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NONLINEAR DECAY OF A DISTURBANCE IN AN UNBOUNDED VISCOUS FLUID

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

The nonlinear decay of a disturbance was calculated by an iterative solution to the Navier-Stokes equations. The general expressions obtained for successive approximations were suitable for machine computations, within the limitations of the computer capacity. For the initial condition, a three-dimensional cosine distribution with two harmonic terms was assumed. The nonlinear interaction of these harmonic terms then produced new harmonics. As the iteration process proceeded, a large number of harmonics, at wave numbers higher than, lower than, or the same as the original ones were generated. This process appeared to be similar to that occurring in the early stages of the development of turbulence by flow through a grid.

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SUMMARY

The nonlinear decay of a disturbance was calculated by an iterative solution to the Navier-Stokes equations. The general expressions obtained for successive approximations were suitable for machine computations, within the limitations of the computer capacity. For the initial condition, a three-dimensional cosine distribution with two harmonic terms was assumed. The nonlinear interaction of these harmonic terms then produced new harmonics. As the iteration process proceeded, a large number of harmonics, at wave numbers higher than, lower than, or the same as the original ones were generated. This process appeared to be similar to that occurring in the early stages of the development of turbulence by flow through a grid.

INTRODUCTION

A fundamental problem in fluid mechanics is to predict what happens to a disturbance in a viscous fluid. Besides being of interest in itself, the problem is important as a first step in understanding the dynamics of turbulence. In particular, it should give an indication of the early stages of the development of turbulence from a regular disturbance, such as that produced by the flow of a fluid through a grid.

Taylor and Green (ref. 1), starting with an initial three-dimensional sine-cosine velocity distribution, obtained successive approximations for the velocity as a function of position and time by substituting known approximations into the Navier-Stokes equations. The results, which showed that small eddies can be generated from larger ones, were limited to small times and to the first few approximations for velocities. Jain (ref. 2) obtained similar results by using the initial conditions from reference 1 in a numerical finite-difference solution of the Navier-Stokes equations.

In this analysis, an initial three-dimensional cosine velocity distribution, which is more convenient to work with than that of Taylor and Green, is used to obtain iterative solutions of the Navier-Stokes equations. Of particular importance is the fact that these

initial conditions can be conveniently used with the compact Cartesian-tensor form of the Navier-Stokes equations. Analytical expressions are first obtained for the first three approximations. In each approximation, the linear terms of the Navier-Stokes equations are considered as unknown, and the nonlinear terms as known from the preceding approximation. General expressions are then obtained which give the terms in an approximation of any order in terms of those in the preceding approximation. By using these expressions, a computer can be programmed to generate successively higher-order approximations from lower-order ones. It will later be seen that, in this process, new harmonics or eddy sizes are generated from existing ones. As the order of the approximation increases, the Reynolds number for which the solution is applicable increases. Accordingly, the upper Reynolds number for which the solution is applicable will be limited only by the computer capacity. Thus, the scope of the solution that can be feasibly obtained will be increased over that which could be obtained by carrying out the iterations algebraically by hand. The basic equations for the analysis will be considered in the next section.

BASIC EQUATIONS AND INITIAL CONDITIONS

The following is considered: given the initial velocity distribution in an unbounded viscous fluid, predict the motion at later times. The equations of motion for a viscous fluid with constant properties are, in dimensionless form,

$$\frac{\partial u_i}{\partial t} - \frac{\partial^2 u_i}{\partial x_k \partial x_k} = - \frac{\partial p}{\partial x_i} - \frac{\partial(u_i u_k)}{\partial x_k} \quad (1)$$

$$\frac{\partial u_k}{\partial x_k} = 0 \quad (2)$$

where

$$u_i = \frac{x_0}{\nu} u_i^*$$

$$p = \frac{x_0^2}{\rho \nu^2} p^*$$

$$t = \frac{\nu}{x_0^2} t^*$$

$$x_i = \frac{x_i^*}{x_0}$$

The quantity u_i^* is a velocity component, x_i^* is a space coordinate, x_0 is a characteristic length, t^* is the time, ρ is the density, and ν is the kinematic viscosity. (Note: stars on dimensional quantities are omitted on corresponding dimensionless quantities. Symbols are defined in the appendix.) The subscripts in equations (1) and (2) can take on the values 1, 2, or 3, and a repeated subscript in a term indicates a summation. To obtain an explicit equation for the pressure, the divergence of equation (1) is applied to the continuity equation (2) to get

$$\frac{\partial^2 p}{\partial x_\ell \partial x_\ell} = - \frac{\partial^2 (u_\ell u_\ell)}{\partial x_\ell \partial x_\ell} \quad (3)$$

In the remainder of the analysis, it will be convenient to use the set of equations (1) and (3), rather than (1) and (2).

The expression assumed for the initial disturbance is, in dimensionless form,

$$u_i = a_i^0 \cos \vec{q} \cdot \vec{x} + b_i^0 \cos \vec{r} \cdot \vec{x} \quad (4)$$

where

$$a_i^0 = \frac{x_0}{\nu} a_i^{0*} \quad b_i^0 = \frac{x_0}{\nu} b_i^{0*}$$

$$\vec{q} = x_0 \vec{q}^* \quad \vec{r} = x_0 \vec{r}^*$$

a_i^{0*} and b_i^{0*} are initial velocity amplitudes or Fourier coefficients of the disturbance, and \vec{q}^* and \vec{r}^* are initial wave number vectors. The quantities $\vec{q} \cdot \vec{x}$ and $\vec{r} \cdot \vec{x}$ are dot products (i. e., $\vec{q} \cdot \vec{x} = q_1 x_1 + q_2 x_2 + q_3 x_3$). In order to satisfy the continuity condition (2), the a's, b's, q's, and r's in equation (4) must be related by

and

$$\left. \begin{aligned} a_i^0 q_i &= 0 \\ b_i^0 r_i &= 0 \end{aligned} \right\} \quad (5)$$

Equation (4) could be generalized to include any number of similar harmonic terms. It represents the simplest expression that can give nonzero pressure and inertia terms when substituted into equations (1) and (3). It can be readily verified that, if only one term of equation (4) were used, the continuity conditions (5) would require the pressure and inertia terms (terms on right side of eq. (1)) to be zero, so that nonlinear effects would be absent. Because the present study is mainly concerned with nonlinear effects, two harmonic terms are included in equation (4).

ANALYSIS FOR FIRST THREE APPROXIMATIONS

As a first approximation for u_i at any time

$$u_i = a_i \cos \vec{q} \cdot \vec{x} + b_i \cos \vec{r} \cdot \vec{x} \quad (6)$$

where a_i and b_i are functions of time, the forms for which will be determined from the equations of motion. The form of equation (6) has been chosen so that it will reduce to equation (4) for $t = 0$.

Substituting equation (6) into the right side of (3) yields

$$\frac{\partial^2 (u_\ell u_k)}{\partial x_\ell \partial x_k} = q_\ell b_\ell r_k a_k \cos(\vec{q} - \vec{r}) \cdot \vec{x} - q_\ell b_\ell r_k a_k \cos(\vec{q} + \vec{r}) \cdot \vec{x} \quad (7)$$

For obtaining equation (7) and subsequent equations, the following identities are used:

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (8)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \quad (9)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \quad (10)$$

In order to be consistent with equations (3) and (7), it is assumed that

$$p = a \cos(\vec{q} - \vec{r}) \cdot \vec{x} + b \cos(\vec{q} + \vec{r}) \cdot \vec{x} \quad (11)$$

Substituting equations (7), (11), and (5) into (3) and equating coefficients of like cosine terms yield

$$a = \frac{q_\ell b_\ell r_k a_k}{q^2 - 2q_\ell r_\ell + r^2} \quad (12)$$

$$b = - \frac{q_\ell b_\ell r_k a_k}{q^2 + 2q_\ell r_\ell + r^2} \quad (13)$$

where q and r are the magnitudes of the initial wave number vectors \vec{q} and \vec{r} (i.e., $q^2 = q_i q_i$). Substituting equations (6) and (8) to (13) into the right side of (1) gives

$$\begin{aligned} - \frac{\partial p}{\partial x_i} - \frac{\partial(u_i u_k)}{\partial x_k} = & \left[\frac{q_\ell b_\ell r_k a_k}{q^2 - 2q_\ell r_\ell + r^2} (q_i - r_i) + \frac{1}{2} (a_i q_\ell b_\ell - b_i r_\ell a_\ell) \right] \sin(\vec{q} - \vec{r}) \cdot \vec{x} \\ & + \left[- \frac{q_\ell b_\ell r_k a_k}{q^2 + 2q_\ell r_\ell + r^2} (q_i + r_i) + \frac{1}{2} (a_i q_\ell b_\ell + b_i r_\ell a_\ell) \right] \sin(\vec{q} + \vec{r}) \cdot \vec{x} \end{aligned} \quad (14)$$

In order to be consistent with equations (1), (4), and (14), it is assumed that, for the second approximation for u_i ,

$$u_i = a_i \cos \vec{q} \cdot \vec{x} + b_i \cos \vec{r} \cdot \vec{x} + c_i \sin(\vec{q} - \vec{r}) \cdot \vec{x} + d_i \sin(\vec{q} + \vec{r}) \cdot \vec{x} \quad (15)$$

From the continuity condition (2) and equation (15),

$$a_i q_i = 0$$

$$b_i r_i = 0$$

$$c_i(q_i - r_i) = 0$$

$$d_i(q_i + r_i) = 0$$

To solve equation (1) by iteration, it is assumed that the linear viscous term $\partial^2 u_i / \partial x_k \partial x_k$, as well as $\partial u_i / \partial t$, is unknown at each stage of the iteration. Considering $\partial^2 u_i / \partial x_k \partial x_k$ as unknown should give a more accurate answer than would be obtained by considering it as known from the preceding approximation, as has been done in previous work. It will be shown that the present scheme gives a series of negative exponentials in time, rather than a power series, as was obtained in reference 1. Substituting equations (15) and (14) into the left and right sides of equation (1), respectively, gives, after equating coefficients of like sine and cosine terms,

$$\frac{\partial a_i}{\partial t} + q^2 a_i = 0$$

Using the initial condition (4) yields

$$a_i = a_i^0 e^{-q^2 t} \quad (16)$$

Similarly,

$$b_i = b_i^0 e^{-r^2 t} \quad (17)$$

The use of equations (16) and (17) in the right side of equation (1) (i. e., eq. (14)) then gives

$$\begin{aligned} \frac{\partial c_i}{\partial t} + (q^2 - 2q_k r_k + r^2) c_i \\ = \left[\frac{q_\ell b_\ell^0 r_k a_k^0}{q^2 - 2q_\ell r_\ell + r^2} (q_i - r_i) + \frac{1}{2} (a_i^0 q_\ell b_\ell^0 - b_i^0 r_\ell a_\ell^0) \right] e^{-(q^2 + r^2)t} \end{aligned} \quad (18)$$

or, again by use of initial condition (4),

$$c_i = -\frac{1}{4q_k r_k} \left[\frac{2a_k^0 r_k b_\ell^0 q_\ell}{q^2 - 2q_\ell r_\ell + r^2} (q_i - r_i) + a_i^0 b_\ell^0 q_\ell - b_i^0 a_\ell^0 r_\ell \right] \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 - 2q_k r_k + r^2)t} \right] \quad (19)$$

Similarly,

$$d_i = \frac{1}{4q_k r_k} \left[-\frac{2a_k^0 r_k b_\ell^0 q_\ell}{q^2 + 2q_\ell r_\ell + r^2} (q_i + r_i) + a_i^0 b_\ell^0 q_\ell + b_i^0 a_\ell^0 r_\ell \right] \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 + 2q_k r_k + r^2)t} \right] \quad (20)$$

The third approximation is obtained in similar fashion by substituting the second approximation into the right sides of equations (1) and (3).

The first, second, and third approximations for u_i may be summarized as follows:

First approximation:

$$u_i = a_i^0 e^{-q^2 t} \cos \vec{q} \cdot \vec{x} + b_i^0 e^{-r^2 t} \cos \vec{r} \cdot \vec{x} \quad (21)$$

Second approximation:

$$\begin{aligned}
u_i = & a_i^0 e^{-q^2 t} \cos \vec{q} \cdot \vec{x} + b_i^0 e^{-r^2 t} \cos \vec{r} \cdot \vec{x} \\
& - \frac{1}{4r_k q_k} \left[\frac{2a_k^0 r_k b_\ell^0 q_\ell}{q^2 - 2q_\ell r_\ell + r^2} (q_i - r_i) + a_i^0 b_\ell^0 q_\ell - b_i^0 a_\ell^0 r_\ell \right] \\
& \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 - 2q_k r_k + r^2)t} \right] \sin(\vec{q} - \vec{r}) \cdot \vec{x} \\
& + \frac{1}{4q_k r_k} \left[-\frac{2a_k^0 r_k b_\ell^0 q_\ell}{q^2 + 2q_\ell r_\ell + r^2} (q_i + r_i) + a_i^0 b_\ell^0 q_\ell + b_i^0 a_\ell^0 r_\ell \right] \\
& \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 + 2q_k r_k + r^2)t} \right] \sin(\vec{q} + \vec{r}) \cdot \vec{x} \quad (22)
\end{aligned}$$

Third approximation:

$$\begin{aligned}
u_i = & \left(a_i^0 e^{-q^2 t} + \left[\frac{1}{2q^2} (b_\ell^0 q_\ell c_k^0 q_k + c_\ell^0 q_\ell b_k^0 q_k) q_i - \frac{1}{2} (b_i^0 c_k^0 q_k + c_i^0 b_k^0 q_k) \right] \right. \\
& \times \left\{ -\frac{1}{2r^2} \left[e^{(-q^2 - 2r^2)t} - e^{-q^2 t} \right] - \frac{1}{2q_k r_k - 2r^2} \left[e^{(-q^2 + 2q_k r_k - 2r^2)t} - e^{-q^2 t} \right] \right\} \\
& + \left[\frac{1}{2q^2} (b_\ell^0 q_\ell d_k^0 q_k + d_\ell^0 q_\ell b_k^0 q_k) q_i - \frac{1}{2} (b_i^0 d_k^0 q_k + d_i^0 b_k^0 q_k) \right] \left\{ -\frac{1}{2r^2} \left[e^{(-q^2 - 2r^2)t} - e^{-q^2 t} \right] \right.
\end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{2q_k r_k + 2r^2} \left[e^{(-q^2 - 2q_k r_k - 2r^2)t} - e^{-q^2 t} \right] \Bigg\} \cos \vec{q} \cdot \vec{x} \\
& + \left(b_i^0 e^{-r^2 t} + \left[-\frac{1}{2r^2} \left(a_{\ell}^0 r_{\ell} c_k^0 r_k + c_{\ell}^0 r_{\ell} a_k^0 r_k \right) r_i + \frac{1}{2} \left(a_i^0 c_k^0 r_k + c_i^0 a_k^0 r_k \right) \right] \right. \\
& \times \left\{ -\frac{1}{2q^2} \left[e^{(-2q^2 - r^2)t} - e^{-r^2 t} \right] + \frac{1}{2q^2 - 2q_k r_k} \left[e^{(-2q^2 + 2q_k r_k - r^2)t} - e^{-r^2 t} \right] \right\} \\
& + \left[\frac{1}{2r^2} \left(a_{\ell}^0 r_{\ell} d_k^0 r_k + d_{\ell}^0 r_{\ell} a_k^0 r_k \right) r_i + \frac{1}{2} \left(-a_i^0 d_k^0 r_k - d_i^0 a_k^0 r_k \right) \right] \left\{ -\frac{1}{2q^2} \left[e^{(-2q^2 - r^2)t} - e^{-r^2 t} \right] \right. \\
& + \frac{1}{2q^2 + 2q_k r_k} \left[e^{(-2q^2 - 2q_k r_k - r^2)t} - e^{-r^2 t} \right] \Bigg\} \cos \vec{r} \cdot \vec{x} \\
& - \frac{1}{4q_k r_k} \left[\frac{2b_{\ell}^0 q_{\ell} a_k^0 r_k}{q^2 - 2q_k r_k + r^2} (q_i - r_i) + a_i^0 b_k^0 q_k - b_i^0 a_k^0 r_k \right] \\
& \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 - 2q_k r_k + r^2)t} \right] \sin(\vec{q} - \vec{r}) \cdot \vec{x}
\end{aligned}$$

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$$\begin{aligned}
& + \frac{1}{4q_k r_k} \left[- \frac{2a_{\ell}^0 r_{\ell} b_k^0 q_k}{q^2 + 2q_k r_k + r^2} (q_i + r_i) + a_i^0 b_k^0 q_k + b_i^0 a_k^0 r_k \right] \\
& \times \left[e^{-(q^2 + r^2)t} - e^{-(q^2 + 2q_k r_k + r^2)t} \right] \sin(\vec{q} + \vec{r}) \cdot \vec{x} \\
& + \left[- \frac{a_{\ell}^0 r_{\ell} c_k^0 q_k}{4q^2 - 4q_k r_k + r^2} (2q_i - r_i) - \frac{1}{2} (a_i^0 c_k^0 q_k - c_i^0 a_k^0 r_k) \right] \\
& \times \left\{ \frac{1}{2q^2 - 4q_k r_k} \left[e^{(-2q^2 - r^2)t} - e^{-(4q^2 - 4q_k r_k + r^2)t} \right] \right. \\
& \left. - \frac{1}{2q^2 - 2q_k r_k} \left[e^{(-2q^2 + 2q_k r_k - r^2)t} - e^{-(4q^2 - 4q_k r_k + r^2)t} \right] \right\} \cos(2\vec{q} - \vec{r}) \cdot \vec{x} \\
& + \left[\frac{a_{\ell}^0 r_{\ell} d_k^0 q_k}{4q^2 + 4q_k r_k + r^2} (2q_i + r_i) - \frac{1}{2} (a_i^0 d_k^0 q_k - d_i^0 a_k^0 r_k) \right] \\
& \times \left\{ \frac{1}{2q^2 + 4q_k r_k} \left[e^{(-2q^2 - r^2)t} - e^{-(4q^2 + 4q_k r_k + r^2)t} \right] \right.
\end{aligned}$$

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$$\begin{aligned}
& - \frac{1}{2q^2 + 2q_k r_k} \left[e^{(-2q^2 - 2q_k r_k - r^2)t} - e^{-(4q^2 + 4q_k r_k + r^2)t} \right] \left\{ \cos(2\vec{q} + \vec{r}) \cdot \vec{x} \right. \\
& + \left[- \frac{c_\ell^0 r_\ell b_k^0 q_k}{q^2 - 4q_k r_k + 4r^2} (q_i - 2r_i) - \frac{1}{2} \left(c_i^0 b_k^0 q_k - b_i^0 c_k^0 r_k \right) \right] \\
& \times \left\{ \frac{1}{-4q_k r_k + 2r^2} \left[e^{(-q^2 - 2r^2)t} - e^{-(q^2 - 4q_k r_k + 4r^2)t} \right] \right. \\
& + \frac{1}{2q_k r_k - 2r^2} \left[e^{(-q^2 + 2q_k r_k - 2r^2)t} - e^{-(q^2 - 4q_k r_k + 4r^2)t} \right] \left. \right\} \cos(\vec{q} - 2\vec{r}) \cdot \vec{x} \\
& + \left[\frac{d_\ell^0 r_\ell b_k^0 q_k}{q^2 + 4q_k r_k + 4r^2} (q_i + 2r_i) - \frac{1}{2} \left(d_i^0 b_k^0 q_k + b_i^0 d_k^0 r_k \right) \right] \\
& \times \left\{ \frac{1}{4q_k r_k + 2r^2} \left[e^{(-q^2 - 2r^2)t} - e^{-(q^2 + 4q_k r_k + 4r^2)t} \right] \right. \\
& - \frac{1}{2q_k r_k + 2r^2} \left[e^{(-q^2 - 2q_k r_k - 2r^2)t} - e^{-(q^2 + 4q_k r_k + 4r^2)t} \right] \left. \right\} \cos(\vec{q} + 2\vec{r}) \cdot \vec{x}
\end{aligned}$$

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$$\begin{aligned}
& + \left[-\frac{2r_i}{r^2} c_{\ell}^0 r_{\ell} d_k^0 r_k + (c_i^0 d_k^0 + d_i^0 c_k^0) r_k \right] \left[\frac{e^{(-2q^2-2r^2)t} - e^{-4r^2t}}{-q^2 + r^2} \right. \\
& - \frac{e^{(-2q^2-2q_k r_k-2r^2)t} - e^{-4r^2t}}{-2q^2 - 2q_k r_k + 2r^2} - \frac{e^{(-2q^2+2q_k r_k-2r^2)t} - e^{-4r^2t}}{-2q^2 + 2q_k r_k + 2r^2} \left. \right] \sin 2\vec{r} \cdot \vec{x} \\
& + \left[\frac{2q_i}{q^2} c_{\ell}^0 q_{\ell} d_k^0 q_k - (c_i^0 d_k^0 + d_i^0 c_k^0) q_k \right] \left[\frac{e^{(-2q^2-2r^2)t} - e^{-4q^2t}}{q^2 - r^2} \right. \\
& - \frac{e^{(-2q^2-2q_k r_k-2r^2)t} - e^{-4q^2t}}{2q^2 - 2q_k r_k - 2r^2} - \frac{e^{(-2q^2+2q_k r_k-2r^2)t} - e^{-4q^2t}}{2q^2 + 2q_k r_k - 2r^2} \left. \right] \sin 2\vec{q} \cdot \vec{x} \quad (23)
\end{aligned}$$

where a_i^0 and b_i^0 are given by the initial condition (4), and

$$c_i^0 = -\frac{1}{4q_k r_k} \left[\frac{2a_k^0 r_k b_{\ell}^0 q_{\ell}}{q^2 - 2q_{\ell} r_{\ell} + r^2} (q_i - r_i) + a_i^0 b_{\ell}^0 q_{\ell} - b_i^0 a_{\ell}^0 r_{\ell} \right] \quad (24)$$

$$d_i^0 = \frac{1}{4q_k r_k} \left[-\frac{2a_k^0 r_k b_{\ell}^0 q_{\ell}}{q^2 + 2q_{\ell} r_{\ell} + r^2} (q_i + r_i) + a_i^0 b_{\ell}^0 q_{\ell} + b_i^0 a_{\ell}^0 r_{\ell} \right] \quad (25)$$

It is evident that the number and complexity of terms in the expression for u_i increase rapidly as the order of the approximation increases. This happens because each term in a particular approximation is multiplied by every other one when it is substituted into the nonlinear terms of equations (1) and (3); that is, each sine or cosine term interacts with every other one to produce more terms. If, on the other hand, the nonlinear terms in equations (1) and (3) were neglected (low Reynolds number), the first approximation (eq. (21)) would be an exact solution of (1) and (3).

GENERAL EXPRESSIONS FOR NONLINEAR PRODUCTION OF HARMONICS

Because of the proliferation of new harmonic terms as the order of the approximation increases, it appears impracticable to generate approximations much higher than the third analytically. Rather, it would seem better to obtain general expressions for the new terms produced by the interaction of pairs of harmonic terms in a particular approximation. It should then be possible to program a computer to generate successive approximations from preceding ones. These general expressions for the nonlinear interaction of pairs of harmonic terms will be considered next.

Examination of the terms in the first three approximations for u_i (eqs. (21) to (23)) shows that they are of the general form

$$u_i = \sum_{\vec{k}} \left(A_{i, \vec{k}}^c \cos \vec{k} \cdot \vec{x} + A_{i, \vec{k}}^s \sin \vec{k} \cdot \vec{x} \right) \quad (26)$$

where

$$A_{i, \vec{k}}^c = \sum_q a_{i, \vec{k}, q}^c e^{-b_{\vec{k}, q}^c t} \quad (27)$$

and

$$A_{i, \vec{k}}^s = \sum_r a_{i, \vec{k}, r}^s e^{-b_{\vec{k}, r}^s t}$$

The superscripts s and c appear in coefficients of sine and cosine terms, respectively. The indices \vec{k} , q , and r take on as many values as are required to represent the terms of the particular approximation.

When equation (26) is substituted into the nonlinear terms of equations (1) and (3), each term of equation (26) is multiplied by (or interacts with) every other term. Thus, sine - sine, sine - cosine, and cosine - cosine products must be considered. Consider, first, the terms generated by the interaction of two terms

$$\left. \begin{aligned} & A_{i, \vec{n}}^c \cos \vec{n} \cdot \vec{x} \\ & A_{i, \vec{m}}^c \cos \vec{m} \cdot \vec{x} \end{aligned} \right\} \quad (29)$$

Thus, the sum

$$A_{i, \vec{n}}^c \cos \vec{n} \cdot \vec{x} + A_{i, \vec{m}}^c \cos \vec{m} \cdot \vec{x} \quad (30)$$

where

$$A_{i, \vec{n}}^c = \sum_q a_{i, \vec{n}, q}^c e^{-b_{\vec{n}, q}^c t} \quad (31)$$

$$A_{i, \vec{m}}^c = \sum_r a_{i, \vec{m}, r}^c e^{-b_{\vec{m}, r}^c t} \quad (32)$$

is substituted for u_i in the right sides of equations (3) and (1). The manipulations are similar to those used for obtaining the first and second approximations in the last section. Because it is necessary to obtain the additive contribution to u_i , which results from the interaction of the two harmonic terms in equation (30), u_i is replaced on the left side of equation (1) by $u_{i, \vec{n}, \vec{m}}^{c, c}$, which represents that contribution. For the initial condition on $u_{i, \vec{n}, \vec{m}}^{c, c}$, $u_{i, \vec{n}, \vec{m}}^{c, c}$ is set equal to zero when $t = 0$. (Initial condition (4) will be satisfied later when the totality of terms contributing to u_i is considered.) The expression for $u_{i, \vec{n}, \vec{m}}^{c, c}$ is, then,

$$\begin{aligned} u_{i, \vec{n}, \vec{m}}^{c, c} = & \sum_{q, r} \left[\frac{1}{2} \left[\frac{2m_{\ell} a_{\ell, \vec{n}, q}^c n_k a_{k, \vec{m}, r}^c}{n^2 - 2n_k m_k + m^2} (n_i - m_i) + a_{i, \vec{n}, q}^c n_k a_{k, \vec{m}, r}^c - a_{i, \vec{m}, r}^c m_k a_{k, \vec{n}, q}^c \right] \right. \\ & \times \frac{e^{-\left(b_{\vec{m}, r}^c + b_{\vec{n}, q}^c\right)t} - e^{-\left(n^2 - 2n_k m_k + m^2\right)t}}{n^2 - 2n_k m_k + m^2 - b_{\vec{m}, r}^c - b_{\vec{n}, q}^c} \sin(\vec{n} - \vec{m}) \cdot \vec{x} \\ & + \sum_{q, r} \left[\frac{1}{2} \left[\frac{2m_{\ell} a_{\ell, \vec{n}, q}^c n_k a_{k, \vec{m}, r}^c}{n^2 + 2n_k m_k + m^2} (n_i + m_i) + a_{i, \vec{n}, q}^c n_k a_{k, \vec{m}, r}^c + a_{i, \vec{m}, r}^c m_k a_{k, \vec{n}, q}^c \right] \right. \\ & \times \frac{e^{-\left(b_{\vec{m}, r}^c + b_{\vec{n}, q}^c\right)t} - e^{-\left(n^2 + 2n_k m_k + m^2\right)t}}{n^2 + 2n_k m_k + m^2 - b_{\vec{m}, r}^c - b_{\vec{n}, q}^c} \sin(\vec{n} + \vec{m}) \cdot \vec{x} \end{aligned} \quad (33)$$

Similarly, the interaction of terms like

$$\left. \begin{aligned} & A_{i, \vec{n}}^S \sin \vec{n} \cdot \vec{x} \\ & A_{i, \vec{m}}^S \sin \vec{m} \cdot \vec{x} \end{aligned} \right\} \quad (34)$$

where

$$A_{i, \vec{n}}^S = \sum_{\vec{q}} a_{i, \vec{n}, \vec{q}}^S e^{-b_{\vec{n}, \vec{q}}^S t} \quad (35)$$

and

$$A_{i, \vec{m}}^S = \sum_{\vec{r}} a_{i, \vec{m}, \vec{r}}^S e^{-b_{\vec{m}, \vec{r}}^S t} \quad (36)$$

results in

$$\begin{aligned} u_{i, \vec{n}, \vec{m}}^{S, S} = & \sum_{\vec{q}, \vec{r}} \frac{1}{2} \left[\frac{2m_{\vec{\ell}} a_{\vec{\ell}, \vec{n}, \vec{q}}^S n_{\vec{k}} a_{\vec{k}, \vec{m}, \vec{r}}^S}{n^2 - 2n_{\vec{k}} m_{\vec{k}} + m^2} (n_i - m_i) + a_{i, \vec{n}, \vec{q}}^S n_{\vec{k}} a_{\vec{k}, \vec{m}, \vec{r}}^S - a_{i, \vec{m}, \vec{r}}^S m_{\vec{k}} a_{\vec{k}, \vec{n}, \vec{q}}^S \right] \\ & \times \left[\frac{e^{-\left(b_{\vec{n}, \vec{q}}^S + b_{\vec{m}, \vec{r}}^S\right)t} - e^{-\left(n^2 - 2n_{\vec{k}} m_{\vec{k}} + m^2\right)t}}{n^2 - 2n_{\vec{k}} m_{\vec{k}} + m^2 - b_{\vec{n}, \vec{q}}^S - b_{\vec{m}, \vec{r}}^S} \right] \sin(\vec{n} - \vec{m}) \cdot \vec{x} \\ & + \sum_{\vec{q}, \vec{r}} \frac{1}{2} \left[\frac{2m_{\vec{\ell}} a_{\vec{\ell}, \vec{n}, \vec{q}}^S n_{\vec{k}} a_{\vec{k}, \vec{m}, \vec{r}}^S}{n^2 + 2n_{\vec{k}} m_{\vec{k}} + m^2} (n_i + m_i) - a_{i, \vec{n}, \vec{q}}^S n_{\vec{k}} a_{\vec{k}, \vec{m}, \vec{r}}^S - a_{i, \vec{m}, \vec{r}}^S m_{\vec{k}} a_{\vec{k}, \vec{n}, \vec{q}}^S \right] \\ & \times \left[\frac{e^{-\left(b_{\vec{n}, \vec{q}}^S + b_{\vec{m}, \vec{r}}^S\right)t} - e^{-\left(n^2 + 2n_{\vec{k}} m_{\vec{k}} + m^2\right)t}}{n^2 + 2n_{\vec{k}} m_{\vec{k}} + m^2 - b_{\vec{n}, \vec{q}}^S - b_{\vec{m}, \vec{r}}^S} \right] \sin(\vec{n} + \vec{m}) \cdot \vec{x} \end{aligned} \quad (37)$$

Finally, the interaction of terms like

$$\left. \begin{aligned} & A_{i, \vec{n}}^S \sin \vec{n} \cdot \vec{x} \\ & A_{i, \vec{m}}^C \cos \vec{m} \cdot \vec{x} \end{aligned} \right\} \quad (38)$$

where $A_{i, \vec{n}}^S$ and $A_{i, \vec{m}}^C$ are defined by equations (34) and (30), results in

$$\begin{aligned} u_{i, \vec{n}, \vec{m}}^{S, C} = & - \sum_{q, r} \frac{1}{2} \left[\frac{2m_{\ell} a_{\ell, \vec{n}, q}^S a_{k, \vec{m}, r}^C}{n^2 - 2n_k m_k + m^2} (n_i - m_i) + a_{i, \vec{n}, q}^S a_{k, \vec{m}, r}^C - a_{i, \vec{m}, r}^C a_{k, \vec{n}, q}^S \right] \\ & \times \left[\frac{e^{-\left(b_{\vec{n}, q}^S + b_{\vec{m}, r}^C\right)t} - e^{-\left(n^2 - 2n_k m_k + m^2\right)t}}{n^2 - 2n_k m_k + m^2 - b_{\vec{n}, q}^S - b_{\vec{m}, r}^C} \right] \cos(\vec{n} - \vec{m}) \cdot \vec{x} \\ & + \sum_{q, r} \frac{1}{2} \left[\frac{2m_{\ell} a_{\ell, \vec{n}, q}^S a_{k, \vec{m}, r}^C}{n^2 + 2n_k m_k + m^2} (n_i + m_i) - a_{i, \vec{n}, q}^S a_{k, \vec{m}, r}^C - a_{i, \vec{m}, r}^C a_{k, \vec{n}, q}^S \right] \\ & \times \left[\frac{e^{-\left(b_{\vec{n}, q}^S + b_{\vec{m}, r}^C\right)t} - e^{-\left(n^2 + 2n_k m_k + m^2\right)t}}{n^2 + 2n_k m_k + m^2 - b_{\vec{n}, q}^S - b_{\vec{m}, r}^C} \right] \cos(\vec{n} + \vec{m}) \cdot \vec{x} \end{aligned} \quad (39)$$

The successive approximations to the velocity components u_i may now be calculated. If the v^{th} approximation is known, equations (33), (37), and (39) give contributions to the $v + 1^{\text{st}}$ approximation, which are due to the presence (or interaction) of pairs of harmonic terms in the v^{th} approximation. Thus, if the v^{th} approximation for u_i is given by

$$\begin{aligned}
u_i^{(v)} = & \dots + A_{i, \vec{n}}^c \cos \vec{n} \cdot \vec{x} + A_{i, \vec{n}}^s \sin \vec{n} \cdot \vec{x} \\
& + \dots + \dots + A_{i, \vec{m}}^c \cos \vec{m} \cdot \vec{x} + A_{i, \vec{m}}^s \sin \vec{m} \cdot \vec{x} + \dots
\end{aligned} \quad (40)$$

then the $v + 1^{\text{st}}$ approximation is

$$\begin{aligned}
u_i^{(v+1)} = & \dots + u_{i, \vec{n}, \vec{m}}^{c, c} + \dots + u_{i, \vec{n}, \vec{m}}^{s, s} + \dots + u_{i, \vec{n}, \vec{m}}^{s, c} + \dots \\
& + u_{i, \vec{m}, \vec{n}}^{s, c} + \dots + a_i^0 e^{-q^2 t} \cos \vec{q} \cdot \vec{x} + b_i^0 e^{-r^2 t} \cos \vec{r} \cdot \vec{x}
\end{aligned} \quad (41)$$

where the $u_{i, \vec{n}, \vec{m}}^{()}$'s are given by the equations (33), (37), and (39).

The last two terms in equation (41) are the first approximation (eq. (21)) and are included in equation (41) because, whereas every other term is regenerated at each stage in the iteration process, the terms of the first approximation are not so regenerated. That is the case because of the assumed initial condition $u_{i, \vec{n}, \vec{m}}^{c, c} = 0$ when $t = 0$. The same result could be obtained by letting $u_{i, \vec{n}, \vec{m}}^{c, c} = a_i^0 \cos \vec{q} \cdot \vec{x} + b_i^0 \cos \vec{r} \cdot \vec{x}$ at $t = 0$ when the two interacting terms include the first approximation. However, it is simpler to use equation (33) throughout as it stands and to include the first approximation in equation (41).

In obtaining equation (41) from (40), the presence (or interaction) of each pair of harmonic terms in equation (40) produces a term $u_{i, \vec{m}, \vec{n}}^{()}$ in (41). Thus, each term in equation (40) interacts once with every other term. Because of the continuity condition (2) (see also eq. (5)) no contributions are obtained by the interaction of a term with itself. Thus, if there are s terms in the v^{th} approximation, the $v + 1^{\text{st}}$ approximation will have $(s - 1) + (s - 2) + \dots + 1 = (s/2)(s - 1)$ contributions, like $u_{i, \vec{n}, \vec{m}}^{()}$, plus the first approximation.

With the velocity components u_i known to some degree of approximation, the space-averaged values of u_i^2 can be calculated. Because u_i is given by equation (26), squaring, integrating over a cycle, and using the orthogonality property of sines and cosines result in

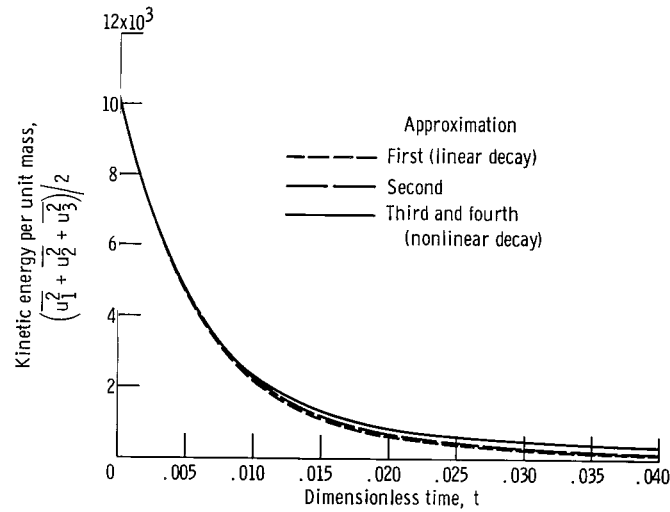
$$\begin{aligned}
\overline{u_i^2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\vec{k}} A_{i,\vec{k}}^c \cos \vec{k} \cdot \vec{x} + A_{i,\vec{k}}^s \sin \vec{k} \cdot \vec{x} \right)^2 d(\vec{k} \cdot \vec{x}) \\
&= \sum_{\vec{k}} \frac{1}{2} \left[\left(A_{i,\vec{k}}^c \right)^2 + \left(A_{i,\vec{k}}^s \right)^2 \right]
\end{aligned} \tag{42}$$

RESULTS AND DISCUSSION

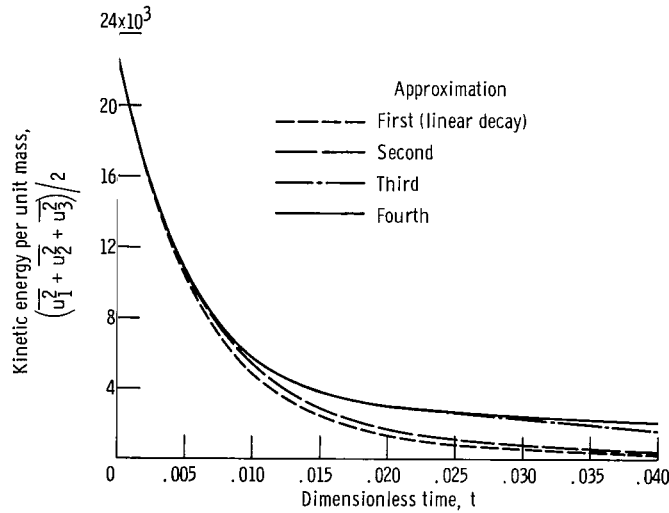
If the velocity distribution in a fluid is given by a sum of harmonic terms as in equation (26), then the presence (or nonlinear interaction) of each pair of terms produces new harmonics given by equations (29) to (39). Each pair of terms then present interacts to produce still more harmonic terms or eddy sizes. It is evident that there will result a very large number of eddy sizes, or a situation resembling turbulent flow. It should be pointed out that, because of wave number vectors $\vec{n} - \vec{m}$ and $\vec{n} + \vec{m}$, which occur, for instance, in equation (33), the generated harmonics can have either higher or lower wave numbers than the original ones. Moreover, when some of the generated harmonics interact with the original ones, new harmonics can be generated with the same wave numbers as the original ones.

One might ask how a random turbulence could develop from a regular disturbance produced, for example, by flow through a grid with equal spacings. It has been pointed out earlier that because of continuity, harmonics of the same wave number and/or intensity vectors will not interact nonlinearly. However, if there are small random disturbances in the flow upstream of the grid or small imperfections in the grid, the eddies produced by the grid will not be identical; therefore, they can interact to produce new harmonics. Because of the random nature of the small upstream disturbances, or grid imperfections, the resulting flow will be random or turbulent.

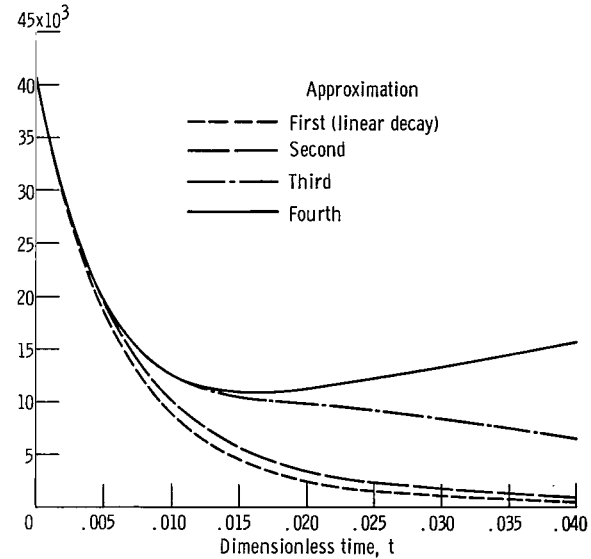
Figure 1 shows calculated kinetic energy per unit mass for a disturbance plotted as a function of dimensionless time. First, second, third, and fourth approximations are shown. The initial condition for the disturbance is given by equation (4), where the values of a_i^0 , b_i^0 , q_i , and r_i are shown in the figure. The values of the initial vectors were chosen to satisfy continuity and to give nonzero interaction terms ($a_i^0 q_i = b_i^0 r_i = 0$, $a_i^0 r_i \neq 0$, and $b_i^0 q_i \neq 0$). Disturbances having three different initial intensities are compared. The first approximation (eq. (21)) gives the linear decay that occurs when the nonlinear or inertia terms in the Navier-Stokes equation are neglected. Comparison of



(a) Weaker disturbance. Initial dimensionless Fourier coefficients:
 $a_1^0 = 20(1, -1, -2, 1)$; $b_1^0 = 80(1, -2, -1)$.



(b) Moderate disturbance. Initial dimensionless Fourier coefficients:
 $a_1^0 = 30(1, -1, -2)$; $b_1^0 = 120(1, -2, -1)$.



(c) Stronger disturbance. Initial dimensionless Fourier coefficients:
 $a_1^0 = 40(1, -1, -2)$; $b_1^0 = 160(1, -2, -1)$.

Figure 1. - Time decay of mean kinetic energy per unit mass. Components of initial wave number vectors: $q_1 = (4, 2, 1)$; $r_1 = 2(4, 1, 2)$.

the nonlinear with the linear solutions shows that the nonlinear effects cause the disturbances to decay more slowly. It is noted that a similar result is obtained for random turbulent flow. Strongly nonlinear turbulence decays approximately as t^{-1} , whereas the decay law for linear turbulence is closer to $t^{-5/2}$.

Comparison of the various approximations indicates that at large times, higher order approximations are required for the iteration process to converge (as indicated by the agreement of the third and fourth approximations) than are required at smaller times. Larger values of the initial intensity parameters indicate convergence only at small times for the first four approximations. On the other hand, smaller values of the initial intensity show convergence even at very large times. The poorer convergence of a given approximation at larger intensities is, of course, due to the stronger inertia effects that occur at high Reynolds numbers; therefore, higher approximations are required to take them into account.

To show how the energy is distributed among the various directional components, the ratios $\overline{u_2^2}/\overline{u_1^2}$ and $\overline{u_3^2}/\overline{u_1^2}$ are plotted against dimensionless time in figure 2. The case shown corresponds to that in figure 1(b). Comparison of the curves in figure 1(b) with those in figure 2 shows that the third and fourth approximations in the curves in figure 2 converge for a smaller range of dimensionless times than do those of figure 1. Evidently, a given order approximation at a time t is less accurate for the individual components than it is for the total energy. Comparison of the first (linear approximation for $\overline{u_2^2}/\overline{u_1^2}$ and for $\overline{u_3^2}/\overline{u_1^2}$ shows that the ratios can either increase or decrease with time because the different components decay viscously at different rates, depending on the initial conditions.

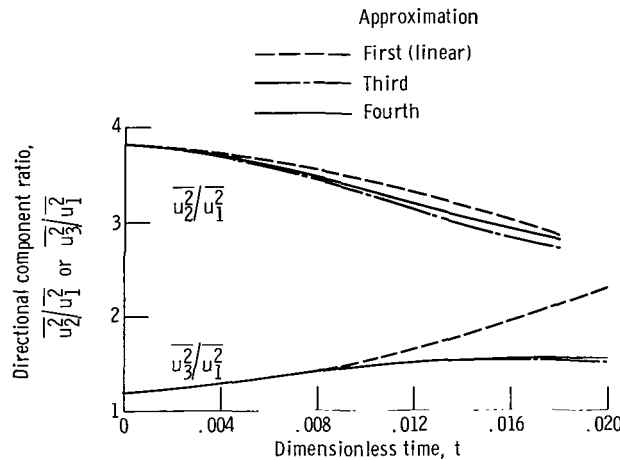


Figure 2. - Variation of mean square velocity component ratios with time. Initial dimensionless wave number vectors: $q_i = (4, 2, 1)$; $r_i = 2(4, 1, 2)$. Initial dimensionless Fourier coefficients $a_i^0 = 30(1, -1, -2)$; $b_i^0 = 120(1, -2, -1)$. Moderate disturbance.

The fourth (nonlinear) approximations show the same general trends as the first approximations, but the nonlinear effects tend to bring the ratios closer to one (this is also true for $\overline{u_3^2}/\overline{u_2^2}$); that is, the nonlinear effects tend to make the disturbance more isotropic.

To show how various harmonics contribute to the kinetic energy of the disturbance, kinetic energy spectra were calculated from equation (42) and plotted against dimensionless wave number κ in figure 3. The cases shown here are the same as those in figure 1. Results are given for several dimensionless times for which the iteration process has converged (see fig. 1). For $t = 0$, there are, of course, only two spikes or harmonics in the spectrum because of the assumed initial condition (eq. (4)). For the later times shown, particularly for the stronger disturbances, a large number of other har-

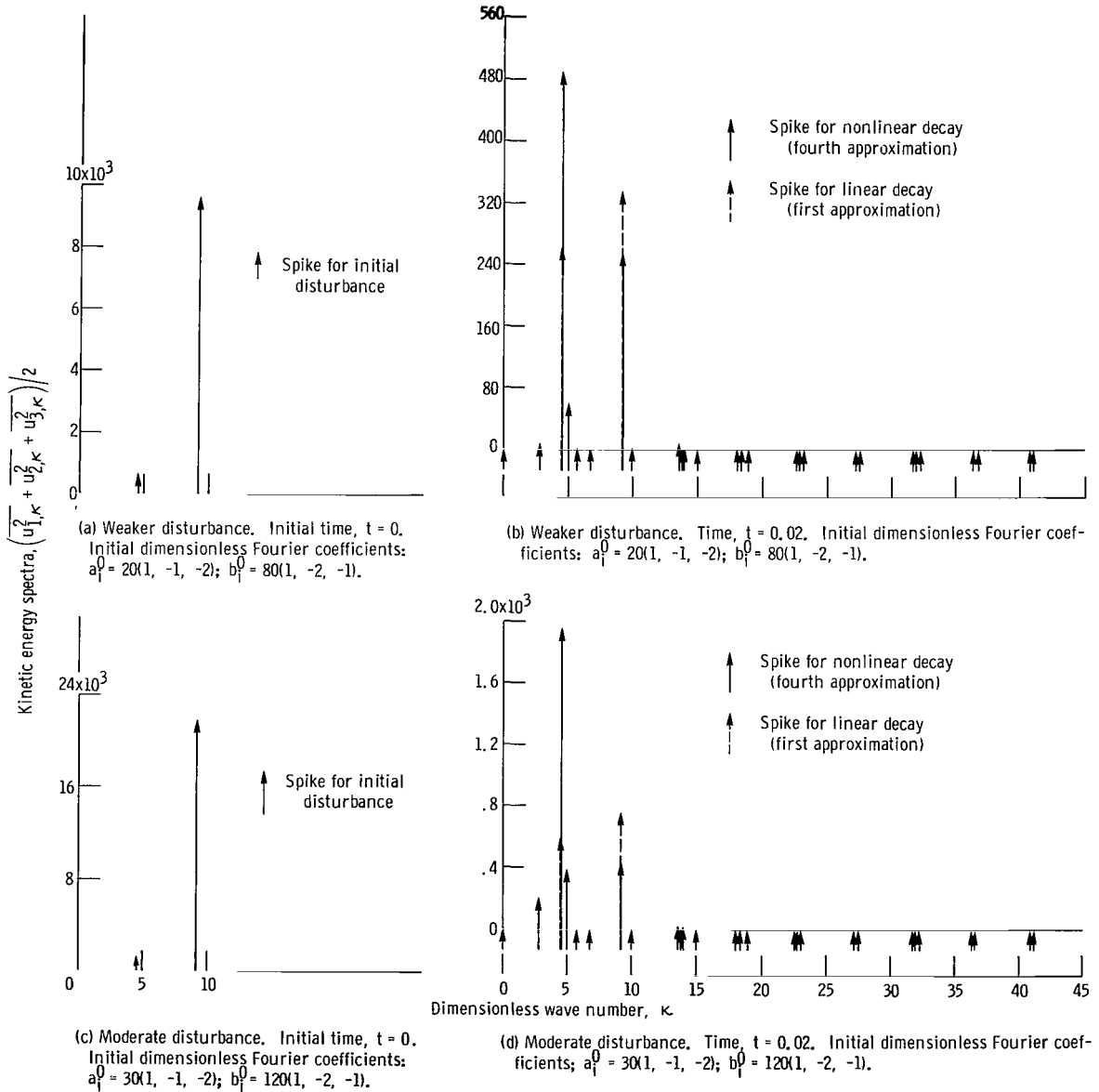


Figure 3. - Spectrum of mean kinetic energy per unit mass. Components of initial wave number vectors: $q_1 = (4, 2, 1)$; $r_1 = 2(4, 1, 2)$.

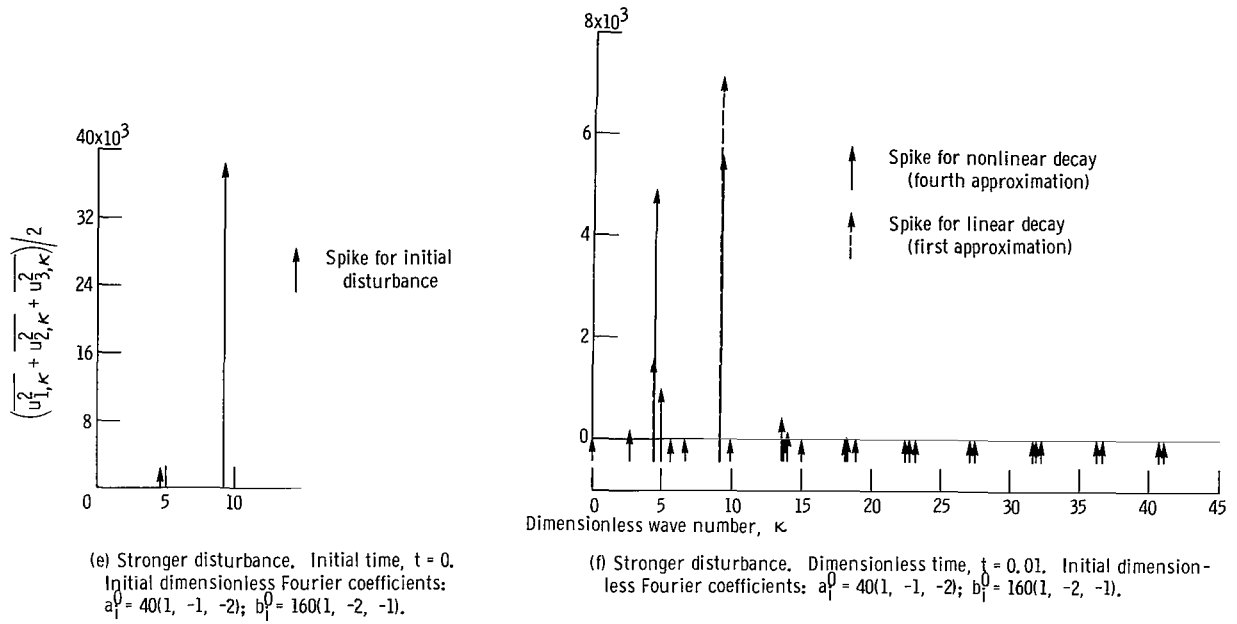


Figure 3. - Concluded.

monics have developed because of the interaction between contributions at various wave numbers. Although some of the generated harmonics are strong, many of them are too weak to have an appreciable effect on the disturbance, so that their representations lie close to the horizontal axis. By increasing the strength of the disturbance, more of the harmonics will have appreciable strengths. However, the fourth approximation will then not be sufficient to obtain convergence of the iterative process. Unfortunately, the computer used for the calculations did not have enough storage capacity to carry out the approximations higher than the fourth.

For comparison with the nonlinear results just discussed, the spikes for the linear solution are shown dashed in figures 3(b), (d), and (f). In this case, no new harmonics are generated. The original harmonics simply decay independently. Comparison of the solid and dashed lines shows that the harmonics at the original wave numbers are greatly affected by nonlinear interactions; that is, many of the eddies, generated or destroyed by nonlinear effects, have the same wave numbers as those in the original disturbance. It might be mentioned that, in order for that to occur, it is necessary to go to at least the third approximation because the second approximation produces only contributions that are at higher or lower wave numbers than the original ones (eq. (22)).

Although the harmonics generated by nonlinear effects can occur at either high or low wave numbers, most of the significant generated harmonics in the calculated results, particularly those for the weaker disturbances, tend to lie in the low wave number re-

gion. The difficulty of generating significant harmonics at the higher wave numbers appears to be related to the fact that viscous effects in the high wave number region tend to prevent their formation. As the disturbance strength increases, however, the tendency to form significant harmonics in the higher wave number region appears to increase.

CONCLUDING REMARKS

An initial three-dimensional disturbance consisting of two harmonics can decay nonlinearly and generate new harmonics at wave numbers greater than, smaller than, or the same as, those of the original harmonics. If either the wave number vectors or the amplitude vectors of the two initial harmonics have the same direction, however, the continuity condition requires that there be no interaction. In that case or if nonlinear effects are ignored by neglecting the inertia terms in the Navier-Stokes equation, the original harmonics will simply decay independently.

In general, nonlinear effects cause the disturbance to decay more slowly. At low Reynolds numbers there is, of course, no significant difference between the linear and nonlinear solutions.

The ratios of the directional components of the mean square velocities changed with time. This was caused by differences in the rates of viscous (linear) decay of the various directional components and by transfer of energy between the components by nonlinear action. Whereas the viscous effects could change the component ratios in either direction (depending on the initial conditions) the nonlinear effects always tended to bring the ratios closer to one or to make the disturbance more isotropic.

For a given approximation, the iterative solution for the nonlinear case was most accurate at small times and/or small Reynolds numbers of the disturbance. As either time or the Reynolds number increased, the order of the approximation that was required to obtain convergence of the iterative process increased.

The generation of a range of harmonics in the early stages of the development of turbulence by flow through a grid appears to be similar to the production of harmonics in the present analysis. Random turbulence can evidently develop by flow through a grid because of small random disturbances in the flow upstream of the grid or small imperfections in the grid. Random harmonics are then generated as a result of the interaction of the slightly different intensities and wave numbers of the disturbances in the flow from the grid.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, September 9, 1968,
129-01-05-20-22.

APPENDIX - SYMBOLS

$A_{i, \vec{k}}^c, A_{i, \vec{k}}^s, A_{i, \vec{n}}^c,$ $A_{i, \vec{m}}^c, A_{i, \vec{n}}^s, A_{i, \vec{m}}^s$	dimensionless Fourier coefficients for velocity components, $x_0 A_{i, \vec{k}}^{c*} / \nu$
$A_{i, \vec{k}}^{c*}, A_{i, \vec{k}}^{s*}$	Fourier coefficients for velocity component
a, b	dimensionless Fourier coefficients for pressure fluctuation, $x_0^2 a^* / \rho \nu^2, x_0^2 b^* / \rho \nu^2$ (see eq. (11))
a_i, b_i, c_i, d_i	dimensionless Fourier coefficients for velocity component, $x_0 a_i^* / \nu, x_0 b_i^* / \nu$, etc. (see eqs. (6) and (15))
$a_{i, \vec{k}, q}^c, a_{i, \vec{k}, r}^s, a_{i, \vec{n}, q}^c,$ $a_{i, \vec{m}, r}^c, a_{i, \vec{n}, q}^s, a_{i, \vec{m}, r}^s$	constants given by eqs. (27), (28), (31), (32), (35), and (36)
a_i^0, b_i^0	initial dimensionless Fourier coefficients for velocity components, $x_0 a_i^{0*} / \nu$ and $x_0 b_i^{0*} / \nu$ (see eq. (4))
a_i^{0*}, b_i^{0*}	initial Fourier coefficients for velocity components
a_i^*, b_i^*	Fourier coefficients for velocity component
a^*, b^*	Fourier coefficients for pressure fluctuation
$b_{\vec{k}, q}^c, b_{\vec{k}, r}^s, b_{\vec{n}, q}^c,$ $b_{\vec{m}, r}^c, b_{\vec{n}, q}^s, b_{\vec{m}, r}^s$	constants given by eqs. (27), (28), (31), (32), (35), and (36)
m, n	magnitudes of dimensionless wave number vectors
m_i, n_i	components of dimensionless wave number vectors, $x_0 m_i^*,$ $x_0 n_i^*$
m_i^*, n_i^*	components of wave number vectors
\vec{m}, \vec{n}	dimensionless wave number vectors
p	dimensionless pressure, $x_0^2 p^* / \rho \nu^2$
p^*	pressure
q_i, r_i	components of initial dimensionless wave number vectors, $x_0 q_i^*, x_0 r_i^*$

q_i^*, r_i^*	components of initial wave number vectors
\vec{q}, \vec{r}	initial dimensionless wave number vectors
t	dimensionless time, $\nu t^*/x_0^2$
t^*	time
u_i	component of dimensionless velocity, $x_0 u_i^*/\nu$
u_i^*	component of velocity
$u_{i, \vec{n}, \vec{m}}^{c, c}, u_{i, \vec{n}, \vec{m}}^{s, s}, u_{i, \vec{n}, \vec{m}}^{s, c}$	interaction terms given by eqs. (33), (37), and (39)
x_i	dimensionless position coordinate, x_i^*/x_0
x_i^*	position coordinate
\vec{x}	dimensionless position vector
x_0	characteristic length
\vec{k}	dimensionless wave number vector, $x_0 \vec{k}^*$
\vec{k}^*	wave number vector
ν	kinematic viscosity
ρ	density
Superscripts:	
c	cosine terms
s	sine term
*	dimensional quantities
—	averaged quantities

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