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A MATHEMATICAL MODEL FOR CALCULATING THE FLIGHT DYNAMICS OF A GENERAL PARACHUTE-PAYLOAD SYSTEM

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ABSTRACT

An analysis yielding six-degree-of-freedom equations of motion is presented for predicting the dynamic behavior of a general parachute-payload system. The parachute canopy and associated air-mass are approximated as a rigid body, and separate equations of motion are derived for the canopy and payload subject to the constraint of the risers and suspension lines. The analysis determines the forces and the response of various riser and suspension-line geometries subjected to large displacements, under the assumption that these lines are linearly elastic. The equations are readily adaptable to computer solutions and should be of interest in analyzing the dynamic performance of lifting-parachute payload systems.

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SUMMARY

An analysis yielding six-degree-of-freedom equations of motion is presented for predicting the dynamic behavior of a general parachute-payload system. The parachute canopy and associated airmass were approximated as a rigid body, and separate equations of motion were derived for the canopy and payload subject to the constraint of the risers and suspension lines. Equations were developed for solving for the force distribution in space frames composed of various riser and suspension-line geometries under the assumption that these lines are linearly elastic. The Newton-Raphson iterative solution was used to solve equations for large displacements, and the solution is readily adaptable to computer solutions. The model should be of interest for analyses of the performance of lifting-type parachutes, particularly when the six-degree-of-freedom mass and aerodynamic properties of these parachutes have been better defined.

INTRODUCTION

The proposed use of lifting parachutes for future land landing of manned spacecraft will impose a greater requirement on computer simulations of the flight dynamics of the parachute-payload system. Dynamic analysis requirements for the present water-landing systems are primarily to determine that the system is stable and that the oscillation limits during descent are acceptable for water-entry of the spacecraft.

Although full-scale simulations have primarily been used to evaluate the dynamic performance of manned parachute systems in the past, a few attempts have been made to theoretically predict the dynamics. According to Ludwig and Heins (ref. 1), the first attempt to consider dynamic stability of a parachute-payload system was performed by Brodetsky (ref. 2), but few analyses followed in the interim period to 1960. Since then, considerable work has been done at the University of Minnesota under the direction of H. G. Heinrich (refs. 3 and 4), by Ludwig and Heins (refs. 1 and 5), and by personnel of a NASA contractor (refs. 6 and 7).

Most of the dynamic analyses have considered the parachute-payload system to be a single rigid body and have linearized the equations of motion, which limits the oscillation amplitudes to small values. Ludwig and Heins employed computer solutions to solve the six-degree-of-freedom motion, using the nonlinear equations of motion; but their analysis also assumed a single rigid-body system.

Proposals have been made for using a lifting parachute with aerodynamic reference lengths on the order of 100 feet which would require suspension lines of 100 feet or more in length. Since the suspension lines used for most parachute-payload systems have a relatively low spring constant, it is possible for a suspension line to lengthen several inches without appreciably increasing the forces or moments on the parachute. The suspension-line geometry is usually such that the vertical component of the suspension-line length is large compared to the lateral component. This geometry permits large lateral displacements between the payload and parachute with only small changes in the suspension-line lengths and thus only small changes in the forces and moments acting on the parachute. Since lifting parachutes are designed for lateral maneuvers, large lateral displacements between the parachute and payload are likely. These displacements result because the high inertia of the parachute (which must include the apparent inertia effects of the surrounding and included airmass) will require large rolling moments to produce a roll attitude change during yaw maneuvers, and the momentum of the payload will be such as to produce a relative lateral displacement during the yaw maneuver. The relative displacements will have an effect on the accuracy of studies which assume the parachute and payload are a single rigid body, since the inertia matrix will be significantly affected by such displacements. This suggests the need for considering the parachute and payload as separate, elastically coupled systems when analyzing the dynamics of a lifting-parachute system.

Another advantage of such an approach is that the aerodynamic damping of the parachute can be considered separately. This can result in a more accurate analysis, since the instantaneous aerodynamic angles of the parachute are calculated; thus, the effect of the damping terms C_{m_q} and C_{ℓ_p} are much less than when the oscillation center is the combined center of mass of the parachute-payload system. With the single rigid-body approach, it is also impossible to account for the effect on aerodynamic damping of coupling of the angular rates caused by relative rotation between the two bodies. The two-body approach also allows the input of wind gusts acting at the actual parachute altitude instead of at the combined center of mass. This could be important in landing studies.

Neustadt, et al., (refs. 6 and 7) have extended the parachute analysis to include relative motion of the parachute and payload, with the elastic effect of the risers and suspension lines included. However, the analysis is restricted to planar motion and only considers very simplified riser geometry.

The need obviously exists for an analysis which will predict the dynamics of a general parachute-payload system. This paper develops a mathematical model for analyzing the dynamics of such a system. The technique used assumes the parachute and payload to be two independent rigid bodies subject to the constraints of the suspension lines.

The forces in the individual suspension lines and risers are calculated by solving the indeterminant space frames composed of these lines. These forces are then used to calculate the forces and moments acting on the parachute and payload. The six-degree-of-freedom equations of motion of the parachute and payload are then solved using these forces and the aerodynamic forces.

The number of individual space frames included in an analysis is limited only by the storage capacity and time requirements for a particular computer. A FORTRAN IV computer program developed at the Manned Spacecraft Center (MSC) provides for four space frames of up to 10 members each.

The author wishes to acknowledge the assistance of William E. Thomas, MSC Structures and Mechanics Division, and Theodore F. Hughes, ITT/Federal Electric Corporation, in the preparation of this paper.

SYMBOLS

The following symbols are referenced to the lifting-parachute (parawing) system as shown in figures 1 to 5. In general, aerodynamic coefficients are functions of both angle of attack and angle of sideslip and are input to the computer in tabular form. If control inputs are available, they can be accounted for in the tables.

$[A]$	direction cosine matrix
$[A]^T$	transpose of direction cosine matrix
C_ℓ	aerodynamic rolling moment coefficient
C_{ℓ_p}	$\dfrac{\partial C_\ell}{\partial \frac{pd}{2V_\infty}}$
C_{ℓ_q}	$\dfrac{\partial C_\ell}{\partial \frac{qd}{2V_\infty}}$
C_{ℓ_r}	$\dfrac{\partial C_\ell}{\partial \frac{rd}{2V_\infty}}$
C_m	aerodynamic pitching moment coefficient

C_{m_p} $\qquad \dfrac{\partial C_m}{\partial \dfrac{pd}{2V_\infty}}$

C_{m_q} $\qquad \dfrac{\partial C_m}{\partial \dfrac{qd}{2V_\infty}}$

C_n $\qquad$ aerodynamic yawing moment coefficient

C_{n_p} $\qquad \dfrac{\partial C_n}{\partial \dfrac{pd}{2V_\infty}}$

C_{n_r} $\qquad \dfrac{\partial C_n}{\partial \dfrac{rd}{2V_\infty}}$

C_x $\qquad$ aerodynamic axial force coefficient

C_y $\qquad$ aerodynamic side force coefficient

C_{y_p} $\qquad \dfrac{\partial C_y}{\partial \dfrac{pd}{2V_\infty}}$

C_{y_r} $\qquad \dfrac{\partial C_y}{\partial \dfrac{rd}{2V_\infty}}$

C_z $\qquad$ aerodynamic normal force coefficient

C_{z_p} $\qquad \dfrac{\partial C_z}{\partial \dfrac{pd}{2V_\infty}}$

C_{z_q} $\qquad \dfrac{\partial C_z}{\partial \dfrac{qd}{2V_\infty}}$

d	aerodynamic reference length
$\{d\}$	column matrix composed of elements dx_i, dy_i, dz_i
dx_i, dy_i, dz_i	displacement of point i with respect to initial unstressed position
dx_o, dy_o, dz_o	displacement of point o with respect to initial unstressed position
$[F]$	force matrix
F_1, F_2, F_3	functional equations for Fx_o, Fy_o, Fz_o
Fx_i, Fy_i, Fz_i	force components at point i required to hold frame in equilibrium
Fx_o, Fy_o, Fz_o	virtual forces applied at frame junction
F_x^A, F_y^A, F_z^A	body aerodynamic forces
F_x^C, F_y^C, F_z^C	body cable forces
G_x, G_y, G_z	gravity components along body axes
I_{xx}, I_{yy}, I_{zz}	moments of inertia about body axis system
I_{xy}, I_{xz}, I_{yz}	products of inertia about body axis system
i	arbitrary frame member
i_2, j_2, k_2	fixed points on parachute with respect to parachute axis system
$\hat{i}_1, \hat{j}_1, \hat{k}_1$	unit vector in payload axis system
$\hat{i}_2, \hat{j}_2, \hat{k}_2$	unit vector in parachute axis system
$[K]$	matrix of the spring constants

K_i	spring constant of frame member i
k	current step in the iteration process
L_i	current length of frame member i
L_{i_0}	unstressed length of frame member i
$\{L_0\}$	row matrix composed of unstressed lengths of frame members
M	mass
M_x^A, M_y^A, M_z^A	body aerodynamic moments
M_x^C, M_y^C, M_z^C	body cable moments
m	total members in frame
n	number of members from payload to junction point o
o	space frame confluence point
$\{P\}$	column matrix composed of the element P_i
P_i	axial force in member i
p, q, r	body axis components of roll, pitch, and yaw inertial angular velocities = $\bar{\omega}_{x_I}, \bar{\omega}_{y_I}, \bar{\omega}_{z_I}$
q_∞	free stream dynamic pressure
$\bar{R}_1$	inertial position vector of payload center of gravity
$\bar{R}_2$	inertial position vector of parachute center of gravity
$r_{ix,2}, r_{iy,2}, r_{iz,2}$	scalar components of a vector from parachute to attach points on payload in parachute axis system
$\bar{r}_i$	vector from parachute to payload attach point
S	aerodynamic reference area
t	time

$V_{x,1}, V_{y,1}, V_{z,1}$	x, y, z velocity components of payload center of gravity along payload axes
$V_{x,2}, V_{y,2}, V_{z,2}$	x, y, z velocity components of parachute center of gravity along parachute axes
V_∞	free stream velocity of center of gravity of payload or parachute
$\bar{v}_1$	inertial velocity vector of payload center of gravity
$\bar{v}_2$	inertial velocity vector of parachute center of gravity
X_i, Y_i, Z_i	coordinates of attach point i
$X_{i_0}, Y_{i_0}, Z_{i_0}$	original X, Y, Z coordinates of attach point i
$X_{o_0}, Y_{o_0}, Z_{o_0}$	original X, Y, Z coordinates of point o
$\dot{x}_{B_I}, \dot{y}_{B_I}, \dot{z}_{B_I}$	inertial velocity of body center of gravity along body axes
$\ddot{x}_{B_I}, \ddot{y}_{B_I}, \ddot{z}_{B_I}$	inertial acceleration of body center of gravity along body axes
$\alpha_i, \beta_i, \gamma_i$	X, Y, Z direction cosines of frame member i in parachute axis system
$[\Gamma]$	3 by 3 transformation matrix for transforming from axis system of body 1 to axis system of body 2
$[\Gamma]^T$	transpose of $[\Gamma]$
$\Gamma_{i,j}$	elements of matrix Γ
ΔX_i	$X_i - X_{o_0} - dx_o$
ΔY_i	$Y_i - Y_{o_0} - dy_o$
ΔZ_i	$Z_i - Z_{o_0} - dz_o$
$\Delta\theta$	total angle change of frame member i from unstressed position

δ	correction to previous estimate in Newton-Raphson iterative solution
δ_i	correction to previous estimate of frame member i
ξ_i, ψ_i, ϕ_i	angles between frame member i and parachute X-, Y-, and Z-axes
$\xi_{i_0}, \psi_{i_0}, \phi_{i_0}$	original angles between frame member i in unstressed position and parachute X-, Y-, and Z-axes
ρ_i, τ_i, σ_i	$\left(\cos \xi_i - \cos \xi_{i_0}\right), \left(\cos \psi_i - \cos \psi_{i_0}\right), \left(\cos \phi_i - \cos \phi_{i_0}\right)$
$\rho_{ix,1}, \rho_{iy,1}, \rho_{iz,1}$	X, Y, Z coordinates of payload attach point of riser i in payload axes system
$\rho_{ix,2}, \rho_{iy,2}, \rho_{iz,2}$	X, Y, Z coordinates of parachute attach point of suspension line i in parachute axes system
$\left(\rho^2 + \tau^2 + \sigma^2\right)$	column matrix composed of the elements ρ, τ, and σ (eq. (21))
$\bar{\rho}_i$	vector from center of gravity to riser or suspension-line attach point
ϕ	3 by 3 matrix composed of elements of the Jacobian
ϕ_{ij}	elements of the Jacobian
$\bar{\Phi}, \bar{\theta}, \bar{\Psi}$	relative Euler angles of body 2 with respect to body 1, applied in the rotation $\bar{\Psi}, \bar{\theta}, \bar{\Phi}$
$\dot{\bar{\Phi}}, \dot{\bar{\theta}}, \dot{\bar{\Psi}}$	time rate of change of $\bar{\Phi}, \bar{\theta}, \bar{\Psi}$
$\omega_{x,1}, \omega_{y,1}, \omega_{z,1}$	angular velocity components of payload about payload axes
$\omega_{x,2}, \omega_{y,2}, \omega_{z,2}$	angular velocity components of parachute about parachute axes
$\bar{\omega}_1$	angular velocity vector of payload
$\bar{\omega}_2$	angular velocity vector of parachute
$\bar{\omega}_{x,1}, \bar{\omega}_{y,1}, \bar{\omega}_{z,1}$	angular velocity component of payload transformed to parachute axes

$\bar{\omega}_{x_I}, \bar{\omega}_{y_I}, \bar{\omega}_{z_I}$ inertial angular velocity components about body axes = p, q, r

$\dot{\omega}_{x_I}, \dot{\omega}_{y_I}, \dot{\omega}_{z_I}$ time rate of change of velocity components about body axes

ANALYSIS

Equations of Motion

The basic approach to this analysis is to assume that the parachute can be approximated as a rigid body, and then to calculate the dynamics of the parachute and payload separately, subject to the constraint of connecting lines. Six-degree-of-freedom equations of motion can be written for each body if the forces and moments caused by aerodynamics as well as the forces and moments in the connecting lines are known. Integration of these equations will yield the motion of each body.

The three translational equations of motion for the center of mass of a rigid body are

$$\begin{Bmatrix} \ddot{x}_{B_I} \\ \ddot{y}_{B_I} \\ \ddot{z}_{B_I} \end{Bmatrix} = \frac{1}{M} \begin{Bmatrix} F_x^A + F_x^C \\ F_y^A + F_y^C \\ F_z^A + F_z^C \end{Bmatrix} + \begin{Bmatrix} G_x \\ G_y \\ G_z \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_{B_I} \\ \dot{y}_{B_I} \\ \dot{z}_{B_I} \end{Bmatrix} \tag{1}$$

The aerodynamic forces are determined by

$$\left.\begin{aligned} F_x^A &= C_x q_\infty S \\ F_y^A &= \left[C_y + C_{y_r}\left(\frac{rd}{2V_\infty}\right) + C_{y_p}\left(\frac{pd}{2V_\infty}\right)\right] q_\infty S \\ F_z^A &= \left[C_z + C_{z_q}\left(\frac{qd}{2V_\infty}\right) + C_{z_p}\left(\frac{pd}{2V_\infty}\right)\right] q_\infty S \end{aligned}\right\} \tag{2}$$

The rotational equation of motion about the body axis is determined by

$$\begin{Bmatrix} \dot{\omega}_{x_I} \\ \dot{\omega}_{y_I} \\ \dot{\omega}_{z_I} \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}^{-1} \left(\begin{Bmatrix} M_x{}^A + M_x{}^C \\ M_y{}^A + M_y{}^C \\ M_z{}^A + M_z{}^C \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_{x_I} \\ \omega_{y_I} \\ \omega_{z_I} \end{Bmatrix} \right) \tag{3}$$

When the aerodynamic reference center coincides with the center of mass, the body aerodynamic moments are given by

$$\left.\begin{aligned} M_x{}^A &= \left[C_\ell + C_{\ell_p}\left(\frac{pd}{2V_\infty}\right) + C_{\ell_q}\left(\frac{qd}{2V_\infty}\right) + C_{\ell_r}\left(\frac{rd}{2V_\infty}\right)\right] q_\infty S d \\ M_y{}^A &= \left[C_m + C_{m_q}\left(\frac{qd}{2V_\infty}\right) + C_{m_p}\left(\frac{pd}{2V_\infty}\right)\right] q_\infty S d \\ M_z{}^A &= \left[C_n + C_{n_r}\left(\frac{rd}{2V_\infty}\right) + C_{n_p}\left(\frac{pd}{2V_\infty}\right)\right] q_\infty S d \end{aligned}\right\} \tag{4}$$

Riser and Suspension-Line Forces

With the aerodynamic forces and moments known, the problem now is finding the forces and moments caused by the connecting lines. For this, consider a vector $\bar{r}_1$ (from the parachute to the payload as shown in fig. 1) which results in the following equations as developed in reference 8.

$$\bar{r}_1 = \bar{R}_1 + \bar{\rho}_1 - \bar{R}_2 - \bar{\rho}_2 \tag{5}$$

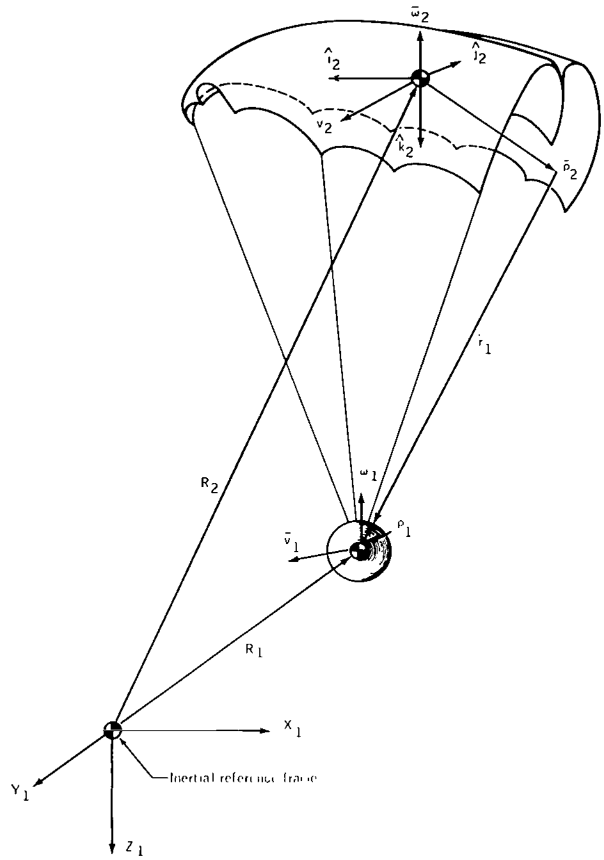

Figure 1. - Vector diagram of parawing configuration.

The time rate of change of $\bar r_1$ is given by

$$\frac{d\bar r_1}{dt} = \dot{\bar r}_1 = \dot{\bar R}_1 + \dot{\bar \rho}_1 - \dot{\bar R}_2 - \dot{\bar \rho}_2 \tag{6}$$

The vector $\bar\rho_1$ is a vector fixed in the payload and can be expressed in the payload referenced frame as

$$\bar\rho_1 = X_1\hat i_1 + Y_1\hat j_1 + Z_1\hat k_1 \tag{7}$$

By taking the derivative of $\bar\rho_1$ with respect to time

$$\dot{\bar\rho}_1 = X_1\dot{\hat i}_1 + Y_1\dot{\hat j}_1 + Z_1\dot{\hat k}_1 = \bar\omega_1 \times \bar\rho_1 \tag{8}$$

Similarly

$$\dot{\bar\rho}_2 = \bar\omega_2 \times \bar\rho_2 \tag{9}$$

The time rate of change of $\bar R_1$ and $\bar R_2$ is equal to the inertial velocity of points 1 and 2.

$$\left.\begin{aligned} \dot{\bar R}_1 &= \bar v_1 \\ \dot{\bar R}_2 &= \bar v_2 \end{aligned}\right\} \tag{10}$$

By combining equations (6), (8), and (10)

$$\dot{\bar r}_1 = \bar v_1 - \bar v_2 + \bar\omega_1 \times \bar\rho_1 - \bar\omega_2 \times \bar\rho_2 \tag{11}$$

The time rate of change of $\bar{r}_1$ also can be expressed in the parachute reference frame as

$$\dot{\bar{r}}_1 = \dot{r}_{1x,2}\hat{i}_2 + \dot{r}_{1y,2}\hat{j}_2 + \dot{r}_{1z,2}\hat{k}_2 \tag{12}$$

By equating the expressions for $\dot{\bar{r}}_1$

$$\dot{r}_{1x,2}\hat{i}_2 + \dot{r}_{1y,2}\hat{j}_2 + \dot{r}_{1z,2}\hat{k}_2 = \bar{v}_1 - \bar{v}_2 + \bar{\omega}_1 \times \bar{\rho}_1 - \bar{\omega}_2 \times \bar{\rho}_2 \tag{13}$$

The vectors $\bar{v}_2$ and $\bar{\omega}_2 \times \bar{\rho}_2$ are already expressed in the axis system of the parachute. Let $[\Gamma]$ be a transformation matrix which transforms vectors from the axis system of body 1 to the axis system of body 2. (The equations for determining $[\Gamma]$ are given in appendix A.) Now

$$\begin{aligned}
\dot{r}_{1x,2}\hat{i}_2 + \dot{r}_{1y,2}\hat{j}_2 + \dot{r}_{1z,2}\hat{k}_2 = {} & \left(-V_{x,2} - \omega_{y,2}\rho_{2z,2} + \omega_{z,2}\rho_{2y,2}\right)\hat{i}_2 \\
& \left(-V_{y,2} - \omega_{z,2}\rho_{2x,2} + \omega_{x,2}\rho_{2z,2}\right)\hat{j}_2 \\
& \left(-V_{z,2} - \omega_{x,2}\rho_{2y,2} + \omega_{y,2}\rho_{2x,2}\right)\hat{k}_2 \\
& + \Big[\Gamma_{11}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
& + \Gamma_{12}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
& + \Gamma_{13}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right)\Big]\hat{i}_2 \\
& + \Big[\Gamma_{21}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
& + \Gamma_{22}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
& + \Gamma_{23}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right)\Big]\hat{j}_2 \\
& + \Big[\Gamma_{31}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
& + \Gamma_{32}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
& + \Gamma_{33}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right)\Big]\hat{k}_2
\end{aligned} \tag{14}$$

Equating the coefficients of $\hat{i}_2$, $\hat{j}_2$, and $\hat{k}_2$ results in the following three scalar equations for the time rate of change of the components of each vector from the parachute to the payload.

$$\left.\begin{aligned}
\dot{r}_{1x,2} &= -V_{x,2} - \omega_{y,2}\rho_{2z,2} + \omega_{z,2}\rho_{2y,2} \\
&\quad + \Gamma_{11}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
&\quad + \Gamma_{12}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
&\quad + \Gamma_{13}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right) \\[2ex]
\dot{r}_{1y,2} &= -V_{y,2} - \omega_{z,2}\rho_{2x,2} + \omega_{x,2}\rho_{2z,2} \\
&\quad + \Gamma_{21}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
&\quad + \Gamma_{22}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
&\quad + \Gamma_{23}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right) \\[2ex]
\dot{r}_{1z,2} &= -V_{z,2} - \omega_{x,2}\rho_{2y,2} + \omega_{y,2}\rho_{2x,2} \\
&\quad + \Gamma_{31}\left(V_{x,1} + \omega_{y,1}\rho_{1z,1} - \omega_{z,1}\rho_{1y,1}\right) \\
&\quad + \Gamma_{32}\left(V_{y,1} + \omega_{z,1}\rho_{1x,1} - \omega_{x,1}\rho_{1z,1}\right) \\
&\quad + \Gamma_{33}\left(V_{z,1} + \omega_{x,1}\rho_{1y,1} - \omega_{y,1}\rho_{1x,1}\right)
\end{aligned}\right\} \tag{15}$$

The integration of these equations yields the displacement of each point on the payload with respect to the parachute reference frame.

If the two points were connected by individual elastic lines such as $\bar{r}_1$ in figure 1, then the resulting forces transmitted to the parachute and payload are determined by multiplying the spring constant of the line by its elongation. The dynamics of the system are then determined. The equations of motion of both the parachute and payload are integrated to yield the velocities and angular rates which are used to determine the elongation of the connecting lines, thus giving the forces in these lines.

In most systems, however, the parachute and payload are not connected by individual lines but by space frames composed of suspension lines with risers or harnesses connecting the confluence point of the suspension lines to the payload (fig. 2). In order to accurately obtain the system dynamics, it is necessary to determine the force distribution in these space frames.

Space Frame Forces

To solve for these forces, consider the typical space frame as shown in figure 3. Points 1 to 3 represent attach points on the payload, and the displacement of these points with respect to the parachute can be found by integrating the time rate of change of a vector from some arbitrary point on the parachute to the payload attach point. Points 4 to 8 are attach points of the suspension lines on the parachute. One of the basic assumptions of the analysis is that the canopy maintains a rigid shape such that the skirt of the canopy, where the suspension lines attach, maintains a fairly constant shape. A review of flight test films indicates this to be a valid assumption for most canopy shapes over their normal angle-of-attack range. Small movements of the canopy attach point will have little effect on the analysis because of the long suspension-line lengths and the low spring constant of the suspension lines. Thus, the parachute attach points can be assumed to be fixed with respect to the parachute axis system. To reduce computations, the parachute suspension lines are represented in the analysis by a reduced amount having an equivalent spring constant. The method developed, however, is applicable to any number of suspension lines and is limited only by the capacity of the computer used in the analysis.

For a given frame, the forces in the members caused by a small displacement of the payload attach points can be determined by one of the methods of indeterminant structures, if the frame members are

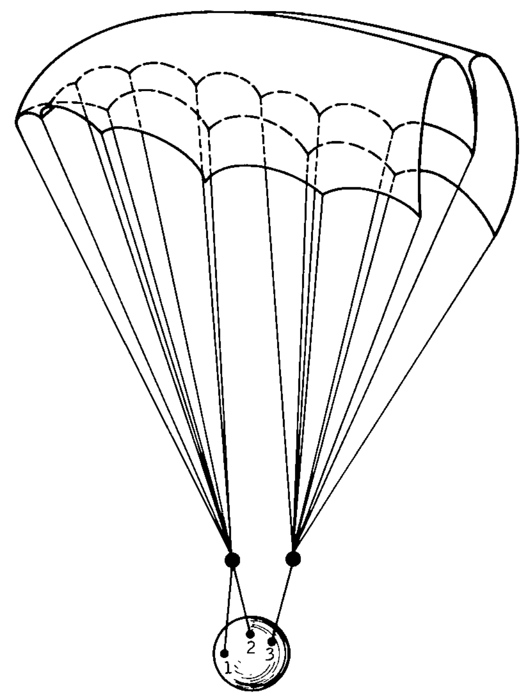

Figure 2. - Suspension-line geometry of parawing configuration.

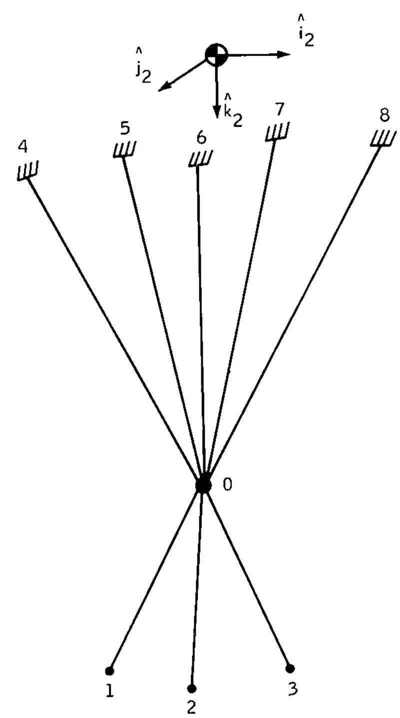

Figure 3. - Riser and suspension-line space frame.

considered to be linearly elastic. This approach assumes that, at any instant of time, the frame is in static equilibrium. This is a valid assumption as long as the frequency of oscillation between the parachute and payload is fairly low so that the acceleration of the frame does not prohibit its maintaining near-equilibrium conditions. The small-displacement theory of indeterminant structures is not valid, however, because the dynamics of the parachute payload system are such that the displacements of the payload attach points with respect to the parachute axis are not always small.

Forces in Space Frames Subjected to Large Displacements

An analytical technique is now developed which solves for the force distribution in the space frame subject to large displacements. This method follows that developed for small-displacement theory by Argyris (as discussed in ref. 9) but is expanded to incorporate large displacements. The following analysis considers a limited number of frame members, but can be readily expanded to incorporate any number of members.

Let points 3 to 5 of figure 4 represent points on the parachute which are fixed with respect to the parachute axis system $\hat{i}_2$, $\hat{j}_2$, $\hat{k}_2$. Points 1 and 2 represent points on the payload which undergo displacements dx_i, dy_i, and dz_i with respect to their initial position in the parachute reference frame. The displacements of point o from its original position are dx_o, dy_o, and dz_o.

In the parachute reference frame, let Fx_i, Fy_i, and Fz_i represent the three components of force at each movable point required to hold the frame in equilibrium when it is displaced from its unstressed position. Let α_i, β_i, and γ_i be the direction cosines of frame member i in the parachute reference frame and P_i be the axial force in member i. The forces Fx_i, Fy_i, and Fz_i can then be written in terms

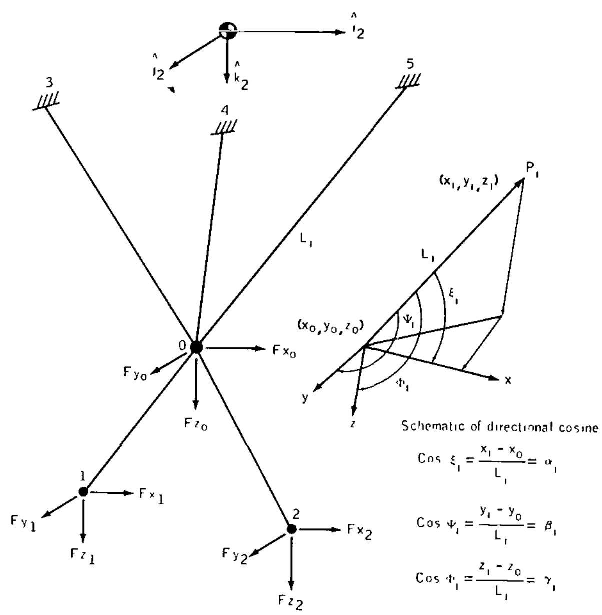

Figure 4. - Space frame equilibrium forces.

of the axial forces and direction cosines as follows

$$
\begin{Bmatrix} Fx_1 \\ Fy_1 \\ Fz_1 \\ Fx_2 \\ Fy_2 \\ Fz_2 \\ Fx_0 \\ Fy_0 \\ Fz_0 \end{Bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 \\ -\beta_1 & -\beta_2 & -\beta_3 & -\beta_4 & -\beta_5 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 & -\gamma_5 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} \tag{16}
$$

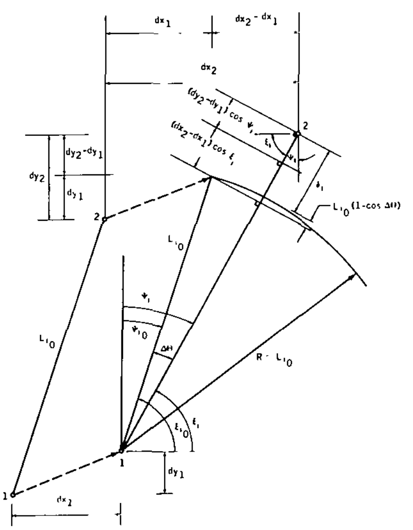

Figure 5. - Large displacement diagram.

To solve for the force P_i in terms of the large displacements, consider figure 5. A single member in planar motion is shown, but the theory readily extends to several members in three dimensional space.

The axial force in member i is a function of its linear spring constant and elongation; thus

$$
\begin{aligned} P_i = K_i\delta_i = K_i\Big\{&\Big[dx_2(\cos \xi_i) - dx_1(\cos \xi_i) \\ &+ dy_2(\cos \psi_i) - dy_1(\cos \psi_i)\Big] \\ &- \Big[L_{i_0}(1 - \cos \Delta\theta)\Big] \end{aligned} \tag{17}
$$

In matrix form, equation (17) becomes

$$P_i = \left[K_i\right] \left\{ \left[\left(\cos \xi_i\right) \left(\cos \psi_i\right) \left(-\cos \xi_i\right) \left(-\cos \psi_i\right) \right] \begin{Bmatrix} dx_2 \\ dy_2 \\ dx_1 \\ dy_1 \end{Bmatrix} - L_{i_0} \left[(1 - \cos \Delta\theta) \right] \right\} \tag{18}$$

The total angle change of frame member i from its unstressed condition is determined from

$$1 - \cos \Delta\theta = \frac{1}{2} \left[\left(\cos \xi_i - \cos \xi_{i_0}\right)^2 + \left(\cos \psi_i - \cos \psi_{i_0}\right)^2 \right] \tag{19}$$

By making this substitution and extending equation (17) to the three dimensional frame of figure 4

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} = \begin{bmatrix} K_1 & & & & \\ & K_2 & & 0 & \\ & & K_3 & & \\ & 0 & & K_4 & \\ & & & & K_5 \end{bmatrix} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 & -\alpha_1 & -\beta_1 & -\gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 & -\alpha_2 & -\beta_2 & -\gamma_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_3 & -\beta_3 & -\gamma_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_4 & -\beta_4 & -\gamma_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\alpha_5 & -\beta_5 & -\gamma_5 \end{bmatrix} \right.$$

$$\left. \begin{Bmatrix} dx_1 \\ dy_1 \\ dz_1 \\ dx_2 \\ dy_2 \\ dz_2 \\ dx_o \\ dy_o \\ dz_o \end{Bmatrix} - \frac{1}{2} \begin{bmatrix} L_{1_0}\left(\rho_1{}^2 + \tau_1{}^2 + \sigma_1{}^2\right) \\ L_{2_0}\left(\rho_2{}^2 + \tau_2{}^2 + \sigma_2{}^2\right) \\ L_{3_0}\left(\rho_3{}^2 + \tau_3{}^2 + \sigma_3{}^2\right) \\ L_{4_0}\left(\rho_4{}^2 + \tau_4{}^2 + \sigma_4{}^2\right) \\ L_{5_0}\left(\rho_5{}^2 + \tau_5{}^2 + \sigma_5{}^2\right) \end{bmatrix} \right\} \tag{20}$$

By substituting for P in equation (16), the force matrix can now be symbolically written as

$$\{F\} = [A][K]\left[[A]^T\{d\} - \left\{\frac{1}{2}\{L_0\}\left(\rho^2 + \tau^2 + \sigma^2\right)\right\}\right] \tag{21}$$

Thus, it is seen that the forces required to hold the space frame in equilibrium for a given displacement can be expressed in terms of the displacements of the points where the forces are applied and the original length and direction cosines of the frame members.

Consider again the frame in figure 4. Points 1 and 2 are the payload attach points, and external forces are required at these points to displace the frame. Point o, however, is an interior point in the frame and no external forces are applied here. Thus, Fx_o, Fy_o, and Fz_o are virtual forces and vanish when the frame is in static equilibrium.

Performing the matrix multiplication required for finding Fx_o, Fy_o, and Fz_o yields the following three equations for a frame with m total members and n members from the payload to confluence point.

$$\left.\begin{aligned}
Fx_o = 0 = &-\sum_{i=1}^{n}\left(K_i\alpha_i^2 dx_i + K_i\alpha_i\beta_i dy_i + K_i\alpha_i\gamma_i dz_i\right) \\
&+ \sum_{i=1}^{m}\left[K_i\alpha_i^2 dx_o + K_i\alpha_i\beta_i dy_o + K_i\alpha_i\gamma_i dz_o + \frac{1}{2}K_i\alpha_i L_{i_0}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right] \\
Fy_o = 0 = &-\sum_{i=1}^{n}\left(K_i\alpha_i\beta_i dx_i + K_i\beta_i^2 dy_i + K_i\beta_i\gamma_i dz_i\right) \\
&+ \sum_{i=1}^{m}\left[K_i\alpha_i\beta_i dx_o + K_i\beta_i^2 dy_o + K_i\beta_i\gamma_i dz_o + \frac{1}{2}K_i\beta_i L_{i_0}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right] \\
Fz_o = 0 = &-\sum_{i=1}^{n}\left(K_i\alpha_i\gamma_i dx_i + K_i\beta_i\gamma_i dy_i + K_i\gamma_i^2 dz_i\right) \\
&+ \sum_{i=1}^{m}\left[K_i\alpha_i\gamma_i dx_o + K_i\beta_i\gamma_i dy_o + K_i\gamma_i^2 dz_o + \frac{1}{2}K_i\gamma_i L_{i_0}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right]
\end{aligned}\right\} \tag{22}$$

Now assume that points 1 and 2 are given known displacements dx_i, dy_i, and dz_i. The parameters α_i, β_i, γ_i, ρ_i, τ_i, and σ_i are all functions of the variables dx_o, dy_o, and dz_o; and equation (22) yields three equations of the form

$$
\begin{aligned}
F_1 = 0 = &-\sum_{i=1}^{n}\left[\frac{K_i\left(X_i - X_{o_0} - dx_o\right)^2 dx_i}{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}\right. \\
&+ \frac{K_i\left(X_i - X_{o_0} - dx_o\right)\left(Y_i - Y_{o_0} - dy_o\right)dy_i}{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2} + \left.\frac{K_i\left(X_i - X_{o_0} - dx_o\right)\left(Z_i - Z_{o_0} - dz_o\right)dz_i}{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}\right] \\
&+ \sum_{i=1}^{m}\left[\frac{K_i\left(X_i - X_{o_0} - dx_o\right)^2 dx_o}{\left(X_j - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2} + \frac{K_i\left(X_i - X_{o_0} - dx_o\right)\left(Y_i - Y_{o_0} - dy_o\right)dy_o}{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}\right. \\
&+ \left.\frac{K_i\left(X_i - X_{o_0} - dx_o\right)\left(Z_i - Z_{o_0} - dz_o\right)dz_o}{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}\right] + \sum_{i=1}^{m}\left[\frac{K_i\left(X_i - X_{o_0} - dx_o\right)L_{i_0}}{2\sqrt{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}}\right] \\
&\left\{\left[\frac{\left(X_i - X_{o_0} - dx_o\right)}{\sqrt{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}} - \frac{\left(X_{i_0} - X_{o_0}\right)}{\sqrt{\left(X_{i_0} - X_{o_0}\right)^2 + \left(Y_{i_0} - Y_{o_0}\right)^2 + \left(Z_{i_0} - Z_{o_0}\right)^2}}\right]^2\right. \\
&+ \left[\frac{\left(Y_i - Y_{o_0} - dy_o\right)}{\sqrt{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}} - \frac{\left(Y_{i_0} - Y_{o_0}\right)}{\sqrt{\left(X_{i_0} - X_{o_0}\right)^2 + \left(Y_{i_0} - Y_{o_0}\right)^2 + \left(Z_{i_0} - Z_{o_0}\right)^2}}\right]^2 \\
&+ \left.\left[\frac{\left(Z_i - Z_{o_0} - dz_o\right)}{\sqrt{\left(X_i - X_{o_0} - dx_o\right)^2 + \left(Y_i - Y_{o_0} - dy_o\right)^2 + \left(Z_i - Z_{o_0} - dz_o\right)^2}} - \frac{\left(Z_{i_0} - Z_{o_0}\right)}{\sqrt{\left(X_{i_0} - X_{o_0}\right)^2 + \left(Y_{i_0} - Y_{o_0}\right)^2 + \left(Z_{i_0} - Z_{o_0}\right)^2}}\right]^2\right\} \qquad (23)
\end{aligned}
$$

Thus, there are three equations in the three unknowns dx_o, dy_o, and dz_o. However, these equations are very nonlinear in the three unknowns, and their solution requires the use of an iterative method. The iterative substitution method of Gauss-Seidel was first attempted, but failed to converge for large displacements.

Newton-Raphson Iterative Solution

The Newton-Raphson method was next used and very good convergence for all displacements was obtained. Reference 10 explains the principles of the Newton-Raphson method, and its application to the present problem is explained below. Let the equations for Fx_o, Fy_o, and Fz_o be denoted as the functions F_1, F_2, and F_3, or in matrix form as $\{F\}$. Define as ϕ the 3 by 3 matrix composed of elements of the Jacobian

$$\frac{\partial\left(F_1, F_2, F_3\right)}{\partial\left(dx_o, dy_o, dz_o\right)} \tag{24}$$

Then

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \tag{25}$$

The determinant of $[\phi]_k$ is the Jacobian of the system evaluated at step k. Equations for all F terms and all ϕ terms are given in appendix B. The equations in appendix B were obtained by hand calculation and were later checked with a computer program using the FORMAC computer languages as described in reference 11.

Now assume that initially at step k, dx_o, dy_o, and dz_o are known. If the payload now experiences an additional displacement, the new values of dx_o, dy_o, and dz_o can be computed as follows.

Let

$$\{\delta\} = \begin{Bmatrix} \delta dx_o \\ \delta dy_o \\ \delta dz_o \end{Bmatrix} = \text{corrections to previous values of } \begin{Bmatrix} dx_o \\ dy_o \\ dz_o \end{Bmatrix} \tag{26}$$

Then by the Newton-Raphson method

$$\begin{Bmatrix} dx_o \\ dy_o \\ dz_o \end{Bmatrix}_{k+1} = \begin{Bmatrix} dx_o \\ dy_o \\ dz_o \end{Bmatrix}_k + \begin{Bmatrix} \delta dx_o \\ \delta dy_o \\ \delta dz_o \end{Bmatrix}_k = \{d\}_{k+1} \tag{27}$$

where

$$\begin{Bmatrix} \delta dx_o \\ \delta dy_o \\ \delta dz_o \end{Bmatrix}_k = -[\phi]_k^{-1}\{F\}_k \tag{28}$$

The matrices $[\phi]_k^{-1}$ and $\{F\}_k$ are evaluated using the values of $\begin{Bmatrix} dx_o \\ dy_o \\ dz_o \end{Bmatrix}$ at step k.

Using the new values of $\{d\}$, the process can be repeated until the solution converges as evidenced by the residual of the functions approaching zero.

The Newton-Raphson method converges rapidly, usually requiring only three or four iterations for a typical 10- to 20-member frame. The initial approximation for $\{d\}_k$ is taken as the unstressed position of the junction, and subsequent approximations are taken from the position at the end of the previous integration step. In general, these are good approximations and are a factor in the rapid convergence of the Newton-Raphson method. The primary problem is determining the partial derivatives of $\{F\}$, but once they are found, the equations can be adapted to handle any number of frame members as long as only one junction exists per frame. Equations for the partial derivatives are given in appendix B and are valid for any frame so long as only one confluence point exists in the frame.

After the displacements of point o are found, the forces in each frame member can be determined by the matrix operation of equation (20).

$$\{P\} = [K]\left[[A]^T\{d\} - \left\{\frac{1}{2}\{L_0\}\left(\rho^2 + \tau^2 + \sigma^2\right)\right\}\right] \tag{29}$$

Since the direction cosines of each frame member in the parachute axis system have been computed, it is possible to determine the force components of each member in this system. The force components of the frame members attached to the payload can be found by multiplying by the transformation matrix $[\Gamma]^T$. If these force components are now multiplied by the moment arms from the parachute and payload centers of mass, the moment contributions caused by the frames can be calculated. These forces and moments can now be added to the aerodynamic forces and moments to calculate the six-degree-of-freedom motion of both the payload and parachute.

CONCLUDING REMARKS

The mathematical model as presented will permit a more detailed analysis of the flight dynamics of general parachute-payload systems than has been possible in the past. A particular advantage of the two-body approach over a single rigid-body analysis is that the aerodynamic damping coefficients are much less critical because of the large distance between the canopy center of gravity and the combined canopy-payload center of gravities. In the single-body analysis, this distance is combined with the angular velocity of the system to give the induced velocity term necessary for damping equations. In the present analysis, damping coefficients have less effect because the induced velocity of the canopy is calculated in the equation of motion. Since damping coefficients are by far the most difficult to determine in wind tunnel testing, the advantage of using this method is obvious. In addition, the two-body approach allows relative motion between the parachute and payload which can result in significant changes in the inertia matrix of the equivalent single rigid body. The single-body approach does not account for these inertia changes.

The mathematical model as presented has been programed in FORTRAN IV and V languages for the IBM 7094 and Univac 1108 computers. The qualitative results appeared very good, but lack of experimental data prevents a complete evaluation of the program. Although studies are currently underway to determine aerodynamic and mass properties of lifting parachutes (refs. 12 to 15), as yet, they have not been defined well enough to have complete confidence for use in six-degree-of-freedom simulations. A program listing of the model will be supplied on request.

Manned Spacecraft Center
National Aeronautics and Space Administration
Houston, Texas, July 12, 1968
961-21-30-09-72

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APPENDIX A

EQUATIONS FOR $[\Gamma]$

The transformation matrix $[\Gamma]$ which transforms vectors from the axis system of body 1 to the axis system of body 2 is determined from

$$[\Gamma] = \begin{bmatrix} (\cos\bar{\theta}\cos\bar{\psi}) & (\cos\bar{\theta}\sin\bar{\psi}) & (-\sin\bar{\theta}) \\ \begin{matrix}(-\sin\bar{\psi}\cos\bar{\Phi} \\ +\sin\bar{\Phi}\sin\bar{\theta}\cos\bar{\psi})\end{matrix} & \begin{matrix}(\cos\bar{\Phi}\cos\bar{\psi} \\ +\sin\bar{\Phi}\sin\bar{\theta}\sin\bar{\psi})\end{matrix} & (\sin\bar{\Phi}\cos\bar{\theta}) \\ \begin{matrix}(\sin\bar{\psi}\sin\bar{\Phi} \\ +\cos\bar{\Phi}\sin\bar{\theta}\cos\bar{\psi})\end{matrix} & \begin{matrix}(-\sin\bar{\Phi}\cos\bar{\psi} \\ +\cos\bar{\Phi}\sin\bar{\theta}\sin\bar{\psi})\end{matrix} & (\cos\bar{\Phi}\cos\bar{\theta}) \end{bmatrix} \tag{A1}$$

The angles $\bar{\Phi}$, $\bar{\theta}$, and $\bar{\psi}$ are the relative Euler angles of body 2 with respect to body 1 applied in the order of rotation $\bar{\psi}, \bar{\theta}, \bar{\Phi}$. These relative Euler angles are determined by integrating the equations

$$\dot{\bar{\Phi}} = \left(\omega_{x,2} - \bar{\omega}_{x,1}\right) + \tan\bar{\theta}\left[\left(\omega_{y,2} - \bar{\omega}_{y,1}\right)\sin\bar{\Phi} + \left(\omega_{z,2} - \bar{\omega}_{z,1}\right)\cos\bar{\Phi}\right] \tag{A2}$$

$$\dot{\bar{\theta}} = \left(\omega_{y,2} - \bar{\omega}_{y,1}\right)\cos\bar{\Phi} - \left(\omega_{z,2} - \bar{\omega}_{z,1}\right)\sin\bar{\Phi} \tag{A3}$$

$$\dot{\bar{\psi}} = \left[\left(\omega_{y,2} - \omega_{y,1}\right)\sin\bar{\Phi} + \left(\omega_{z,2} - \bar{\omega}_{z,1}\right)\cos\bar{\Phi}\right] \div \cos\bar{\theta} \tag{A4}$$

where

$$\begin{Bmatrix} \bar{\omega}_{x,1} \\ \bar{\omega}_{y,1} \\ \bar{\omega}_{z,1} \end{Bmatrix} = [\Gamma]\begin{Bmatrix} \omega_{x,1} \\ \omega_{y,1} \\ \omega_{z,1} \end{Bmatrix} \tag{A5}$$

APPENDIX B

EQUATIONS FOR EVALUATING ELEMENTS OF THE JACOBIAN

$$F_1 = -\sum_{i=1}^{n}\left(\frac{K_i \Delta X_i^2 dx_i}{L_i^2} + \frac{K_i \Delta X_i \Delta Y_i dy_i}{L_i^2} + \frac{K_i \Delta X_i \Delta Z_i dz_i}{L_i^2}\right)$$

$$+ \sum_{i=1}^{m}\left[\frac{K_i \Delta X_i^2 dx_o}{L_i^2} + \frac{K_i \Delta X_i \Delta Y_i dy_o}{L_i^2} + \frac{K_i \Delta X_i \Delta Z_i dz_o}{L_i^2} + \frac{K_i \Delta X_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right] \tag{B1}$$

$$F_2 = -\sum_{i=1}^{n}\left(\frac{K_i \Delta X_i \Delta Y_i dx_i}{L_i^2} + \frac{K_i \Delta Y_i^2 dy_i}{L_i^2} + \frac{K_i \Delta Y_i \Delta Z_i dz_i}{L_i^2}\right)$$

$$+ \sum_{i=1}^{m}\left[\frac{K_i \Delta X_i \Delta Y_i dx_o}{L_i^2} + \frac{K_i \Delta Y_i^2 dy_o}{L_i^2} + \frac{K_i \Delta Y_i \Delta Z_i dz_o}{L_i^2} + \frac{K_i \Delta Y_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right] \tag{B2}$$

$$F_3 = -\sum_{i=1}^{n}\left(\frac{K_i \Delta X_i \Delta Z_i dx_i}{L_i^2} + \frac{K_i \Delta Y_i \Delta Z_i dy_i}{L_i^2} + \frac{K_i \Delta Z_i^2 dz_i}{L_i^2}\right)$$

$$+ \sum_{i=1}^{m}\left[\frac{K_i \Delta X_i \Delta Z_i dx_o}{L_i^2} + \frac{K_i \Delta Y_i \Delta Z_i dy_o}{L_i^2} + \frac{K_i \Delta Z_i^2 dz_o}{L_i^2} + \frac{K_i \Delta Z_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)\right] \tag{B3}$$

$$
\begin{aligned}
\phi_{11} = \sum_{i=1}^{n} \Bigg(& \frac{2K_i \Delta X_i dx_i}{L_i^2} - \frac{2K_i \Delta X_i^3 dx_i}{L_i^4} + \frac{K_i \Delta Y_i dy_i}{L_i^2} - \frac{2K_i \Delta X_i^2 dy_i \Delta Y_i}{L_i^4} \\
& + \frac{K_i \Delta Z_i dz_i}{L_i^2} - \frac{2K_i \Delta X_i^2 \Delta Z_i dz_i}{L_i^4} \Bigg) + \sum_{i=1}^{m} \Bigg\{ - \frac{2K_i \Delta X_i dx_o}{L_i^2} \\
& + \frac{2K_i \Delta X_i^3 dx_o}{L_i^4} - \frac{K_i \Delta Y_i dy_o}{L_i^2} + \frac{K_i \Delta X_i^2}{L_i^2} + \frac{2K_i \Delta X_i^2 \Delta Y_i dy_o}{L_i^4} - \frac{K_i \Delta Z_i dz_o}{L_i^2} \\
& + \frac{2K_i \Delta X_i^2 \Delta Z_i dz_o}{L_i^4} - \frac{K_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right) + \frac{K_i \Delta X_i^2 L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right) \\
& + \frac{K_i \Delta X_i L_{i_0}}{L_i}\left[\tau_i \frac{\Delta X_i \Delta Y_i}{L_i^3} + \rho_i\left(\frac{\Delta X_i^2}{L_i^3} - \frac{1}{L_i}\right) + \frac{\sigma_i \Delta X_i \Delta Z_i}{L_i^3}\right]\Bigg\}
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
\phi_{22} = \sum_{i=1}^{n} \Bigg(& \frac{K_i \Delta X_i dx_i}{L_i^2} - \frac{2K_i \Delta X_i \Delta Y_i^2 dx_i}{L_i^4} + \frac{2K_i \Delta Y_i dy_i}{L_i^2} - \frac{2K_i \Delta Y_i^3 dy_i}{L_i^4} \\
& + \frac{K_i \Delta Z_i dz_i}{L_i^2} - \frac{2K_i \Delta Y_i^2 \Delta Z_i dz_i}{L_i^4} \Bigg) + \sum_{i=1}^{m} \Bigg\{ \frac{2K_i \Delta Y_i^3 dy_o}{L_i^4} - \frac{K_i \Delta Z_i dz_o}{L_i^2} \\
& - \frac{K_i \Delta X_i dx_o}{L_i^2} + \frac{2K_i \Delta X_i \Delta Y_i^2 dx_o}{L_i^4} - \frac{2K_i \Delta Y_i dy_o}{L_i^2} + \frac{K_i \Delta Y_i^2}{L_i^2} \\
& + \frac{2K_i \Delta Y_i^2 \Delta Z_i dz_o}{L_i^4} - \frac{K_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right) + \frac{K_i \Delta Y_i^2 L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right) \\
& + \frac{K_i \Delta Y_i L_{i_0}}{L_i}\left[\rho_i \frac{\Delta X_i \Delta Y_i}{L_i^3} + \tau_i\left(\frac{\Delta Y_i^2}{L_i^3} - \frac{1}{L_i}\right) + \sigma_i \frac{\Delta Y_i \Delta Z_i}{L_i^3}\right]\Bigg\}
\end{aligned}
\tag{B5}
$$

$$
\begin{aligned}
\phi_{12} = & \sum_{i=1}^{n}\left(-\frac{2K_i\Delta X_i^{2}\Delta Y_i dx_i}{L_i^{4}} + \frac{K_i\Delta X_i dy_i}{L_i^{2}} - \frac{2K_i\Delta X_i\Delta Y_i^{2}dy_i}{L_i^{4}} - \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dz_i}{L_i^{4}}\right) \\
& + \sum_{i=1}^{m}\Bigg\{\frac{2K_i\Delta X_i^{2}\Delta Y_i dx_o}{L_i^{4}} - \frac{K_i\Delta X_i dy_o}{L_i^{2}} + \frac{K_i\Delta X_i\Delta Y_i}{L_i^{2}} + \frac{2K_i\Delta X_i\Delta Y_i^{2}dy_o}{L_i^{4}} \\
& + \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dz_o}{L_i^{4}} + \frac{K_i\Delta X_i\Delta Y_i L_{i_0}}{2L_i^{3}}\left(\rho_i^{2} + \tau_i^{2} + \sigma_i^{2}\right) \\
& + \frac{K_i\Delta X_i L_{i_0}}{L_i}\left[\rho_i\frac{\Delta X_i\Delta Y_i}{L_i^{3}} + \tau_i\left(\frac{\Delta Y_i^{2}}{L_i^{3}} - \frac{1}{L_i}\right) + \sigma_i\frac{\Delta Y_i\Delta Z_i}{L_i^{3}}\right]\Bigg\}
\end{aligned}
\qquad \text{(B6)}
$$

$$
\begin{aligned}
\phi_{21} = & \sum_{i=1}^{n}\left(\frac{K_i\Delta Y_i dx_i}{L_i^{2}} - \frac{2K_i\Delta X_i^{2}\Delta Y_i dx_i}{L_i^{4}} - \frac{2K_i\Delta X_i\Delta Y_i^{2}dy_i}{L_i^{4}} - \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dz_i}{L_i^{4}}\right) \\
& + \sum_{i=1}^{m}\Bigg\{-\frac{K_i\Delta Y_i dx_o}{L_i^{2}} + \frac{K_i\Delta X_i\Delta Y_i}{L_i^{2}} + \frac{2K_i\Delta X_i^{2}\Delta Y_i dx_o}{L_i^{4}} + \frac{2K_i\Delta X_i\Delta Y_i^{2}dy_o}{L_i^{4}} \\
& + \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dz_o}{L_i^{4}} + \frac{K_i\Delta X_i\Delta Y_i L_{i_0}}{2L_i^{3}}\left(\rho_i^{2} + \tau_i^{2} + \sigma_i^{2}\right) \\
& + \frac{K_i\Delta Y_i L_{i_0}}{L_i}\left[\rho_i\left(\frac{\Delta X_i^{2}}{L_i^{3}} - \frac{1}{L_i}\right) + \tau_i\frac{\Delta X_i\Delta Y_i}{L_i^{3}} + \sigma_i\frac{\Delta X_i\Delta Z_i}{L_i^{3}}\right]\Bigg\}
\end{aligned}
\qquad \text{(B7)}
$$

$$\phi_{23} = \sum_{i=1}^{n}\left(-\frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dx_i}{L_i^4} - \frac{2K_i\Delta Y_i^2\Delta Z_i dy_i}{L_i^4} + \frac{K_i\Delta Y_i dz_i}{L_i^2} - \frac{2K_i\Delta Y_i\Delta Z_i^2 dz_i}{L_i^4}\right)$$

$$+ \sum_{i=1}^{m}\left\{\frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dx_o}{L_i^4} + \frac{2K_i\Delta Y_i^2\Delta Z_i dy_o}{L_i^4} - \frac{K_i\Delta Y_i dz_o}{L_i^2} + \frac{K_i\Delta Y_i\Delta Z_i}{L_i^2}\right.$$

$$+ \frac{2K_i\Delta Y_i\Delta Z_i^2 dz_o}{L_i^4} + \frac{K_i\Delta Y_i\Delta Z_i L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)$$

$$\left. + \frac{K_i\Delta Y_i L_{i_0}}{L_i}\left[\rho_i \frac{\Delta X_i\Delta Z_i}{L_i^3} + \tau_i \frac{\Delta Y_i\Delta Z_i}{L_i^3} + \sigma_i\left(\frac{\Delta Z_i^2}{L_i^3} - \frac{1}{L_i}\right)\right]\right\} \tag{B8}$$

$$\phi_{33} = \sum_{i=1}^{n}\left(\frac{K_i\Delta X_i dx_i}{L_i^2} - \frac{2K_i\Delta X_i\Delta Z_i^2 dx_i}{L_i^4} + \frac{K_i\Delta Y_i dy_i}{L_i^2} - \frac{2K_i\Delta Y_i\Delta Z_i^2 dy_i}{L_i^4}\right.$$

$$\left. + \frac{2K_i\Delta Z_i dz_i}{L_i^2} - \frac{2K_i\Delta Z_i^3 dz_i}{L_i^4}\right) + \sum_{i=1}^{m}\left\{-\frac{K_i\Delta X_i dx_o}{L_i^2} + \frac{2K_i\Delta X_i\Delta Z_i^2 dx_o}{L_i^4}\right.$$

$$- \frac{K_i\Delta Y_i dy_o}{L_i^2} + \frac{2K_i\Delta Y_i\Delta Z_i^2 dy_o}{L_i^4} - \frac{2K_i\Delta Z_i dz_o}{L_i^2} + \frac{K_i\Delta Z_i^2}{L_i^2} + \frac{2K_i\Delta Z_i^3 dz_o}{L_i^4}$$

$$- \frac{K_i L_{i_0}}{2L_i}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right) + \frac{K_i\Delta Z_i^2 L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)$$

$$\left. + \frac{K_i\Delta Z_i L_{i_0}}{L_i}\left[\rho_i \frac{\Delta X_i\Delta Z_i}{L_i^3} + \tau_i \frac{\Delta Y_i\Delta Z_i}{L_i^3} + \sigma_i\left(\frac{\Delta Z_i^2}{L_i^3} - \frac{1}{L_i}\right)\right]\right\} \tag{B9}$$

$$\phi_{32} = \sum_{i=1}^{n}\left(-\frac{2K_i\Delta X_i\Delta Z_i\Delta Y_i dx_i}{L_i^4} + \frac{K_i\Delta Z_i dy_i}{L_i^2} - \frac{2K_i\Delta Y_i^2\Delta Z_i dy_i}{L_i^4} - \frac{2K_i\Delta Z_i^2\Delta Y_i dz_i}{L_i^4}\right)$$

$$+ \sum_{i=1}^{m}\left\{\frac{2K_i\Delta X_i\Delta Z_i\Delta Y_i dx_o}{L_i^4} - \frac{K_i\Delta Z_i dy_o}{L_i^2} + \frac{K_i\Delta Y_i\Delta Z_i}{L_i^2} + \frac{2K_i\Delta Y_i^2\Delta Z_i dy_o}{L_i^4}\right.$$

$$+ \frac{2K_i\Delta Z_i^2\Delta Y_i dz_o}{L_i^4} + \frac{K_i\Delta Y_i\Delta Z_i L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)$$

$$\left. + \frac{K_i\Delta Z_i L_{i_0}}{L_i}\left[\frac{\rho_i\Delta X_i\Delta Y_i}{L_i^3} + \tau_i\left(\frac{\Delta Y_i^2}{L_i^3} - \frac{1}{L_i}\right) + \sigma_i\frac{\Delta Y_i\Delta Z_i}{L_i^3}\right]\right\} \tag{B10}$$

$$\phi_{13} = \sum_{i=1}^{n}\left(-\frac{2K_i\Delta X_i^2\Delta Z_i dx_i}{L_i^4} - \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dy_i}{L_i^4} + \frac{K_i\Delta X_i dz_i}{L_i^2} - \frac{2K_i\Delta X_i\Delta Z_i^2 dz_i}{L_i^4}\right)$$

$$+ \sum_{i=1}^{m}\left\{\frac{2K_i\Delta X_i^2\Delta Z_i dx_o}{L_i^4} + \frac{2K_i\Delta X_i\Delta Y_i\Delta Z_i dy_o}{L_i^4} - \frac{K_i\Delta X_i dz_o}{L_i^2} + \frac{K_i\Delta X_i\Delta Z_i}{L_i^2}\right.$$

$$+ \frac{2K_i\Delta X_i\Delta Z_i^2 dz_o}{L_i^4} + \frac{K_i\Delta X_i\Delta Z_i L_{i_0}}{2L_i^3}\left(\rho_i^2 + \tau_i^2 + \sigma_i^2\right)$$

$$\left. + \frac{K_i\Delta X_i L_{i_0}}{L_i}\left[\rho_i\frac{\Delta X_i\Delta Z_i}{L_i^3} + \tau_i\frac{\Delta Y_i\Delta Z_i}{L_i^3} + \sigma_i\left(\frac{\Delta Z_i^2}{L_i^3} - \frac{1}{L_i}\right)\right]\right\} \tag{B11}$$

$$\phi_{31} = \sum_{i=1}^{n}\left(\frac{K_i \Delta Z_i dx_i}{L_i^{2}} - \frac{2K_i \Delta X_i^{2} \Delta Z_i dx_i}{L_i^{4}} - \frac{2K_i \Delta X_i \Delta Y_i \Delta Z_i dy_i}{L_i^{4}} - \frac{2K_i \Delta X_i \Delta Z_i^{2} dz_i}{L_i^{4}}\right)$$

$$+ \sum_{i=1}^{m}\left\{- \frac{K_i \Delta Z_i dx_o}{L_i^{2}} + \frac{K_i \Delta X_i \Delta Z_i}{L_i^{2}} + \frac{2K_i \Delta X_i^{2} \Delta Z_i dx_o}{L_i^{4}} + \frac{2K_i \Delta X_i \Delta Y_i \Delta Z_i dy_o}{L_i^{4}}\right.$$

$$+ \frac{2K_i \Delta X_i \Delta Z_i^{2} dz_o}{L_i^{4}} + \frac{K_i \Delta X_i \Delta Z_i L_{i_0}}{2L_i^{3}}\left(\rho_i^{2} + \tau_i^{2} + \sigma_i^{2}\right)$$

$$\left. + \frac{K_i \Delta Z_i L_{i_0}}{L_i}\left[\rho_i\left(\frac{\Delta X_i^{2}}{L_i^{3}} - \frac{1}{L_i}\right) + \tau_i \frac{\Delta X_i \Delta Y_i}{L_i^{3}} + \sigma_i \frac{\Delta X_i \Delta Z_i}{L_i^{3}}\right]\right\} \tag{B12}$$

where

$$\Delta X_i = X_i - X_{o_0} - dx_o \tag{B13}$$

$$\Delta Y_i = Y_i - Y_{o_0} - dy_o \tag{B14}$$

and

$$\Delta Z_i = Z_i - Z_{o_0} - dz_o \tag{B15}$$

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