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ABSTRACT

As a step toward understanding relativistic rescattering singularities we examine n_{max}, the maximum number of binary collisions allowed by special relativity among three particles with zero-range forces. We find that for three equal-mass particles $n_{max} = 3$ but for three particles with masses $m_1 = m_3 = 1$ and $m_2 = m < 1$, n_{max} decreases with increasing total energy, W, and finally attains the value of four when $W > 1/m$.

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Since the kernals of the exact non-relativistic integral equations for the three-paticle T-matrix have singularities whenever rescattering for point particles with zero-range forces is allowed by energy and momentum conservation, [1] one expects similar singularities in an exact relativistic formulation. As a step toward understanding these singularities, we examine here n_{max}, the maximum number of binary collisions among three such particles allowed by special relativity, as a function of the total energy and the masses of the three particles. Three algebraic results concerning n_{max} are obtained and two cases are examined numerically.

NOTATION AND EQUATIONS

Let W be the total energy in the three-particle center of momentum (C.O.M.) system. Let ω_i = energy of particles j and k in their C.O.M. system and let $\mathbf{p}_i$ be their relative momentum in this system. $\mathbf{q}_i$ is the momentum of the i^{th} particle in the three particle C.O.M. system. Let $\cos\theta_i = -\hat{\mathbf{p}}_i \cdot \hat{\mathbf{q}}_i'$, where $\hat{\mathbf{q}}_i'$ is the momentum of the i^{th} particle in the C.O.M. system of particles j and k. $\hat{\mathbf{p}}_i = \widehat{\mathbf{q}_j' - \mathbf{q}_k'}$.

[1] G. Doolen, Phys. Rev. 166, 1651 (1968)

Here i, j, k are cyclic and assume the values 1, 2 or 3. m_i is the mass of the i^{th} particle. The hat, ^, indicates a unit vector. Using the invariant dot products of four-vectors, one can show [2]

that

$$\cos\theta_3 = \frac{1}{4W\omega_3 p_3 q_3}\left[(W^2 + m_1^2 + m_2^2 + m_3^2 - \omega_3^2)\,\omega_3^2 - (W^2 - m_3^2)(m_1^2 - m_2^2) - 2\omega_3^2\omega_1^2\right] \tag{1}$$

$$\equiv f_3(\omega_3, \omega_1)$$

and

$$\cos\theta_1 = \frac{-1}{4W\omega_1 p_1 q_1}\left[(W^2 + m_1^2 + m_2^2 + m_3^2 - \omega_1^2)\,\omega_1^2 - (W^2 - m_1^2)(m_3^2 - m_2^2) - 2\omega_3^2\omega_1^2\right]. \tag{2}$$

$$\equiv f_1(\omega_1, \omega_3)$$

We assume that the maximum number of relativistic binary collisions is attained by the same initial configuration as the non-relativistic case; namely collinear scatterings in which two particles collide with almost zero relative momentum and in which this pair of particles is approaching the third particle. The particle with the lowest mass then scatters back-and-forth between the two heavier ones until all three particles are diverging from each other.

Consider the case in which the lowest mass particle, 3, scatters back-and-forth between the heavier particles, 1 and 2. In each binary collision, the relative momentum of the two particles involved changes sign in their C.O.M. system: $\bar{p}_i(N) = -\bar{p}_i(N+1)$. The number in parentheses is the number of collisions that

[2] Equation (1) is the same as Eq. (31) of G.C. Wick, Annals of Phys. 18, 65 (1962).

have occured. Hence $\cos\theta_i$, whose magnitude is one, also changes sign. Using this fact along with (1) and (2), we obtain

$$\omega_1^2(N) = W^2 + m_1^2 + m_2^2 + m_3^2 - \omega_3^2(N-1) - (W^2 - m_3^2)(m_1^2 - m_2^2)/\omega_3^2(N-1) - \omega_1^2(N-1) \quad (3)$$

and

$$\omega_3(N) = \omega_3(N-1) \quad . \quad (4)$$

Eqs. (3) and (4) relate the variables before a collision between particles 1 and 2 to the variables afterwards.

Similarly,

$$\omega_3^2(N+1) = W^2 + m_1^2 + m_2^2 + m_3^2 - \omega_1^2(N) - (W^2 - m_1^2)(m_3^2 - m_2^2)/\omega_1^2(N) - \omega_3^2(N) \quad (5)$$

and

$$\omega_1(N+1) = \omega_1(N) \quad (6)$$

relate the variables before a collision between particles 2 and 3 to the variabies afterwards.

In the initial configuration, particles (1) and (3) are directed toward the center of mass. In the final configuration they are directed away from it. Hence q_1' and q_3' both change sign sometime during the collision sequence.

Choosing the unit vector $\mathbf{i}$ as indicated in Fig. I, we see that a necessary and sufficient condition for no further scattering is

$$\hat{\mathbf{p}}_3 = \mathbf{i} \quad (7)$$

and

$$\hat{\mathbf{p}}_1 = -\mathbf{i} \quad . \quad (8)$$

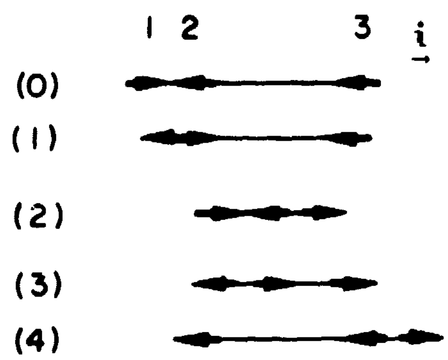

Fig. 1–Schematic Representation of Three Point Particles After each Collision in a Particular Scattering Sequence (See Text)

In the initial configuration, $\hat{\mathbf{p}}_3 = \mathbf{i}$, $\hat{\mathbf{p}}_1 = \mathbf{i}$, $\hat{\mathbf{q}}'_1 = \mathbf{i}$, and $\hat{\mathbf{q}}'_3 = -\mathbf{i}$. Hence $\cos\theta_1 = -1$ and $\cos\theta_3 = 1$ initially.

Since $\hat{\mathbf{q}}'_1 = -\mathbf{i}$ and $\hat{\mathbf{q}}'_3 = \mathbf{i}$ in the final configuration,

$$\cos\theta_1 = -1 \text{ and } \cos\theta_3 = +1 \text{ there also.} \qquad (9)$$

After about half of the collisions have occured, $\cos\theta_1$ and $\cos\theta_3$ change sign due to the change in direction of $\mathbf{q}'_1$ and $\mathbf{q}'_3$. We now note the sequence of collisions illustrated in Fig. 1 which is used below to obtain algebraic results. The two adjacent arrows indicate the momentum of the two particles in their center-of-momentum system. The third arrow represents the momentum of the other particle in the three-particle center-of-momentum system. The particles are ordered 1, 2, 3 along the axis whose positive direction is indicated by the unit vector, $\mathbf{i}$. In the initial state, particles 1 and 2, are on a collision course with negligible relative momentum.

ALGEBRAIC RESULTS

1.) If $m_1 = m_2 = m_3$, $n_{max} = 3$ independent of the total energy W.

Proof: Using Equations (3) - (6), one obtains $\omega_1^2(3) = \omega_3^2(0)$ and $\omega_3^2(3) = \omega_1^2(0)$.

Substituting this into (1), one finds

$$\cos\theta_1(3) = -\cos\theta_3(0) \quad . \qquad (10)$$

Now the initial conditions $\hat{p}_3(0) = i$ and $\hat{q}'_3(0) = -i$ imply that $\cos\theta_3(0) = -1$ by definition. Hence (10) implies $\cos\theta_1(3) = -\hat{p}_1 \cdot \hat{q}'_1 = 1$. If we can show that $\hat{q}'_1(3) = -i$, then $\hat{p}_1$ will equal $-i$. (This means that particles 2 and 3 would be diverging because of their order in Fig. 1 so that the fourth collision could not occur.)

In the equal mass case,

$$\cos\theta_3(2) = \frac{\alpha_3^2(2)}{4W\alpha_3 p_3 q_3}\,\{\alpha_3^2(0) - \alpha_1^2(0)\}\,. \tag{11}$$

Since $\alpha_3^2(2)/4W\alpha_3 p_3 q_3$ is always positive, the sign of $\cos\theta_3(2)$ is determined by the relative size of $p_3(0)$ and $p_1(0)$. Our initial conditions specify that $p_3(0)$ is negligible. Hence $\cos\theta_3(2) < 0$ and since $|\cos\theta_3(2)| = 1$, $\cos\theta_3(2) = -1$. Now $\hat{p}_3(2) = i$ because if $\hat{p}_3(2) = -i$, then particles 1 and 2 would be divergin and since particles 2 and 3 are diverging after their recent collision, there wou be no further collisions. However, $\cos\theta_3(2) = -1$ and $\hat{p}_3(2) = -i$ imply that $\hat{q}'_3 = -i$, i.e. particle 3 would be moving toward the C.O.M. of particles 1 and 2 causing another collision to occur which contradicts the previous sentence.

Now $\cos\theta_3(2) = -1$ and $\hat{p}_3(2) = i$ imply $\hat{q}'_3(2) = i$. $\hat{q}'_3(2) = i$ implies that $\hat{q}_3(2) = i$. This is true because if the Lorentz boost from the two-particle C.O.M. to the three-particle C.O.M. caused $\hat{q}_3(2)$ to equal $-i$, it would also cause the two-particle C.O.M. system to move in the $-i$ direction violating the restriction of the three-particle C.O.M., namely $(q_1 + q_2) + q_3 = 0$.

Hence $\hat{q}_3(2) = i$ and the C.O.M. of particles 1 and 2 moves in the opposite direction (to the left in Fig. 1). Since the C.O.M. motion of particles 1 and 2 is not affected by collision 3 and since particle 1 is moving to the left faster than this C.O.M., $\hat{q}_1(3) = -i$. $\hat{q}_1(3) = -i$ implies that the two-particle C.O.M. system of particles 2 and 3 is moving to the right. Hence $\hat{q}_1'(3) = -i$. Q.E.D.

2.) If $m_1 = m_3 > m_2$, $n_{max} = 4$ when $W > m_1^2/m_2$.

A full proof requires the solution of a fourth order equation in W^2. Since such a solution is too lengthy to produce here, we only note that if $\omega_3^2(0) = (m_2 + m_1)^2$ and $W = m_1^2/m_2$, then $\omega_3^2(4) = (m_2 + m_1)^2$. This says that if the initial relative momentum of particles 1 and 2 is zero, then after four transformations using Eq. (3) - (6) their relative momentum will again be zero so that no more collisions occur. The solution of the fourth order equation is necessary to show that for a small but finite initial relative momentum, the particle will be diverging after four collisions. To show that four collisions are always possible, merely note that it is true non-relativistically.[1]

3.) If $m_1 = m_2 < m_3$, $n_{max} = 4$ independent of W.

To show this is true, note that it is true non-relativistically [1] where the particles are diverging after four collisions. Then note that $\omega_3^2(4)$ increases as W^2 increases so that the relative momentum of the diverging particles 1 and 2 increases also.

Because no simple analytic form for n_{max} could be found for the relativistic case, the above procedure was programmed on an IBM-360. In Fig. 2, the n_{max} obtained from the program is plotted as a function of $\log_{10} W$ for masses corresponding to an electron and two nucleons. W is expressed in units of the nucleon mass. In the non-relativistic limit, 95 collisions are allowed. When the kinetic energy in the center-of-mass system reaches about 130 kev, only 94 collisions are allowed. n_{max} continues to drop rapidly until about $W = 3$ where the curve begins to flatten out. Not until $W > 1836.4$ is the asymptotic limit of $n_{max} = 4$ attained. The same plot is presented for masses corresponding to a pion and two nucleons. Here $n_{max} = 4$ is attained when $W > 1/m_\pi = 6.72$.

It is interesting to note that as the total energy increases, any complications due to the number of binary rescatterings of three unequal mass particles will decrease. In particular, one would expect their relativistic scattering amplitude to reach its asymptotic form sooner than the non-relativistic amplitude if particle production processes did not enter.

Although it would be useful to have a simple analytic form for n_{max} as a function of W and the three masses, such a solution was not found. As an example of the way the complexity of n_{max} increases, one can show that $n_{max} = 5$ when

$$\frac{1}{m^2} > W^2 > \frac{1+4m-2m^2-2m^3-m^4+\sqrt{1-8m+12m^2+24m^3+6m^4-24m^5-20m^6+8m^7+m^8}}{2m-m^2} .$$

Here $m_1 = m_3 = 1$ and $m = m_2$. As n_{max} increases the order of the polynomial to be solved also increases so that analytic solutions might not even exist.

Acknowledgments

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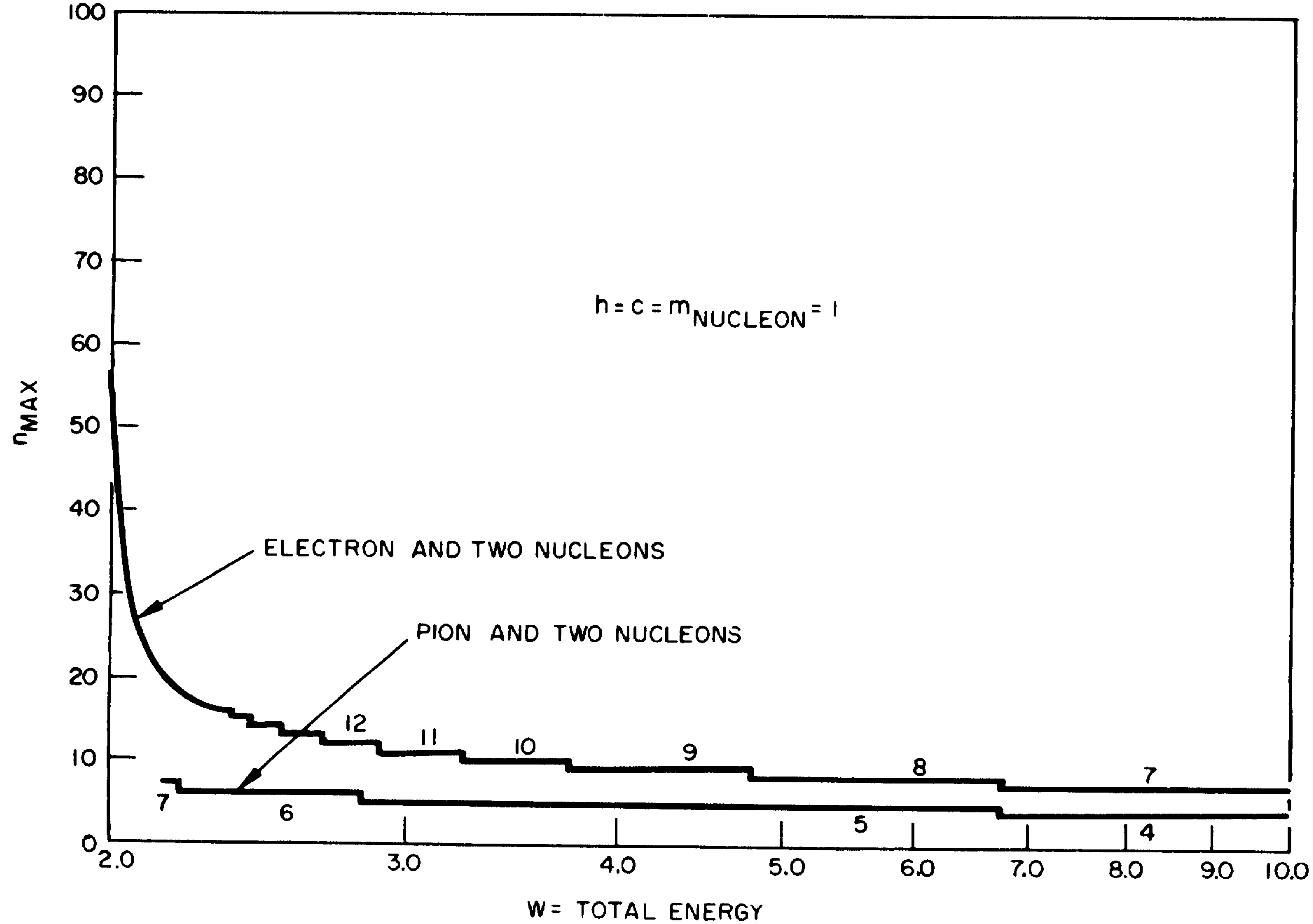

Figure 2-Energy dependence of the Maximum Number of Binary Collisons Allowed by Point Particles with Zero-range Forces for Two Choices of the Mass Values