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REGRESSION ANALYSIS PROCEDURES FOR THE EVALUATION OF TRACKING SYSTEM MEASUREMENT ERRORS

by Bobby G. Junkin

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DEFINITION OF SYMBOLS

Symbol	Definition
$\Delta R, \Delta A, \Delta E$	functional expressions for the systematic errors in range, azimuth, and elevation, respectively
$\Delta R^0, \Delta A^0, \Delta E^0$	observed tracking errors in range, azimuth, and elevation, respectively
V_R, V_A, V_E	residuals in range, azimuth, and elevation, respectively
$V_{C_0}, V_{C_1}, \ldots, V_{F_{12}}$	coefficient observational residuals
n	number of observations
TEMS	acronym for Tracking System Error Model Studies
$C_0, C_1, \ldots$	coefficients in range error model
$D_0, D_1, \ldots$	coefficients in azimuth error model
$F_0, F_1, \ldots$	coefficients in elevation error model
R^0, A^0, E^0	measured tracking parameters in range, azimuth, and elevation, respectively
R^r, A^r, E^r	reference tracking parameters in range, azimuth, and elevation, respectively
$\dot{R}, \dot{A}, \dot{E}$	first derivatives of range, azimuth, and elevation, respectively, with respect to time
$\ddot{A}, \ddot{E}$	second derivatives of azimuth and elevation, respectively, with respect to time
X_e, Y_e, Z_e	reference position of vehicle in an earth-fixed plumbline coordinate system with origin at the launch site
$X_{es} Y_{es}, Z_{es}$	reference position of vehicle in an earth-fixed plumbline coordinate system with origin at the tracking site

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
X, Y, Z	reference position of vehicle in an earth-fixed ephemeris coordinate system with origin at the tracking site
Φ_L, λ_L	geodetic latitude and geocentric longitude, respectively, of launch site
Φ_T, λ_T	geodetic latitude and geocentric longitude, respectively, of tracking site
r_L, r_T	radius of earth at launch site and tracking site, respectively
K_L	firing azimuth of vehicle
K_T	tracker azimuth
B_L, B_T	difference between geodetic and geocentric latitude at launch site and tracking site, respectively
$\hat{a}$, $\hat{b}$	semimajor and semiminor axes, respectively, of earth
ψ_L, ψ_T	geocentric latitude of launch site and tracking site, respectively
h_L, h_T	height of launch site and tracking site, respectively, above reference ellipsoid
σ_R^2, σ_A^2, σ_E^2	variances in range, azimuth, and elevation, respectively
σ_{VR}^2, σ_{VA}^2, σ_{VE}^2	least squares residual variances in range, azimuth, and elevation, respectively
σ_0^2	unit variance
$\bar{\dot{W}}$	parameter weight matrix

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
$\overline{W}$	observational weight matrix
$\tilde{C}_0, \tilde{C}_1, \ldots$	parameter approximation values
$\delta C_0, \delta C_1 \ldots$	parameter corrections
$\sigma^2_{C_0}, \sigma^2_{C_1}, \ldots$	parameter variances
$C_0^{\infty}, C_1^{\infty}, \ldots$	parameter a priori values
$\sigma_0^2 (\overline{B}^T \overline{W}\, \overline{B} + \dot{\overline{W}})^{-1}$	variance-covariance matrix of the regression parameters
Y^o	observed response variable
Y^c	computed response variable
$Z_1, Z_2, \ldots, Z_p$	independent variables
$\overline{Z}_1, \overline{Z}_2, \ldots, \overline{Z}_p$	means of the p independent variables
$b_0, b_1, b_2, \ldots, b_p$	partial regression coefficients
σ_Y	standard deviation of the response variable
$\sigma_{b_0}, \sigma_{b_1}, \ldots, \sigma_{b_p}$	standard deviations of the partial regression coefficients
$\tilde{y}$	observed response variable, transformed
y^c	computed response variable, transformed
$\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_p$	transformed independent variables
$\alpha_1, \alpha_2, \ldots, \alpha_p$	standard partial regression coefficients (referred to as standard when each independent variable in the regression equation is a deviation from the mean in units of its standard deviation)

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
r_{IJ}	linear correlation coefficient of i-th and j-th variables
$\tilde{\sigma}_Y$	standard deviation of the transformed response variable
$\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots, \sigma_{\alpha_p}$	standard deviations of the standard partial regression coefficients
d	number of independent variables in the regression equation plus 1
$R^2_{Y.12\ldots p}$	multiple correlation coefficient
$\rho_{qY.12\ldots p}$	partial correlation coefficient for the variable Z_q not in the regression equation
S_{YY}	total sum of squares about the mean
S(RES)	sum of squares about regression (residual)
S(REG)	sum of squares due to regression
F	ratio for determining the statistical significance of a regression equation
$F_{i(OUT)}$	F value used to determine if the i-th variable should be deleted from the regression equation
$F_{q(IN)}$	F value used to determine if the q-th variable should be entered into the regression equation
$\tilde{d}$	number of independent variables in the regression equation plus 2
F_{IN}	F value for entering a variable
F_{OUT}	F value for deleting a variable

REGRESSION ANALYSIS PROCEDURES FOR THE EVALUATION OF TRACKING SYSTEM MEASUREMENT ERRORS

SUMMARY

The TEMS Multiple Regression Analysis Method for evaluating systematic errors in measurements obtained from various tracking systems is presented. The mathematical procedures in the method include a rigorous least squares adjustment of error model parameters with constraints. The mathematics for using a priori values for these parameters and their variances is also included. Truncated tracker error models for representing the systematic errors are established using the TEMS method in conjunction with a stepwise regression procedure. The basic approach in the stepwise regression procedure involves examining at every step the variables incorporated in the error model in previous steps. A specific variable is deleted from or entered into the model by using the Gaussian Elimination Method for solving the linear system of normal equations in the regression.

Although C-band radar error models are used in the development, the procedures can be adapted to other types of tracking systems. Results from application to the Apollo-Saturn 501 (AS-501) flight test data are presented and indicate generally acceptable truncated C-band radar error models.

INTRODUCTION

The errors in a given measurement obtained from a tracking system can be classified as random and systematic. The random errors are random in character and thus cannot be predicted. The effects of random errors can, however, be minimized by using a sufficiently large number of observations in a least squares reduction. The combined effects of sources of significant systematic error on each quantity measured by a tracking system are frequently represented by analytical expressions referred to as error models. The problem of evaluating these systematic errors represented by error models is of importance in determining an accurate flight trajectory from the basic tracking measurements.

A method for accomplishing this evaluation is provided in TEMS, an acronym for Tracking System Error Model Studies. The overall objectives of the studies are twofold:

(1) To evaluate systematic errors in the tracking system measurements used in determining a postflight trajectory.

(2) To conduct analyses to establish truncated tracker error models to represent the systematic errors.

Basically, the TEMS Multiple Regression Analysis Method [1, 2, 3] involves establishing the tracker errors and then determining, in the least squares sense, error model expressions to describe these established errors. Results from application of the method to Apollo-Saturn flight test data are presented in Reference 4. In these references, truncating the total error models was required because of highly correlated coefficients that made an insignificant error contribution. The approach to constructing truncated error models using the TEMS results has been based on the significance of an individual variable and its correlation with other variables. This approach actually constitutes a qualitative examination of a subset of regressions from the all possible regressions approach.

There are other model building approaches. These include: backward elimination procedure, forward selection procedure, stepwise regression procedure, and stagewise regression procedure. These are not unique, however, so far as selecting which of several independent variables should be used in a regression equation to describe a response variable. The different procedures mentioned above do not necessarily give the same results when applied to the same problem. After a careful study of all the procedures, it was concluded that the stepwise regression procedure had the greatest potential for useful application to the TEMS error model construction problem. Other applications of the stepwise procedure can be found in References 5 and 6. Additional information on the other procedures is also contained in Reference 5.

The basic approach in the Stepwise Regression Analysis involves examining at every step the variables incorporated into the regression model in previous steps. At a given step in the analysis, a specific variable is deleted from or entered into the regression model by using the Gaussian Elimination Method for solving the linear system of normal equations. This method is ideally suited for application of the stepwise procedures since it obtains elements of the solution vector one at a time. The final regression model results in only the most significant variables being retained in the model.

Application of the TEMS method and the Stepwise Regression Analysis to C-band radar tracking systems operating on the Apollo-Saturn V AS-501 flight is presented later in the report. A summary of the C-band radar tracking data utilization on the first burn flight phase (launch to parking orbit insertion) and the S-IVB second burn flight phase (S-IVB reignition to S-IVB/CSM separation) is included. The approach given by the guidelines in this report for obtaining truncated error models to describe the systematic errors has generally resulted in acceptable models for the AS-501 first and second burn data. Several cases are noted on the stepwise results for the AS-501 data where the introduction of additional variables into the regression does not improve significantly the σ_Y curve fit value. It appears that a rather critical examination of results from application of the stepwise procedures is required in order to obtain meaningful and useful information for input to the TEMS program.

ACKNOWLEDGMENTS

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THE TEMS MULTIPLE REGRESSION ANALYSIS METHOD

Introduction

The development of mathematical procedures in the TEMS Multiple Regression Analysis Method is presented in this section. The development provides for a comprehensive evaluation of systematic errors in measurements obtained from various radar tracking systems. Detailed mathematics for a rigorous least squares adjustment of the radar error model parameters (coefficients) is presented. Included are provisions for the use of a priori information about the coefficients of the error model expressions. Coefficient variances can also be constrained to be consistent with a priori values. In addition, a priori estimates for the coefficients can be entered into the adjustment. Functional relations between the coefficients are also considered in the development. An important by-product of the adjustment is the variance-covariance matrix of the estimated error model coefficients.

The C-band radar tracking system error models used in the development are discussed in Appendix A. It should be noted, however, that the development of the method for application to tracking systems other than radars is analogous to that in this section. A similar development for the AZUSA (Glotrac Station I) tracking system is contained in Reference 3.

Observational Equations for the Least Squares Adjustment with Parameter Constraints

The basic tracking data which serve as input consists of the radar measured tracking parameters (R^0, A^0, E^0) and of a reference trajectory representing the best estimate of the trajectory from a composite of data. These composite data are from various tracking systems such as AZUSA, Radar, ODOP, Glotrac, and fixed cameras. The reference trajectory is in an earth-fixed plumbline coordinate system (X_e, Y_e, Z_e) with origin at the launch site. These data are transformed into radar reference tracking parameters by the two transformations:

(a) $(X_e, Y_e, Z_e) \rightarrow (X_{es}, Y_{es}, Z_{es})$

(b) $(X_{es}, Y_{es}, Z_{es}) \rightarrow (R^r, A^r, E^r)$

Detailed mathematics involved in these transformations are given in Appendix B. The tracking errors for the particular system under consideration and for the i-th observation are thus determined from the equations:

$$\left.\begin{aligned} \Delta R_i^0 &= R_i^r - R_i^0 \\ \Delta A_i^0 &= A_i^r - A_i^0 \\ \Delta E_i^0 &= E_i^r - E_i^0 \end{aligned}\right\} \qquad (1)$$

An additional transformation of the form:

(c) $(X_{es}, Y_{es}, Z_{es}) \rightarrow (X, Y, Z)$

is required to determine the position of the vehicle in an earth-fixed ephemeris coordinate system with origin at the tracking site. This transformation is also given in Appendix B.

The fundamental observational equations are expressed in the following form $(i = 1, 2, \ldots, n)$:

$$\left.\begin{aligned} \Delta R_i^0 - \Delta R_i - V_{Ri} &= 0 \\ \Delta A_i^0 - \Delta A_i - V_{Ai} &= 0 \\ \Delta E_i^0 - \Delta E_i - V_{Ei} &= 0 \end{aligned}\right\} \qquad (2)$$

The functional expressions for the systematic errors in equation (2) are given by $(i = 1, 2, \ldots, n)$:

$$
\left.
\begin{aligned}
\Delta R_i &= C_0 + C_1 R_i + C_2 \dot{R}_i + C_4(-.022 \operatorname{cosec} E_i) + C_5 (X_i/R_i) \\
&\quad + C_6 (Y_i/R_i) + C_7 (Z_i/R_i) + C_8 t_i \\
\Delta A_i &= D_0 + D_1 \dot{A}_i + D_3 \ddot{A}_i + D_5 \tan E_i + D_6 \sec E_i + D_7 \tan E_i \sin A_i \\
&\quad + D_8 \tan E_i \cos A_i + D_9 [(\sin A_i \cos A_i)/X_i] \\
&\quad + D_{10} [(-\sin A_i \cos A_i)/Y_i] + D_{11} \dot{A}_i \sec E_i \\
\Delta E_i &= F_0 + F_1 \dot{E}_i + F_3 \ddot{E} + F_5(-\sin A_i) + F_6 \cos A_i \\
&\quad + F_7 \left[\left(\frac{.022}{R_i \sin E_i} - 10^{-6} \right) \operatorname{cotan} E_i \right] + F_9[(-X_i \tan E_i)/R_i^2] \\
&\quad + F_{10}[(-Y_i \tan E_i)/R_i^2] + F_{11}[(\cos E_i)/R_i] + F_{12} \dot{E}_i \cos E_i
\end{aligned}
\right\} \quad (3)
$$

The following constraints in the form of functional relations between the coefficients are imposed upon equations (3):

$$
\left.
\begin{aligned}
C_2 &= D_1 = F_1 \\
C_4 &= F_7 \\
C_5 &= D_9 = F_9 \\
C_6 &= D_{10} = F_{10} \\
C_7 &= F_{11} \\
D_8 &= F_5 \\
D_7 &= F_6
\end{aligned}
\right\} \quad (4)
$$

Equations (3) are thus rewritten as $(i = 1, 2, \ldots, n)$

$$\left.\begin{aligned}
\Delta R_i &= C_0 + C_1\tilde{r}_{1i} + C_2\tilde{r}_{2i} + C_4\tilde{r}_{3i} + C_5\tilde{r}_{4i} + C_6\tilde{r}_{5i} + C_7\tilde{r}_{6i} + C_8\tilde{r}_{7i} \\
\Delta A_i &= D_0 + C_2 a_{1i} + D_3 a_{2i} + D_5 a_{3i} + D_6 a_{4i} + D_7 a_{5i} + D_8 a_{6i} \\
&\quad + C_5 a_{7i} + C_6 a_{8i} + D_{11} a_{9i} \\
\Delta E_i &= F_0 + C_2 e_{1i} + F_3 e_{2i} + D_8 e_{3i} + D_7 e_{4i} + C_4 e_{5i} + C_5 e_{6i} \\
&\quad + C_6 e_{7i} + C_7 e_{8i} + F_{12} e_{9i}
\end{aligned}\right\} \qquad (5)$$

where

$$\tilde{r}_{1i} = R_i \qquad \tilde{r}_{4i} = X_i/R_i$$

$$\tilde{r}_{2i} = \dot{R}_i \qquad \tilde{r}_{5i} = Y_i/R_i$$

$$\tilde{r}_{3i} = -.022 \operatorname{cosec} E_i \qquad \tilde{r}_{6i} = Z_i/R_i$$

$$\tilde{r}_{7i} = t_i$$

$$a_{1i} = \dot{A}_i \qquad e_{1i} = \dot{E}_i$$

$$a_{2i} = \ddot{A}_i \qquad e_{2i} = \ddot{E}_i$$

$$a_{3i} = \tan E_i \qquad e_{3i} = -\sin A_i$$

$$a_{4i} = \sec E_i \qquad e_{4i} = \cos A_i$$

$$a_{5i} = \tan E_i \sin A_i \qquad e_{5i} = \left[\frac{.022}{R_i \sin E_i} - 10^{-6}\right] \cot E_i$$

$$a_{6i} = \tan E_i \cos A_i \qquad e_{6i} = (-X_i \tan E_i)/R_i^2$$

$$a_{7i} = (\sin A_i \cos A_i)/X_i \qquad e_{7i} = (-Y_i \tan E_i)/R_i^2$$

$$a_{8i} = (-\sin A_i \cos A_i)/Y_i \qquad e_{8i} = (\cos E_i)/R_i$$

$$a_{9i} = \dot{A}_i \sec E_i \qquad e_{9i} = \dot{E}_i \cos E_i$$

The coefficients in equation (5) are written as:

$$\left.\begin{aligned} C_0 &= \tilde{C}_0 + \delta C_0 \\ C_1 &= \tilde{C}_1 + \delta C_1 \\ &\cdot \\ &\cdot \\ &\cdot \\ F_{12} &= \tilde{F}_{12} + \delta F_{12} \end{aligned}\right\} \qquad (6)$$

where $\tilde{C}_0$, $\tilde{C}_1$, ..., $\tilde{F}_{12}$ are approximations (which can be chosen equal to zero if desired) to C_0, C_1, . . . , F_{12}, respectively; and δC_0, δC_1, . . . , δF_{12} are unknown corrections which will be determined in the least squares sense. The systematic errors can then be written as ($i = 1, 2, \ldots, n$):

$$\left.\begin{aligned} \Delta R_i &= \widetilde{\Delta R}_i + \delta \Delta R_i \\ \Delta A_i &= \widetilde{\Delta A}_i + \delta \Delta A_i \\ \Delta E_i &= \widetilde{\Delta E}_i + \delta \Delta E_i \end{aligned}\right\} \qquad (7)$$

where

$$\left.\begin{aligned} \widetilde{\Delta R}_i &= \tilde{C}_0 + \tilde{C}_1 \tilde{r}_{1i} + \tilde{C}_2 \tilde{r}_{2i} + \ldots + \tilde{C}_8 \tilde{r}_{7i} \\ \widetilde{\Delta A}_i &= \tilde{D}_0 + \tilde{C}_2 a_{1i} + \tilde{D}_3 a_{2i} + \ldots + \tilde{D}_{11} a_{9i} \\ \widetilde{\Delta E}_i &= \tilde{F}_0 + \tilde{C}_2 e_{1i} + \tilde{F}_3 e_{2i} + \ldots + \tilde{F}_{12} e_{9i} \end{aligned}\right\} \qquad (8)$$

and

$$\left.\begin{aligned}
\delta\Delta R_i &= \delta C_0 + \delta C_1\tilde{r}_{1i} + \delta C_2\tilde{r}_{2i} + \ldots + \delta C_8\tilde{r}_{7i} \\
\delta\Delta A_i &= \delta D_0 + \delta C_2 a_{1i} + \delta D_3 a_{2i} + \ldots + \delta D_{11} a_{9i} \\
\delta\Delta E_i &= \delta F_0 + \delta C_2 e_{1i} + \delta F_3 e_{2i} + \ldots + \delta F_{12} e_{9i}
\end{aligned}\right\} \tag{9}$$

Substituting equations (8) and (9) in equations (7) and the result in equations (2) yields the following observational equations (1 = 1, 2, . . . , n):

$$\left.\begin{aligned}
&\Delta R_i^0 - [(\tilde{C}_0 + \tilde{C}_1\tilde{r}_{1i} + \ldots + \tilde{C}_8\tilde{r}_{7i}) \\
&\qquad + (\delta C_0 + \delta C_1\tilde{r}_{1i} + \ldots + \delta C_8\tilde{r}_{7i})] - V_{Ri} = 0 \\
&\Delta A_i^0 - [(\tilde{D}_0 + \tilde{C}_2 a_{1i} + \ldots + \tilde{D}_{11} a_{9i}) \\
&\qquad + (\delta D_0 + \delta C_2 a_{1i} + \ldots + \delta D_{11} a_{9i})] - V_{Ai} = 0 \\
&\Delta E_i^0 - [(\tilde{F}_0 + \tilde{C}_2 e_{1i} + \ldots + \tilde{F}_{12} e_{9i}) \\
&\qquad + (\delta F_0 + \delta C_2 e_{1i} + \ldots + \delta F_{12} e_{9i})] - V_{Ei} = 0
\end{aligned}\right\} \tag{10}$$

The system given by equations (10) can be written in matrix notation as:

$$\overline{N} - (\overline{B}\,\overline{C} + \overline{B}\,\overline{\delta}) - \overline{V} = 0 \tag{11}$$

where

$$\underset{[18x1]}{\overline{C}} = \begin{bmatrix} \tilde{C}_0 \\ \tilde{C}_1 \\ \tilde{C}_2 \\ \tilde{C}_4 \\ \tilde{C}_5 \\ \tilde{C}_6 \\ \tilde{C}_7 \\ \tilde{C}_8 \\ \tilde{D}_0 \\ \tilde{D}_3 \\ \tilde{D}_5 \\ \tilde{D}_6 \\ \tilde{D}_7 \\ \tilde{D}_8 \\ \tilde{D}_{11} \\ \tilde{F}_0 \\ \tilde{F}_3 \\ \tilde{F}_{12} \end{bmatrix} \qquad (12)$$

$$\underset{[18x1]}{\overline{\delta}} = \begin{bmatrix} \delta C_0 \\ \delta C_1 \\ \delta C_2 \\ \delta C_4 \\ \delta C_5 \\ \delta C_6 \\ \delta C_7 \\ \delta C_8 \\ \delta D_0 \\ \delta D_3 \\ \delta D_5 \\ \delta D_6 \\ \delta D_7 \\ \delta D_8 \\ \delta D_{11} \\ \delta F_0 \\ \delta F_3 \\ \delta F_{12} \end{bmatrix} \qquad (13)$$

$$\underset{[3nx1]}{\overline{N}} = \begin{bmatrix} \Delta R_1^0 \\ \Delta A_1^0 \\ \Delta E_1^0 \\ \Delta R_2^0 \\ \Delta A_2^0 \\ \Delta E_2^0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \Delta R_n^0 \\ \Delta A_n^0 \\ \Delta E_n^0 \end{bmatrix} \tag{14}$$

$$\underset{[3nx1]}{\overline{V}} = \begin{bmatrix} V_{R1} \\ V_{A1} \\ V_{E1} \\ V_{R2} \\ V_{A2} \\ V_{E2} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{Rn} \\ V_{An} \\ V_{En} \end{bmatrix} \tag{15}$$

$$
\underset{[3n\mathrm{x}18]}{\bar{B}} = \begin{bmatrix}
1 & \tilde{r}_{11} & \tilde{r}_{21} & \tilde{r}_{31} & \tilde{r}_{41} & \tilde{r}_{51} & \tilde{r}_{61} & \tilde{r}_{71} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{11} & 0 & a_{71} & a_{81} & 0 & 0 & 1 & a_{21} & a_{31} & a_{41} & a_{51} & a_{61} & a_{91} & 0 & 0 & 0 \\
0 & 0 & e_{11} & e_{51} & e_{61} & e_{71} & e_{81} & 0 & 0 & 0 & 0 & 0 & e_{41} & e_{31} & 0 & 1 & e_{21} & e_{91} \\
1 & \tilde{r}_{12} & \tilde{r}_{22} & \tilde{r}_{32} & \tilde{r}_{42} & \tilde{r}_{52} & \tilde{r}_{62} & \tilde{r}_{72} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{12} & 0 & a_{72} & a_{82} & 0 & 0 & 1 & a_{22} & a_{32} & a_{42} & a_{52} & a_{62} & a_{92} & 0 & 0 & 0 \\
0 & 0 & e_{12} & e_{52} & e_{62} & e_{72} & e_{82} & 0 & 0 & 0 & 0 & 0 & e_{42} & e_{32} & 0 & 1 & e_{22} & e_{92} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \tilde{r}_{1n} & \tilde{r}_{2n} & \tilde{r}_{3n} & \tilde{r}_{4n} & \tilde{r}_{5n} & \tilde{r}_{6n} & \tilde{r}_{7n} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{1n} & 0 & a_{7n} & a_{8n} & 0 & 0 & 1 & a_{2n} & a_{3n} & a_{4n} & a_{5n} & a_{6n} & a_{9n} & 0 & 0 & 0 \\
0 & 0 & e_{1n} & e_{5n} & e_{6n} & e_{7n} & e_{8n} & 0 & 0 & 0 & 0 & 0 & e_{4n} & e_{3n} & 0 & 1 & e_{2n} & e_{9n}
\end{bmatrix} \tag{16}
$$

In addition to the observational equations (11), error model coefficient observational equations are introduced into the adjustment using the following equations:

$$\left.\begin{aligned} C_0 &= C_0^{\infty} + V_{C_0} \\ C_1 &= C_1^{\infty} + V_{C_1} \\ &\vdots \\ F_{12} &= F_{12}^{\infty} + V_{F_{12}} \end{aligned}\right\} \tag{17}$$

where the superscript ∞ denotes an a priori value of the coefficient and $V_{C_0}, V_{C_1}, \ldots, V_{F_{12}}$ are the corresponding observational residuals. Substituting equations (17) into equation (6) and rearranging these equations, the following observational equations for the error model coefficients are obtained:

$$\left.\begin{aligned} V_{C_0} - \delta C_0 - \tilde{C}_0 + C_0^{\infty} &= 0 \\ V_{C_1} - \delta C_1 - \tilde{C}_1 + C_1^{\infty} &= 0 \\ &\vdots \\ V_{F_{12}} - \delta F_{12} - \tilde{F}_{12} + F_{12}^{\infty} &= 0 \end{aligned}\right\} \tag{18}$$

Equations (18) are written in matrix notation as:

$$\bar{\dot{V}} - \bar{\delta} - \bar{\epsilon} = 0 \tag{19}$$

where

$$\underset{[18\text{x}1]}{\bar{\dot{V}}} = \begin{bmatrix} V_{C_0} \\ V_{C_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ V_{F_{12}} \end{bmatrix} \quad (20) \qquad \underset{[18\text{x}1]}{\bar{\epsilon}} = \begin{bmatrix} \tilde{C}_0 - C_0^{\infty} \\ \tilde{C}_1 - C_1^{\infty} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \tilde{F}_{12} - F_{12}^{\infty} \end{bmatrix} \tag{21}$$

The overall set of observational equations is then given by:

$$\left.\begin{aligned} &\overline{N} - (\overline{B}\,\overline{C} + \overline{B}\,\overline{\delta}) - \overline{V} = 0 \\ &\dot{\overline{V}} - \overline{\delta} - \overline{\epsilon} = 0 \end{aligned}\right\} . \tag{22}$$

The Normal Equations

The final system of observational equations given by the matrix equations (22) consists of (3n + 18) linear equations in the 18 unknown corrections to the approximated error model coefficients. In general, for the application herein the total number of observational equations (3n + 18) is large compared to the number of unknown corrections. Thus the system consists of an over-determined set of equations wherein the effects of random observational errors can be minimized in a least squares reduction. According to the principle of least squares, the best representation of the data is that which makes the weighted sum of the squares of the residuals a minimum. Thus the function to minimize is:

$$\begin{aligned} f(\delta C_0, \delta C_1, \ldots, \delta F_{12}) &= \sum_{i=1}^{n} (W_{3i-2} V_{Ri}^2) + \sum_{i=1}^{n} (W_{3i-1} V_{Ai}^2) \\ &+ \sum_{i=1}^{n} (W_{3i} V_{Ei}^2) + \dot{W}_1 V_{C_0}^2 + \dot{W}_2 V_{C_1}^2 + \ldots + \dot{W}_{18} V_{F_{12}}^2 \quad . \end{aligned} \tag{23}$$

The condition which fulfills the minimizing requirements is that the partial derivatives with respect to each of the 18 unknown corrections be zero. Thus the following 18 normal equations are obtained:

$$\left.\begin{aligned} \frac{\partial f}{\partial(\delta C_0)} &= \sum_{i=1}^{n} \left[W_{3i-2} V_{Ri} \frac{\partial V_{Ri}}{\partial(\delta C_0)} \right] + \sum_{i=1}^{n} \left[W_{3i-1} V_{Ai} \frac{\partial V_{Ai}}{\partial(\delta C_0)} \right] \\ &+ \sum_{i=1}^{n} \left[W_{3i} V_{Ei} \frac{\partial V_{Ei}}{\partial(\delta C_0)} \right] + \dot{W}_1 V_{C_0} \frac{\partial V_{C_0}}{\partial(\delta C_0)} = 0 \end{aligned}\right\} \tag{24}$$

Note: Equation (24) is concluded on following page.

$$
\left.
\begin{aligned}
\frac{\partial f}{\partial(\delta C_1)} &= \sum_{i=1}^{n}\left[W_{3i-2}V_{Ri}\frac{\partial V_{Ri}}{\partial(\delta C_1)}\right] + \sum_{i=1}^{n}\left[W_{3i-1}V_{Ai}\frac{\partial V_{Ai}}{\partial(\delta C_1)}\right] \\
&+ \sum_{i=1}^{n}\left[W_{3i}V_{Ei}\frac{\partial V_{Ei}}{\partial(\delta C_1)}\right] + \dot{W}_2 V_{C_1}\frac{\partial V_{C_1}}{\partial(\delta C_1)} = 0 \\
&\vdots \\
\frac{\partial f}{\partial(\delta F_{12})} &= \sum_{i=1}^{n}\left[W_{3i-2}V_{Ri}\frac{\partial V_{Ri}}{\partial(\delta F_{12})}\right] + \sum_{i=1}^{n}\left[W_{3i-1}V_{Ai}\frac{\partial V_{Ai}}{\partial(\delta F_{12})}\right] \\
&+ \sum_{i=1}^{n}\left[W_{3i}V_{Ei}\frac{\partial V_{Ei}}{\partial(\delta F_{12})}\right] + \dot{W}_{18} V_{F_{12}}\frac{\partial V_{F_{12}}}{\partial(\delta F_{12})} = 0
\end{aligned}
\right\} \quad \text{(24) (Concluded)}
$$

The system of normal equations given by equations (24) can be conveniently written in matrix notation as:

$$
\overline{P}\,\overline{W}\,\overline{V} + \overline{\dot{P}}\,\overline{\dot{W}}\,\overline{\dot{V}} = 0 \tag{25}
$$

where

$$
\underset{[3nx3n]}{\overline{W}} = \begin{bmatrix}
\sigma_0^2/\sigma_{R1}^2 & 0 & 0 \cdots\cdots 0 & 0 & 0 \\
0 & \sigma_0^2/\sigma_{A1}^2 & 0 \cdots\cdots 0 & 0 & 0 \\
0 & 0 & \sigma_0^2/\sigma_{E1}^2 \quad 0 & 0 & 0 \\
\vdots & \vdots & \ddots \quad \vdots & \vdots & \vdots \\
0 & 0 & 0 \cdots\cdots \sigma_0^2/\sigma_{Rn}^2 & 0 & 0 \\
0 & 0 & 0 \cdots\cdots 0 & \sigma_0^2/\sigma_{An}^2 & 0 \\
0 & 0 & 0 \cdots\cdots 0 & 0 & \sigma_0^2/\sigma_{En}^2
\end{bmatrix} \tag{26}
$$

$$\underset{[18x3n]}{\overline{P}} = \begin{bmatrix} \frac{\partial V_{R1}}{\partial(\delta C_0)} & \frac{\partial V_{A1}}{\partial(\delta C_0)} & \frac{\partial V_{E1}}{\partial(\delta C_0)} & \cdots & \frac{\partial V_{Rn}}{\partial(\delta C_0)} & \frac{\partial V_{An}}{\partial(\delta C_0)} & \frac{\partial V_{En}}{\partial(\delta C_0)} \\ \frac{\partial V_{R1}}{\partial(\delta C_1)} & \frac{\partial V_{A1}}{\partial(\delta C_1)} & \frac{\partial V_{E1}}{\partial(\delta C_1)} & \cdots & \frac{\partial V_{Rn}}{\partial(\delta C_1)} & \frac{\partial V_{An}}{\partial(\delta C_1)} & \frac{\partial V_{En}}{\partial(\delta C_1)} \\ \frac{\partial V_{R1}}{\partial(\delta C_2)} & \frac{\partial V_{A1}}{\partial(\delta C_2)} & \frac{\partial V_{E1}}{\partial(\delta C_2)} & \cdots & \frac{\partial V_{Rn}}{\partial(\delta C_2)} & \frac{\partial V_{An}}{\partial(\delta C_2)} & \frac{\partial V_{En}}{\partial(\delta C_2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \frac{\partial V_{R1}}{\partial(\delta F_{12})} & \frac{\partial V_{A1}}{\partial(\delta F_{12})} & \frac{\partial V_{E1}}{\partial(\delta F_{12})} & \cdots & \frac{\partial V_{Rn}}{\partial(\delta F_{12})} & \frac{\partial V_{An}}{\partial(\delta F_{12})} & \frac{\partial V_{En}}{\partial(\delta F_{12})} \end{bmatrix} \tag{27}$$

$$\underset{[18x18]}{\overline{\dot{P}}} = \begin{bmatrix} \frac{\partial V_{C_0}}{\partial(\delta C_0)} & 0 & \cdots & 0 \\ 0 & \frac{\partial V_{C_1}}{\partial(\delta C_1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial V_{F_{12}}}{\partial(\delta F_{12})} \end{bmatrix} \tag{28}$$

$$\underset{[18x18]}{\overline{\dot{W}}} = \begin{bmatrix} \sigma_0^2/\sigma_{C_0}^2 & 0 & \cdots & 0 \\ 0 & \sigma_0^2/\sigma_{C_1}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_0^2/\sigma_{F_{12}}^2 \end{bmatrix} \tag{29}$$

It is perhaps important to point out that if the j-th error model coefficient is to be unconstrained, then the j-th diagonal element of equation (29) is equated to zero. It has also been found from experience that if the error model fits the physical situation and there is no significant correlation between the coefficients, then erroneous (within reasonable limits) observational weighting in equation (26) does not have a significant effect upon the determination of the coefficients. The associated estimates of variances and covariances become erroneous, however. Additional information on the subject of observational and parameter weighting in the least squares adjustment is contained in Appendix C of Reference 2.

The matrix expressions for the residuals $\overline{V}$ and $\overline{\dot{V}}$ in equation (25) can be obtained directly from the observational equations (22) and written as:

$$\left.\begin{aligned} \overline{V} &= \overline{N} - (\overline{B}\,\overline{C} + \overline{B}\,\overline{\delta}) \\ \overline{\dot{V}} &= \overline{\delta} + \overline{\epsilon} \end{aligned}\right\} \tag{30}$$

Substituting equations (30) in equation (25) yields the following matrix equation for the normal equations:

$$\overline{P}\,\overline{W}\,[\overline{N} - (\overline{B}\,\overline{C} + \overline{B}\,\overline{\delta})] + \overline{\dot{P}}\,\overline{\dot{W}}\,(\overline{\epsilon} + \overline{\delta}) = 0 \tag{31}$$

Solving for $\overline{\delta}$:

$$\overline{\delta} = (-\overline{P}\,\overline{W}\,\overline{B} + \overline{\dot{P}}\,\overline{\dot{W}})^{-1}\,[-\,\overline{P}\,\overline{W}\,(\overline{N} - \overline{B}\,\overline{C}) - \overline{\dot{P}}\,\overline{\dot{W}}\,\overline{\epsilon}] \tag{32}$$

Detailed examination of $\overline{B}$ and $\overline{\dot{P}}$ reveals that the solution given by equation (32) is equivalent to:

$$\overline{\delta} = (\overline{B}^T\,\overline{W}\,\overline{B} + \overline{\dot{W}})^{-1}\,(\overline{B}^T\,\overline{W}\,\overline{N} - \overline{B}^T\,\overline{W}\,\overline{B}\,\overline{C} - \overline{\dot{W}}\,\overline{\epsilon}) \tag{33}$$

After $\overline{\delta}$ has been found from (33), this value is added to the initial approximation $\overline{C}$ to obtain the adjusted error model coefficients. The matrix $(\overline{B}^T\,\overline{W}\,\overline{B} + \overline{\dot{W}})$ is the coefficient matrix of the reduced normal equations. The inverse of this matrix multiplied by the unit variance provides an estimate of the output variance-covariance matrix of the error model coefficients resulting from the regression.

Properties of the Adjustment

The vector of least squares residuals for the observations may be computed directly from the observational equations or from equations (30) after the normal equations have been solved for δ. The least squares residuals thus obtained can be rewritten in the more familiar quadratic form as:

$$\bar{S} = \bar{V}^T \bar{W} \bar{V} + \bar{\dot{V}}^T \bar{\dot{W}} \bar{\dot{V}} \tag{34}$$

An estimate of the unit variance resulting from the adjustment is given by:

$$\sigma_0^2 = \bar{S}/f \tag{35}$$

where f is the degrees of freedom involved and is the number of observations in excess of the minimum required for a unique solution.

This is given by:

$$f = 3n \tag{36}$$

where 3n is the total number of observations less the number of unknowns.

An estimate of the variance-covariance matrix for the parameters resulting from the regression analysis is given by:

$$\bar{\sigma}_C = \sigma_0^2 \, [\bar{B}^T \bar{W} \bar{B} + \bar{\dot{W}}]^{-1} \tag{37}$$

For the 18 x 18 case, the matrix $\bar{\sigma}_C$ is given by:

$$\bar{\sigma}_C = \begin{bmatrix} \sigma^2_{C_0} & \sigma_{C_0C_1} & \cdots & \sigma_{C_0F_{12}} \\ \sigma_{C_0C_1} & \sigma^2_{C_1} & \cdots & \sigma_{C_1F_{12}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{C_0F_{12}} & \sigma_{C_1F_{12}} & & \sigma^2_{F_{12}} \end{bmatrix} \qquad (38)$$

The elements of the covariance matrix of the error model coefficients thus permits the accuracies of the final results to be estimated.

MATHEMATICAL DEVELOPMENTS IN A STEPWISE REGRESSION ANALYSIS

Basic Stepwise Approach

The basic approach in a stepwise regression analysis involves examining at every step the variables incorporated into the regression model in previous steps. A variable determined to be significant at an earlier step may be insignificant at a later step because of its relation with other variables in the current regression. Results from a given step in the analysis provide statistical F tests whereby it can be determined if a specific variable should be deleted from or entered into the regression model. The test for deletion is made before a variable is added to the regression. The procedure is terminated when no more variables will be admitted and no more will be rejected. A summary of the basic stepwise approach is given in Figure 1.

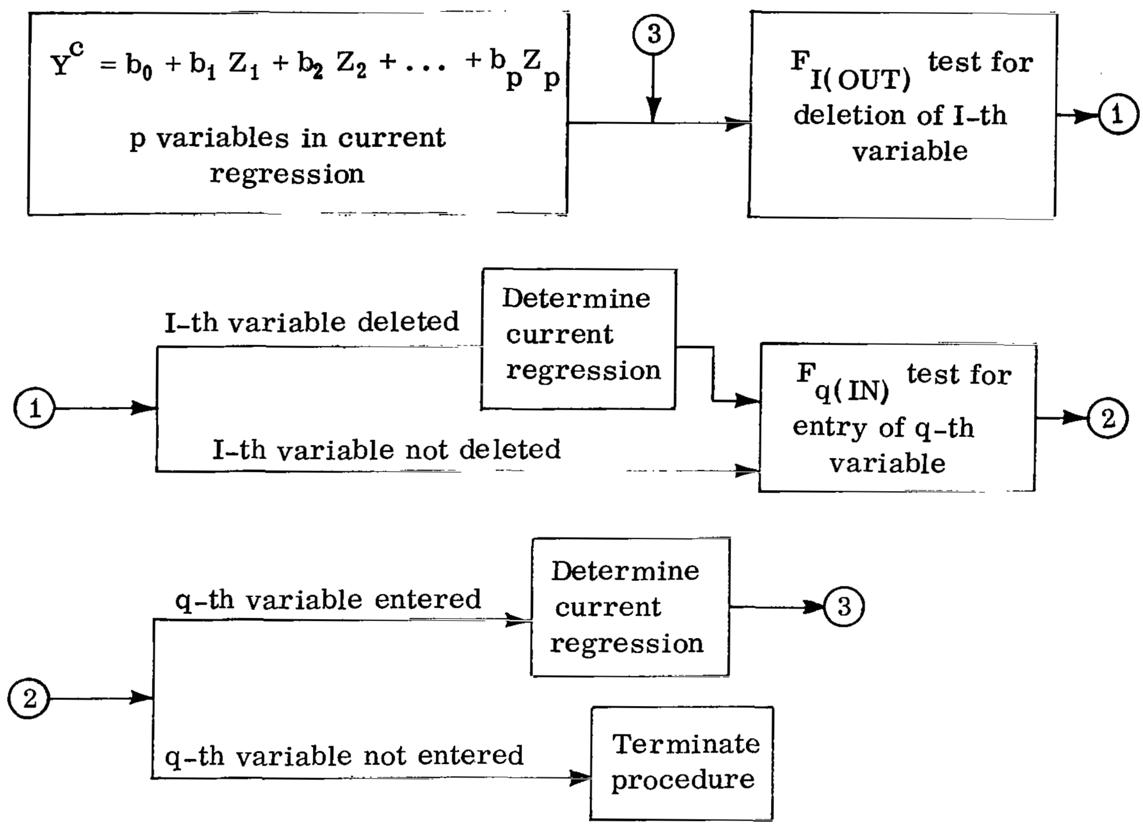

FIGURE 1. BASIC STEPWISE APPROACH

Regression Analysis for Data Centered about the Mean

Assume that the observed response variable Y_i^o is to be estimated by the model:

$$Y_i^c = b_0 + b_1 Z_{1i} + b_2 Z_{2i} + \ldots + b_p Z_{pi} \tag{39}$$

where i = 1, 2, . . . , n. This model is rewritten as:

$$Y_i^c = b_0 + b_1 Z_{1i} + (b_1 \bar{Z}_1 - b_1 \bar{Z}_1) + b_2 Z_{2i} + (b_2 \bar{Z}_2 - b_2 \bar{Z}_2) + \ldots + b_p Z_{pi} + (b_p \bar{Z}_p - b_p \bar{Z}_p) \tag{40}$$

Regrouping the terms in equation (40), we obtain the following equation:

$$Y_i^c = (b_0 + b_1 \overline{Z}_1 + b_2 \overline{Z}_2 + \ldots + b_p \overline{Z}_p) + b_1 (Z_{1i} - \overline{Z}_1)$$

$$+ b_2 (Z_{2i} - \overline{Z}_2) + \ldots + b_p (Z_{pi} - \overline{Z}_p) \quad . \tag{41}$$

If we let:

$$\left.\begin{aligned} z_{1i} &= Z_{1i} - \overline{Z}_1 \\ z_{2i} &= Z_{2i} - \overline{Z}_2 \\ &\vdots \\ z_{pi} &= Z_{pi} - \overline{Z}_p \\ b_0' &= b_0 + b_1 \overline{Z}_1 + b_2 \overline{Z}_2 + \ldots + b_p \overline{Z}_p \end{aligned}\right\} \quad . \tag{42}$$

Then equation (41) can be written as:

$$Y_i^c = b_0' + b_1 z_{1i} + b_2 z_{2i} + \ldots + b_p z_{pi} \tag{43}$$

At this point the input data matrix of observations appears as shown in Table I.

TABLE I. INPUT DATA CENTERED ABOUT MEAN

z_{1i}	z_{2i}	...	z_{pi}	y_i
$Z_{11} - \overline{Z}_1$	$Z_{21} - \overline{Z}_2$	...	$Z_{p1} - \overline{Z}_p$	$Y_1^o - \overline{Y}$
$Z_{12} - \overline{Z}_1$	$Z_{22} - \overline{Z}_2$	...	$Z_{p2} - \overline{Z}_p$	$Y_2^o - \overline{Y}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$Z_{1n} - Z_1$	$Z_{2n} - \overline{Z}_2$	...	$Z_{pn} - \overline{Z}_p$	$Y_n^o - \overline{Y}$

The sample estimates $b'_0, b_1, b_2, \ldots, b_p$ are obtained by minimizing the weighted sum of squares of deviations between the observed and predicted response values. Since observations of the response are assumed to have equal reliability, unit weighting factors are used. The minimizing function is then of the form:

$$M = \sum_{i=1}^{n} (Y_i^o - Y_i^c)^2 \tag{44}$$

The requirement for M a minimum yields the following normal equations:

$$\left.\begin{aligned} \frac{\partial M}{\partial b'_0} &= 0 \\ \frac{\partial M}{\partial b_1} &= 0 \\ &\vdots \\ \frac{\partial M}{\partial b_p} &= 0 \end{aligned}\right\} \tag{45}$$

Note the exact form of the first normal equation obtained by the indicated partial differentiation:

$$\frac{\partial M}{\partial b'_0} = \left[\sum_{i=1}^{n} (Y_i^o - b'_0 - b_1 z_{1i} - b_2 z_{2i} - \ldots - b_p z_{pi})\right](-1) = 0$$

$$= -\sum_{i=1}^{n} Y_i^o + nb'_0 + b_1 \sum_{i=1}^{n} z_{i1} + b_2 \sum_{i=1}^{n} z_{2i} + \ldots + b_p \sum_{i=1}^{n} z_{pi} = 0$$

$$= -n\bar{Y} + nb'_0 + nb_1 \bar{z}_1 + n\, b_2 \bar{z}_2 + \ldots + nb_p \bar{z}_p = 0$$

$$= -\bar{Y} + b'_0 + b_1 \bar{z}_1 + b_2 \bar{z}_2 + \ldots + b_p \bar{z}_p = 0 \tag{46}$$

But:

$$\bar{z}_1 = \frac{\sum_{i=1}^{n} z_{1i}}{n} = \frac{\sum_{i=1}^{n} (Z_{1i} - \bar{Z}_1)}{n}$$

$$= \frac{(Z_{11} - \bar{Z}_1) + (Z_{12} - \bar{Z}_1) + \ldots + (Z_{1n} - \bar{Z}_1)}{n}$$

$$= \frac{\sum_{i=1}^{n} Z_{1i}}{n} - \frac{n\bar{Z}_1}{n} = 0 \quad , \tag{47}$$

and the same holds for $\bar{z}_2$, $\bar{z}_3$, ..., $\bar{z}_p$. Thus, equation (46) becomes:

$$b_0' = \bar{Y} \quad . \tag{48}$$

Substituting equation (48) into equation (42), the constant term b_0 in equation (39) is given by:

$$b_0 = \bar{Y} - b_1 \bar{Z}_1 - b_2 \bar{Z}_2 - \ldots - b_p \bar{Z}_p \quad . \tag{49}$$

A similar substitution in equation (43) yields:

$$Y_i^c = \bar{Y} + b_1 z_{1i} + b_2 z_{2i} + \ldots + b_p z_{pi} \quad . \tag{50}$$

This is an alternate form to equation (39) and we have to estimate one less parameter since equations (48) and (49) hold regardless of the values for b_1, b_2, ..., b_p in equation (46). The remaining normal equations are given by:

$$\left.\begin{aligned}
\frac{\partial M}{\partial b_1} &= \sum_{i=1}^{n} \left[z_{1i} \left(y_i - b_1 z_{1i} - b_2 z_{2i} - \ldots - b_p z_{pi}\right)\right] = 0 \\
\frac{\partial M}{\partial b_2} &= \sum_{i=1}^{n} \left[z_{2i} \left(y_i - b_1 z_{1i} - b_2 z_{2i} - \ldots - b_p z_{pi}\right)\right] = 0 \\
&\vdots \\
\frac{\partial M}{\partial b_p} &= \sum_{i=1}^{n} \left[z_{pi} \left(y_i - b_1 z_{1i} - b_2 z_{2i} - \ldots - b_p z_{pi}\right)\right] = 0
\end{aligned}\right\} \quad (51)$$

Let:

$$S_{IJ} = \sum_{i=1}^{n} z_{Ii}\, z_{Ji} = \sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)(Z_{Ji} - \bar{Z}_J) \qquad \begin{aligned} I &= 1, 2, \ldots, p \text{ for each} \\ J &= 1, 2, \ldots, p \end{aligned} \quad (52)$$

$$S_{IY} = \sum_{i=1}^{n} z_{Ii}\, y_i = \sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)(Y_i^o - \bar{Y}) \qquad I = 1, 2, \ldots, p \quad (53)$$

$$S_{YY} = \sum_{i=1}^{n} y_i\, y_i = \sum_{i=1}^{n} (Y_i^o - \bar{Y})(Y_i^o - \bar{Y}) \quad (54)$$

Then the normal equations (51) can be written as:

$$\left.\begin{aligned}
b_1 S_{11} + b_2 S_{12} + \ldots + b_p S_{1p} &= S_{1Y} \\
b_1 S_{21} + b_2 S_{22} + \ldots + b_p S_{2p} &= S_{2Y} \\
&\vdots \\
b_1 S_{pi} + b_2 S_{p2} + \ldots + b_p S_{pp} &= S_{pY}
\end{aligned}\right\} \quad (55)$$

The solution of equations (55) for the b's yields:

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \cdots & S_{1p} \\ S_{21} & S_{22} \cdots & S_{2p} \\ \vdots & \vdots & \vdots \\ S_{p1} & S_{p2} & S_{pp} \end{bmatrix}^{-1} \begin{bmatrix} S_{1Y} \\ S_{2Y} \\ \vdots \\ S_{pY} \end{bmatrix} \tag{56}$$

The estimate of the standard deviation of the response variable is obtained from the residual deviations and is given by:

$$\sigma_Y = \left[\frac{\sum_{i=1}^{n} (Y_i^o - Y_i^c)^2}{n - d} \right]^{1/2} . \tag{57}$$

It can be shown [7] that the standard errors of the estimated partial regression coefficients given by equations (56) are given by:

$$\begin{bmatrix} \sigma_{b_1} \\ \sigma_{b_2} \\ \vdots \\ \sigma_{b_p} \end{bmatrix} = \sigma_Y \begin{bmatrix} \sqrt{c_{11}} \\ \sqrt{c_{22}} \\ \vdots \\ \sqrt{c_{pp}} \end{bmatrix} \tag{58}$$

where the c's are elements in the inverse of the S matrix in equations (56).

The standard error of the term $b'_0 = \bar{Y}$ in the regression equation is given by:

$$\sigma_{b'_0} = \sigma_Y / \sqrt{n} . \tag{59}$$

Confidence intervals [8] for the estimated coefficients are obtained from:

$$\begin{bmatrix} b'_0 \\ b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix} \pm \delta \begin{bmatrix} \sigma_{b'_0} \\ \sigma_{b_1} \\ \sigma_{b_2} \\ \vdots \\ \sigma_{b_p} \end{bmatrix} \tag{60}$$

The specific value to use for δ can be obtained from a "t" table containing the percentiles of the "t" distribution. It is dependent on the number of degrees of freedom $(n - d)$ and the particular confidence limits selected. If 95 percent confidence limits are selected, then the confidence intervals given by equation (60) can be interpreted as follows. If the experiment is repeated with another set of data, then another set of limits would be obtained. Continuing the process in this manner, then 95 percent of the limits obtained would cover the particular coefficient.

Centering the input data about the mean as shown in Table I reduces the absolute size of the numbers entering the computations. Since the centered data are required in obtaining the linear correlation matrix, no additional computations are required that would not otherwise be available.

Regression Analysis for Data in Correlation Form

The regression analysis can be transformed into a form which involves linear correlations [8]. This is accomplished by making the following transformations on the centered data of Table I.

$$\left.\begin{aligned} \tilde{y}_i &= (Y_i^o - \bar{Y}) / \sqrt{S_{YY}} \\ \tilde{z}_{1i} &= (Z_{1i} - \bar{Z}_1) / \sqrt{S_{11}} \\ \tilde{z}_{2i} &= (Z_{2i} - \bar{Z}_2) / \sqrt{S_{22}} \\ &\vdots \\ \tilde{z}_{pi} &= (Z_{pi} - \bar{Z}_p) / \sqrt{S_{pp}} \end{aligned}\right\} \tag{61}$$

The same transformations on the dependent and independent variables in the model given by equation (50) yields the following model:

$$\tilde{y}_i^c \sqrt{S_{YY}} = b_1 \sqrt{S_{11}}\, \tilde{z}_{1i} + b_2 \sqrt{S_{22}}\, \tilde{z}_{2i} + \ldots + b_p \sqrt{S_{pp}}\, \tilde{z}_{pi} \quad . \tag{62}$$

Or:

$$\tilde{y}_i^c = \alpha_1 \tilde{z}_{1i} + \alpha_2 \tilde{z}_{2i} + \ldots + \alpha_p \tilde{z}_{pi} \tag{63}$$

where

$$\left.\begin{aligned} \alpha_1 &= b_1 \sqrt{\frac{S_{11}}{S_{YY}}} \\ \alpha_2 &= b_2 \sqrt{\frac{S_{22}}{S_{YY}}} \\ &\vdots \\ \alpha_p &= b_p \sqrt{\frac{S_{pp}}{S_{YY}}} \end{aligned}\right\} \quad . \tag{64}$$

The new coefficients $\alpha_1, \alpha_2, \ldots, \alpha_p$ are to be estimated from the transformed data $\tilde{y}_i, \tilde{z}_{1i}, \tilde{z}_{2i}, \ldots, \tilde{z}_{pi}$. The minimizing function is:

$$V = \sum_{i=1}^{n} (\tilde{y}_i - \tilde{y}_i^c)^2 \quad . \tag{65}$$

The normal equations are:

$$\left.\begin{aligned} \frac{\partial V}{\partial \alpha_1} &= 0 \\ \frac{\partial V}{\partial \alpha_2} &= 0 \\ &\vdots \\ \frac{\partial V}{\partial \alpha_p} &= 0 \end{aligned}\right\} \quad . \tag{66}$$

Evaluation of these equations yields:

$$\left.\begin{aligned}
\frac{\partial V}{\partial \alpha_1} &= \sum_{i=1}^{n} \left[\tilde{z}_{1i} \left(\tilde{y}_i - \alpha_1 \tilde{z}_{1i} - \alpha_2 \tilde{z}_{2i} - \ldots - \alpha_p \tilde{z}_{pi} \right) \right] = 0 \\
\frac{\partial V}{\partial \alpha_2} &= \sum_{i=1}^{n} \left[\tilde{z}_{2i} \left(\tilde{y}_i - \alpha_1 \tilde{z}_{1i} - \alpha_2 \tilde{z}_{2i} - \ldots - \alpha_p \tilde{z}_{pi} \right) \right] = 0 \\
&\vdots \\
\frac{\partial V}{\partial \alpha_p} &= \sum_{i=1}^{n} \left[\tilde{z}_{pi} \left(\tilde{y}_i - \alpha_1 \tilde{z}_{1i} - \alpha_2 \tilde{z}_{2i} - \ldots - \alpha_p \tilde{z}_{pi} \right) \right] = 0 \ .
\end{aligned}\right\} \tag{67}$$

Let:

$$\tilde{S}_{IJ} = \sum_{i=1}^{n} \tilde{z}_{Ii} \tilde{z}_{Ji} \qquad I = 1, 2, \ldots, p \text{ for each } J = 1, 2, \ldots, p \tag{68}$$

$$\tilde{S}_{IY} = \sum_{i=1}^{n} \tilde{z}_{Ii} \tilde{y}_i \quad , \ I = 1, 2, \ldots, p \ . \tag{69}$$

Then equations (67) can be written as:

$$\left.\begin{aligned}
\alpha_1 \tilde{S}_{11} + \alpha_2 \tilde{S}_{12} + \ldots + \alpha_p \tilde{S}_{1p} &= \tilde{S}_{1Y} \\
\alpha_1 \tilde{S}_{21} + \alpha_2 \tilde{S}_{22} + \ldots + \alpha_p \tilde{S}_{2p} &= \tilde{S}_{2Y} \\
&\vdots \\
\alpha_1 \tilde{S}_{pi} + \alpha_2 \tilde{S}_{p2} + \ldots + \alpha_p \tilde{S}_{pp} &= \tilde{S}_{pY}
\end{aligned}\right\} \ . \tag{70}$$

The solution for the α's becomes:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} & \cdots & \tilde{S}_{1p} \\ \tilde{S}_{21} & \tilde{S}_{22} & \cdots & \tilde{S}_{2p} \\ \vdots & \vdots & & \vdots \\ \tilde{S}_{p1} & \tilde{S}_{p2} & \cdots & \tilde{S}_{pp} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{S}_{1Y} \\ \tilde{S}_{2Y} \\ \vdots \\ \tilde{S}_{pY} \end{bmatrix} . \tag{71}$$

Examination of equation (68) reveals that:

$$\begin{aligned} \tilde{S}_{IJ} &= \frac{S_{IJ}}{\sqrt{S_{II}\, S_{JJ}}} \\ &= \frac{\sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)(Z_{Ji} - \bar{Z}_J)}{\sqrt{\sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)^2 \sum_{i=1}^{n} (Z_{Ji} - \bar{Z}_J)^2}} \\ &= r_{IJ} \quad . \end{aligned} \tag{72}$$

Also:

$$\begin{aligned} \tilde{S}_{IY} &= \frac{S_{IY}}{\sqrt{S_{II}\, S_{YY}}} \\ &= \frac{\sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)(Y_i^o - \bar{Y})}{\sqrt{\sum_{i=1}^{n} (Z_{Ii} - \bar{Z}_I)^2 \sum_{i=1}^{n} (Y_i^o = \bar{Y})^2}} \\ &= r_{IY} \quad . \end{aligned} \tag{73}$$

Thus, the $\tilde{S}$ elements of equations (71) are the linear correlation coefficients for the independent variables and the response. Equations (71) can then be written as:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \cdots & r_{1p} \\ r_{21} & r_{22} \cdots & r_{2p} \\ \vdots & \vdots & \vdots \\ r_{p1} & r_{p2} \cdots & r_{pp} \end{bmatrix}^{-1} \begin{bmatrix} r_{1Y} \\ r_{2Y} \\ \vdots \\ r_{pY} \end{bmatrix} \tag{74}$$

The constant term is obtained by solving equations (64) for $b_1, b_2, \ldots, b_p$ and substituting in equation (49):

$$b_0 = \bar{Y} - \alpha_1 \sqrt{\frac{S_{YY}}{S_{11}}}\,\bar{Z}_1 - \alpha_2 \sqrt{\frac{S_{YY}}{S_{22}}}\,\bar{Z}_2 - \ldots - \alpha_p \sqrt{\frac{S_{YY}}{S_{pp}}}\,\bar{Z}_p \,. \tag{75}$$

The estimate of the standard deviation of the transformed response variable is given by:

$$\tilde{\sigma}_Y = \left[\frac{\sum_{i=1}^{n} (\tilde{y}_i - \tilde{y}_i^{\,c})^2}{n - d} \right]^{\frac{1}{2}} . \tag{76}$$

The standard errors of the new regression coefficients are given by:

$$\begin{bmatrix} \sigma_{\alpha_1} \\ \sigma_{\alpha_2} \\ \vdots \\ \sigma_{\alpha_p} \end{bmatrix} = \tilde{\sigma}_Y \begin{bmatrix} \sqrt{\tilde{c}_{11}} \\ \sqrt{\tilde{c}_{22}} \\ \vdots \\ \sqrt{\tilde{c}_{pp}} \end{bmatrix} \tag{77}$$

where the $\tilde{c}$'s are elements in the inverse of the r matrix in equation (74). The standard error of the term $b'_0 = \bar{Y}$ is obtained by first writing equation (85), which appears later in the discussion, as:

$$\sigma_Y = \sqrt{S_{YY}} \ \tilde{\sigma}_Y \quad . \tag{78}$$

Using this result in equation (59), we see that:

$$\sigma_{b'_0} = \frac{\tilde{\sigma}_Y \sqrt{S_{YY}}}{\sqrt{n}} \quad . \tag{79}$$

This equation is necessary if the input data are in correlation form. Equation (59), however, is sufficient when the input data are centered about the mean.

Alternate Form for the Standard Deviations of the Estimated Partial Regression Coefficients

The standard errors of the estimated partial regression coefficients as given by equations (58) can be expressed in another form. Equation (76) is first written as:

$$\tilde{\sigma}_Y^2 = \frac{(\tilde{y}_1 - \tilde{y}_1^c)^2 + (\tilde{y}_2 - \tilde{y}_2^c)^2 + \ldots + (\tilde{y}_n - \tilde{y}_n^c)^2}{n - d} \quad . \tag{80}$$

Substituting $\tilde{z}_{1i}, \tilde{z}_{2i}, \ldots, \tilde{z}_{pi}$ of equations (61) in (63) and the result in equation (80) and $\tilde{y}_i$ of equations (61) in equation (80) we see that:

$$\tilde{\sigma}_Y^2 = \frac{1}{n-d}\left\{\left[\frac{(Y_1^o - \bar{Y})}{\sqrt{S_{YY}}} - \frac{\alpha_1(Z_{11} - \bar{Z}_1)}{\sqrt{S_{11}}} - \frac{\alpha_2(Z_{21} - \bar{Z}_2)}{\sqrt{S_{22}}} - \dots - \frac{\alpha_p(Z_{p1} - \bar{Z}_p)}{\sqrt{S_{pp}}}\right]^2\right.$$

$$+\left[\frac{(Y_2^o - \bar{Y})}{\sqrt{S_{YY}}} - \frac{\alpha_1(Z_{12} - \bar{Z}_1)}{\sqrt{S_{11}}} - \frac{\alpha_2(Z_{22} - \bar{Z}_2)}{\sqrt{S_{22}}} - \dots - \frac{\alpha_p(Z_{p2} - \bar{Z}_p)}{\sqrt{S_{pp}}}\right]^2$$

$$\left.+ \dots + \left[\frac{(Y_n^o - \bar{Y})}{\sqrt{S_{YY}}} - \frac{\alpha_1(Z_{1n} - \bar{Z}_1)}{\sqrt{S_{11}}} - \frac{\alpha_2(Z_{2n} - \bar{Z}_2)}{\sqrt{S_{22}}} - \dots - \frac{\alpha_p(Z_{pn} - \bar{Z}_p)}{\sqrt{S_{pp}}}\right]^2\right\}. \tag{81}$$

Substituting equation (64) for the standard partial regression coefficients in equation (81) and rearranging the equation, we find:

$$\tilde{\sigma}_Y^2 = \frac{1}{n-d}\left\{\frac{1}{S_{YY}}\left[(Y_1^o - \bar{Y}) - b_1(Z_{11} - \bar{Z}_1) - b_2(Z_{21} - \bar{Z}_2) - \dots - b_p(Z_{p1} - \bar{Z}_p)\right]^2\right.$$

$$+\frac{1}{S_{YY}}\left[(Y_2^o - \bar{Y}) - b_1(Z_{12} - \bar{Z}_1) - b_2(Z_{22} - \bar{Z}_2) - \dots - b_p(Z_{p2} - \bar{Z}_p)\right]^2$$

$$\left.+ \dots + \frac{1}{S_{YY}}\left[(Y_n^o - \bar{Y}) - b_1(Z_{1n} - \bar{Z}_1) - b_2(Z_{2n} - \bar{Z}_2) - \dots - b_p(Z_{pn} - \bar{Z}_p)\right]^2\right\}. \tag{82}$$

Then, by substituting $(Y_i^c - \bar{Y})$ as obtained from equation (50) in equation (82), the following equation is obtained:

$$\tilde{\sigma}_Y^2 = \frac{1}{(n-d)\,S_{YY}} \left\{ \left[(Y_1^o - \bar{Y}) - (Y_1^c - \bar{Y}) \right]^2 + \left[(Y_2^o - \bar{Y}) - (Y_2^c - \bar{Y}) \right]^2 \right.$$

$$\left. + \dots + \left[(Y_n^o - \bar{Y}) - (Y_n^c - \bar{Y}) \right]^2 \right\}$$

$$= \frac{1}{(n-d)\,S_{YY}} \left\{ \sum_{i=1}^{n} (Y_i^o - Y_i^c)^2 \right\}. \tag{83}$$

But from equation (57), equation (83) can be written as:

$$\tilde{\sigma}_Y^2 = \sigma_Y^2 / S_{YY} \quad . \tag{84}$$

Or:

$$\tilde{\sigma}_Y = \sigma_Y / \sqrt{S_{YY}} \quad . \tag{85}$$

This equation gives the relation between the response variable and the transformed response variable.

This result can be substituted in equation (77) for a typical j-th coefficient:

$$\sigma_{\alpha_j} = \frac{\sigma_Y}{\sqrt{S_{YY}}} \sqrt{\tilde{c}_{jj}} \quad . \tag{86}$$

Since α_j is a linear function of b_j, then the variance in α_j can be written as:

$$\sigma_{\alpha_j}^2 = \sigma_{b_j}^2 \left(\frac{\partial \alpha_j}{\partial b_j} \right)^2 = \sigma_{b_j}^2 \left(\frac{S_{jj}}{S_{YY}} \right) \quad . \tag{87}$$

By substituting σ_{α_j} from equation (87) in equation (86), the following equation is obtained for the standard error of a typical j-th partial regression coefficient:

$$\sigma_{b_j} = \frac{\sigma_Y}{\sqrt{S_{YY}}} \sqrt{\tilde{c}_{jj}} \quad . \tag{88}$$

Equation (88) is used in place of equation (58) when the regression analysis has been transformed into the form involving correlations (input data in correlation form) wherein the inverse elements of the S matrix are not available.

Alternate Form for the Standard Deviation of the Response Variable

The residual sum of squares can be written as (see Reference 5):

$$\sum_{i=1}^{n} (Y_i^o - Y_i^c)^2 = \sum_{i=1}^{n} (Y_i^o - \bar{Y})^2 - \sum_{i=1}^{n} (Y_i^c - \bar{Y})^2. \tag{89}$$

That is, the residual sum of squares is equal to the total sum of squares about the mean minus the sum of squares due to regression. But from the definition of the multiple correlation coefficient:

$$\sum_{i=1}^{n} (Y_i^c - \bar{Y})^2 = \sum_{i=1}^{n} (Y_i^o - \bar{Y})^2 \; R^2_{Y.12\ldots p} \quad . \tag{90}$$

Substituting equation (90) in equation (89), we find:

$$\sum_{i=1}^{n} (Y_i^o - Y_i^c)^2 = \sum_{i=1}^{n} (Y_i^o - \bar{Y})^2 \left[1 - R^2_{Y \,.\, 12\ldots p}\right]$$

$$= S_{YY} \left[1 - R^2_{Y \,.\, 12\ldots p}\right]. \tag{91}$$

Then by substituting equation (91) in equation (57):

$$\sigma_Y = \left[\frac{S_{YY}\left(1 - R^2_{Y \,.\, 12\ldots p}\right)}{n - d}\right]^{\frac{1}{2}} \tag{92}$$

This expression enables the estimate of the standard deviation of the response variable to be expressed in a more convenient computational form. Its use has an advantage over equation (57) when the Gaussian Elimination Procedure is used to solve the normal equations. This is discussed later on in this report.

Partial Correlation Coefficients

The partial correlation coefficient of the variable Z_q (not in the regression) with the response Y — given that $Z_1, Z_2, \ldots, Z_p$ are already in the regression — is denoted as $\rho_{qY \,.\, 12\ldots p}$. This statistic is used in the stepwise procedure to determine which of several variables not in the regression should be considered for entry into the model at a given step. The variable considered for entry is the one whose partial correlation with the response is highest.

Mathematically, $\rho_{qY \,.\, 12\ldots p}$ is found by determining the simple linear correlation between the residuals from the regression:

$$Y_i^c = b_0 + b_1 Z_{1i} + b_2 Z_{2i} + \ldots + b_p Z_{pi} \tag{93}$$

and the residuals from the regression:

$$Z_{qi}^{c} = \hat{b}_0 + \hat{b}_1 Z_{1i} + \hat{b}_2 Z_{2i} + \ldots + \hat{b}_p Z_{pi} \quad . \tag{94}$$

Let the residuals from these regressions be denoted as:

$$\left.\begin{aligned} VR_{1i} &= Y_i^{o} - Y_i^{c} \\ VR_{2i} &= Z_{qi} - Z_{qi}^{c} \end{aligned}\right\} \quad . \tag{95}$$

Then:

$$\rho_{qY\,.\,12\ldots p} = \frac{\sum_{i=1}^{n} (VR_{1i}\ VR_{2i})}{\sqrt{\sum_{i=1}^{n} VR_{1i}^{2} \sum_{i=1}^{n} VR_{2i}^{2}}} \quad . \tag{96}$$

An alternate form for the partial correlation coefficient involves the linear correlation coefficients. Let:

$$\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} & r_{1q} & r_{1Y} \\ r_{21} & r_{22} & \cdots & r_{2p} & r_{2q} & r_{2Y} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} & r_{pq} & r_{py} \\ r_{q1} & r_{q2} & \cdots & r_{qp} & r_{qq} & r_{qY} \\ r_{Y1} & r_{Y2} & \cdots & r_{Yp} & r_{Yq} & r_{YY} \end{bmatrix}^{-1} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} & c_{1q} & c_{1Y} \\ c_{21} & c_{22} & \cdots & c_{2p} & c_{2q} & c_{2Y} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pp} & c_{pq} & c_{pY} \\ c_{q1} & c_{q2} & \cdots & c_{qp} & c_{qq} & c_{qY} \\ c_{Y1} & c_{Y2} & \cdots & c_{Yp} & c_{Yq} & c_{YY} \end{bmatrix} \tag{97}$$

Then from Reference 9:

$$\rho_{qY\,.\,12\ldots p} = \frac{-c_{qY}}{\sqrt{c_{qq}\, c_{YY}}} \quad . \tag{98}$$

Other partial correlation coefficients can be determined having once determined the inverse equation (97). For example:

$$\rho_{1Y.23\ldots qp} = \frac{-c_{1Y}}{\sqrt{c_{11}\, c_{YY}}} \quad . \tag{99}$$

Significance of the Estimated Regression Equation

The significance associated with an estimated regression equation can be considered from an analysis of variance viewpoint. If the estimated regression equation is of the form given by equation (50), then the analysis of variance is summarized in Table II.

TABLE II. ANALYSIS OF VARIANCE

d f	Type Variation	SS	MS	F Value
$n - 1$	Total	$S_{YY} = \sum_{i=1}^{n} (Y_i^o - \bar{Y})^2$		
$n - d$	Residual	$S(RES) = \sum_{i=1}^{n} (Y_i^o - Y_i^c)^2$	$M(RES) = \frac{S(RES)}{n - d}$	
$d - 1$	Regression	$S(REG) = \sum_{i=1}^{n} (Y_i^c - \bar{Y})^2$	$M(REG) = \frac{S(REG)}{d-1}$	$\frac{M(REG)}{M(RES)}$

df = degrees of freedom
SS = sum of squares
MS = mean square = SS/df
n = number of observations
p = number of independent variables
d = p + 1

From equation (89), the total sum of squares can be written as:

$$\sum_{i=1}^{n} (Y_i^o - \bar{Y})^2 = \sum_{i=1}^{n} (Y_i^o - Y_i^c)^2 + \sum_{i=1}^{n} (Y_i^c - \bar{Y})^2$$

$$S_{YY} = S(RES) + S(REG) \tag{100}$$

That is, the total sum of squares about the mean is equal to the sum of squares about regression (residual) plus the sum of squares due to regression. This shows that a way of assessing how useful the regression equation will be for prediction purposes is to see how much of S_{YY} is given by S(RES) and how much is given by S(REG). If S(RES) = 0, then the actual observations are described exactly by the regression equation (50). The ratio defined by:

$$R^2_{Y.12...p} = S(REG)/S_{YY} \tag{101}$$

is known as the multiple correlation coefficient and measures the closeness with which the regression equation describes the observed data.

Another useful ratio in regression analysis is given by the following equation (Reference 5):

$$F = \frac{S(REG)\ /(d-1)}{S(RES)\ /(n-d)} . \tag{102}$$

It follows an F distribution with (d-1) and (n-d) degrees of freedom. This quantity is used to determine the statistical significance of the regression equation under consideration by comparing it with the appropriate F table value. For example, determine the 95-percent percentage points F[(d-1), (n-d), 0.95] in the appropriate F table. If the computed F value given by equation (102) is greater than the table value, then the regression equation is statistically significant. An alternate expression for equation (102) can be obtained by substituting the residual sum of squares as given by equation (91) into equation (100) to obtain:

$$S(REG) = S_{YY} - S_{YY} (1 - R^2_{Y.12...p}) . \tag{103}$$

Substituting equation (91) and equation (103) into equation (102), we see that:

$$F = \frac{R^2_{Y.12\ldots p}(n-d)}{(1 - R^2_{Y.12\ldots p})(d-1)} \quad . \tag{104}$$

This equation is a more convenient computational form for the F quantity when performing a stepwise regression analysis. This is because the quantity $(1 - R^2_{Y.12\ldots p})$ is available at a given step in the analysis. This is discussed later on in this report.

It can be shown (Reference 10, Chapter 14), that the F distribution with 1 and (n-d) degrees of freedom is equivalent to the t^2 distribution with (n-d) degrees of freedom. That is,

$$F\ [\ 1,\ (n-d)\ ] = t^2\ [n-d] \quad . \tag{105}$$

The significance of the individual partial regression coefficients $(b_1, b_2, \ldots, b_p)$ is determined from (Reference 10, Chapter 20):

$$t_i = b_i / \sigma_{b_i} \tag{106}$$

which is distributed as t with (n-d) degrees of freedom. By virtue of equation (105), this can be written as:

$$F_{i(OUT)} = b_i^2 / \sigma^2_{b_i} \tag{107}$$

with 1 and (n-d) degrees of freedom. The F value is denoted as $F_{i(OUT)}$ to indicate its use in the stepwise procedure for leaving an insignificant variable out of the regression equation. If the computed $F_{i(OUT)}$ value given by equation (107) is greater than the appropriate table value, then the i-th coefficient is statistically significant. If the regression analysis has been transformed into the form involving correlations (input data in correlation form), then the appropriate form of equation (107) to use is obtained by substituting equation (92) in equation (88) to obtain for σ_{b_i}:

$$\sigma_{b_i} = \left[\frac{S_{YY}\,(1 - R^2_{Y.12\ldots p})}{n - d}\right]^{\frac{1}{2}} \sqrt{\frac{\tilde{c}_{ii}}{S_{ii}}} \quad . \tag{108}$$

The partial regression coefficients b_i can be expressed in terms of the standard partial regression coefficients α_i from equations (64). This result and equation (108) are substituted in equation (107) to obtain:

$$F_{i(OUT)} = \frac{\alpha_i^2\,(n - d)}{\tilde{c}_{ii}(1 - R^2_{Y.12\ldots p})} \, . \tag{109}$$

This is the form of equation (107) to use when the input data are in correlation form.

It was pointed out earlier in this report that of several variables not in the regression, the variable considered for entry is the one whose partial correlation with the response is highest. If this is determined to be $\rho_{qY.12\ldots p}$ (variable Z_q not in the regression), then the significance of $\rho_{qY.12\ldots p}$ is determined from (Reference 10):

$$t = \frac{\rho_{qY.12\ldots p}\sqrt{(n-\tilde{d})}}{\sqrt{1 - \rho^2_{qY.12\ldots p}}} \tag{110}$$

which is distributed as t with $(n-\tilde{d})$ degrees of freedom and where $\tilde{d}$ is the total number of variables. From equation (105) this can be written as

$$F_{q(IN)} = \frac{\rho^2_{qY.12\ldots p}\,(n-\tilde{d})}{1 - \rho^2_{qY.12\ldots p}} \tag{111}$$

with 1 and $(n-\tilde{d})$ degrees of freedom. The F value is denoted as $F_{q(IN)}$ to indicate its use in the stepwise procedure for entering a significant variable into the regression equation. If the computed $F_{q(IN)}$ value given by equation (111) is greater than the appropriate table value, then the coefficient corresponding to variable Z_q is statistically significant. In the stepwise process the coefficient corresponding to the smallest of $F_{i(OUT)}$ is tested for deletion before a new variable is considered for entry. The coefficient corresponding to the largest of $F_{q(IN)}$ is tested for entry after the deletion test.

The Gaussian Elimination Method as Applied to a Stepwise Regression Analysis

The normal equations given by equations (70) are a system of simultaneous linear algebraic equations in $\alpha_1, \alpha_2, \ldots, \alpha_p$. These equations can be written as:

$$\left.\begin{aligned} \alpha_1 r_{11} + \alpha_2 r_{12} + \ldots + \alpha_p r_{1p} &= r_{1Y} \\ \alpha_1 r_{21} + \alpha_2 r_{22} + \ldots + \alpha_p r_{2p} &= r_{2Y} \\ \alpha_1 r_{p1} + \alpha_2 r_{p2} + \ldots + \alpha_p r_{pp} &= r_{pY} \end{aligned}\right\} \tag{112}$$

Or in matrix notation:

$$\bar{r}_{pp} \bar{\alpha}_p = \bar{r}_{pY} \tag{113}$$

where

$$\bar{\alpha}_p = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix} \quad (114) \qquad \bar{r}_{pY} = \begin{bmatrix} r_{1Y} \\ r_{2Y} \\ \vdots \\ r_{pY} \end{bmatrix} \quad (115)$$

$$\bar{r}_{pp} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1p} \\ r_{21} & r_{22} & \cdots & r_{2p} \\ \vdots & \vdots & & \vdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} \end{bmatrix} . \tag{116}$$

One method of solving the system given by equations (112) involves linear transformations and is referred to as the Gaussian Elimination Method [11, 12]. The inverse of the linear correlation matrix $\bar{r}_{pp}$ and the solution vector $\bar{\alpha}_p$ can be obtained from the method. One by-product as applied to the regression analysis is the availability of partial correlation coefficients for the variables not in the regression. Another advantage involves the estimate of the standard deviation of the response variable. Normally this estimate is computed from the summation given by equation (57). A more convenient computational form is given by equation (92). The quantity $(1 - R^2_{Y.12...p})$ in equation (92) for the p variables in the regression is easily obtained when the Gaussian Elimination Method is used.

Basically the underlying principle of the method involves two elementary operations:

(1) multiplication (or division) of any equation (any row) by a nonzero scalar.

(2) addition of one equation (one row) to another.

When these operations are performed on the system of equations (112), a system is obtained which is equivalent to the original system. The basic approach is illustrated by augmenting the correlation matrix $\bar{r}_{pp}$ as follows:

$$\bar{r}_{YY} = \left[\begin{array}{ccccc|cccc} r_{11} & r_{12} & \cdots & r_{1p} & r_{1Y} & 1 & 0 & \cdots & 0 \\ r_{21} & r_{22} & \cdots & r_{2p} & r_{2Y} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ r_{p1} & r_{p2} & \cdots & r_{pp} & r_{pY} & 0 & 0 & \cdots & 1 \\ r_{Y1} & r_{Y2} & \cdots & r_{Yp} & r_{YY} & 0 & 0 & \cdots & 0 \end{array}\right] \tag{117}$$

$$\qquad (p+1) \times (p+1) \qquad\qquad (p+1) \times p$$

By performing the operations (1) and (2) on rows of this matrix, the following arrangement can eventually be obtained:

$$
r_{YY}^{-1} = \left[\begin{array}{cccccc|cccc}
1 & 0 & \cdots & 0 & \alpha_1 & & \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1p} \\
0 & 1 & \cdots & 0 & \alpha_2 & & \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2p} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 1 & \alpha_p & & \tilde{c}_{p1} & \tilde{c}_{p2} & \cdots & \tilde{c}_{pp} \\
0 & 0 & \cdots & 0 & (1 - R^2_{Y.12\ldots p}) & & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_p
\end{array}\right]
$$

$$(p+1) \times (p+1) \qquad\qquad (p+1) \times p$$

(118)

It is seen that appropriate row operations on the $\bar{r}_{pp}$ matrix to the left of the line in equation (117) produces the identity matrix. The same row operations performed on the identity matrix to the right of the line in equation (117) produces the inverse of the $\bar{r}_{pp}$ matrix. The same row operations on the appropriate elements of the (p+ 1) -th column to the left of the line in equation (117) produces the solution vector $\bar{\alpha}_p$ and the quantity $(1 - R^2_{Y.12\ldots p})$.

Elements of the solution vector $\bar{\alpha}_p$ are actually obtained one at a time. This amounts to entering the variables into the regression equation one at a time. Thus, the Gaussian Elimination Method is ideally suited for a stepwise regression analysis where all p of the independent variables may not be desired in the final regression equation. At a given step in the method (before equation (118) is obtained), the array formed by the first p rows and columns contains nonzero off-diagonal elements. These nonzero elements indicate the variables not in the regression. The partial correlation coefficients of the variables, not in regression, with the response can be easily obtained using the appropriate diagonal element in this p x p matrix, the diagonal element in the (p+ 1) x (p+ 1) position, and the appropriate element in the (p+ 1) -th column. This is clarified in the example in Appendix C. It is pointed out to indicate the availability of the partial correlation coefficient required in equation (111) for determining $F_{q(IN)}$. This equation provides the test for determining the variable to next enter the regression and is discussed further in the second example.

The following simple example illustrates the basic principles of the method. Assume the normal equations are given by:

$$\begin{bmatrix} r_{11} & r_{14} \\ r_{41} & r_{44} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} r_{1Y} \\ r_{4Y} \end{bmatrix} . \tag{119}$$

The augmented matrix similar to equation (117) is:

$$\bar{r}_{33} = \left[\begin{array}{ccc|cc} r_{11} & r_{14} & r_{1Y} & 1 & 0 \\ r_{41} & r_{44} & r_{4Y} & 0 & 1 \\ r_{Y1} & r_{Y4} & r_{YY} & 0 & 0 \end{array}\right] . \tag{120}$$

The nondiagonal elements of the first and second columns must be zeroed out. This is accomplished by the following series of operations. Divide the first row by r_{11}, multiply the resulting row by $(-r_{41})$, and add the result to the second row (zeros second element of first column):

$$\bar{r}'_{33} = \left[\begin{array}{ccc|cc} 1 & r'_{14} = r_{14}/r_{11} & r'_{1Y} = r_{1Y}/r_{11} & \tilde{c}'_{11} = 1/r_{11} & \tilde{c}'_{14} = 0 \\ 0 & r'_{44} = r_{44} - \dfrac{r_{41}r_{14}}{r_{11}} & r'_{4Y} = r_{4Y} - \dfrac{r_{41}r_{1Y}}{r_{11}} & \tilde{c}'_{41} = -r_{41}/r_{11} & \tilde{c}'_{44} = 1 \\ r_{Y1} & r_{Y4} & r_{YY} & 0 & 0 \end{array}\right] \tag{121}$$

Multiply the first row of equation (121) by $(-r_{Y1})$ and add the resulting row to the third row (zeros third element of first column):

$$\bar{r}''_{33} = \left[\begin{array}{ccc|cc} 1 & r'_{14} & r'_{1Y} & \tilde{c}'_{11} & \tilde{c}'_{14} \\ 0 & r'_{44} & r'_{4Y} & \tilde{c}'_{41} & \tilde{c}'_{44} \\ 0 & r'_{Y4}=r_{Y4}-\dfrac{r_{Y1}r_{14}}{r_{11}} & r'_{YY}=r_{YY}-\dfrac{r_{Y1}r_{1Y}}{r_{11}} & \tilde{c}'_{Y1}=-r_{41}/r_{11} & \tilde{c}'_{Y4}=0 \end{array}\right] \tag{122}$$

Divide second row of equation (122) by r'_{44}, multiply the resulting row by $(-r'_{14})$, and add the result to the first row (zeros first element of second column):

$$\bar{r}'''_{33} = \left[\begin{array}{cccc|cc} 1 & 0 & r''_{1Y}=r'_{1Y}-\dfrac{r'_{14}r'_{4Y}}{r'_{44}} & & \tilde{c}''_{11}=\tilde{c}'_{11}-\dfrac{r'_{14}\tilde{c}'_{41}}{r'_{44}} & \tilde{c}''_{14}=\dfrac{-r'_{14}\tilde{c}'_{44}}{r'_{44}} \\ 0 & 1 & r''_{4Y}=r'_{4Y}/r'_{44} & & \tilde{c}''_{41}=\tilde{c}'_{41}/r'_{44} & \tilde{c}''_{44}=\tilde{c}'_{44}/r'_{44} \\ 0 & r'_{Y4} & r'_{YY} & & \tilde{c}'_{Y1} & \tilde{c}'_{Y4} \end{array}\right] \tag{123}$$

Multiply the second row of equation (123) by $(-r'_{Y4})$ and add the result to the third row (zeros third element of second column):

$$\bar{r}^{IV}_{33} = \left[\begin{array}{ccc|cc} 1 & 0 & r''_{1Y} & \tilde{c}''_{11} & \tilde{c}''_{14} \\ 0 & 1 & r''_{4Y} & \tilde{c}''_{41} & \tilde{c}''_{44} \\ 0 & 0 & r''_{YY}=r'_{YY}-\dfrac{r'_{Y4}r'_{Y4}}{r'_{44}} & \tilde{c}''_{Y1}=\tilde{c}'_{Y1}-\dfrac{r'_{Y4}\tilde{c}'_{41}}{r'_{44}} & \tilde{c}''_{Y4}=\tilde{c}'_{Y1}-\dfrac{r'_{Y4}\tilde{c}'_{44}}{r'_{44}} \end{array}\right] \tag{124}$$

It can be verified that equation (124) is equivalent to:

$$\bar{r}^{V}_{33} = \left[\begin{array}{ccc|cc} 1 & 0 & \alpha_1 & \tilde{c}_{11} & \tilde{c}_{14} \\ 0 & 1 & \alpha_4 & \tilde{c}_{41} & \tilde{c}_{44} \\ 0 & 0 & (1 - R^2_{Y.14}) & -\alpha_1 & -\alpha_4 \end{array}\right] \tag{125}$$

where:

$$\begin{bmatrix} r_{11} & r_{14} \\ r_{41} & r_{44} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{14} \\ \tilde{c}_{41} & \tilde{c}_{44} \end{bmatrix} \quad . \tag{126}$$

It should be pointed out that the method is reversible. That is, the first and third elements of the fifth column in equation (124) can be zeroed out to obtain equation (122). This is especially useful for variable deletion purposes when it is desired to leave an insignificant variable out of the regression equation. The variable deletion is accomplished by zeroing the appropriate elements in the column to the right of the line. The column selected corresponds to the particular variable being deleted. This is discussed further in the example in Appendix C.

It is thus seen that zeroing a column in the r matrix results in a regression equation with one more variable. Similarly, zeroing a column in the c matrix results in a regression equation with one less variable. At a given step, the last column of the r matrix contains the regression coefficients for the particular variables in the regression. The $\tilde{c}$ matrix contains the inverse of that part of the r matrix corresponding to the variables in the regression at a given step. Specific details in the development of the complete stepwise procedure are given in Appendix C.

Computational Algorithm in the General Stepwise Application

Variable Entry Algorithm. Succeeding elements of the $\bar{r}$ matrix for a variable being entered or deleted from the regression can be generated from general algorithms. Let r'_{ij} and $\tilde{c}'_{ij}$ denote elements of the current matrix. If K denotes the independent variable being entered, then the new r''_{ij} elements can be obtained from the following equations:

$$\left.\begin{array}{ll} \left.\begin{array}{l} r''_{KK} = r'_{KK}/r'_{KK} = 1 \;;\; i = j = K \\ r''_{Kj} = r'_{Kj}/r'_{KK} \;;\; i = K,\ j \neq K \end{array}\right\} & \text{elements of K-th row} \\ r''_{ij} = r'_{ij} - (r'_{iK}\, r'_{Kj})/\, r'_{KK} \;; & \text{all other elements} \end{array}\right\} . \qquad (127)$$

The new c''_{ij} elements can be obtained from:

$$\left.\begin{array}{ll} \left.\begin{array}{l} \tilde{c}''_{KK} = \tilde{c}'_{KK}/r'_{KK};\; i = j = K \\ \tilde{c}''_{Kj} = \tilde{c}'_{Kj}/r'_{KK} \;;\; i = K,\ j \neq K \end{array}\right\} & \text{elements of K-th row} \\ \tilde{c}''_{ij} = \tilde{c}'_{ij} - (r'_{iK}\, \tilde{c}'_{Kj})/\, r'_{KK}; & \text{all other elements} \end{array}\right\} . \qquad (128)$$

Variable Deletion Algorithm. An independent variable can be deleted from the regression using somewhat similar algorithms. Assume the current matrix consists of the r''_{ij} and $\tilde{c}''_{ij}$ elements. If the K-th independent variable is to be deleted, then the new $\tilde{r}'''_{ij}$ elements can be obtained from:

$$\left.\begin{array}{ll}\left.\begin{array}{l} r'''_{KK} = r''_{KK}/\tilde{c}''_{KK}\ ;\ i = j = K \\ r'''_{Kj} = r''_{Kj}/\tilde{c}''_{KK}\ ;\ i = K,\ j \neq K \end{array}\right\} & \begin{array}{l}\text{elements of}\\ \text{K-th row}\end{array} \\ r'''_{ij} = r''_{ij} - (\tilde{c}''_{iK}\, r''_{Kj})/\tilde{c}''_{KK} & \begin{array}{l}\text{all other}\\ \text{elements}\end{array} \end{array}\right\} . \tag{129}$$

The new $\tilde{c}'''_{ij}$ elements can be obtained from:

$$\left.\begin{array}{ll}\left.\begin{array}{l} \tilde{c}'''_{KK} = \tilde{c}''_{KK}/\tilde{c}''_{KK} = 1\ ;\ i = j = K \\ \tilde{c}'''_{Kj} = \tilde{c}''_{Kj}/\tilde{c}''_{KK}\ ;\ i = K,\ j \neq K \end{array}\right\} & \begin{array}{l}\text{elements of}\\ \text{K-th row}\end{array} \\ \tilde{c}'''_{ij} = \tilde{c}''_{ij} - (\tilde{c}''_{iK}\, \tilde{c}''_{Kj})/\tilde{c}''_{KK} & \begin{array}{l}\text{all other}\\ \text{elements}\end{array} \end{array}\right\} . \tag{130}$$

The algorithms given by equations (127) through (130) result from a generalization of the equations in the Gaussian Elimination Procedure used for solving the normal equations.

Compact Computational Algorithm for Variable Entry and Deletion. The order of the augmented matrix at a given step is $(p+1) \times (2p+1)$. It is noted, however, that p of the columns contain zeros and the unity element. For every column zeroed in the r matrix, the corresponding column in the $\tilde{c}$ matrix contains nonzero elements. The nonzero columns of the $\tilde{c}$ matrix can be placed in the r positions corresponding to the column in the r matrix that is zeroed out. If this is done from the first step on, then an r matrix is set up that can be computed using a compact computational algorithm since the other elements are zero and unity and can be generated as needed. For example, in equation C-2 of Appendix C, the elements in the $\tilde{c}'_{i4}$ column would be placed in the corresponding positions in the r'_{i4} column. The i-th element $\tilde{c}'_{i4}$ would then be designated as r'_{i4}. The complete r' matrix for the variable Z_4 in the regression would then be given by:

$$
r' = \begin{array}{c}
\begin{array}{ccccc}
\underline{r'_{i1}} & \underline{r'_{i2}} & \underline{r'_{i3}} & \underline{r'_{i4}} & \underline{r'_{iY}}
\end{array} \\
\begin{bmatrix}
r_{11}-\frac{r_{14}r_{41}}{r_{44}} & r_{12}-\frac{r_{14}r_{42}}{r_{44}} & r_{13}-\frac{r_{14}r_{43}}{r_{44}} & -r_{14}/r_{44} & r_{1Y}-\frac{r_{14}r_{4Y}}{r_{44}} \\
r_{21}-\frac{r_{24}r_{41}}{r_{44}} & r_{22}-\frac{r_{24}r_{42}}{r_{44}} & r_{23}-\frac{r_{24}r_{43}}{r_{44}} & -r_{24}/r_{44} & r_{2Y}-\frac{r_{24}r_{4Y}}{r_{44}} \\
r_{31}-\frac{r_{34}r_{41}}{r_{44}} & r_{32}-\frac{r_{34}r_{42}}{r_{44}} & r_{33}-\frac{r_{34}r_{43}}{r_{44}} & -r_{34}/r_{44} & r_{3Y}-\frac{r_{34}r_{4Y}}{r_{44}} \\
r_{41}/r_{44} & r_{42}/r_{44} & r_{43}/r_{44} & 1/r_{44} & r_{4Y}/r_{44} \\
r_{Y1}-\frac{r_{Y4}r_{41}}{r_{44}} & r_{Y2}-\frac{r_{Y4}r_{42}}{r_{44}} & r_{Y3}-\frac{r_{Y4}r_{43}}{r_{44}} & -r_{Y4}/r_{44} & r_{YY}-\frac{r_{Y4}r_{4Y}}{r_{44}}
\end{bmatrix}
\end{array} . \tag{131}
$$

The r'_{ij} elements of this matrix can be computed from the following algorithms with $K = 4$:

$$
\left.
\begin{array}{lll}
r'_{KK} = 1/r_{KK}\,;\ i = j = k & & \text{(intersection of row K and column K)} \\
r'_{Kj} = r_{Kj}/r_{KK}\,;\ i = K,\ j \neq K & & \text{(other elements of row K)} \\
r'_{iK} = -r_{iK}/r_{KK}\,;\ i \neq K,\ j = K & & \text{(other elements of column K)} \\
r'_{ij} = r_{ij} - (r_{iK}\, r_{Kj})/r_{KK};\ i \neq K,\ j \neq K & & \text{(all others)}
\end{array}
\right\} . \tag{132}
$$

At a given step a succeeding matrix (that is, one obtained by entering or deleting a variable) can be generated using expressions analogous to equations (132). The subscript K corresponds to the variable being entered or deleted. Note, however, that from the first step on, the $\tilde{c}$ elements must be designated as r elements in the manner discussed in the previous example. The algorithm given by equation (132) for entering or deleting a variable gives the same results as equations (127) and (128) for variable entry and equations (129) and (130) for variable deletion. The obvious advantage in using equations (132) is that it involves handling a (p+1) x (p+1) matrix rather than a (p+1) x (2p+1) augmented matrix.

RESULTS FROM THE APOLLO-SATURN AS-501 VEHICLE FLIGHT TEST

Introduction

The Apollo-Saturn AS-501 vehicle was launched at 0700:01 Eastern Standard Time on November 9, 1967, from KSC Launch Complex 39, Pad A. This section presents results from application of the TEMS Method and the stepwise regression procedure to tracking data obtained from this launch. Tracking data from seven C-band radars providing coverage on the first burn flight phase (launch to parking orbit insertion) and four providing coverage on the second burn flight phase (S-IVB reignition to S-IVB/CSM separation) were used in the reduction. The postflight reference trajectory used as the standard is presented in Reference 13.

The relation between the vehicle trajectory for the first burn flight phase and the various C-band radar tracking sites is shown in Figure 2. The overall first burn tracking data utilization and significant event times are also shown in this figure. The second burn flight phase summary with the overall tracking data utilization and significant event times is shown in Figure 3. The specific tracking data utilization for the first and second burn flight phases is shown in Figures 4 and 5. These two figures show the individual time spans of usable data as determined from an edit pass through the TEMS program. Location data for the launch site and the various tracking stations are given in Table III. The minimum elevation angles corresponding to tracking coverage at the begin and end times are given in Table IV.

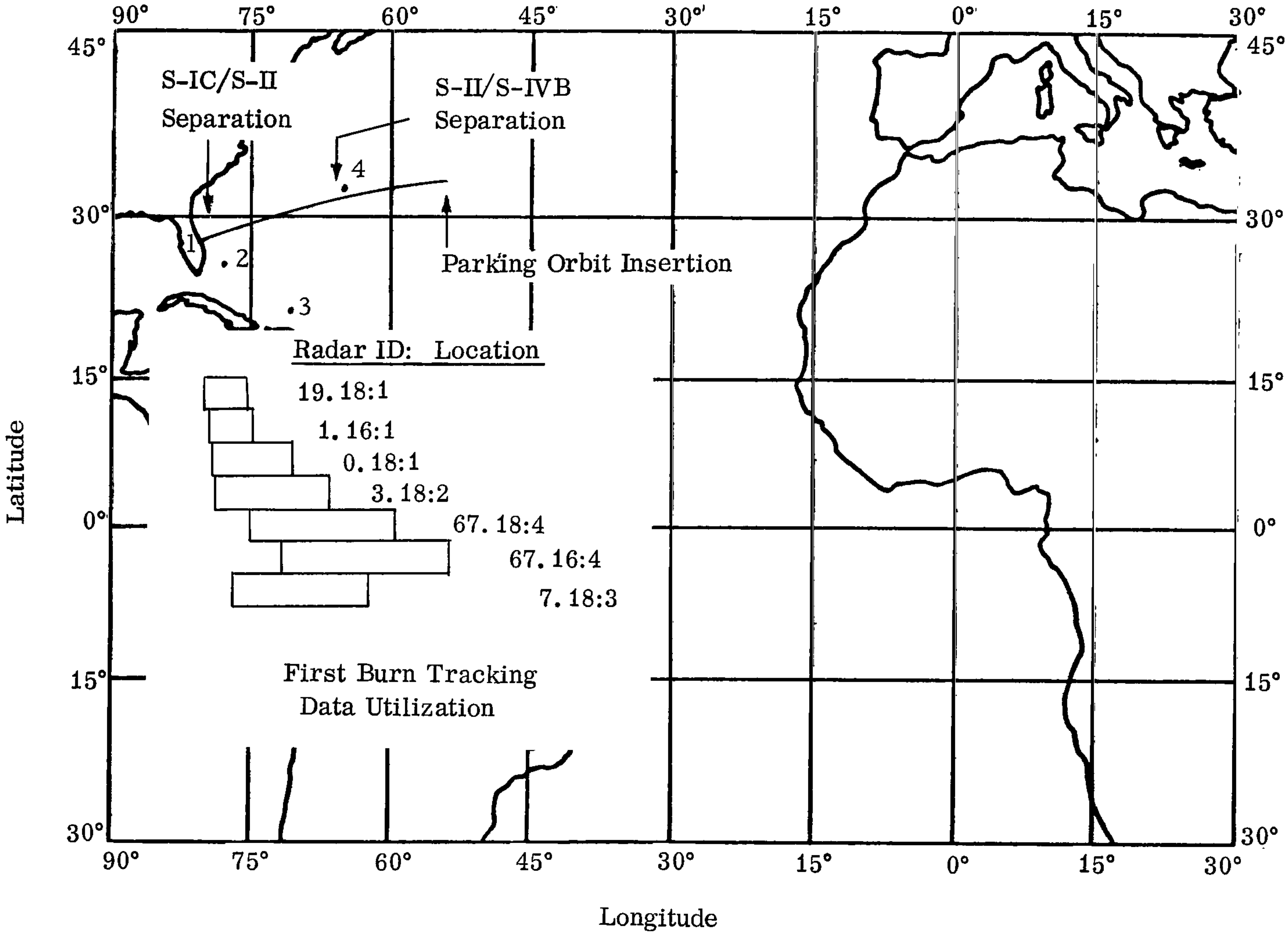

FIGURE 2. AS-501 FIRST BURN FLIGHT PHASE AND TRACKING STATION DATA

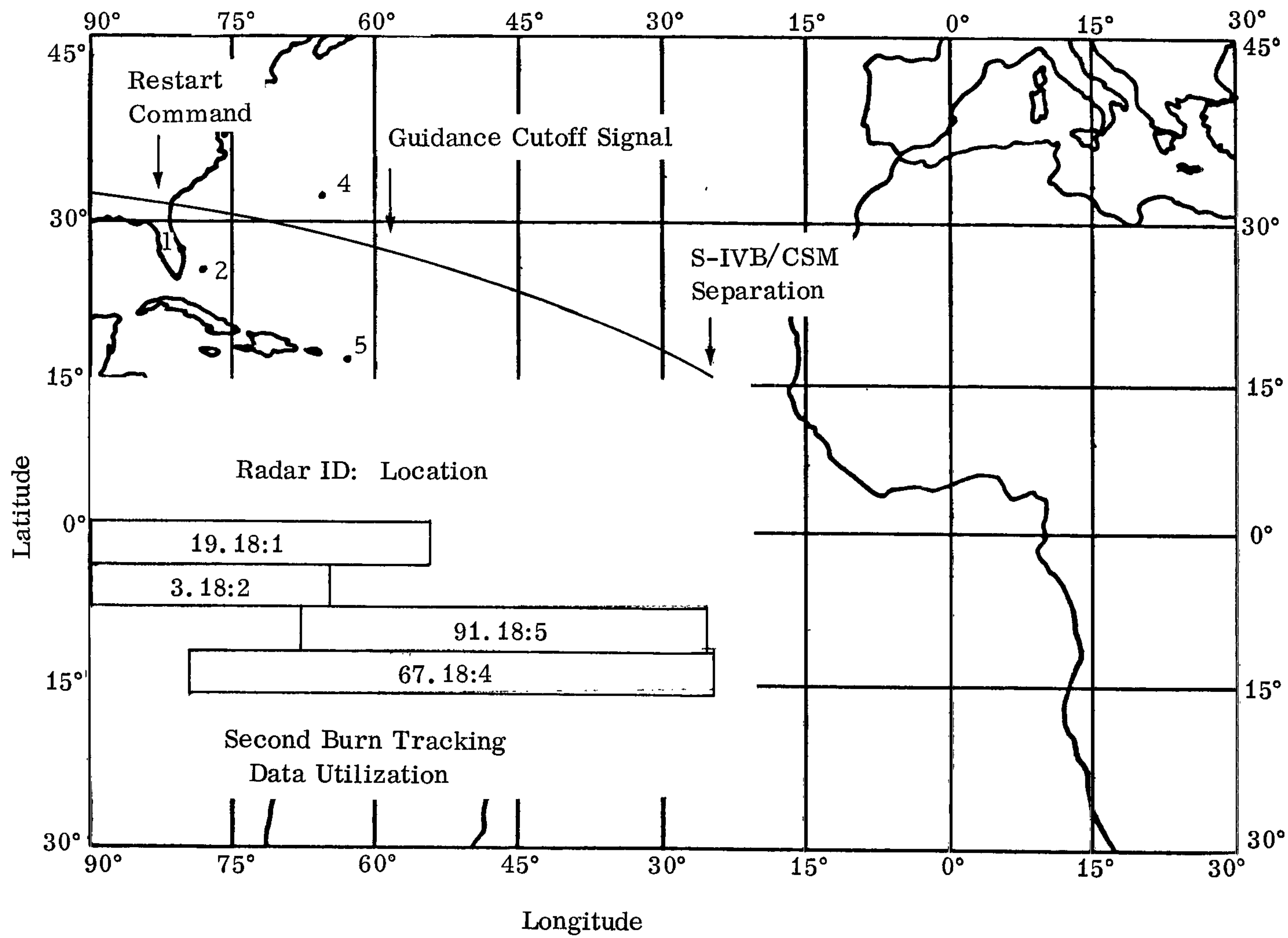

FIGURE 3. AS-501 SECOND BURN FLIGHT PHASE AND TRACKING STATION DATA

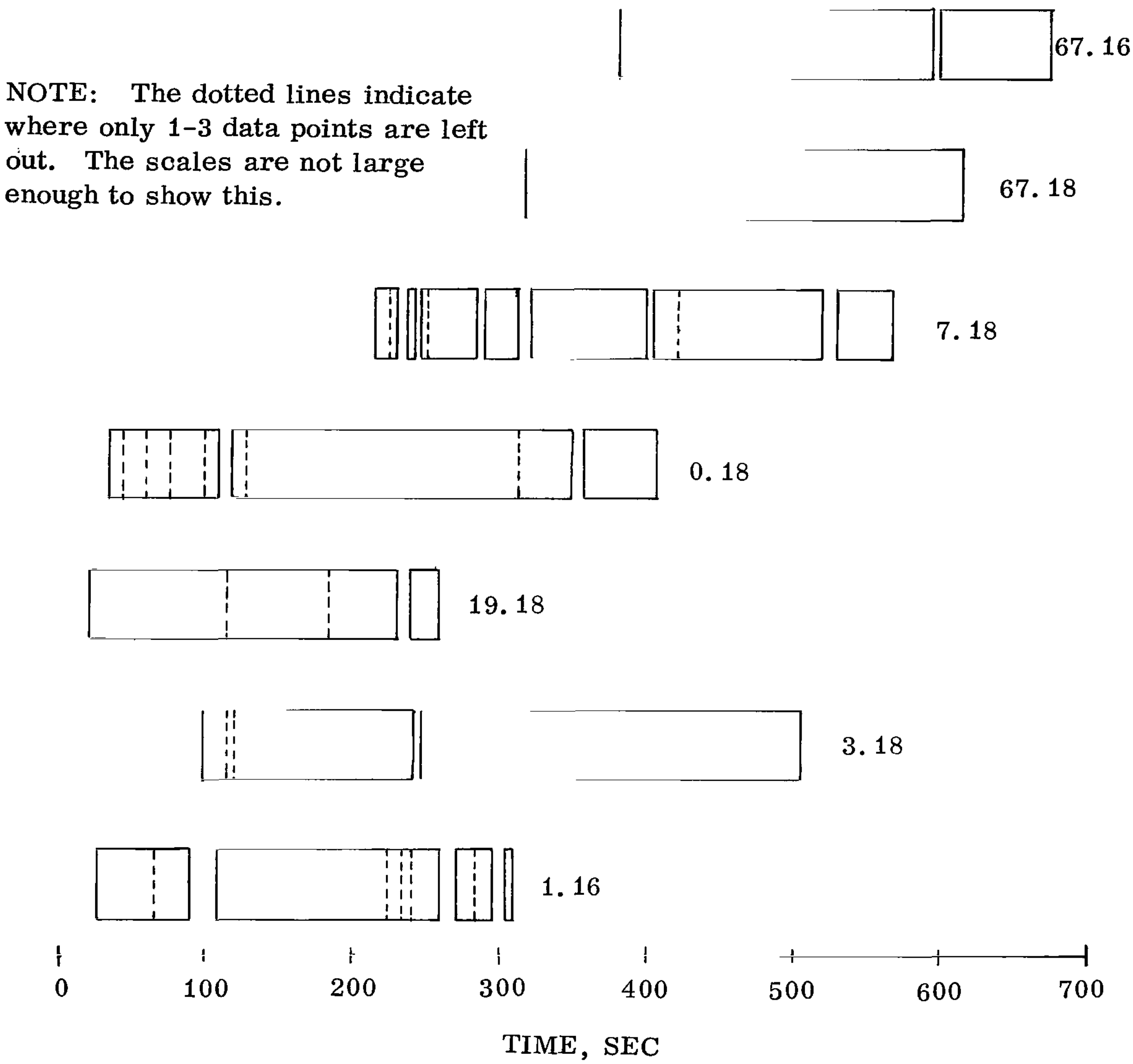

FIGURE 4. TEMS AS-501 FIRST BURN RADAR TRACKING DATA UTILIZATION

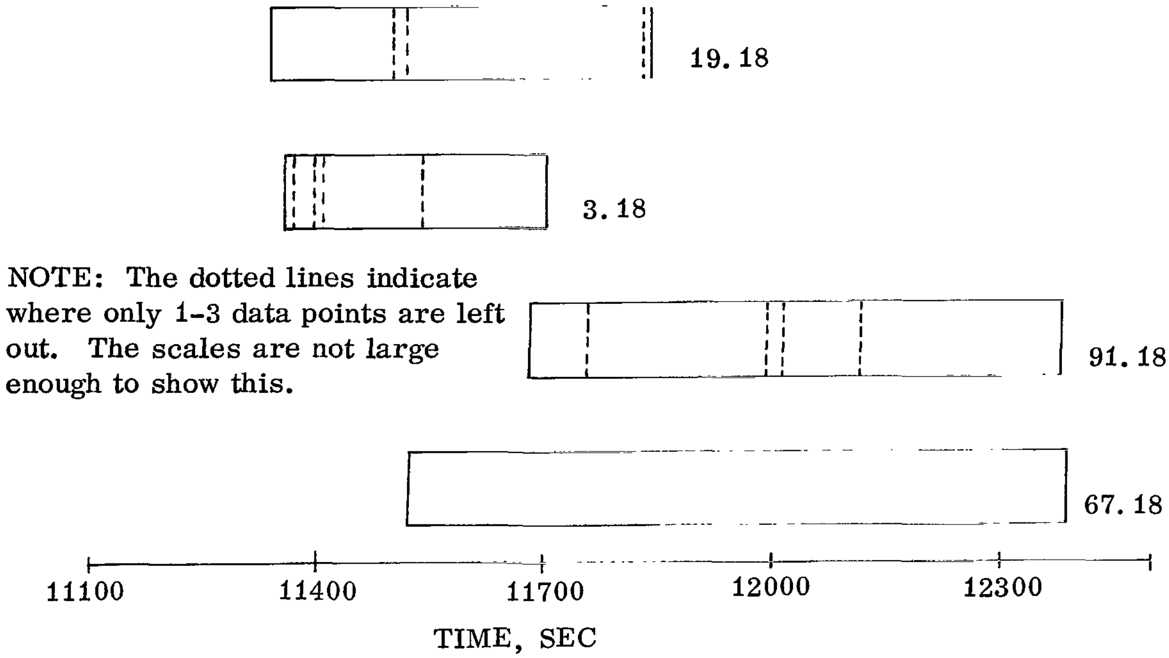

FIGURE 5. TEMS AS-501 SECOND BURN RADAR TRACKING DATA UTILIZATION

TABLE III. LOCATION OF LAUNCH SITE AND C-BAND TRACKING RADARS USED IN TEMS AS-501 REDUCTION

Site	Latitude, deg	Longitude, deg	Height,[a] m
Launch Complex - 39, Pad A	28.608422	80.604133	111.58[b]
Patrick Radar (0.18)	28.226553	80.599293	15.51
Merritt Island Radar (19.18)	28.424862	80.664404	12.02
Grand Bahama Radar (3.18)	26.636350	78.267708	12.05
Grand Turk Radar (7.18)	21.462890	71.132114	28.45
67.16 (FPS-16)	32.348103	64.653801	24.31
67.18 (FPQ-6)	32.347964	64.653742	25.51
Cape Kennedy (1.16)	28.481766	80.576515	14.40
Antigua Radar (91.18)	17.144032	61.792859	49.37

a. Elevation above the Fischer Ellipsoid

b. Elevation of the C-band radar antenna above the Fischer Ellipsoid

TABLE IV. ELEVATION ANGLES AT BEGIN AND END TIMES FOR FIRST AND SECOND BURN DATA ON AS-501

Radar	First Burn		Second Burn	
	Time, sec	E°, deg	Time, sec	E°, deg
0.18	35. 407.	2.53 7.38	——	——
19.18	22. 257.	2.07 19.63	11338. 11840.	2.61 2.85
3.18	99. 506.	2.89 2.75	11364. 11705.	1.31 10.07
7.18	215. 567.	0.6 0.9	——	——
67.16	379. 675.	8.29 7.25	——	——
67.18	316. 614.	3.87 18.25	11521. 12386.	1.61 4.18
1.16	26. 309.	4.27 15.29	——	——
91.18	——	——	11684. 12380.	5.69 9.26

Detailed information on the IBM 7094 computer program for application of the TEMS method is given in Appendix D. The computer program for application of the stepwise regression analysis procedures developed earlier in this report is discussed in Appendix E. The utilization of these two programs on the AS-501 first and second burn data is illustrated in Figure 6.

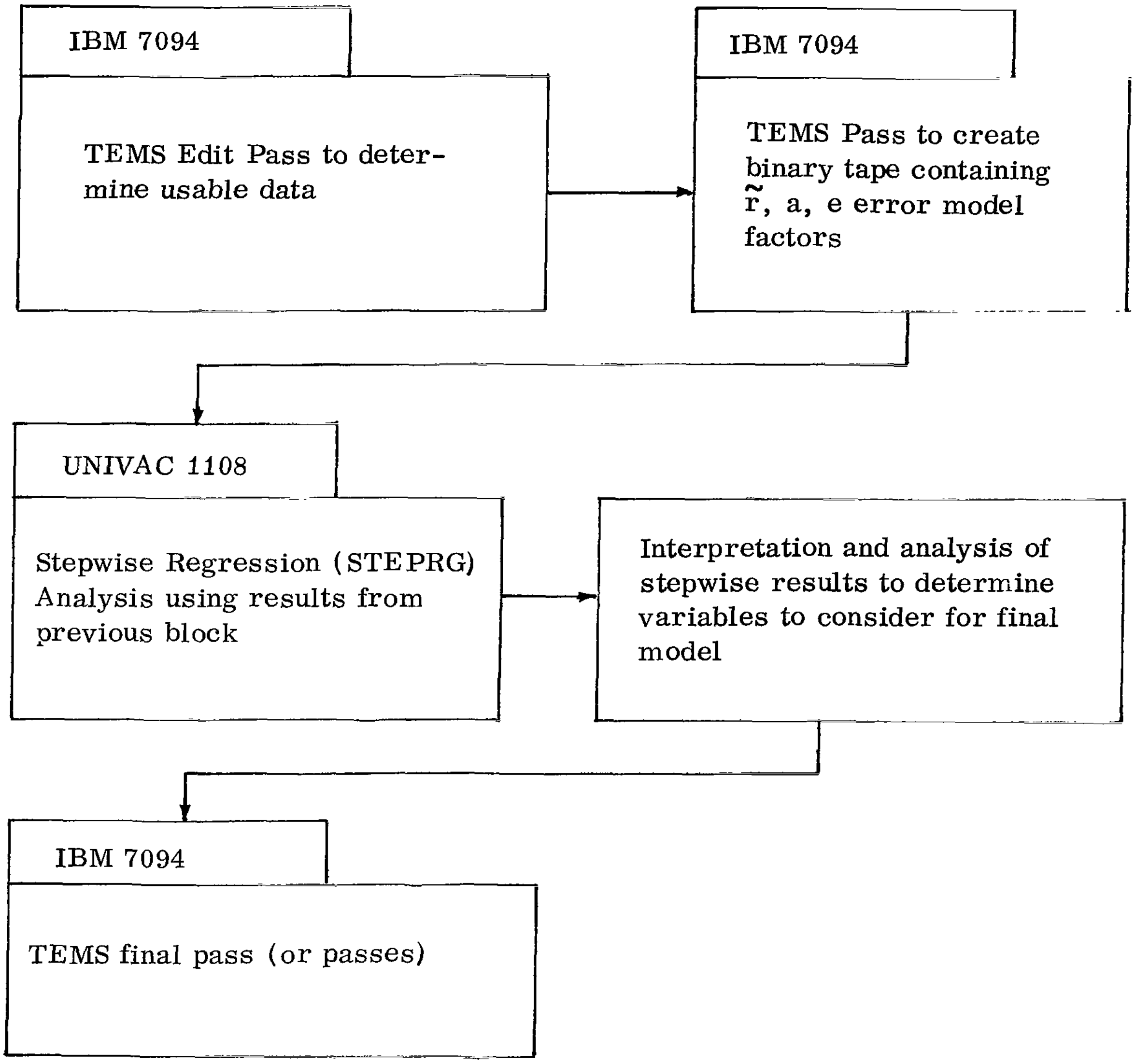

FIGURE 6. UTILIZATION OF THE TEMS AND STEPRG COMPUTER PROGRAMS

It should be pointed out that a certain amount of caution has been used in the interpretation and analysis of the stepwise regression results. The stepwise regression results in Appendix F indicate several cases where the σ_Y curve fit value is not improved significantly by the introduction of additional variables into the regression. The F_{IN} and F_{OUT} levels selected for variable entry and deletion are also critical in the number of variables entered into the regression model. A value of 3.5 for F_{IN} and F_{OUT} was arbitrarily selected and used in obtaining the stepwise results in Appendix C. A higher value would reduce the number of steps and result in a final regression model with a lower number of variables. A rather critical examination of the stepwise regression analysis results thus seems to be required in order to obtain meaningful and useful information for input to the TEMS program.

Truncated Error Model Regression Analysis Results

The general approach for obtaining truncated error models to describe the AS-501 range, azimuth, and elevation response variables is summarized in the following guidelines:

(1) It was assumed that the survey terms, the rate bias term, and the azimuth and elevation velocity lag terms, were not essential in obtaining truncated error models to describe the response variables.

(2) The first two variables entered in the stepwise regression (excluding those left out under the above assumption) were selected for consideration in the final TEMS error model.

(3) A third variable was considered if an adequate description of the response variable was not obtained with the first two or if a constraining condition required an additional variable in the model.

This approach actually results in entering the most significant variables into the error model. It should be pointed out that the third variable selected in guideline (3) often involved selecting one of two variables that represented borderline cases so far as the order of entry in the stepwise regression was concerned; i.e., the two variables had ρ^2 values nearly equal.

The AS-501 first and second burn truncated error model results obtained using guidelines (1) - (3) are presented in Tables V through VIII. Plots of the observed and computed response variables and the least squares residuals for the truncated models are given in Appendix F. This appendix also includes a summary of the stepwise regression analysis results. Coefficient correlations from the TEMS program for the truncated models are also presented. The average standard deviations of the least squares residuals in Tables V and VII

TABLE V. TRUNCATED RADAR ERROR MODEL MULTIPLE REGRESSION RESULTS FOR FIRST BURN DATA ON AS-501 VEHICLE FLIGHT TEST

Radar	Coefficient											σ_{VR}	σ_{VA}	σ_{VE}	No. of
	C_0	C_1	C_2	C_4	D_0	D_3	D_5	D_7	D_8	F_0	F_3	m	Deg	Deg	Data Points
0.18	-19.92	—	0.0091	23.71	0.0087	0.6915	—	-0.0202	—	0.0194	0.1791	3.96	0.0082	0.0072	335
19.18	-18.11	—	0.0055	-36.03	0.72E-3	—	0.0697	-0.0761	—	0.0330	-0.4390	5.23	0.0046	0.0062	219
3.18	5.21	—	0.0066	93.25	0.0054	0.5517	—	—	—	-0.84E-3	2.10	4.02	0.0027	0.0055	395
7.18	-12.28	—	0.0024	36.20	-0.0176	-2.84	—	—	—	-0.0085	—	6.09	0.0038	0.0165	297
67.16	58.47	-1.14E-4	-0.0022	—	0.34E-3	0.1632	—	—	0.0083	0.0073	0.2380	9.75	0.0097	0.0051	289
67.18	84.34	-0.97E-4	-0.0049	—	0.0056	0.0192	—	—	0.0067	0.0021	-0.0027	9.16	0.0045	0.0057	297
1.16	-36.91	-0.59E-4	0.0171	—	0.0171	0.3386	—	-0.0177	—	0.0042	—	4.31	0.0121	0.0106	225
											Average σ	6.07	0.0065	0.0081	

TABLE VI. COEFFICIENT STANDARD DEVIATIONS FOR TRUNCATED RADAR ERROR MODELS FOR FIRST BURN DATA ON AS-501 VEHICLE FLIGHT TEST

Radar	σ_K For Indicated Coefficient											Terms
	C_0	C_1	C_2	C_4	D_0	D_3	D_5	D_7	D_8	F_0	F_3	
0.18	0.84	—	0.25E-3	5.21	0.54E-3	0.074	—	0.0015	—	0.93E-3	0.100	8
19.18	0.72	—	0.33E-3	3.75	0.0010	—	0.0024	0.0016	—	0.0010	0.050	8
3.18	0.36	—	0.11E-3	2.27	0.23E-3	0.086	—	—	—	0.24E-3	0.173	7
7.18	0.95	—	0.30E-3	1.97	0.64E-3	0.971	—	—	—	0.61E-3		6
67.16	1.15	0.21E-5	0.09E-3	—	0.56E-3	0.005	—	—	0.48E-3	0.54E-3	0.014	8
67.18	0.84	0.15E-5	0.10E-3	—	0.46E-3	0.004	—	—	0.43E-3	0.48E-3	0.012	8
1.16	1.03	0.67E-5	0.89E-3	—	1.14E-3	0.073	—	0.0025	—	1.57E-3		7
No. Occurrences	7	3	7	4	7	6	1	3	2	7	5	

TABLE VII. TRUNCATED RADAR ERROR MODEL MULTIPLE REGRESSION RESULTS FOR SECOND BURN DATA ON AS-501 VEHICLE FLIGHT TEST

Radar	Coefficient									σ_{VR} m	σ_{VA} Deg	σ_{VE} Deg	No. of Data Points
	C_0	C_1	C_2	D_0	D_3	D_7	D_8	F_0	F_3				
19.18	-25.89	-73E-5	-0.0059	-0.86E-3	-0.0117	-0.0223	——	0.0018	0.4277	7.30	0.0050	0.0073	492
3.18	8.46	——	-0.0061	0.0091	0.2989	——	——	-0.0182	2.5233	7.16	0.0038	0.0055	322
91.18	34.84	-2.43E-5	0.0043	0.0032	0.4009	——	——	-0.7E-4	-4.1971	2.22	0.0038	0.0059	684
67.18	8.83	-2.87E-5	-0.0011	0.0055	——	——	0.0012	-0.0116	——	7.28	0.0053	0.0058	864
									Average σ	5.99	0.0045	0.0061	

TABLE VIII. COEFFICIENT STANDARD DEVIATIONS FOR TRUNCATED RADAR ERROR MODELS FOR SECOND BURN DATA ON AS-501 VEHICLE FLIGHT TEST

Radar	σ_K For Indicated Coefficient									
	C_0	C_1	C_2	D_0	D_3	D_7	D_8	F_0	F_3	Terms
19.18	0.60	0.05E-5	0.52E-4	0.33E-3	0.057	0.0012	——	0.58E-3	0.157	8
3.18	0.29	——	0.55E-4	0.35E-3	0.075	——	——	0.36E-3	0.256	6
91.18	0.35	0.02E-5	0.61E-4	0.18E-3	0.201	——	——	0.18E-3	0.434	7
67.18	0.36	0.01E-5	0.43E-4	0.24E-3	——	——	0.29E-3	0.27E-3	——	6
No. Occurrences	4	3	4	4	3	1	1	4	3	

indicate fairly close agreement with the accuracy estimates of 5 meters in range and 0.006 degrees in azimuth and elevation. The σ_{VE} value of 0.0165 degrees in Table V for Radar 7.18 probably results from roughness in the data associated with the low elevation coverage ($E^\circ \leq 5.33^\circ$).

CONCLUSIONS

The TEMS Multiple Regression Analysis Method provides for a comprehensive evaluation of systematic errors in measurements obtained from various tracking systems. The error model equations used in the development of the method are for C-band radar tracking systems. It should be noted, however, that the development for application to tracking systems other than radars is analogous to that presented herein. Truncated tracker error models for representing the systematic errors are established using the TEMS method in conjunction with a stepwise regression procedure. The stepwise procedure involves examining at every step the variables incorporated into the error model in previous steps. A specific variable is deleted from or entered into the model by using the Gaussian Elimination Method for solving the linear system of normal equations. The procedure shows considerable promise in solving the TEMS error model construction problem.

The approach given by guidelines (1) - (3) in the last section for obtaining truncated error models to describe the systematic errors has generally resulted in acceptable models for the AS-501 first and second burn data. Guideline (2) uses the stepwise regression procedure to determine the variables for consideration in the truncated TEMS error models. It is noted that the average random errors remaining in the first and second burn residuals for the truncated error models indicate fairly close agreement with the input accuracy estimates of 5 meters in range, and 0.006 degrees in azimuth and elevation. The stepwise results on the AS-501 data indicate several cases where the σ_Y curve fit value is not improved significantly by the introduction of additional variables into the regression. It appears that a rather critical examination of results from applications of the stepwise regression procedure is required in order to obtain meaningful and useful information for input to the TEMS program.

The scope of future work will be concerned with the reduction and analysis of tracking data on each Saturn IB and Saturn V flight tests. A comprehensive utilization of the stepwise regression analysis results will be stressed in this work.

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama, June 20, 1968
933-50-04-00-62

APPENDIX A

THE C-BAND RADAR TRACKING SYSTEM ERROR MODELS

The Basic Error Models

The basic radar error model equations [2] in range, azimuth, and elevation are given by:

Range

$$\Delta R = C_0 + C_1 R + C_2 \dot{R} + C_3 t + C_4 (-.022 \operatorname{cosec} E) + C_5 \left(\frac{X}{R}\right) + C_6 \left(\frac{Y}{R}\right) + C_7 \left(\frac{Z}{R}\right) \tag{A-1}$$

Azimuth

$$\Delta A = D_0 + D_1 \dot{A} + D_3 \ddot{A} + D_5 \tan E + D_6 \sec E + D_7 \tan E \sin A + D_8 \tan E \cos A + D_9 \left(\frac{\sin A \cos A}{X}\right) + D_{10} \left(-\frac{\sin A \cos A}{Y}\right) + D_{11} \dot{A} \sec E \tag{A-2}$$

Elevation

$$\Delta E = F_0 + F_1 \dot{E} + F_3 \ddot{E} + F_5 (-\sin A) + F_6 \cos A + F_7 \left[\left(\frac{.022}{R \sin E} - 10^{-6}\right) \operatorname{cotan} E\right] + F_9 \left(\frac{-X \tan E}{R^2}\right) + F_{10}\left(\frac{-Y \tan E}{R^2}\right) + F_{11}\left(\frac{\cos E}{R}\right) + F_{12} \dot{E} \cos E \tag{A-3}$$

The error model terms appearing in equations (A-1), (A-2), and (A-3) are subject to specific physical interpretations. These interpretations are listed below:

<u>In ΔR</u>

(1) C_0 - Zero set (bias) error which arises because of inaccuracies in calibration.

(2) C_1R - Range is obtained by multiplying the measured quantity by a scale factor. This is error due to error in the scale factor.

(3) $C_2\dot{R}$ - Error due to timing delays and is essentially the error in station time clock.

(4) C_3t - Rate bias error due to linear drift of oscillator frequency with time.

(5) $C_4(-0.022 \text{ cosec } E)$ - Error in correction made for tropospheric refraction.

(6) $C_5(X/R)$, $C_6(Y/R)$, $C_7(Z/R)$ - Survey errors in the radar site location.

<u>In ΔA</u>

(1) D_0 - Same interpretation as for C_0.

(2) $D_1\dot{A}$ - Same interpretation for $C_2\dot{R}$.

(3) $D_3\ddot{A}$ - Error due to dynamic lag in azimuth serve system.

(4) $D_5 \tan E$ - Error which accounts for the nonparallellism between the elevation shaft and the plane of the azimuth table.

(5) $D_6 \sec E$ - Error due to nonperpendicularity of the R-F axis and elevation shaft.

(6) $D_7 \tan E \sin A$, $D_8 \tan E \cos A$ - Errors in azimuth due to azimuth plane tilt about orthogonal axes in the azimuth plane.

(7) $D_9(\sin A \cos A/X)$, $D_{10}(-\sin A \cos A/Y)$ - Errors in azimuth due to north and east components of survey error.

(8) $D_{11} \dot{A} \sec E$ - Azimuth velocity lag.

<u>In ΔE</u>

(1) F_0 - Same interpretation as for C_0.

(2) $F_1\dot{E}$ - Same interpretation as for $C_2\dot{R}$.

(3) $F_3\ddot{E}$ - Error due to dynamic lag in elevation servo system.

(4) $F_5\,(-\sin A)$, $F_6 \cos A$ - Error in elevation caused by azimuth plane tilt - same interpretation as for D_7, D_8.

(5) $F_7 \left[\left(\frac{0.022}{R \sin E} - 10^{-6}\right) \cot E\right]$ - Error in correction made for tropospheric refraction.

(6) $F_9(-X \tan E/R^2)$, $F_{10}(-Y \tan E/R^2)$, $F_{11}(\cos E/R)$, - Survey errors in radar site location.

(7) $F_{12}\, \dot{E} \cos E$ - Elevation velocity lag.

Derivation of Survey Error Terms in Range Error Model

The error model equations contain terms which can, theoretically, be determined on the basis of physically established relationships. For example, consider the survey error coefficients C_5, C_6, and C_7 in the range error model equation. Let C_5 denote the error in the X direction, C_6 the error in Y direction, and C_7 the error in the Z direction. Let the correctly measured range be given by:

$$R_m = (X^2 + Y^2 + Z^2)^{\frac{1}{2}} \quad . \tag{A-4}$$

If the radar site location is displaced by the distances ΔX, ΔY, and ΔZ in the respective coordinates, then the range from the site location $(X + \Delta X,\ Y + \Delta Y,\ Z + \Delta Z)$ is given by:

$$R = [(X + \Delta X)^2 + (Y + \Delta Y)^2 + (Z + \Delta Z)^2]^{\frac{1}{2}} \quad . \tag{A-5}$$

Expanding (A-5):

$$R = [X^2 + Y^2 + Z^2 + 2 (X \Delta X + Y \Delta Y + Z \Delta Z)$$

$$+ (\Delta X^2 + \Delta Y^2 + \Delta Z^2)]^{\frac{1}{2}} \quad . \tag{A-6}$$

For the application herein, the last three terms in equation (A-6) are small compared to the terms $(X^2 + Y^2 + Z^2)$ and $2(X \Delta X + Y \Delta Y + Z \Delta Z)$. Thus, these terms can be neglected and equation (A-6) written as:

$$R = [X^2 + Y^2 + Z^2 + 2 (X \Delta X + Y \Delta Y + Z \Delta Z)]^{\frac{1}{2}} \quad . \tag{A-7}$$

Expanding the right side of equation (A-7) in the binomial series and retaining the first two terms yields:

$$R = (X^2 + Y^2 + Z^2)^{\frac{1}{2}} + \frac{X \Delta X + Y \Delta Y + Z \Delta Z}{(X^2 + Y^2 + Z^2)^{\frac{1}{2}}} \quad . \tag{A-8}$$

The error in R is then given by:

$$\Delta R = R - R_m \quad . \tag{A-9}$$

Substituting equations (A-8) and (A-4) in (A-9):

$$\Delta R = \left(\frac{X}{R_m}\right) \Delta X + \left(\frac{Y}{R_m}\right) \Delta Y + \left(\frac{Z}{R_m}\right) \Delta Z \tag{A-10}$$

where

$$\Delta X = C_5$$

$$\Delta Y = C_6$$

$$\Delta Z = C_7 \quad .$$

APPENDIX B

COORDINATE SYSTEM TRANSFORMATIONS

The reference trajectory used as the standard in the TEMS method is in an earth-fixed plumbline coordinate system (X_e, Y_e, Z_e) with origin at the launch site. It is required to transform the (X_e, Y_e, Z_e) reference trajectory data into radar reference tracking parameters (R^r, A^r, E^r). The radar measured tracking parameters can then be compared with the radar reference tracking parameters to establish the tracking errors for the particular system under consideration. There are two transformations required before this comparison can be made. These are:

(1) Transformation of earth-fixed plumbline coordinates (X_e, Y_e, Z_e) of vehicle with origin at the launch site to earth-fixed plumbline coordinates (X_{es}, Y_{es}, Z_{es}) of vehicle with origin at the tracking site.

(2) Transformation of earth-fixed plumbline coordinate (X_{es}, Y_{es}, Z_{es}) of vehicle with origin at tracking site to spherical coordinates (R^r, A^r, E^r) of vehicle with origin at tracking site.

The first of these transformations for the i-th observation is given by the following equation:

$$\bar{X}_{es_i} = \bar{K}_T^T \, \bar{\Phi}_T^T \, \bar{\lambda}_T^T \, \bar{\lambda}_L \, \bar{\Phi}_L \, \bar{K}_L (\bar{X}_{e_i} + \bar{r}_{OL}) - \bar{r}_{OT} \qquad \text{(B-1)}$$

where

$$\bar{X}_{es_i} = \begin{bmatrix} X_{es} \\ Y_{es} \\ Z_{es} \end{bmatrix}_i \qquad \text{(B-2)} \qquad\qquad \bar{X}_{e_i} = \begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix}_i \qquad \text{(B-3)}$$

$$\bar{r}_{OL} = \begin{bmatrix} -r_L \sin B_L \cos K_L \\ r_L \cos B_L + h_L \\ r_L \sin B_L \sin K_L \end{bmatrix} \quad (B-4) \qquad \bar{r}_{OT} = \begin{bmatrix} -r_T \sin B_T \cos K_T \\ r_T \cos B_T + h_T \\ r_T \sin B_T \sin K_T \end{bmatrix} \quad (B-5)$$

$$\bar{K}_L = \begin{bmatrix} \sin K_L & 0 & \cos K_L \\ 0 & 1 & 0 \\ -\cos K_L & 0 & \sin K_L \end{bmatrix} \quad (B-6) \qquad \bar{\Phi}_L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos \Phi_L & -\sin \Phi_L \\ 0 & \sin\Phi_L & -\cos \Phi_L \end{bmatrix} \quad (B-7)$$

$$\bar{\lambda}_L = \begin{bmatrix} \sin \lambda_L & -\cos \lambda_L & 0 \\ \cos \lambda_L & \sin \lambda_L & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (B-8) \qquad \bar{K}_T = \begin{bmatrix} \sin K_T & 0 & \cos K_T \\ 0 & 1 & 0 \\ -\cos K_T & 0 & \sin K_T \end{bmatrix} \quad (B-9)$$

$$\bar{\Phi}_T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos \Phi_T & -\sin \Phi_T \\ 0 & \sin \Phi_T & -\cos \Phi_T \end{bmatrix} \quad (B-10) \qquad \bar{\lambda}_T = \begin{bmatrix} \sin \lambda_T & -\cos \lambda_T & 0 \\ \cos \lambda_T & \sin \lambda_T & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (B-11)$$

The equations required for calculating r_L, r_T, B_L, and B_T appearing in equations (B-4) and (B-5) are given by:

$$B_L = \Phi_L - \Psi_L \qquad (B-12)$$

$$r_L = \hat{a}/(\cos^2 \Psi_L + \frac{\hat{a}^2}{\hat{b}^2} \sin^2 \Psi_L)^{\frac{1}{2}} \tag{B-13}$$

$$B_T = \Phi_T - \Psi_T \tag{B-14}$$

$$r_T = \hat{a}/(\cos^2 \Psi_T + \frac{\hat{a}^2}{\hat{b}^2} \sin^2 \Psi_T)^{\frac{1}{2}} \tag{B-15}$$

where

$$\Psi_L = \tan^{-1} \left[\frac{\hat{b}^2}{\hat{a}^2} \tan \Phi_L \right] \tag{B-16}$$

$$\Psi_T = \tan^{-1} \left[\frac{\hat{b}^2}{\hat{a}^2} \tan \Phi_T \right] . \tag{B-17}$$

The spherical coordinate transformation for the i-th observation is given by:

$$R_i^r = (X_{es_i}^2 + Y_{es_i}^2 + Z_{es_i}^2)^{\frac{1}{2}} \tag{B-18}$$

$$A_i^r = [\tan^{-1} (Z_{es_i}/X_{es_i})] + K_T, \; 0 \leq A_i^r \leq 360° \tag{B-19}$$

$$E_i^r = \tan^{-1} \left[\frac{Y_{es_i}}{(X_{es_i}^2 + Z_{es_i}^2)^{\frac{1}{2}}} \right], \; 0 \leq E_i^r \leq 90° \quad . \tag{B-20}$$

An additional transformation is required to determine the (X, Y, Z) position of the vehicle in an earth-fixed ephemeris coordinate system with origin at the tracking site. This transformation is given by:

$$\overline{X}_i = \overline{\lambda}_T \, \overline{\Phi}_T \, \overline{K}_T \, \overline{X}_{es_i} \tag{B-21}$$

where

$$\overline{X}_i = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i \tag{B-22}$$

and X_{es_i} is obtained from the transformation given by equation (B-1).

APPENDIX C

A STEPWISE REGRESSION EXAMPLE ILLUSTRATING THE GAUSSIAN ELIMINATION METHOD

As pointed out earlier, the Gaussian Elimination Method is ideally suited for a stepwise regression analysis since it actually enters variables into the regression equation one at a time. A significant variable entered into the regression equation at an early step may, after several other variables are entered, be found to be insignificant. This insignificant variable is then deleted from the regression equation before another variable is considered for entry. Thus, the final regression equation includes only the significant variables. The complete method is best illustrated by taking a specific example and performing the step-by-step operations. Assume there are p = 4 independent variables to be considered for entry into the regression equation. The augmented matrix similar to equation (117) is:

$$\bar{A} = \begin{array}{c} \begin{array}{ccccc|cccc} \underline{r_{i1}} & \underline{r_{i2}} & \underline{r_{i3}} & \underline{r_{i4}} & \underline{r_{iY}} & \underline{\tilde{c}_{i1}} & \underline{\tilde{c}_{i2}} & \underline{\tilde{c}_{i3}} & \underline{\tilde{c}_{i4}} \end{array} \\ \left[\begin{array}{ccccc|cccc} r_{11} & r_{12} & r_{13} & r_{14} & r_{1Y} & 1 & 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{2Y} & 0 & 1 & 0 & 0 \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{3Y} & 0 & 0 & 1 & 0 \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{4Y} & 0 & 0 & 0 & 1 \\ r_{Y1} & r_{Y2} & r_{Y3} & r_{Y4} & r_{YY} & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad (C-1)$$

Step 1

The first step consists of entering into the regression the variable most highly correlated with the response. Thus the variable corresponding to the largest of r^2_{1Y}, r^2_{2Y}, r^2_{3Y}, and r^2_{4Y} is selected for entry. Assume that r^2_{4Y} is the largest. Then the nondiagonal elements r_{14}, r_{24}, r_{34}, and r_{Y4} of the fourth column are zeroed out. This is accomplished by dividing the fourth row by r_{44}; multiplying the new fourth row by $(-r_{14})$ and adding the result to the first row; multiplying the new fourth row by $(-r_{24})$ and adding the result to the second row; multiplying the new fourth row by $(-r_{34})$ and adding the result to the third row; and finally, multiplying the new fourth row by $(-r_{Y4})$ and adding the result to the fifth row:

$$
\begin{array}{c}
\begin{array}{ccccc|cccc}
\underline{r'_{i1}} & \underline{r'_{i2}} & \underline{r'_{i3}} & \underline{r'_{i4}} & \underline{r'_{iY}} & \underline{\tilde{c}'_{i1}} & \underline{\tilde{c}'_{i2}} & \underline{\tilde{c}'_{i3}} & \underline{\tilde{c}'_{i4}}
\end{array}
\\
\bar{A}' = \left[\begin{array}{ccccc|cccc}
\dfrac{r_{11}-r_{14}r_{41}}{r_{44}} & \dfrac{r_{12}-r_{14}r_{42}}{r_{44}} & \dfrac{r_{13}-r_{14}r_{43}}{r_{44}} & 0 & \dfrac{r_{1Y}-r_{14}r_{4Y}}{r_{44}} & 1 & 0 & 0 & -r_{14}/r_{44} \\
\dfrac{r_{21}-r_{24}r_{41}}{r_{44}} & \dfrac{r_{22}-r_{24}r_{42}}{r_{44}} & \dfrac{r_{23}-r_{24}r_{43}}{r_{44}} & 0 & \dfrac{r_{2Y}-r_{24}r_{4Y}}{r_{44}} & 0 & 1 & 0 & -r_{24}/r_{44} \\
\dfrac{r_{31}-r_{34}r_{41}}{r_{44}} & \dfrac{r_{32}-r_{34}r_{42}}{r_{44}} & \dfrac{r_{33}-r_{34}r_{43}}{r_{44}} & 0 & \dfrac{r_{3Y}-r_{34}r_{4Y}}{r_{44}} & 0 & 0 & 1 & -r_{34}/r_{44} \\
r_{41}/r_{44} & r_{42}/r_{44} & r_{43}/r_{44} & 1 & r_{4Y}/r_{44} & 0 & 0 & 0 & 1/r_{44} \\
\dfrac{r_{Y1}-r_{Y4}r_{41}}{r_{44}} & \dfrac{r_{Y2}-r_{Y4}r_{42}}{r_{44}} & \dfrac{r_{Y3}-r_{Y4}r_{43}}{r_{44}} & 0 & \dfrac{r_{YY}-r_{Y4}{}^{4}{}_{4Y}}{r_{44}} & 0 & 0 & 0 & -r_{Y4}/r_{44}
\end{array}\right]
\end{array}
\tag{C-2}
$$

At this step the fourth element in the fifth column is the solution α_4 in the normal equation:

$$\alpha_4 \, r_{44} = r_{4Y} \, . \tag{C-3}$$

The fourth element in the fourth row of the matrix $\tilde{c}'$ to the right of the line in equation (C-2) contains the inverse of r_{44}. Of the three variables not in the regression the variable next considered for entry is the one whose partial correlation with the response is highest; that is, the variable corresponding to the largest of $\rho^2_{1Y.4}$, $\rho^2_{2Y.4}$, $\rho^2_{3Y.4}$. These would normally be determined as indicated earlier in this report. It can be verified, however, that $\rho^2_{iY.4}$ $(i = 1, 2, 3)$ can be obtained using elements of equation (C-2) by the equation:

$$\rho^2_{iY.4} = \frac{(r'_{iY})^2}{r'_{11}\ r'_{YY}} \quad . \tag{C-4}$$

Where i = 1, 2, 3 and r'_{ii} refers to the i-th diagonal element, r'_{iY} refers to the element in the i-th row, fifth column and r'_{YY} refers to the element in the fifth row, fifth column of equation (C-2). Assume $\rho^2_{1Y.4}$ is found to be the largest. Then the test for entering the variable Z_1 is given by:

$$F_{1(IN)} = \frac{\rho^2_{1Y.4}(n-d)}{1-\rho^2_{1Y.4}} \quad . \tag{C-5}$$

If this $F_{1(IN)}$ value is less than or equal to the appropriate table value, then the variable Z_1 is not entered and the process is terminated.

Step 2

If, however, it is found that $F_{1(IN)}$ is greater than the table value, then the variable Z_1 is entered. This is done analogous to the manner in which Z_4 was entered in the first step. That is, the nondiagonal elements r'_{21}, r'_{31}, r'_{41}, and r'_{Y1} of the first column in equation (C-2) are zeroed out and the following new matrix is obtained:

$$\begin{array}{c}
\begin{array}{ccccc|cccc}
\underline{r''_{i1}} & \underline{r''_{i2}} & \underline{r''_{i3}} & \underline{r''_{i4}} & \underline{r''_{iY}} & \underline{\tilde{c}''_{i1}} & \underline{\tilde{c}''_{i2}} & \underline{\tilde{c}''_{i3}} & \underline{\tilde{c}''_{i4}}
\end{array} \\
\bar{A}'' = \left[\begin{array}{ccccc|cccc}
1 & r'_{12}/r'_{11} & r'_{13}/r'_{11} & 0 & r'_{1Y}/r'_{11} & 1/r'_{11} & 0 & 0 & \tilde{c}'_{14}/r'_{11} \\
0 & r'_{22}-\frac{r'_{21}r'_{12}}{r'_{11}} & r'_{23}-\frac{r'_{21}r'_{13}}{r'_{11}} & 0 & r'_{2Y}-\frac{r'_{21}r'_{1Y}}{r'_{11}} & -r'_{21}/r'_{11} & 1 & 0 & \tilde{c}'_{24}-\frac{r'_{21}\tilde{c}'_{14}}{r'_{11}} \\
0 & r'_{32}-\frac{r'_{31}r'_{12}}{r'_{11}} & r'_{33}-\frac{r'_{31}r'_{13}}{r'_{11}} & 0 & r'_{3Y}-\frac{r'_{31}r'_{1Y}}{r'_{11}} & -r'_{31}/r'_{11} & 0 & 1 & \tilde{c}'_{34}-\frac{r'_{31}\tilde{c}'_{14}}{r'_{11}} \\
0 & r'_{42}-\frac{r'_{41}r'_{12}}{r'_{11}} & r'_{43}-\frac{r'_{41}r'_{13}}{r'_{11}} & 1 & r'_{4Y}-\frac{r'_{41}r'_{1Y}}{r'_{11}} & -r'_{41}/r'_{11} & 0 & 0 & \tilde{c}'_{44}-\frac{r'_{41}\tilde{c}'_{14}}{r'_{11}} \\
0 & r'_{Y2}-\frac{r'_{Y1}r'_{12}}{r'_{11}} & r'_{Y3}-\frac{r'_{Y1}r'_{13}}{r'_{11}} & 0 & r'_{YY}-\frac{r'_{Y1}r'_{1Y}}{r'_{11}} & -r'_{Y1}/r'_{11} & 0 & 0 & \tilde{c}'_{Y4}-\frac{r'_{Y1}\tilde{c}'_{14}}{r'_{11}}
\end{array}\right]
\end{array} \tag{C-6}$$

At this step the first and fourth elements in the fifth column are the solutions α_1 and α_4, respectively, in the normal equations:

$$\begin{bmatrix} r_{11} & r_{14} \\ r_{41} & r_{44} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} r_{1Y} \\ r_{4Y} \end{bmatrix} . \tag{C-7}$$

The elements in the 11, 14, 41, and 44 positions of the $\tilde{c}''$ matrix are elements of the inverse matrix:

$$\begin{bmatrix} r_{11} & r_{14} \\ r_{41} & r_{44} \end{bmatrix}^{-1} .$$

Before the variables corresponding to α_2 and α_3 are considered for entry into the regression, the variables in the regression are tested for significance to determine if one should be deleted. This test for deletion is always made before a variable is added to the regression. The two variables in the regression corresponding to the coefficients α_1 and α_4 are considered for deletion by computing $F_{1(OUT)}$ and $F_{4(OUT)}$ from:

$$\begin{aligned} F_{i(OUT)} &= \frac{\alpha_i^2 (n-d)}{\tilde{c}''_{ii} (1-R^2_{Y.14})} \\ &= \frac{(r''_{iY})^2 (n-d)}{\tilde{c}''_{ii}\, r''_{YY}} \end{aligned} \tag{C-8}$$

where i = 1, 4. The smallest of $F_{1(OUT)}$ and $F_{4(OUT)}$ is tested for significance by comparison with the appropriate F table value. Assume the smallest is found to be greater than the appropriate table value. Then both Z_1 and Z_4

remain in the regression and one of the two remaining variables is considered for entry into the regression. The variable considered for entry is the one whose partial correlation with the response is highest; that is, the variable corresponding to the largest of $\rho^2_{2Y.14}$ and $\rho^2_{3Y.14}$. It can be verified that $\rho^2_{iY.14}$ (i= 2,3) can be obtained using elements of equation (C-6) by the equation:

$$\rho^2_{iY.14} = \frac{(r''_{iY})^2}{r''_{ii}\, r''_{YY}} \tag{C-9}$$

where i = 2,3, and r'' elements are the appropriate elements to the left of the line in equation (C-6). Assume $\rho^2_{2Y.14}$ is found to be the largest. Then the test for entering Z_2 is given by:

$$F_{2(IN)} = \frac{\rho^2_{2Y.14}(n-d)}{1-\rho^2_{2Y.14}} \quad . \tag{C-10}$$

Step 3

Assume $F_{2(IN)}$ is indicated to be significant from the comparison with the appropriate F table value. Then the variable Z_2 corresponding to α_2 is entered. The nondiagonal elements r''_{12}, r''_{32}, r''_{42}, and r''_{Y2} of the second column in equation (C-6) are zeroed out and the following new matrix is obtained:

$$
\overline{A}''' = \begin{array}{c}
\begin{array}{ccccc|cccc}
\underline{r'''_{i1}} & \underline{r'''_{i2}} & \underline{r'''_{i3}} & \underline{r'''_{i4}} & \underline{r'''_{iY}} & \underline{\tilde{c}'''_{i1}} & \underline{\tilde{c}'''_{i2}} & \underline{\tilde{c}'''_{i3}} & \underline{\tilde{c}'''_{i4}}
\end{array} \\
\left[\begin{array}{ccccc|cccc}
1 & 0 & \dfrac{r''_{13}-r''_{12}r''_{23}}{r''_{22}} & 0 & \dfrac{r''_{1Y}-r''_{12}r''_{2Y}}{r''_{22}} & \dfrac{\tilde{c}''_{11}-r''_{12}\tilde{c}''_{21}}{r''_{22}} & -r''_{12}/r''_{22} & 0 & \dfrac{\tilde{c}''_{14}-r''_{12}\tilde{c}''_{24}}{r''_{22}} \\
0 & 1 & r''_{23}/r''_{22} & 0 & r''_{2Y}/r''_{22} & \tilde{c}''_{21}/r''_{22} & 1/r''_{22} & 0 & \tilde{c}''_{24}/r''_{22} \\
0 & 0 & \dfrac{r''_{33}-r''_{32}r''_{23}}{r''_{22}} & 0 & \dfrac{r''_{3Y}-r''_{32}r''_{2Y}}{r''_{22}} & \dfrac{\tilde{c}''_{31}-r''_{32}\tilde{c}''_{21}}{r''_{22}} & -r''_{32}/r''_{22} & 1 & \dfrac{\tilde{c}''_{34}-r''_{32}\tilde{c}''_{24}}{r''_{22}} \\
0 & 0 & \dfrac{r''_{43}-r''_{42}r''_{23}}{r''_{22}} & 1 & \dfrac{r''_{4Y}-r''_{42}r''_{2Y}}{r''_{22}} & \dfrac{\tilde{c}''_{41}-r''_{42}\tilde{c}''_{21}}{r''_{22}} & -r''_{42}/r''_{22} & 0 & \dfrac{\tilde{c}''_{44}-r''_{42}\tilde{c}''_{24}}{r''_{22}} \\
0 & 0 & \dfrac{r''_{Y3}-r''_{Y2}r''_{23}}{r''_{22}} & 0 & \dfrac{r''_{YY}-r''_{Y2}r''_{2Y}}{r''_{22}} & \dfrac{\tilde{c}''_{Y1}-r''_{Y2}\tilde{c}''_{21}}{r''_{22}} & -r''_{Y2}/r''_{22} & 0 & \dfrac{\tilde{c}''_{Y4}-r''_{Y2}\tilde{c}''_{24}}{r''_{22}}
\end{array}\right]
\end{array}
\tag{C-11}
$$

At this step the first, second, and fourth elements in the fifth column are the solutions α_1, α_2, and α_4, respectively, in the normal equations:

$$
\begin{bmatrix} r_{11} & r_{12} & r_{14} \\ r_{21} & r_{22} & r_{24} \\ r_{41} & r_{42} & r_{44} \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_4 \end{bmatrix}
=
\begin{bmatrix} r_{1Y} \\ r_{2Y} \\ r_{4Y} \end{bmatrix}
\tag{C-12}
$$

Elements of the inverse matrix

$$
\begin{bmatrix} r_{11} & r_{12} & r_{14} \\ r_{21} & r_{22} & r_{24} \\ r_{41} & r_{42} & r_{44} \end{bmatrix}^{-1}
$$

are contained in the appropriate positions of the $\tilde{c}'''$ matrix in equation (C-11). The three variables in the regression are considered for deletion by computing $F_{1(OUT)}$, $F_{2(OUT)}$, and $F_{4(OUT)}$ from:

$$F_{i(OUT)} = \frac{\alpha_i^2 \, (n-d)}{\tilde{c}'''_{ii} \, (1 - R^2_{Y.124})}$$

$$= \frac{(r'''_{1Y})^2 \, (n-d)}{\tilde{c}'''_{ii} \; r'''_{YY}} \qquad \text{(C-13)}$$

where i = 1,2,4. The smallest $F_{i(OUT)}$ (i=1,2,4) is tested for significance by comparison with the appropriate F table value. If the smallest is found to be greater than the appropriate F table value then all three variables corresponding to α_1, α_2, α_4 would remain in the regression. The remaining variable would then be tested for entry by first computing:

$$\rho^2_{3Y.124} = \frac{(r'''_{3Y})^2}{r'''_{33} \, r'''_{YY}} \quad . \qquad \text{(C-14)}$$

The r''' elements are the appropriate elements in the matrix to the left of the line in equation (C-11). The test for entering Z_3 is then given by:

$$F_{3(IN)} = \frac{\rho^2_{3Y.124} \, (n-d)}{1 - \rho^2_{3Y.124}} \quad . \qquad \text{(C-15)}$$

If $F_{3(IN)}$ is found to be insignificant, then the process is terminated and the final regression equation contains α_1, α_2, and α_4.

Step 4

If, however, $F_{3(IN)}$ is found to be significant, then the variable corresponding to α_3 is entered by zeroing the nondiagonal elements r'''_{13}, r'''_{23}, r'''_{43}, and r'''_{Y3} of the third column in equation (C-11). The following new matrix is then obtained:

$$
\begin{array}{cc}
 & \begin{array}{ccccc|cccc} r^{IV}_{i1} & r^{IV}_{i2} & r^{IV}_{i3} & r^{IV}_{i4} & r^{IV}_{iY} & \tilde{c}^{IV}_{i1} & \tilde{c}^{IV}_{i2} & \tilde{c}^{IV}_{i3} & \tilde{c}^{IV}_{i4} \end{array} \\
\bar{A}^{IV} = & \left[\begin{array}{ccccc|cccc}
1 & 0 & 0 & 0 & \frac{r'''_{1Y}-r'''_{13}r'''_{3Y}}{r'''_{33}} & \frac{\tilde{c}'''_{11}-r'''_{13}\tilde{c}'''_{31}}{r'''_{33}} & \frac{\tilde{c}'''_{12}-r'''_{13}\tilde{c}'''_{32}}{r'''_{33}} & -r'''_{13}/r'''_{33} & \frac{\tilde{c}'''_{14}-r'''_{13}\tilde{c}'''_{34}}{r'''_{33}} \\
0 & 1 & 0 & 0 & \frac{r'''_{2Y}-r'''_{23}r'''_{3Y}}{r'''_{33}} & \frac{\tilde{c}'''_{21}-r'''_{23}\tilde{c}'''_{32}}{r'''_{33}} & \frac{\tilde{c}'''_{22}-r'''_{23}\tilde{c}'''_{32}}{r'''_{33}} & -r'''_{23}/r'''_{33} & \frac{\tilde{c}'''_{24}-r'''_{23}\tilde{c}'''_{34}}{r'''_{33}} \\
0 & 0 & 1 & 0 & r'''_{3Y}/r'''_{33} & \tilde{c}'''_{31}/r'''_{33} & \tilde{c}'''_{32}/r'''_{33} & 1/r'''_{33} & \tilde{c}'''_{34}/r'''_{33} \\
0 & 0 & 0 & 1 & \frac{r'''_{4Y}-r'''_{43}r'''_{3Y}}{r'''_{33}} & \frac{\tilde{c}'''_{41}-r'''_{43}\tilde{c}'''_{31}}{r'''_{33}} & \frac{\tilde{c}'''_{42}-r'''_{43}\tilde{c}'''_{32}}{r'''_{33}} & -r'''_{43}/r'''_{33} & \frac{\tilde{c}'''_{44}-r'''_{43}\tilde{c}'''_{32}}{r'''_{33}} \\
0 & 0 & 0 & 0 & \frac{r'''_{YY}-r'''_{Y3}r'''_{3Y}}{r'''_{33}} & \frac{\tilde{c}'''_{Y1}-r'''_{Y3}\tilde{c}'''_{31}}{r'''_{33}} & \frac{\tilde{c}'''_{Y2}-r'''_{Y3}\tilde{c}'''_{32}}{r'''_{33}} & -r'''_{Y3}/r'''_{33} & \frac{\tilde{c}'''_{Y4}-r'''_{Y3}\tilde{c}'''_{32}}{r'''_{33}}
\end{array}\right]
\end{array}
\qquad (C-16)
$$

At this step all the variables are in the regression and equation (C-16) is given by:

$$\bar{A}^{IV} = \left[\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & \alpha_1 & \tilde{c}^{IV}_{11} & \tilde{c}^{IV}_{12} & \tilde{c}^{IV}_{13} & \tilde{c}^{IV}_{14} \\ 0 & 1 & 0 & 0 & \alpha_2 & \tilde{c}^{IV}_{21} & \tilde{c}^{IV}_{22} & \tilde{c}^{IV}_{23} & \tilde{c}^{IV}_{24} \\ 0 & 0 & 1 & 0 & \alpha_3 & \tilde{c}^{IV}_{31} & \tilde{c}^{IV}_{32} & \tilde{c}^{IV}_{33} & \tilde{c}^{IV}_{34} \\ 0 & 0 & 0 & 1 & \alpha_4 & \tilde{c}^{IV}_{41} & \tilde{c}^{IV}_{42} & \tilde{c}^{IV}_{43} & \tilde{c}^{IV}_{44} \\ 0 & 0 & 0 & 0 & (1-R^2_{Y.1234}) & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \end{array}\right] \tag{C-17}$$

where:

$$\begin{bmatrix} \tilde{c}^{IV}_{11} & \tilde{c}^{IV}_{12} & \tilde{c}^{IV}_{13} & \tilde{c}^{IV}_{14} \\ \tilde{c}^{IV}_{21} & \tilde{c}^{IV}_{22} & \tilde{c}^{IV}_{23} & \tilde{c}^{IV}_{24} \\ \tilde{c}^{IV}_{31} & \tilde{c}^{IV}_{32} & \tilde{c}^{IV}_{33} & \tilde{c}^{IV}_{34} \\ \tilde{c}^{IV}_{41} & \tilde{c}^{IV}_{42} & \tilde{c}^{IV}_{43} & \tilde{c}^{IV}_{44} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix}^{-1} \tag{C-18}$$

The four variables in the regression are considered for deletion by computing:

$$F_{i(OUT)} = \frac{\alpha_i^2 \,(n - d)}{\tilde{c}^{IV}_{ii}\,(1-R^2_{Y.1234})}$$

$$= \frac{(r^{IV}_{YY})^2 \,(n-d)}{\tilde{c}^{IV}_{ii}\; r^{IV}_{YY}} \tag{C-19}$$

where i = 1, 2, 3, 4. If the smallest $F_{i(OUT)}$ value is found to be greater than the appropriate F table value, then all four variables remain in the regression and the process is terminated. If, however, the smallest $F_{i(OUT)}$ value is found to be insignificant, then the variable corresponding to this $F_{i(OUT)}$ value is deleted from the regression by zeroing the appropriate elements in the column to the right of the line in equation (C-16). The column selected corresponds to the particular variable being deleted.

The deletion process will be illustrated by going back to equation (C-11). This equation contains the solution for the coefficients α_1, α_2, and α_4. Assume $F_{2(OUT)}$ as computed by equation (C-13) is smaller than $F_{1(OUT)}$ and $F_4(OUT)$ and is found to be insignificant. Then Z_2 is deleted by zeroing the first, third, fourth, and fifth elements of the $\tilde{c}'''_{i2}$ column. This is accomplished by dividing the entire second row by $\tilde{c}'''_{22}$; multiplying the new second row by $(-\tilde{c}'''_{12})$ and adding the result to the first row (zeros first element); multiplying the new second row by $(-\tilde{c}'''_{32})$ and adding the result to the third row (zeros the third element); multiplying the new second row by $(-\tilde{c}'''_{42})$ and adding the result to the fourth row (zeros the fourth element); and finally, multiplying the new second row by $(-\tilde{c}'''_{Y2})$ and adding the result to the fifth row (zeros the fifth element):

$$
\begin{array}{cc}
 & \begin{array}{ccccc|cccc}
\underline{r^V_{ii}} & \underline{r^V_{i2}} & \underline{r^V_{i3}} & \underline{r^V_{i4}} & \underline{r^V_{iY}} & \underline{\tilde{c}^V_{ii}} & \underline{\tilde{c}^V_{i2}} & \underline{\tilde{c}^V_{i3}} & \underline{\tilde{c}^V_{i4}}
\end{array} \\
\bar{A}^V = & \left[\begin{array}{ccccc|cccc}
1 & -\tilde{c}'''_{12}/\tilde{c}'''_{22} & \frac{r'''_{13}-\tilde{c}'''_{12}r'''_{23}}{\tilde{c}'''_{22}} & 0 & \frac{r'''_{1Y}-\tilde{c}'''_{12}r'''_{2Y}}{\tilde{c}'''_{22}} & \frac{\tilde{c}'''_{11}-\tilde{c}'''_{12}\tilde{c}'''_{21}}{\tilde{c}'''_{22}} & 0 & 0 & \frac{\tilde{c}'''_{14}-\tilde{c}'''_{12}\tilde{c}'''_{24}}{\tilde{c}'''_{22}} \\
0 & 1/\tilde{c}'''_{22} & r'''_{23}/\tilde{c}'''_{22} & 0 & r'''_{2Y}/\tilde{c}'''_{22} & \tilde{c}'''_{21}/\tilde{c}'''_{22} & 1 & 0 & \tilde{c}'''_{24}/\tilde{c}'''_{22} \\
0 & -\tilde{c}'''_{32}/\tilde{c}'''_{22} & \frac{r'''_{33}-\tilde{c}'''_{32}r'''_{23}}{\tilde{c}'''_{22}} & 0 & \frac{r'''_{3Y}-\tilde{c}'''_{32}r'''_{2Y}}{\tilde{c}'''_{22}} & \frac{\tilde{c}'''_{31}-\tilde{c}'''_{32}\tilde{c}'''_{21}}{\tilde{c}'''_{22}} & 0 & 1 & \frac{\tilde{c}'''_{34}-\tilde{c}'''_{32}\tilde{c}'''_{24}}{\tilde{c}'''_{22}} \\
0 & -\tilde{c}'''_{42}/\tilde{c}'''_{22} & \frac{r'''_{43}-\tilde{c}'''_{42}r'''_{23}}{\tilde{c}'''_{22}} & 1 & \frac{r'''_{4Y}-\tilde{c}'''_{42}r'''_{2Y}}{\tilde{c}'''_{22}} & \frac{\tilde{c}'''_{41}-\tilde{c}'''_{42}\tilde{c}'''_{21}}{\tilde{c}'''_{22}} & 0 & 0 & \frac{\tilde{c}'''_{44}-\tilde{c}'''_{42}\tilde{c}'''_{24}}{\tilde{c}'''_{22}} \\
0 & -\tilde{c}'''_{Y2}/\tilde{c}'''_{22} & \frac{r'''_{Y3}-\tilde{c}'''_{Y2}r'''_{23}}{\tilde{c}'''_{22}} & 0 & \frac{r'''_{YY}-\tilde{c}'''_{Y2}r'''_{2Y}}{\tilde{c}'''_{22}} & \frac{\tilde{c}'''_{Y1}-\tilde{c}'''_{Y2}\tilde{c}'''_{21}}{\tilde{c}'''_{22}} & 0 & 0 & \frac{\tilde{c}'''_{Y4}-\tilde{c}'''_{Y2}\tilde{c}'''_{24}}{\tilde{c}'''_{22}}
\end{array}\right]
\end{array}
\qquad (C\text{-}20)
$$

By substituting the elements $\tilde{c}'''_{i1}$, $\tilde{c}'''_{i2}$, $\tilde{c}'''_{i4}$, r'''_{i3} and r'''_{iY} of equation (C-11) (where i = 1, 2, 3, 4, Y) into equation (C-20), the following equation is obtained:

$$\bar{A}^V = \left[\begin{array}{ccccc|cccc} 1 & r''_{12} & r''_{13} & 0 & r''_{1Y} & \tilde{c}''_{11} & 0 & 0 & \tilde{c}''_{14} \\ 0 & r''_{22} & r''_{23} & 0 & r''_{2Y} & \tilde{c}''_{21} & 1 & 0 & \tilde{c}''_{24} \\ 0 & r''_{32} & r''_{33} & 0 & r''_{3Y} & \tilde{c}''_{31} & 0 & 1 & \tilde{c}''_{34} \\ 0 & r''_{42} & r''_{43} & 1 & r''_{4Y} & \tilde{c}''_{41} & 0 & 0 & \tilde{c}''_{44} \\ 0 & r''_{Y2} & r''_{Y3} & 0 & r''_{YY} & \tilde{c}''_{Y1} & 0 & 0 & \tilde{c}''_{Y4} \end{array}\right] \tag{C-21}$$

This equation is seen to be equivalent to equation (C-6), which contains the solution for the variables Z_1 and Z_4 in the regression.

APPENDIX D

TEMS IBM 7094 COMPUTER PROGRAM FOR C-BAND RADAR TRACKING SYSTEMS

The various computational procedures in the TEMS method are implemented through the use of a highly flexible IBM 7094 computer program. Basically, the program consists of a main program and five subroutines. Control parameters are input by way of data cards in the main program to indicate the specific subroutines required for the reduction. Any combination of terms in the total error models can be retained in a given regression through the use of program control matrices. Trajectory data are processed independent of storage capacity, and bad data or discrete points can be deleted by the use of program control cards. A variable sampling rate can also be accepted. Additional features of flexibility include variable plot scaling and double precision computations.

One of the areas with considerable flexibility is in the selection of the specific error models for the reduction. Although constraints interrelate the equations for ΔR, ΔA, and ΔE, the results for a single equation regression can be obtained through the use of appropriate program control matrices. An 18 x 18 control matrix AA(I, J), consisting of 1's and 0's, is used for selecting the elements of $(\bar{B}^T\bar{W}\bar{B}+\dot{\bar{W}})$ to use in a specific computer run. An 18 x 1 control matrix BB(I, 1) is also used for selecting elements of $(\bar{B}^T\bar{W}\ \bar{N} - \bar{B}^T\bar{W}\bar{B}\bar{C} - \dot{\bar{W}}\ \bar{\epsilon})$ to use. The use of these matrices enables any combination of terms in the total error models to be retained in a given regression. A summary of the basic logic and computations in the program is presented in the flow chart herein (Fig. D-1). Additional information on all aspects of the program development is given in Reference 1.

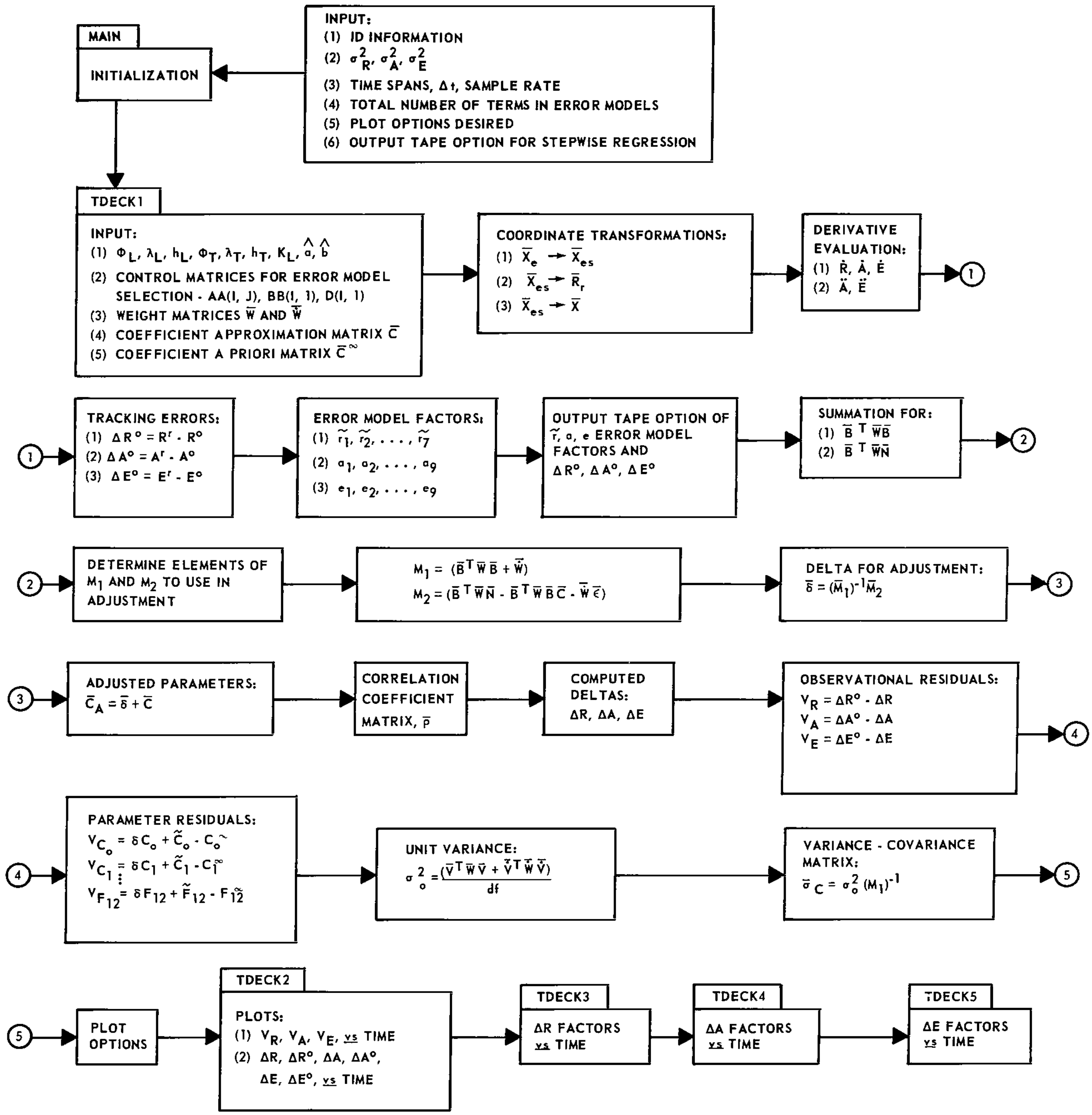

FIGURE D-1. TEMS PROGRAM FLOW CHART

APPENDIX E

UNIVAC 1108 COMPUTER PROGRAM FOR APPLICATION OF THE STEPWISE REGRESSION ANALYSIS PROCEDURES

This appendix contains a summary of the basic logic and computations in the UNIVAC 1108 MAIN and STEPRG programs (Fig. E-1). The STEPRG program was developed by debugging and modifying a UNIVAC 1108 regression subroutine. It was made stepwise by incorporating into the subroutine the stepwise procedures presented eariler in this report and by appropriate use of the output.

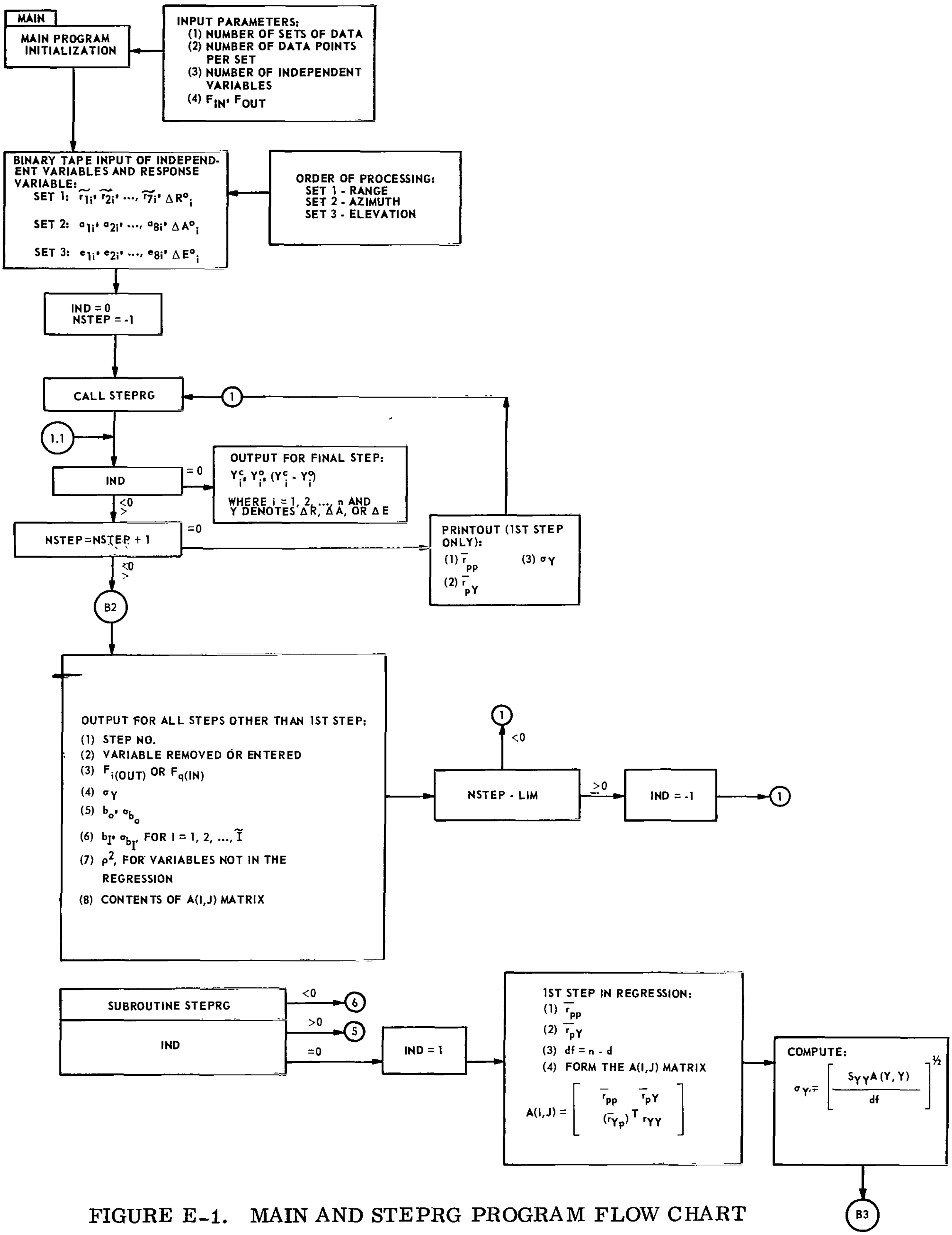

FIGURE E-1. MAIN AND STEPRG PROGRAM FLOW CHART

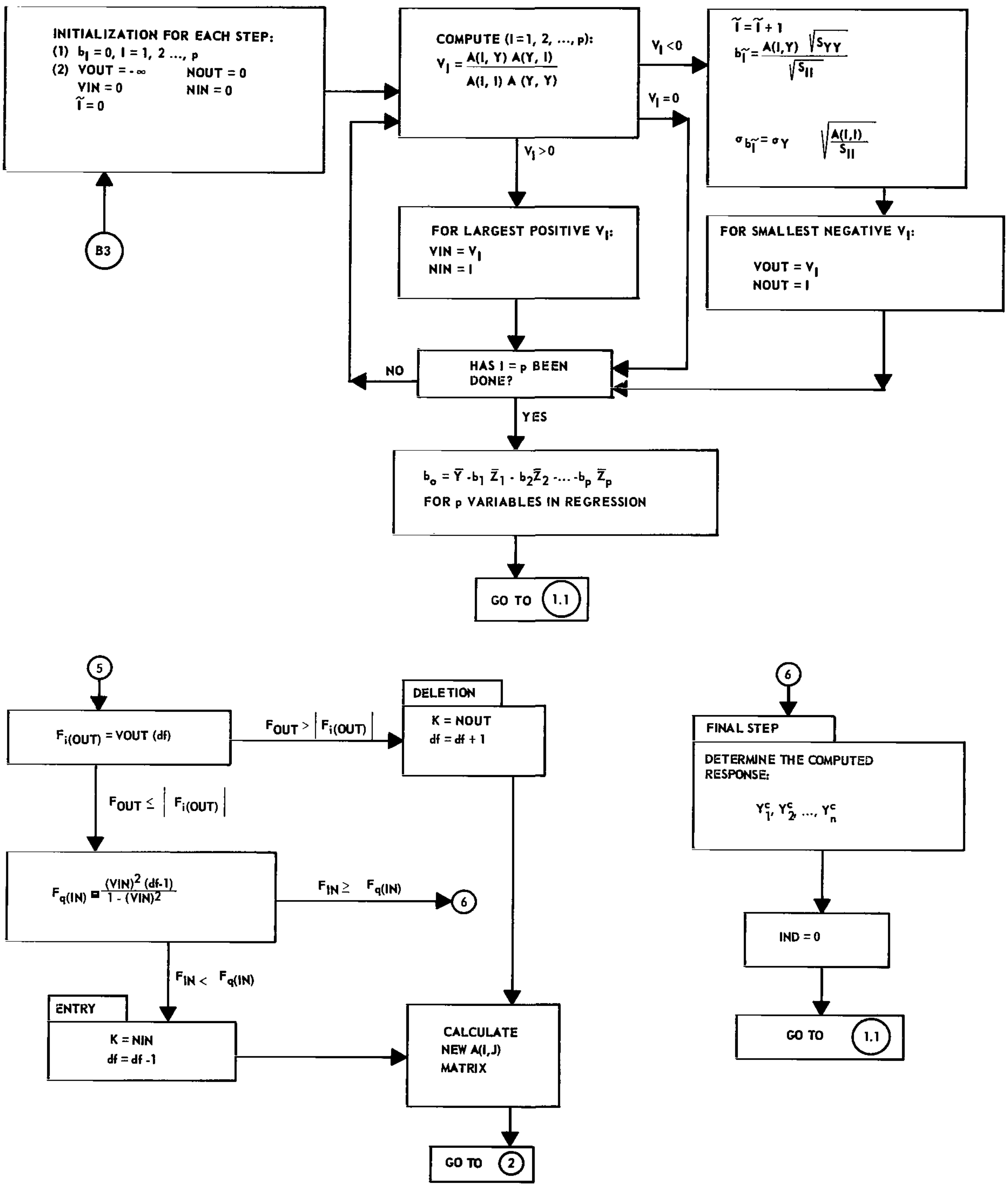

FIGURE E-1. (Concluded)

APPENDIX F

RESULTS FROM THE APOLLO-SATURN 501 VEHICLE FLIGHT TEST

This appendix presents a summary of the results from the Apollo-Saturn 501 Vehicle Flight Test launched on November 9, 1967. The Stepwise Regression Analysis results for the first and second burn data are presented in Tables F-I through F-VII and Tables F-VIII through F-XI, respectively. Coefficient correlations for the truncated error models for the first and second burn data are given in Tables F-XII and F-XIII, respectively.

In the figures which follow these tables (F-1 through F-22), the tracking errors for the various radars are represented by dots. The description of these errors as obtained from the TEMS least squares adjustment program is represented by the solid computed curves.

The least squares residuals for the truncated error models presented in this appendix can be thought of as being composed of random errors and unmodeled systematic errors. A high random error content in the data may prevent a systematic error of comparable magnitude from being determined. The latter errors are those that can be attributed to uncertainties in the standard used in establishing the tracking errors, unknown systematic errors not absorbed by those that are modeled, or to geometry limitations. The presence of a significant unmodeled systematic error may prevent an adequate description of the data from being obtained.

TABLE F-I. RADAR 0.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = 0.76$	1	C_0, C_5	3.06	4459.1
2	$r[C_2 \mid \Delta R] = 0.94$	2	C_0, C_5, C_7	2.22	301.3
3	$r[C_4 \mid \Delta R] = 0.16$	3	C_0, C_5, C_7, C_8	1.74	204.1
4	$r[C_5 \mid \Delta R] = 0.96$	4	C_0, C_5, C_7, C_8, C_4	1.73	9.3
5	$r[C_6 \mid \Delta R] = -0.33$	5	$C_0, C_5, C_7, C_8, C_4, C_6$	1.72	4.9
6	$r[C_7 \mid \Delta R] = -0.96$				
7	$r[C_8 \mid \Delta R] = 0.87$	Final	$C_0. C_5, C_8, C_4, C_6$	1.71	-0.1
1	$r[C_2 \mid \Delta A] = 0.60$	1	D_0, C_2	0.0089	191.5
2	$r[D_3 \mid \Delta A] = 0.55$	2	D_0, C_2, D_3	0.0062	353.9
3	$r[D_5 \mid \Delta A] = 0.09$	3	D_0, C_2, D_3, D_7	0.0061	13.6
4	$r[D_6 \mid \Delta A] = 0.12$	4	D_0, C_2, D_3, D_7, D_8	0.0058	30.9
5	$r[D_7 \mid \Delta A] = -0.43$	5	D_0, C_2, D_7, D_8	0.0058	-0.2
6	$r[D_8 \mid \Delta A] = 0.50$				
7	$r[C_5 \mid \Delta A] = -0.03$				
8	$r[C_6 \mid \Delta A] = -0.13$	Final	D_0, C_2, D_7, D_8, C_6	0.0058	6.6
1	$r[C_2 \mid \Delta E] = -0.22$	1	F_0, C_5	0.0074	104.5
2	$r[F_3 \mid \Delta E] = 0.15$	2	F_0, C_5, C_6	0.0067	68.8
3	$r[D_8 \mid \Delta E] = -0.31$	3	F_0, C_5, C_6, C_2	0.0061	74.3
4	$r[D_7 \mid \Delta E] = -0.44$	4	F_0, C_5, C_6, C_2, D_7	0.0060	9.9
5	$r[C_4 \mid \Delta E] = 0.01$	5	$F_0, CE, C_6, C_2, D_7, D_8$	0.0060	3.6
6	$r[C_5 \mid \Delta E] = 0.49$				
7	$r[C_6 \mid \Delta E] = 0.06$				
8	$r[C_7 \mid \Delta E] = -0.35$				
		Final	F_0, C_6, C_2, D_7, D_8	0.0060	-1.0

TABLE F-II. RADAR 19.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = 0.65$	1	C_0, C_1	5.96	162.2
2	$r[C_2 \mid \Delta R] = 0.57$	2	C_0, C_1, C_6	5.04	86.9
3	$r[C_4 \mid \Delta R] = -0.22$	3	C_0, C_1, C_6, C_5	2.78	494.5
4	$r[C_5 \mid \Delta R] = 0.42$				
5	$r[C_6 \mid \Delta R] = 0.17$				
6	$r[C_7 \mid \Delta R] = -0.64$				
7	$r[C_8 \mid \Delta R] = 0.53$	Final	C_0, C_1, C_6, C_5, C_7	2.39	76.9
1	$r[C_2 \mid \Delta A] = 0.78$	1	D_0, C_2	0.0061	358.7
2	$r[D_3 \mid \Delta A] = 0.41$				
3	$r[D_5 \mid \Delta A] = 0.18$				
4	$r[D_6 \mid \Delta A] = 0.25$				
5	$r[D_7 \mid \Delta A] = -0.24$				
6	$r[D_8 \mid \Delta A] = 0.71$				
7	$r[C_5 \mid \Delta A] = 0.56$				
8	$r[C_6 \mid \Delta A] = 0.10$	Final	D_0, C_2, D_7	0.0042	234.9
1	$r[C_2 \mid \Delta E] = -0.89$	1	F_0, C_7	0.0050	2780.7
2	$e[F_3 \mid \Delta E] = -0.29$	2	F_0, C_7, C_2	0.0049	13.1
3	$r[D_8 \mid \Delta E] = -0.96$				
4	$r[D_7 \mid \Delta E] = -0.93$				
5	$r[C_4 \mid \Delta E] = -0.45$				
6	$r[C_5 \mid \Delta E] = 0.08$				
7	$r[C_6 \mid \Delta E] = 0.65$				
8	$r[C_7 \mid \Delta E] = -0.96$				
		Final	F_0, C_7, C_2, F_3	0.0048	9.5

TABLE F-III. RADAR 7.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.36$	1	C_0, C_7	3.35	3089.3
2	$r[C_2 \mid \Delta R] = 0.15$	2	C_0, C_7, C_6	2.86	110.7
3	$r[C_4 \mid \Delta R] = 0.77$	3	C_0, C_7, C_6, C_1	1.16	1493.9
4	$r[C_5 \mid \Delta R] = 0.31$	4	C_0, C_7, C_6, C_1, C_8	1.15	4.9
5	$r[C_6 \mid \Delta R] = 0.32$				
6	$r[C_7 \mid \Delta R] = 0.96$				
7	$r[C_8 \mid \Delta R] = 0.36$	Final	C_0, C_7, C_6, C_1, C_8, C_2	1.13	15.0
1	$r[C_2 \mid \Delta A] = 0.28$	1	D_0, D_7	0.0037	76.6
2	$r[D_3 \mid \Delta A] = -0.44$				
3	$r[D_5 \mid \Delta A] = -0.09$				
4	$r[D_6 \mid \Delta A] = -0.09$				
5	$r[D_7 \mid \Delta A] = 0.45$				
6	$r[D_8 \mid \Delta A] = -0.06$				
7	$r[C_5 \mid \Delta A] = -0.05$				
8	$r[C_6 \mid \Delta A] = -0.37$	Final	Only 1 step.		
1	$r[C_2 \mid \Delta E] = 0.03$	1	F_0, D_7	0.0158	22.9
2	$r[F_3 \mid \Delta E] = 0.22$				
3	$r[D_8 \mid \Delta E] = 0.01$				
4	$r[D_7 \mid \Delta E] = -0.27$				
5	$r[C_4 \mid \Delta E] = 0.07$				
6	$r[C_5 \mid \Delta E] = 0.004$				
7	$r[C_6 \mid \Delta E] = 0.24$				
8	$r[C_7 \mid \Delta E] = -0.16$				
		Final	Only 1 step.		

TABLE F-IV. RADAR 3.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = 0.47$	1	C_0, C_5	5.61	1669.0
2	$r[C_2 \mid \Delta R] = 0.73$	2	C_0, C_5, C_1	2.36	1834.7
3	$r[C_4 \mid \Delta R] = 0.25$	3	C_0, C_5, C_1, C_7	2.03	136.1
4	$r[C_5 \mid \Delta R] = 0.90$	4	C_0, C_5, C_1, C_7, C_8	1.93	45.3
5	$r[C_6 \mid \Delta R] = 0.62$	5	$C_0, C_5, C_1, C_7, C_8, C_2$	1.87	24.2
6	$r[C_7 \mid \Delta R] = -0.26$				
7	$r[C_8 \mid \Delta R] = 0.74$	Final	$C_0, C_5, C_1, C_7, C_8, C_2, C_6$	1.82	22.8
1	$r[C_2 \mid \Delta A] = 0.62$	1	D_0, C_2	0.0028	243.7
2	$r[D_3 \mid \Delta A] = 0.43$	2	D_0, C_2, D_3	0.0025	117.1
3	$r[D_5 \mid \Delta A] = 0.30$	3	D_0, C_2, D_3, D_6	0.0024	7.9
4	$r[D_6 \mid \Delta A] = 0.32$				
5	$r[D_7 \mid \Delta A] = -0.60$				
6	$r[D_8 \mid \Delta A] = 0.41$				
7	$r[C_5 \mid \Delta A] = 0.07$				
8	$r[C_6 \mid \Delta A] = 0.51$	Final	D_0, C_2, D_3, D_6, D_7	0.0024	35.0
1	$r[C_2 \mid \Delta E] = -0.11$	1	F_0, C_6	0.0053	104.1
2	$r[F_3 \mid \Delta E] = 0.44$	2	F_0, C_6, C_5	0.0051	25.2
3	$r[D_8 \mid \Delta E] = -0.08$	3	F_0, C_6, C_5, D_7	0.0044	133.9
4	$r[D_7 \mid \Delta E] = -0.42$	4	F_0, C_6, C_5, D_7, C_4		
5	$r[C_4 \mid \Delta E] = -0.13$				
6	$r[C_5 \mid \Delta E] = -0.01$				
7	$r[C_6 \mid \Delta E] = 0.46$				
8	$r[C_7 \mid \Delta E] = -0.23$				
		Final	$F_0, C_6, C_5, D_7, C_4, F_3$	0.0043	16.8

TABLE F-V. RADAR 67.16 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.88$	1	C_0, C_1	14.66	990.2
2	$r[C_2 \mid \Delta R] = -0.49$	2	C_0, C_1, C_6	7.96	687.8
3	$r[C_4 \mid \Delta R] = 0.83$	3	C_0, C_1, C_6, C_7	3.91	898.3
4	$r[C_5 \mid \Delta R] = -0.25$	4	C_0, C_6, C_7	3.91	-1.6
5	$r[C_6 \mid \Delta R] = -0.78$	5	C_0, C_6, C_7, C_8	2.76	291.5
6	$r[C_7 \mid \Delta R] = 0.23$				
7	$r[C_8 \mid \Delta R] = -0.31$	Final	C_0, C_6, C_7, C_8, C_1	2.73	7.5
1	$r[C_2 \mid \Delta A] = 0.07$	1	D_0, D_3	0.0099	1178.0
2	$r[D_3 \mid \Delta A] = 0.90$	2	D_0, D_3, D_7	0.0078	182.9
3	$r[D_5 \mid \Delta A] = -0.10$	3	D_0, D_3, D_7, D_5	0.0072	47.7
4	$r[D_6 \mid \Delta A] = -0.10$				
5	$r[D_7 \mid \Delta A] = 0.84$				
6	$r[D_8 \mid \Delta A] = 0.25$				
7	$r[C_5 \mid \Delta A] = -0.11$				
8	$r[C_6 \mid \Delta A] = 0.03$	Final	D_0, D_3, D_7, D_5, C_6	0.0072	3.8
1	$r[C_2 \mid \Delta E] = 0.46$	1	F_0, F_3	0.0092	299.1
2	$r[F_3 \mid \Delta E] = 0.71$	2	F_0, F_3, D_8	0.0050	696.4
3	$r[D_8 \mid \Delta E] = 0.63$	3	F_0, F_3, D_8, C_7	0.0045	65.0
4	$r[D_7 \mid \Delta E] = 0.05$				
5	$r[C_4 \mid \Delta E] = 0.02$				
6	$r[C_5 \mid \Delta E] = 0.70$				
7	$r[C_6 \mid \Delta E] = -0.42$				
8	$r[C_7 \mid \Delta E] = 0.41$				
		Final	F_0, F_3, D_8, C_7, D_7	0.0044	10.1

TABLE F-VI. RADAR 67.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.68$	1	C_0, C_6	18.03	394.5
2	$r[C_2 \mid \Delta R] = -0.32$	2	C_0, C_6, C_1	3.61	7061.2
3	$r[C_4 \mid \Delta R] = 0.60$	3	C_0, C_6, C_1, C_7	2.18	511.4
4	$r[C_5 \mid \Delta R] = -0.004$	4	C_0, C_6, C_1, C_7, C_4	2.04	41.9
5	$r[C_6 \mid \Delta R] = -0.76$	5	$C_0, C_6, C_1, C_7, C_4, C_2$	1.95	30.0
6	$r[C_7 \mid \Delta R] = 0.33$				
7	$r[C_8 \mid \Delta R] = 0.26$	Final	C_0, C_1, C_7, C_4, C_2	1.95	0.8
1	$r[C_2 \mid \Delta A] = -0.14$	1	D_0, D_3	0.0043	131.3
2	$r[D_3 \mid \Delta A] = 0.56$	2	D_0, D_3, D_5	0.0042	14.6
3	$r[D_5 \mid \Delta A] = 0.15$				
4	$r[D_6 \mid \Delta A] = 0.13$				
5	$r[D_7 \mid \Delta A] = 0.38$				
6	$r[D_8 \mid \Delta A] = -0.06$				
7	$r[C_5 \mid \Delta A] = -0.08$				
8	$r[C_6 \mid \Delta A] = -0.05$	Final	D_0, D_3, D_5, D_7	0.0042	5.6
1	$r[C_2 \mid \Delta E] = 0.53$	1	F_0, C_2	0.0046	118.2
2	$r[F_3 \mid \Delta E] = 0.05$	2	F_0, C_2, D_7	0.0044	34.0
3	$r[D_8 \mid \Delta E] = 0.49$	3	F_0, C_2, D_7, F_3	0.0038	94.6
4	$r[D_7 \mid \Delta E] = -0.47$	4	F_0, C_2, D_7, F_3, C_5	0.0035	50.7
5	$r[C_4 \mid \Delta E] = 0.15$	5	$F_0, C_2, D_7, F_3, C_5, C_7$	0.0035	12.4
6	$r[C_5 \mid \Delta E] = 0.19$				
7	$r[C_6 \mid \Delta E] = 0.26$				
8	$r[C_7 \mid \Delta E] = 0.21$				
		Final	$F_0, C_2, D_7, F_3, C_5, C_7, C_4$	0.0034	7.9

TABLE F-VII. RADAR 1.16 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 FIRST BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = 0.64$	1	C_0, C_5	4.55	1190.1
2	$r[C_2 \mid \Delta R] = 0.89$	2	C_0, C_5, C_4	4.51	5.2
3	$r[C_4 \mid \Delta R] = 0.36$	3	C_0, C_5, C_4, C_8	4.34	18.3
4	$r[C_5 \mid \Delta R] = 0.92$	4	C_0, C_5, C_4, C_8, C_2	4.06	32.8
5	$r[C_6 \mid \Delta R] = -0.42$				
6	$r[C_7 \mid \Delta R] = -0.89$				
7	$r[C_8 \mid \Delta R] = 0.79$	Final	C_0, C_5, C_8, C_2	4.05	-0.06
1	$r[C_2 \mid \Delta A] = 0.70$	1	D_0, C_2	0.0126	216.2
2	$r[D_3 \mid \Delta A] = 0.53$	2	D_0, C_2, D_7	0.0106	95.6
3	$r[D_5 \mid \Delta A] = 0.28$	3	D_0, C_2, D_7, D_8	0.0103	12.8
4	$r[D_6 \mid \Delta A] = 0.31$				
5	$r[D_7 \mid \Delta A] = -0.41$				
6	$r[D_8 \mid \Delta A] = 0.67$				
7	$r[C_5 \mid \Delta A] = 0.43$				
8	$r[C_6 \mid \Delta A] = 0.12$	Final	D_0, C_2, D_7, D_8, C_6	0.0102	5.9
1	$r[C_2 \mid \Delta E] = 0.001$	1	F_0	0.0101	
2	$r[F_3 \mid \Delta E] = -0.03$				
3	$r[D_8 \mid \Delta E] = -0.004$				
4	$r[D_7 \mid \Delta E] = -0.01$				
5	$r[C_4 \mid \Delta E] = 0.06$				
6	$r[C_5 \mid \Delta E] = 0.02$				
7	$r[C_6 \mid \Delta E] = 0.01$				
8	$r[C_7 \mid \Delta E] = -0.01$				
		Final	Only 1 Step.		

TABLE F-VIII. RADAR 91.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 SECOND BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.97$	1	C_0, C_8	4.12	63402.7
2	$r[C_2 \mid \Delta R] = -0.85$	2	C_0, C_8, C_7	2.61	1024.5
3	$r[C_4 \mid \Delta R] = 0.21$	3	C_0, C_8, C_7, C_2	2.04	438.3
4	$r[C_5 \mid \Delta R] = -0.88$	4	C_0, C_8, C_7, C_2, C_4	1.90	99.4
5	$r[C_6 \mid \Delta R] = -0.97$				
6	$r[C_7 \mid \Delta R] = 0.99$				
7	$r[C_8 \mid \Delta R] = -0.99$	Final	$C_0, C_8, C_7, C_2, C_4, C_1$	1.77	105.8
1	$r[C_2 \mid \Delta A] = 0.14$	1	D_0, C_2	0.0037	13.5
2	$r[D_3 \mid \Delta A] = 0.08$	2	D_0, C_2, D_5	0.0037	3.9
3	$r[D_5 \mid \Delta A] = -0.01$	3	D_0, C_2, D_5, D_6	0.0037	4.7
4	$r[D_6 \mid \Delta A] = 0.005$	4	D_0, C_2, D_5, D_6, D_7	0.0036	7.8
5	$r[D_7 \mid \Delta A] = -0.14$	5	D_0, C_2, D_6, D_7	0.0036	-0.05
6	$r[D_8 \mid \Delta A] = 0.11$				
7	$r[C_5 \mid \Delta A] = 0.09$				
8	$r[C_6 \mid \Delta A] = 0.002$	Final	D_0, C_2, D_6, D_7, D_3	0.0035	36.7
1	$r[C_2 \mid \Delta E] = 0.38$	1	F_0, C_2	0.0055	118.4
2	$r[F_3 \mid \Delta E] = -0.25$				
3	$r[D_8 \mid \Delta E] = 0.37$				
4	$r[D_7 \mid \Delta E] = 0.35$				
5	$r[C_4 \mid \Delta E] = -0.02$				
6	$r[C_5 \mid \Delta E] = 0.18$				
7	$r[C_6 \mid \Delta E] = 0.14$				
8	$r[C_7 \mid \Delta E] = 0.34$				
		Final	F_0, C_2, F_3	0.0054	11.8

TABLE F-IX. RADAR 3.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 SECOND BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.03$	1	C_0, C_2	7.16	6504.7
2	$r[C_2 \mid \Delta R] = -0.98$	2	C_0, C_2, C_6	5.85	160.9
3	$r[C_4 \mid \Delta R] = -0.47$	3	C_0, C_2, C_6, C_5	4.53	214.1
4	$r[C_5 \mid \Delta R] = -0.97$	4	C_0, C_2, C_6, C_5, C_1	3.48	221.1
5	$r[C_6 \mid \Delta R] = -0.87$				
6	$r[C_7 \mid \Delta R] = 0.44$				
7	$r[C_8 \mid \Delta R] = -0.96$	Final	$C_0, C_2, C_6, C_5, C_1, C_7$	3.37	22.3
1	$r[C_2 \mid \Delta A] = -0.16$	1	D_0, D_3	0.0038	44.1
2	$r[D_3 \mid \Delta A] = 0.34$	2	D_0, D_3, D_7	0.0037	17.8
3	$r[D_5 \mid \Delta A] = -0.21$	3	D_0, D_3, D_7, D_5	0.0036	21.5
4	$r[D_6 \mid \Delta A] = -0.22$	4	D_0, D_3, D_7, D_5, D_6	0.0035	8.2
5	$r[D_7 \mid \Delta A] = -0.24$				
6	$r[D_8 \mid \Delta A] = -0.14$				
7	$r[C_5 \mid \Delta A] = -0.01$				
8	$r[C_6 \mid \Delta A] = 0.19$	Final	D_0, D_3, D_5, D_6	0.0035	-0.9
1	$r[C_2 \mid \Delta E] = -0.08$	1	F_0, C_7	0.0048	267.6
2	$r[F_3 \mid \Delta E] = 0.53$				
3	$r[D_8 \mid \Delta E] = 0.08$				
4	$r[D_7 \mid \Delta E] = -0.58$				
5	$r[C_4 \mid \Delta E] = -0.51$				
6	$r[C_5 \mid \Delta E] = -0.001$				
7	$r[C_6 \mid \Delta E] = 0.52$				
8	$r[C_7 \mid \Delta E] = -0.67$				
		Final	Only 1 step		

TABLE F-X. RADAR 19.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 SECOND BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.63$	1	C_0, C_2	8.29	11057.6
2	$r[C_2 \mid \Delta R] = -0.98$	2	C_0, C_2, C_8	4.73	1019.6
3	$r[C_4 \mid \Delta R] = 0.27$				
4	$r[C_5 \mid \Delta R] = -0.97$				
5	$r[C_6 \mid \Delta R] = -0.97$				
6	$r[C_7 \mid \Delta R] = 0.66$				
7	$r[C_8 \mid \Delta R] = -0.98$	Final	C_0, C_2, C_8, C_7	4.12	153.8
1	$r[C_2 \mid \Delta A] = -0.48$	1	D_0, D_5	0.0051	185.9
2	$r[D_3 \mid \Delta A] = 0.48$	2	D_0, D_5, D_3	0.0043	198.9
3	$r[D_5 \mid \Delta A] = -0.52$	3	D_0, D_5, D_3, D_7	0.0042	26.5
4	$r[D_6 \mid \Delta A] = -0.50$	4	D_0, D_5, D_3, D_7, D_8	0.0042	3.9
5	$r[D_7 \mid \Delta A] = -0.37$	5	D_0, D_5, D_3, D_7, D_8, D_6	0.0039	84.3
6	$r[D_8 \mid \Delta A] = -0.42$				
7	$r[C_5 \mid \Delta A] = -0.41$				
8	$r[C_6 \mid \Delta A] = -0.00$	Final	D_0, D_3, D_7, D_8, D_6	0.0039	-1.5
1	$r[C_2 \mid \Delta E] = -0.18$	1	F_0, C_7	0.0068	739.6
2	$r[F_3 \mid \Delta E] = 0.40$	2	F_0, C_7, C_4	0.0065	44.3
3	$r[D_8 \mid \Delta E] = -0.35$	3	F_0, C_7, C_4, D_7	0.0064	26.9
4	$r[D_7 \mid \Delta E] = -0.75$	4	F_0, C_4, D_7	0.0064	-1.4
5	$r[C_4 \mid \Delta E] = -0.76$	5	F_0, C_4, D_7, F_3	0.0063	14.7
6	$r[C_5 \mid \Delta E] = 0.02$	6	F_0, C_4, D_7, F_3, D_8	0.0062	13.4
7	$r[C_6 \mid \Delta E] = 0.33$				
8	$r[C_7 \mid \Delta E] = -0.76$				
		Final	F_0, C_4, F_3, D_8	0.0062	-0.2

TABLE F-XI. RADAR 67.18 STEPWISE REGRESSION ANALYSIS RESULTS FOR AS-501 SECOND BURN DATA

Variable No.	Linear Correlation	Step No.	Variables in Regression	σ_Y	F Level
1	$r[C_1 \mid \Delta R] = -0.98$	1	C_0, C_1	8.73	28286.9
2	$r[C_2 \mid \Delta R] = -0.62$	2	C_0, C_1, C_6	4.89	1889.6
3	$r[C_4 \mid \Delta R] = 0.44$	3	C_0, C_1, C_6, C_4	4.76	46.9
4	$r[C_5 \mid \Delta R] = -0.62$	4	C_0, C_1, C_6, C_4, C_2	4.58	70.1
5	$r[C_6 \mid \Delta R] = -0.94$	5	$C_0, C_1, C_6, C_4, C_2, C_5$	4.23	148.4
6	$r[C_7 \mid \Delta R] = 0.30$	6	C_0, C_6, C_4, C_2, C_5	4.23	-0.05
7	$r[C_8 \mid \Delta R] = -0.96$	7	$C_0, C_6, C_4, C_2, C_5, C_8$	4.13	39.7
		8	$C_0, C_6, C_4, C_2, C_5, C_8, C_7$	3.91	103.8
		Final	All	3.85	28.4
1	$r[C_2 \mid \Delta A] = -0.53$	1	D_0, C_2	0.0046	333.5
2	$r[D_3 \mid \Delta A] = -0.04$	2	D_0, C_2, D_8	0.0046	6.4
3	$r[D_5 \mid \Delta A] = 0.47$	3	D_0, C_2, D_8, D_7	0.0042	171.6
4	$r[D_6 \mid \Delta A] = 0.48$	4	D_0, C_2, D_8, D_7, D_5	0.0041	18.8
5	$r[D_7 \mid \Delta A] = -0.22$	5	$D_0, C_2, D_8, D_7, D_5, D_6$	0.0040	50.1
6	$r[D_8 \mid \Delta A] = -0.46$				
7	$r[C_5 \mid \Delta A] = 0.02$				
8	$r[C_6 \mid \Delta A] = -0.01$	Final	$D_0, C_2, D_8, D_7, D_5, D_6, D_3$	0.0040	26.9
1	$r[C_2 \mid \Delta E] = 0.001$	1	F_0, D_8	0.0057	64.8
2	$r[F_3 \mid \Delta E] = -0.001$	2	F_0, D_8, C_2	0.0054	114.8
3	$r[D_8 \mid \Delta E] = 0.26$				
4	$r[D_7 \mid \Delta E] = 0.10$				
5	$r[C_4 \mid \Delta E] = -0.16$				
6	$r[C_5 \mid \Delta E] = -0.08$				
7	$r[C_6 \mid \Delta E] = 0.02$				
8	$r[C_7 \mid \Delta E] = 0.13$	Final	F_0, D_8, C_2, C_4	0.0053	43.0

TABLE F-XII. COEFFICIENT CORRELATIONS FOR THE TRUNCATED AS-501 FIRST BURN RADAR ERROR MODELS

	C_0	C_1	C_2	D_0	D_3	D_8	F_0	F_3
C_0	1.00	-0.92	0.12	0.01	0.	-0.02	0.	0.
C_1		1.00	-0.16	-0.01	0.	0.03	0.	0.
C_2			1.00	0.05	0.01	-0.18	0.01	0.
D_0				1.00	-0.03	0.25	-0.01	-0.01
D_3					1.00	-0.12	0.01	0.01
D_8						1.00	-0.04	-0.05
F_0							1.00	0.03
F_3								1.00

Radar 67.16

	C_0	C_1	C_2	D_0	D_3	D_8	F_0	F_3
C_0	1.00	-0.88	-0.18	-0.01	0.	0.05	-0.02	0.
C_1		1.00	0.41	0.02	0.01	-0.10	0.04	0.01
C_2			1.00	0.05	0.02	-0.25	0.09	0.01
D_0				1.00	-0.03	0.26	-0.10	-0.02
D_3					1.00	-0.13	0.05	0.01
D_8						1.00	-0.39	-0.07
F_0							1.00	0.06
F_3								1.00

Radar 67.18

TABLE F-XII. (Continued

	C_0	C_2	C_4	D_0	D_3	F_0
C_0	1.00	-0.15	0.77	0.01	0.	0.02
C_2		1.00	0.25	-0.10	0.01	0.01
C_4			1.00	-0.03	0.	0.03
D_0				1.00	-0.24	0.
D_3					1.00	0.
F_0						1.00

Radar 7.18

	C_0	C_2	C_4	D_0	D_3	D_7	F_0	F_3
C_0	1.00	-0.75	0.58	0.06	0.	0.	-0.01	-0.06
C_2		1.00	-0.05	-0.01	-0.04	-0.10	0.09	0.
C_4			1.00	0.10	-0.06	-0.13	0.10	-0.12
D_0				1.00	-0.27	-0.72	0.66	-0.06
D_3					1.00	0.43	-0.39	0.03
D_7						1.00	-0.91	0.08
F_0							1.00	-0.03
F_3								1.00

Radar 0.18

	C_0	C_2	C_4	D_0	D_5	D_7	F_0	F_3
C_0	1.00	-0.82	0.63	-0.03	0.09	-0.02	-0.01	-0.17
C_2		1.00	-0.37	0.03	-0.05	-0.07	0.07	0.10
C_4			1.00	-0.02	0.13	-0.18	0.10	-0.26
D_0				1.00	-0.80	0.06	-0.06	0.
D_5					1.00	-0.56	0.52	-0.01
D_7						1.00	-0.92	0.
F_0							1.00	0.07
F_3								1.00

Radar 19.18

TABLE F-XII. (Concluded)

	C_0	C_2	C_4	D_0	D_3	F_0	F_3
C_0	1.00	-0.37	0.56	0.04	0.	0.02	0.
C_2		1.00	0.41	-0.12	0.01	0.02	0.01
C_4			1.00	-0.05	0.	0.04	0.01
D_0				1.00	0.04	0.	0.
D_3					1.00	0.	0.
F_0						1.00	0.30
F_3							1.00

Radar 3.18

	C_0	C_1	C_2	D_0	D_3	D_7	F_0
C_0	1.00	0.19	-0.61	-0.11	0.21	0.27	-0.23
C_1		1.00	0.84	-0.15	0.29	0.37	-0.31
C_2			1.00	0.18	-0.35	-0.44	0.37
D_0				1.00	-0.56	-0.80	0.73
D_3					1.00	0.68	-0.62
D_7						1.00	-0.91
F_0							1.00

Radar 1.16

TABLE F-XIII. COEFFICIENT RADAR CORRELATIONS FOR THE TRUNCATED AS-501 SECOND BURN ERROR MODELS

	C_0	C_1	C_2	D_0	D_3	D_7	F_0	F_3
C_0	1.00	-0.87	0.33	-0.01	0.	0.	0.	0.
C_1		1.00	-0.56	0.03	0.	0.01	0.	0.
C_2			1.00	-0.05	-0.01	-0.01	0.01	0.
D_0				1.00	-0.07	-0.20	0.17	-0.10
D_3					1.00	0.45	-0.37	0.21
D_7						1.00	-0.83	0.48
F_0							1.00	-0.38
F_3								1.00

Radar 19.18

	C_0	C_2	D_0	D_3	F_0	F_3
C_0	1.00	0.01	0.	0.	0.	0.
C_2		1.00	-0.06	0.	0.	0.
D_0			1.00	0.	0.	0.
D_3				1.00	0.	0.
F_0					1.00	0.25
F_3						1.00

Radar 3.18

	C_0	C_1	C_2	D_0	D_8	F_0
C_0	1.00	-0.69	-0.02	0.	0.	0.
C_1		1.00	-0.55	-0.01	0.02	0.01
C_2			1.00	0.03	-0.04	-0.02
D_0				1.00	0.22	0.12
D_8					1.00	0.53
F_0						1.00

Radar 67.18

TABLE F-XIII. (Concluded)

	C_0	C_1	C_2	D_0	D_3	F_0	F_3
C_0	1.00	-0.67	0.01	0.	0.	0.	0.
C_1		-1.00	-0.69	0.04	0.	0.	0.
C_2			1.00	-0.05	0.	0.	0.01
D_0				1.00	0.35	0.	0.
D_3					1.00	0.	0.
F_0						1.00	0.44
F_3							1.00

Radar 91.18

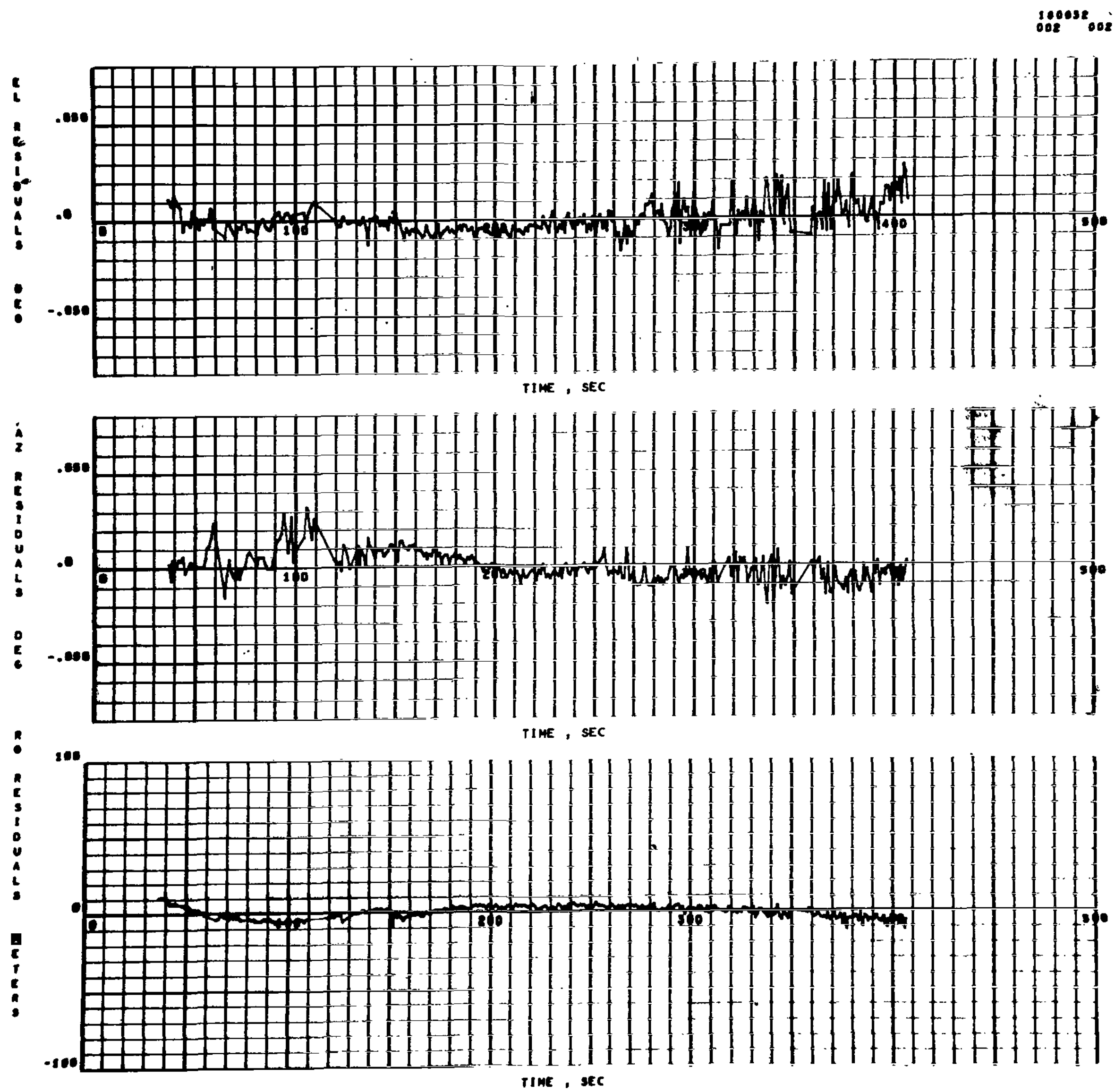

FIGURE F-1. RADAR 0.18 RESIDUALS ON AS-501 FIRST BURN DATA

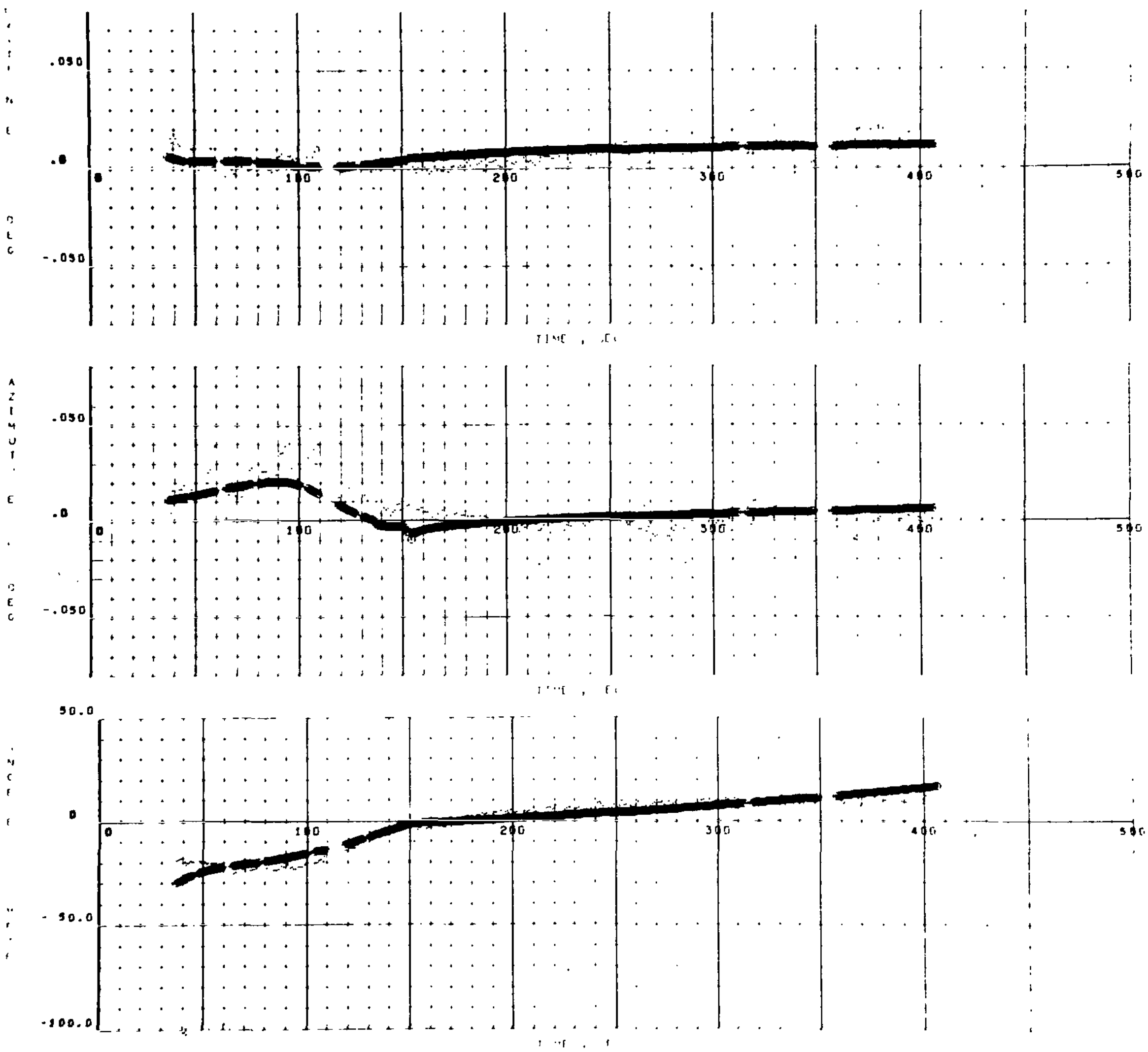

FIGURE F-2. RADAR 0.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

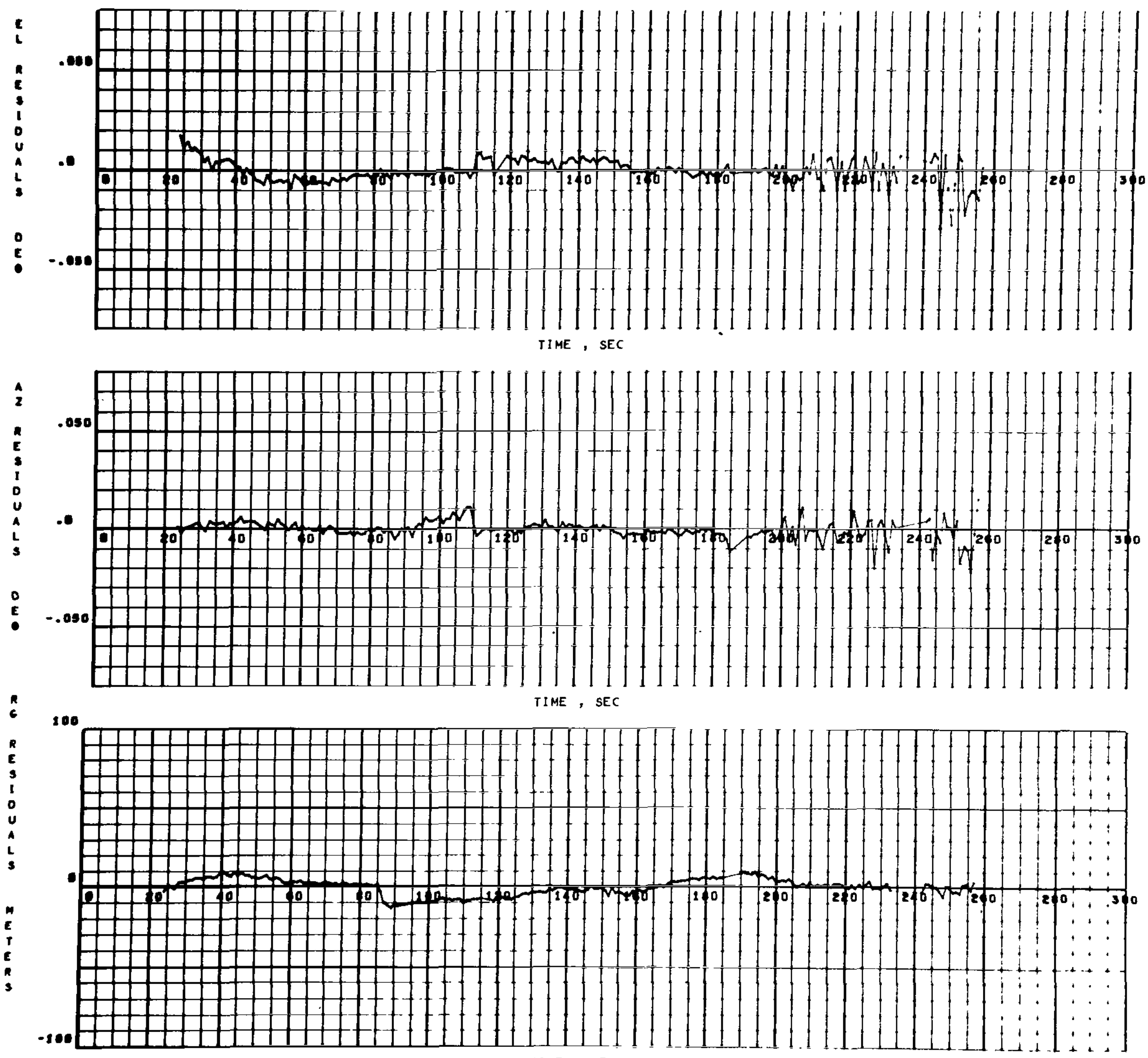

FIGURE F-3. RADAR 19.18 RESIDUALS ON AS-501 FIRST BURN DATA

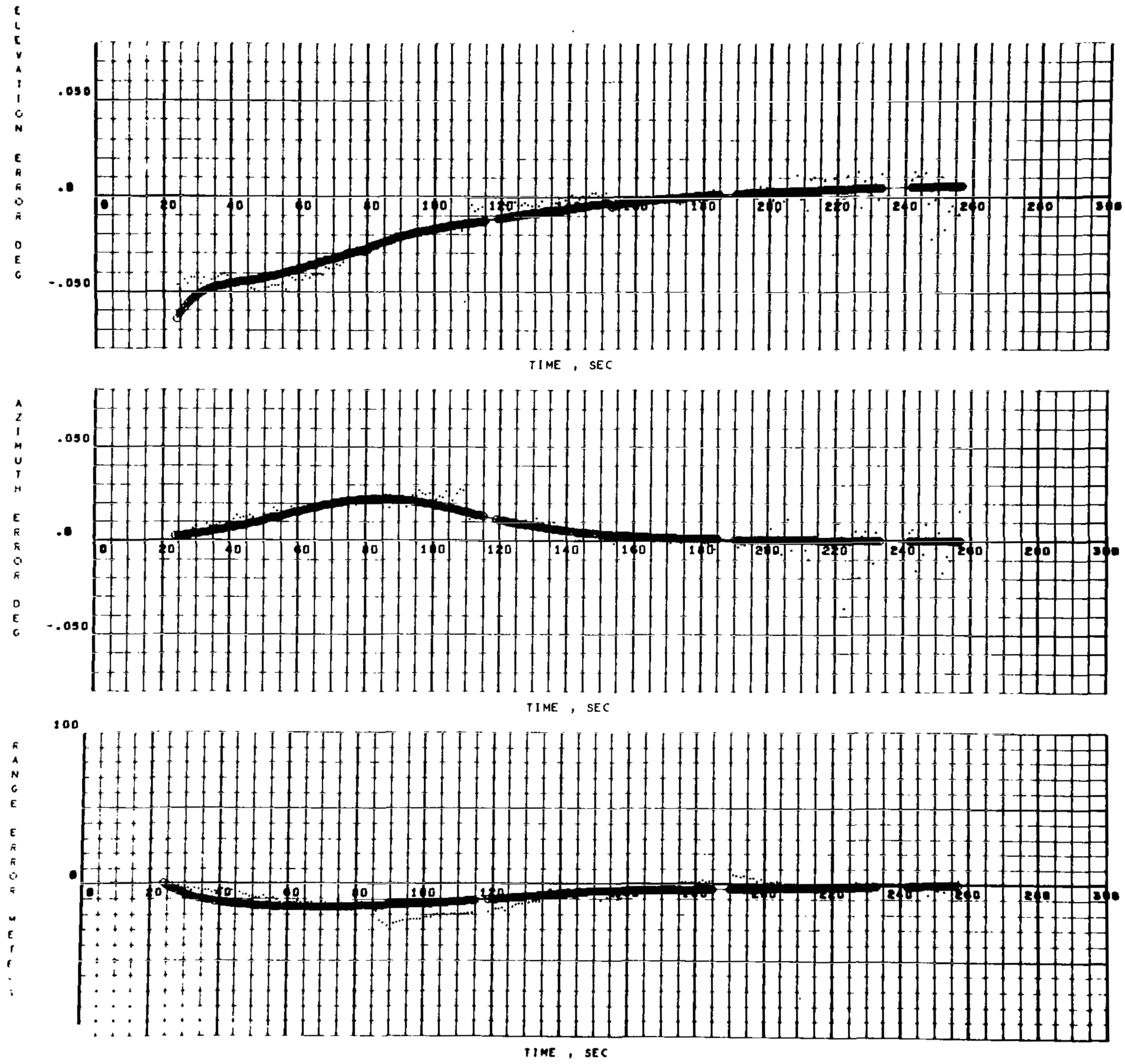

FIGURE F-4. RADAR 19.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

FIGURE F-5. RADAR 3.18 RESIDUALS ON AS-501 FIRST BURN DATA

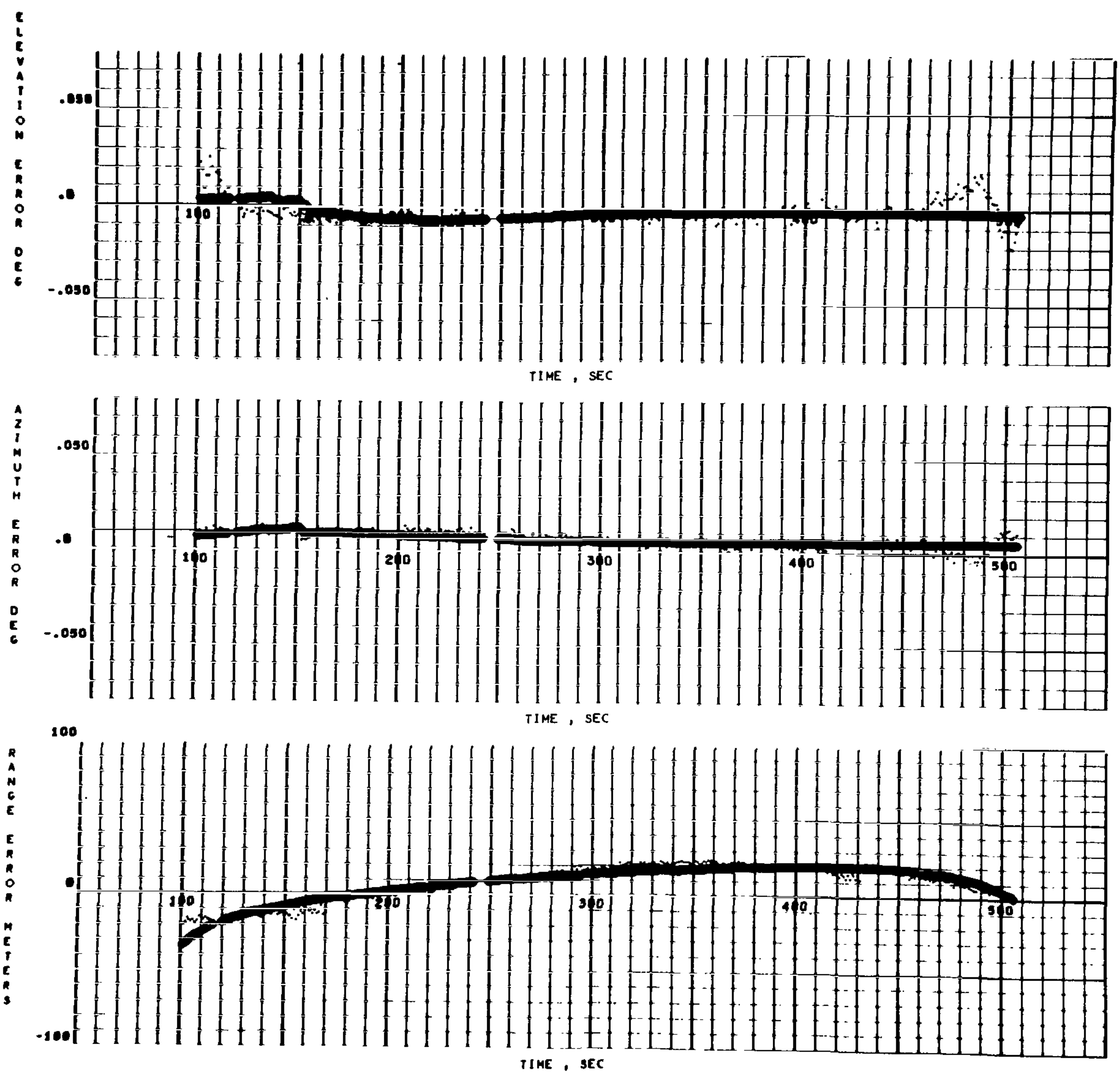

FIGURE F-6. RADAR 3.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

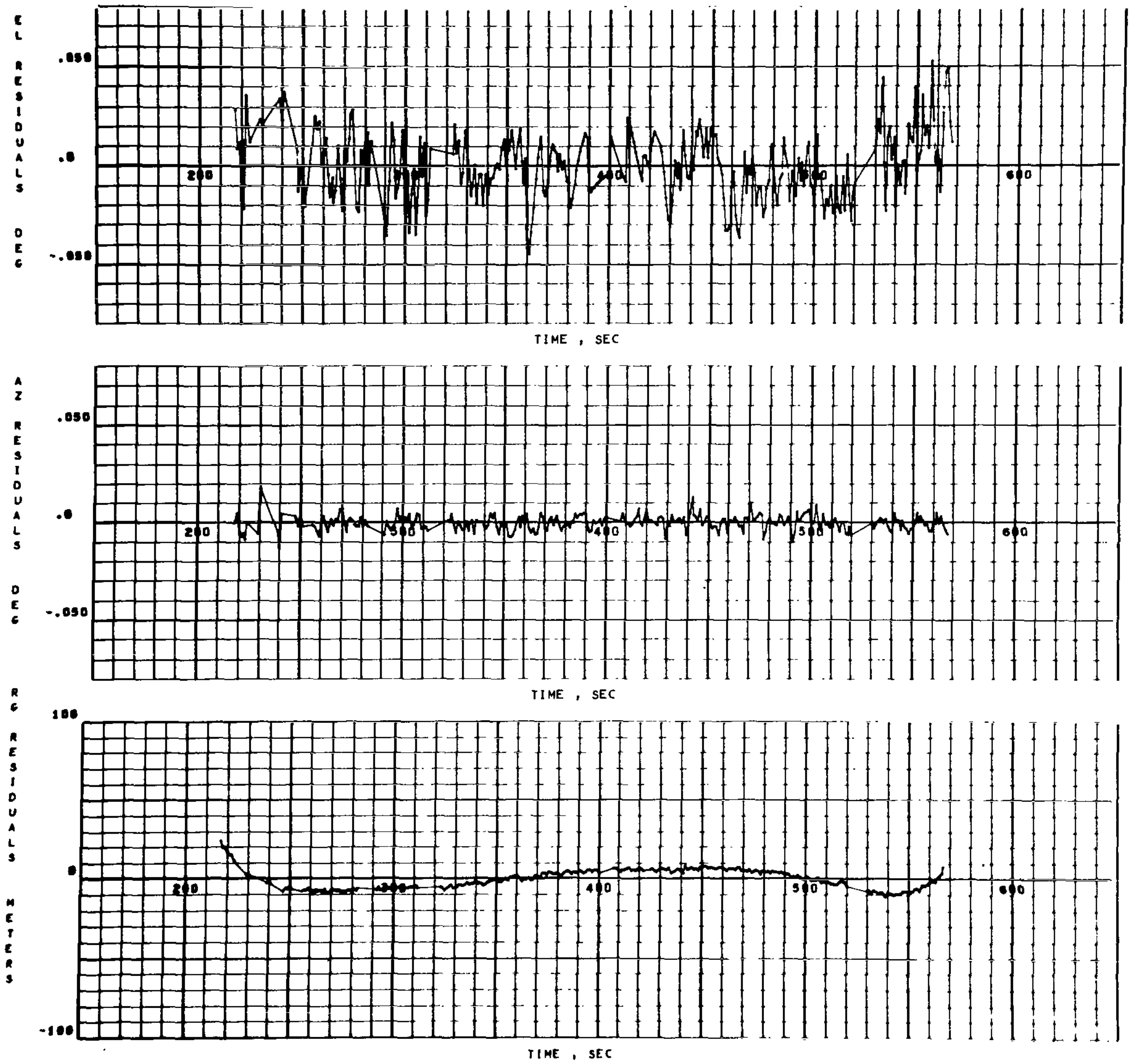

FIGURE F-7. RADAR 7.18 RESIDUALS ON AS-501
FIRST BURN DATA

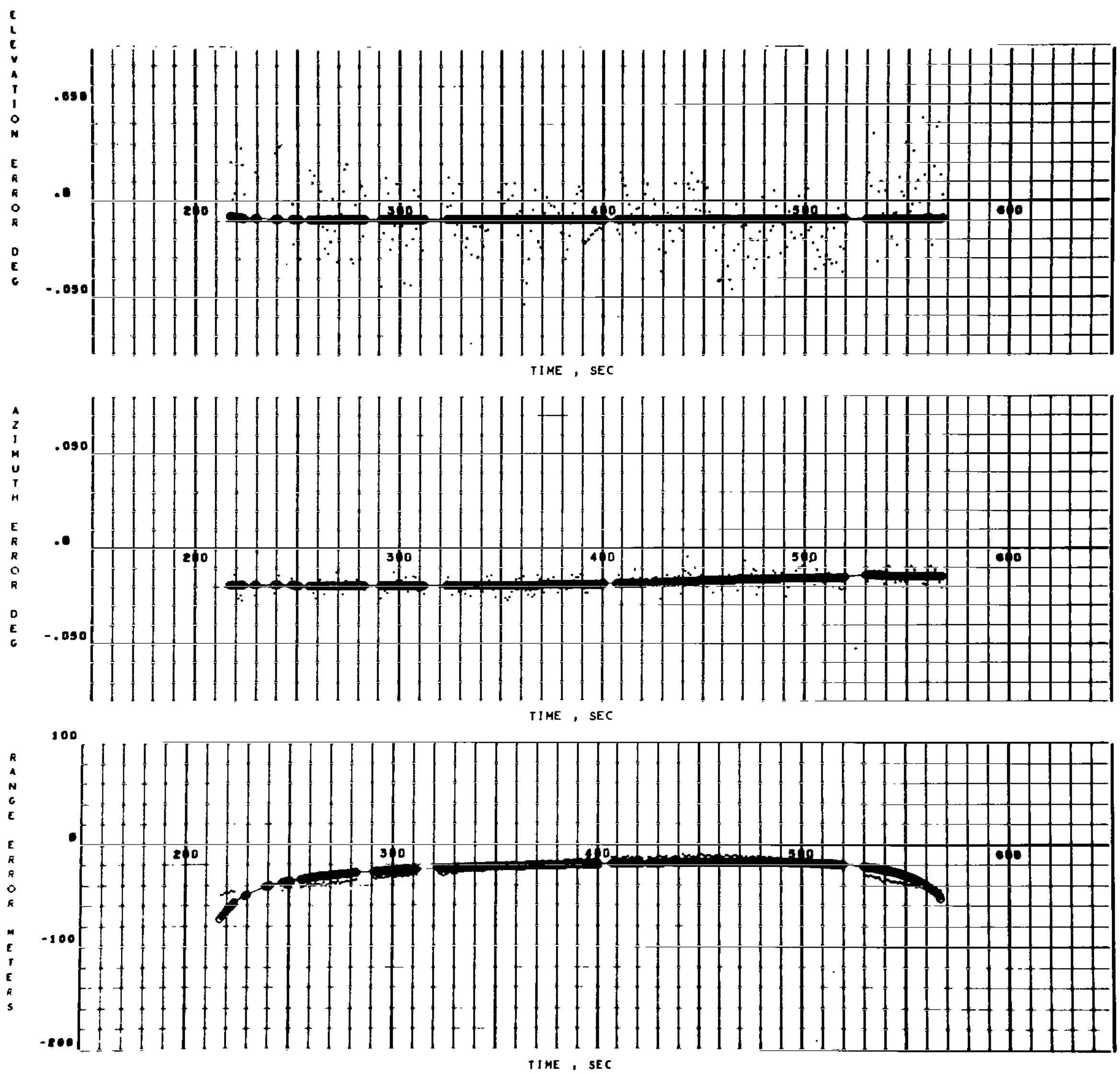

FIGURE F-8. RADAR 7.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

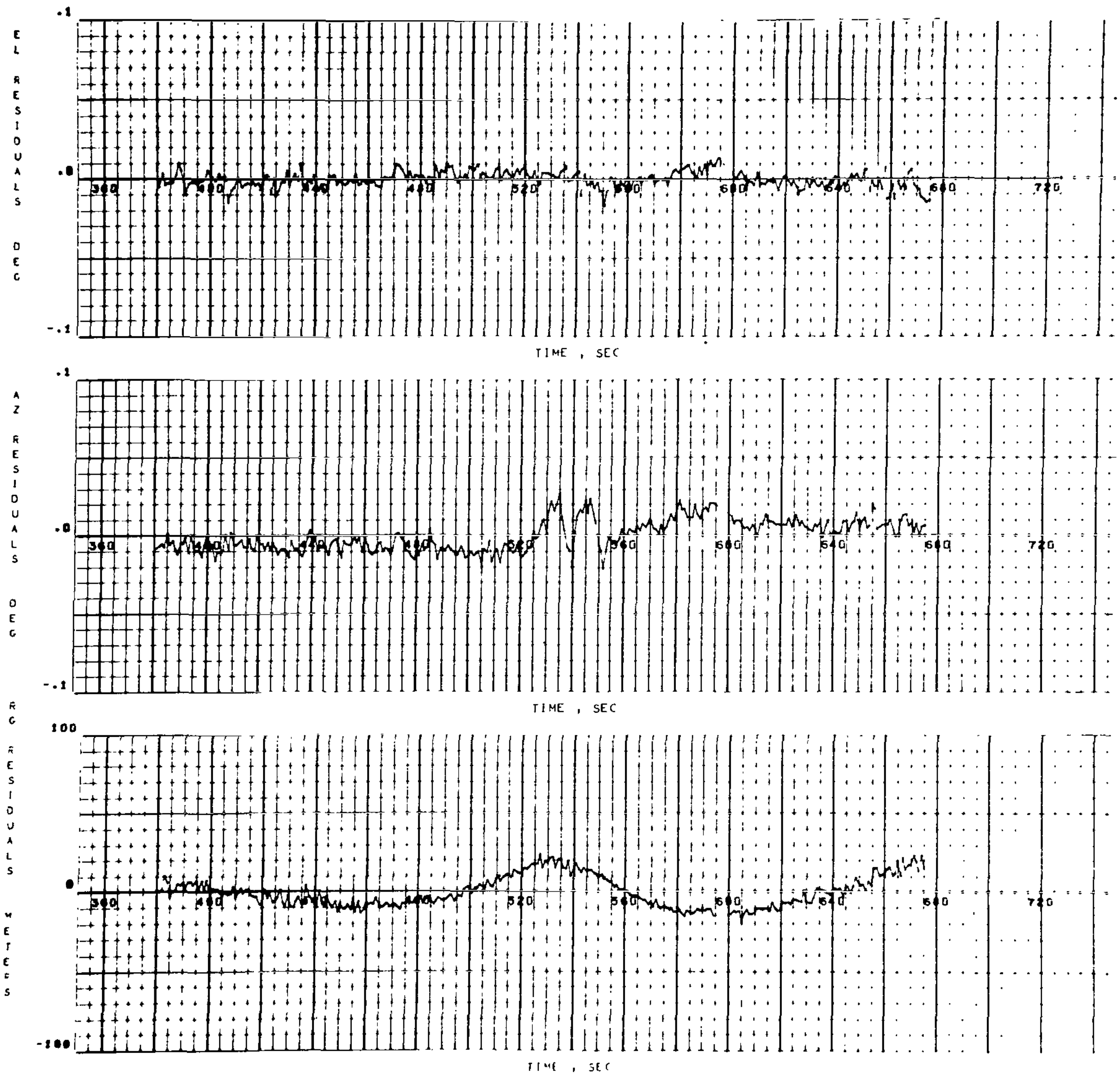

FIGURE F-9. RADAR 67.16 RESIDUALS ON AS-501 FIRST BURN DATA

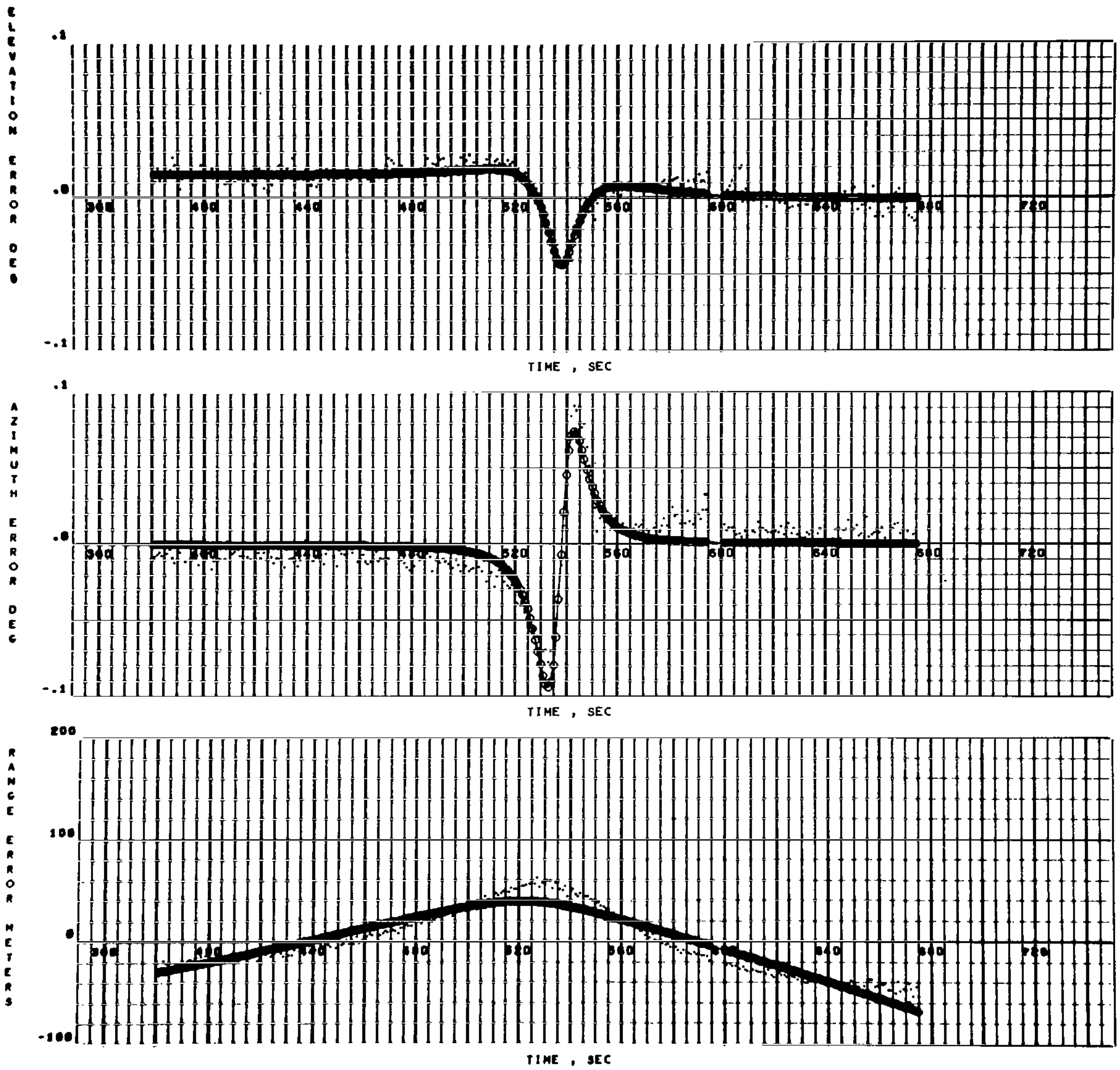

FIGURE F-10. RADAR 67.16 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

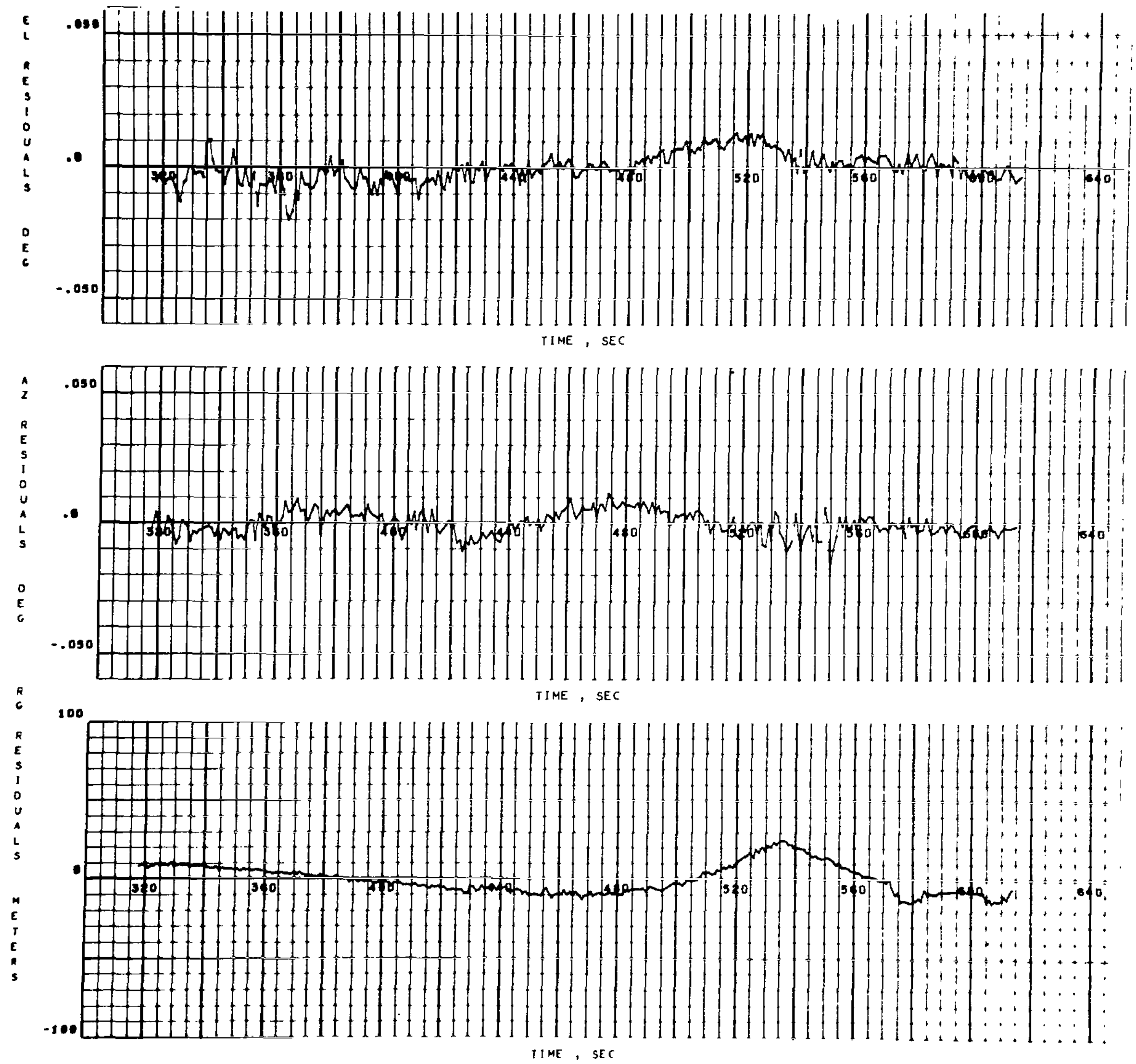

FIGURE F-11. RADAR 67.18 RESIDUALS ON AS-501 FIRST BURN DATA

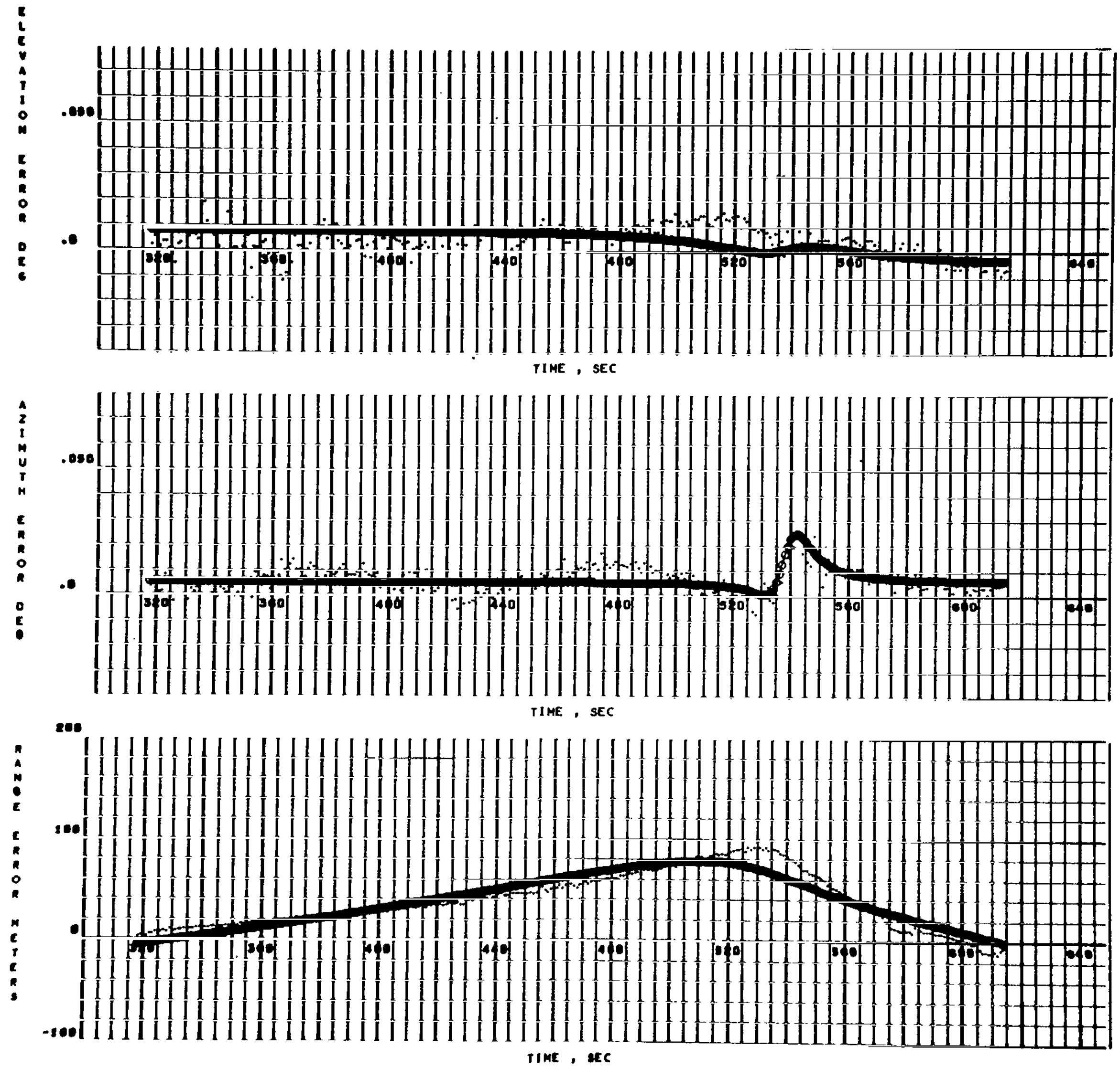

FIGURE F-12. RADAR 67.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

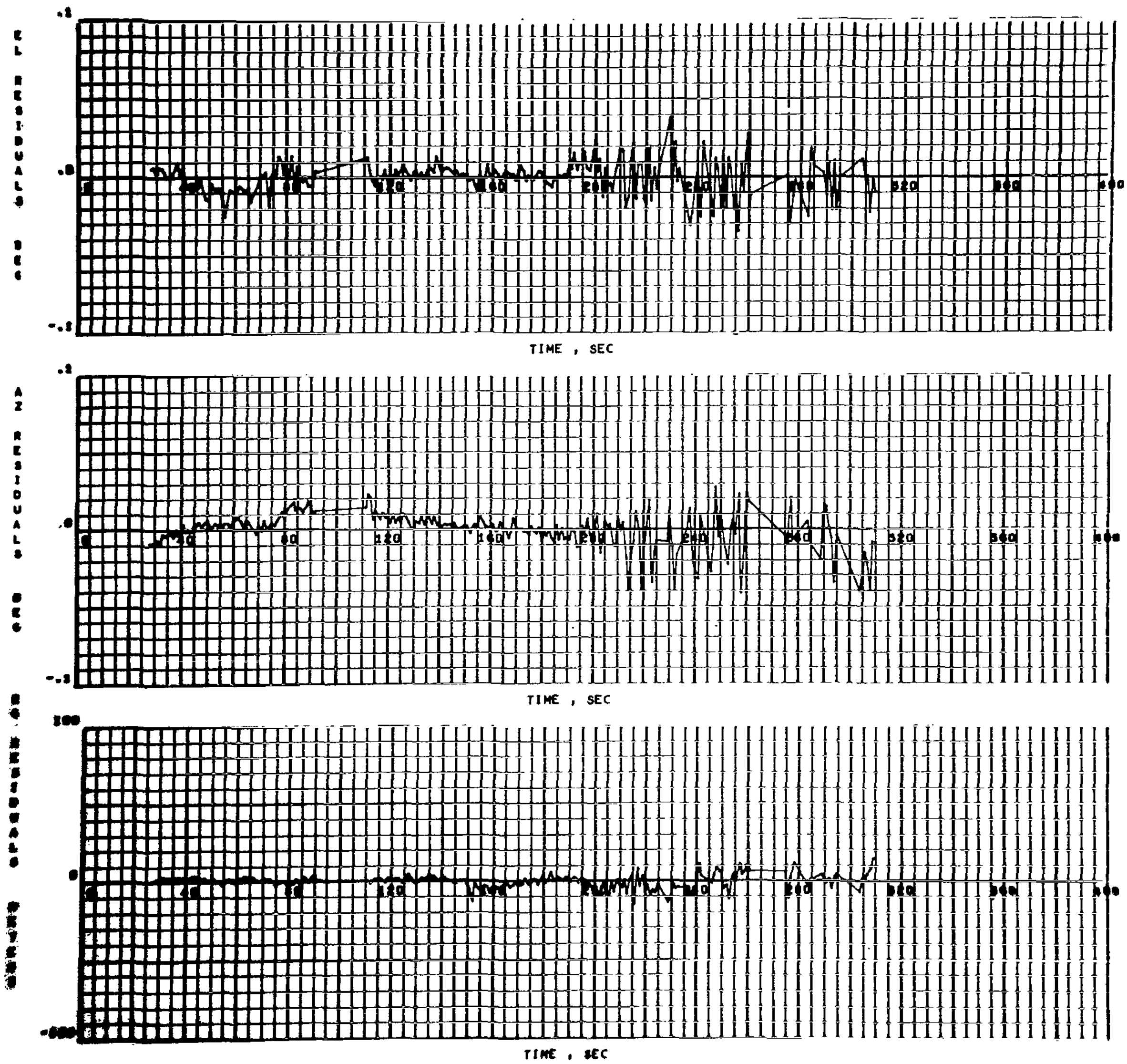

FIGURE F-13. RADAR 1.16 RESIDUALS ON AS-501
FIRST BURN DATA

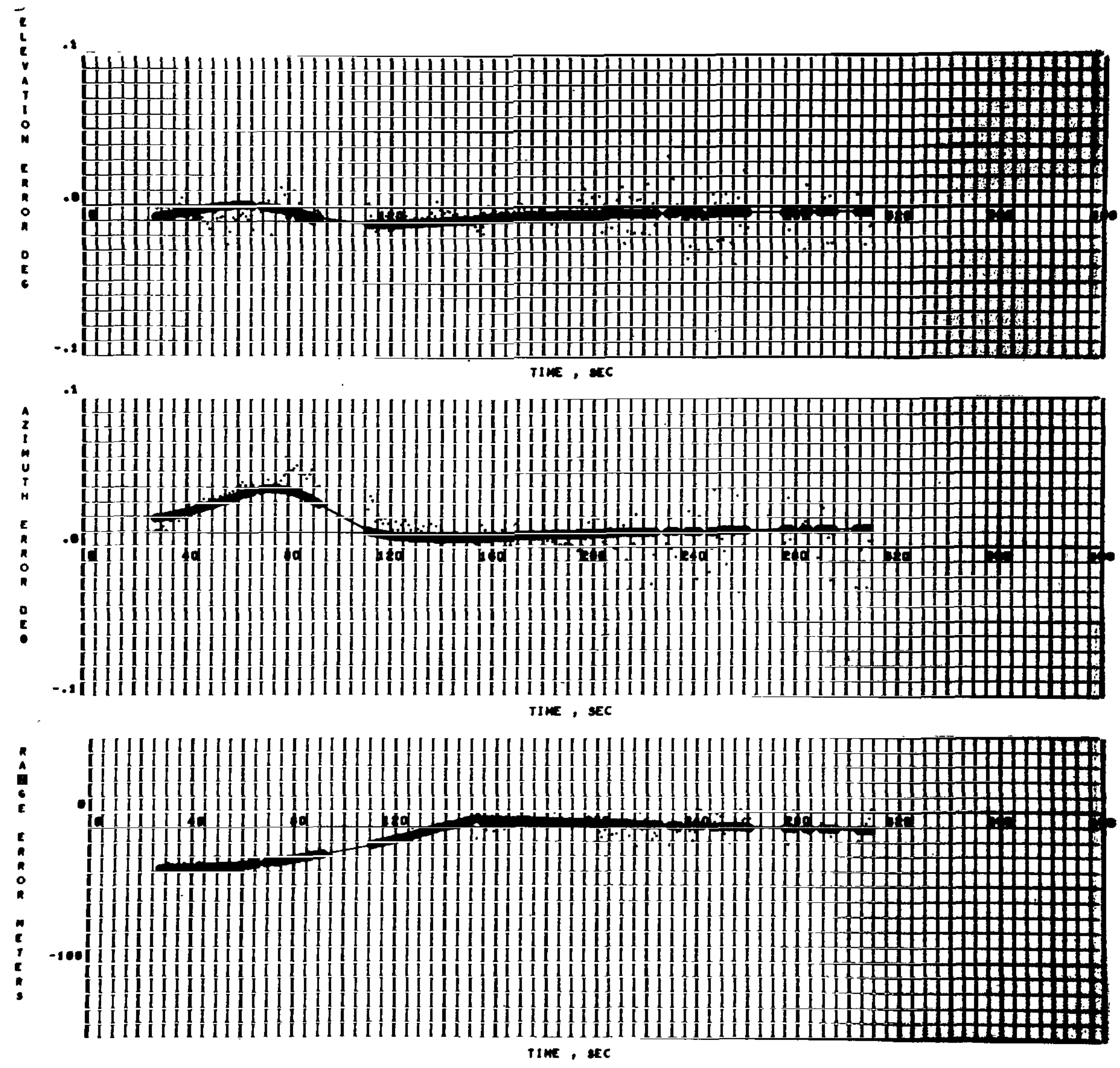

FIGURE F-14. RADAR 1.16 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 FIRST BURN DATA

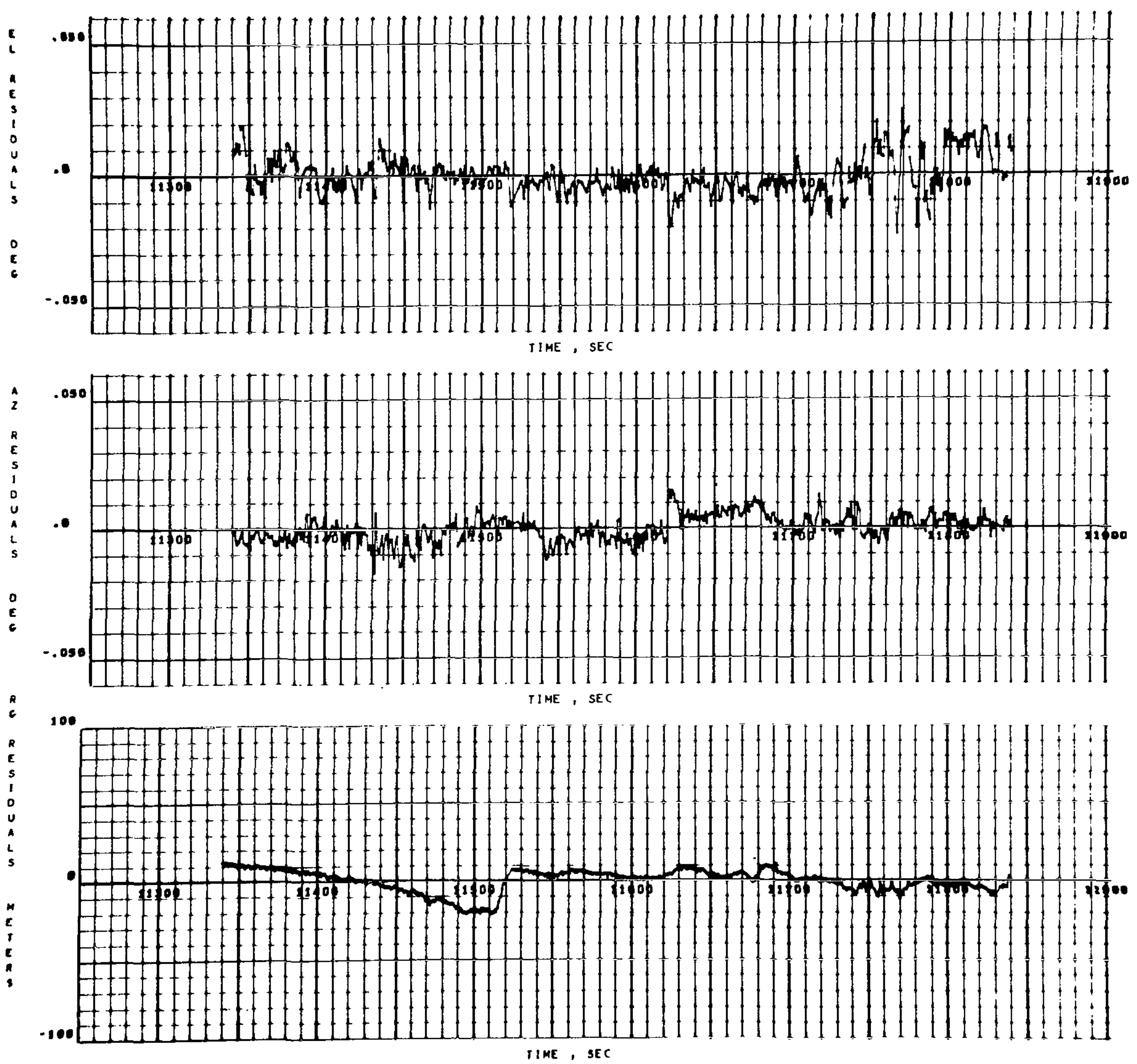

FIGURE F-15. RADAR 19.18 RESIDUALS ON AS-501 SECOND BURN DATA

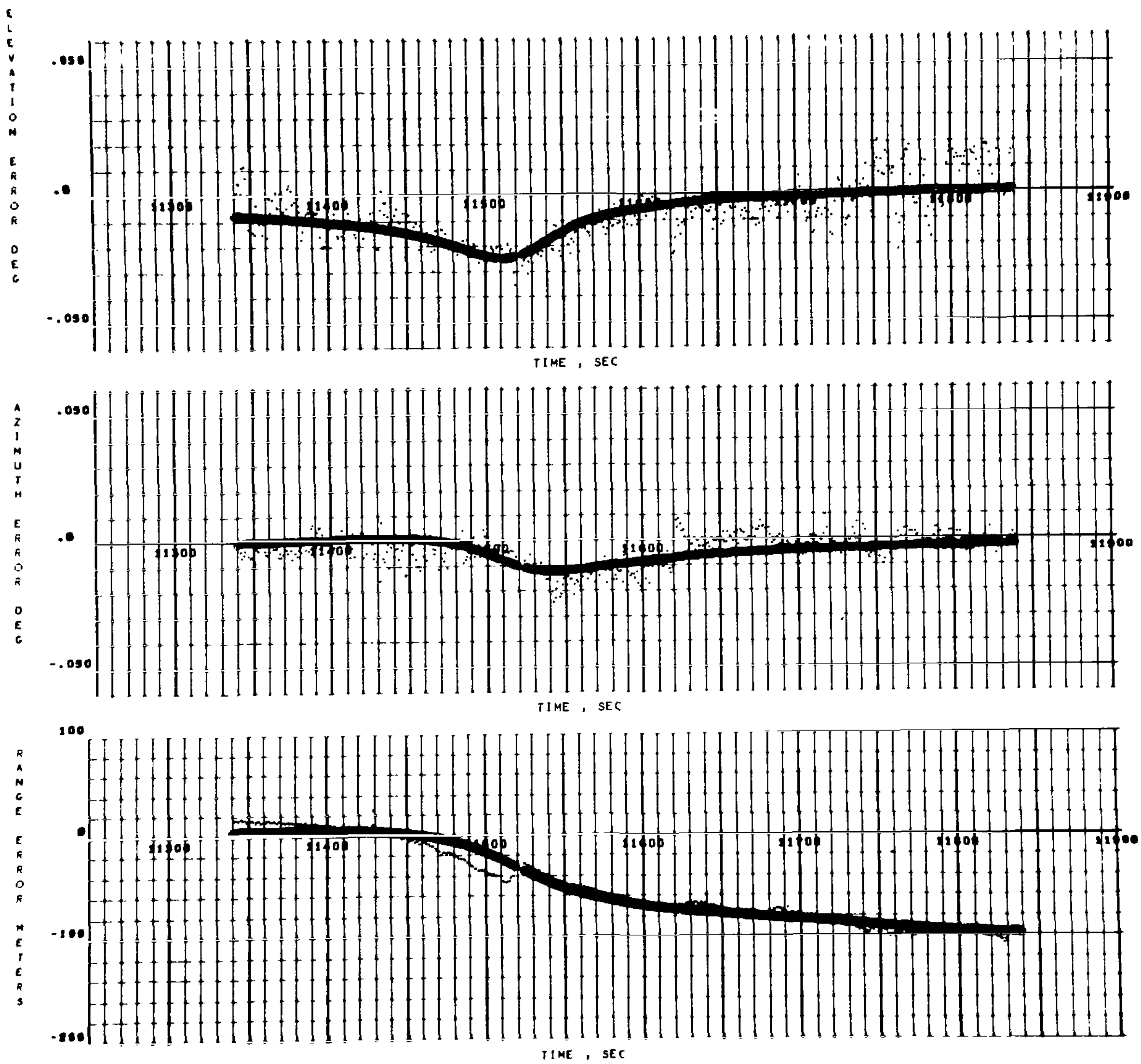

FIGURE F-16. RADAR 19.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 SECOND BURN DATA

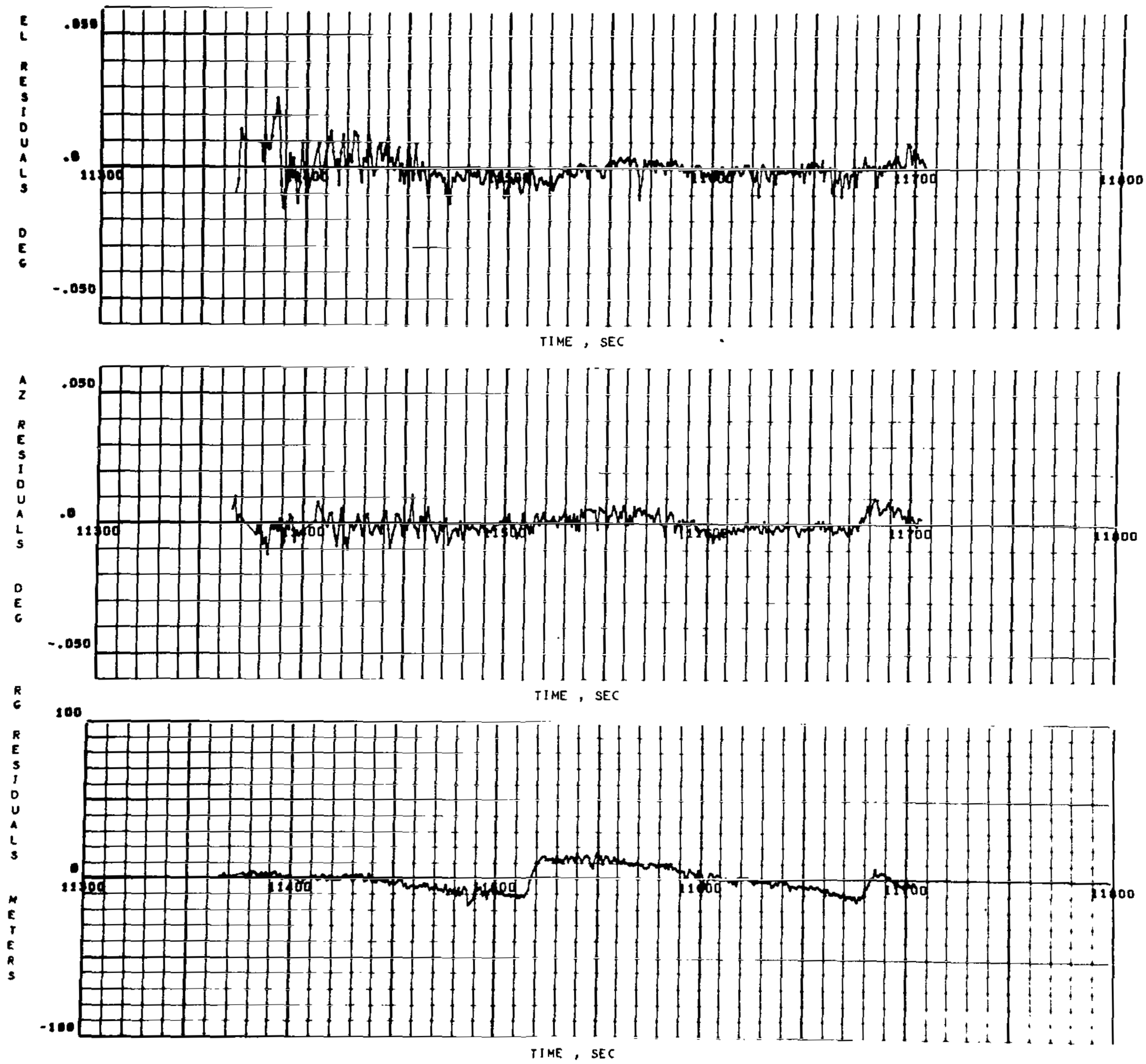

FIGURE F-17. RADAR 3.18 RESIDUALS ON AS-501 SECOND BURN DATA

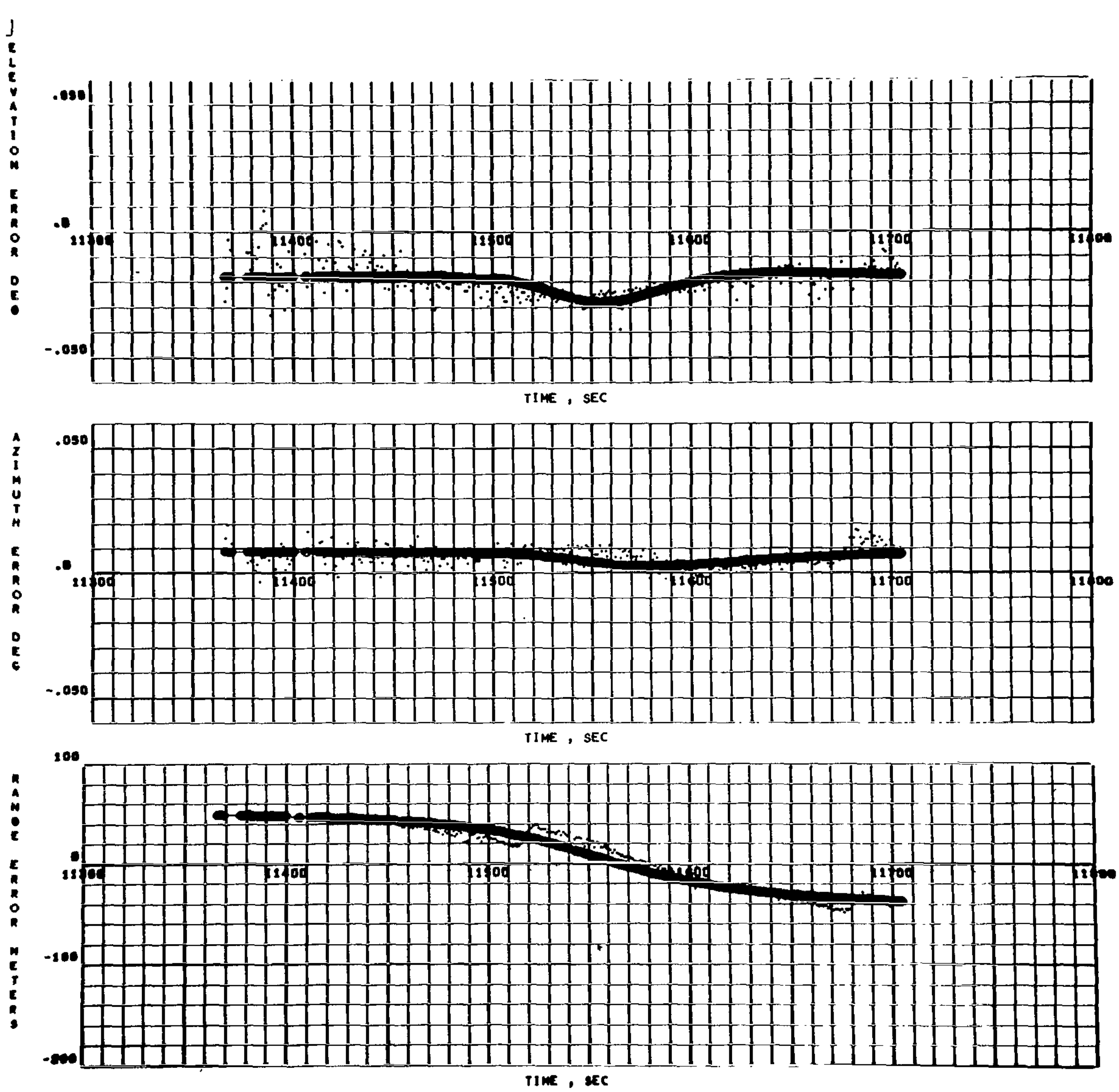

FIGURE F-18. RADAR 3.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 SECOND BURN DATA

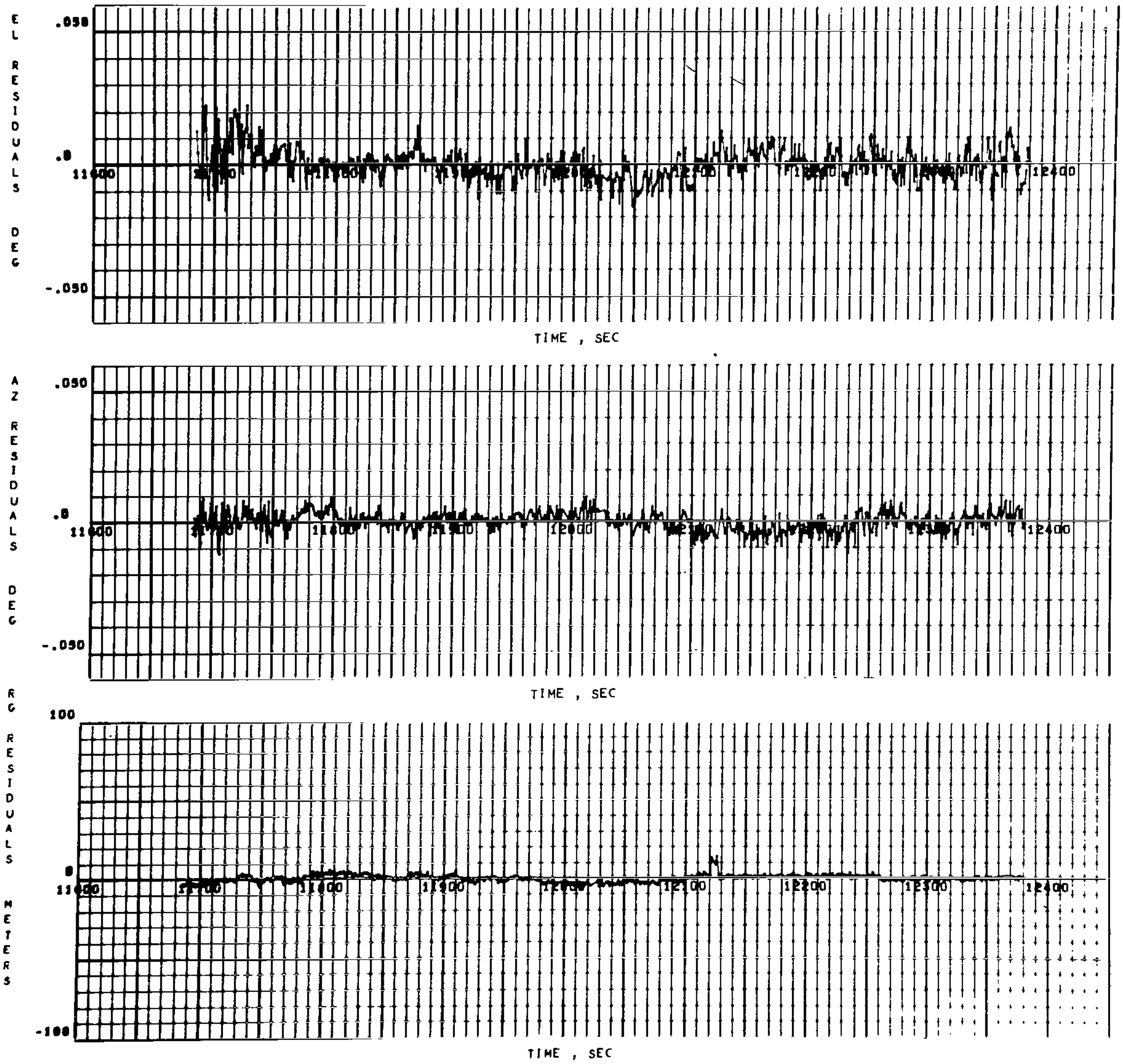

FIGURE F-19. RADAR 19.18 RESIDUALS ON AS-501 SECOND BURN DATA

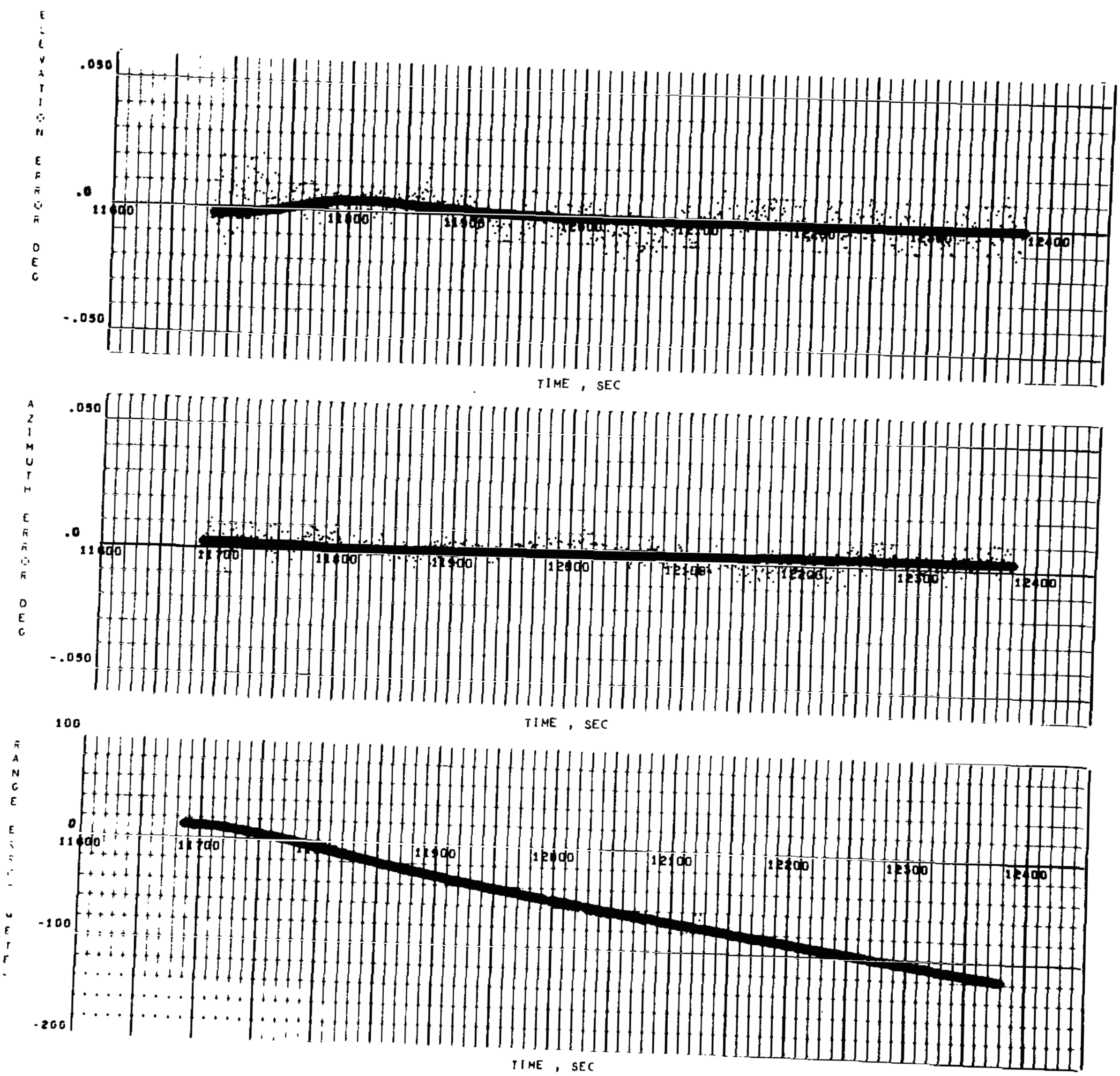

FIGURE F-20. RADAR 19.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 SECOND BURN DATA

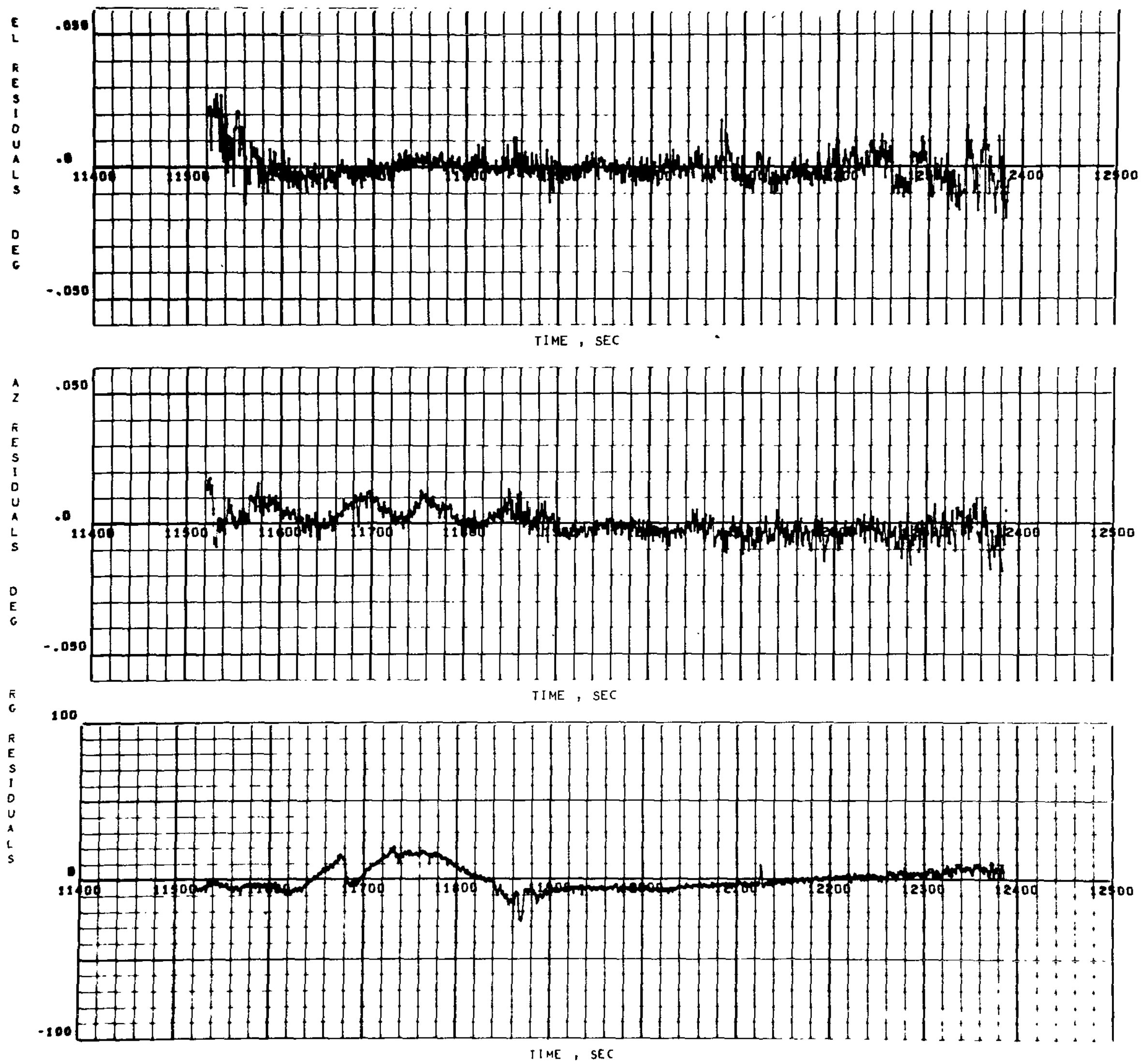

FIGURE F-21. RADAR 67.18 RESIDUALS ON AS-501 SECOND BURN DATA

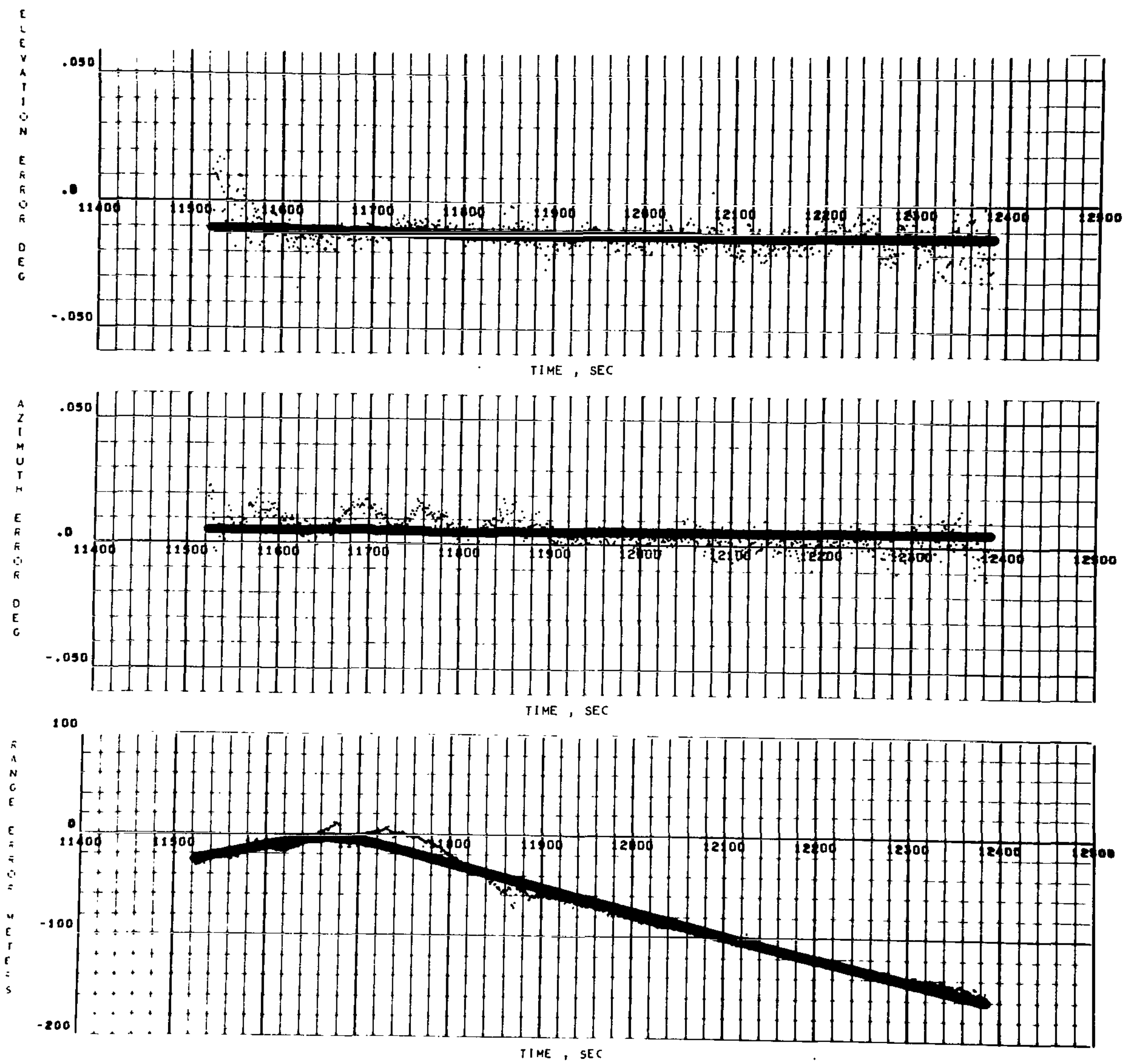

FIGURE F-22. RADAR 67.18 RANGE, AZIMUTH, AND ELEVATION ERRORS ON AS-501 SECOND BURN DATA

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