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MINIMUM THRUST FOR CORRECTING KEPLERIAN ORBITS WITH APPLICATION TO INTERPLANETARY GUIDANCE

by Frederick W. Boltz

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SUMMARY

An analytical investigation has been conducted to determine minimum thrust requirements for effecting small changes in several parameters of Keplerian orbits. Impulsive thrust and the approximations of linear perturbation theory have been assumed. The results are generally applicable to essentially two-body planar motion where the natural deviations of the vehicle flight path from a conic section are sufficiently small.

The first-order thrust requirements obtained in the analysis are presented in terms of the optimum thrust angle and the minimum derivative of thrust-produced velocity impulse with respect to the particular orbital parameter to be corrected. Minimum thrust requirements are found for correction at a given point along an orbit of the semimajor axis, the eccentricity, and the true anomaly or orientation of the line of apsides (argument of pericenter) within the orbital plane. Minimum thrust requirements are also found for correcting the orbital radius at an arbitrary point and the elapsed flight time to arrive at that point. In addition, thrust requirements are obtained for correction at a given point along an orbit of the elapsed flight time to a second arbitrary point having a fixed value of orbital radius.

A general formulation of the thrust requirements for simultaneous correction of any pair of the orbital parameters that define the in-plane motion is presented in terms of the minimum thrust requirements for correction of each of the parameters separately. Application of this result to the problem of implicit in-plane guidance during the midcourse and approach phases of interplanetary flight is indicated. In particular, it is shown how the minimum-thrust results obtained in the present analysis may be applied to both fixed and variable time of arrival as well as fixed radius of pericenter types of guidance which make use of flight-path deviations from a reference trajectory.

INTRODUCTION

The problem of spacecraft guidance has been the subject of considerable research in recent years. This research has produced a number of methods for simplifying the onboard computation of the vehicle velocity correction. These methods may be generally classified as either implicit or explicit guidance, depending on whether or not a reference trajectory is used. Explicit guidance requires numerical integration of the equations of motion but has the advantage of greater flexibility in regard to off-design conditions. In implicit guidance, numerical integration is avoided by utilizing a reference trajectory. It is this type of guidance which is of primary interest in the present analysis. Corrections to the spacecraft velocity are calculated using perturbation expressions evaluated on the reference orbit in conjunction with projected flight-path deviations from the reference orbit at the terminal point. The method is based on the assumptions that the velocity corrections are impulsive and that the variant motion about the reference trajectory is adequately described by linearized equations of motion. If two-body dynamics can be assumed, the required corrective velocity impulse can be obtained analytically from parameter perturbation expressions of the type found in references 1 through 5. When only a single orbital parameter is to be corrected, the thrust angle can be selected to minimize the velocity impulse. The optimum use of impulsive thrust to correct the pericenter radius during approach to a planet has been considered in references 6 and 7. In general, it is possible to correct three independent orbital parameters or three components of flight-path deviation from a target point with a single application of thrust. The problem of finding the required thrust impulse for this type of correction has been extensively treated in references 8 through 14.

The present investigation is concerned with the derivation of minimum thrust requirements for single-parameter corrections and with the application of these results to several types of spacecraft guidance. A general formulation of minimum thrust requirements for correcting any orbital parameter has been obtained along with specific formulas for a number of parameters of interest. These results were obtained by applying the condition for minima given in the elementary calculus to general expressions for parameter perturbations due to impulsive velocity changes. This method is the same as that used in reference 7 for correcting pericenter radius and provides only first-order solutions. While the minimum-thrust requirements for correcting some parameters (such as period of orbit and pericenter radius) are well known, this is not the case for other parameters considered in the analysis (such as orbital radius and flight time to an arbitrary point on the orbit).

With the admission of linearized two-body dynamics the general guidance correction can be separated into a correction of the plane of motion and an independent correction of motion within the orbital plane. The in-plane correction can be generally treated as the correction of two independent orbital parameters. It is shown in the present analysis how the in-plane corrective velocity vector can be calculated from minimum thrust requirements for the various independent parameters. Several different types of midcourse and approach guidance for use during interplanetary flights considered include both fixed and variable time of arrival and fixed radius of pericenter (cf. refs. 11-14).

Along with the assumptions of impulsive thrust and two-body dynamics, it is assumed in the analysis that the desired changes in the parameters to be controlled are sufficiently small to justify the use of linear perturbation theory. In most instances these assumptions provide sufficient accuracy in the computation of required velocity corrections and are almost a necessity when analytic solutions are desired.

NOTATION

a	semimajor axis of orbit, positive for elliptic orbits, negative for hyperbolic orbits
b	semiminor axis of orbit, positive for elliptic orbits, imaginary for hyperbolic orbits
e	eccentricity of orbit
E	eccentric anomaly, positive for ascent ($0 \leq E \leq \pi$), negative for descent ($-\pi \geq E \geq 0$)
H	hyperbolic anomaly, positive for ascent, negative for descent
p	semilatus rectum of conic section
P	general symbol for any orbital parameter of interest
r	radial distance from central body
t	time
T	period of orbit
u	horizontal component of velocity
u_c	circular orbital velocity
v	vertical component of velocity
V	resultant velocity relative to inertial coordinates
V_τ	thrust-produced velocity impulse
γ	flight-path angle relative to local horizontal, positive for ascent, negative for descent
$\delta(\ )$	operator signifying small deviation
$\Delta(\ )$	operator signifying increment or difference
θ	true anomaly, positive for ascent ($0 \geq \theta \geq \pi$), negative for descent ($-\pi \leq \theta \leq 0$)
μ	product of universal constant of gravitation and mass of celestial body
τ	thrust angle relative to local horizontal, positive for thrust vector increasing vertical component of velocity, negative for thrust vector decreasing vertical component of velocity

τ_a thrust angle to maximize change in semimajor axis or period of orbit

τ_{a_o} thrust angle to minimize change in semimajor axis or period of orbit

τ_e thrust angle to maximize change in eccentricity

τ_{e_o} thrust angle to minimize change in eccentricity

τ_θ thrust angle to maximize change in true anomaly

τ_{θ_o} thrust angle to minimize change in true anomaly

τ_r thrust angle to maximize change in orbital radius at fixed central angle from thrusting point

τ_{r_o} thrust angle to minimize change in orbital radius at fixed central angle from thrusting point; also, thrust angle to correct flight time to fixed point on orbit

τ_r' thrust angle to maximize change in orbital radius having fixed value of true anomaly

τ_t thrust angle to maximize change in flight time to traverse fixed central angle

τ_{t_o} thrust angle to minimize change in flight time to traverse fixed central angle

τ_t' thrust angle to maximize change in elapsed flight time to arrive at fixed value of true anomaly

ϕ angle from pericenter to any point on orbit measured at center of orbit

$(\bar{\ })$ quantity normalized with local circular orbital velocity, $\sqrt{\mu/r}$

Subscripts

a value at apocenter

p value at pericenter

1 value at orbital position where thrust impulse is to be applied

2 value at orbital position where primary change in orbit is to be effected

ANALYSIS

Assumptions and Restrictions

It is assumed in the analysis that vehicle thrust produces an impulsive velocity increment ΔV_T which is added vectorially to the velocity vector of the vehicle. It is also assumed that small perturbations in the vehicle motion due to impulsive thrust may be calculated with sufficient accuracy by means of linearized two-body, point-mass equations of motion. The restrictions on orbital conditions in the case of a noncentral force field require simply that the perturbations of the vehicle flight path from a conic section be relatively small. Within the limitation of these restrictions, the linearized two-body orbital relations may also be used as a basis for determining minimum thrust requirements for effecting small changes in certain orbital elements or trajectory variables.

One effect of the restriction in size of impulsive thrust is that perturbations of the vehicle motion within and normal to the orbital plane are independent. Since thrust directed normal to the orbital plane does not affect motion within the plane of orbit (to first order), the present analysis is restricted to in-plane thrusting only and the resulting effects on the in-plane motion.

The convention of notation used in the present analysis is illustrated in figure 1, which indicates the geometric parameters of elliptic and hyperbolic orbits pertinent to the analysis. Parabolic orbits may be considered a special case of elliptic orbits in which the semimajor axis, a, is infinitely large. For hyperbolic orbits a is negative but has a positive counterpart in the semitransverse axis, -a.

In the derivation of analytical expressions that follows, intermediate steps are often omitted or indicated only briefly. Since in many instances the auxiliary relations used in the derivation are not given, a short summary of two-body point-mass relations pertinent to the present analysis is presented in appendix A. (A more detailed summary is to be found in ref. 2.)

General Formulas for Optimum Correction of Orbital Parameters

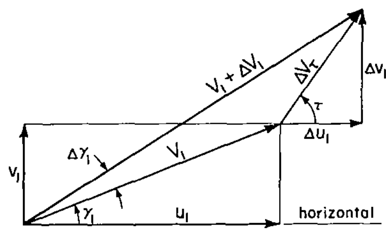
Impulsive thrust applied to an orbiting vehicle produces an instantaneous change in the local velocity vector and related changes in all orbit parameters. If P is any parameter of interest, it can be expressed as a function of the local velocity vector and the orbital radius at the point of thrust application. Thus, with the velocity vector described by its magnitude and direction (V_1, γ_1) or by its horizontal and vertical components (u_1, v_1),

$$P = f(V_1, \gamma_1, r_1) = g(u_1, v_1, r_1) \tag{1}$$

The perturbation of P due to a small change in the local velocity vector is given by

$$dP = \frac{\partial f}{\partial V_1} dV_1 + \frac{\partial f}{\partial \gamma_1} d\gamma_1 = \frac{\partial g}{\partial u_1} du_1 + \frac{\partial g}{\partial v_1} dv_1 \tag{2}$$

since the orbital radius at the thrusting point is considered fixed.



Sketch (a)

The changes in vehicle velocity due to the application of impulsive thrust are illustrated in sketch (a). The velocity increment associated with the thrust impulse is described by its magnitude, ΔV_τ, and direction, τ, referred to the local horizontal. It is readily seen that, in the limit as ΔV_τ goes to 0, changes in the various components of the velocity vector are given by

$$\left.\begin{aligned} dV_1 &= dV_\tau \cos(\tau - \gamma_1) \\ d\gamma_1 &= \frac{dV_\tau}{V_1} \sin(\tau - \gamma_1) \\ du_1 &= dV_\tau \cos \tau \\ dv_1 &= dV_\tau \sin \tau \end{aligned}\right\} \tag{3}$$

where $\tau - \gamma_1$ is the thrust angle relative to the flight path or velocity vector. When these differential expressions are substituted into equation (2), it is found that

$$dP = \left[\frac{\partial f}{\partial V_1} \cos(\tau - \gamma_1) + \frac{1}{V_1} \frac{\partial f}{\partial \gamma_1} \sin(\tau - \gamma_1)\right] dV_\tau \tag{4a}$$

or

$$dP = \left[\frac{\partial g}{\partial u_1} \cos \tau + \frac{\partial g}{\partial v_1} \sin \tau\right] dV_\tau \tag{4b}$$

These equations provide alternate expressions for the derivative dV_τ/dP in terms of thrust angle relative to the flight path or relative to the horizontal. This derivative may be used to obtain a first-order estimate of the thrust magnitude, ΔV_τ, required to change the parameter P an amount ΔP according to

$$\Delta V_\tau = \left(\frac{dV_\tau}{dP}\right) \Delta P$$

The thrust magnitude will be a minimum when the thrust angle, τ, is selected so as to minimize the derivative. This optimum thrust angle is found by setting the derivative of equation (4a) or (4b) with respect to τ equal to 0 and solving for τ. If τ_* represents the optimum thrust angle, it is found that

$$\tan(\tau_* - \gamma_1) = \frac{\partial f/\partial \gamma_1}{V_1 \, \partial f/\partial V_1} \tag{5a}$$

or

$$\tan \tau_* = \frac{\partial g/\partial v_1}{\partial g/\partial u_1} \tag{5b}$$

from which two values of $\tau_* - \gamma_1$ and of τ_* (differing by 180°) are obtained.

The corresponding minimum value of dV_{τ_*}/dP is obtained by substituting τ_* for τ in equation (4a) or (4b) with the result that

$$\frac{dV_{\tau_*}}{dP} = \pm\left[\left(\frac{\partial f}{\partial V_1}\right) + \frac{1}{V_1^2}\left(\frac{\partial f}{\partial \gamma_1}\right)^2\right]^{-1/2} \tag{6a}$$

or

$$\frac{dV_{\tau_*}}{dP} = \pm\left[\left(\frac{\partial g}{\partial u_1}\right)^2 + \left(\frac{\partial g}{\partial v_1}\right)^2\right]^{-1/2} \tag{6b}$$

where the plus and minus signs correspond to thrusting in opposite directions.

The thrust angle for which the parameter P is invariant is found by setting equation (4a) or (4b) equal to 0 and solving for τ. If τ_{*_o} represents this thrust angle, it is found that

$$\tan \tau_{*_o} = -\frac{\partial g/\partial u_1}{\partial g/\partial v_1} = -\cot \tau_* \tag{7}$$

so that, to first order, the thrust angle for no change in a given parameter is different by 90° from that for maximum change. It should be noted that for thrust at this angle the parameter does in fact change, but by an amount proportional to second- or higher-order derivatives.

It follows from the above results that the general linearized expression for the parameter perturbation (eq. (4)) can also be given, in terms of the optimum thrust angle and the maximum derivative of the parameter with respect to the thrust-produced velocity impulse, by

$$dP = \frac{dP}{dV_{\tau_*}} dV_\tau \cos(\tau - \tau_*) \tag{8}$$

since, to first order, only the component of thrust in the optimum-thrust direction contributes to the perturbation.

The first part of the following analysis is concerned with developing linear perturbation expressions for various parameters of interest. In the second part minimum thrust requirements are determined for the control of these parameters.

Perturbation Formulas for Various Orbital Parameters

The perturbative effects of impulsive thrust on various elements or parameters of an orbit are obtained from the general linearized perturbation expressions of the previous section and the appropriate two-body relations for these parameters. The orbit elements considered in the present analysis are the semimajor axis (or period of orbit), the eccentricity, and the true anomaly at the point of thrust application. Perturbations of the orbital radius at any other point on the orbit and of the elapsed flight time to any other point are also considered. Perturbations of the semimajor axis and the eccentricity reflect changes in the orbit size and shape. The perturbation of true anomaly at a given point is equal to the rotation of the line of apsides of the orbit within the orbital plane.

In the following presentation of perturbation expressions, alternate forms of the results are given in the case of semimajor axis (or period of orbit), eccentricity, and true anomaly. These forms differ in the specification of thrust angle (either measured with respect to the flight path or the horizontal) and in the particular component of velocity with which the thrust-produced velocity differential is normalized. For perturbations of orbital radius and elapsed flight time, only one form is presented here; the other in each case is given in appendixes B and C, respectively.

Perturbation of semimajor axis or period of orbit.- The perturbative effects of impulsive thrust at orbital position (r_1, θ_1) on the semimajor axis, a, or the period of orbit, T, are obtained by applying equation (4) to the two-body relations for these elements given in appendix A. It is found that

$$\frac{da}{a} = \frac{2}{3}\frac{dT}{T} = \frac{2}{1 - e^2}\left[(1 + 2e \cos \theta_1 + e^2)\cos(\tau - \gamma_1)\right]\frac{dV_\tau}{V_1} \tag{9a}$$

or

$$\frac{da}{a} = \frac{2}{3}\frac{dT}{T} = \frac{2(1 + e \cos \theta_1)}{1 - e^2}\left[(1 + e \cos \theta_1)\cos \tau + e \sin \theta_1 \sin \tau\right]\frac{dV_\tau}{u_1} \tag{9b}$$

The value of a is positive for elliptic orbits and negative for hyperbolic orbits. Both a and da are undefined for parabolic orbits.

Perturbation of eccentricity.- The perturbative effect of impulsive thrust at orbital position (r_1, θ_1) on the eccentricity of orbit, e, is obtained by applying equation (4) to the two-body relations for this element given in appendix A. It is found that

$$de = \left[2(e + \cos\theta_1)\cos(\tau - \gamma_1) + \frac{(1 - e^2)\sin\theta_1}{1 + e\cos\theta_1}\sin(\tau - \gamma_1)\right]\frac{dV_\tau}{V_1} \tag{10a}$$

or

$$de = \{[(2 + e\cos\theta_1)\cos\theta_1 + e]\cos\tau + (1 + e\cos\theta_1)\sin\theta_1\sin\tau\}\frac{dV_\tau}{u_1} \tag{10b}$$

In the case of nearly circular orbits $(e \to 0)$ equations (10a) and (10b) reduce to

$$de = (2\cos\theta_1\cos\tau + \sin\theta_1\sin\tau)d\overline{V}_\tau \tag{10c}$$

This expression cannot be used directly for perfectly circular orbits, since the true anomaly is undefined. However, since the change of eccentricity is the value of eccentricity after thrusting, it is given by

$$de = \sqrt{1 - (2 - \overline{V}^2)\overline{u}^2} \tag{10d}$$

Here $\overline{V}$ and $\overline{u}$ are the normalized values of velocity and horizontal component of velocity after thrusting which, with $\overline{V}_1 = \overline{u}_1 = 1$ and $\gamma_1 = 0$ (see sketch (a)), are given by

$$\overline{V} = \sqrt{1 + (d\overline{V}_\tau)^2 + 2d\overline{V}_\tau\cos\tau}$$

$$\overline{u} = 1 + d\overline{V}_\tau\cos\tau$$

Thus, for a circular orbit it is found that, to first order,

$$de = \sqrt{1 + 3\cos^2\tau}\, d\overline{V}_\tau \tag{10e}$$

It is interesting to note that this result can also be obtained from equation (10c), if the value of true anomaly attained at the thrusting point after the application of thrust is substituted for θ_1. With $\overline{V}$ and $\overline{u}$ as given above, this value of true anomaly is readily obtained from the two-body relation

$$\cos \theta = \frac{\bar{u}^2 - 1}{\sqrt{1 - (2 - \bar{V}^2)\bar{u}^2}}$$

It is found that, to first order,

$$\cos \theta = \frac{2 \cos \tau}{\sqrt{1 + 3 \cos^2 \tau}}$$

with dependence only on thrust angle.

Perturbation of true anomaly.- The perturbative effect of impulsive thrust at orbital position (r_1, θ_1) on the true anomaly, θ_1, is obtained by applying equation (4) to the two-body relations for this parameter given in appendix A. It is found that

$$d\theta_1 = -\frac{1}{e}\left[2 \sin \theta_1 \cos(\tau - \gamma_1) - \frac{2e + (1 + e^2) \cos \theta_1}{1 + e \cos \theta_1} \sin(\tau - \gamma_1)\right] \frac{dV_\tau}{V_1} \tag{11a}$$

or

$$d\theta_1 = -\frac{1}{e} \left[(2 + e \cos \theta_1)\sin \theta_1 \cos \tau - (1 + e \cos \theta_1)\cos \theta_1 \sin \tau\right] \frac{dV_\tau}{u_1} \tag{11b}$$

This change of true anomaly at the thrusting point is equal to the change of true anomaly at any other point, since it is due to a slight rotation of the line of apsides of the orbit within the orbital plane.

Perturbation of orbital radius.- The perturbative effect of impulsive thrust at orbital position (r_1, θ_1) on the orbital radius at position (r_2, θ_2) is obtained from the conic relation

$$r_2 = r_1 \left(\frac{1 + e \cos \theta_1}{1 + e \cos \theta_2}\right)$$

The logarithmic derivative of this expression, with r_1 constant, gives

$$\frac{dr_2}{r_2} = \left(\frac{\cos \theta_1}{1 + e \cos \theta_1} - \frac{\cos \theta_2}{1 + e \cos \theta_2}\right) de - \frac{e \sin \theta_1 \, d\theta_1}{1 + e \cos \theta_1} + \frac{e \sin \theta_2 \, d\theta_2}{1 + e \cos \theta_2} \tag{12}$$

For r_1 fixed, the differentials de and $d\theta_1$ are given by equations (10) and (11), respectively, in terms of e, θ_1, τ, and dV_τ/V_1. The differential $d\theta_2$ can be set equal to $d\theta_1$ or 0 depending on the particular problem under consideration. If the central angle $\theta_2 - \theta_1$ is considered fixed, then $d\theta_2$ is equal to $d\theta_1$ as given by equation (11). In this case equation (12) gives the perturbation of the orbital radius r_2 along an inertially fixed radial line in the orbital plane. If the value of θ_2 is considered fixed, then $d\theta_2$ is equal to 0, and equation (12) gives the perturbation of the orbital radius r_2 along a line which, in general, undergoes rotation with thrust so as to maintain a fixed value of true anomaly, θ_2. Since the coefficient of $d\theta_2$ in equation (12) vanishes for θ_2 equal to 180° or 0, perturbations of apocenter and pericenter radii can be determined without specification of $d\theta_2$.

For fixed central angle, $\theta_2 - \theta_1$, equation (12) can be expressed as

$$\frac{dr_2}{r_2} = \frac{1}{1 + e \cos \theta_2} \{[2(1 - \cos \Delta\theta) - e \sin \Delta\theta \sin \theta_1]\cos \tau + (1 + e \cos \theta_1)\sin \Delta\theta \sin \tau\} \frac{dV_\tau}{u_1} \tag{13a}$$

and, for fixed true anomaly, θ_2, as

$$\frac{dr_2}{r_2} = \frac{1}{1 + e \cos \theta_2} \{[2(1 - \cos \theta_1 \cos \theta_2) + e \sin^2 \theta_1 \cos \theta_2]\cos \tau + (1 + e \cos \theta_1)\sin \theta_1 \cos \theta_2\} \frac{dV_\tau}{u_1} \tag{13b}$$

Alternate forms of these perturbation expressions in terms of thrust angle relative to the flight path, $\tau - \gamma_1$, are given in appendix B.

Perturbation of elapsed flight time.- The perturbative effect of impulsive thrust at orbital position (r_1, θ_1) on the elapsed time of flight to position (r_2, θ_2) is obtained from Kepler's equation expressed as

$$t_2 - t_1 = \sqrt{\frac{a^3}{\mu}} [E_2 - E_1 - e(\sin E_2 - \sin E_1)]$$

where E is the eccentric anomaly. The logarithmic derivative of this equation gives

$$\frac{d(t_2 - t_1)}{t_2 - t_1} = \frac{3}{2}\frac{da}{a} + \frac{\sqrt{a^3/\mu}}{t_2 - t_1} [(1 - e \cos E_2)dE_2 - (1 - e \cos E_1)dE_1 - (\sin E_2 - \sin E_1)de] \tag{14}$$

The differentials da and de are given, for r_1 fixed, by equations (9) and (10), respectively, in terms of e, θ_1, τ, and dV_τ/V_1. The remaining differentials, dE_2 and dE_1, are obtained from formulas given in appendix A as

$$dE_j = \frac{\sqrt{1 - e^2}}{1 + e \cos \theta_j} \left(-\sin \theta_j \frac{de}{1 - e^2} + d\theta_j\right), \qquad j = 1,2$$

where de and $d\theta_1$ are given by equations (10) and (11), respectively. The single unspecified differential, $d\theta_2$, can be set equal to $d\theta_1$ or to 0 depending on whether $\theta_2 - \theta_1$ or θ_2 is considered fixed. In the former case the terminal point remains inertially fixed, and in the latter case it does not. Both constraints are useful in guidance computations as is subsequently shown.

For fixed central angle, $\theta_2 - \theta_1$, equation (14) can be expressed as

$$\frac{d(t_2 - t_1)}{t_2 - t_1} = \frac{\sqrt{p^3/\mu}}{t_2 - t_1} \frac{1 + e \cos \theta_1}{e} \left(X_1 \cos \tau + X_2 \sin \tau\right) \frac{dV_\tau}{u_1} \tag{15}$$

where

$$X_1 = \frac{A - B(2 + e \cos \theta_1)\sin \theta_1}{1 + e \cos \theta_1} + K(1 + e \cos \theta_1)$$

$$X_2 = B \cos \theta_1 + Ke \sin \theta_1$$

with

$$A = \frac{(2 + e \cos \theta_2)\sin \theta_2}{(1 + e \cos \theta_2)^2} - \frac{(2 + e \cos \theta_1)\sin \theta_1}{(1 + e \cos \theta_1)^2}$$

$$B = (1 + e \cos \theta_2)^{-2} - (1 + e \cos \theta_1)^{-2}$$

$$K = \left(3e \frac{t_2 - t_1}{\sqrt{p^3/\mu}} - A\right) \frac{1}{1 - e^2}$$

For fixed true anomaly, θ_2, B is replaced in the above expressions by

$$B' = -(1 + e \cos \theta_1)^{-2}$$

These results are valid for both elliptic and hyperbolic orbits, since Kepler's equation is applicable to any type of orbit. It is only the form of this equation that need be changed for use in the different cases. This is simply done by using the identity $E = -iH$ or $H = iE$ (cf. appendix A).

However, the above results, which do not contain any imaginary terms for certain values of eccentricity, are independent of the usual restriction on type of orbit specified.

In the case of parabolic orbits, the factor K in equation (15) is indeterminate, and a limiting value must be obtained. If this limiting value is represented by L, it is found that

$$L = \lim_{e \to 1} K = -\frac{1}{2}\left[\tan\frac{\theta_2}{2} - \tan\frac{\theta_1}{2} - \frac{1}{5}\left(\tan^5\frac{\theta_2}{2} - \tan^5\frac{\theta_1}{2}\right)\right]$$

In the case of circular orbits, equation (15) reduces to the limiting value

$$\frac{d(t_2 - t_1)}{t_2 - t_1} = \frac{1}{\Delta\theta}\left[3\ \Delta\theta - 4 \sin \Delta\theta)\cos \tau + 2(1 - \cos \Delta\theta)\sin \tau\right]\frac{dV_\tau}{V}$$

Alternate forms of these perturbation expressions in terms of thrust angle relative to the flight path, $\tau - \gamma_1$, are given in appendix C.

Minimum Thrust Requirements for Correction of Various Orbital Parameters

The minimum thrust requirement as used throughout this report refers to the optimum thrust angle for correcting a particular orbital parameter and the minimum derivative of thrust-produced velocity impulse with respect to that parameter. Such minimum thrust requirements for correcting various parameters of an orbit are obtained by applying equations (5) and (6) to the appropriate two-body relations for these parameters. The partial derivatives appearing in equations (5) and (6) may be obtained from a comparison of equation (4) with the perturbation expressions for the various parameters given in the previous section.

In general, there is a particular point along an orbit where the minimum derivative, dV_{τ_*}/dP, attains an absolute minimum value. This optimum orbital location for parameter correction is found by setting the derivative of dV_{τ_*}/dP with respect to θ_1 equal to 0 and solving for θ_1. Such optimum values of θ_1 are also presented for several of the orbital parameters considered in the analysis.

As in the previous section, alternate forms of the results are given in the case of semimajor axis (or period of orbit), eccentricity, and true anomaly. In the case of orbital radius and elapsed flight time, only one form is presented here; the other in each case is given in appendixes B and C, respectively.

Correction of semimajor axis or period of orbit.- If the thrust angle for maximum change in the semimajor axis (or period of orbit), for a given small amount of thrust, is represented by τ_a, it is found from equation (5) that

$$\tan(\tau_a - \gamma_1) = 0 \tag{16a}$$

or

$$\tan \tau_a = \tan \gamma_1 = \frac{e \sin \theta_1}{1 + e \cos \theta_1} \tag{16b}$$

It is readily seen that this optimum thrust direction is valid regardless of the amount of thrust applied, since in all cases a maximum change in V_1 is desired for a given value of dV_τ. No change in semimajor axis or period occurs if the speed, V_1, is unchanged by thrusting; this requires thrust normal to the flight path or velocity vector for small amounts of thrust.

The minimum dimensionless[1] derivative of thrust-produced velocity impulse with respect to semimajor axis or period of orbit is found by substituting τ_a for τ in equation (9) to obtain

$$\frac{a}{V_1}\frac{dV_{\tau_a}}{da} = \frac{3}{2}\frac{T}{V_1}\frac{dV_{\tau_a}}{dT} = \pm\frac{1 - e^2}{2(1 + 2e \cos \theta_1 + e^2)} \tag{17a}$$

or

$$\frac{a}{V_p}\frac{dV_{\tau_a}}{da} = \frac{3}{2}\frac{T}{V_p}\frac{dV_{\tau_a}}{dT} = \pm\frac{1 - e}{2\sqrt{1 + 2e \cos \theta_1 + e^2}} \tag{17b}$$

From equation (17b) it is found that the absolute minimum values of dV_{τ_a}/da and dV_{τ_a}/dT occur at a value of θ_1 equal to 0.

Correction of eccentricity.- If the thrust angle for maximum change in the eccentricity of orbit, for a given small amount of thrust, is represented by τ_e, it is found from equation (5) that

$$\tan(\tau_e - \gamma_1) = \frac{(1 - e^2)\sin \theta_1}{2(1 + e \cos \theta_1)(e + \cos \theta_1)} \tag{18a}$$

or

$$\tan \tau_e = \frac{(1 + e \cos \theta_1)\sin \theta_1}{(2 + e \cos \theta_1)\cos \theta_1 + e} \tag{18b}$$

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[1]The differential dV_τ can be normalized with any convenient speed parameter. In this report a variety of such parameters are used for this purpose including V_1, V_2, u_1, V_{c_p}, etc.

The thrust angle for which the eccentricity is invariant, to first order, is different from τ_e by 90°.

The minimum dimensionless derivative of thrust-produced velocity impulse with respect to eccentricity is found by substituting τ_e for τ in equation (10) to obtain

$$\frac{1}{V_1}\frac{dV_{\tau_e}}{de} = \pm\left\{4(e + \cos\theta_1)^2 + \left[\frac{(1 - e^2)\sin\theta_1}{1 + e\cos\theta_1}\right]^2\right\}^{-1/2} \tag{19a}$$

or

$$\frac{1}{u_1}\frac{dV_{\tau_e}}{de} = \pm\{[(2 + e\cos\theta_1)\cos\theta_1 + e]^2 + (1 + e\cos\theta_1)^2\sin^2\theta_1\}^{-1/2} \tag{19b}$$

For circular orbits, it can be shown from equation (10e) that τ_e is equal to 0 or 180°; consequently,

$$\frac{1}{V_1}\frac{dV_{\tau_e}}{de} = \frac{d\bar{V}_{\tau_e}}{de} = \pm 0.5 \tag{19c}$$

From equation (19b) it is found that the absolute minimum value of dV_{τ_e}/de occurs at θ_1 equal to 0 or 180° with the absolute maximum value occurring at a value of θ_1 given by

$$e^2\cos^3\theta_1 + 3e\cos^2\theta_1 + 3\cos\theta_1 + e(2 - e^2) = 0$$

The one real root of this cubic equation for positive values of eccentricity is

$$\cos\theta_1 = \frac{1}{e}\left[(1 - e^2)^{2/3} - 1\right]$$

Correction of true anomaly.- If the thrust angle for maximum change in the true anomaly (i.e., maximum rotation of the line of apsides within the orbital plane), for a given small amount of thrust, is represented by τ_θ, it is found from equation (5) that

$$\tan(\tau_\theta - \gamma_1) = -\frac{2e + (1 + e^2)\cos\theta_1}{2(1 + e\cos\theta_1)\sin\theta} \tag{20a}$$

or

$$\tan \tau_\theta = -\frac{(1 + e\cos\theta_1)\cos\theta_1}{(2 + e\cos\theta_1)\sin\theta_1} \tag{20b}$$

The thrust angle for which the true anomaly is invariant, to first order, is different from τ_θ by 90°.

The minimum dimensionless derivative of thrust-produced velocity impulse with respect to true anomaly is found by substituting τ_θ for τ in equation (11) to obtain

$$\frac{1}{V_1}\frac{dV_{\tau_\theta}}{d\theta_1} = \pm e\left\{4\sin^2\theta_1 + \left[\frac{2e + (1 + e^2)\cos\theta_1}{1 + e\cos\theta_1}\right]^2\right\}^{-1/2} \tag{21a}$$

or

$$\frac{1}{u_1}\frac{dV_{\tau_\theta}}{d\theta_1} = \pm e\left[(3 + 2e\cos\theta_1)\sin^2\theta_1 + (1 + e\cos\theta_1)^2\right]^{-1/2} \tag{21b}$$

From equation (21b) it is found that the absolute minimum value of $dV_{\tau_\theta}/d\theta_1$ occurs at a value of θ_1 given by

$$e^2\cos^3\theta_1 + 3e\cos^2\theta_1 + (3 + e^2)\cos\theta_1 + 2e = 0$$

The one real root of this cubic equation for positive values of eccentricity is

$$\cos\theta_1 = \frac{1}{e}\left[\sqrt[3]{\frac{1 - e^2}{2} + \sqrt{\frac{(1 - e^2)^2}{4} + \frac{e^6}{27}}} + \sqrt[3]{\frac{1 - e^2}{2} - \sqrt{\frac{(1 - e^2)^2}{4} + \frac{e^6}{27}}} - 1\right]$$

Correction of orbital radius.- The thrust angles resulting in minimum required thrust for making a correction at a given point (r_1, θ_1) along an orbit of the orbital radius at a second point (r_2, θ_2) are denoted by τ_r and τ_r', depending on whether $\theta_2 - \theta_1$ or θ_2 is considered fixed. These optimum thrust angles and corresponding minimum dimensionless derivatives of thrust-produced velocity impulse with respect to orbital radius are obtained from equations (5) and (6) with reference to equations (4) and (13). For fixed central angle, $\theta_2 - \theta_1$, it is found that

$$\tan \tau_r = \frac{(1 + e\cos\theta_1)\sin\Delta\theta}{2(1 - \cos\Delta\theta) - e\sin\Delta\theta\sin\theta_1} \tag{22}$$

and

$$\frac{r_2}{u_1}\frac{dV_{\tau_r}}{dr_2} = \pm \frac{1 + e \cos \theta_2}{\sqrt{[2(1 - \cos \Delta\theta) - e \sin \Delta\theta \sin \theta_1]^2 + (1 + e \cos \theta_1)^2 \sin^2 \Delta\theta}} \tag{23a}$$

or

$$\frac{r_2}{u_2}\frac{dV_{\tau_r}}{dr_2} = \pm \frac{1 + e \cos \theta_1}{\sqrt{[2(1 - \cos \Delta\theta) - e \sin \Delta\theta \sin \theta_1]^2 + (1 + e \cos \theta_1)^2 \sin^2 \Delta\theta}} \tag{23b}$$

In the case of fixed true anomaly, θ_2, it is found that

$$\tan \tau_r' = -\frac{(1 + e \cos \theta_1)\sin \theta_1 \cos \theta_2}{2(1 - \cos \theta_1 \cos \theta_2) + e \sin^2 \theta_1 \cos \theta_2} \tag{24}$$

and

$$\frac{r_2}{u_1}\frac{dV_{\tau_r'}}{dr_2}$$

$$= \pm \frac{1 + e \cos \theta_2}{\sqrt{[2(1 - \cos \theta_1 \cos \theta_2) + e \sin^2 \theta_1 \cos \theta_2]^2 + (1 + e \cos \theta_1)^2 \sin^2 \theta_1 \cos^2 \theta_2}} \tag{25a}$$

or

$$\frac{r_2}{u_2}\frac{dV_{\tau_r'}}{dr_2}$$

$$= \pm \frac{1 + e \cos \theta_1}{\sqrt{[2(1 - \cos \theta_1 \cos \theta_2) + e \sin^2 \theta_1 \cos \theta_2]^2 + (1 + e \cos \theta_1)^2 \sin^2 \theta_1 \cos^2 \theta_2}} \tag{25b}$$

Alternate formulations of these minimum thrust requirements for orbital radius correction are given in appendix B. Formulas for the special cases of apocenter-radius and pericenter-radius correction are listed in appendix D.

Correction of elapsed flight time.- The thrust angles resulting in minimum required thrust for making a correction at a given point (r_1, θ_1) along an orbit of the elapsed flight time to a second point (r_2, θ_2) are denoted by τ_t and $\tau_{t'}$, depending on whether $\theta_2 - \theta_1$ or θ_2 is considered fixed. These optimum thrust angles and corresponding minimum dimensionless derivatives of thrust-produced velocity impulse with respect to elapsed flight time are obtained from equations (5) and (6) with reference to equations (4) and (15). For fixed central angle, $\theta_2 - \theta_1$, it is found that

$$\tan \tau_t = \frac{X_2}{X_1} = \frac{B \cos \theta_1 + Ke \sin \theta_1}{\frac{A - B(2 + e \cos \theta_1)\sin \theta_1}{1 + e \cos \theta_1} + K(1 + e \cos \theta_1)} \tag{26}$$

and

$$\frac{t_2 - t_1}{u_1} \frac{dV_{\tau_t}}{d(t_2 - t_1)} = \pm \frac{t_2 - t_1}{\sqrt{p^3/\mu}} \frac{e/(1 + e \cos \theta_1)}{\sqrt{X_1^2 + X_2^2}} \tag{27}$$

where X_1, X_2, A, B, and K are functions of e, θ_1, and θ_2 previously defined with equation (15). In the case of fixed true anomaly, θ_2, the optimum thrust requirements are the same as those above except that B is replaced by B'. For parabolic orbits K is indeterminate and is replaced by its limiting value, L, given with equation (15). For circular orbits equations (26) and (27) in the case of fixed central angle reduce to the limiting forms

$$\tan \tau_t = \frac{2(1 - \cos \Delta\theta)}{3 \Delta\theta - 4 \sin \Delta\theta}$$

and

$$\frac{t_2 - t_1}{V_1} \frac{dV_{\tau_t}}{d(t_2 - t_1)} = \pm\Delta\theta / \sqrt{(3 \Delta\theta - 4 \sin \Delta\theta)^2 + 4(1 - \cos \Delta\theta)^2}$$

with dependence only on the central angle, $\Delta\theta$. The thrust requirements for correction of elapsed flight time with fixed true anomaly at the terminal point are ambiguous for exactly circular orbits, since the true anomaly is undefined.

In general, correction of elapsed flight time with the above minimum thrust specifications will result in a change in the orbital radius at the terminal point. However, there is one direction in which thrust will not change the terminal radius (to first order). This thrust direction is normal to that required for maximizing the change in orbital radius. Thus, the thrust angle for correction of elapsed flight time with invariant terminal radius is either τ_{r_o} or τ_{r_o}', depending on whether $\theta_2 - \theta_1$ or θ_2 is considered fixed. These thrust angles can be expressed as

$$\tan \tau_{r_o} = -\cot \tau_r \tag{28a}$$

$$\tan \tau_{r_o}' = -\cot \tau_r' \tag{28b}$$

where τ_r and τ_r' are given by equations (22) and (24), respectively.

When the thrust angles τ_{r_o} and τ_{r_o}' are substituted for τ in equation (15), it is found that, for fixed central angle, $\theta_2 - \theta_1$,

$$\frac{t_2 - t_1}{u_1} \frac{dV_{\tau_{r_o}}}{d(t_2 - t_1)} = \pm \frac{t_2 - t_1}{\sqrt{p^3/\mu}} \frac{e\sqrt{\sin^2 \Delta\theta + \left[\frac{2(1 - \cos \Delta\theta) - e \sin \Delta\theta \sin \theta_1}{1 + e \cos \theta_1}\right]^2}}{A \sin \Delta\theta + 2B(\cos \theta_2 - \cos \theta_1) + K[2e(\sin \theta_2 - \sin \theta_1) + (1 + e^2)\sin \Delta\theta]} \tag{29a}$$

and, for fixed true anomaly, θ_2,

$$\frac{t_2 - t_1}{u_1} \frac{dV_{\tau_{r_o}'}}{d(t_2 - t_1)} = \pm \frac{t_2 - t_1}{\sqrt{p^3/\mu}} \frac{e\sqrt{\sin^2 \theta_1 \cos^2 \theta_2 + \left[\frac{2(1 - \cos \theta_1 \cos \theta_2) + e \sin^2 \theta_1 \cos \theta_2}{1 + e \cos \theta_1}\right]^2}}{A \sin \theta_1 \cos \theta_2 - 2B'(\cos \theta_2 - \cos \theta_1) + K[2e + (1 + e^2)\cos \theta_2]\sin \theta_1} \tag{29b}$$

where, again, A, B, B', and K are functions of e, θ_1, and θ_2 previously defined, with K replaced by its limiting value, L, in the case of parabolic orbits. For circular orbits equation (29a) reduces to the limiting form

$$\frac{t_2 - t_1}{V_1} \frac{dV_{\tau_{r_o}}}{d(t_2 - t_1)} = \pm \frac{\Delta\theta\sqrt{\sin^2 \Delta\theta + 4(1 - \cos \Delta\theta)^2}}{3\, \Delta\theta \sin \Delta\theta - 8(1 - \cos \Delta\theta)}$$

with dependence only on the central angle, $\Delta\theta$. The derivative expressed by equation (29b) is 0 for perfectly circular orbits but is meaningless in this case due to the ambiguity of the true anomaly.

Alternate formulations of these thrust requirements for correction of elapsed flight time are given in appendix C. Formulas for the special cases of correction of flight time to apocenter and pericenter are listed in appendix E.

Thrust Requirements for Simultaneous Correction of Two Orbital Parameters

In general, it is possible to correct two independent orbital elements or parameters with a single application of impulsive thrust, since two independent scalar quantities are required to define the corrective thrust vector in the orbital plane. The specification of the corrective thrust vector in terms of the thrust angle, τ, and the thrust magnitude, ΔV_τ, is obtained from the simultaneous solution of the perturbation expressions for the two parameters. Two different formulations for τ and ΔV_τ based on the use of equations (4) and (8) are developed below.

General formulation of thrust requirements.- Consider two independent orbital parameters, P_1 and P_2, requiring corrections δP_1 and δP_2. From equation (4b) it follows that

$$\Delta V_\tau = \frac{\delta P_1}{(\partial g_1/\partial u_s)\cos\tau + (\partial g_1/\partial v_s)\sin\tau} = \frac{\delta P_2}{(\partial g_2/\partial u_s)\cos\tau + (\partial g_2/\partial v_s)\sin\tau}$$

where u_s and v_s are the horizontal and vertical (radial) components of the vehicle velocity at the time of thrusting. When these two equations are solved for τ and ΔV_τ, it is found that

$$\tan\tau = \frac{\dfrac{\partial g_1}{\partial u_s}\delta P_2 - \dfrac{\partial g_2}{\partial u_s}\delta P_1}{\dfrac{\partial g_2}{\partial v_s}\delta P_1 - \dfrac{\partial g_1}{\partial v_s}\delta P_2} \tag{30}$$

and

$$\Delta V_\tau = \frac{\sqrt{\left(\dfrac{\partial g_2}{\partial u_s}\delta P_1 - \dfrac{\partial g_1}{\partial u_s}\delta P_2\right)^2 + \left(\dfrac{\partial g_2}{\partial v_s}\delta P_1 - \dfrac{\partial g_1}{\partial v_s}\delta P_2\right)^2}}{\left|\dfrac{\partial g_1}{\partial u_s}\dfrac{\partial g_2}{\partial v_s} - \dfrac{\partial g_2}{\partial u_s}\dfrac{\partial g_1}{\partial v_s}\right|} \tag{31}$$

It is to be noted that two values of thrust angle differing by 180° are obtained from equation (30). The value of τ corresponding to a positive value of ΔV_τ is that for which the sign of $(\partial g_j/\partial u_s)\cos\tau + (\partial g_j/\partial v_s)\sin\tau$ is equal to the sign of δP_j. The condition for the two orbital parameters being independent is that the denominator in equation (31) is not 0. In general, two parameters which are usually independent can become dependent at one or more points on an orbit. These points are specified by the conditions under which the denominator of equation (31) is 0.

An alternate formulation of the thrust requirement for simultaneous correction of two independent orbital parameters may be obtained in terms of optimum thrust angles and minimum derivatives of thrust-produced velocity impulse with respect to each of the parameters. From equation (8) it follows that

$$\Delta V_\tau = \frac{dV_{\tau_{*1}}}{dP_1}\frac{\delta P_1}{\cos(\tau - \tau_{*1})} = \frac{dV_{\tau_{*2}}}{dP_2}\frac{\delta P_2}{\cos(\tau - \tau_{*2})}$$

where τ_{*_1} and τ_{*_2} are the optimum thrust angles and $dV_{\tau_{*_1}}/dP_1$ and $dV_{\tau_{*_2}}/dP_2$ are the minimum derivatives. When these two equations are solved for τ and ΔV_τ, it is found that

$$\tan \tau = \frac{\Delta V_{\tau_{*_1}} \cos \tau_{*_2} - \Delta V_{\tau_{*_2}} \cos \tau_{*_1}}{\Delta V_{\tau_{*_2}} \sin \tau_{*_1} - \Delta V_{\tau_{*_1}} \sin \tau_{*_2}} \tag{32}$$

and

$$\Delta V_\tau = \frac{\sqrt{\left(\Delta V_{\tau_{*_1}}\right)^2 + \left(\Delta V_{\tau_{*_2}}\right)^2 - 2\,\Delta V_{\tau_{*_1}} \Delta V_{\tau_{*_2}} \cos(\tau_{*_2} - \tau_{*_1})}}{\left|\sin(\tau_{*_2} - \tau_{*_1})\right|} \tag{33}$$

where

$$\Delta V_{\tau_{*_j}} = \frac{dV_{\tau_{*_j}}}{dP_j}\,\delta P_j, \qquad j = 1, 2$$

This thrust requirement can be expressed more concisely as

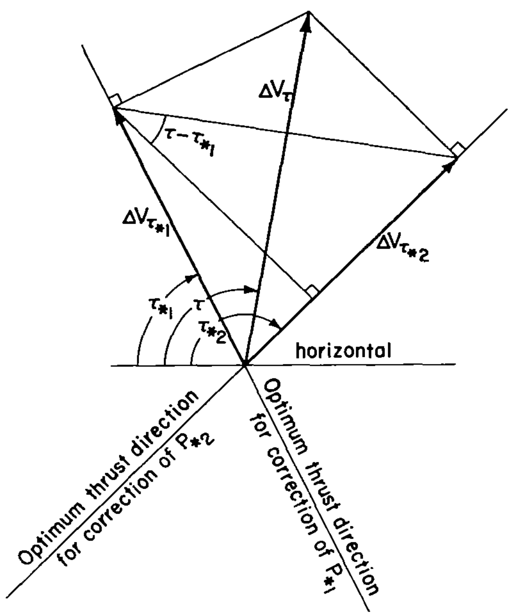
$$\tan(\tau - \tau_{*_1}) = \frac{\Delta V_{\tau_{*_2}} - \Delta V_{\tau_{*_1}} \cos(\tau_{*_2} - \tau_{*_1})}{\Delta V_{\tau_{*_1}} \sin(\tau_{*_2} - \tau_{*_1})} \tag{34}$$

and

$$\Delta V_\tau = \frac{\Delta V_{\tau_{*_1}}}{\cos(\tau - \tau_{*_1})} \tag{35}$$

As before, two values of τ differing by 180° are obtained from either equation (32) or equation (34). The value of τ corresponding to a positive value of ΔV_τ is that for which $|\tau - \tau_{*_1}|$ or $|\tau - \tau_{*_2}|$ is less than 90°. The condition for the two orbital parameters being independent is that the optimum thrust directions for correction of the two parameters separately do not coincide. It is also to be noted that when the required correction in one of the parameters is 0, the thrust specification is in the direction which, to first order, does not change this parameter. Thus, the thrust requirement in this case is different from the minimum thrust requirement for correction of the other parameter singly.

Geometric interpretation of results.- A geometric interpretation of the results obtained above for computing the single velocity impulse required to correct two different orbital parameters using their minimum thrust requirements is obtained from consideration of the vector diagram presented in sketch (b). As is indicated in this sketch, the required velocity impulse may be resolved into either of two particular pairs of orthogonal components. One of the two components in each pair is in the thrust direction required for maximum change of one of the two parameters being corrected. The other component in each pair, being orthogonal to the first, has no effect, to first order, on the pertinent parameter. Thus, it is clear that, when the two thrust directions for maximum thrust effectiveness in the case of the two parameters are specified along with the velocity impulses $\Delta V_{\tau_{*1}}$ and $\Delta V_{\tau_{*2}}$ required in these respective directions to correct each parameter separately, the velocity vector for simultaneous correction of the two parameters is uniquely defined.



Sketch (b)

From sketch (b) it is apparent that the thrust vector for simultaneous correction of two parameters can be easily obtained graphically from the minimum thrust vectors for correction of the two parameters separately. All that is required is the erection of perpendiculars to the ends of these minimum thrust vectors. The intersection of these perpendiculars is the terminus of the desired single thrust vector correcting both parameters simultaneously.

Also indicated in the vector diagram of sketch (b) is the geometric basis for the relations expressed in equations (34) and (35). It can easily be shown that, as is indicated, the angle $\tau - \tau_{*1}$ between the vectors of magnitude ΔV_τ and $\Delta V_{\tau_{*1}}$ is the same as the angle between the line joining the ends of the vectors of magnitude $\Delta V_{\tau_{*1}}$ and $\Delta V_{\tau_{*2}}$ and the perpendicular to the vector of magnitude $\Delta V_{\tau_{*2}}$. With this correspondence in angles the relation given by equation (34) is readily obtained.

RESULTS AND DISCUSSION

Graphical Presentation of Results

Minimum thrust requirements for correcting various orbital parameters are presented in figures 2 through 8. These results are given in dimensionless form and can be used for analyzing thrust-perturbed orbital motion around any celestial body. The formulas comprising the minimum thrust requirements (optimum thrust angle and minimum derivative of thrust-produced velocity impulse with respect to the parameter of interest) are listed in table I for convenience. These thrust specifications are generally expressed as functions of the eccentricity of orbit and the true anomaly at the thrusting point. Values of eccentricity from 0 to ∞ have been considered as well as the full range of positive values of true anomaly.

Optimum correction of semimajor axis or period.- The minimum thrust requirements for correcting the semimajor axis of various orbits are presented in figure 2. They are proportional to those for correcting the period of orbit. The optimum thrust angle for increasing both the semimajor axis and the period is equal to the flight-path angle. Thrust in the opposite direction maximizes the reduction of these parameters. The variation of the flight-path angle with true anomaly is presented in figure 2(a) for various values of eccentricity. The extreme values of flight-path angle for elliptic orbits occur at the two midpoints of orbit (where the orbital radius is equal to the semimajor axis and the local velocity is equal to the local circular speed). The variation with true anomaly of the minimum dimensionless derivative[2] of thrust-produced velocity impulse with respect to the semimajor axis is presented in figure 2(b). The corresponding minimum derivative with respect to the period of orbit is obtained by applying the factor 2/3 to the derivatives indicated in this figure. It is seen that the minimum local values of these derivatives occur at pericenter and the maximum at the point of greatest orbital radius. For elliptic orbits the values of either derivative at apocenter and pericenter differ by the factor $(1 + e)/(1 - e)$.

Optimum correction of eccentricity.- The minimum thrust requirements for correcting the eccentricity of various orbits are presented in figure 3. From

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[2]The dimensionless derivative shown in figure 2(b) has been normalized with the circular orbital speed at pericenter, V_{c_p}. This dimensionless form is obtained by multiplying equation (17a) by the factor

$$\frac{V_1}{V_{c_p}} = \sqrt{\frac{1 + 2e \cos \theta_1 + e^2}{1 + e}}$$

Since V_{c_p} is constant for a given orbit and is the same for all orbits having the same value of pericenter radius, the orbital position of absolute minimum thrust for correction of semimajor axis is obtained from figure 2(b) along with a comparison of minimum thrust requirements for various orbits having a common pericenter.

figure 3(a) it is seen that for elliptic orbits the optimum thrust angle for increasing eccentricity varies from 0 at pericenter to 180° at apocenter. It is noted that the variation of τ_e with θ_1 shown for $e \to 0$ is not valid for exactly circular orbits for which τ_e is 0 or 180° at all points on the orbit. The optimum thrust direction for parabolic orbits is along the flight path; for hyperbolic orbits it varies from along the flight path at pericenter to normal to the flight path at infinite radial distance.

The variation with true anomaly of the minimum dimensionless derivative of thrust-produced velocity impulse with respect to the eccentricity is presented in figure 3(b). It is seen that the minimum local value of this derivative for elliptic and parabolic orbits occurs at pericenter with the maximum local value occurring at or near the midpoints of the orbit. For hyperbolic orbits the derivative tends toward 0 as the orbital radius tends toward infinity. In the case of nearly circular orbits the absolute value of $(1/V_{c_p})(dV_{\tau_e}/de)$ varies from about 0.5 at apocenter and pericenter to about 1.0 at the midpoints of orbit. This derivative has a constant value of 0.5 at all points for exactly circular orbits.

Optimum correction of true anomaly.- The minimum thrust requirements for correcting the true anomaly (i.e., for slight rotation of the line of apsides within the orbital plane) are presented in figure 4. From figure 4(a) it is seen that the optimum thrust angle for increase of true anomaly varies from -90° at pericenter to 0 at the latus rectum and is rather insensitive to variations in eccentricity in this region. In the true anomaly range beyond 90° the optimum thrust angle increases to 90° at apocenter for elliptic orbits and returns to 0 as the radial distance becomes infinitely large for hyperbolic orbits.

The variation with true anomaly of the minimum dimensionless derivative of thrust-produced velocity impulse with respect to true anomaly is presented in figure 4(b). For elliptic orbits the minimum value of this derivative occurs at or near the midpoint of the orbit with a local maximum occurring at both apocenter and pericenter. For parabolic and hyperbolic orbits the absolute value of this derivative decreases monotonically with increasing absolute value of true anomaly at the thrusting point.

Optimum correction of orbital radius.- The minimum thrust requirements for correcting apocenter or pericenter radius are presented in figure 5. The optimum thrust direction for correcting either of these radii (fig. 5(a)) varies from the horizontal to the vertical as the point of thrust application moves toward the point of orbital radius correction from the opposite point on the orbit. Good approximations of the optimum thrust directions in the two cases are available along certain portions of the orbit. When $r_1/r_a << 1$ the optimum thrust direction for correcting apocenter radius is nearly along the flight path, and when $r_1/r_p >> 1$ the optimum thrust direction for correcting pericenter radius is nearly horizontal (cf. appendix D, eqs. (D5) and (D6)).

The variation with thrusting-point true anomaly, θ_1, of the minimum dimensionless derivative of thrust-produced velocity impulse with respect to apocenter or pericenter radius is presented in figure 5(b). In both cases this derivative decreases monotonically with increasing angular separation between the thrusting and correcting points and attains an absolute minimum at the opposite point on the orbit.

The minimum thrust requirements for correcting orbital radii at the apses are special cases of the minimum thrust specifications for correcting any orbital radius, r_2. These general results for correcting orbital radius are given by equations (22) and (23) for fixed central angle, $\theta_2 - \theta_1$, between the thrusting and correction points, and by equations (24) and (25) for fixed true anomaly, θ_2, at the correction point. The thrust requirements obtained from these expressions are the same when the orbital radius to be corrected is at one of the apses.

It can be shown from equation (22) that the optimum thrust direction for correcting orbital radius at fixed central angle from the thrusting point varies from the horizontal to a direction normal to the flight path as the central angle is reduced from its maximum value to 0. The thrust point location on the orbit for the absolute minimum thrust requirement, however, does not generally occur at maximum angular separation from the orbital radius to be corrected, as is the case for correcting the apses. This optimum thrust point location, in terms of the optimum central angle, $\theta_2 - \theta_1{}^*$, is presented in figure 6 as a function of the true anomaly of the orbital radius to be corrected, θ_2. In general, the point of minimum required thrust along the orbit is disposed toward apocenter from the nadir of the correction point (i.e., point of maximum angular separation where $\theta_2 - \theta_1 = 180°$). In the case of parabolic and hyperbolic orbits the optimum thrust point is located at infinite radial distance.

The abrupt change in location of the minimum thrust point indicated in figure 6 at the higher values of eccentricity can be explained in terms of the locations of the thrusting and correction points relative to the midpoints of orbit. It is found in all cases that the minimum thrust point moves toward pericenter as the point of orbital radius correction approaches apocenter. At some value of θ_2 the absolute values of $\theta_1{}^*$ and θ_2 are equal as indicated in figure 6. When this equality occurs in the inner part of the orbit as delineated by the minor axis (i.e., with $r_1 = r_2 < a$), there is a smooth progression of the minimum thrust point toward pericenter as the absolute value of θ_2 increases. When this equality occurs in the outer part of the orbit (i.e., with $r_1 = r_2 > a$), the minimum thrust point suddenly shifts toward pericenter as the absolute value of θ_2 increases. This effect is shown in the curves for the higher values of eccentricity in figure 6. It is indicated that simultaneous midpoint locations of the minimum thrust and correction points occur for an orbit of eccentricity about 0.7.

The absolute minimum thrust specifications for correction of orbital radius presented in figure 7 were obtained from equations (22) and (23) using the optimum values of θ_1 indicated in figure 6. In figure 7(a) the optimum thrust angle results reflect the discontinuous changes in optimum thrust point location shown in figure 6. The effects of these changes in the location of the optimum thrust point are less apparent in the minimum velocity impulse derivative results shown in figure 7(b). The primary effect is that the curves level off rather abruptly at the higher values of eccentricity.

For correction of orbital radius at fixed true anomaly, the optimum thrust angle is 0 when the location of the thrust point is at either apsis. From equation (25) it is found that the absolute minimum thrust requirement occurs at one or the other apsis, depending on whether the point of orbital radius correction is on the inner or the outer half of the orbit. For correcting orbital radii on the inner half ($r_2 < a$), the optimum thrust location is at apocenter, and for correcting orbital radii on the outer half ($r_2 > a$) the optimum thrust location is at pericenter. Absolute minimum values of the velocity impulse derivative given by equation (25) and based on apocenter or pericenter location of the thrust point are indicated by the dashed curves in figure 7(b). It is noted that the values of this minimum derivative for correction of apocenter or pericenter radius are the same as those presented for the case of fixed central angle.

Correction of elapsed flight time.- Thrust requirements for correcting the elapsed flight time to pericenter are presented in figures 8 and 9. The results presented in figure 8 are minimum thrust requirements for correcting flight time with no constraint on pericenter radius, and those in figure 9 are thrust requirements for correcting flight time with fixed orbital radius at the terminal point. In both cases two different thrust specifications are shown for the constraints of fixed central angle, $\theta_2 - \theta_1$, and fixed true anomaly, θ_2, at the terminal point.

In figure 8 are shown the variation with thrusting point true anomaly, θ_1, of both the optimum thrust angles, τ_t and τ_t', and the corresponding minimum dimensionless derivatives of thrust-produced velocity impulse with respect to elapsed flight time to pericenter. It is clear from figure 8(a) that the optimum thrust directions in the two cases tend toward and normal to the flight-path direction independent of eccentricity as the thrusting point approaches pericenter. There is seen to be less variation of the thrust angle τ_t with eccentricity than of the thrust angle τ_t'. It is shown that the two thrust angles tend toward the same values for the higher values of eccentricity as the angular separation and, therefore, the radial distance from pericenter increase. From figure 8(b) it is seen that the minimum dimensionless derivatives tend toward unity and 0 for thrust at angles τ_t and τ_t', respectively, as pericenter is approached. As for the optimum thrust angle, there is less variation of the minimum derivative with eccentricity for thrust at angle τ_t than for thrust at angle τ_t'. It is apparent that the two derivatives tend toward the same values for the higher values of eccentricity as the distance from pericenter increases.

Somewhat similar effects are to be seen in the results presented in figure 9 for correction of the elapsed flight time to pericenter under the constraint of fixed pericenter radius. In this case the thrust angles τ_{r_o} and τ_{r_o}' are equal and both differ by 90° from the values of τ_{r_p} given in figure 5(a). Although the dimensionless derivatives also tend toward unity as pericenter is approached under the constraint of fixed central angle, they tend toward finite values other than 0 under the constraint of fixed true anomaly at the terminal point. However, as in the results of figure 8 the dimensionless derivatives for the two different terminal constraints tend toward the same values as the angular separation from pericenter increases.

Application of Minimum-Thrust Results to Interplanetary Guidance

The guidance problems considered here occur during interplanetary missions when the spacecraft has not been precisely injected into the planned transfer orbit, and subsequent flight-path corrections are needed. It is convenient to separate guidance for such missions into a midcourse phase and an approach phase depending on whether the vehicle is outside or inside the sphere of influence of the target planet. When the guidance is based on the use of a reference trajectory, it is termed implicit guidance in contrast to explicit guidance for which no reference trajectory is required.

During the midcourse phase flight-path corrections are made to ensure that the spacecraft is on a collision course with the target planet or a point slightly removed from the planet. Both fixed and variable time of arrival guidance have been considered for this purpose (e.g., see refs. 8-14). In the case of implicit guidance there is little difference in the guidance computations whether the time of planet intercept is considered fixed or variable. In either case it is necessary to know the flight-path deviations from the reference trajectory at or near the terminal point.

During the approach phase final corrections of the vehicle flight path are made prior to actual interception of the target planet. Along with correction of the orbital plane, control of the time and/or location of pericenter in a planetocentric frame of reference will usually be required. For this purpose a form of fixed radius of pericenter guidance, based on the in-plane pericenter deviations from a reference trajectory, may be used. As is the case whenever impulsive thrust corrections are applied, three elements or parameters of the orbit can be corrected with each application of thrust.

The manner in which the minimum thrust results of the present analysis apply to several specific types of implicit guidance during the midcourse and approach phases of interplanetary flight is next described. In both phases only the in-plane components of the total required thrust correction are considered, since the out-of-plane corrections controlling the plane of orbit are generally independent of the in-plane corrections and can be treated separately. It is assumed that navigation information is available as to the in-plane flight-path deviations of the uncorrected spacecraft orbit from the reference trajectory at the terminal point. A summary of the guidance computations outlined in this section is presented in table II.

Midcourse guidance.- During the midcourse phase of interplanetary flight the spacecraft is between the spheres of influence of the launch and target planets. For this phase of flight the spacecraft motion relative to the Sun is essentially two-body planar motion so that the first-order minimum thrust relationships of the present analysis are suitable in connection with linear guidance about a reference trajectory. This guidance may be such as to result in interception of the target planet at the reference intercept time (fixed time of arrival) or at some time before or after the reference intercept time (variable time of arrival) depending on the particular constraints to be satisfied. Both cases are considered below.

The general situation prior to midcourse guidance action is illustrated in figure 10 which shows the orbits of the spacecraft and the planet in the vicinity of the planned intercept point. Interception of the target planet is designed to occur at time t_2 with the planet at position P_2. The uncorrected flight path of the spacecraft is radially displaced a distance δr_2 from the design intercept point and crosses the radial line through this point at a time different by δt_2 from the design value. In fixed time of arrival guidance the in-plane flight path corrections required are precisely these deviations from the reference trajectory at the terminal point. The direction and magnitude of the in-plane velocity impulse required to correct the flight path, in terms of the end-point deviations, are obtained from equations (34) and (35) expressed as

$$\tan(\tau - \tau_r) = \frac{\Delta V_{\tau_t} - \Delta V_{\tau_r} \cos(\tau_t - \tau_r)}{\Delta V_{\tau_r} \sin(\tau_t - \tau_r)} = \frac{\Delta V_{\tau_{r_o}}}{\Delta V_{\tau_r}} + \tan(\tau_t - \tau_{r_o}) \tag{36}$$

$$\Delta V_\tau = \Delta V_{\tau_r}/\cos(\tau - \tau_r) \tag{37}$$

where, since $d(t_2 - t_1) = dt_2$,

$$\Delta V_{\tau_r} = \frac{dV_{\tau_r}}{dr_2}\,\delta r_2, \qquad \Delta V_{\tau_t} = \frac{dV_{\tau_t}}{dt_2}\,\delta t_2, \qquad \Delta V_{\tau_{r_o}} = \frac{dV_{\tau_{r_o}}}{dt_2}\,\delta t_2$$

The optimum thrust angles τ_r and τ_t are given in the text by equations (22) and (26) and the minimum derivatives dV_{τ_r}/dr_2 and dV_{τ_t}/dt_2 by equations (23) and (27). The derivative $dV_{\tau_{r_o}}/dt_2$ is given by equation (29a) with the thrust angle τ_{r_o} different from τ_r by 90°. In all cases the thrust angles and derivatives are evaluated on the reference trajectory.

In variable time of arrival guidance about a reference orbit, interception of the target planet can occur at any point along its orbit in the vicinity of the nominal intercept point. Such a point is represented in figure 10 by P_2' separated from the nominal intercept point, P_2, by the central angle $\delta\theta_2$. The in-plane components of orbit deviation from planet interception at point P_2' can be expressed in terms of the components of deviation from interception at point P_2 approximately as

$$\left.\begin{aligned} \delta r_2' &= \delta r_2 - r_2(\tan\gamma_s - \tan\gamma_b)\delta\theta_2 \\ \delta t_2' &= \delta t_2 - \frac{r_2}{u_b}\left(1 - \frac{u_b}{u_s}\right)\delta\theta_2 \end{aligned}\right\} \tag{38}$$

where γ_s, u_s and γ_b, u_b are nominal values of flight-path angle and horizontal component of velocity for the spacecraft and planet, respectively, at the nominal intercept time, t_2. With substitution of these adjusted values of δr_2 and δt_2 in the thrust vector computation above, the required in-plane corrective velocity impulse is obtained for interception of the target planet at point P_2'. It is assumed in this procedure that the optimum thrust angles and minimum derivatives in the computation are invariant, to first order, for small values of $\delta\theta_2$. The computation reduces to the fixed-time-of-arrival result when $\delta\theta_2$ is 0.

For given values of the components of deviation, δr_2 and δt_2, from planet interception at the nominal or design intercept point, P_2, there is a particular value of $\delta\theta_2$ at each point along the uncorrected vehicle flight path for which the magnitude of the in-plane corrective velocity impulse for planet interception is a minimum. This value of $\delta\theta_2$ is obtained, after introduction of equation (38) into equation (37), by setting the derivative of ΔV_τ with respect to $\delta\theta_2$ equal to 0 and solving for $\delta\theta_2$. It is found that

$$\delta\theta_2 = \frac{\left[D_r - D_t \cos(\tau_r - \tau_t)\right] \dfrac{dV_{\tau_r}}{dr_2}\,\delta r_2 + \left[D_t - D_r \cos(\tau_r - \tau_t)\right] \dfrac{dV_{\tau_t}}{dt_2}\,\delta t_2}{D_r^2 + D_t^2 - 2D_rD_t \cos(\tau_r - \tau_t)} \tag{39}$$

where

$$D_r = r_2(\tan\gamma_s - \tan\gamma_b)\,\frac{dV_{\tau_r}}{dr_2}$$

$$D_t = \frac{r_2}{u_b}\left(1 - \frac{u_b}{u_s}\right)\frac{dV_{\tau_t}}{dt_2}$$

The in-plane thrust requirements (magnitude and direction of velocity impulse) for the minimum-thrust maneuver are obtained by simply inserting this value of $\delta\theta_2$ into the expressions for $\delta r_2'$ and $\delta t_2'$ and substituting these adjusted terminal point deviations for δr_2 and δt_2 in equations (36) and (37).

Approach guidance.- During the approach phase of interplanetary flight the spacecraft is within the sphere of influence of the target planet. Here the vehicle motion relative to the planet is adequately defined by the two-body equations of motion, and minimum thrust relationships derived from these two-body equations can be used in connection with linear guidance about a reference trajectory. The type of implicit guidance obtained is essentially the same as that given for the midcourse phase except that additional simplification is possible.

Various forms of implicit guidance have been considered in the literature for the approach phase. These include fixed time of arrival guidance wherein the spacecraft is constrained to pass through the reference terminal point (usually pericenter of the reference trajectory) at the design arrival time. In this case corrective thrust computations are identical to those already described for this form of guidance in the midcourse phase. Although the pericenter radius is not explicitly controlled with this type of guidance, the deviation of pericenter radius from its reference value will usually be small, and the time of pericenter passage will not differ greatly from the reference arrival time.

When explicit control of the pericenter radius is desired, some form of fixed radius of pericenter guidance is used. Along with the constraint on pericenter radius a second constraint must be imposed to uniquely define the in-plane corrective thrust vector. Three such secondary constraints which merit consideration are: (1) minimum corrective thrust, (2) fixed time of pericenter passage, and (3) fixed location of pericenter. The computation of in-plane corrective velocity impulse vectors for approach guidance with these constraints is indicated below.

The general situation prior to approach guidance action is illustrated in figure 11 which shows the in-plane deviation of the uncorrected flight path of the spacecraft from the reference trajectory. Pericenter of the uncorrected orbit has components of deviation δr_p and $\delta\theta_p$ from the reference location along with an error, δt_p, in the time of pericenter passage. From equation (8) corrections to these pericenter parameters due to a velocity impulse at angle τ of magnitude ΔV_τ can be expressed as

$$\left.\begin{aligned}
\delta r_p &= \frac{\Delta V_\tau \cos(\tau - \tau_{r_p'})}{dV_{\tau_{r_p'}}/dr_p} \\
\delta\theta_p &= \frac{\Delta V_\tau \cos(\tau - \tau_\theta)}{dV_{\tau_\theta}/d\theta_1} \\
\delta t_p &= \frac{\Delta V_\tau \cos(\tau - \tau_{t_p'})}{dV_{\tau_{t_p'}}/dt_p} = \frac{\Delta V_\tau \cos(\tau - \tau_{t_p'})}{\dfrac{dV_{\tau_{r_o'}}}{dt_p}\cos(\tau_{r_o'} - \tau_{t_p'})}
\end{aligned}\right\} \tag{40}$$

where the optimum thrust angles, $\tau_{r_p'}$, $\tau_{t_p'}$, and τ_θ, are given by equations (D6), (E3), and (20), respectively, and the minimum derivatives $dV_{\tau_{r_p'}}/dr_p$, $dV_{\tau_{t_p'}}/dt_p$, and $dV_{\tau_\theta}/d\theta_1$ by equations (D10), (E5), and (21). In the alternate expression for δt_p the thrust angle $\tau_{r_o'}$ is different from $\tau_{r_p'}$ by 90°, and the derivative $dV_{\tau_{r_o'}}/dt_p$ is obtained from equation (E8).

It is possible to correct any pair of the three pericenter parameters expressed in equation (40) with a single application of impulsive thrust. The thrust requirements for such a correction are obtained by solving simultaneously the pertinent pair of equations for τ and ΔV_τ with the result given in the form of equations (34) and (35). In the following three cases of approach guidance considered, one of the two parameters being corrected in each case is the pericenter radius. The other parameter being corrected or the secondary constraint imposed in each instance is one of the three listed above.

For approach guidance which corrects the pericenter radius deviation with minimum thrust, the direction and magnitude of the in-plane corrective velocity impulse are given by

$$\tau = \tau_{r_p'}$$

$$\Delta V_\tau = \Delta V_{\tau_{r_p'}} = \frac{dV_{\tau_{r_p'}}}{dr_p}\,\delta r_p$$

In this case the time of pericenter passage and the angular position of pericenter after thrusting will generally differ from the reference values. These differences are readily computed by introducing the above values of τ and ΔV_τ into equations (40) and adding the resulting changes in t_p and θ_p to the initial errors.

For approach guidance which corrects the deviations in pericenter radius and time of pericenter passage, the direction and magnitude of the in-plane corrective velocity impulse are given by

$$\tan(\tau - \tau_{r_p'}) = \frac{\Delta V_{\tau_{t_p'}} - \Delta V_{\tau_{r_p'}}\cos(\tau_{t_p'} - \tau_{r_p'})}{\Delta V_{\tau_{r_p'}}\sin(\tau_{t_p'} - \tau_{r_p'})} = \frac{dV_{\tau_{r_o'}}}{dt_p}\,\frac{dr_p}{dV_{\tau_{r_p'}}}\,\frac{\delta t_p}{\delta r_p} - \cot(\tau_{t_p'} - \tau_{r_p'})$$

$$\Delta V_\tau = \frac{\Delta V_{\tau_{r_p'}}}{\cos(\tau - \tau_{r_p'})} = \frac{dV_{\tau_{r_p'}}}{dr_p}\,\frac{\delta r_p}{\cos(\tau - \tau_{r_p'})}$$

In this case the angular position of pericenter after thrusting will generally differ from the reference value. This difference is readily computed using equations (40) and the initial error in orbit orientation.

For approach guidance which corrects the location of pericenter in the plane of orbit, the direction and magnitude of the in-plane corrective velocity impulse are given by

$$\tan(\tau - \tau_{r_p}') = \frac{\Delta V_{\tau_\theta} - \Delta V_{\tau_{r_p}'} \cos(\tau_\theta - \tau_{r_p}')}{\Delta V_{\tau_{r_p}'} \sin(\tau_\theta - \tau_{r_p}')} = \frac{dV_{\tau_\theta}}{d\theta_1} \frac{dr_p}{dV_{\tau_{r_p}'}} \frac{\delta\theta_p}{\delta r_p} - \cot(\tau_\theta - \tau_{r_p}')$$

$$\Delta V_\tau = \frac{\Delta V_{\tau_{r_p}'}}{\cos(\tau - \tau_{r_p}')} = \frac{dV_{\tau_{r_p}'}}{dr_p} \frac{\delta r_p}{\cos(\tau - \tau_{r_p}')}$$

In this case the time of pericenter passage will generally differ from the reference value. This difference is readily computed using equations (40) and the initial error in t_p.

SUMMARY OF RESULTS

From a first-order analysis of the Keplerian-Newtonian equations of motion, expressions have been developed that give the minimum thrust requirements for effecting small changes in certain elements or parameters of two-body orbits. The thrust requirements are presented in terms of the optimum thrust angle and the derivative of the thrust-produced velocity impulse with respect to the orbital parameter to be corrected. A summary of such thrust specifications for single parameter correction is presented in table I.

The optimum thrust angles for correction of various orbital parameters are invariant, to first order, with the amount of thrust applied. For correcting either the semimajor axis or the period of orbit, the optimum thrust direction is along the flight path at all points along the orbit for all values of eccentricity. For correcting the eccentricity of elliptic orbits, the optimum thrust direction is along the flight path at apocenter and pericenter. At other points along elliptic orbits optimum-angle thrust for correction of eccentricity tends toward the flight-path direction as the value of eccentricity increases to 1 (parabolic orbit). A minimum amount of optimum-angle thrust is required at pericenter for correcting either the eccentricity or the orbital period. In the case of correcting true anomaly or rotating the line of apsides of an orbit, the absolute value of the optimum thrust angle varies from 90° at apocenter and pericenter to 0 on the semilatus rectum. For small amounts of rotation, a minimum amount of optimum-angle thrust is required near the midpoints of the orbit. To first order, the optimum thrust angle for minimizing the change in any parameter is different by 90° from the optimum thrust angle for maximizing the change in that parameter.

The optimum thrust direction for maximum change in the orbital radius at a given point along an orbit varies from the horizontal at the nadir point on the orbit to a direction normal to the flight path at the given point. The amount of optimum-angle thrust required in correcting the orbital radius at a given point along the orbit increases from a minimum at some point on the opposite side of the orbit (which is generally disposed toward apocenter from the nadir point) to a maximum (infinite value) at the point of desired radius change. The thrust direction required in correcting the elapsed flight time to a given point along an orbit is normal to the thrust direction required in maximizing the orbital radius change at this point. The thrust magnitude required in correcting the elapsed flight time to a given point (as a percent of the value of orbital velocity at the given point) tends toward the percentage change in elapsed flight time as the given point is approached.

In general, it is possible to correct two independent parameters of a given orbit with a single application of impulsive thrust. The thrust requirements for such a correction can be formulated in terms of the minimum thrust requirements for correcting each parameter separately. These results are found to have application to the problem of spacecraft guidance during interplanetary flight. The application has specific relevance to implicit guidance within the plane of orbit and is essentially the same whether the spacecraft is in the intermediate stage of its flight or is in the final stage within the sphere of influence of the target planet. In either case navigation information is required of the in-plane deviations of the projected vehicle flight path from a design or reference trajectory at the terminal point. For midcourse guidance these terminal-point deviations consist of the differences in orbital radius and elapsed flight time from reference values at the desired point of planet interception. This desired point of interception can be any point along the orbit of the planet in the vicinity of the nominal or design intercept point. For approach guidance, after penetration of the sphere of influence of the target planet, the terminal-point deviations consist of the differences from reference values in pericenter location or in pericenter radius and time of pericenter passage. A summary of in-plane guidance computations based on use of the minimum-thrust results of the present analysis is given in table II.

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APPENDIX A

SUMMARY OF USEFUL KEPLERIAN-NEWTONIAN RELATIONS FOR TWO-BODY MOTION

$$a = \frac{r}{2 - \overline{V}^2} = \frac{r}{2 - (\overline{u}^2 + \overline{v}^2)}, \qquad T = 2\pi\sqrt{\frac{a^3}{\mu}} = \frac{2\pi}{\sqrt{1 + e}}\,\frac{a}{u_{c_a}} = \frac{2\pi}{\sqrt{1 - e}}\,\frac{a}{u_{c_p}}$$

$$e = \sqrt{1 - (2 - \overline{V}^2)\overline{V}^2 \cos^2 \gamma} = \sqrt{1 - (2 - \overline{V}^2)\overline{u}^2} = \sqrt{(1 - \overline{u}^2)^2 + \overline{u}^2\overline{v}^2} = \sqrt{1 - (b/a)^2}$$

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{p}{1 + e \cos \theta} = \frac{r_a(1 - e)}{1 + e \cos \theta} = \frac{r_p(1 + e)}{1 + e \cos \theta} = a\left[1 - \frac{e\sqrt{1 - e^2} \cos \phi}{\sqrt{1 - e^2 \cos^2 \phi}}\right] = a(1 - e \cos E)$$

$$= (-a)(e \cosh H - 1)$$

$$t - t_p = \frac{T}{2\pi}(E - e \sin E) = \sqrt{\frac{a^3}{\mu}}(E - e \sin E) = \sqrt{\frac{p^3}{\mu}}\,\frac{E - e \sin E}{(1 - e^2)^{3/2}}, \qquad 0 \le e < 1$$

$$= \sqrt{\frac{(-a)^3}{\mu}}(e \sinh H - H) = \sqrt{\frac{p^3}{\mu}}\,\frac{e \sinh H - H}{(e^2 - 1)^{3/2}}, \qquad e > 1$$

$$= \frac{1}{2}\sqrt{\frac{p^3}{\mu}}\left(\tan\frac{\theta}{2} - \frac{1}{3}\tan^3\frac{\theta}{2}\right), \qquad e = 1$$

$$\overline{u} = u\sqrt{\frac{r}{\mu}} = \sqrt{1 + e \cos \theta} = \sqrt{\frac{(1 - e^2)\sqrt{1 - e^2 \cos^2 \phi}}{\sqrt{1 - e^2 \cos^2 \phi} - e\sqrt{1 - e^2} \cos \phi}} = \sqrt{\frac{1 - e^2}{1 - e \cos E}} = \sqrt{\frac{e^2 - 1}{e \cosh H - 1}}$$

$$\bar{v} = v\sqrt{\frac{r}{\mu}} = \frac{e \sin \theta}{\sqrt{1 + e \cos \theta}} = \frac{e \sin \phi}{\left[1 - e^2 \cos^2 \phi - e\sqrt{(1 - e^2)(1 - e^2 \cos^2 \phi)}\cos \phi\right]^{1/2}} = \frac{e \sin E}{\sqrt{1 - e \cos E}}$$

$$= \frac{e \sinh H}{\sqrt{e \cosh H - 1}}$$

$$\bar{V} = V\sqrt{\frac{r}{\mu}} = \sqrt{\bar{u}^2 + \bar{v}^2} = \sqrt{\frac{1 + 2e \cos \theta + e^2}{1 + e \cos \theta}} = \left[1 + \frac{e\sqrt{1 - e^2} \cos \phi}{\sqrt{1 - e^2 \cos^2 \phi}}\right]^{1/2} = \sqrt{1 + e \cos E} = \sqrt{1 + e \cosh H}$$

$$E = -iH, \qquad H = iE$$

$$\sin \gamma = \frac{\bar{v}}{\bar{V}} = \frac{e \sin \theta}{\sqrt{1 + 2e \cos \theta + e^2}} = \frac{e \sin \phi}{\sqrt{1 - (2 - e^2)e^2 \cos^2 \phi}} = \frac{e \sin E}{\sqrt{1 - e^2 \cos^2 E}} = \frac{e \sinh H}{\sqrt{e^2 \cosh^2 H - 1}}$$

$$\cos \gamma = \frac{\bar{u}}{\bar{V}} = \frac{1 + e \cos \theta}{\sqrt{1 + 2e \cos \theta + e^2}} = \left[\frac{(1 - e^2)(1 - e^2 \cos^2 \phi)}{1 - (2 - e^2)e^2 \cos^2 \phi}\right]^{1/2} = \left[\frac{1 - e^2}{1 - e^2 \cos^2 E}\right]^{1/2} = \left[\frac{e^2 - 1}{e^2 \cosh^2 H - 1}\right]^{1/2}$$

$$\tan \gamma = \frac{\bar{v}}{\bar{u}} = \frac{e \sin \theta}{1 + e \cos \theta} = \frac{e \sin \phi}{\sqrt{(1 - e^2)(1 - e^2 \cos^2 \phi)}} = \frac{e \sin E}{\sqrt{1 - e^2}} = \frac{e \sinh H}{\sqrt{e^2 - 1}}$$

$$\sin \theta = \frac{\bar{u}\bar{v}}{e} = \frac{\sqrt{1 - e^2} \sin \phi}{\sqrt{1 - e^2 \cos^2 \phi} - e\sqrt{1 - e^2} \cos \phi} = \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E} = \frac{\sqrt{e^2 - 1} \sinh H}{e \cosh H - 1}$$

$$\cos \theta = \frac{\bar{u}^2 - 1}{e} = \frac{\sqrt{1 - e^2} \cos \phi - e\sqrt{1 - e^2 \cos^2 \phi}}{\sqrt{1 - e^2 \cos^2 \phi} - e\sqrt{1 - e^2} \cos \phi} = \frac{\cos E - e}{1 - e \cos E} = \frac{e - \cosh H}{e \cosh H - 1}$$

$$\tan \theta = \frac{\bar{u}\bar{v}}{\bar{u}^2 - 1} = \frac{\sqrt{1 - e^2} \sin \phi}{\sqrt{1 - e^2} \cos \phi - e\sqrt{1 - e^2 \cos^2 \phi}} = \frac{\sqrt{1 - e^2} \sin E}{\cos E - e} = \frac{\sqrt{e^2 - 1} \sinh H}{e - \cosh H}$$

$$\sin\phi = \frac{\bar{u}\bar{v}}{\left[\left(\dfrac{\bar{u}^2}{1-e^2} - 1\right)^2 + \bar{u}^2\bar{v}^2\right]^{1/2}} = \frac{(1-e^2)\sin\theta}{\sqrt{(1+e\cos\theta)^2 - e^2(1-e^2)\sin^2\theta}} = \frac{\sqrt{1-e^2}\,\sin E}{\sqrt{1-e^2\sin^2 E}}$$

$$= -\frac{\sqrt{e^2-1}\,\sinh H}{\sqrt{e^2\sinh^2 H - 1}}$$

$$\cos\phi = \frac{\dfrac{\bar{u}^2}{1-e^2} - 1}{\left[\left(\dfrac{\bar{u}^2}{1-e^2} - 1\right)^2 + \bar{u}^2\bar{v}^2\right]^{1/2}} = \frac{e+\cos\theta}{\sqrt{(1+e\cos\theta)^2 - e^2(1-e^2)\sin^2\theta}} = \frac{\cos E}{\sqrt{1-e^2\sin^2 E}}$$

$$= \frac{\cosh H}{\sqrt{e^2\sinh^2 H - 1}}$$

$$\tan\phi = \frac{\bar{u}\bar{v}}{\dfrac{\bar{u}^2}{1-e^2} - 1} = \frac{(1-e^2)\sin\theta}{e+\cos\theta} = \sqrt{1-e^2}\,\tan E = -\sqrt{e^2-1}\,\tanh H$$

$$\sin E = -i\sinh H = \frac{\sqrt{1-e^2}}{e}\,\frac{\bar{v}}{\bar{u}} = \frac{\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta} = \frac{\sin\phi}{\sqrt{1-e^2\cos^2\phi}}$$

$$\cos E = \cosh H = \frac{\bar{V}^2 - 1}{e} = \frac{e+\cos\theta}{1+e\cos\theta} = \frac{\sqrt{1-e^2}\,\cos\phi}{\sqrt{1-e^2\cos^2\phi}}$$

$$\tan E = -i\tanh H = \frac{\sqrt{1-e^2}\,\dfrac{\bar{v}}{\bar{u}}}{\bar{V}^2 - 1} = \frac{\sqrt{1-e^2}\,\sin\theta}{e+\cos\theta} = \frac{\tan\phi}{\sqrt{1-e^2}}$$

APPENDIX B

ALTERNATE FORMULAS FOR CORRECTING ORBITAL RADIUS

The following expressions are alternates to equations (13) and (22) through (25) for thrust requirements for correcting orbital radius r_2. These results are obtained using V_1 and γ_1 as dependent variables defining the spacecraft velocity vector at the thrusting point with the thrust angle, $\tau - \gamma_1$, specified relative to the flight path. The results consist of thrust specifications for both arbitrary and optimum angles of thrust. The general expression of these results is given by equations (4a), (5a), and (6a) when the terminal point constraint on θ_2 is specified (either $d\theta_2 = d\theta_1$ or $d\theta_2 = 0$).

For correcting orbital radius at a fixed central angle, $\theta_2 - \theta_1$,

$$\frac{V_1}{r_2}\frac{dr_2}{dV_\tau} = \frac{2(1 - \cos\Delta\theta)}{1 + e\cos\theta_2}\cos(\tau - \gamma_1) + \frac{(1 + e^2)\sin\Delta\theta + 2e(\sin\theta_2 - \sin\theta_1)}{(1 + e\cos\theta_2)(1 + e\cos\theta_1)}\sin(\tau - \gamma_1) \tag{B1}$$

$$\tan(\tau_r - \gamma_1) = \frac{(1 + e^2)\sin\Delta\theta + 2e(\sin\theta_2 - \sin\theta_1)}{2(1 - \cos\Delta\theta)(1 + e\cos\theta_1)} \tag{B2}$$

$$\frac{r_2}{V_1}\frac{dV_{\tau_r}}{dr_2} = \pm\frac{1 + e\cos\theta_2}{\sqrt{4(1 - \cos\Delta\theta)^2 + \left[\dfrac{(1 + e^2)\sin\Delta\theta + 2e(\sin\theta_2 - \sin\theta_1)}{1 + e\cos\theta_1}\right]^2}} \tag{B3}$$

For correcting orbital radius at fixed true anomaly, θ_2,

$$\frac{V_1}{r_2}\frac{dr_2}{dV_{\tau'}} = \frac{2(1 - \cos\theta_1\cos\theta_2)}{1 + e\cos\theta_2}\cos(\tau' - \gamma_1) - \frac{[(1 + e^2)\cos\theta_2 + 2e]\sin\theta_1}{(1 + e\cos\theta_2)(1 + e\cos\theta_1)}\sin(\tau' - \gamma_1) \tag{B4}$$

$$\tan(\tau_{r'} - \gamma_1) = -\frac{[(1 + e^2)\cos\theta_2 + 2e]\sin\theta_1}{2(1 - \cos\theta_1 \cos\theta_2)(1 + e\cos\theta_1)} \tag{B5}$$

$$\frac{r_2}{V_1}\frac{dV_{\tau_r}'}{dr_2} = \pm\frac{1 + e\cos\theta_2}{\sqrt{4(1 - \cos\theta_1 \cos\theta_2)^2 + \left\{\frac{[(1 + e^2)\cos\theta_2 + 2e]\sin\theta_1}{1 + e\cos\theta_1}\right\}^2}} \tag{B6}$$

APPENDIX C

ALTERNATE FORMULAS FOR CORRECTING ELAPSED FLIGHT TIME

The following expressions are alternates to equations (15) and (26) through (29) for correcting elapsed flight time $t_2 - t_1$. As in appendix B the results are obtained using V_1 and γ_1 as dependent variables defining the spacecraft velocity vector at the thrusting point with the thrust angle, $\tau - \gamma_1$, specified relative to the flight path. Again the results consist of thrust specifications for both arbitrary and optimum angles of thrust as given in general form by equations (4a), (5a), and (6a) when the terminal point constraint on θ_2 is specified (either $d\theta_2 = d\theta_1$ or $d\theta_2 = 0$).

$$\frac{V_1}{t_2 - t_1}\,\frac{d(t_2 - t_1)}{dV_\tau} = \frac{\sqrt{p^3/\mu}}{t_2 - t_1}\left[Y_1 \cos(\tau - \gamma_1) + Y_2 \sin(\tau - \gamma_1)\right] \tag{C1}$$

where

$$Y_1 = J - \frac{2B}{e} \sin \theta_1$$

$$Y_2 = -\frac{A \sin \theta_1 - B\left[2 + \frac{1 + e^2}{e} \cos \theta_1\right]}{1 + e \cos \theta_1}$$

$$A = \frac{(2 + e \cos \theta_2)\sin \theta_2}{(1 + e \cos \theta_2)^2} - \frac{(2 + e \cos \theta_1)\sin \theta_1}{(1 + e \cos \theta_1)^2}$$

$$B = (1 + e \cos \theta_2)^{-2} - (1 + e \cos \theta_1)^{-2}$$

$$J = \frac{3(1 + 2e \cos \theta_1 + e^2)}{1 - e^2}\,\frac{t_2 - t_1}{\sqrt{p^3/\mu}} - \frac{2A(e + \cos \theta_1)}{1 - e^2}$$

In the case of parabolic orbits J is replaced by its limiting value defined by

$$\lim_{e \to 1} J = 2L(1 + \cos \theta_1)$$

where L is given in the text with equation (15).

The minimum thrust requirements for correcting elapsed flight time to traverse a fixed central angle with optimum-angle thrust are given by

$$\tan(\tau_t - \gamma_1) = \frac{Y_2}{Y_1} = \frac{Ae \sin\theta_1 - B[2e + (1 + e^2)\cos\theta_1]}{(2B \sin\theta_1 - Je)(1 + e \cos\theta_1)} \tag{C2}$$

$$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_t}}{d(t_2 - t_1)} = \pm\frac{e(t_2 - t_1)/\sqrt{p^3/\mu}}{\sqrt{Y_1^2 + Y_2^2}} \tag{C3}$$

For correcting elapsed flight time to arrive at a fixed true anomaly, θ_2, B is replaced in the above expressions by B' where

$$B' = -(1 + e \cos\theta_1)^{-2}$$

As is indicated in the text, thrust at angle τ_{r_o} or τ_{r_o}' will correct the elapsed flight time to a given terminal point on the orbit without changing the orbital radius at the terminal point to first order. In the one case (τ_{r_o}) the terminal point is at a fixed central angle, $\theta_2 - \theta_1$, while in the other case (τ_{r_o}') the terminal point is at a fixed true anomaly, θ_2. When these thrust angles are inserted in equation (C1), it is found that

$$\frac{t_2-t_1}{V_1}\frac{dV_{\tau_{r_o}}}{d(t_2-t_1)} = \pm\frac{\frac{t_2-t_1}{\sqrt{p^3/\mu}}\sqrt{4(1-\cos\Delta\theta)^2(1+e\cos\theta_1)^2+[1+e^2)\sin\Delta\theta+2e(\sin\theta_2-\sin\theta_1)}}{2A(1-\cos\Delta\theta)\sin\theta_1+\frac{2B}{e}(1+2e\cos\theta_1+e^2)(\cos\theta_2-\cos\theta_1)+J[(1+e^2)\sin\Delta\theta+2e(\sin\theta_2-\sin\theta_1)]} \tag{C4}$$

$$\frac{t_2-t_1}{V_1}\frac{dV_{\tau_{r_o}'}}{d(t_2-t_1)} = \pm\frac{\frac{t_2-t_1}{\sqrt{p^3/\mu}}\sqrt{4(1-\cos\theta_1\cos\theta_2)^2(1+e\cos\theta_1)^2+[(1+e^2)\cos\theta_2+2e]^2\sin^2\theta_1}}{2A(1-\cos\theta_1\cos\theta_2)\sin\theta_1+\frac{2B'}{e}[(1+e^2)\cos\theta_2-(1+2e\cos\theta_1+e^2)\cos\theta_1]-J[(1+e^2)\cos\theta_2+2e]\sin\theta_1} \tag{C5}$$

where A, B, B', and J are defined above.

APPENDIX D

FORMULAS FOR CORRECTING APOCENTER OR PERICENTER RADIUS

The following expressions are special cases ($\theta_2 = \pi$, 0) of the general thrust requirements for orbital radius correction presented in the text and in appendix B. These thrust specifications for correction of apocenter and pericenter radii are independent of the terminal point constraint on θ_2 due to the insensitivity of orbital radius at $\theta_2 = \pi$ or 0 to small changes in θ_2.

For thrust at arbitrary angle there are obtained from equation (13)

$$\frac{V_a}{r_a}\frac{dr_a}{dV_\tau} = \frac{2(1 + \cos\theta_1) - e\sin^2\theta_1}{1 + e\cos\theta_1}\cos\tau + \sin\theta_1 \sin\tau \qquad \text{(D1)}$$

$$\frac{V_p}{r_p}\frac{dr_p}{dV_\tau} = \frac{2(1 - \cos\theta_1) + e\sin^2\theta_1}{1 + e\cos\theta_1}\cos\tau - \sin\theta_1 \sin\tau \qquad \text{(D2)}$$

and from equation (B1)

$$\frac{V_1}{r_a}\frac{dr_a}{dV_\tau} = \frac{2(1 + \cos\theta_1)}{1 - e}\cos(\tau - \gamma_1) + \frac{(1 - e)\sin\theta_1}{1 + e\cos\theta_1}\sin(\tau - \gamma_1) \qquad \text{(D3)}$$

$$\frac{V_1}{r_p}\frac{dr_p}{dV_\tau} = \frac{2(1 - \cos\theta_1)}{1 + e}\cos(\tau - \gamma_1) - \frac{(1 + e)\sin\theta_1}{1 + e\cos\theta_1}\sin(\tau - \gamma_1) \qquad \text{(D4)}$$

Optimum thrust angles for correcting apocenter or pericenter radius are obtained from equation (24) as

$$\tan\tau_{r_a}' = \frac{(1 + e\cos\theta_1)\sin\theta_1}{2(1 + \cos\theta_1) - e\sin^2\theta_1} = \frac{\tan\gamma_1}{1 - (r_1/r_a)^2} \qquad \text{(D5)}$$

$$\tan\tau_{r_p}' = -\frac{(1 + e\cos\theta_1)\sin\theta_1}{2(1 - \cos\theta_1) + e\sin^2\theta_1} = \frac{\tan\gamma_1}{1 - (r_1/r_p)^2} \qquad \text{(D6)}$$

and from equation (B5) as

$$\tan(\tau_{r_a}' - \gamma_1) = \frac{(1 - e)^2\sin\theta_1}{2(1 + \cos\theta_1)(1 + e\cos\theta_1)} = \frac{\tan\gamma_1}{(r_a/r_1)^2\sec^2\gamma_1 - 1} \qquad \text{(D7)}$$

$$\tan(\tau_{r_p}' - \gamma_1) = -\frac{(1 + e)^2\sin\theta_1}{2(1 - \cos\theta_1)(1 + e\cos\theta_1)} = \frac{\tan\gamma_1}{(r_p/r_1)^2\sec^2\gamma_1 - 1} \qquad \text{(D8)}$$

Minimum dimensionless derivatives of thrust-produced velocity impulse with respect to apocenter and pericenter radii are obtained from equation (25b) as

$$\frac{r_a}{V_a}\frac{dV_{\tau r_a'}}{dr_a} = \pm \frac{1 + e\cos\theta_1}{\sqrt{[2(1 + \cos\theta_1) - e\sin^2\theta_1]^2 + (1 + e\cos\theta_1)^2\sin^2\theta_1}} \tag{D9}$$

$$\frac{r_p}{V_p}\frac{dV_{\tau r_p'}}{dr_p} = \pm \frac{1 + e\cos\theta_1}{\sqrt{[2(1 - \cos\theta_1) + e\sin^2\theta_1]^2 + (1 + e\cos\theta_1)^2\sin^2\theta_1}} \tag{D10}$$

and from equation (B6) as

$$\frac{r_a}{V_1}\frac{dV_{\tau r_a'}}{dr_a} = \pm \frac{(1 - e)(1 + e\cos\theta_1)}{\sqrt{4(1 + \cos\theta_1)^2(1 + e\cos\theta_1)^2 + (1 - e)^2\sin^2\theta_1}} \tag{D11}$$

$$\frac{r_p}{V_1}\frac{dV_{\tau r_p'}}{dr_p} = \pm \frac{(1 + e)(1 + e\cos\theta_1)}{\sqrt{4(1 - \cos\theta_1)^2(1 + e\cos\theta_1)^2 + (1 + e)^2\sin^2\theta_1}} \tag{D12}$$

APPENDIX E

FORMULAS FOR CORRECTING FLIGHT TIME TO APOCENTER OR PERICENTER

The following expressions are special cases ($\theta_2 = \pi$, 0) of the general thrust requirements for elapsed flight time correction presented in the text and in appendix C. In obtaining these expressions the constraint of fixed true anomaly at the terminal point, θ_2, has been imposed. For the sake of brevity, in each instance a single formula is presented that is valid for both cases of terminal point location. It is understood that the subscript 2 is to be replaced by the subscript a when the terminal point is apocenter and by the subscript p when the terminal point is pericenter.

For thrust at arbitrary angle there is obtained from equation (15)

$$\frac{V_2}{t_2 - t_1} \frac{d(t_2 - t_1)}{dV_{\tau'}} = \frac{1 + e \cos \theta_2}{(1 - e^2)} \left\{ \left[\frac{(2 + e \cos \theta_1) \sin \theta_1}{e(1 + e \cos \theta_1)} \frac{\sqrt{p^3/\mu}}{t_2 - t_1} + 3(1 + e \cos \theta_1) \right] \cos \tau' + \left[\frac{e(1 + \sin^2 \theta_1) - \cos \theta_1}{e(1 + e \cos \theta_1)} \frac{\sqrt{p^3/\mu}}{t_2 - t_1} + 3e \sin \theta_1 \right] \sin \tau' \right\} \tag{E1}$$

and from equation (C1)

$$\frac{V_1}{t_2 - t_1} \frac{d(t_2 - t_1)}{dV_{\tau'}} = \frac{\sqrt{p^3/\mu}}{e(t_2 - t_1)} \left\{ \frac{1 + 2e \cos \theta_1 + e^2}{1 - e^2} \left[\frac{2 \sin \theta_1}{1 + e \cos \theta_1} + \frac{3e(t_2 - t_1)}{\sqrt{p^3/\mu}} \right] \cos(\tau - \gamma_1) - \frac{\cos \theta_1}{1 + e \cos \theta_1} \sin(\tau - \gamma_1) \right\} \tag{E2}$$

Optimum thrust angles for correcting elapsed flight time to apocenter or pericenter are obtained from equations (26) and (C2) as

$$\tan \tau_{t_2}' = \frac{e(1 + \sin^2 \theta_1) - \cos \theta_1 + 3e^2(1 + e \cos \theta_1)\sin \theta_1(t_2 - t_1)/\sqrt{p^3/\mu}}{(2 + e \cos \theta_1)\sin \theta_1 + 3e(1 + e \cos \theta_1)^2(t_2 - t_1)/\sqrt{p^3/\mu}} \tag{E3}$$

$$\tan(\tau_{t_2}' - \gamma_1) = -\frac{(1 - e^2)\cos \theta_1}{2(1 + e \cos \theta_1 + e^2)\sin \theta_1 + 3e(1 + 2e \cos \theta_1 + e^2)(1 + e \cos \theta_1)(t_2 - t_1)/\sqrt{p^3/\mu}} \tag{E4}$$

Minimum dimensionless derivatives of thrust-produced velocity impulse with respect to elapsed flight time to apocenter or pericenter are obtained from equations (27) and (C3) as

$$\frac{t_2-t_1}{V_2}\frac{dV_{\tau_{t_2}'}}{d(t_2-t_1)} = \pm\frac{\dfrac{e(1-e^2)}{1+e \cos \theta_2}}{\sqrt{\left[\dfrac{(2+e \cos \theta_1)\sin \theta_1}{1+e \cos \theta_1}\dfrac{\sqrt{p^3/\mu}}{t_2-t_1} + 3e(1+e \cos \theta_1)\right]^2 + \left[\dfrac{e(1+\sin^2 \theta_1)-\cos \theta_1}{1+e \cos \theta_1}\dfrac{\sqrt{p^3/\mu}}{t_2-t_1} + 3e^2 \sin \theta_1\right]^2}} \tag{E5}$$

$$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_{t_2}'}}{d(t_2 - t_1)} = \pm\frac{e(t_2 - t_1)/\sqrt{p^3/\mu}}{\sqrt{\left[\dfrac{1 + 2e \cos \theta_1 + e^2}{1 - e^2}\right]^2\left[\dfrac{2 \sin \theta_1}{1 + e \cos \theta_1} + \dfrac{3e(t_2 - t_1)}{\sqrt{p^3/\mu}}\right]^2 + \dfrac{\cos^2 \theta_1}{(1 + e \cos \theta_1)^2}}} \tag{E6}$$

The corresponding dimensionless derivatives for thrust in the direction that, to first order, does not change the orbital radius at the terminal point ($\tau = \tau_{r_o}'$) are obtained from equation (29b) as

$$\frac{t_a - t_1}{V_a}\frac{dV_{\tau_{r_o}'}}{d(t_a - t_1)} = \pm\frac{\dfrac{1 + e}{1 - e}\dfrac{t_a - t_1}{\sqrt{p^3/\mu}}\sqrt{[2(1 + \cos \theta_1) - e \sin^2 \theta_1]^2 + (1 + e \cos \theta_1)^2 \sin^2 \theta_1}}{\dfrac{2}{e}\left(\dfrac{1 + \cos \theta_1 - e \sin^2 \theta_1}{1 + e \cos \theta_1}\right) + 3(1 - e)\dfrac{t_a - t_1}{\sqrt{p^3/\mu}}\sin \theta_1} \tag{E7}$$

$$\frac{t_p - t_1}{V_p}\frac{dV_{\tau_{r_o}'}}{d(t_p - t_1)} = \pm\frac{\dfrac{1 - e}{1 + e}\dfrac{t_p - t_1}{\sqrt{p^3/\mu}}\sqrt{[2(1 - \cos \theta_1) + e \sin^2 \theta_1]^2 + (1 + e \cos \theta_1)^2 \sin^2 \theta_1}}{\dfrac{2}{e}\left(\dfrac{1 - \cos \theta_1 + e \sin^2 \theta_1}{1 + e \cos \theta_1}\right) + 3(1 + e)\dfrac{t_p - t_1}{\sqrt{p^3/\mu}}\sin \theta_1} \tag{E8}$$

and from equation (C5) as

$$\frac{t_a - t_1}{V_1} \frac{dV_{\tau r_o'}}{d(t_a - t_1)} = \pm \frac{\frac{t_a - t_1}{\sqrt{p^3/\mu}} \sqrt{4(1 + \cos\theta_1)^2(1 + e\cos\theta_1)^2 + (1 - e)^4 \sin^2\theta_1}}{\frac{1 + 2e\cos\theta_1 + e^2}{1 + e}\left[\frac{2}{e}\left(\frac{1 + \cos\theta_1 - e\sin^2\theta_1}{1 + e\cos\theta_1}\right) + 3(1 - e)\frac{t_a - t_1}{\sqrt{p^3/\mu}}\sin\theta_1\right]} \tag{E9}$$

$$\frac{t_p - t_1}{V_1} \frac{dV_{\tau r_o'}}{d(t_p - t_1)} = \pm \frac{\frac{t_p - t_1}{\sqrt{p^3/\mu}} \sqrt{4(1 - \cos\theta_1)^2(1 + e\cos\theta_1)^2 + (1 + e)^4 \sin^2\theta_1}}{\frac{1 + 2e\cos\theta_1 + e^2}{1 - e}\left[\frac{2}{e}\left(\frac{1 - \cos\theta_1 + e\sin^2\theta_1}{1 + e\cos\theta_1}\right) + 3(1 + e)\frac{t_p - t_1}{\sqrt{p^3/\mu}}\sin\theta_1\right]} \tag{E10}$$

REFERENCES

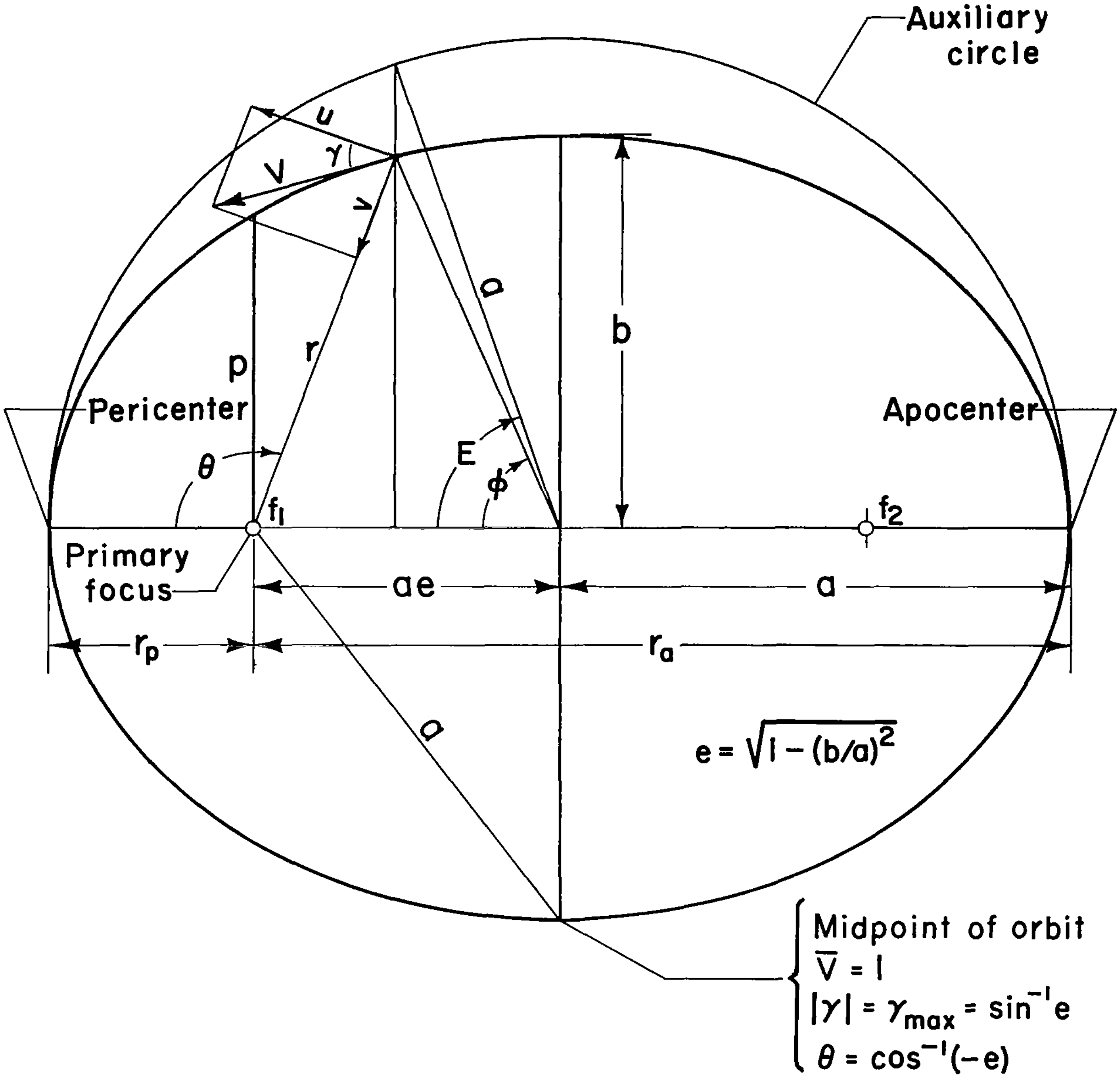
1. Ramsey, A. S.: Dynamics. Part II. The University Press, Cambridge, 1956.
2. Martin Marietta Corp.: Design Guide to Orbital Flight. McGraw-Hill, N. Y., 1962.
3. Orbital Flight Handbook. NASA SP-33, 1963.
4. Lunar Flight Handbook. NASA SP-34, 1963.
5. Planetary Flight Handbook. NASA SP-35, 1963.
6. Friedlander, Alan L.; and Harry, David P., III: Requirements of Trajectory Corrective Impulses During the Approach Phase of an Interplanetary Mission. NASA TN D-255, 1960.
7. Wong, Thomas J.; and Slye, Robert E.: The Effect of Lift on Entry Corridor Depth and Guidance Requirements for the Return Lunar Flight. NASA TR R-80, 1961.
8. Stern, R. G.: Analytic Solution of the Equations of Motion of an Interplanetary Space Vehicle in the Midcourse Phase of Its Flight. MIT Rep. RE-4, Experimental Astronomy Lab., Nov. 1963.
9. Slater, G. L.; and Stern, R. G.: Simplified Midcourse Guidance Techniques. MIT Rep. RE-18, Experimental Astronomy Lab., Feb. 1966.
10. Cicolani, Luigi S.: Linear Theory of Impulsive Velocity Corrections for Space Mission Guidance. NASA TN D-3365, 1966.
11. White, John S.; Callas, George P.; and Cicolani, Luigi S.: Application of Statistical Filter Theory to the Interplanetary Navigation and Guidance Problem. NASA TN D-2697, 1965.
12. Ehricke, K. A.: Space Flight. Vol. I - Environment and Celestial Mechanics. D. Van Nostrand Co., Inc., Princeton, N. J., 1960.
13. Leondes, C. T., ed.: Guidance and Control of Aerospace Vehicles. McGraw-Hill, N. Y., 1963.
14. Battin, R. H.: Astronautical Guidance. McGraw-Hill, N. Y., 1964.

TABLE I.- THRUST SPECIFICATIONS FOR CORRECTION OF ORBITAL PARAMETERS SINGLY

Parameter	Constraint	Thrust angle		Velocity impulse derivative		Minimum thrust point on orbit
		Function	Equation	Function	Equation	
Semimajor axis, a, or period of orbit, T	Minimum thrust	$\tan(\tau_a - \gamma_1)$	(16a)	$\frac{a}{V_1}\frac{dV_{\tau_a}}{da}, \frac{T}{V_1}\frac{dV_{\tau_a}}{dT}$	(17a)	Pericenter
		$\tan \tau_a$-	(16b)	$\frac{a}{V_p}\frac{dV_{\tau_a}}{da}, \frac{T}{V_p}\frac{dV_{\tau_a}}{dT}$	(17b)	
Eccentricity, e	Minimum thrust	$\tan(\tau_e - \gamma_1)$	(18a)	$\frac{1}{V_1}\frac{dV_{\tau_e}}{de}$	(19a)	Pericenter (for $e < 1$)
		$\tan \tau_e$	(18b)	$\frac{1}{u_1}\frac{dV_{\tau_e}}{de}$	(19b)	Infinity (for $e \geq 1$)
True anomaly, θ_1	Minimum thrust	$\tan(\tau_\theta - \gamma_1)$	(20a)	$\frac{1}{V_1}\frac{dV_{\tau_\theta}}{d\theta_1}$	(21a)	Near midpoint (for $e < 1$)
		$\tan \tau_\theta$	(20b)	$\frac{1}{u_1}\frac{dV_{\tau_\theta}}{d\theta_1}$	(21b)	Infinity (for $e \geq 1$)
Orbital radius, r_2	Minimum thrust, $\theta_2 - \theta_1$ = const.	$\tan(\tau_r - \gamma_1)$	(B2)	$\frac{r_2}{V_1}\frac{dV_{\tau_r}}{dr_2}$	(B3)	$180° > \theta_1 > \theta_2 + 180°$ (for $\theta_2 \leq 0$)
		$\tan \tau_r$	(22)	$\frac{r_2}{u_1}\frac{dV_{\tau_r}}{dr_2}$	(23a)	$-180° < \theta_1 < \theta_2 - 180°$ (for $\theta_2 \geq 0$)
				$\frac{r_2}{u_2}\frac{dV_{\tau_r}}{dr_2}$	(23b)	
	Minimum thrust, θ_2 = const.	$\tan(\tau_r' - \gamma_1)$	(B5)	$\frac{r_2}{V_1}\frac{dV_{\tau_r'}}{dr_2}$	(B6)	Apocenter (for $r_2 \leq a$)
		$\tan \tau_r'$	(24)	$\frac{r_2}{u_1}\frac{dV_{\tau_r'}}{dr_2}$	(25a)	Pericenter (for $r_2 \geq a$)
				$\frac{r_2}{u_2}\frac{dV_{\tau_r'}}{dr_2}$	(25b)	
Elapsed flight time, $t_2 - t_1$	Minimum thrust, $\theta_2 - \theta_1$ = const.	$\tan(\tau_t - \gamma_1)$	(C2)	$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_t}}{d(t_2 - t_1)}$	(C3)	
		$\tan \tau_t$	(26)	$\frac{t_2 - t_1}{u_1}\frac{dV_{\tau_t'}}{d(t_2 - t_1)}$	(27)	
	Minimum thrust, θ_2 = const.	$\tan(\tau_t' - \gamma_1)$	(C2), B = B'	$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_t'}}{d(t_2 - t_1)}$	(C3), B = B'	
		$\tan \tau_t'$	(26), B = B'	$\frac{t_2 - t_1}{u_1}\frac{dV_{\tau_t'}}{d(t_2 - t_1)}$	(27), B = B'	
	r_2 = const., $\theta_2 - \theta_1$ = const.	$\tan(\tau_{r_o} - \gamma_1)$	---	$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_{r_o}}}{d(t_2 - t_1)}$	(C4)	
		$\tan \tau_{r_o}$	(28a)	$\frac{t_2 - t_1}{u_1}\frac{dV_{\tau_{r_o}}}{d(t_2 - t_1)}$	(29a)	
	r_2 = const., θ_2 = const.	$\tan(\tau_{r_o}' - \gamma_1)$	---	$\frac{t_2 - t_1}{V_1}\frac{dV_{\tau_{r_o}'}}{d(t_2 - t_1)}$	(C5)	
		$\tan \tau_{r_o}'$	(28b)	$\frac{t_2 - t_1}{u_1}\frac{dV_{\tau_{r_o}}}{d(t_2 - t_1)}$	(29b)	

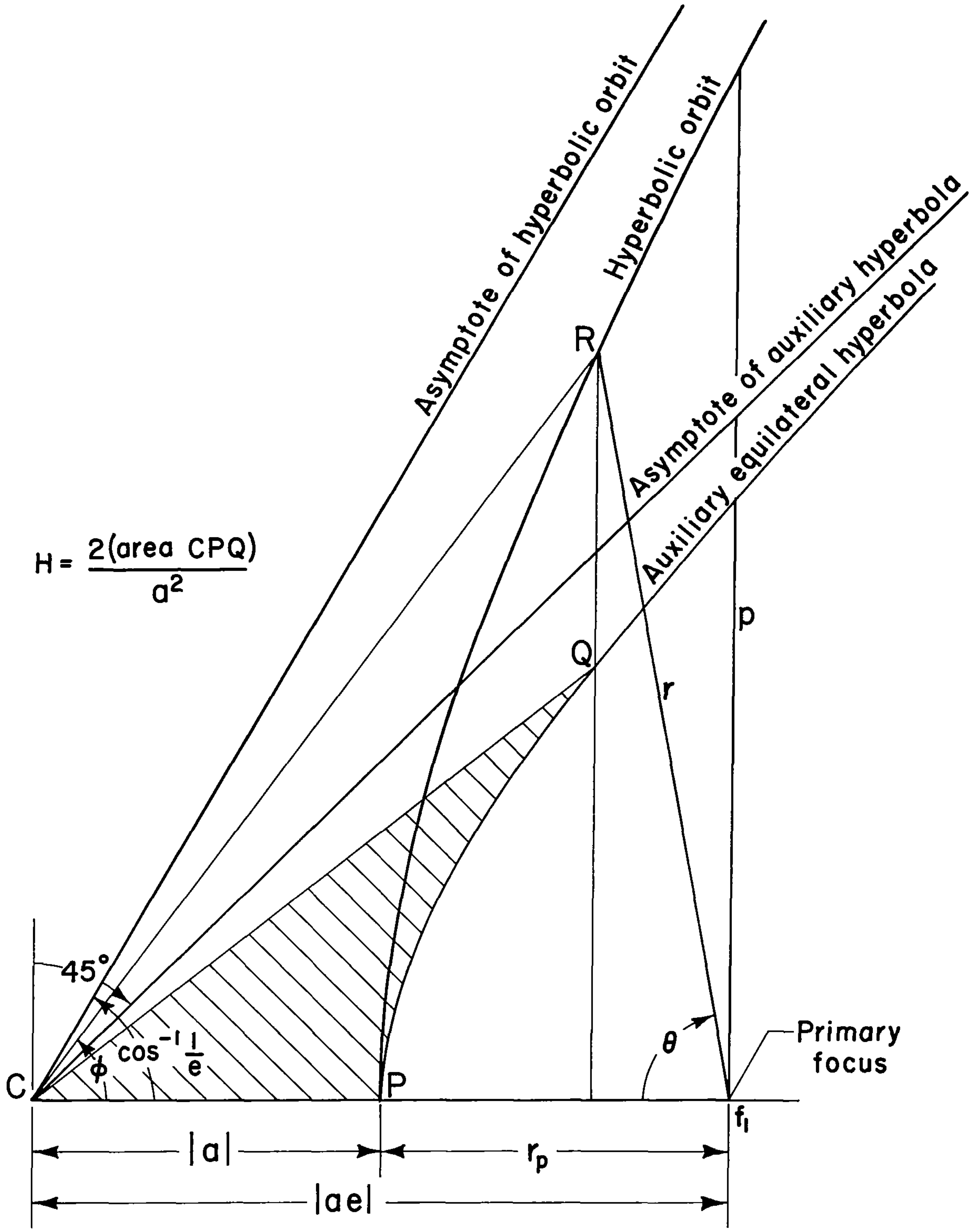
TABLE II.- APPLICATION OF MINIMUM THRUST RESULTS TO IN-PLANE INTERPLANETARY GUIDANCE

	Type of guidance	Guidance objectives or trajectory constraints	Orbital parameter corrections required	Direction and magnitude of corrective velocity impulse	Source of quantities used in guidance computations		
					Optimum thrust angle and minimum velocity impulse for single parameter correction		Minimum thrust produced velocity impulse derivative
Midcourse phase	Fixed time of arrival	Planet interception at time t_2	Terminal point deviations, δr_2 and δt_2	$\tau = \tau_r + \tan^{-1}\left[\frac{\Delta V_{\tau_t}/\Delta V_{\tau_r}}{\sin(\tau_t - \tau_r)} - \cot(\tau_t - \tau_r)\right]$ $\Delta V_\tau = \frac{\Delta V_{\tau_r}}{\cos(\tau - \tau_r)}$	τ_r from eq. (22) τ_t from eq. (26)	$\Delta V_{\tau_r} = \frac{dV_{\tau_r}}{dr_2}\delta r_2$ $\Delta V_{\tau t} = \frac{dV_{\tau_t}}{dt_2}\delta t_2$	$\frac{dV_{\tau_r}}{dr_2}$ from eq. (23) $\frac{dV_{\tau_t}}{dt_2}$ from eq. (27)
	Variable time of arrival	Planet interception at time near t_2	$\delta r_2' = \delta r_2 - r_2(\tan\gamma_s - \tan\gamma_b)\delta\theta_2$ $\delta t_2' = \delta t_2 - \frac{r_2}{u_b}\left(1 - \frac{u_b}{u_s}\right)\delta\theta_2$	Same as above	Same as above	$\Delta V_{\tau_r} = \frac{dV_{\tau_r}}{dr_2}\delta r_2'$ $\Delta V_{\tau_t} = \frac{dV_{\tau_t}}{dt_2}\delta t_2'$	Same as above
		Planet interception with minimum thrust	Same as above, $\delta\theta_2$ from eq. (39)	Same as above	Same as above	Same as above	Same as above
Approach phase	Fixed radius of pericenter	Pericenter radius control with minimum thrust	Pericenter radius deviation, δr_p	$\tau = \tau_{r_p}'$ $\Delta V_\tau = \Delta V_{\tau_{r_p}'}$	τ_{r_p}' from eq. (D6)	$\Delta V_{\tau_{r_p}'} = \frac{dV_{\tau_{r_p}'}}{dr_p}\delta r_p$	$\frac{dV_{\tau_{r_p}'}}{dr_p}$ from eq. (D10) or eq. (D12)
		Fixed values of pericenter radius and time of pericenter passage	Pericenter deviations, δr_p and δt_p	$\tau = \tau_{r_p}' + \tan^{-1}\left[\frac{\Delta V_{\tau_{t_p}'}/\Delta V_{\tau_{r_p}'}}{\sin(\tau_{t_p}' - \tau_{r_p}')} - \cot(\tau_{t_p}' - \tau_{r_p}')\right]$ $\Delta V_\tau = \frac{\Delta V_{\tau_{r_p}'}}{\cos(\tau - \tau_{r_p}')}$	τ_{r_p}' from eq. (D6) τ_{t_p}' from eq. (E3)	$\Delta V_{\tau_{r_p}'} = \frac{dV_{\tau_{r_p}'}}{dr_p}\delta r_p$ $\Delta V_{\tau_{t_p}'} = \frac{dV_{\tau_{t_p}'}}{dt_p}\delta t_p$	$\frac{dV_{\tau_{r_p}'}}{dr_p}$ from eq. (D10) or eq. (D12) $\frac{dV_{\tau_{t_p}'}}{dt_p}$ from eq. (E5) or eq. (E6)
		Fixed location of pericenter	Pericenter deviations, δr_p and $\delta\theta_p$	$\tau = \tau_{r_p}' + \tan^{-1}\left[\frac{\Delta V_{\tau_\theta}/\Delta V_{\tau_{r_p}'}}{\sin(\tau_\theta - \tau_{r_p}')} - \cot(\tau_\theta - \tau_{r_p}')\right]$ $\Delta V_\tau = \frac{\Delta V_{\tau_{r_p}'}}{\cos(\tau - \tau_{r_p}')}$	τ_{r_p}' from eq. (D6) τ_θ from eq. (20)	$\Delta V_{\tau_{r_p}'} = \frac{dV_{\tau_{r_p}'}}{dr_p}\delta r_p$ $\Delta V_{\tau_\theta} = \frac{dV_{\tau_\theta}}{d\theta_1}\delta\theta_p$	$\frac{dV_{\tau_{r_p}'}}{dr_p}$ from eq. (D10) or eq. (D12) $\frac{dV_{\tau_\theta}}{d\theta_1}$ from eq. (21)



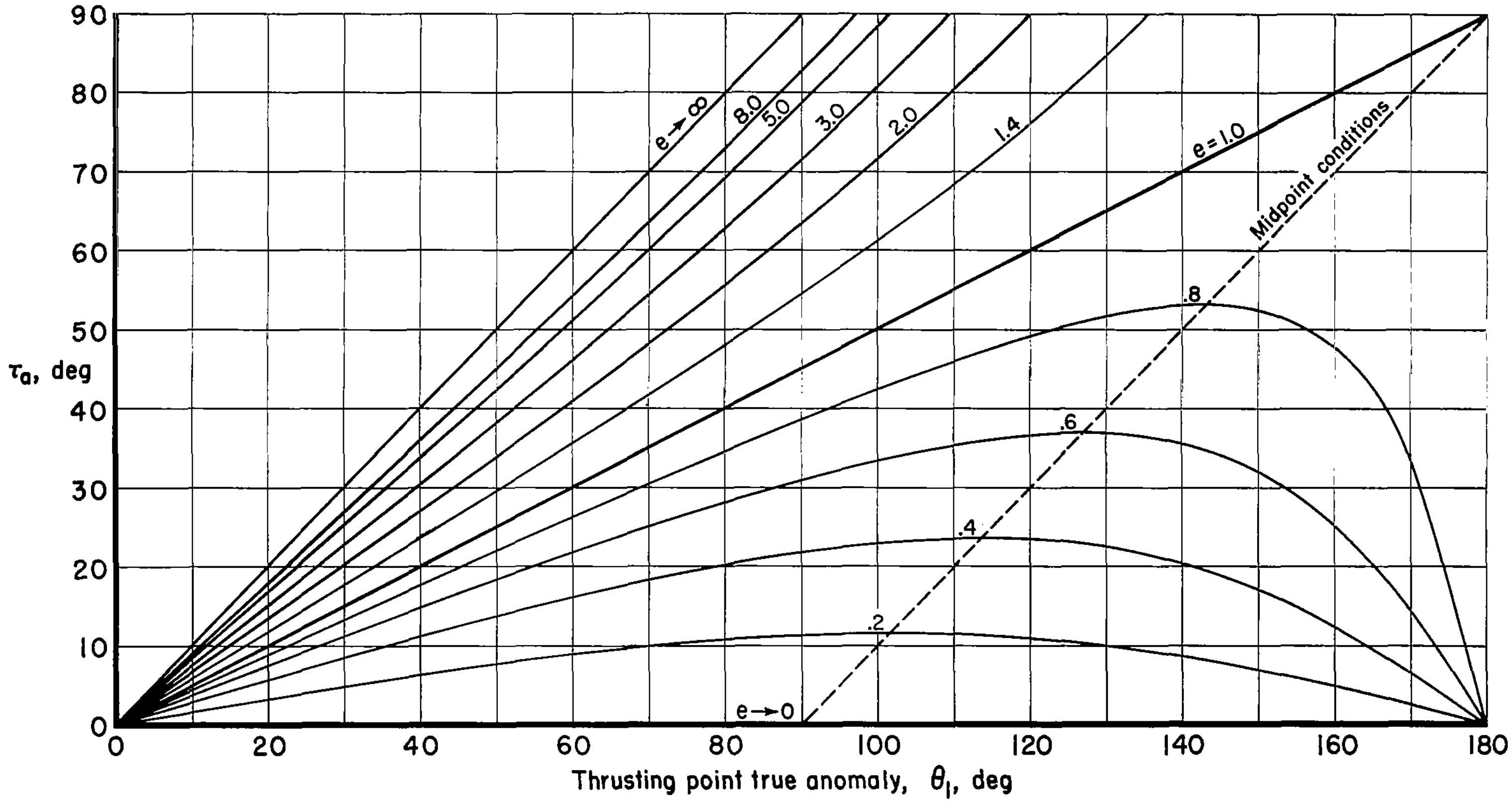
(a) Parameters of elliptic orbits.

Figure 1.- Convention of notation used in analysis.



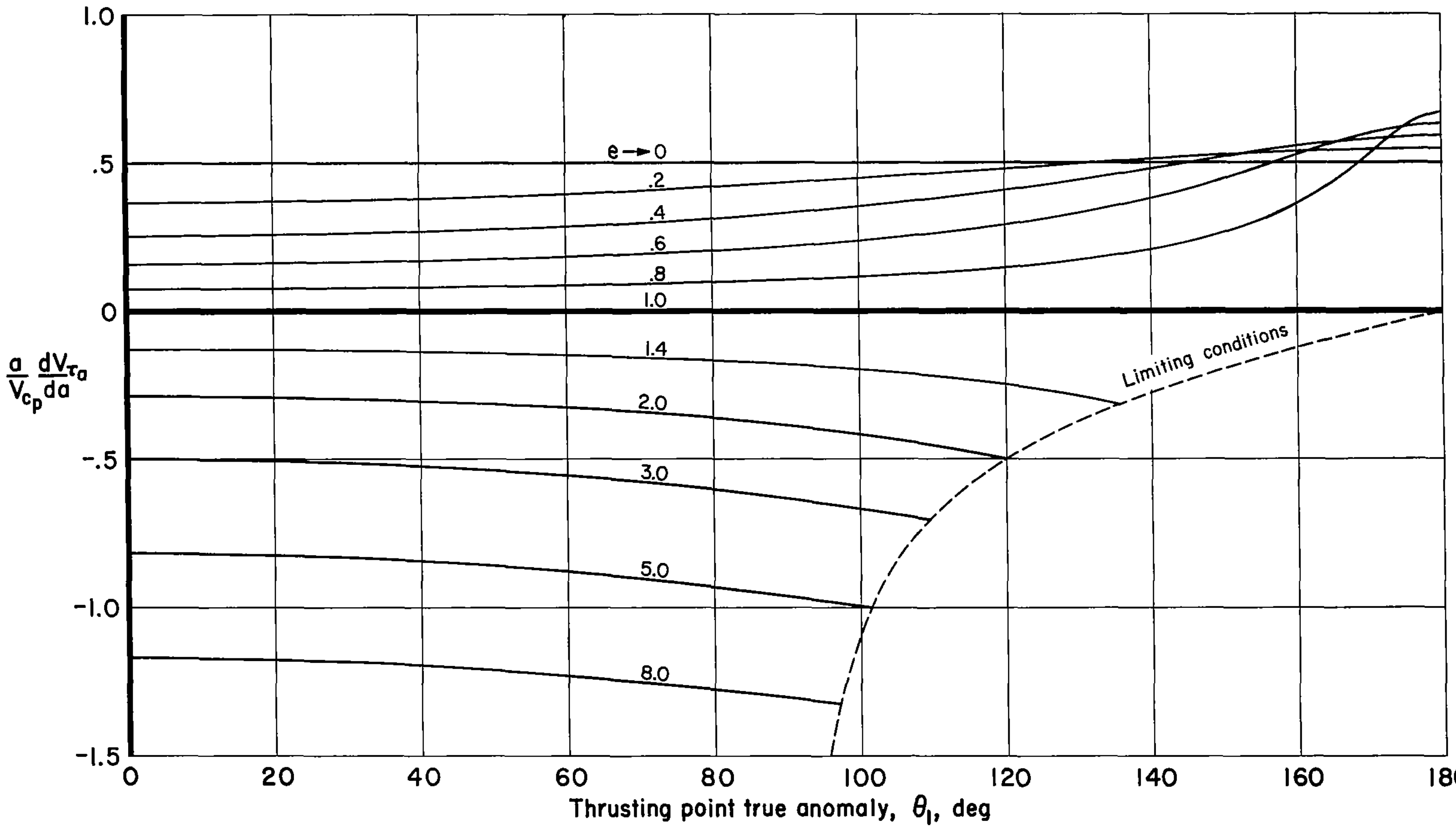
(b) Parameters of hyperbolic orbits.

Figure 1.- Concluded.



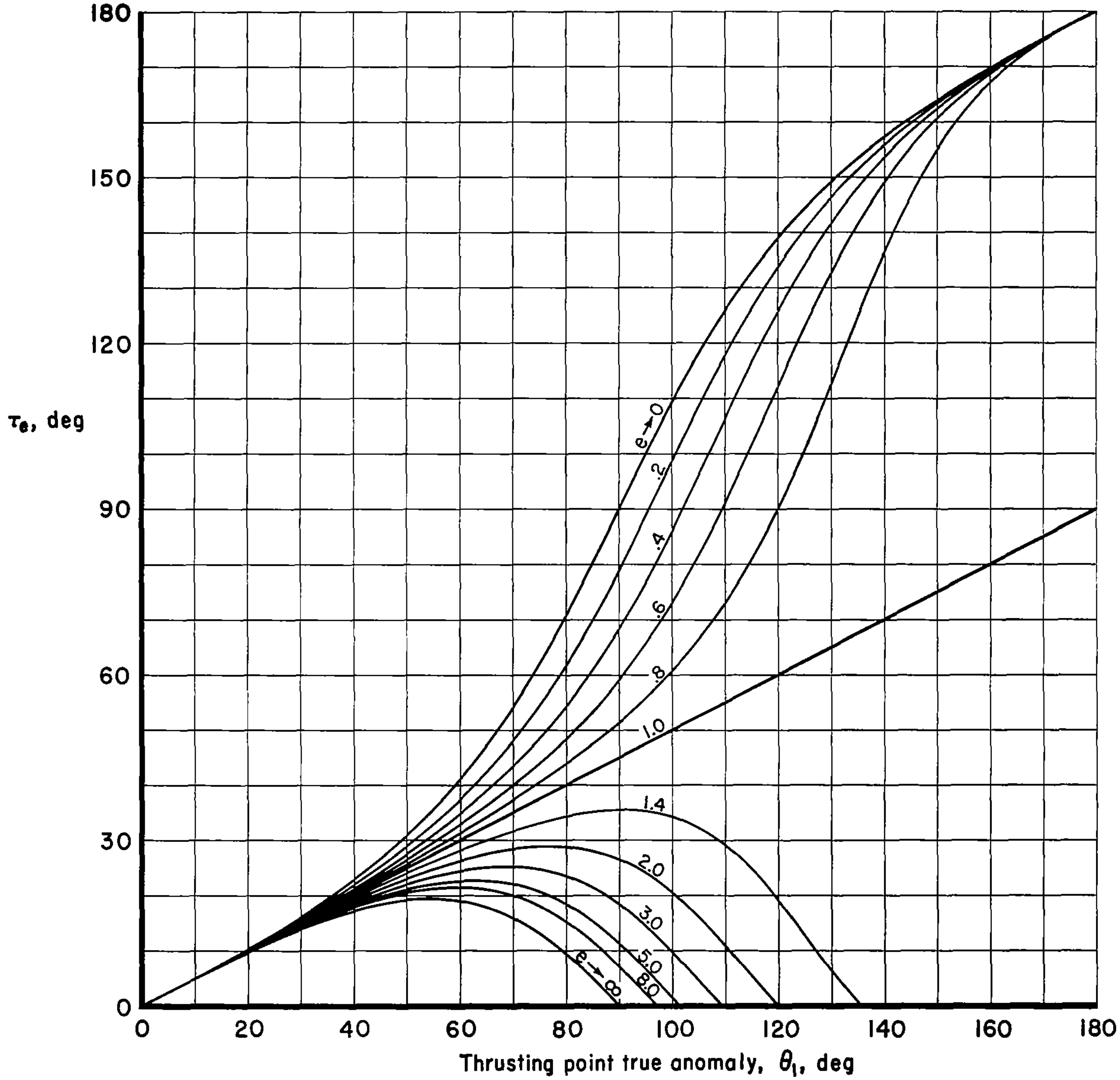
(a) Optimum thrust angle, τ_a.

Figure 2.- Minimum thrust requirements for correction of semimajor axis or period of orbit.



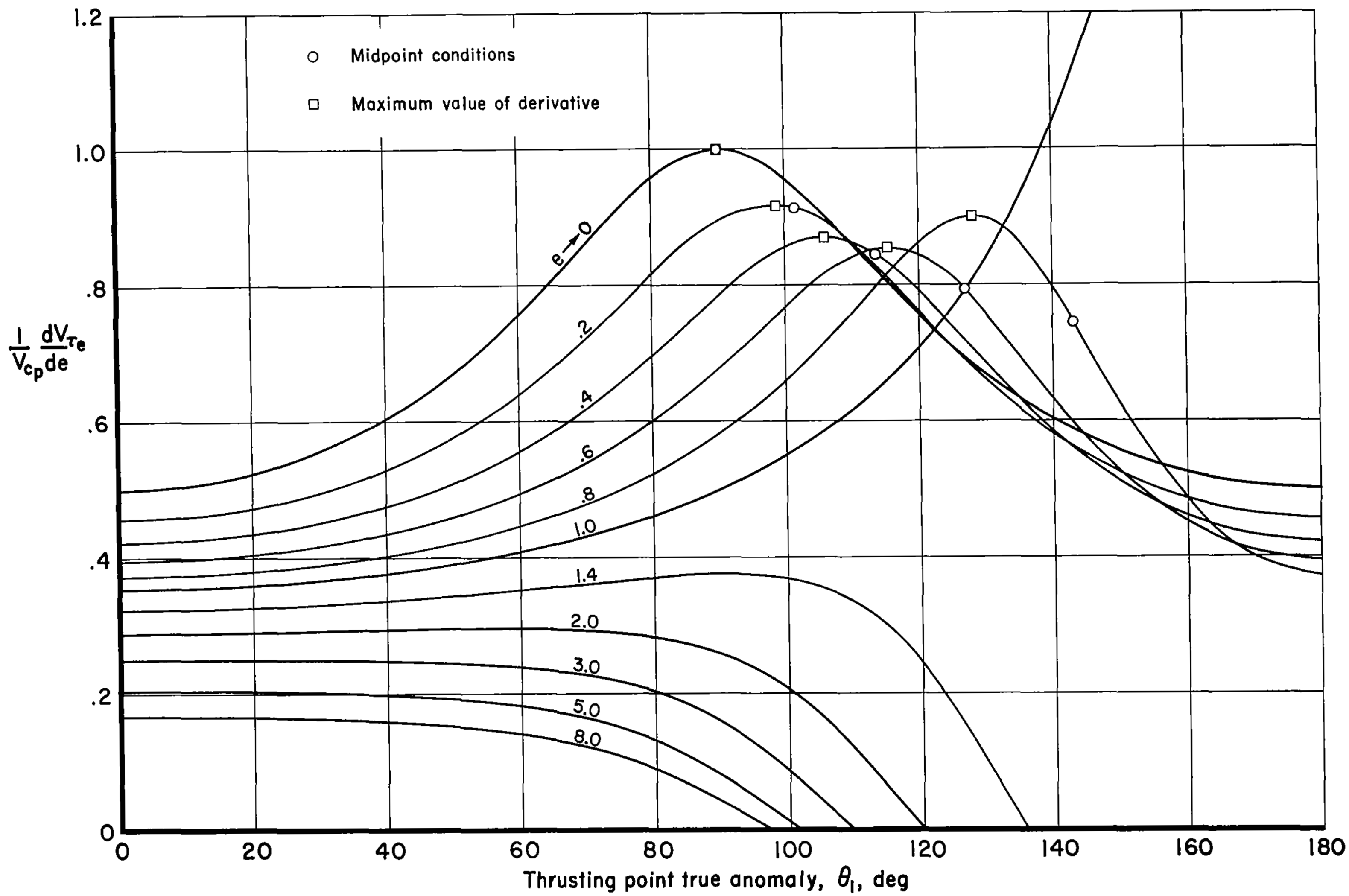
(b) Minimum derivative of thrust-produced velocity impulse with respect to semimajor axis.

Figure 2.- Concluded.



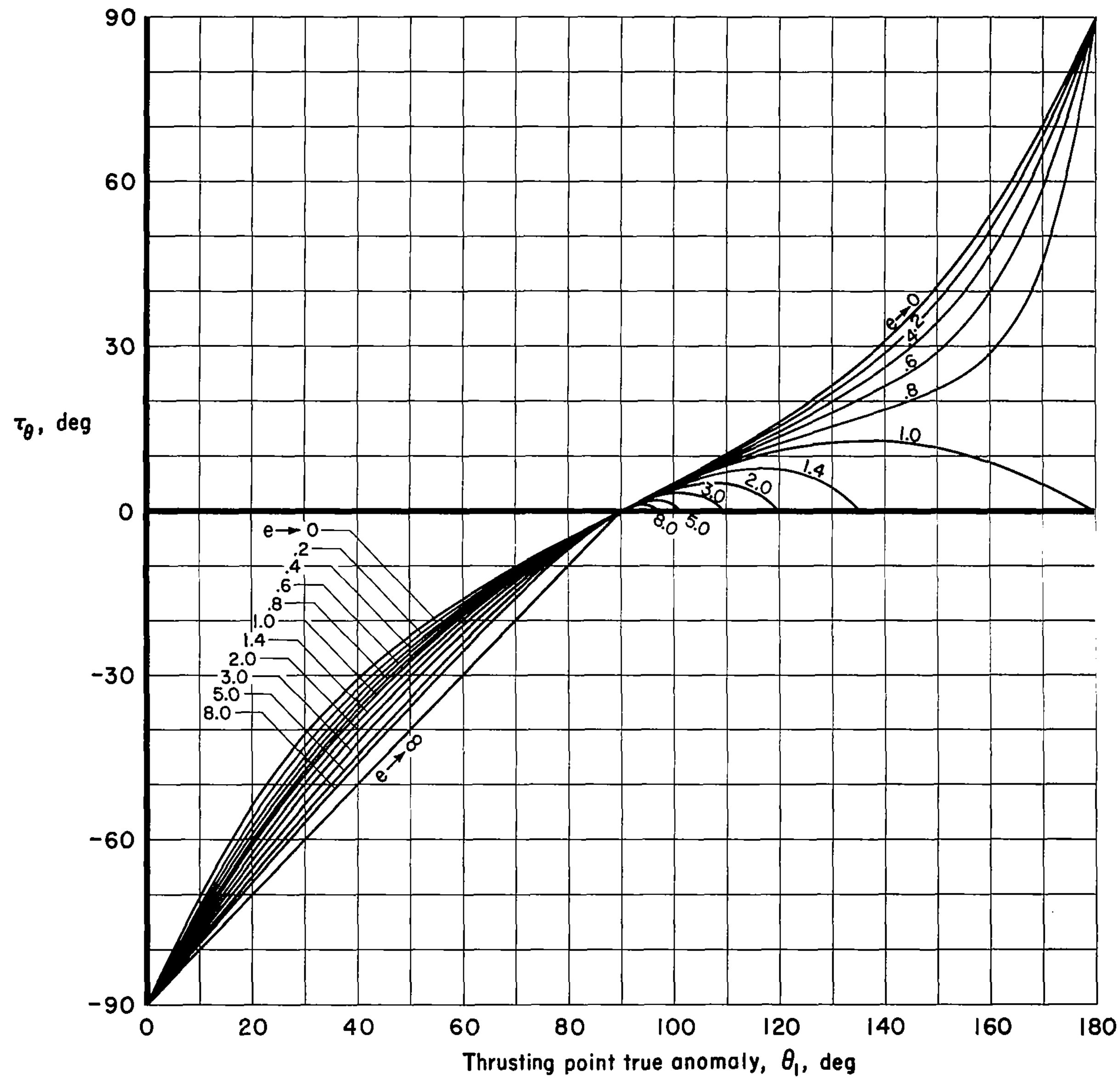
(a) Optimum thrust angle, τ_e.

Figure 3.- Minimum thrust requirements for correction of eccentricity.



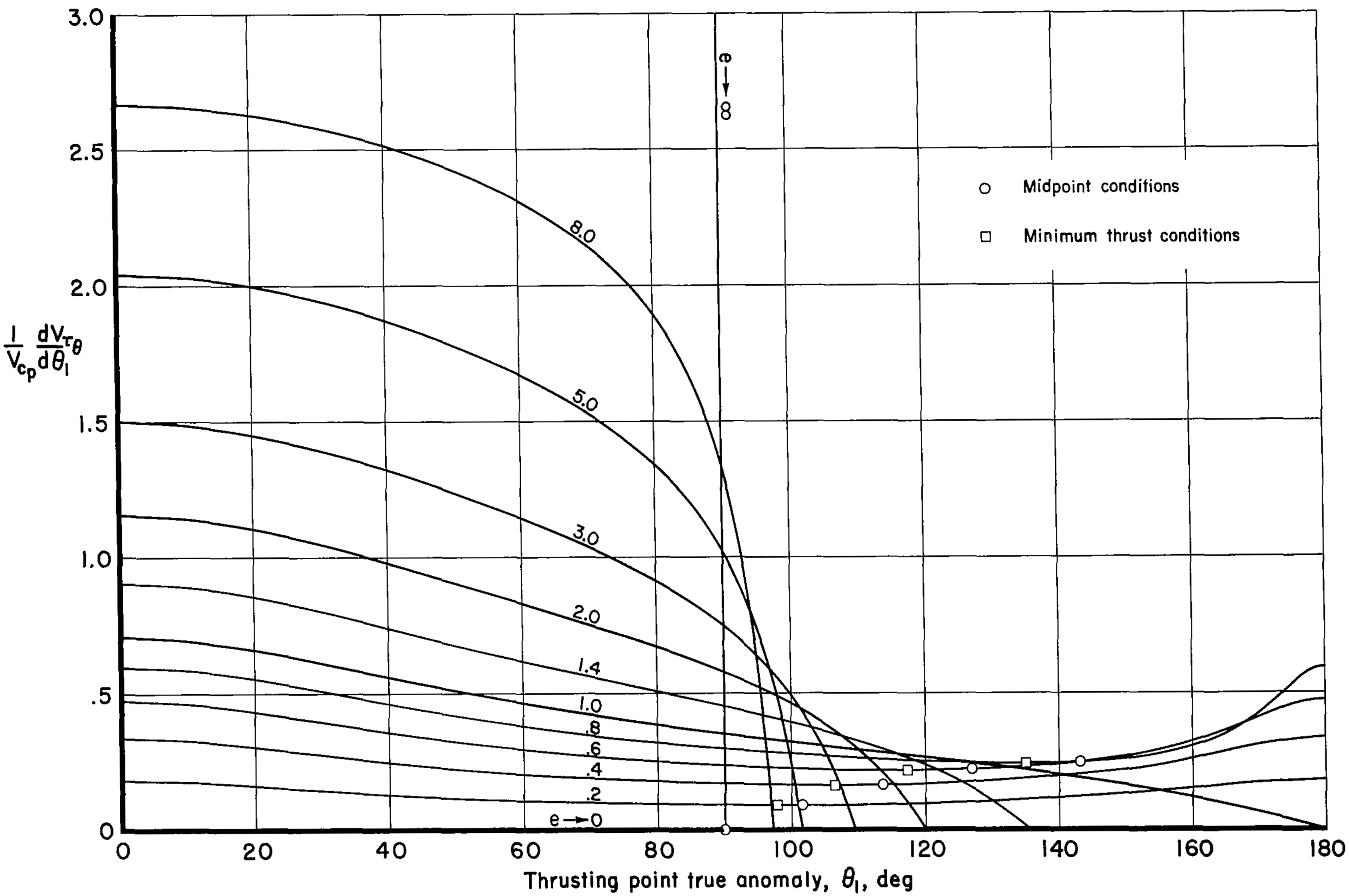
(b) Minimum derivative of thrust-produced velocity impulse with respect to eccentricity.

Figure 3.- Concluded.



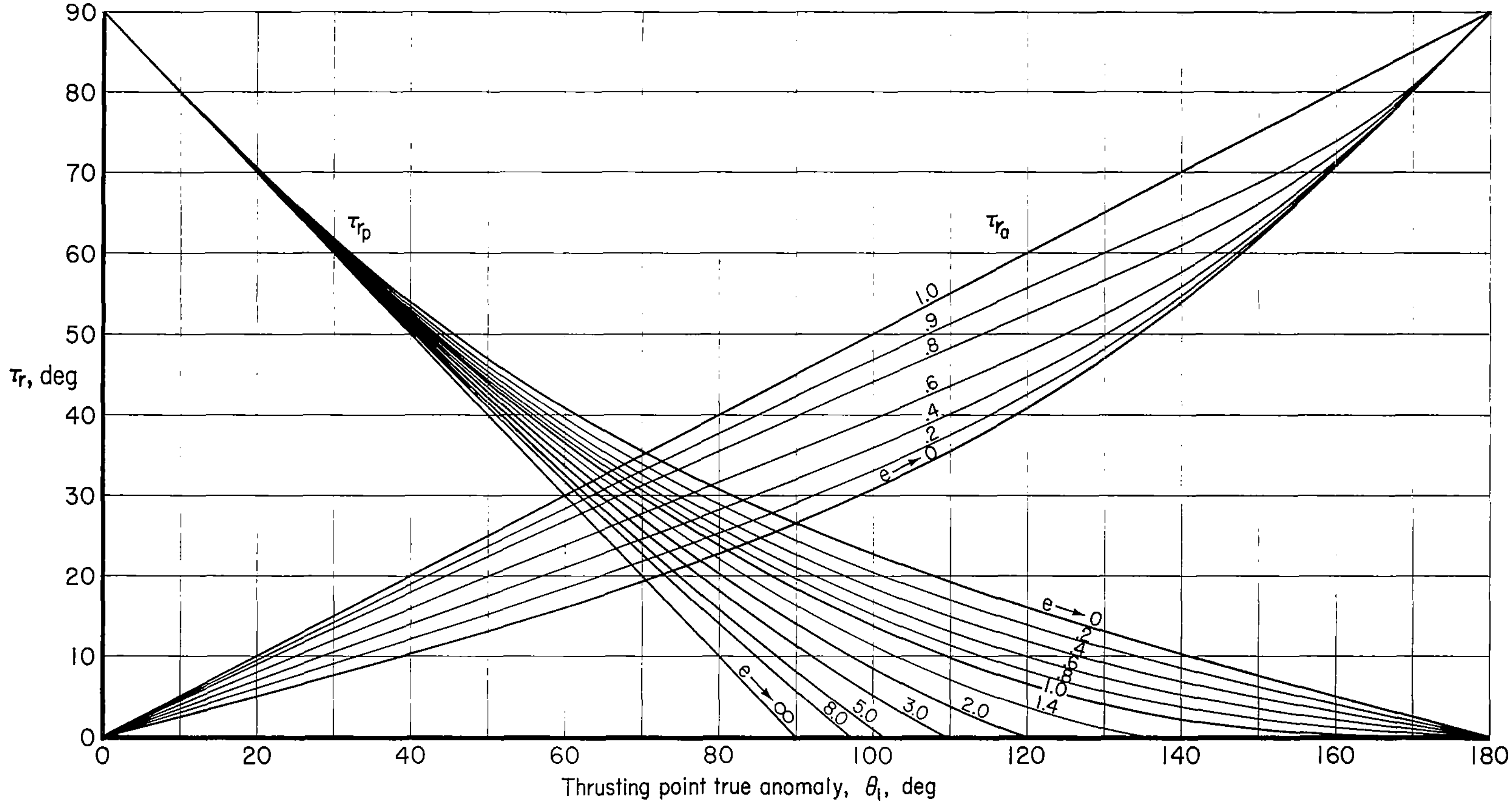
(a) Optimum thrust angle, τ_θ.

Figure 4.- Minimum thrust requirements for correction of true anomaly.



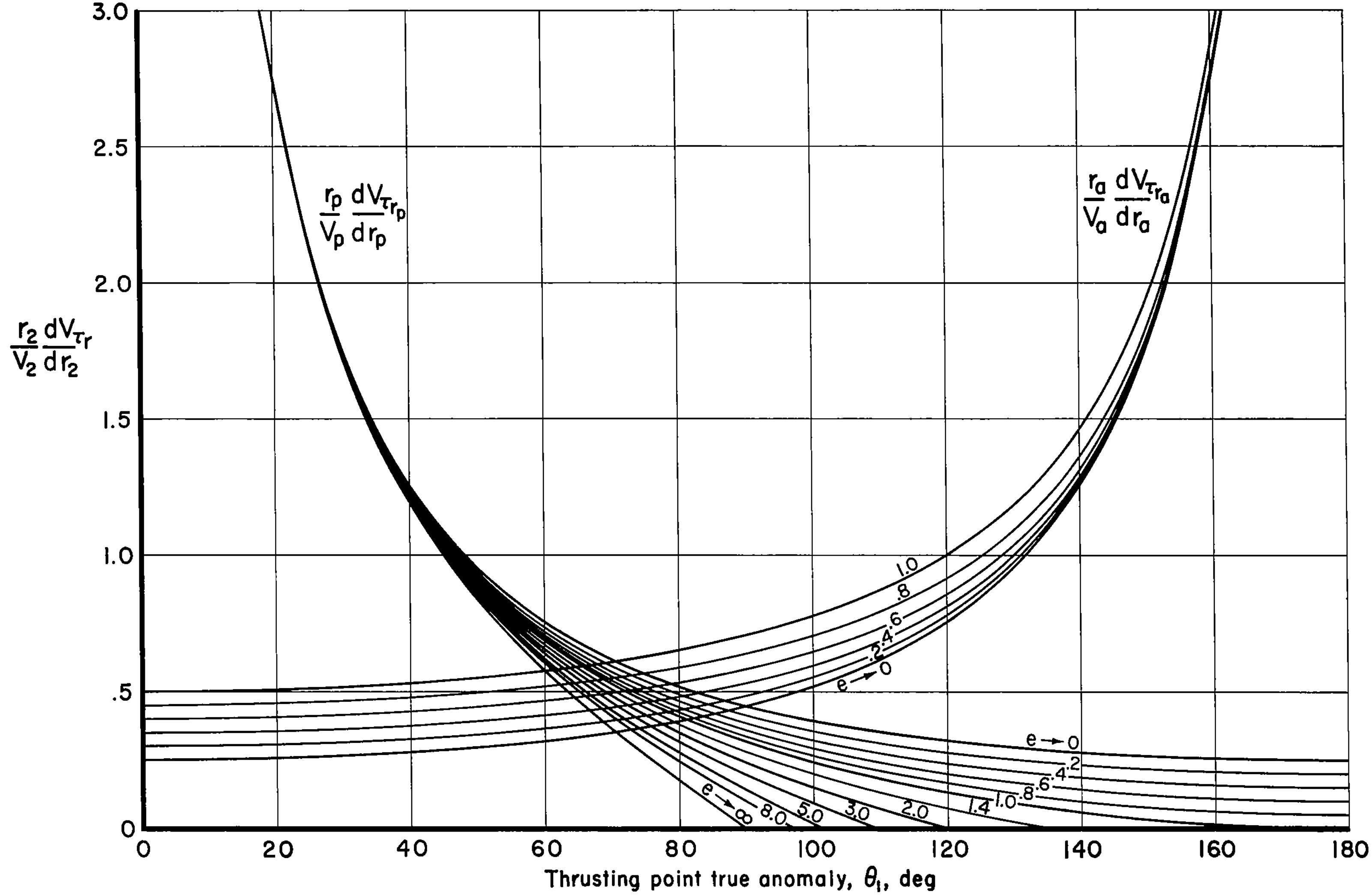
(b) Minimum derivative of thrust-produced velocity impulse with respect to true anomaly.

Figure 4.- Concluded.



(a) Optimum thrust angles, τ_{r_a} and τ_{r_p}.

Figure 5.- Minimum thrust requirements for correction of apocenter and pericenter radii.



(b) Minimum derivative of thrust-produced velocity impulse with respect to apocenter or pericenter radius.

Figure 5.- Concluded.

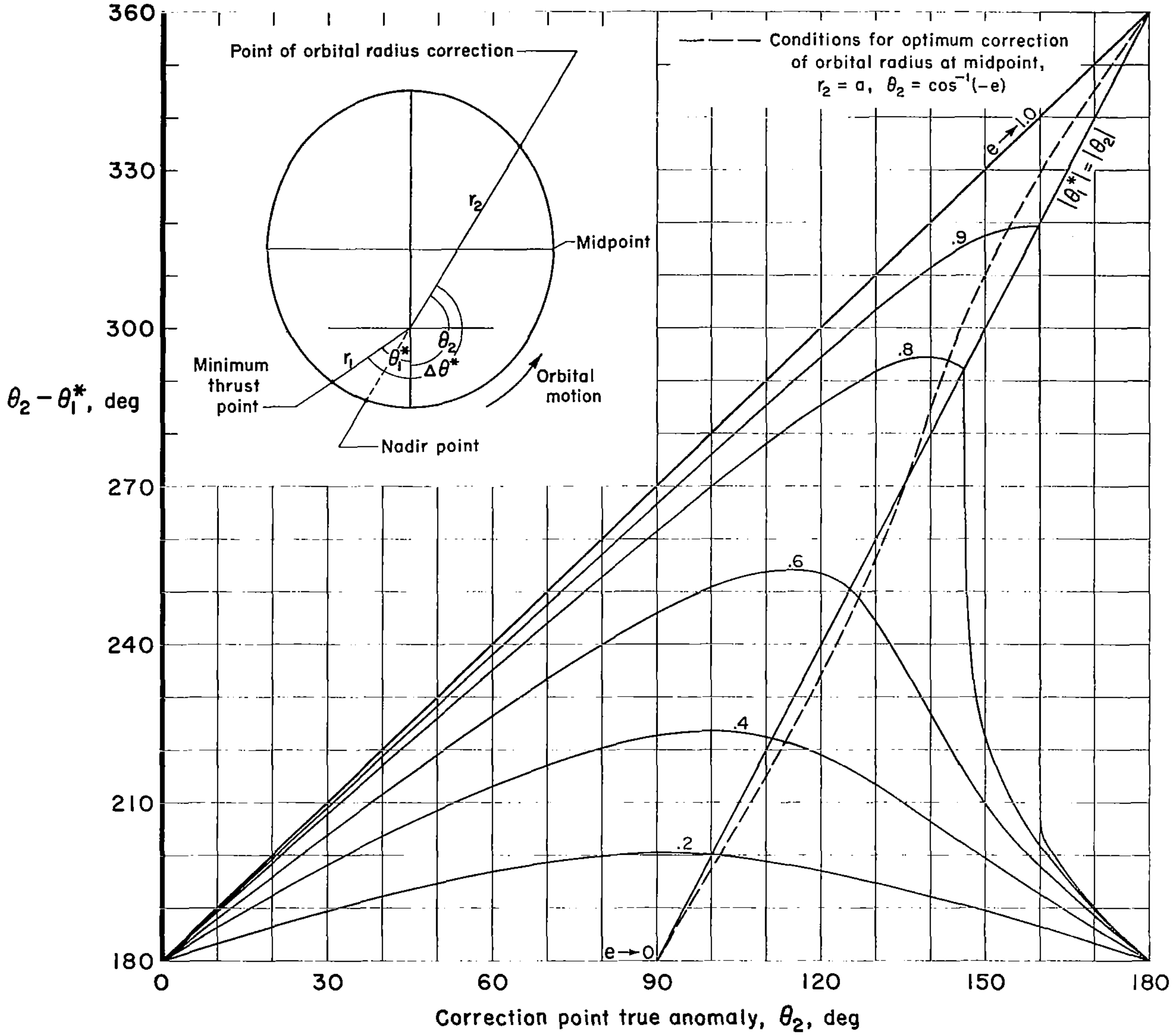
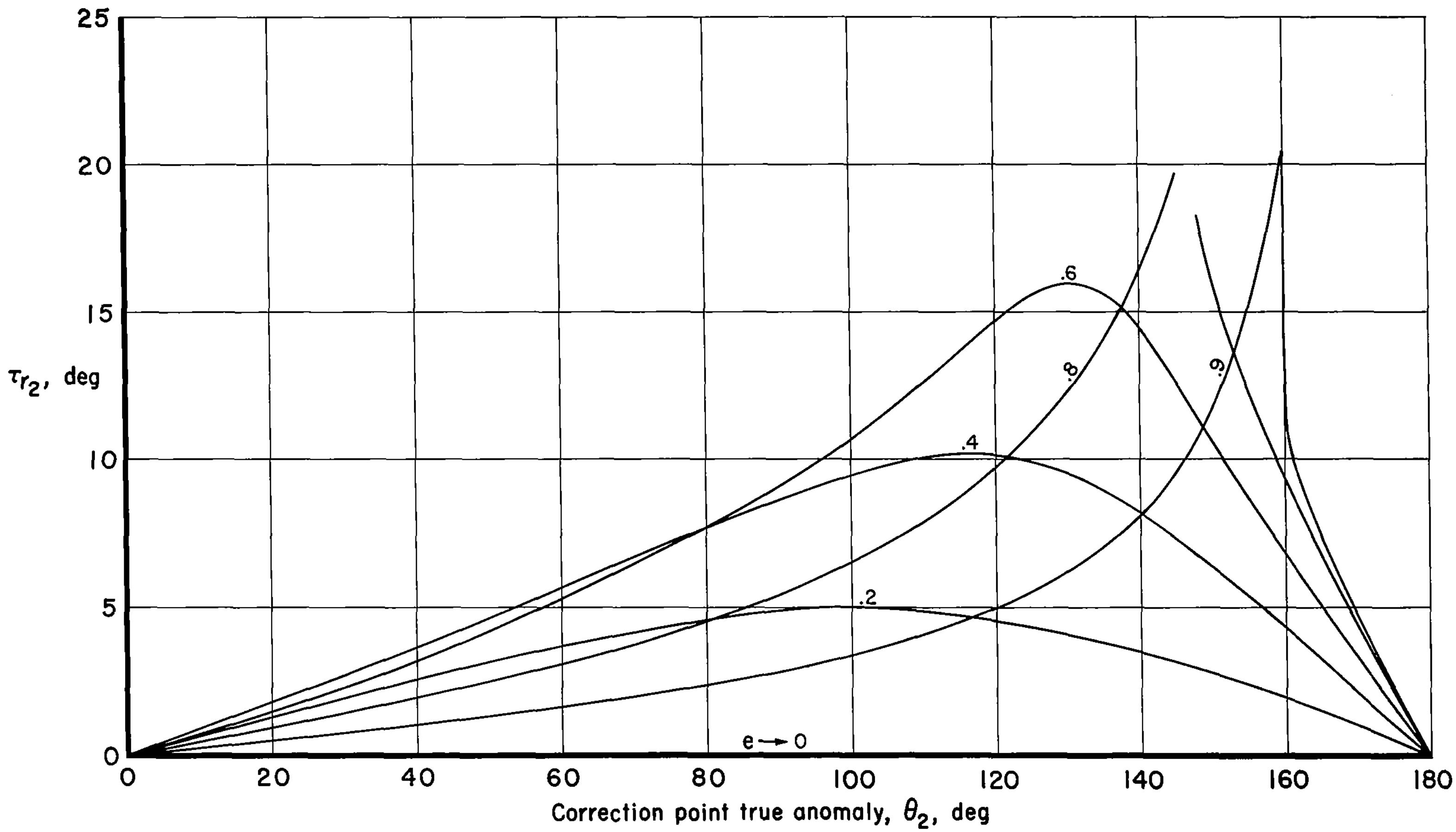
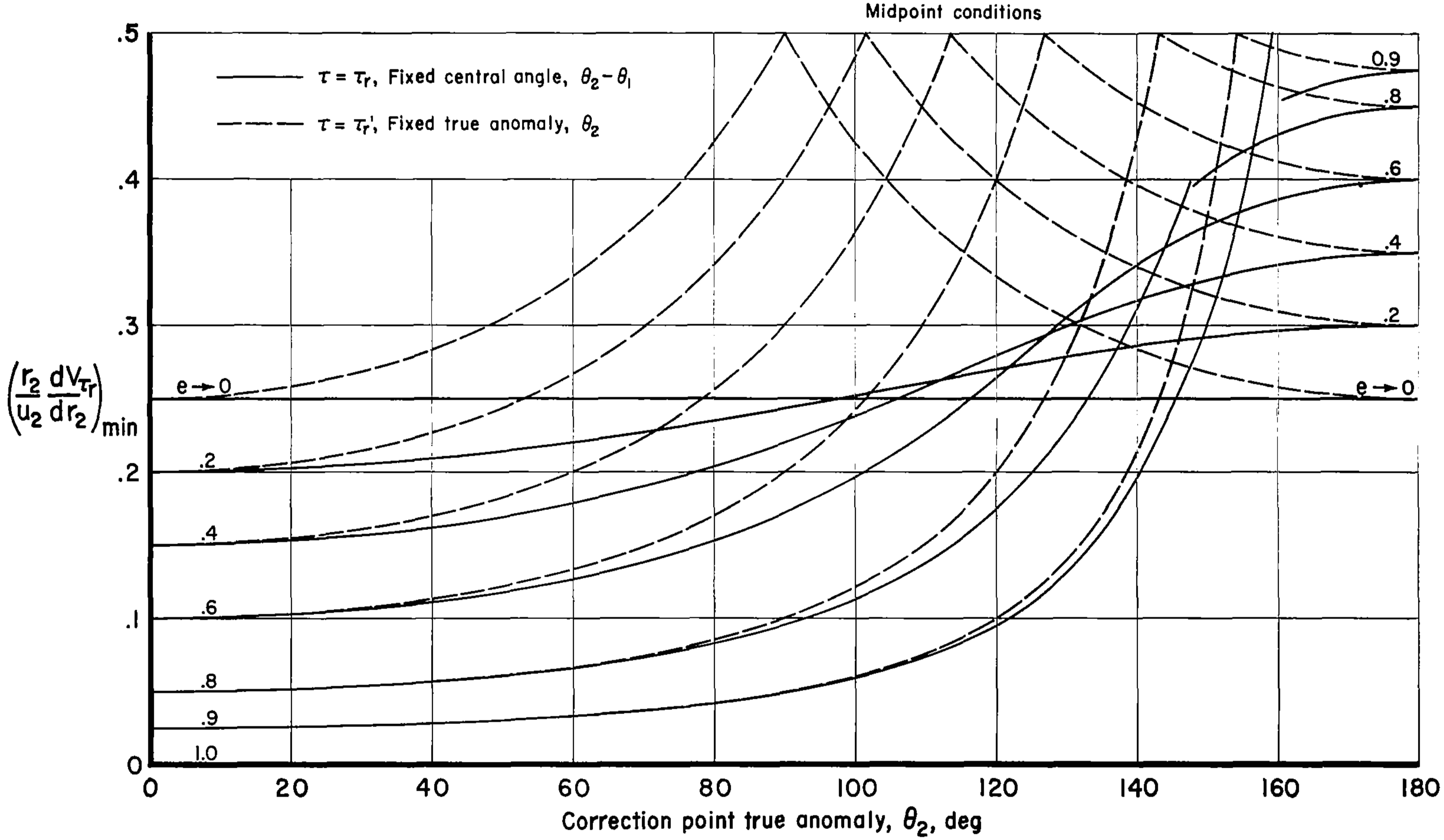


Figure 6.- Optimum location in orbit for correction of orbital radius under constraint of fixed central angle, $\theta_2 - \theta_1$.



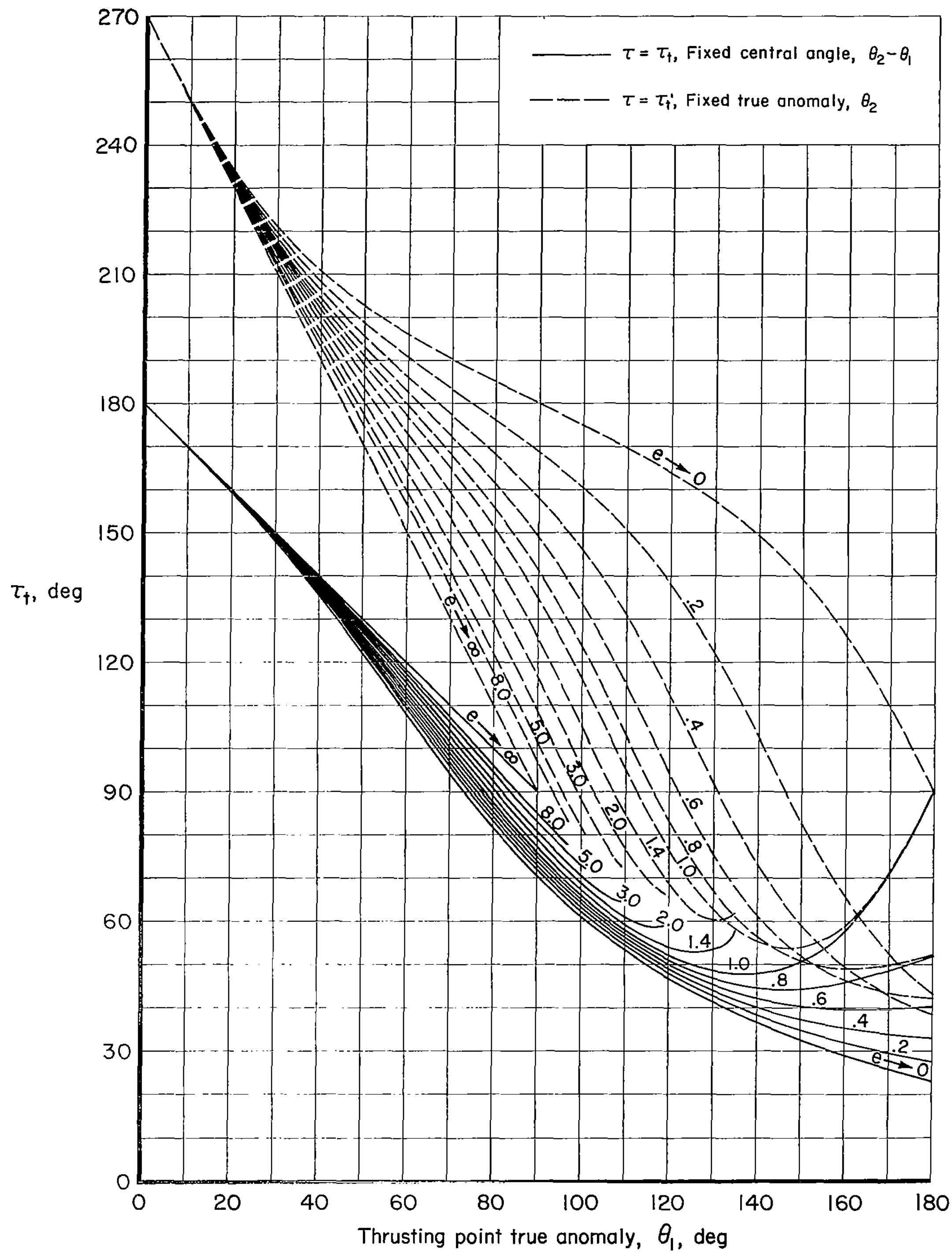
(a) Optimum thrust angle, τ_{r_2}.

Figure 7.- Absolute minimum thrust requirements for correction of orbital radius at any point along orbit.



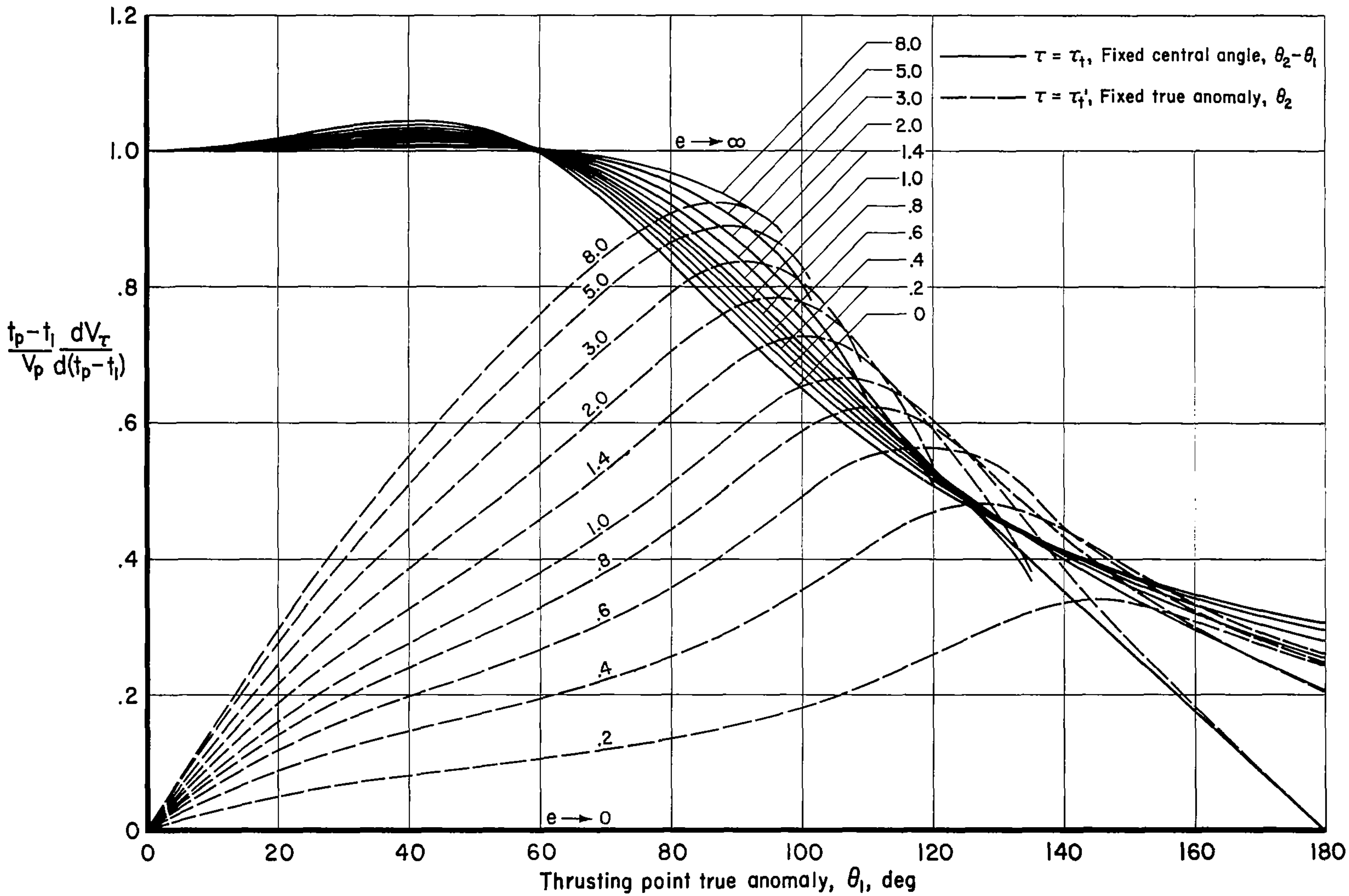
(b) Absolute minimum derivative of thrust-produced velocity impulse with respect to orbital radius at any point along orbit.

Figure 7.- Concluded.



(a) Optimum thrust angles, τ_t and τ_t'.

Figure 8.- Minimum thrust requirements for correction of elapsed flight time to pericenter.



(b) Minimum derivative of thrust-produced velocity impulse with respect to elapsed flight time to pericenter.

Figure 8.- Concluded.

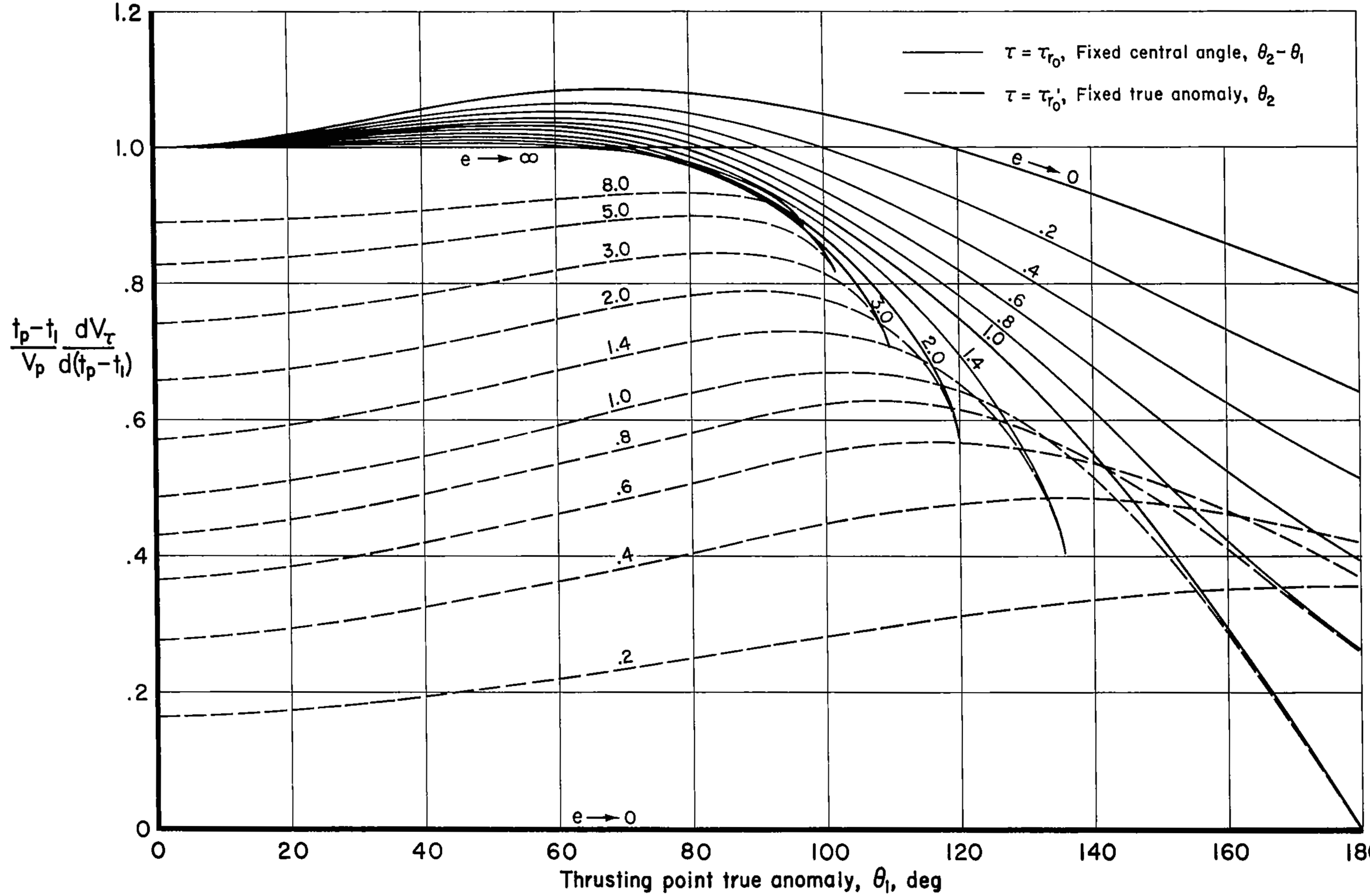


Figure 9.- Dimensionless derivative of thrust-produced velocity impulse with respect to elapsed flight time to pericenter under constraint of fixed value of pericenter radius.

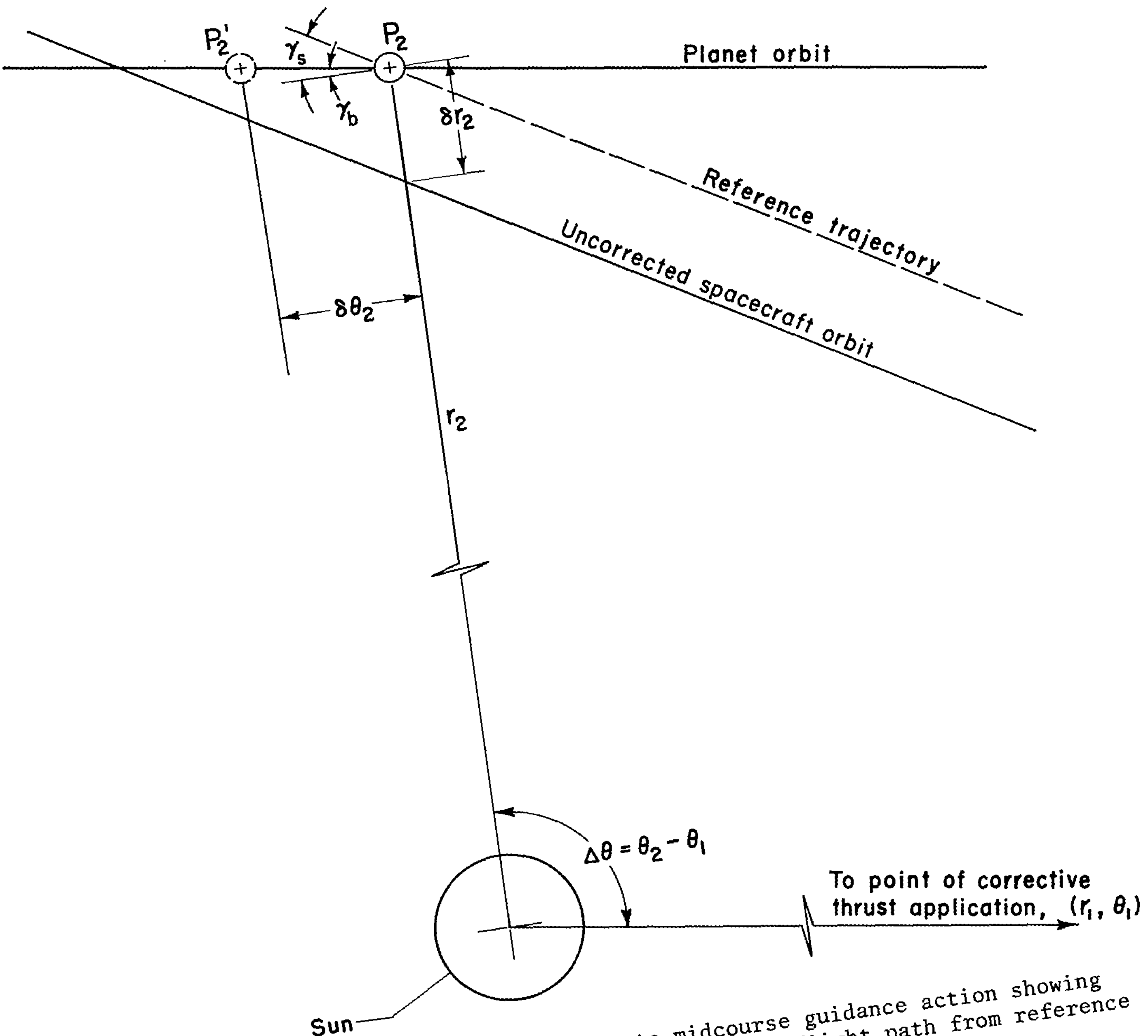


Figure 10.- General situation prior to midcourse guidance action showing in-plane deviation of uncorrected spacecraft flight path from reference trajectory.

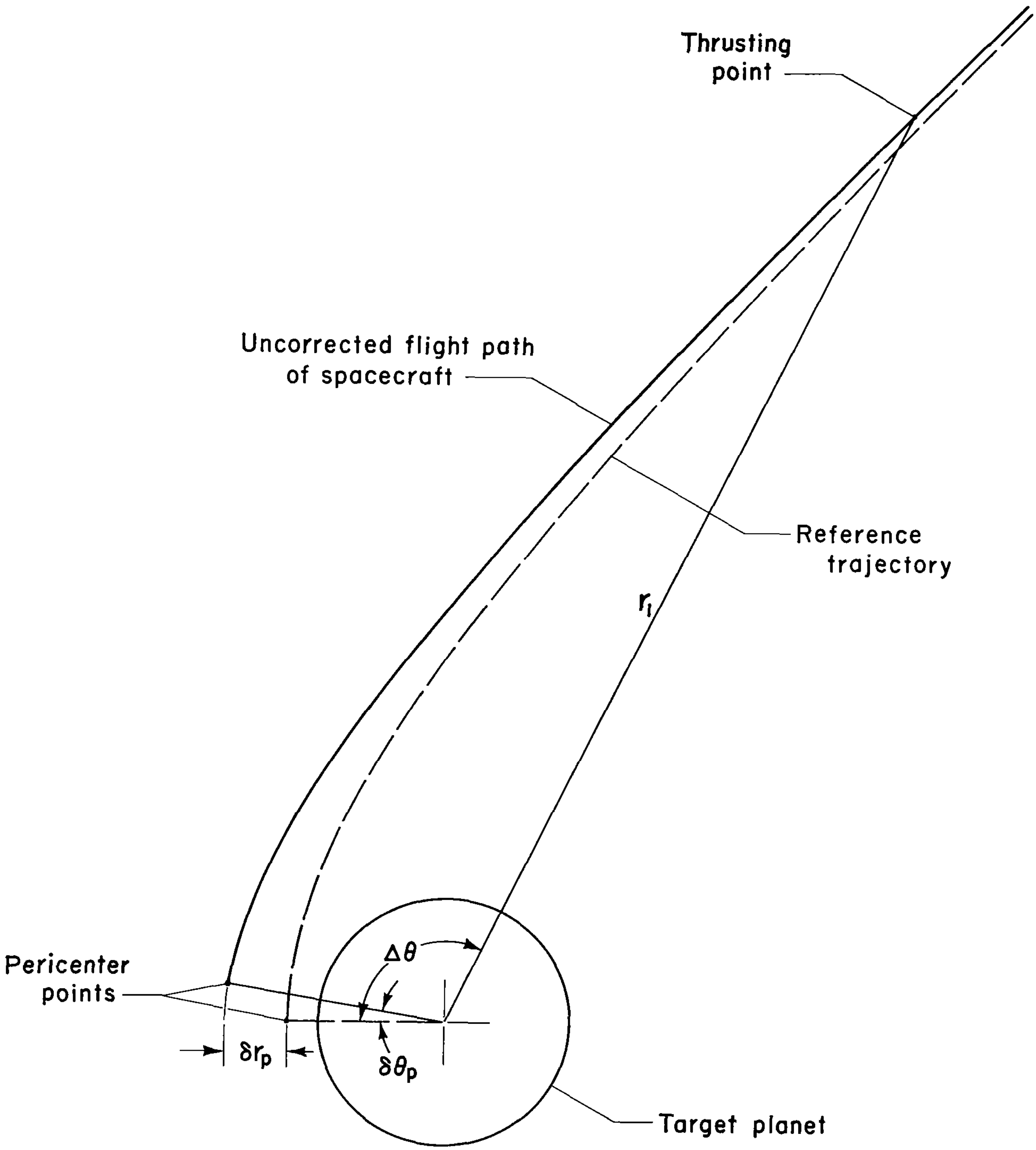


Figure 11.- General situation prior to approach guidance action showing in-plane deviation of uncorrected spacecraft flight path from reference trajectory.

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