-b)/\delta] \left\{ \vec{u}_T(\theta,\lambda)\cos[k_o(b-r)] - \frac{\cos\ \theta}{|\cos\ \theta|}\ \hat{i}_r \times \vec{u}_T(\theta,\lambda)\sin\ [k_o(b-r)] \right\} \qquad \text{(A.24)}$$

where $\vec{u}_T(\theta,\lambda)$ is a transverse vector, which is determined by the boundary condition (A.7) to be

$$\vec{u}_T(\theta,\lambda) = \vec{V}_M(b,\theta,\lambda) \qquad \text{(A.25)}$$

We can now estimate the variation in the tangential magnetic field, $\vec{B}_T$, across the boundary layer in terms of the current $\vec{J}_{BT}$ obtained from Eqs. (A.17) and (A.24):

$$\vec{B}_T(b) - \vec{B}_T(b-\bar{\varepsilon}) \equiv \Delta \vec{B}_T = -4\pi \int_b^{b-\bar{\varepsilon}} \hat{i}_r \times \vec{J} dr \quad \frac{4\pi\sigma\delta B_r}{\left(1+k_o^2\delta^2\right)} \left(-\vec{u}_T + \frac{k_o \delta \cos\theta \; \hat{i}_r \times \vec{u}_T}{|\cos\theta|} \right) \tag{A.26}$$

where $\bar{\varepsilon} > \delta$. In order of magnitude,

$$\frac{|\Delta \vec{B}_T|}{|B_r|} \sim 4\pi \; \sigma \; \delta \; |\vec{u}_T| \sim 4 \times 10^{-3} \tag{A.27}$$

where we have made the pessimistic assumption that $\nu \sim 10^3$; so that $\delta \sim 4 \times 10^3$cm. Near the equator we would get $\delta \sim 10^5$cm, so the ratio increases to about 10^{-1} at worst. Thus the original assumption of continuity of $\vec{B}$ appears reasonable.

Finally, we may recombine the velocities into the single solution [letting $\vec{V}_{MT}(b,\theta,\lambda) \simeq \vec{V}_{MT}(r,\theta,\lambda)$ near the boundary layer]

$$\vec{v}(r,\theta,\lambda) = \hat{i}_r v_r(r,\theta,\lambda) + \vec{V}_{MT}(r,\theta,\lambda) \left\{ 1 - \exp[(r-b)/\delta] \cos[k_o(b-r)] \right\}$$
$$+ \hat{i}_r \times \vec{V}_{MT}(r,\theta,\lambda) \exp[(r-b)/\delta] \sin[k_o(b-r)] \left(\frac{\cos\theta}{|\cos\theta|} \right) \tag{A.28}$$

REFERENCES

Backus, G. E., Kinematics of geomagnetic secular variation in a perfectly conducting core, (abstract) Trans. Amer. Geophys. Union, 49, 150, 1968.

Braginskiy, S. A., Principles of the theory of the earth's hydromagnetic dynamo, Geomagnetism and Aeronomy, VII, 323-329, 1967.

Bullard, E. C., and H. Gellman, Homogeneous dynamos and terrestrial magnetism, Phil. Trans. Roy. Soc., 247A, 213-278, 1954.

Cain, J. C., S. J. Hendricks, R. A. Langel, and W. V. Hudson, A proposed model for the international geomagnetic reference field-1965, NASA, Goddard Space Flight Center Rept # X-612-67-173, 1967.

Elsasser, W. M., The earth's interior and geomagnetism, Rev. Mod. Phys., 22, 1-35, 1950.

Elsasser, W. M., Hydromagnetism II. A review, Amer. J. Phys., 24, 85-110, 1956a. Hydromagnetic Dynamo Theory, b.

Hide, R., Free hydromagnetic oscillations of the earth's core and the theory of the geomagnetic secular variation, Phil Trans. Roy. Soc., 259A, 615-650, 1966.

Hide, R., and P. H. Roberts, The origin of the main geomagnetic field, Physics and Chemistry of the Earth, 4, Pergamon Press, London, 27-98, 1961.

Kahle, Anne B., R. H. Ball, and E. H. Vestine, Comparison of estimates of fluid motions at the surface of the earth's core for various epochs, J. Geophys. Res., 72, 4917-4924, 1967.

Kahle, Anne B., E. H. Vestine, and R. H. Ball, Estimated surface motions of the earth's core, J. Geophys. Res., 72, 1095-1108, 1967.

Kern, J. W., A hypothetical connection between the non-dipole field and the earth's gravitational field, (abstract), Trans Amer. Geophys. Union, 46, 68-69, 1965.

Malkus, W. V. R., Precessional torques as the cause of geomagnetism, J. Geophys. Res., 68, 2871-2886, 1963.

Richmond, A. D., Relation of the westward drift of the geomagnetic field to the rotation of the earth's core, The RAND Corp., RM-5861-NASA, Dec. 1968.

Rikitake, T., Non-dipole field and fluid motion in the earth's core, J. Geomagnet. Geoelec., 19, 129-142, 1967.

Roberts, P. H., and S. Scott, On analysis of the secular variation (I) A hydromagnetic constraint: theory, J. Geomagnet. Geoelec., 17, 137-151, 1965.

Stewartson, K., The dispersion of a current on the surface of a highly conducting fluid, Proc. Camb. Phil. Soc., 53, 774-775, 1957.

Stewartson, K., On the motion of a non-conducting body through a perfectly conducting fluid, J. Fluid Mech., 8, 82-96, 1960a.

Stewartson, K., Motion of bodies through conducting fluids, Rev. Mod Phys., 32, 855-859, 1960b.

Vestine, E. H., R. H. Ball, and Anne B. Kahle, Nature of surface flow in the earth's central core, J. Geophys. Res., 72, 4927-4936, 1967.

Vestine, E. H., and Anne B. Kahle, On the small amplitude of magnetic secular change in the Pacific area, J. Geophys. Res., 71, 527-530, 1966.