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STUDY OF ELASTIC AND THERMODYNAMIC CONTINUOUS MEDIA IN GENERAL RELATIVITY

Jean-Francois Bennoun

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Study of Elastic and Thermodynamic Continuous
Media in General Relativity

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ABSTRACT: This work is devoted the formulation of a relativistic theory of continuous media, due consideration being taken of the fact that, within the general theory of relativity, only the variation of physical quantities is operationally accessible. In the first part the structure of elastic media is dealt with: linking equations are established which describe the variation of states; the problem of elasticity in general relativity is then fully described. The study is then extended to relativistic thermodynamics, on the basis of a tensorial interpretation of the second principle; a continuity hypothesis related to the rate of change of the entropy density permits of study of characteristic varieties and derivation of the general form of the relativistic equation for thermodynamic wave fronts. The work concludes with an application of the theory to the isotropic use.

Introduction

The problem of representing the gravitational field sources in general relativity is fundamentally linked to the solution of a difficulty of a conceptual nature: General relativity, based rigorously on the principle of slight equivalence, assumes the validity of the laws of special relativity within a local inertial frame of reference on condition that the value of the local measurement standards at each space-time point is modified. The consequence of this situation, from the point of view of the description of the material media which create the field, is that the definition of a natural reference state of the medium, which is necessary to establish an absolute scale to measure the conditions of state, loses all physical significance and, therefore, that the total representation of the media with the aid of equations of state can no longer be considered.

This difficulty may, however, be overcome on condition that the classical description of the states of a material medium is abandoned in favor of the description of the variation of the states of the medium in the course of its changes. This point of view constitutes the central idea of this work, the object of which is to study a representation of the continuous media compatible with the theory of general relativity.

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In Chapter I of this work I shall endeavor to show from an analysis of the basic principles of general relativity the need to introduce the point of view of the variation of states in the formulation of a relativistic theory of continuous media.

In Chapter II, I consider a precise physical situation related to adiabatic elastic media, and I formulate the group of correlating equations describing the variation of states of these media. I then present the problem of linking the gravitational field to its sources according to a diagram adapted to the point of view of the variations of states.

I then extend this representation to the thermodynamic continuous media stating the fundamental postulates of relativistic thermodynamics; this study is the object of Chapter III.

During the investigation of the characteristic variations of the problem of thermodynamic continuous media, developed in Chapter IV, I established the general equation of thermodynamic wave fronts.

As an illustration of this general study, in Chapter V, I apply the preceding results to the specific case of isotropic media. After proposing a definition of these media adapted to general relativity, I set forth certain properties of the thermodynamic wave fronts propagating in isotropic media; in closing I studied by means of an example the structure of the gravitational field created by a particular class of isotropic thermodynamic continuous media.

Chapter I. The Problem of Representing Continuous Media in General Relativity

1. Basic Principles of the Theory of General Relativity.

The problem of the representation of the gravitational field sources is closely linked to the significance of the principles of the general theory of relativity; for the purpose of presenting this problem in all its generality, we propose briefly to retrace the road which leads from the classical Newtonian mechanics to Einstein's gravitation theory.

The Newtonian mechanics assume essentially[1]:

(i). The equivalence of reference systems in repose in relation to one another, thus ensuring the existance of inertial systems.

(ii). The equivalence of the inertial systems in the determination of space and time.

[1]Cf. S. Kichenassamy (1964)

(iii). The principle of action at a distance.

From these postulates we deduce the following consequences:

(a) To each reference system S is associated a three-dimensional space which may be provided with an Euclidian structure, and a single time valid throughout this space. Clocks have been synchronized by means of a signal travelling at infinite velocity.

(b) At a point of such a Newtonian frame of reference S, measurement of the gravitational field as described by Newton's gravitational theory (1679) becomes identified with the measurement of the acceleration of a test body placed at this same point, this field having the remarkable empirical characteristic of being independent of the nature of the test body (Eotvös (1889), Dicke (1962)). This is the fact which is at the source of the purely contingent result of the proportionality between the passive gravitational mass, a concept of theoretical origin, and the inertial mass of a body; from the postulate of the identity between these two masses, and by virtue of the active and passive identity between the gravitational masses, a rigorous consequence of the principles of mechanics, the process of unification of the concept of mass is achieved, just as it appears in the statement of the fundamental law of dynamics.

(c) When the space associated with S is not the source of a gravitational effect on the bodies in the presence of each other, every free material point assumes, in regard to S, a uniform motion of translation, this fact constituting the background of the principle of inertia; the inertial systems S are defined in an absolute manner.

The invariance of the Newtonian mechanics in the Galileo group therefore asserts that it is impossible to show the uniform translation movement of a reference system S exclusively through mechanical experiments conducted in the interior of S; this is the situation described by the Newtonian principle of relativity.

We know that Einstein (1905) was led by a critical analysis of the notion of remote control simultaneity to extend this principle to all experiments, both mechanical and optical, conducted in the interior of S; such a special principle of relativity led to the adoption of the Lorentz group to be adopted as the invariance group of the laws of physics.

The special theory of relativity assumes[2]:

(i) Like classical mechanics, the equivalence of frames of reference in relative repose for the determination of space and time; this postulate maintains for space-time the properties attributed to the Newtonian space and time, and in the same manner introduces the existence of inertial frames of reference.

(ii) The propagation of isotropic light, at a constant speed in relation to the entire inertial system.

This theory led us to abondon Newton's absolute space and time; the possibility of bringing into evidence the accelerated motion of a reference system S in relation to an inertial system, by means of physical experiments inside S and causing fictitious inertial forces to participate, does however remain.

However, Einstein's principle of equivalence (1916), founded on the identity between gravitational mass (passive) and inertial mass, does not permit the inertial field and the homogeneous gravitational field to be distinguished locally; this situation means that in a sufficiently small space-time region in which it is impossible to detect the gravitational field source, the reference system S may be compared to an inertial system. In return, if we consider an expanded space-time region, S no longer constitutes an inertial system, so that in this case we can actually describe the gravitational field. This means that at each point of the reference system S there are two events which are no longer characterized by the same intervals of length and duration. The effect of the presence of a gravitational field is therefore to modify the geometrical space-time structure at each point; the reference system S is constituted by an infinity of inertial systems, and space-time is represented by a line graph.

Thus, the representation of space-time by a line graph expresses the situation created by the local equivalence between the inertial field and the homogeneous gravitational field. Does it follow that this equivalence is substantial, i.e., that the laws of the special theory of relativity, their numerical content included, are valid in an inertial frame of reference independently of the position of said frame of reference in space-time? Actually that is not so, because S is not a true inertial frame of reference; the

[2]These two postulates suffice to substantiate the entire special theory of relativity, the character of reciprocity of the reference systems in uniform relative translation being rigorously deduced; cf. S. Kichenemassy (1964).

measurement standards do not coincide with those of the special theory of relativity except in extremely small areas. The equivalence between inertial field and gravitational field is slight and this situation resembles the only one described by the Eotvös experiment[3].

2. The Postulates of the General Theory of Relativity.

The presence of a gravitational field leads to the representation of space-time by a line graph. Einstein stated this concept providing this V_4 variation with a Riemann structure defined by the normal hyperbolic type metric equation (signature: -2), which is assumed to be regular:

$$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \quad (\alpha \text{ and all Greek sub- and superscripts } = 0, 1, 2, 3). \qquad 2.1$$

The local coordinates (x^γ) have only a purely topological meaning; the geometrical characteristics of V_4 and the definition of the local measurement standards can only be obtained through the metric equation (2.1).

According to the interpretation of the theory, the $g_{\alpha\beta}$ are indentified with the gravitational potentials. The local determination of these potentials is obtained by assuming that:

(i) In the domains of V_4 vacuums of non-gravitational energy, the metric equation is regular and meets Einstein's external equations:

$$\sqrt{-g}G^{\alpha\beta} = 0, \qquad 2.2$$

or:

$$g = \text{dét}\,(g_{\alpha\beta}) \qquad 2.3$$

and $G^{\alpha\beta}$ designates Einstein's tensor V_4:

$$G^{\alpha\beta} = R^{\alpha\beta} - 1/2 g^{\alpha\beta}R, \qquad 2.4$$

$R^{\alpha\beta}$ being Ricci's tensor, and R the scalar curvature of V_4.

(ii) In the V_4 domain provided with an energy distribution with which the pulse-rate-energy $T^{\alpha\beta}$ is associated, the $g_{\alpha\beta}$ which are assumed to

[3] Cf. S. Kichenassamy (1964).

[4] The choice of units adopted in this work is such that c = 1, c designating the speed of light in a vacuum and $X = 1$, or $X = 8\pi G_n/c^2$ is Einstein's constant, and Gn Newton's gravitational constant.

be regular meet Einstein's internal equations[4].

$$\sqrt{-g}G^{\alpha\beta} = T^{\alpha\beta}. \qquad 2.5$$

(iii) At the boundary hypersurfaces which limit the energy distributions, the gravitational potentials $g_{\alpha\beta}$ and their first derivatives are continuous, according to the conditions of Schwarzschild's line.

These postulates are compatible with an hypothesis of differentiability of V_4 which we assume to be of the $c^2 - c^4$ class by bits[5].

The arbitrariness of the choice of coordinates is ensured by the existence of four identities:

$$\nabla_\beta G^{\alpha\beta} = 0, \qquad 2.6$$

where ∇ designates the operator of covariant derivation relating to the metric $g_{\alpha\beta}$; these identities, a consequence of the Bianchi identities, as a result of (2.4) introduce the four equations of conservation of the pulse-energy density:

$$\nabla_\beta T^{\alpha\beta} = 0. \qquad 2.7$$

These equations play the role of conditions of compatibility during the search for accurate solutions for the field equations; at the time of the calculation of approximate solutions, their integration over the domain of V_4 representing the variation of the material medium furnished the total conservation of energy and the motion of the field sources.

3. The Problem of Representation of Gravitational Field Sources.

The description of the gravitational field sources is ensured by the known quantity of the pulse-energy density $T^{\alpha\beta}$ which appears in the second member of Einstein's internal equations. The formulation of a relativistic theory of matter consists in the characterization of the structure $T^{\alpha\beta}$ with the aid of hypotheses of phenomenologic origin adapted to the properties of the material medium under consideration. The precise adjustment of such a problem requires that we revert to the significance of the principles of general relativity.

[5]Cf. A. Lichnerowicz (1955).

We have seen that this theory is rigorously based on the principle of slight equivalence, which leads us to assume that if the laws of special relativity remain valid within a local inertial frame of reference, the effect of the presence of a gravitational field is, however, to modify either the local measurement standards, or if these standards remain identical to those of special relativity, the value of the physical quantities. Needless to say that these two interpretations are rigorously equivalent from the point of view of numerical content of the laws of general relativity, but the choice of one of them is profoundly linked to the comprehension of Einstein's theory of gravitation.

We know that S. Kichenassamy (1964) was led to assume that the presence of the gravitational field changes the value of the local measurement standards, to the extent that the phenomena of a small space-time region are described by the same numbers attributed to them in a true inertial frame of reference[6 7].

Such an interpretation is based on the point of view adopted for the study of the problem of representing continuous media in general relativity. In order to properly understand this point, it is necessary to revert to the manner in which a similar problem is presented in classical mechanics: In this case we revert to the determination of the equations of state of the material medium which link the conditions of state characteristic of the physical properties of the medium to the characteristic values of the configuration of said medium. The conditions of state have an absolute significance, to the extent that they characterize the state of the material medium in relation to a reference state defined a priori in an absolute manner, i.e., prior to any influence external to the medium. The hypothesis which might define such a natural reference state is quite fundamental in the classical theory of continuous media, even though it is not always formulated explicitly.

Now, the definition of a natural reference state ceases to be possible in general relativity when the medium considered is itself the source of a gravitational field which contributes to alter the value of the measurement standards intended to characterize such a state. It follows that it is impossible to attribute absolute values to the conditions of state characteristic of the medium, and therefore that the representation of material media in

[6]The link between the proper time associated to a reference system S linked to a test particle and the acceleration of S in relation to an inertial system S_0 linked to the laboratory could be expressed in the case of a uniformly super-accelerated movement of S in relation to S_0; cf. S. Kichenassamy (1965).
[7]This interpretation makes it especially possible to provide a satisfactory explanation of the phenomenon of drift towards red which, in total effect distinguishes the classical Doppler effect from the purely gravitational effect.

general relativity can no longer be made with the aid of the equations of state of the medium. This conceptual difficulty which was first brought to light by Synge (1959) must lead to a radical change in the classical ideas concerning the problem of the representation of continuous media.

If representation of a material medium in the classical sense escapes us, we can always describe the latter from the known characteristics of an extremely close prior state which acts as a reference state; in other words, it is possible to attribute an intrinsic physical significance to the variation of states of a material medium in the course of the variation of said medium, i.e., independent of a natural reference state[8]. The pulse-energy density $T^{\alpha\beta}$ ceases to have an absolute physical significance, and only the variation of this quantity remains operationally accessible. The problem of the representation of material media therefore concerns the problem of the determination of the variation of states of the medium under consideration compatible with the assumed interpretation of general relativity.

This point of view, initially suggested by Synge (1959), is the one we are adopting here, and we systematically exploit its consequences so as to evolve a relativistic theory of continuous media.

Chapter II. Representation of Elastic Continuous Media.

4. Introduction

Our goal is the formulation of a relativistic theory of elasticity according to the point of view of the variation of states.

The problem of elasticity already appears to be of great importance in the special theory of relativity as a result of the great peculiarity of the class of rigid motions within the scope of a Minkowski space-time, since said movements have only three degrees of freedom[9]; particularly as shown by Herglotz (1910) and Noëther (1910), the rigid non-irrotational motions always constitute and isometry of M_4.

The situation appears to be less restrictive in general relativity, where the rigid motions in the Born Meaning[10] do have the required six degrees of freedom; however, it has been shown[11] that the rigid motions of the material

[8] This point of view is also comtemplated in the classical mechanics in a different sense, especially by C. Truesdell (1955), W. Noll (1955, 1958), B. Bernstein and J. L. Ericksen (1958); cf. B. Bernstein (1960).

[9] M. Born (1909), G. Herglotz (1910), F. Noëther (1910); cf. W. Pauli (1958) Par. 45.

[10] N. Rosen (1947), G. Salzman and A. H. Taub (1954); cf. Pirani and Williams (1962).

[11] C. B. Rayner (1959), F. A. E. Pirani and G. Williams (1962), R. H. Boyer (1965).

media represented with aid of a normal type pulse-energy density are always produced with a constant angular velocity. Consideration of given physical situations makes it therefore necessary to study more general types of motions.

The problem of elasticity in general relativity was initially presented by Synge[12] (1959). In order to eliminate the difficulties connected with the action of the gravitational field, this author introduces a group of correlating equations intended to describe the variations of state of elastic media; these equations, however, do not appear to be entirely satisfactory due to the non-equivalence between the definition of the rate of change of the constraints and that of the rate of change of deformations which this author adopts[13]. The theory proposed shortly afterwards by Rayner (1963) reintroduces the difficulties connected with the definition of a natural reference state of the elastic medium in the presence of a gravitational field, and this author is being forced to introduce an auxiliary metric of space which is completely alien to the general theory of relativity[12] in order to characterize such a state; by changing the Rayner equations it is however possible to deduce a group of correlating equations which no longer present the difficulties of the Synge equations[13]. In any event, said theories have an arbitrary character in the sense that they do not specify the hypotheses of the structure of the medium subjacent to the proposed equations.

In the course of the following work, we propose to formulate a theory of elasticity which is compatible with the general theory of relativity according to the ideas developed above. We shall first set forth the structure of the pulse-energy density describing the elastic character of the medium under consideration in order to deduce, by a process of variation, the correlated equations of the medium linking the rates of change of the constraints with the rate of change of the deformations. We are confining this study to the case of adiabatic elastic media, without interaction with an external field other than the gravitational field, so as to clearly disengage the characteristic properties of these media; subsequently we shall extend this representation to a larger class of continuous media.

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[12]The problem of elasticity in general relativity has been remarkably dealt with by J. L. Synge (1959) and C. B. Rayner (1963); we are also citing the works of J. M. Souriau (1958), A. Bressan (1963), B. de Witt (1963), who adopt a rigorously classical point of view.

[13]Cf. J. F. Bennoun (1963).

A. Determination of the Structure of $T^{\alpha\beta}$.

In this section we propose to show that, under a simple hypothesis making the transposition in relativity of the Cauchy-Green hypothesis of elastic media, the pulse-energy density revealing this structure is of the normal type, with a density of constraints described by a group of six equations of state which we shall determine.

5. Geometric Representation of a Material Medium in V_4.

Let us consider, in the Rieman variation V_4 of general relativity, a class of point transformations, assumed to be correct and of the C^2 class[14].

$$x^\alpha = x^\alpha(y^\mu), \quad \det(x^\alpha{}_{.\mu}) \neq 0, \quad x^\alpha{}_{.\mu} = \partial x^\alpha/\partial y^\mu, \tag{5.1}$$

the trajectories of which form a three-parameter family (y^a) (a and all Latin sub- and superscripts = 1, 2, 3) of lines oriented in time, and parametrized by (y^0), this scalar being by hypothesis a rigorously increasing function of the proper times; the Lagrangian variables (y^μ) behaving like scalars of V_4; the four vectors $(x^\alpha{}_{.\mu})$ define a frame of reference at each point (x) of the Ω domain of V_4 filled by world lines; if we choose s as the variable parameter along these world lines, they may be considered as trajectories of the field of velocity vectors:

$$u^\alpha = \partial x^\alpha/\partial s = (\partial s/\partial y^0)^{-1} x^\alpha{}_{.0}, \quad (\partial s/\partial x^0)^2 = g_{\alpha\beta} x^\alpha{}_{.0} x^\beta{}_{.0} \tag{5.2}$$

unitary, and oriented in time:

$$g_{\alpha\beta} u^\alpha u^\beta = 1. \tag{5.3}$$

Let Σ be the triplanar field of orthogonal space directions in $a\ \vec{u}$; P designates the space projection operator which enables us to associate to the total value U of V_4, its space component P(U) over Σ; for example, the space component $\bar{g}_{\alpha\beta}$ = P($g_{\alpha\beta}$) of the metric V_4 is defined by the expression:

$$\bar{g}_{\alpha\beta} = P^\gamma_\alpha P^\delta_\beta g_{\gamma\delta} = g_{\alpha\beta} - u_\alpha u_\beta, \quad \bar{g}_{\alpha\beta} u^\alpha = 0, \quad P^\beta_\alpha = \delta^\beta_\alpha - u_\alpha u^\beta. \tag{5.4}$$

[14]Cf. Pham Mau Quan (1954)

Let us consider an infinitesimal displacement at Σ:

$$\overline{dx}^\alpha = \overline{x}^\alpha{}_{.\mu} dy^\mu, \qquad \overline{x}^\alpha{}_{.\mu} = P(x^\alpha{}_{.\mu}). \tag{5.5}$$

Noting that $\overline{dx}^\alpha$ depends only on the differentials dy^α, since:

$$\overline{x}^\alpha{}_{.0} = x^\alpha{}_{.0} - u_\beta u^\alpha x^\beta{}_{.0} = 0. \tag{5.6}$$

The relations (5.5) define a type of point transformations of Ω, which depend on the parameter (y^0), by applying the points (y^α) of the natural reference state of a material medium to the points of the triplanar field Σ associated with the variation of the states of the medium under consideration. Let us state certain characteristics of these types of transformations.

(i) Let $\overline{ds}$ be the elementary length interval of Σ, defined by:

$$\overline{ds}^2 = \overline{g}_{\alpha\beta} \overline{dx}^\alpha \overline{dx}^\beta = \gamma_{\mu\nu} dy^\mu dy^\nu, \tag{5.7}$$

where the Pfaff forms $\overline{dx}^\alpha = P(dx^\alpha)$ are not exact differentials except when Σ is integrable, and where $\gamma_{\mu\nu}$ designates the Lagrangian metric:

$$\gamma_{ab} = \overline{g}_{\alpha\beta} \overline{x}^\alpha{}_{.a} \overline{x}^\beta{}_{.b}, \qquad \gamma_{0\alpha} = 0. \tag{5.8}$$

The six γ_{ab} quantities define the deformation of the material medium; it is immediately evident that such a concept depends on the definition of the natural reference state of the medium.

In turn, the deformation variation may be characterized in the absence of the natural reference medium with the aid of the deformation variation rate tensor[10 15]:

$$E_{\alpha\beta} = 1/2 \mathcal{L} \overline{g}_{\alpha\beta} = P(\nabla_{(\alpha} u_{\beta)}), \tag{5.9}$$

introduced by the expression:

$$\partial / \partial s \overline{ds}^2 = 2 E_{\alpha\beta} \overline{dx}^\alpha \overline{dx}^\beta. \tag{5.10}$$

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[15] $\mathcal{L}$ designates the Lie derivative operator for the vector field $\vec{u}$; cf. A. Lichnerowicz (1958).

This situation constitutes a first illustration of the fact that only the variations of state quantities have an intrinsic physical significance, i.e., that they do not depend on the definition of a natural reference state.

(ii) Let us now consider the volume element defined by Σ:

$$\sqrt{-g}d^3x = \sqrt{-g}1/3!\,\varepsilon_{\alpha\beta\gamma\delta}u^\alpha dx^\beta dx^\gamma dx^\delta = \sqrt{-g}\,\text{dét}\,(\bar{x}^\alpha{}_{.\mu})d^3y, \qquad 5.11$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ designates the Kronecker indicator, and :

$$\text{dét}\,(\bar{x}^\alpha{}_{.\mu}) = 1/3!\,\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{abc}u^\alpha\bar{x}^\beta{}_{.a}\bar{x}^\gamma{}_{.b}\bar{x}^\delta{}_{.c} = (\partial s/\partial y^0)^{-1}\,\text{dét}\,(x^\alpha{}_{.\mu}). \qquad 5.12$$

The deformation of this volume element, or dilation, is characterized by the quantity:

$$\sqrt{-\gamma} = \sqrt{-g}\,\text{dét}\,(\bar{x}^\alpha{}_{.\mu}), \qquad \gamma = \text{dét}\,(\gamma_{\mu\nu}), \qquad 5.13$$

where by virtue of the original hypothesis (5.1) and of the fact that the world lines are assumed rigorously oriented in time, we have:

$$\text{dét}\,(\bar{x}^\alpha{}_{.\mu}) \neq 0, \qquad 5.14$$

which ensures the regularity of the point transformation (5.5); the three vectors $\bar{x}^\alpha{}_{.\mu}$ form a space frame of reference which characterize the principle directions of deformation of the medium.

This concept of dilation depends on the definition of the natural state of reference of the medium; but in the absence of the existence of such a state we can, as heretofore, define the dilation variation rate θ:

$$\mathcal{L}(\sqrt{-g}d^3x) = \theta\sqrt{-g}d^3x, \qquad \theta = g^{\alpha\beta}E_{\alpha\beta} = \nabla_\alpha u^\alpha. \qquad 5.15$$

We have just determined the characteristics of the geometrical representation of a continuous medium in V_4; in the following paragraphs we are going to set forth the properties of the dynamic representation of such a medium.

6. A Lemma on the Conditions of State Quantities.

Let us establish a preliminary result related to the behavior of the conditions of state; we assume that these quantities depend only on the configuration of the continuous medium under consideration, i.e., that they are entirely determined by the known quantity of the variations of state $(g_{\alpha\beta}, y^{\mu}, x^{\alpha}{}_{.\mu})$.

Let U be such a quantity; U is referred to the following hypotheses, which suffice for the purpose of our work:

(i) U is a contravariant tensor density of space of the nth order:

$$U^{\alpha_1 \ldots \alpha_i \ldots \alpha_n} u_{\alpha_i} = 0, \qquad i = 1, \ldots, n. \tag{6.1}$$

(ii) The Lagrangian components of U, which may be considered either as tensor densities $\overset{\circ}{U}$ of the same type as U in the group of parameter transformations (y^{α}) or as an ensemble of n! scalar densities of V_4 defined by:

$$U^{\alpha_1 \ldots \alpha_n} = \underset{\bullet}{U}^{\mu_1 \ldots \mu_n} \bar{x}^{\alpha_1}{}_{.\mu_1} \ldots \bar{x}^{\alpha_n}{}_{.\mu_n} \,\text{dét}^{-1}(\bar{x}^{\alpha}{}_{.\mu}), \tag{6.2}$$

depending only on the Lagrangian metric $\gamma_{\mu\nu}$:

$$\underset{\bullet}{U} = \underset{\bullet\bullet}{U}(\gamma_{\mu\nu}). \tag{6.3}$$

(a) Let us take the Lie derivative through the two-member vector field $\vec{u}$ of Eq. (6.2) bearing in mind the identity:

$$\mathcal{L} x^{\alpha}{}_{.a} = 0 \tag{6.4}$$

arising from the fact that $\mathcal{L}$ does not change the numerical values of the coordinates (x^{α}) and commutates with $\partial/\partial y^{\alpha}$ since the (y^{α}) values are constants all along the length of each world line; we thus obtain:

$$P(\mathcal{L}U^{\alpha_1 \ldots \alpha_n}) = \mathcal{L}(\underset{\circ}{U}^{\mu_1 \ldots \mu_n}) \bar{x}^{\alpha_1}{}_{.\mu_1} . \bar{x}^{\alpha_n}{}_{.\mu_n} \,\text{dét}^{-1}(\bar{x}^{\alpha}{}_{.\mu}). \tag{6.5}$$

(b) Let us bear in mind the second hypothesis (ii) concerning U, taking the two-member variational derivative of Eq. (6.2) in relation to the metric $g_{\alpha\beta}$[16]; we arrive at:

$$\bar{\delta}/\delta g_{\alpha\beta} U^{\gamma_1 \cdots \gamma_n} = \delta/\delta \gamma_{\mu\nu}(\underset{\bullet}{U}^{\rho_1 \cdots \rho_n})\bar{x}^{\gamma_1}{}_{.\rho_1} \cdots \bar{x}^{\gamma_n}{}_{.\beta}\bar{x}^{\alpha}{}_{.\mu}\bar{x}^{\beta}{}_{.\nu} \, \text{dét}^{-1}(\bar{x}^{\alpha}{}_{.\mu}), \qquad 6.6$$

where we have introduced this notation:

$$\bar{\delta}/\delta g_{\alpha\beta}(U^{\gamma_1 \cdots \gamma_n}) = P(\delta/\delta g_{\alpha\beta}(U^{\gamma_1 \cdots \gamma_n})). \qquad 6.7$$

(c) Finally, by virtue of the hypotheses (i) and (ii), we have identically:

$$\mathfrak{L}(\underset{\bullet}{U}^{\gamma_1 \cdots \gamma_n}) = \delta/\delta \gamma_{\mu\nu}(\underset{\bullet}{U}^{\gamma_1 \cdots \gamma_n})\mathfrak{L}\gamma_{\mu\nu}. \qquad 6.8$$

Bringing together the thrèe results (a, b, c) set forth we obtain the final result sought:

$$P(\mathfrak{L}U^{\gamma_1 \cdots \gamma_n}) = \bar{\delta}/\delta g_{\alpha\beta}(U^{\gamma_1 \cdots \gamma_n})\mathfrak{L}\bar{g}_{\alpha\beta}, \qquad 6.9$$

bearing in mind:

$$\mathfrak{L}\gamma_{\mu\nu} = (\mathfrak{L}\bar{g}_{\alpha\beta})\bar{x}^{\alpha}{}_{.\mu}\bar{x}^{\beta}{}_{.\nu}. \qquad 6.10$$

The initial structure hypothesis concerning $\underset{\circ}{U}$ is therefore equivalent, from the point of view of the variation of U, to the hypothesis that U depends only on the dynamic variables $g_{\alpha\beta}$:

$$U = U(\bar{g}_{\alpha\beta}). \qquad 6.11$$

Let us summarize these results in the following lemma, which will be very useful:

LEMMA. Let U be a countervariant tensor density of space (weight + 1) and of the arbitrary order n, and let $\underset{\circ}{U}$ be the Lagrangian component of U:

$$U^{\alpha_1 \cdots \alpha_n} = \underset{\circ}{U}^{\mu_1 \cdots \mu_n}\bar{x}^{\alpha_1}{}_{.\mu_1} \cdots \bar{x}^{\alpha_n}{}_{.\mu_n} \, \text{dét}^{-1}(\bar{x}^{\alpha}{}_{.\mu}), \qquad U^{\alpha_1 \cdots \alpha_l \cdots \alpha_n} u_{\alpha_l} = 0.$$

[16]Cf. Appendix I.

The two following hypotheses are equivalent from the variational point of view:

1. $\underset{\circ}{U}$ depends only on the Langrangian metric:

$$\underset{\bullet}{U} = \underset{\bullet}{U}(\gamma_{\mu\nu}) \qquad (\gamma_{\mu\nu} = \bar{g}_{\alpha\beta}\bar{x}^{\alpha}{}_{.\mu}\bar{x}^{\beta}{}_{.\nu}).$$

2. U is a function of the space component of the metric $g_{\alpha\beta}$.

$$U = U(\bar{g}_{\alpha\beta})$$

which means, likewise, that the variation rate of U along a world line is characterized by the relation:

$$P(\mathcal{L}U) = \bar{\delta}U/\delta g_{\alpha\beta}\mathcal{L}\bar{g}_{\alpha\beta}.$$

7. Dynamic Representation of a Material Medium in V_4.

In this paragraph we are going to set forth the characteristics of the representation of the state of a continuous medium.

We propose the hypothesis, which is fundamental later on in this study, that the structural properties of the medium are entirely deductible from the known quantity of the energy density L attached to the unit volume of this medium; it is assumed that this quantity meets the following conditions:

(i) L is a scalar density

(ii) L is a function of the C^0-C^2 class of the variables of the state of the medium.

For subsequent developments it will suffice to consider functions of the type:

$$L = L(g_{\alpha\beta}\,;\,x^{\alpha}{}_{.\mu}). \tag{7.1}$$

(a) Having presented this, let us apply to the total Lagrangian of general relativity, field + matter, a variation principle:

$$\delta\int(\sqrt{-g}R + 2L)d^4x = 0\,; \tag{7.2}$$

we arrive at:

(α) Einstein's gravitational field equations:

$$\sqrt{-g}G^{\alpha\beta} = T^{\alpha\beta}. \qquad 2.5$$

(β) The equations of state of the medium, by definition of the pulse-energy density $T^{\alpha\beta}$:

$$T^{\alpha\beta} = 2\delta L/\delta g_{\alpha\beta}. \qquad 7.3$$

(γ) The equations of conservation of $T^{\alpha\beta}$ as condequences of the invariance of L in the group of general transformations of coordinates:

$$\nabla_\beta T^{\alpha\beta} = 0. \qquad 2.7$$

(b) The postulated structure (7.1) of L ensures, by virtue of the invariance of this quantity under the group of transformations of coordinates, the validity of the identities:

$$\delta L/\delta g_{\alpha\beta}\mathcal{L}(\vec{\xi})g_{\alpha\beta} + \delta L/\delta x^{\alpha}{}_{.\mu}\mathcal{L}(\vec{\xi})x^{\alpha}{}_{.\mu} - \partial_\alpha(L\xi^\alpha) = 0, \qquad 7.4$$

where $\vec{\xi}$ is a field of arbitrary vectors of V_4; from this arbitrary characteristic it follows that the terms of the first member of (7.4) which are factors of $\vec{\xi}$, and the terms which are the first derivatives of said vector can be cancelled separately; we are thus led to the group of identities[17]:

$$\delta L/\delta x^{\alpha}{}_{.\mu}\nabla_\gamma x^{\alpha}{}_{.\mu} - \nabla_\gamma L = 0. \qquad 7.5\text{-a}$$

$$T^{\alpha\beta} = t^{\alpha\beta}. \qquad 7.5\text{-b}$$

$$g^{\gamma[\alpha}x^{\beta]}{}_{.\mu}\delta L/\delta x^{\gamma}{}_{.\mu} = 0, \qquad 7.5\text{-c}$$

[17]E. Noether (1918); Cf. A. Trautman (1963)

where the marks ( ) and [ ] designate the symmetrization and antisymmetrizarion of the sub- and superscripts; $T^{\alpha\beta}$ designates the pulse-energy density (2.7), and $t^{\alpha\beta}$ the canonical pulse energy density:

$$t^{\beta}_{\alpha} = \delta L/\delta x^{\alpha}{}_{.\mu} x^{\beta}{}_{.\mu} + L\delta^{\beta}_{\alpha}. \qquad 7.6$$

In conclusion, the dynamic representation of a continuous medium in relativity, which is deduced from the known quantity of the energy density of the medium L, is obtained from the 10 equations of state (7.3); the invariance properties of L are set forth, bearing in mind, the hypothesis of the particular structure (7.1), from the group of Noëther's identities (7.5).

These results will be applied to the specific study of the characterization of the structure of elastic-type media.

Remarks. In the contemplated formalism, the quantity which is naturally attached to the description of pulse-energy of the contemplated continuous medium is the pulse-energy density $T^{\alpha\beta}$, i.e., a symmetric countervariant tensorial density of the second order, to which the pulse tensor-energy $T^{\alpha\beta}/\sqrt{-g}$ corresponds.

8. Structure of Elastic Continuous Media.

As in every theory of phenomenological origin, we must distinguish between the different types of energy associated with the properties of the material medium under consideration.

An elastic continuous medium is characterized by the energy density:

$$L = m + E, \qquad 8.1$$

where the scalar density m designates the density of the mass of conservation, and the scalar density E the elastic energy density which characterizes the action of constraint forces, and identical in the case under consideration to the internal energy density of the medium. Densities m and E are each supposed to satisfy the hypotheses (i) and (ii) of paragraph 7.

(a-α). Let us designate with $\overset{}{\underset{\circ}{m}}$ the Lagrangian component of m; the conservation quality of m is ensured by the relation.

$$\partial/\partial y^{\circ} \underset{\circ}{m} = 0, \qquad 8.2$$

which means, by integration, and by referring to a natural frame of reference of local coordinates (cf. 6.2):

$$m = \underset{\bullet}{m}(y^a)\, \text{dét}^{-1}\,(\bar{x}^{a}{}_{.\mu}). \tag{8.3}$$

(β) By virtue of (6.5) the initial Eq. (8.2) is written in this frame of reference.

$$\underset{\bullet}{\mathfrak{L}}m = 0. \tag{8.4}$$

(γ) The canonical pulse-energy density (7.6) associated with m is written (18):

$$t_{\alpha}^{\beta}(m) = m u_{\alpha} u^{\beta}. \tag{8.5}$$

This is the pulse-energy density associated with a free particle.

(b-α) Let us propose a structural hypothesis on E adapted to the elastic character of the medium under consideration, expanding in the relativistic case the Cauchy-Green hypothesis of the classical theory of elasticity:

Definition. The internal energy density associated with an adiabatic elastic medium in general relativity is of the type:

$$E = E(g_{\alpha\beta}\,;\bar{x}^{a}{}_{,\mu}). \tag{8.6}$$

From the study of the Noëther identities (7.5) we shall determine the form of the function E (8.6). Let us say first that these identities are met independently by m as a result of Eq. (8.5)[18], so that we merely require that they also be independently met by E; this will actually be so if the following relations are also met[18]:

$$\delta E/\delta \bar{x}^{\gamma}{}_{.\mu}\,\bar{x}^{[\beta}{}_{.\mu}\,\bar{g}^{\alpha]\gamma} = 0. \tag{8.7-a}$$

$$u^{\gamma}\delta E/\delta \bar{x}^{\gamma}{}_{.\mu} = 0. \tag{8.7-b}$$

$$2\delta E/\delta g_{\alpha\beta} = \bar{g}^{\alpha\gamma}\bar{x}^{\beta}{}_{.\mu}\,\delta E/\delta \bar{x}^{\gamma}{}_{.\mu} + E g^{\alpha\beta}, \tag{8.7-c}$$

[18]Cf. Appendix I.

bearing in mind:

$$t^{\beta}_{\alpha}(E) = \bar{g}^{\gamma}_{\alpha}\bar{x}^{\beta}{}_{.\mu}\,\delta E/\delta\bar{x}^{\gamma}{}_{.\mu} + E\delta^{\beta}_{\alpha}. \qquad 8.8$$

A sufficient condition for these conditions (8.7) to be likewise is that the Lagrangian component $\underset{\circ}{E}$ of E be of the type[18]:

$$\underset{\circ}{E} = \underset{\circ}{E}(\gamma_{\mu\nu}). \qquad 8.9$$

By accepting this condition, we can express it with the following equivalent form (cf. par. 6):

Proposition. The elastic density energy associated with an isothermal and adiabatic elastic medium in general relativity is of the type:

$$E = E(\bar{g}_{\alpha\beta}). \qquad 8.10$$

(β) Bearing in mind (6.2, 6.6) the hypothesis (8.9) means that the canonical density (8.8) associated with E is written under the form of[18]:

$$t^{\alpha\beta}(E) = 2\bar{\delta}E/\delta g_{\alpha\beta} + E u^{\alpha}u^{\beta}. \qquad 8.11$$

(γ) Energy density E of the type (8.10) duly meets all the conditions for the application of the lemma of paragraph 6; we thus obtain directly, with the aid of Eq. (6.9), the equation which expresses the value of the variation rate of the elastic energy density at the time of the variation of the medium:

$$\mathcal{L}E = \bar{\delta}E/\delta g_{\alpha\beta}\,\mathcal{L}\bar{g}_{\alpha\beta}. \qquad 8.12$$

(c). Let us bring together the results (8.5, 8.11), bearing in mind the identity between pulse-energy and canonical density; we thus obtain the equations of state which express the form of pulse-energy associated with an elastic continuous medium:

$$T^{\alpha\beta} = (m + E)u^{\alpha}u^{\beta} + 2\bar{\delta}E/\delta g_{\alpha\beta}. \qquad 8.13$$

(α) $T^{\alpha\beta}$ is of the normal type; the constraint density associated with the medium is defined with the aid of six equations of state:

$$\theta^{\alpha\beta} = -2\bar{\delta}E/\delta g_{\alpha\beta}, \tag{8.14}$$

in which the integrability conditions are written:

$$\bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta} = \bar{\delta}/\delta g_{\alpha\beta}\theta^{\gamma\delta}. \tag{8.15}$$

(β). The density of mass p is identical to the energy density associated with the medium.

$$\rho = L = m + E. \tag{8.16}$$

(γ) The eigen vector, oriented in time, of $T^{\alpha\beta}$ is identical to the world velocity vector associated with the elastic medium:

$$u^{\alpha} = \partial x^{\alpha}/\partial s. \tag{8.17}$$

Let us summarize the results:

Proposition. The structure of an elastic medium, which is assumed to be isothermal and adiabatic, is represented in general relativity by a pulse-energy density $T^{\alpha\beta}$ which meets the following conditions:

(i). $T^{\alpha\beta}$ is of the normal type:

$$T^{\alpha\beta} = \rho u^{\alpha}u^{\beta} - \theta^{\alpha\beta}, \qquad u_{\alpha}u^{\alpha} = 1, \qquad \theta^{\alpha\beta}u_{\alpha} = 0. \tag{8.18}$$

(ii). Its eigen vector, oriented in time, is identical to the world velocity vector of the medium.

(iii). The associated eigen value, i.e., the density of mass is identical to the energy density of the medium.

(iv). The space component of $T^{\alpha\beta}$ is identified with the constraint densities $\theta^{\alpha\beta}$ of the elastic medium; $\theta^{\alpha\beta}$ is determined from the elastic energy density E with the aid of six equations of state:

$$\theta^{\alpha\beta} = -2\bar{\delta}E/\delta g_{\alpha\beta},$$

and must satisfy the integrability conditions of these equations:

$$\bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta} = \bar{\delta}/\delta g_{\alpha\beta}\theta^{\gamma\delta}.$$

B. Representaticn of the Variation of States of Elastic Media.

Taking as a base the representation of states of an elastic medium established in the preceding section, we propose to set forth the representation of the variation of states of the medium in accordance with the assumed principles of general relativity; we shall make a special effort:

(i). To introduce the concept of internal energy in a form adapted to the point of view of the variation of states, which will not introduce the mass of conservation of the medium, but which reintroduces the Newtonian concept of mass already abandoned by the special theory of relativity;

(ii). To establish the form of linking equations of the medium which describing the variation of states of said medium and are compatible with the known quantity of the internal energy density.

9. The Concept of Rate of Change of Internal Energy.

Let us consider the equations of conservation (2.7) of the pulse-energy density (8.18) attached to a medium of the elastic type under consideration; these equations are equivalent to the group:

$$\mathfrak{L}\rho = -1/2\theta^{\alpha\beta}\mathfrak{L}\bar{g}_{\alpha\beta}. \qquad 9.1\text{-a}$$

$$\rho\dot{u}_\alpha = P(\nabla_\beta\theta^\beta_\alpha), \qquad 9.1\text{-b}$$

where $\dot{u}_\alpha$ designates the acceleration vector attached to the medium:

$$\dot{u}_\alpha = \mathfrak{L}u_\alpha = u^\beta\nabla_\beta u_\alpha, \qquad u^\alpha\dot{u}_\alpha = 0. \qquad 9.2$$

These equations express locally the conservation of total energy and pulses of the elastic medium.

Let us refer now to certain results established in the preceding paragraph; first of all, by virtue of (8.4, 8.16) we have:

$$\mathfrak{L}E = \mathfrak{L}\rho ; \qquad 9.3$$

furthermore, according to (8.12, 8.14) the rate of change of E is characterized by the equation:

$$\mathfrak{L}E = -1/2\theta^{\alpha\beta}\mathfrak{L}\bar{g}_{\alpha\beta}. \qquad 9.4$$

One of the three equations (9.1-a), (9.3) or (9.4) is rigorously deduced from the other two; we can therefore equally define the rate of change of the elastic energy E by Eqs. (9.3) or (9.4). The expression (9.3), which links the variations of two quantities, appears to us to be the only way of introducing the concept of internal energy in relativity under a form accessible to physical measurements; this definition does not introduce the theoretical concept of a mass of conservation of the medium directly; it concerns continuous media of any type whatsoever and expresses the relativistic principle of the unification of mass and energy according to the point of view of the variations of state.

These arguments lead us to define the concept of rate of change of the internal energy in general relativity as follows:

Definition. The rate of change of the internal energy density associated with a medium in general relativity is identical to the rate of change of the mass density of said medium:

$$\mathfrak{L}E = \mathfrak{L}\rho.$$

10. Determination of the Correlating Equations of the Elastic Medium.

According to the ideas developed above we propose to deduce from the representation of states of an elastic medium the representation of the variation of states of the medium compatible with general relativity.

The correlating equations which provides the description of the variation of states will be obtained by direct application of the lemma of paragraph 6 to the constraint density equations (8.14), as this quantity duly meets the conditions of application of this lemma. By changing two members of Eq. (8.14) we thus obtain the equations:

$$P(\mathfrak{L}\theta^{\alpha\beta}) = 1/2C^{\alpha\beta\gamma\delta}\mathfrak{L}\bar{g}_{\gamma\delta}, \tag{10.1}$$

where we have made:

$$C^{\alpha\beta\gamma\delta} = 2\bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta} = -4\bar{\delta}^2E/\delta g_{\gamma\delta}\delta g_{\alpha\beta}. \tag{10.2}$$

The contravariant density with four superscripts $C^{\alpha\beta\gamma\delta}$ characterizes the structural properties of the elastic medium; by analogy with the classical theory of elasticity, we give this quantity the name of density of modulus of elasticity $C^{\alpha\beta\gamma\delta}$ is subject, by definition, to the following conditions:

(i). It is a space tensor density.

(ii). It depends on the state variables $\bar{g}^{\alpha\beta}$:

$$C^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta}(\bar{g}_{\mu\nu}). \tag{10.3}$$

(iii). It is subject to the conditions of symmetry derived, on the one hand, from its definition (10.2) and, on the other, from the integrability conditions (8.15) of the equations of state (8.14):

$$C^{\alpha\beta\gamma\delta} = C^{(\alpha\beta)(\gamma\delta)}, \tag{10.4-a}$$

$$C^{\alpha\beta\gamma\delta} = C^{\gamma\delta\alpha\beta}. \tag{10.4-b}$$

(iv). Finally, it meets all the requirements of the differential conditions:

$$\bar{\delta}/\delta g_{\rho\sigma}C^{\alpha\beta\gamma\delta} = \bar{\delta}/\delta g_{\gamma\delta}C^{\alpha\beta\rho\sigma} \tag{10.5}$$

ensuring the integrability of (10.1).

By virtue of (i) and (iii) $C^{\alpha\beta\gamma\delta}$ has $N(N + 1)/2 = 21$ independent components of the type $(N = n(n + 1)/2,\ n = 3)$, which agrees with the results of the classical theory of elasticity; but $C^{\alpha\beta\gamma\delta}$ also agrees with the $(N - 1)^3 + 3(N - 1))/2 = 70$ differential conditions (iv) which leaves 56 supplementary conditions of the same type of phenomenological origin which must be imposed on $C^{\alpha\beta\gamma\delta}$ in order to determine this quantity perfectly.

Let us bring together these results in the following proposition:

Proposition. The correlating equations of elastic media in general relativity assume the form:

$$P(\mathfrak{L}\theta^{\alpha\beta}) = 1/2\, C^{\alpha\beta\gamma\delta}\mathfrak{L}\bar{g}_{\gamma\delta},$$

where the density of the modulus of elasticity $C^{\alpha\beta\gamma\delta}$ meets the following conditions:

(i). It is a space tensor density.

(ii). It is a C^0 - C^2 p.m. type function of the variations of state $\bar{g}^{\alpha\beta}$.

(iii). It has the symmetries compatible with the existence of an elastic energy density:

$$C^{\alpha\beta\gamma\delta} = C^{(\alpha\beta)(\gamma\delta)}, \qquad C^{\alpha\beta\gamma\delta} = C^{\gamma\delta\alpha\beta}.$$

(iv). It meets the conditions which ensure the integrability of the correlating equations:

$$\bar{\delta}/\delta g_{\rho\sigma} C^{\alpha\beta\gamma\delta} = \bar{\delta}/\delta g_{\gamma\delta} C^{\alpha\beta\rho\sigma}.$$

C. The Problem of Elasticity in General Relativity

We now propose to outline the problem of linking the gravitational field to its sources according to the point of view of the variation of states adopted in this work.

11. Equations of the Variation of States

We are primarily interested in the field equations; having assumed that only the variations of the conditions of state have physical significance, we must assume that the Einstein equations play the role of initial conditions, and the determination of the field itself must introduce the variation of these equations.

To this end let us take the Lie derivative in the two-member vector field $\vec{u}$ of the field equations (2.5) taking into account the normal character of $T^{\alpha\beta}$; let us immediately express the variation of constraints from the second member of the correlating equations (10.1); we thus obtain the sought form of the equation of variation of states:

$$\mathfrak{L}(\sqrt{-g}G^{\alpha\beta}) = \mathfrak{L}\rho u^\alpha u^\beta + 2u^{(\alpha}\theta^{\beta\gamma)}u_\gamma - C^{\alpha\beta\gamma\delta}E_{\gamma\delta}. \qquad 11.1$$

We shall not express here the form of the first member of (11.1).

12. Various Equations of Conservation.

The second member of the correlating equations (10.1) of the elastic media must meet the requirements of a group of three conditions which we now propose to establish.

For this purpose, we start with the following identities[19], applied to the constraint density $\theta^{\alpha\beta}$:

$$\nabla_\beta \mathfrak{L}\theta^{\alpha\beta} - \mathfrak{L}\nabla_\beta\theta^{\alpha\beta} = -\theta^{\beta\gamma}\mathfrak{L}\Gamma^\alpha_{\beta\gamma}, \qquad 12.1$$

where the $T^\alpha_{\beta\gamma}$ designate the Christoffel symbols.

Due to the spatial character of $\theta^{\alpha\beta}$, only the components $P(\mathfrak{L}\Gamma^\alpha_{\beta\gamma})$ and $P(u_\alpha\Gamma^\alpha_{\beta\gamma})$ of $\mathfrak{L}\Gamma^\alpha_{\beta\gamma}$ intervene in the calculation of (12.1); we shall calculate these two quantities as a function of the acceleration vector u_α (9.2), of the tensor representing the rate of change of deformations $E_{\alpha\beta}$ (5.9) and of the tensor representing the rate of change of the vortices.

$$\Omega_{\alpha\beta} = u_\lambda \bar{\partial}_{[\alpha}(u_{\beta]}u^\lambda) = P(\nabla_{[\alpha}u_{\beta]}), \qquad 12.2$$

where the explicit components are given by the identity:

$$\nabla_\alpha u_\beta = E_{\alpha\beta} + \Omega_{\alpha\beta} + u_\alpha u_\beta. \qquad 12.3$$

$\bar{\partial}t = P(\partial t)$ and $\bar{\nabla}t = P(\nabla t)$ designate the operations of partial and covariant space derivative acting on the geometric quantities $t \in V_4$.

From the well known identity:

$$\mathfrak{L}\Gamma^\alpha_{\beta\gamma} = 1/2g^{\alpha\delta}(\nabla_\beta\mathfrak{L}g_{\delta\gamma} + \nabla_\gamma\mathfrak{L}g_{\beta\delta} - \nabla_\delta\mathfrak{L}g_{\beta\gamma}). \qquad 12.4$$

we arrive at the results:

[19] Cf. A. Lichnerowicz (1958), Chapter II.

$$P(\mathcal{L}\Gamma^{\gamma}_{\alpha\beta}) = \overline{\nabla}_{\alpha}E^{\gamma}_{\beta} + \overline{\nabla}_{\beta}E^{\gamma}_{\alpha} - \overline{\nabla}^{\gamma}E_{\alpha\beta} + 2\dot{u}_{(\alpha}\Omega_{\beta)}{}^{\gamma} + \dot{u}^{\gamma}E_{\alpha\beta}. \qquad 12.5\text{-a}$$

$$P(u_{\gamma}\mathcal{L}\Gamma^{\gamma}_{\alpha\beta}) = -\mathcal{L}E_{\alpha\beta} - \dot{u}_{\alpha}\dot{u}_{\beta} + \overline{\nabla}_{(\alpha}\dot{u}_{\beta)}. \qquad 12.5\text{-b}$$

By inserting these expressions in the second member of (12.1) we obtain a group of three identities equivalent to (12.1); we now have to express in the first member of these new identities the term $P(\mathcal{L}\theta^{\alpha\beta})$ from its value deduced from the correlating equations (10.1) in order to arrive at the group of three desired conditions of compatibility of (10.1); these various equations of conservation are expressed as:

$$P(\overline{\nabla}_{\beta}(C^{\alpha\beta\gamma\delta}E_{\gamma\delta})) = P(\mathcal{L}\overline{\nabla}_{\beta}\theta^{\alpha\beta} - \theta^{\beta\gamma}\nabla_{\delta}(E^{\alpha}_{\beta}\overline{g}^{\delta}_{\gamma} + E^{\alpha}_{\gamma}\overline{g}^{\delta}_{\beta} - E_{\gamma\beta}\overline{g}^{\alpha\delta})) \qquad 12.6$$

where, by definition:

$$\overline{\nabla}_{\beta}\theta^{\alpha\beta} = P(\overline{g}^{\beta\gamma}\overline{\nabla}_{\beta}\theta^{\alpha}_{\gamma}), \text{ etc.}$$

13. Equations of Variation of the Tensors Representing the Rate of Change of Deformations and Vortices.

We propose to establish a last group of equations describing the variation of the change of deformations and vortices of the continuous media under consideration which are necessary for the complete formulation of the problem of elasticity in general relativity.

To this end we start from the group of identities:

$$\mathcal{L}E_{\alpha\beta} = \overline{\nabla}_{(\alpha}\dot{u}_{\beta)} - \dot{u}_{\alpha}\dot{u}_{\beta} + (E^{\gamma}_{\alpha} + \Omega^{\gamma}_{\alpha})(E_{\beta\gamma} + \Omega_{\beta\gamma}) + u^{\gamma}u^{\delta}R_{\alpha\gamma\beta\delta}. \qquad 13.1\text{-a}$$

$$\mathcal{L}\Omega_{\alpha\beta} = \overline{\nabla}_{[\alpha}\dot{u}_{\beta]}. \qquad 13.1\text{-b}$$

When the density of constraints $\theta_{\alpha\beta}$ is known through integration of the correlating equations (10.1) it is possible to deduce from the equations of conservation (9.1) the value of the acceleration vector $\dot{u}_\alpha$ as a function of the density of mass, of the metric and of its principal derivatives; when this value of $\dot{u}_\alpha$ is inserted in the second members of the identities (13.1) we obtain the desired equations deducing the variation of deformations and vortices of the continuous medium under consideration[20 21].

14. The Problems of Elasticity in General Relativity

The combination of results obtained causes us to present the problem of the determination of the gravitational field created by continuous elastic medium of the type under consideration in the following manner[21].

Proposition:

—The 10 equations of the variation of fields,

—The 4 equations of conservation of the pulse-energy density (9.1),

—The 6 correlating equations (10.1),

—The 3 equations of variations of conservation (12.6),

—The 9 equations of variation of the tensors representing the rates of change of deformations and vortices,

form a group of 32 equations which should allow us to determine the 32 quantities consisting of:

—The 10 gravitational potentials,

—The 3 independent components of the world velocity vector, bearing in mind the unitary character of this vector,

—The density of mass,

—The 6 components of the constraint density,

—The 3 components of the acceleration vector,

—The 9 independent components of the tensors representing the rate of change of deformations and vortices.

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[20]The equations of variation of the vortices (J.-F. Bennoun (1964) constitute the relativistic extension of Helmholtz's equations in classical hydrodynamics; these equations especially comprise, as specific cases, the analogous equations established by Y. Bruhat (1958) in the case of perfect fluids and of perfect fluids interacting with a Maxwell electromagnetic field, and by A. Lichnerowicz (1964-1965) in the case of perfect adiabatic fluids.

[21]Cf. J.-F. Bennoun (1964); we must approach the point of view considered here from the one adopted by A. Lichnerowicz (1964-1965) when he established the theorems of the existence of equations of perfect adiabatic fluids.

Chapter III. Representation of Thermodynamic Continuous Media

15. Introduction

The phenomenological macroscopic thermodynamics established on a purely axiomatic basis may be developed coherently and autonomously, even if its profound significance can be attained only by a statistical interpretation of the basic concepts.

This theory assumes[22]:

(i) The conservation of the total energy of the system.

(ii) The strictly non-decreasing character of total entropy at the time of variation of the system under consideration.

It is this axiomatic character of the theory which enables it to be successfully absorbed by the special theory of relativity and, as we shall see, to be integrated into the general theory of relativity, this synthesis being at the origin of a mutual enrichment of the two theories.

We shall attempt to set a basis for the study of relativistic thermodynamics through an anlysis of the ideas presented in prior works carried out in this field[23].

a. It is necessary at first to determine the structure of the pulse-energy density associated with a thermodynamic continuous medium; such a quantity must include:

(i) A purely temporal term associated with the mass, i.e., to the energy of the system under consideration.

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[22]We are not considering here the zero principle, which introduces the concept of temperature by means of the postulate of its identity among the different parts of a system in thermodynamic equilibrium (Fowler (1931), cited by A. Sommerfeld (1956)), or the third principle (Nernst's Law (1906)), which defined an absolute scale for measuring entropy, since these postulates do not have an immediate significance in the assumed interpretation of the theory of general relativity.

[23]The early studies related to the formulation of relativistic thermodynamics (Planck (1907, 1908), Einstein (1907)) concern the study of the variance of certain thermodynamic quantities in the Lorentz group. Investigation of an axiomatic formulation of the theory appears with Tolman who became interested in the study of the second principle but limited himself to the study of stationary world models in thermodynamic equilibrium. A clear formulation of the two principles is due to Eckart (1940), within the special theory of relativity. The later works of Pham Mau Quan (1954), then Stuelckelberg (1962) and Ehlers (1961), tending to find a thermodynamics within the framework theory of general relativity, are based on the principles initially postulated by Eckert. More recently, Lichnerowicz (1964-1965) established a series of existence theorems for the equations describing the perfect adiabatic fluids originally studied by Van Dantzig (1939-1940), then by A. H. Taub (1948).

(ii) ... of the momentum type, linked to the existence of a heat flow within the medium.

(iii) A purely spatial term describing the action of the constraints of the continuous medium under consideration.

These requirements lead to the adoption of the Eckart-type (1940) of pulse-energy density.

$$T^{\alpha\beta} = \rho u^\alpha u^\beta - 2u^{(\alpha}q^{\beta)} - \theta^{\alpha\beta}, \qquad u_\alpha q^\alpha = 0, \qquad \theta^{\alpha\beta}u_\alpha = 0. \qquad 15.1$$

Contrary to the extremely clear physical situation presented by the initial study of elastic media, the pulse-energy density $T^{\alpha\beta}$ is no longer of the normal type; this situation involves the following consequence: $T^{\alpha\beta}$ is formed by 13 quantities constituted by the three independent components of the world velocity vector u^α, the density of mass (energy) p, the 3 components of the density of heat flow q^α and the 6 independent components of the constraint density $\theta^{\alpha\beta}$; now $T^{\alpha\beta}$ contains only 10 independent components; it follows that these 13 quantities must necessarily be linked by a group of 3 relations. The investigation of these 3 equations constitutes one of the important problems of relativistic thermodynamics, closely linked to the relativistic formulation of the second principle.

b. From the fact of the unification of the concepts of pulse and energy in relativity we must think that the most normal relativistic expansion of the first principle of thermodynamics must consist of 4 equations of the total conservation of pulse-energy; we know that said equations are the immediate consequence of the field equations in general relativity; in this respect, we can say that the first principle of the relativistic thermodynamics is contained in the basic hypotheses of Einstein's theory of gravitation.

Let us consider the temporal portion of these equations:

$$\mathfrak{L}\rho = \nabla_\alpha q^\alpha - q^\alpha \dot{u}_\alpha - \theta^{\alpha\beta}E_{\alpha\beta} \qquad 15.2$$

which describe the total conservation of energy during the variation of the continuous medium, and which is therefore the analogue, in a rigorous sense, of the first principle of classical thermodynamics; the second member of this equation, in its contribution to the rate of change of the heat energy, reveals the existence of a term linked to the acceleration of the medium, term which does not disappear unless the heat flow vector is orthogonal to the accelerating field. The existence of this term, with a purely relativistic origin, is of particular importance as we shall see in a moment.

c. Let T be the scalar field of the material system under consideration, and S the entropy density associated therewith. The second principle of the relativistic thermodynamics of Eckart et al assumes that:

(i) A vector density is with the total entropy of the system:

$$S^{\alpha} = Su^{\alpha} - T^{-1}q^{\alpha}. \qquad 15.3$$

(ii) The flux of the total entropy density along the edge of the system under consideration is rigorously non-decreasing during the variation of said system; this condition is expressed locally by the inequality:

$$\nabla_{\alpha}S^{\alpha} = \mathfrak{L}S - \nabla_{\alpha}(T^{-1}q^{\alpha}) \geqslant 0. \qquad 15.4$$

A glance at (15.4) brings to light an essential difficulty overcome by this interpretation of the second principle: The first member of the inequality Eq. (15.4), which is characteristic of the rate of change of total entropy does not contain a term linked to the contribution of the acceleration of the medium in the variation of the heat flux; now, we must logically rely on the existence of such a term, by virtue of the interpretation of the first principle. A contradiction thus appears at the very level of the formation of the basic concepts of relativistic thermodynamics which should compel us to reflect anew on the subject of the interpretation of the second principle in relativity.

The statement of Fourier's relativistic law obtained, under convenient hypotheses, as a consequence of the two principles, sheds light on one of the aspects of this difficulty: When we adopt the formulation Eq. (15.4) of the second principle, we are led to a phenomenological law of heat conduction of the type (Eckart, Stuelckelberg, Ehlers):

$$q^{\alpha} = -\chi^{\alpha\beta}(T^{-1}\bar{\partial}_{\beta}T - \dot{u}_{\beta}), \qquad 15.5$$

where $X^{\alpha\beta}$ is a tensor density of the second order, symmetrical and defined as positive; this law is distinguished from Fourier's classical law by the appearance of a supplementary term linked to the acceleration of the medium, a term whose form leads to the following difficulty: During the study of Cauchy's problem the postulated continuity of q^{α}, while a condition of state, involves the disappearance of transverse wave fronts, leaving only the longitudinal waves which is in disagreement with the results of the theory of continuous media.

d. The investigation of the equations of state of the medium was undertaken by Pham Mau Quan (1954) in the particular case of perfect fluids. In addition to an equation of state among mass, pressure and temperature, this author proposes equation of heat conduction, the examination of which calls for certain observations:

(i) Like the statement Eq. (15.4) of the second principle, this law does not take into account the contribution to the rate of change of the heat energy made by the term linked to the acceleration of matter.

(ii) The thermodynamic quantities are defined therein in relation to the mass unit of the medium, and the volume changes are represented with the aid of mass changes; now, these definitions which are admissible in the classical thermodynamics as a result of the direct link between mass and volume resulting from the character of conservation of the mass:

$$\mathfrak{L}\rho = 0, \qquad 15.6$$

must be abandoned in relativity as a result of the principle of unification of energy, since all forms of energy contribute to the variation mass:

$$\mathfrak{L}\rho \neq 0. \qquad 15.7$$

In this respect, let us say that a number of authors define the thermodynamic quantities relative to the unit of conservation of mass of the medium; now, this procedure introduces an artificial complication into the theory, since such a concept, a survival from Newtonian mechanics, has no further definite significance within the relativistic mechanics; as previously shown, only the variation of the mass (energy) has meaning.

This brief analysis of the present state of thermodynamics in relativity dictates the road to follow. After we made an effort to establish a statement of the two principles of thermodynamics rigorously compatible with relativistic ideas, we shall approach the problem of the representation of states of a thermodynamic continuous medium to deduce therefrom a description of the variation of states of said medium adapted to the accepted interpretation of the general theory of relativity.

A. The Two Principles of Relativistic Thermodynamics

In this secion we propose to establish a special formulation of the two principles of thermodynamics compatible with relativity, introducing, on the one hand, a general postulate of unification of mass and energy and, on the other, appealing to a suitable definition of the concept of total entropy of a thermodynamic system.

16. The First Principle of a Relativistic Thermodynamics

We always consider a Ω domain of the V_4 graph of general relativity subjected to specified conditions of differentiability, a domain created by a family of world lined oriented in time, and trajectories of the field of world velocity vectors:

$$u^\alpha = dx^\alpha/ds, \qquad g_{\alpha\beta}u^\alpha u^\beta = 1. \tag{16.1}$$

The geometrical structure of Ω is determined with the aid of the system of internal field equations:

$$\sqrt{-g}G^{\alpha\beta} = T^{\alpha\beta}, \tag{2.5}$$

where, for reasons which have been analyzed in the preceding paragraph, the pulse-energy density $T^{\alpha\beta}$ is of the type[24]:

$$T^{\alpha\beta} = \rho u^\alpha u^\beta - 2u^{(\alpha}q^{\beta)} - \theta^{\alpha\beta}, \qquad u_\alpha q^\alpha = 0, \qquad u_\alpha \theta^{\alpha\beta} = 0, \tag{15.1}$$

where p designates the density of mass (energy) of the system under consideration q^α the heat flow density and $\theta^{\alpha\beta}$ the constraint density. Contrary to the results of the dynamic study of elastic media made in the preceding paragraph, the form

[24]We recall that the quantities (ρ, q^α, $\theta^{\alpha\beta}$, S) which characterize respectively mass (energy) per unit of volume medium, the heat flow, the constraints operating on the surface of a unit of volume element, and the specific entropy of the medium attached to the unit of volume of said medium are of weight tensor densities + 1. In return, the quantities ($\bar{g}_{\alpha\beta}$, T) which characterize the deformation and the temperature of the medium are tensors.

of $T^{\alpha\beta}$ constitutes a postulate which is independent of the theory of thermodynamic continuous media.

Let us clarify the form of these equations of conservation of $T^{\alpha\beta}$:

$$u_\alpha \nabla_\beta T^{\alpha\beta} = \mathcal{L}\rho - \nabla_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + 1/2 \theta^{\alpha\beta} \mathcal{L} \bar{g}_{\alpha\beta} = 0. \qquad 16.2\text{-a}$$

$$P(\nabla_\beta T^\beta_\alpha) = \rho \dot{u}_\alpha - \mathcal{L} q_\alpha + 2 q_\beta \Omega^\beta_\alpha - P(\nabla_\beta \theta^\beta_\alpha) = 0. \qquad 16.2\text{-b}$$

These equations describe the total conservation of pulse-energy associated with the medium during its variation; the characterization of each of the terms which intervene is immediate; their examination, however, calls for certain comments:

(i) The heat flow density $\vec{q}$ behaves like a momentum density; this is especially clear in the examination of the 3 equations of the total conservation of pulse (16.2-b), where this quantity intervenes, on the one hand, due to its variation and, on the other, through a term with a purely relativistic origin, a term linked to the rotation of the medium and formally analogous to a Coriolis acceleration.

(ii) In the equation of conservation of energy Eq. (16.2-a) the contribution to the rate of change of the heat energy introduces, in addition to a divergence linked to the variation of the heat flux and rigorously analogous to the corresponding term in the classical theory, a term which is purely relativistic in origin linked to the acceleration of matter $\dot{u}_\alpha$.

These 4 equations of conservation of pulse rate-energy appear to us to be the foundation of the first principle of relativistic thermodynamics which thus becomes the rigorous consequence of the field equations; the tensorial form of this principle, which had not been set by the classical thermodynamics comes from the unification of the concept of energy in relativity.

For reasons which will soon be clear, we shall not, however, make use of the scalar equation of conservation of energy Eq. (16.2-a) which constitutes the interpretation in the strict sense of the first principle in relativity. Let us introduce the concept of internal energy associated with the continuous medium considered with the aid of the general definition established following the considerations in paragraph 9:

$$\mathcal{L}E = \mathcal{L}\rho. \qquad 16.3$$

Thus presented, with:

—the postulate Eq. (16.3) of unification of mass and energy,

—the equation of conservation of total energy Eq. (16.2-a),

we justify the introduction of the following proposition, which constitutes the special formulation of the first principle of relativistic thermodynamics.

Proposition. The rate of change of the internal energy density associated with a thermodynamic continuous medium is equal to the sum of the contributions of the rate of change of the heat energy, and of the rate of change of the energy of constraint forces acting within the medium under consideration:

$$\mathcal{L}E = \nabla_\alpha q^\alpha - q^\alpha \dot{u}_\alpha - 1/2\theta^{\alpha\beta}\mathcal{L}\bar{g}_{\alpha\beta}. \qquad 16.4$$

17. The Second Principle of Relativistic Thermodynamics.

In the introduction to this chapter we were lead to assume that the formulation of the second Eckart-Pham Mau Quan principle is not adapted to relativistic concepts because it does not take into account the term linked to the acceleration of the medium, a term whose existence is nevertheless strongly suggested by the relativistic interpretation of the first principle, in the expression of the rate of change of total entropy.

Furthermore, we have seen that the univoqual definition of pulse rate-energy, which is no longer of the normal type, requires the introduction of 3 unknown supplementary relations.

We have been led to believe that the solution of the indicated difficulties may be found in a tensorial formulation of the second principle, substituting for the initial vector formulation; such a formulation must include, in addition to an inequality of the scalar type which explicitly introduces the missing term in the initial formulation of the second principle, three new relations which must be in the origin of the 3 equations missing from the complete statement of the problem of the determination of the structure of the thermodynamic continuous media. The investigation of a tensorial formulation of the second principle may be greatly aided by using an analogy with the formulation of the first principle, even though it is evident that we cannot provide immediate justification for such an analogy.

The dynamics of the continuous media in general relativity is deduced from the following hypotheses:

(i) The pulse-energy associated with a material medium is represented by a symmetric contravariant tensor density of the second order $T^{\alpha\beta}$.

(ii) The divergence of $T^{\alpha\beta}$ described the rate of change of pulse-total energy; this quantity of conservation by virtue of the field equations.

In the same manner, we assume that:

(i) A symmetric contravariant tensor density of the second order $S^{\alpha\beta}$ is associated with the characteristic total entropy of a material system.

(ii) The divergence of the total entropy density $S^{\alpha\beta}$ describes the rate of change of the total entropy during the variation of the medium under consideration.

We can readily set forth the structure of $S^{\alpha\beta}$ in the case of thermodynamic continuous media; such a quantity must, in effect, include in said case:

(i) A purely temporal term which characterizes the entropy density S associated with the system under consideration; S is a scalar density, assumed to be rigorously non-negative[24].

(ii) A momentum type of term which characterizes the appearance of the entropy flow density $T^{-1}q^{\alpha}$, due to the existence of the heat density q^{α}; the temperature field T of the medium is a field of rigorously non-negative scalars[24].

It follows from these requirements, that the total entropy density $S^{\alpha\beta}$ characteristic of the thermodynamic continuous medium necessarily assumes the form:

$$S^{\alpha\beta} = S u^{\alpha} u^{\beta} - 2T^{-1} u^{(\alpha} q^{\beta)}. \qquad 17.1$$

The rate of change of the total entropy density is a vector quantity whose time and space components are explicitly set forth by the relations:

17.2-a

$$u_{\alpha} \nabla_{\beta} S^{\alpha\beta} = \mathfrak{L}S - \nabla_{\alpha}(T^{-1}q^{\alpha}) + T^{-1}q^{\alpha}\dot{u}_{\alpha}$$

17.2-b

$$P(\nabla_{\beta} S^{\beta}_{\alpha}) = S\dot{u}_{\alpha} - \mathfrak{L}(T^{-1}q_{\alpha}) + 2T^{-1}q_{\beta}\Omega^{\beta}_{\alpha}.$$

It is very encouraging to notice that the relation Eq. (17.2-a) leads to the appearance of the term missing from the initial interpretation Eq. (15.4) of the second principle, a term characteristic of a contribution to the total entropy variation due to the acceleration of the medium. The hypothesis adopted thus causes the disappearance of the difficulty present in the second principle of Eckart et al; the complete tensor formulation of the second principle requires the prior interpretation of three new quantities Eq. (17.2-b).

We are therefore limiting ourselves at present to a formulation of the second principle in the strict sense, i.e., introducing only the scalar density (17.2-a) which we postulate to be rigorously non-negative:

$$u_\alpha \nabla_\beta S^{\alpha\beta} = \mathfrak{L}S - \nabla_\alpha(T^{-1}q^\alpha) + T^{-1}q^\alpha \dot{u}_\alpha \geqslant 0. \qquad 17.3$$

Summarizing, we have assumed the following fundamental hypotheses:

Proposition. (i) The total entropy associated with a thermodynamic continuous medium is characterized by the total entropy density:

$$S^{\alpha\beta} = Su^\alpha u^\beta - 2T^{-1}u^{(\alpha}q^{\beta)}.$$

(ii) The rate of change of the total entropy density during the variation of the medium is defined by the divergence of $S^{\alpha\beta}$:

$$u_\alpha \nabla_\beta S^{\alpha\beta} = \mathfrak{L}S - \nabla_\alpha(T^{-1}q^\alpha) + 2T^{-1}q^\alpha \dot{u}_\alpha$$

$$P(\nabla_\beta S^\beta_\alpha) = S\dot{u}_\alpha - \mathfrak{L}(T^{-1}q_\alpha) + 2T^{-1}q_\beta \Omega^\beta_\alpha.$$

These hypotheses have led us to the special formulation of the second principle of relativistic thermodynamics:

Proposition. The temporal portion of the rate of change of the total entropy associated with a thermodynamic continuous medium is rigorously non-negative:

$$\mathfrak{L}S - \nabla_\alpha(T^{-1}q^\alpha) + q^\alpha \dot{u}_\alpha \geqslant 0.$$

B. Determination of the Structure of $T^{\alpha\beta}$.

The object of the developments of this section is the deduction of equations of state of a thermodynamic continuous medium which generalize the equations established previously for the elastic media by means of a hypothesis of simple structure regarding the free energy density of the medium.

18. Structure of Thermodynamic Continuous Media

From the special formulation of the two principles of thermodynamics which we have just established, and of a suitable hypothesis of the structure of the continuous medium under consideration, we shall deduce a representation of the thermodynamic states of such a medium.

The fundamental hypothesis is that the structure properties of thermodynamic continuous media are entirely deducible from the known quantity of the free energy density (in the Helmholtz sense) associated with such a medium. Let us introduce this concept in a manner adapted to general relativity: If E designates the internal energy density associated with the medium, the rate of change of the free energy density F is defined by the relation:

$$\mathfrak{L}F = \mathfrak{L}(E - TS). \qquad 18.1$$

a. By extension of the hypotheses concerning the elastic media, we assume that:

(i) The conditions of state $(T^{\alpha\beta}, S^{\alpha\beta}, F)$ are tensor densities of the $C^0 - C^2$ p.m. class.

(ii) These quantities are functions of the independent state variables $(\bar{g}_{\alpha\beta}; T)$.

According to the initial hypothesis, the ensemble of structure properties of the medium is deduced from the known quantity of the function F which satisfies (i) and (ii).

$$F = F(\bar{g}_{\alpha\beta}; T). \qquad 18.2$$

b. From Eq. (16.4) and Eq. (18.1) let us write the special formulation of the first principle:

$$\mathfrak{L}F = \nabla_\alpha q^\alpha - q^\alpha \dot{u}_\alpha - 1/20^{\alpha\beta}\mathfrak{L}\bar{g}_{\alpha\beta} - \mathfrak{L}(TS). \qquad 18.3$$

The second principle under the special form Eq. (17.3) affirms, bearing in mind Eq. (18.3) that:

$$-\mathfrak{L}F - S\dot{T} - 1/2\theta^{\alpha\beta}\mathfrak{L}\bar{g}_{\alpha\beta} + q^{\alpha}T^{-1}\bar{\partial}_{\alpha}T \geqslant 0. \qquad 18.4$$

c. By virtue of the above hypotheses (i) and (ii), and of the known quantity of the function F Eq. (18.2), the preceding relation (18.4) is written:

$$-(\delta F/\delta T + S)\dot{T} - (\bar{\delta}F/\delta g_{\alpha\beta} + 1/2\theta^{\alpha\beta})\mathfrak{L}\bar{g}_{\alpha\beta} + q^{\alpha}T^{-1}\bar{\partial}_{\alpha}T \geqslant 0. \qquad 18.5$$

First, let us consider the case of an adiabatic system in thermodynamic equilibrium; the postulated independence of the state variables $(\bar{g}_{\alpha\beta}, T)$ means that the following relations must also be confirmed:

$$S = -\delta F/\delta T. \qquad 18.6\text{-a}$$

$$\theta^{\alpha\beta} = -2\bar{\delta}F/\delta g_{\alpha\beta}. \qquad 18.6\text{-b}$$

They are the equations of state of the thermodynamic continuous medium; these equations are integrable on condition that:

$$\delta\theta^{\alpha\beta}/\delta T = 2\bar{\delta}S/\delta g_{\alpha\beta}. \qquad 18.7\text{-a}$$

$$\bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta} = \bar{\delta}/\delta g_{\alpha\beta}\theta^{\gamma\delta}. \qquad 18.7\text{-b}$$

So as not to renounce to a description of the thermodynamic system when it undergoes irriversible variation, we must offer the hyposthesis that this variation does not depart too far from a reversible-type variation, so that the equations (18.6) always describe the state of the system under consideration.

Under this condition, according to Eq. (18.6), the second principle Eq. (18.5) required that:

$$q^{\alpha}T^{-1}\bar{\partial}_{\alpha}T \geqslant 0. \qquad 18.8.$$

Remarks. The preceding inequality is particularly satisfied by the equation:

$$q^{\alpha} = -\chi^{\alpha\beta}\bar{\partial}_{\beta} T\, T^{-1}, \tag{18.9}$$

where the symmetric space tensor density and its definition as positive $\chi^{\alpha\beta}$ characterize the thermal conductivity of the medium; this equation is the exact transposition in relativity of Fourier's classical law (1888) of heat conduction. Said result, according to the opinion of Pham Mau Quan causes the form Eq. (15.5) of said law (Eckart, Stuelkelberg, Ehlers) deduced from the formulation of Eckart's second principle to be abandoned, since it led to the difficulties which have been pointed out.

d. Bearing in mind Eq. (18.3) the equations of state Eq. (18.6) we arrive at the equation[25]:

$$T\mathfrak{L}S - \nabla_{\alpha} q^{\alpha} + q^{\alpha}\dot{u}_{\alpha} = 0. \tag{18.10}$$

Let us include this result in the equation of conservation of energy Eq. (16.2-a); we arrive at:

$$\mathfrak{L}\rho - T\mathfrak{L}S + 1/2\,0^{\alpha\beta}\mathfrak{L}\bar{g}_{\alpha\beta} = 0; \tag{18.11}$$

from this equation, and bearing in mind the hypotheses of structure (i) and (ii) of conditions of state, we deduce the relations:

$$\delta\rho/\delta T = T\delta S/\delta T. \tag{18.12-a}$$

$$\bar{\delta}\rho/\delta g_{\alpha\beta} = T\bar{\delta}S/\delta g_{\alpha\beta} - 1/2\,0^{\alpha\beta}. \tag{18.12-b}$$

The group of equations (18.12) is quite integrable by virtue of the conditions of integrability Eq. (18.7) of the equations of state Eq. (18.6) thus ensuring the coherence of the postulates of relativistic thermodynamics.

We can summarize the ensemble of results obtained.

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[25] The adiabatic fluids ($q^{\alpha} = 0$) have been studied in the isotropic case, especially by Van Dantzig (1940), Taub(1948) and Lichnerowicz (1964-1965).

Proposition. The representation of the states of a thermodynamic continuous medium in general relativity, close to thermodynamic equilibrium, is ensured by the ensemble of the following relations:

(i) The equations of state of the medium, linking the densities of constraints and entropy of the medium to the free density energy:

$$S = -\delta F/\delta T, \qquad \theta^{\alpha\beta} = -2\bar{\delta}F/\delta g_{\alpha\beta},$$

and the conditions of integrability of these equations:

$$\delta\theta^{\alpha\beta}/\delta T = 2\bar{\delta}S/\delta g_{\alpha\beta}, \qquad \bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta} = \bar{\delta}/\delta g_{\alpha\beta}\theta^{\gamma\delta}.$$

(ii) The equations of state linking the density of mass (energy) of the medium to the densities of entropy and constraints associated with said medium:

$$\frac{\delta\rho}{\delta T} = T\frac{\delta S}{\delta T} \qquad \frac{\bar{\delta}\rho}{\delta g_{\alpha\beta}} = T\frac{\bar{\delta}S}{\delta g_{\alpha\beta}} - \frac{1}{2}\theta^{\alpha\beta}.$$

(iii) The fundamental inequality, governing the phenomenon of heat conduction within the medium:

$$q^{\alpha}\bar{\partial}_{\alpha}T\,T^{-1} \geqslant 0.$$

C. Representation of the Variation of States of the Thermodynamic Continuous Media.

In this section we propose to deduce the linking equations which ensure the description of the variation of states of the medium, rigorously compatible with the principles of general relativity from the preceding purely theoretical description of the states of a thermodynamic continuous media.

19. Linking Equations of the Thermodynamic Continuous Media

The deduction of the equations ensuring the representation of the variation of states of a thermodynamic continuous medium from equations of state Eq. (18.6) is immediate if we note that the conditions for applications of the lemma in paragraph 6 are properly combined insofar as the entropy density S and the constraints density $\theta^{\alpha\beta}$ attached to said medium are concerned, bearing in mind the hypotheses of the structure of the conditions of state (i) and (ii) set forth in paragraph 18.

Through variation of equations of state Eq. (18.6) we thus arrive at the group[26]:

$$T\mathcal{L}S = 1/2L'^{\alpha\beta}\mathcal{L}\bar{g}_{\alpha\beta} + CT^{-1}\dot{T}. \qquad 19.1\text{-a}$$

$$P(\mathcal{L}\theta^{\alpha\beta}) = 1/2C^{\alpha\beta\gamma\delta}\mathcal{L}\bar{g}_{\gamma\delta} + L''^{\alpha\beta}T^{-1}\dot{T}. \qquad 19.1\text{-b}$$

where, by definition,

$$C = T^2\delta S/\delta T. \qquad 19.2\text{-a}$$

$$L'^{\alpha\beta} = 2T\bar{\delta}S/\delta g_{\alpha\beta}. \qquad 19.2\text{-b}$$

$$L''^{\alpha\beta} = T\delta/\delta T\theta^{\alpha\beta}. \qquad 19.2\text{-c}$$

$$C^{\alpha\beta\gamma\delta} = 2\bar{\delta}/\delta g_{\gamma\delta}\theta^{\alpha\beta}. \qquad 19.2\text{-d}$$

(i) The quantities $C^{\alpha\beta\gamma\delta}$, $L'^{\alpha\beta}$, $L''^{\alpha\beta}$ are space tensor densities characterizing the structure properties of the thermodynamic continuous medium.

(ii) These quantities are C^0 - C^2 p.m. type functions of the independent state variables $(g_{\alpha\beta}, T)$.

(iii) By formulation, the densities $C^{\alpha\beta\gamma\delta}$, $L'^{\alpha\beta}$ et $L''^{\alpha\beta}$ satisfy the symmetry conditions:

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[26]Cf. J. F. Pennoun (1964).

$$L'^{\alpha\beta} = L'^{(\alpha\beta)}.$$ 19.3-a

$$L''^{\alpha\beta} = L''^{(\alpha\beta)}.$$ 19.3-b

$$C^{\alpha\beta\gamma\delta} = C^{(\alpha\beta)(\gamma\delta)}.$$ 19.3-c

(iv) Moreover, they satisfy the following supplementary conditions deduced from the conditions of integrability Eq. (18.7) of the equations of state Eq. (18.6):

$$L'^{\alpha\beta} = L''^{\alpha\beta}.$$ 19.4-a

$$C^{\alpha\beta\gamma\delta} = C^{\gamma\delta\alpha\beta}.$$ 19.4-b

It follows that the structure properties of the thermodynamic continuous media are characterized by a group of 28 quantities constituted by the specific heat density under conditions of rigidity T^{-1} C, the N = 6 components of the density of heat deformation $L^{\alpha\beta}$ Eq. (19.4-a) and the N(N + 1)/2 = 21 components of the density of modulus of elasticity $C^{\alpha\beta\gamma\delta}$.

(v) Said 28 quantities satisfy the requirements of a group of differential conditions which ensure the integrability of the linking equations Eq. (19.1) which are expressed:

$$T^2\delta/\delta(T^{-1}L^{\alpha\beta}) = 2\bar{\delta}C/\delta g_{\alpha\beta}.$$ 19.5-a

$$T\delta/\delta T C^{\alpha\beta\gamma\delta} = 2\bar{\delta}/\delta g_{\gamma\delta} L^{\alpha\beta}.$$ 19.5-b

$$\bar{\delta}/\delta g_{\rho\sigma} C^{\alpha\beta\gamma\delta} = \bar{\delta}/\delta g_{\gamma\delta} C^{\alpha\beta\rho\sigma}. \qquad 19.5\text{-c}$$

These conditions number $(N + N(N+1)/2 + (N-1)^3/2 + 3(N-1)/2 = 97$, which means that there are still 99 conditions of the same type to be established by a phenomenological process so that the structure of the continuous medium under consideration may be perfectly determined.

The representation of the variation of states of a thermodynamic continuous medium is thus completed. We shall now assemble the necessary elements for such a description under the form of the following statement.

Proposition. The linking equations of the thermodynamic continuous media in general relativity, which link the rates of change of the entropy and constraints densities to the rates of change of temperature and deformation of the medium under consideration close to thermodynamic equilibrium, assume the form:

$$T\mathfrak{L}S = 1/2 L^{\alpha\beta} \mathfrak{L}\bar{g}_{\alpha\beta} + CT^{-1}\dot{T}, \qquad P(\mathfrak{L}\theta^{\alpha\beta}) = 1/2 C^{\alpha\beta\gamma\delta} \mathfrak{L}\bar{g}_{\gamma\delta} + L^{\alpha\beta} T^{-1}\dot{T},$$

where the structure coefficients of the medium, constituted by the specific heat density under conditions of rigidity $T^{-1}C$, the deformation heat density $L^{\alpha\beta}$, and the density of modulus of elasticity $C^{\alpha\beta\gamma\delta}$, are subjected to the following conditions:

(i) They are space tensor densities, $C^0 - C^2$ p.m. functions of the state variables $(\bar{g}_{\alpha\beta}, T)$.

(ii) These quantities confirm the symmetry properties which ensure the existence of a free energy density:

$$L^{\alpha\beta} = L^{(\alpha\beta)}, \qquad C^{\alpha\beta\gamma\delta} = C^{(\alpha\beta)(\gamma\delta)}, \qquad C^{\alpha\beta\gamma\delta} = C^{\gamma\delta\alpha\beta}.$$

(iii) They meet the conditions which ensure the integrability of the linking equations.

$$T^2\delta/\delta T(T^{-1}L^{\alpha\beta}) = 2\bar{\delta}C/\delta g_{\alpha\beta}, \qquad T\delta/\delta T C^{\alpha\beta\gamma\delta} = 2\bar{\delta}/\delta g_{\gamma\delta} L^{\alpha\beta},$$
$$\bar{\delta}/\delta g_{\rho\sigma} C^{\alpha\beta\gamma\delta} = \bar{\delta}/\delta g_{\gamma\delta} C^{\alpha\beta\rho\sigma}.$$

Chapter IV. Study of the Characteristic Variations

The following developments concern the study of the characteristic variations of the equations of a thermodynamic continuous medium. By means of a general continuity hypothesis related to the variation of total entropy of the medium, a hypothesis which clarifies one of the aspects of the second principle, we propose to set forth the types of characteristic variations of the problem, which will lead to a study of the phenomenon of propagation of thermodynamic wave fronts.

20. Position of the Problem.

We have just established an ensemble of equations adapted to the description of thermodynamic continuous media in general relativity; however, by reason of the present lack of interpretation of three of the four quantities which must intervene in the tensorial formulation of the second principle, we do not know the form of the four relations which must complete the proposed group of equations. It is not therefore possible to deal here with Cauchy's problem, i.e., to approach the problem of the search of an effective local solution to these equations.

In return, however, we shall be able fully to characterize the characteristic variations of these equations; this situation becomes possible with the aid of a hypothesis of continuity related to the entropy variation in the course of the variation of the medium under consideration, a hypothesis which interprets in the main one of the aspects of the second principle.

We thus consider the following ensemble of relations as they have been set forth throughout the study of the thermodynamic continuous media in relativity, i.e.:

(i) The gravitational field equations:

$$\sqrt{-g}G^{\alpha\beta} = T^{\alpha\beta}, \qquad 2.5$$

where

$$T^{\alpha\beta} = \rho u^\alpha u^\beta - 2u^{(\alpha}q^{\beta)} - \theta^{\alpha\beta}, \quad u_\alpha q^\alpha = 0, \quad u_\alpha \theta^{\alpha\beta} = 0, \quad u_\alpha u^\alpha = 1. \qquad 15.1$$

(ii) The 4 pulse rate-energy conservation equations:

$$u_\alpha \nabla_\beta T^{\alpha\beta} = \mathcal{L}\rho - \nabla_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + 1/2\theta^{\alpha\beta}\mathcal{L}g_{\alpha\beta} = 0. \qquad 16.2\text{-a}$$

$$P(\nabla_\beta T^\beta_\alpha) = \rho \dot{u}_\alpha - \mathcal{L}q_\alpha + 2q_\beta \Omega^\beta_\alpha - P(\nabla_\beta \theta^\beta_\alpha) = 0. \qquad 16.2\text{-b}$$

(iii) The 7 linking equations, which ensure the descripti..n of the variation of states of the thermodynamic continuous media:

$$T\mathcal{L}S = 1/2 L^{\alpha\beta}\mathcal{L}\overline{g}_{\alpha\beta} + CT^{-1}\dot{T}. \qquad 19.1\text{-a}$$

$$P(\mathcal{L}\theta^{\alpha\beta}) = 1/2 C^{\alpha\beta\gamma\delta}\mathcal{L}\overline{g}_{\gamma\delta} + L^{\alpha\beta}T^{-1}\dot{T}. \qquad 19.1\text{-b}$$

(iv) 4 relations, unknown at present, which must come from the second principle of thermodynamics and introduce the rate of change of total entropy density:

$$u_\alpha \nabla_\beta S^{\alpha\beta} = \mathcal{L}S - \nabla_\alpha(T^{-1}q^\alpha) + T^{-1}q^\alpha\dot{u}_\alpha. \qquad 17.2\text{-a}$$

$$P(\nabla_\beta S^\beta_\alpha) = S\dot{u}_\alpha - \mathcal{L}(T^{-1}q_\alpha) + 2T^{-1}q_\beta\Omega^\beta_\alpha. \qquad 17.2\text{-b}$$

We present the following differentiability conditions relating to the ensemble of quantities involved in the problem presented:

(i) The gravitation potentials $g^{\alpha\beta}$ are of the C^1 - C^2 p.m. type.

(ii) The conditions of state constituted by the pulse rate-energy $T^{\alpha\beta}$, the total entropy dens..ty $S^{\alpha\beta}$, and the structure coefficients of the continuous medium under consideration $C^{\alpha\beta\gamma\delta}$, $L^{\gamma\beta}$ and C, are assumed to be of the C^0 - C^2 p.m. type.

(iii) The rate of change of the total entropy density $\overline{\nabla_\beta S^{\alpha\beta}}$ is assumed to be of the C^0 type.

Hypotheses (i) and (ii), to which we refer, have their origin in an analogy with the Newtonian theory of gravitation and classical dynamics.

Hypothesis (iii) is fundamental; it is the hypothesis which enables us completely to solve the problem of the search for the characteristic variations of the equations of the thermodynamic continuous media. We believe that this hypothesis must in the main be a direct consequence of the second principle of relativistic thermodynamics; in the absence of a complete formulation of the second principle, we present it as an independent postulate of the theory under consideration.

The ensemble of these hypotheses leads us to present the problem in the following manner[27 28].

(i) To the initial hypersurface (S) we ascribe the value of the set of values constituted by:

—The 10 gravitational potentials $g^{\alpha\beta}$ and their 10 derivatives normal to (S),

—The 15 conditions of state $(\rho, u^\alpha, q^\alpha, \theta^{\alpha\beta}, T, S)$,

—The 28 structure coefficients of the medium $(C^{\alpha\beta\gamma\delta}, L^{\alpha\beta}, C)$,

which satisfy the conditions of integrability Eq. (19.5) of the linking equations.

(ii) Having presented this, we seek to determine the behavior of the ensemble of the 25 quantities $(g_{\alpha\beta}; u^\alpha, \rho, q^\alpha, \theta^{\alpha\beta}, S, T)$ in the vicinity of the initial hypersurface (S) with the aid of the 25 relations (i) to (iv) which describe the thermodynamic continuous media.

We are going to show that the problem thus presented enables us to establish the general form of the characteristic hypersurface equations related to the thermodynamic continuous media by means of suitable differentiability hypotheses.

21. Nature of the Wave Fronts.

Let us designate with:

$$f(x^\alpha) = 0 \qquad 21.1$$

the hypersurface equation (S) which includes the initial date, and let:

$$l_\alpha = \partial_\alpha f \qquad 21.2$$

be a vector normal to S.

Let Φ be a geometrical quantity of V_4 which constitutes one of the initial data of the problem; the quantity Φ, which is assumed to be of the C^0 type, is therefore continuous over S, but it is not with respect to its derivative $\partial\Phi$,

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[27] Cf. J. Hadamard (1903), T. Lévi-Civita (1931).

[28] When we have a satisfactory formulation of the second principle, we must seek to deal with the problem of the gravitational field link created by a thermodynamic continuous medium in accordance with the point of view of the variation of states, in the same manner that it was possible to deal with this problem in respect to the elastic media (cf. Chapter II, Section C).

continuous by bits in V_4 and which is therefore liable to be discontinuous when it traverses the initial hypersurface S; we will always indicate such a discontinuity in [ ].

We note first of all that by virtue of the assumed continuity of the metric and its first derivatives we also have:

$$[\partial\Phi] = [\nabla\Phi]. \qquad 21.3$$

The components of the Φ derivatives in S are known since this quantity is given over S; we submit the essential hypothesis that these derivatives assume the same conditions of differentiability as Φ, i.e., that they are continuous over (S); it follows that the only possible discontinuities of $\nabla\Phi$ when they traverse (S) are those of the derivative of this quantity along the normal to (S), which leads us to present:

$$[\nabla_\alpha\Phi] = \varphi l_\alpha. \qquad 21.4$$

The discontinuities Eq. (21.4) are said to be of the Hadamard type[27]. If one or several discontinuities of this type are ≠0, (S) is a characteristic variation of the problem presented; by reason of the arbitrary nature of the discontinuities of $\nabla\Phi$, Cauchy's problem does not admit a physically single solution in this case: it is undetermined. Let us assume that (S) is such a characteristic variation; let $\pi = S \cap \Sigma$ be the two-dimensional plane of the space directions orthogonal to the normal to (S), and let $\vec{n}$ be the unit space vector which characterizes the direction of said normal:

$$n_\alpha = (-\bar{l}_\beta \bar{l}^\beta)^{-1/2}\,\bar{l}_\alpha, \quad \bar{l}_\alpha = P(l_\alpha), \quad n_\alpha u^\alpha = 0, \quad n_\alpha n^\alpha = -1. \qquad 21.5$$

(π) is termed the wave front associated with the characteristic variation (S) in relation to the time direction $\vec{u}$ and $\vec{n}$ the wave vector which characterizes the direction of propagation of the wave front.

The existence of the characteristic variations of a given Cauchy problem thus leads to the setting of the phenomenon of wave propagation transporting discontinuities of the material quantities of the field quantities. We now propose to set forth certain general characteristics of these wave fronts.

Let us use $\bar{\varphi} = P(\varphi)$ to designate the space component of the discontinuity ϕ (21.4); $\bar{\phi}$ may be singularly decomposed locally over (π) and the normal direction to (π) with the aid of a projector:

$$\hat{P}^{\beta}_{\alpha} = \bar{g}^{\beta}_{\alpha} + n_{\alpha}n^{\beta}, \quad \hat{P}^{\beta}_{\alpha}n^{\alpha} = 0, \quad \hat{P}^{\beta}_{\alpha}u^{\alpha} = 0. \qquad 21.6$$

Specifically, the component $\hat{\varphi} = \hat{P}(\bar{\varphi})$ of $\bar{\phi}$ in (π) is called the transverse component of $\bar{\phi}$, and its component along the normal to (π) its longitudinal component. Let it be, for example, a vector discontinuity $\bar{\phi}^{\alpha}$; it may be locally decomposed into a transverse discontinuity $\overset{T}{\phi}{}^{\alpha}$ and a longitudinal discontinuity $\overset{L}{\phi}{}^{\alpha}$, defined by:

$$\bar{\varphi}^{\alpha} = \overset{T}{\varphi}{}^{\alpha} + \overset{L}{\varphi}{}^{\alpha}, \quad u_{\alpha}\bar{\varphi}^{\alpha} = 0, \quad n_{\alpha}\overset{T}{\varphi}{}^{\alpha} = 0, \quad n^{[\alpha}\overset{L}{\varphi}{}^{\beta]} = 0, \qquad 21.7$$

or manifestly $\overset{T}{\varphi}{}^{\beta} = (g^{\alpha}_{\beta} + n_{\alpha}n^{\beta})\bar{\varphi}^{\alpha}$ et $\overset{L}{\varphi}{}^{\beta} = -n_{\alpha}n^{\beta}\bar{\varphi}^{\alpha}$. These ideas will be useful when we study the independent propagation of longitudinal and transverse wave fronts.

Following one of A. Lichnerowicz's[29] results, we introduce the ideas of the velocity v of propagation of the wave front (π) by the relation:

$$v^2 = (-\bar{l}_{\alpha}\bar{l}^{\alpha})^{-1}(l_{\beta}u^{\beta})^2. \qquad 21.8$$

According to relativistic ideas, such a velocity cannot be greater than the unit (speed of light in a void):

$$v^2 \leqslant 1. \qquad 21.9$$

This requirement may be expressed equivalently with the aid of the following postulate:

Proposition. Every characteristic hypersurface of a Cauchy problem in general relativity is rigorously non-oriented in space:

$$l_{\alpha}l^{\alpha} \geqslant 0. \qquad 21.10$$

[29] Cf. A. Lichnerowicz (1955), Chapter II.

The introduction of these ideas will enable us directly to undertake the study of the phenomenon of propagation of wave fronts in thermodynamic continuous media.

22. Study of the Characteristic Variations and Determination of the Equation of Propagation of Thermodynamic Wave Fronts.

The study of the characteristic variations of a thermodynamic continuous medium introduces, on the one hand, the conditions of geometrical compatibility of the Eq. (21.4) type of the quantity discontinuities involved, and, on the other, the conditions of dynamic compatibility imposed upon these discontinuities by the ensemble of equations governing the medium under consideration; when the linear and homogeneous system of equations related to the ensemble of these discontinuities is regular, they are also null, and the initial hypersurface is not characteristic; therefore, we now propose explicitly to write such a system in the case of the thermodynamic continuous media under consideration.

a. First of all, let us write the conditions of geometrical compatibility of the quantity discontinuities which intervene in this problem:

$$[\partial_{\gamma\delta} g_{\alpha\beta}] = a_{\alpha\beta} l_\gamma l_\delta, \quad [\nabla_\alpha \rho] = a l_\alpha, \quad [\nabla_\alpha u^\beta] = b^\beta l_\alpha,$$
$$[P(\nabla_{(\alpha)} q^\beta)] = c^\beta l_\alpha, \quad [P(\nabla_{(\alpha)} \theta^{\beta\gamma})] = t^{\beta\gamma} l_\alpha, \quad [T^{-1} \dot{\nabla}_\alpha \dot{T}] = t l_\alpha, \quad [\nabla_\alpha S] = s l_\alpha \qquad 22.1$$

where every sub- and superscript in parentheses is withdrawn from the action of the space projection operator P. By virtue of the unitary nature of u^α, and, on the other hand, as a result of the spatial character of q^α and $\theta^{\alpha\beta}$, the discontinuities (b^α, c^α, $t^{\alpha\beta}$) are space quantities:

$$u_\alpha b^\alpha = 0, \qquad 22.2\text{-a}$$

$$u_\alpha c^\alpha = 0, \qquad 22.2\text{-b}$$

$$u_\alpha t^{\alpha\beta} = 0 \qquad 22.2\text{-c}$$

and we have manifestly:

$$[\nabla_\alpha q^\beta] = (c^\beta - q^\gamma b_\gamma u^\beta) l_\alpha \qquad 22.3\text{-a}$$

$$[\nabla_\alpha \theta^{\beta\gamma}] = (t^{\beta\gamma} - 2u^{(\beta}\theta^{\gamma)\delta} b_\delta) l_\alpha . \qquad 22.3\text{-b}$$

B. By reason of the developments of paragraph 20, we assume that the ensemble of compatibility conditions which the preceding discontinuities must satisfy are derived from the following hypotheses:

(i) The continuity of Einstein's tensor $G^{\alpha\beta}$, a consequence of the postulated continuity of $T^{\alpha\beta}$ and of the field equations Eq. (2.5) means:

$$[G^{\alpha\beta}] = 0. \qquad 22.4$$

(ii) The continuity of the first member of the conservation equations Eq. (16.2), a consequence of the structure of the field equations requires, on the other hand:

$$\begin{aligned}[\nabla_\beta T^\beta_\alpha] = [(\mathcal{L}\rho - \nabla_\beta q^\beta + q^\beta \dot{u}_\beta + 1/2\theta^{\beta\gamma}\mathcal{L}\bar{g}_{\beta\gamma}) u_\alpha \\ + \rho \dot{u}_\alpha - \mathcal{L} q_\alpha + 2 q_\beta \Omega^\beta_\alpha - P(\nabla_\beta \theta^\beta_\alpha)] = 0.\end{aligned} \qquad 22.5$$

(iii) The postulated continuity of the rate of change of the total entropy density Eq. (17.2), assumed as a consequence of the second principle of thermodynamics, imposes, moreover:

$$\begin{aligned}[\nabla_\beta S^\beta_\alpha] = [(\mathcal{L}S - \nabla_\beta(T^{-1}q^\beta) + T^{-1}q^\beta \dot{u}_\beta) u_\alpha \\ + S\dot{u}_\alpha - \mathcal{L}(T^{-1}q_\alpha) + 2T^{-1}q_\beta \Omega^\beta_\alpha] = 0.\end{aligned} \qquad 22.6$$

(iv) Finally, the linking equations Eq. (19.1) of the thermodynamic continuous medium requires that the following conditions be confirmed:

$$[P(\mathcal{L}\theta^{\alpha\beta}) - 1/2C^{\alpha\beta\gamma\delta}\mathcal{L}\bar{g}_{\gamma\delta} - L^{\alpha\beta}T^{-1}\dot{T}] = 0. \qquad 22.7\text{-a}$$

$$[T\mathcal{L}S - 1/2L^{\alpha\beta}\mathcal{L}\bar{g}_{\alpha\beta} - CT^{-1}\dot{T}] = 0. \qquad 22.7\text{-b}$$

The 25 discontinuities Eq. (22.1) must therefore satisfy the entire 25 conditions Eq. (22.4-7).

c. The discontinuities $a_{\alpha\beta}$ of the gravitational field satisfy the entire 10 conditions Eq. (22.4) independently of the conditions imposed upon the other discontinuities; these 10 conditions are not independent since, as a result of the identities:

$$\underset{\alpha\beta\gamma}{S}\, l_\alpha [R_{\beta\gamma\delta\epsilon}] = 0, \qquad 22.8$$

satisfied by the discontinuities of the curvature tensor, the discontinuities of Einstein's tensor likewise satisfy the 4 relations:

$$l_\alpha [G^{\alpha\beta}] = 0. \qquad 22.9$$

The investigation of the initial data on (S) compatible with Eq. (22.9) constitute the problem of initial conditions; these conditions show only 6 of the conditions Eq. (22.4) as independent, those which introduce only the discontinuity components of the gravitational field on the initial hypersurface (S); only these components thus appear able to have a physical significance; they are:

$$[G^{\alpha\beta}] = -1/2 l^\gamma l_\gamma (a'^{\alpha\beta} - 1/2 g'^{\alpha\beta} a'^{\delta}_{\delta}), \quad g'_{\alpha\beta} l^\alpha = 0, \quad a'_{\alpha\beta} l^\alpha = 0, \qquad 22.10$$

where $g'^{\alpha\beta}$ and $a'_{\alpha\beta}$ designate respectively the components of $g^{\alpha\beta}$ and $a_{\alpha\beta}$ in (S). The conditions Eq. (22.4) are likewise confirmed if $a'_{\alpha\beta} = 0$, i.e., if the gravitational field is not discontinued on traversing the hypersurface (S). If in return, one of the discontinuities $a'_{\alpha\beta}$ is $\neq 0$, S is the characteristic variation of the problem presented, and the conditions Eq. (22.4), in that event, impose the condition:

$$l^\alpha l_\alpha = 0; \qquad 22.11\text{-a}$$

the vector field $\vec{l}$ having been assumed to be of the integrable type Eq. (21.2), it is easy to show that the trajectories of S are isotropic geodesic lines of V_4:

$$\mathfrak{L}(\vec{l})l_\alpha = 0. \qquad 22.11\text{-b}$$

The wave fronts associated with the characteristic variations Eq. (22.11) tangent to the elementary cone $ds^2 = 0$ are the gravitational wave fronts; we have seen that the characteristics of these fronts do not appear to depend on the structure of the material media which constitute the field sources.

d. Let us now assume that S is not such a variation, i.e., it does not carry the discontinuities of the gravitational field:

$$l^\alpha l_\alpha \neq 0. \qquad 22.12$$

We are then led to consider the special ensemble of 15 conditions Eq. (22.5-7), which must be satisfied by the 15 discontinuities $(a, b^\alpha, c^\alpha, t^{\alpha\beta}, t, s)$, where b^α, c^α, and $t^{\alpha\beta}$ are space quantities, by virtue of Eq. (22.2).

We are going to show that such a study is reduced to the consideration of a system of 5 conditions related to the behavior of the 5 discontinuities a, b^α, and t.

The contemplated 15 discontinuities are in effect subjected to the ensemble of the following 15 conditions which are rigorously equivalent to the initial conditions Eq. (22.5-7)[30]:

$$av^2u_\alpha + M_\alpha^\beta b_\beta + vN_\alpha t = 0, \qquad 22.13\text{-a}$$

$$P^\alpha b_\alpha + v^2Ct = 0, \qquad 22.13\text{-b}$$

$$vt^{\alpha\beta} = U^{\alpha\beta\gamma\delta}n_\gamma b_\delta + vL^{\alpha\beta}t, \qquad 22.13\text{-c}$$

[30]These relations are obtained by carrying out the following successive operations: (i) Eliminating in Eq. (22.5) the $[\nabla_\alpha q^\beta]$ with the aid of Eq. (22.6). (ii) Then eliminating the $[\nabla_\alpha S]$ in Eq. (22.5-6) with the aid of (22.7-b). (iii) Eliminating $[\nabla_\alpha q^\alpha]$ and $[\nabla_\beta \theta^{\alpha\beta}]$ in the new relations (22.5'-6') with the aid of Eq. (22.7-a) and of the spatial portion of Eq. (22.6').

$$v c_\alpha = V_\alpha^\beta b_\beta + v q_\alpha t, \qquad 22.13\text{-d}$$

$$v T s = W^\alpha b_\alpha + v C t\,; \qquad 22.13\text{-e}$$

v designates the velocity of propagation Eq. (21.8) of the wave front associated with (S), and $\vec{n}$ the unitary space vector Eq. (21.5) which characterizes the direction of propagation of said wave front, and we have noted:

$$M_\alpha^\beta = v n_\gamma(\bar{g}^{\beta\gamma} + \theta^{\beta\gamma} - L^{\beta\gamma}) u_\alpha + v^2((L - TS)\bar{g}_\alpha^\beta + \theta_\alpha^\beta) - C_\alpha^{\gamma\beta\delta} n_\gamma n_\delta - \theta^{\gamma\delta} n_\gamma n_\delta \bar{g}_\alpha^\beta, \qquad 22.14\text{-a}$$

$$N_\alpha = -(vC + q^\beta n_\beta) u_\alpha - (v q_\alpha + n_\beta L_\alpha^\beta), \qquad 22.14\text{-b}$$

$$P^\alpha = v n_\beta (L^{\alpha\beta} - TS\bar{g}^{\alpha\beta}) + 2 n_\beta q^\beta n^\alpha + 2 v^2 q^\alpha, \qquad 22.14\text{-c}$$

$$U^{\alpha\beta\gamma\delta} = C^{\alpha\beta\gamma\delta} + 2\bar{g}^{\alpha(\gamma}\theta^{\delta)\beta} - \theta^{\alpha\beta}\bar{g}^{\gamma\delta}, \qquad 22.14\text{-d}$$

$$V_\alpha^\beta = vTS\bar{g}_\alpha^\beta - q_\alpha n^\beta - q^\gamma n_\gamma \bar{g}_\alpha^\beta, \qquad 22.14\text{-e}$$

$$W^\alpha = n_\beta (L^{\alpha\beta} - TS\bar{g}^{\alpha\beta}). \qquad 22.14\text{-f}$$

The 5 conditions Eq. (22.13-a,b) govern the behavior of the 5 discontinuities (a, t, b^α) independently of the value of the remaining discontinuities; it follows that if said 5 discontinuities are null, the 10 discontinuities ($t^{\alpha\beta}$, c^α, s) are also annulled by virtue of Eq. (22.13-c, d, e) and thus the Cauchy problem may be presented in respect to (S).

If, on the other hand, one of the discontinuities (a, t, b^α) is $\neq 0$, the initial hypersurface (S) is a characteristic variation; we are thus led to the conclusion that the investigation of the characteristic variations of 15-condition system Eq. (22.5-7) linking the 15 discontinuities under consideration reverts to the investigation of the situations for which the 5-conditions system Eq. (22.13-a,b) is noted. By explicitly setting forth the spatial characteristics of the discontinuities b^α with the aid of condition Eq. (22.2-a), we write this system in the form:

$$E^B_A \varphi_B = 0 \qquad (A, \ldots = 0, 1, 2, \ldots, 5), \qquad 22.15$$

or

$$\varphi_\alpha = b_\alpha, \qquad \varphi_4 = a, \qquad \varphi_5 = t, \qquad 22.16\text{-a}$$

$$\begin{array}{lll} E^\beta_\alpha = M^\beta_\alpha, & E^4_\alpha = v^2 u_\alpha, & E^5_\alpha = vN_\alpha, \\ E^\beta_4 = P^\beta, & E^4_4 = 0, & E^5_4 = v^2 C, \\ E^\beta_5 = u^\beta, & E^4_5 = 0, & E^5_5 = 0. \end{array} \qquad 22.16\text{-b}$$

System Eq. (22.15) is singular if:

$$\text{dét}\,(E^B_A) = 1/6!\, \delta^{ABCDEF}_{MNPQRS} E^M_A E^N_B E^P_C E^Q_D E^R_E E^S_F = 0. \qquad 22.17$$

By virtue of Eq. (22.16-b) this condition becomes explicit in the form:

$$v(vC\, \overline{\text{dét}}\,(\overline{M}^\beta_\alpha) - (\overline{\min}\,\overline{M})^\beta_\alpha \overline{N}_\beta P^\alpha) = 0, \qquad 22.18$$

where $\overline{\text{dét}}\,(\overline{M}^\beta_\alpha)$ and $(\overline{\min}\,\overline{M})^\beta_\alpha$ respectively designate the space determinant and the space minor of the matrix 4 x 4 formed by the components of the symmetric space tensor $\overline{M}_{\cdot\alpha}^{\beta}$ [31]:

[31] Cf. Appendix II

$$\overline{\det}\,(\overline{M}^{\beta}_{\alpha}) = 1/3\,!\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} u^{\mu} u_{\alpha} \overline{M}^{\nu}_{\beta} \overline{M}^{\rho}_{\gamma} \overline{M}^{\sigma}_{\delta}, \qquad 22.19\text{-a}$$

$$(\min \overline{M})^{\alpha}_{\mu} = 1/2\,!\,\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} u_{\beta} u^{\nu} \overline{M}^{\rho}_{\gamma} \overline{M}^{\sigma}_{\delta}. \qquad 22.19\text{-b}$$

Thus, only the space quantities appear in the material wave front equation (22.18):

$$\overline{M}^{\beta}_{\alpha} = v^2((\rho - TS)\overline{g}^{\beta}_{\alpha} + \theta^{\beta}_{\alpha}) - C^{\gamma\beta\delta}_{\alpha} n_{\gamma} n_{\delta} - \theta^{\gamma\delta} n_{\gamma} n_{\delta} \overline{g}^{\beta}_{\alpha}, \qquad 22.20$$

$$\overline{N}_{\alpha} = -(vq_{\alpha} + n_{\beta} L^{\beta}_{\alpha}), \qquad 22.21$$

$$P^{\alpha} = vn_{\beta}(L^{\alpha\beta} - TS\overline{g}^{\alpha\beta}) + 2n_{\beta} q^{\beta} n^{\alpha} + 2v^2 q^{\alpha}. \qquad 22.14\text{-c}$$

e. The equation (22.18) is annulled in particular for v = 0, i.e., when:

$$l_{\alpha} u^{\alpha} = 0. \qquad 22.22\text{-a}$$

Since we have assumed $\vec{l}$ to be of the integrable type, we deduce that the trajectories of this hypersurface Eq. (22.22-a) are equation curves:

$$\mathfrak{L} l_{\alpha} = 0. \qquad 22.22\text{-b}$$

The conditions Eq. (22.22) characterize the second type of characteristic variations of the Cauchy problem related to thermodynamic continuous media equations, which are the hypersurfaces tangential to the flow lines or generated by them; the associated wave fronts carry the discontinuities linked to the material medium.

f. Let us suppose that (S) is not such a hypersurface:

$$l_\alpha u^\alpha_s \neq 0. \qquad 22.23$$

Equation (22.18) means that we must also confirm the condition:

$$vC\,\overline{\det}\,(\overline{M}^\beta_\alpha) - (\min\overline{M})^\beta_\alpha \overline{N}_\beta P^\alpha = 0. \qquad 22.24$$

This equation describes the propagation of the thermodynamic wave fronts; it is of the seventh degree in v; we must anticipate what it reveals:

—The propagation of 6 elastic wave fronts, the unfolding of the 3 known wave fronts of the classical theory being due to the existence within the medium under consideration of the privileged direction of the heat flow $\vec{q}$.

—The propagation of a thermal wave front linked to the existence of the heat flow within the medium.

These remarks will be clarified during the particular study of the isotropic media. Let us now bring together the entirely general results established in the course of this paragraph:

Proposition. The characteristic variations of the thermodynamic continuous media equations in general relativity involve:

(i) The hypersurfaces tangential to the elementary cone, propagating at the unit velocity along the isotropic geodesic lines of V_4, which characterize the propagation of the discontinuities of the gravitational field:

$$l^\alpha l_\alpha = 0, \qquad \mathfrak{L}(\overline{l})l_\alpha = 0.$$

(ii) The hypersurfaces tangential to the world lines, or generated by them, which characterize the propagation of discontinuities linked to the continuous medium under consideration,

$$l_\alpha u^\alpha = 0, \qquad \mathfrak{L}l_\alpha = 0.$$

(iii) The hypersurfaces which characterize the 7 thermodynamic wave fronts, having the equation:

$$Cv\,\overline{\det}\,(\overline{M}^\beta_\alpha) - (\overline{\min\,\overline{M}})^\beta_\alpha \overline{N}_\beta P^\alpha = 0,$$

where:

$$\overline{M}_\alpha^\beta = v^2((\rho - TS)\overline{g}_\alpha^\beta + \theta_\alpha^\beta) - C_\alpha^{\gamma\beta\delta} n_\gamma n_\delta - \theta^{\gamma\delta} n_\gamma n_\delta \overline{g}_\alpha^\beta,$$
$$\overline{N}_\alpha = -(vq_\alpha + n_\beta L_\alpha^\beta),$$
$$P^\alpha = vn_\beta(L^{\alpha\beta} - TS\overline{g}^{\alpha\beta}) + 2n_\beta q^\beta n^\alpha + 2v^2 q^\alpha.$$

Chapter V. Application: The Isotropic Thermodynamic Continuous Media

A. Characterization of the Isotropic Media.

In this chapter, we propose to apply the formulism developed with the study of thermodynamic continuous media in general relativity to the particular case of isotropic media.

In this first section we shall introduce the concept of the isotropy of a continuous medium from a particular property of the material wave fronts.

23. Introduction of the Concept of Isotropy

The phenomenon of independent propagation of longitudinal and transverse wave fronts is characteristic of the isotropic media in the classical theory of elasticity; we propose to show that this property suffices to define the isotropic character of an elastic continuous medium in general relativity.

Let us therefore consider an elastic-type medium, i.e., described by the linking equations Eq. (10.1), and let us study the propagation of elastic waves in said media; by restricting the results Eq. (22.13) of the preceding paragraph to the case under consideration, the phenomenon of propagation of elastic waves is governed by the three equations:

$$\overline{M}'^\beta_\alpha b_\beta = 0, \qquad \overline{M}'^\beta_\alpha = v^2(\rho\overline{g}_\alpha^\beta + \theta_\alpha^\beta) - C_\alpha^{\gamma\beta\delta} n_\gamma n_\delta - \theta^{\gamma\delta} n_\gamma n_\delta \overline{g}_\alpha^\beta. \qquad 23.1$$

By definition, we say that there is independent propagation of longitudinal and transverse wave fronts if the three conditions Eq. (23.1) la d to the discontinuities b_α are separable into a group of two conditions which introduce the only transverse discontinuities $\overset{T}{b}_\alpha$, and one condition related to the longitudinal discontinuity $\overset{L}{b}_\alpha$, where, by virtue of the defi s in paragraph 21 these quantities are characterized by the relations:

$$b_\alpha = \overset{T}{b}_\alpha + \overset{L}{b}_\alpha, \qquad n^\alpha \overset{T}{b}_\alpha = 0, \qquad n_{[\alpha}\overset{L}{b}_{\beta]} = 0. \tag{23.2}$$

The three equations (23.1) are equivalent to the group:

$$n^\alpha \overline{M}'^{\beta}_{\alpha}\overset{L}{b}_\beta + n^\alpha \overline{M}'^{\beta}_{\alpha}\overset{T}{b}_\beta = 0. \tag{23.3-a}$$

$$n_{[\alpha}\overline{M}'_{\beta]}{}^{\gamma}\overset{L}{b}_\gamma + n_{[\alpha}\overline{M}'_{\beta]}{}^{\gamma}\overset{T}{b}_\gamma = 0. \tag{23.3-b}$$

The postulate of the independent propagation of wave fronts requires that the system be reduced to:

$$n^\alpha \overline{M}'^{\beta}_{\alpha}\overset{L}{b}_\beta = 0, \tag{23.4-a}$$

$$n_{[\alpha}\overline{M}'_{\beta]}{}^{\gamma}\overset{T}{b}_\gamma = 0, \tag{23.4-b}$$

i.e., by virtue of the arbitrary nature of the discontinuities b_α and of the symmetry of the space tensor $\overline{M}'^{\beta}_{\alpha}$, that the following conditions be also confirmed:

$$n_{[\alpha}\overline{M}'_{\beta]}{}^{\gamma}n_\gamma = 0. \tag{23.5}$$

We are thus led to investigate the structure of the tensors $\overline{M}'^{\beta}_{\alpha}$ of the Eq. (23.5) type which satisfy Eq. (23.5), i.e., such as:

$$n_{[\alpha}(\theta_{\beta]}{}^{\gamma}n_\gamma v^2 - C_{\beta]}{}^{\gamma\delta\epsilon}n_\gamma n_\delta n_\epsilon) = 0. \tag{23.6}$$

As these conditions must also be confirmed by each of the velocities v of wave fronts propagation it is also required that the two types of conditions be carried out simultaneously:

$$n_{[\alpha}\theta_{\beta]}{}^{\gamma}n_\gamma = 0. \tag{23.7-a}$$

$$n_{[\alpha}C_{\beta]}{}^{\gamma\delta\epsilon}n_{\gamma}n_{\delta}n_{\epsilon} = 0. \qquad 23.7\text{-b}$$

(i) The conditions Eq. (23.7-a) mean that $\vec{n}$ is the eigenvector of the constraint density $\theta^{\alpha\beta}$, i.e., it is one of the three principal constraint vectors $\vec{e}$ defined by:

$$\theta^{\alpha\beta} = \eta^{ij} \underset{i}{p} \underset{i}{e^{\alpha}} \underset{j}{e^{\beta}}, \qquad \bar{g}_{\alpha\beta} \underset{i}{e^{\alpha}} \underset{j}{e^{\beta}} = \eta_{ij}, \qquad \eta_{ij} = -\delta_{ij}. \qquad 23.8$$

We are therefore led to present:

$$n_{\alpha} = \underset{i}{e_{\alpha}}. \qquad 23.9\text{-a}$$

(ii) That being the case, the density of modulus of elasticity $C^{\alpha\beta\gamma\delta}$ must satisfy the two conditions Eq. (23.7-b); bearing in mind the symmetry conditions Eq. (10.4-a) of this quantity, a purely algebraic study shows that the most general structure of $C^{\alpha\beta\gamma\delta}$ which satisfy such conditions becomes explicit in the form[32]:

$$C^{\alpha\beta\gamma\delta} = \sum_{\substack{k \\ (i\neq j\neq k)}} \{ \underset{k}{a} \underset{k}{e^{\alpha}} \underset{k}{e^{\beta}} \underset{k}{e^{\gamma}} \underset{k}{e^{\delta}} + \underset{k}{b}(\underset{i}{e^{\alpha}} \underset{i}{e^{\beta}} \underset{j}{e^{\gamma}} \underset{j}{e^{\delta}} + \underset{j}{e^{\alpha}} \underset{j}{e^{\beta}} \underset{i}{e^{\gamma}} \underset{i}{e^{\delta}}) \qquad 23.9\text{-b}$$
$$+ \underset{k}{c}(\underset{i}{e^{(\alpha}} \underset{j}{e^{\beta)}} \underset{i}{e^{(\gamma}} \underset{j}{e^{\delta)}}) + \underset{k}{d}(\underset{k}{e^{(\alpha}} \underset{i}{e^{\beta)}} \underset{k}{e^{(\gamma}} \underset{j}{e^{\delta)}} + \underset{k}{e^{(\alpha}} \underset{j}{e^{\beta)}} \underset{k}{e^{(\gamma}} \underset{i}{e^{\delta)}}) + \underset{k}{e}(\underset{k}{e^{\alpha}} \underset{k}{e^{\beta}} \underset{i}{e^{(\gamma}} \underset{j}{e^{\delta)}} + \underset{i}{e^{(\alpha}} \underset{j}{e^{\beta)}} \underset{k}{e^{\gamma}} \underset{k}{e^{\delta}})\}.$$

The algebraic expressions Eq. (23.9) are such that the conditions Eq. (23.7) of independent wave front propagation are likewise satisfied; however, the structure of the media studied are not entirely determined; in effect, by virtue of the integrability conditions Eq. (10.5) of the linking equations, the 15 scalar densities $(\underset{i}{a}, \underset{i}{b}, \underset{i}{c}, \underset{i}{d}, \underset{i}{e})$ are not independent; the conditions imply that the densities $C^{\alpha\beta\gamma\delta}$ which belong to the Eq. (23.9-b) are not necessarily of the type:

$$C^{\alpha\beta\gamma\delta} = -\lambda \bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta} - 2\mu \bar{g}^{\alpha(\gamma}\bar{g}^{\delta)\beta}, \qquad 23.10\text{-a}$$

where:

$$\lambda \bar{g}^{\alpha\beta} = -2\delta\mu/\delta g_{\alpha\beta}; \qquad \delta\lambda/\delta g_{\alpha\beta}\bar{g}^{\gamma\delta} = \delta\lambda/\delta g_{\gamma\delta}\bar{g}^{\alpha\beta}. \qquad 23.10\text{-b}$$

[32] In the case of more general types of continuous media, the two necessary and sufficient conditions Eq. (23.9) to have independent propagation of longitudinal and transverse wave fronts involve the remarkable peculiarity that each of the transverse wave propagates at a different velocity; cf. J.-F. Bennoun (1964).

By virtue of the Eq. (23.10-b) the structure coefficients can be expressed from a single independent variable, for example, the density of mass p; if we use the consequence of the conservation of energy equation:

$$\mu \bar{g}^{\alpha\beta} = -2\bar{\delta}\rho/\delta g_{\alpha\beta}; \qquad 23.11$$

the desired link is written:

$$\lambda = -\mu + (\rho + \mu)\partial\mu/\partial\rho, \qquad 23.12$$

where:

$$\partial\mu/\partial\rho = \delta(\mu/\sqrt{-g})/\delta(\rho/\sqrt{-g}). \qquad 23.13$$

The medium described is a perfect fluid[33]:

$$\theta^{\alpha\beta} = \mu \bar{g}^{\alpha\beta}. \qquad 23.14$$

It follows that the tensor $\overline{M}_{\alpha}^{\cdot\beta}$ Eq. (23.1) assumes the form:

$$\overline{M}'^{\beta}_{\alpha} = (\rho + \mu)(v^2 \bar{g}^{\beta}_{\alpha} + \partial\mu/\partial\rho\, n_\alpha n^\beta), \qquad 23.15$$

which, from Eq. (23.4), immediately involves the well known result that there is propagation of a single longitudinal wave in the elastic medium under consideration, with the velocity:

$$v_0 = (\partial\mu/\partial\rho)^{1/2}. \qquad 23.16$$

Thus, the mere postulate of the independent propagation of longitudinal and transverse wave fronts suffices fully to define the isotropic character of a medium:

Proposition. Given an adiabatic elastic medium in general relativity, there is a rigorous equivalence between the two hypotheses:

(i) The longitudinal and transverse wave fronts propagate independently:

$$n^{[\alpha}\Theta^{\beta]\gamma}n_{\gamma} = 0, \qquad n^{[\alpha}C^{\beta]\gamma\delta\epsilon}n_{\gamma}n_{\delta}n_{\epsilon} = 0.$$

(ii) The medium is isotropic:

$$C^{\alpha\beta\gamma\delta} = (p - (\rho + p)\partial p/\partial\rho)\bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta} - 2p\bar{g}^{\alpha(\gamma}\bar{g}^{\delta)\beta}, \qquad \Theta^{\alpha\beta} = p\bar{g}^{\alpha\beta}.$$

24. Structure of the Isotropic Thermodynamic Continuous Media.

a. Expanding the preceding result, we define the isotropic character of the thermodynamic continuous media with the aid of the conditions:

$$C^{\alpha\beta\gamma\delta} = -\lambda\bar{g}^{\alpha\beta}\bar{g}^{\gamma\delta} - 2\mu\bar{g}^{\alpha(\gamma}\bar{g}^{\delta)\beta}. \qquad 24.1\text{-a}$$

$$L^{\alpha\beta} = L\bar{g}^{\alpha\beta}. \qquad 24.1\text{-b}$$

By virtue of the equations Eq. (19.2) and Eq. (19.5), consequences of the linking equations and their integrability conditions, the four structure coefficients of the medium under consideration are linked by the ensemble of relations:

$$\lambda\bar{g}^{\alpha\beta} = -2\delta\mu/\delta g_{\alpha\beta}. \qquad 24.2\text{-a}$$

$$L = T\delta\mu/\delta T. \qquad 24.2\text{-b}$$

$$L\bar{g}^{\alpha\beta} = 2T\delta S/\delta g_{\alpha\beta}. \qquad 24.2\text{-c}$$

---

[33] Bernstein (1960) established the following extremely restrictive result in classical theory: The only elastic media which can be described by variation of state equations, and compatible with the known quantity of an elastic energy density, are the isotropic media; such a result cannot however be retained, since it is based on an unsuitable choice of Truesdell's (1955) equations of the variations of [illegible] of elastic media.

$$C = T^2 \delta S/\delta T. \qquad 24.2\text{-d}$$

$$T\delta\lambda/\delta T\bar{g}^{\alpha\beta} = -2\bar{\delta}L/\delta g_{\alpha\beta}. \qquad 24.2\text{-e}$$

$$T^2\delta(T^{-1}L)/\delta T\bar{g}^{\alpha\beta} = 2\bar{\delta}C/\delta g_{\alpha\beta}. \qquad 24.2\text{-f}$$

$$\bar{\delta}\lambda/\delta g_{\alpha\beta}\bar{g}^{\gamma\delta} = \bar{\delta}\lambda/\delta g_{\gamma\delta}\bar{g}^{\alpha\beta}. \qquad 24.2\text{-g}$$

Moreover, the equations (18.12) derived from the conservation of energy equation enables us to link the density of mass to these coefficients:

$$\delta\rho/\delta T = T\delta S/\delta T. \qquad 24.3\text{-a}$$

$$\bar{\delta}\rho/g_{\alpha\beta} = T\bar{\delta}S/\delta g_{\alpha\beta} - 1/2\mu\bar{g}^{\alpha\beta}. \qquad 24.3\text{-b}$$

By integration of the linking equations, we deduce that the medium thus described is a perfect fluid:

$$\theta^{\alpha\beta} = p\bar{g}^{\alpha\beta}, \qquad 24.4\text{-a}$$

where the scalar pressure density p is defined by the relation:

$$p = \mu, \qquad 24.4\text{-b}$$

which justified a posteriori the adopted definition of the isotropy of the thermodynamic continuous media.

b. The isotropic thermodynamic continuous media are thus described with the aid of two independent parameters of state; here we choose as parameters the temperature T, and the scalar V defined by the relation:

$$V = \exp \int \theta ds \qquad 24.5$$

where the integral is extended all along a world line, in such a way that we have:

$$2V^{-1}\bar{\delta}V/\delta g_{\alpha\beta} = \bar{g}^{\alpha\beta}. \qquad 24.6$$

That [being] the case, the structure coefficients of the medium are expressed from the scalar pressure density p by the relations:

$$\mu = p, \qquad 24.7\text{-a}$$

$$\lambda = -\partial(pV)/\partial V \qquad 24.7\text{-b}$$

$$L = T\partial p/\partial T \qquad 24.7\text{-c}$$

$$C = V^{-1}\int T^2\partial^2 p/\partial T^2 dV \qquad 24.7\text{-d}$$

and the densities of mass (energy) and entropy of said medium are deduced from p with the aid of the integrals:

$$\rho = V^{-1}\int T^2\partial(pT^{-1})/\partial T dV \qquad 24.8\text{-a}$$

$$S = V^{-1}\int \partial p/\partial T dV. \qquad 24.8\text{-b}$$

c. We deduce that the free enthalpy density attached to the isotropic medium under consideration assumes the expression:

$$\rho + p - TS = V^{-1}\int V\partial p/\partial V\, dV \qquad 24.9$$

and that the dilation heat density is written:

$$L - TS = -T\partial/\partial T\left(V^{-1}\int V\partial p/\partial V\, dV\right) \qquad 24.10$$

where, by definition:

$$\int V\partial p/\partial V\, dV = \sqrt{-g}\int V\partial/\partial V(p/\sqrt{-g})\, dV.$$

When the conditions of state are not dependent on V, the second members of these last two expressions are also annulled, and we can then say that the system under consideration varies reversibly according to a succession of states extremely close to thermodynamic equilibrium. This situation occurs, particularly, in the case of rigid motions of the medium under consideration.

In the following paragraph we shall set forth the structure of the two classes of insotropic media of this type.

25. Two Particular Classes of Isotropic Thermodynamic Continuous Media

a. Let us propose to set forth the characteristics of the isotropic media, since these can be described with the aid of a single independent variable; the examination of the results of the preceding paragraph immediately reveals that the state of such a medium depends only on its temperature T.

(i) The structure coefficients of the isotropic medium under consideration, expressed, for instance, from the pressure p, are written:

$$\mu = p \qquad 25.1\text{-a}$$

$$\lambda = -p \qquad 25.1\text{-b}$$

$$L = T\delta p/\delta T \tag{25.1-c}$$

$$C = T^2\delta^2 p/\delta T^2. \tag{25.1-d}$$

(ii) The solution to the linking equations is written:

$$\rho = T^2\delta/\delta T(T^{-1}p), \tag{25.2-a}$$

$$S = \delta p/\delta T. \tag{25.2-b}$$

As we have just seen, the five conditions of state (ρ, p, T, S, L) are linked by the relations:

$$L - TS = \rho + p - TS = 0. \tag{25.3}$$

A single relations of phenomenological origin suffices perfectly to determine the structure of these media; we shall immediately give two examples of this situation.

(b-i) Let us first of all assume that the densities of pressure and mass are linked by the relation:

$$p/\sqrt{-g} = m(\rho/\sqrt{-g})^n, \quad m = \text{cte}, n = \text{cte}, \ n \neq 1. \tag{25.4}$$

According to Eq. (25.1-2), in this case the equations of state of the medium assume the form:

$$\rho/\sqrt{-g} = (m^{-1}(aT^{(n-1)/n} - 1))^{1/(n-1)} \tag{25.5-a}$$

$$p/\sqrt{-g} = m(m^{-1}(aT^{(n-1)/n} - 1))^{n/(n-1)} \qquad 25.5\text{-b}$$

$$S/\sqrt{-g} = aT^{-1/n}(m^{-1}(aT^{(n-1)/n} - 1))^{1/(n-1)}, \qquad 25.5\text{-c}$$

where a is an integration constant whose meaning is linked to the physical interpretation of the solution.

(ii) Let us study separately the particular case n = 1 which is not included in the preceding study, i.e., we present:

$$p = m\rho, \qquad m = \text{cte.} \qquad 25.6$$

By the same procedure we obtain the equations of state:

$$\rho/\sqrt{-g} = aT^{(1+1/m)} \qquad 25.7\text{-a}$$

$$p/\sqrt{-g} = maT^{(1+1/m)} \qquad 25.7\text{-b}$$

$$S/\sqrt{-g} = (m+1)aT^{1/m}. \qquad 25.7\text{-c}$$

This second type of solutions to the isotropic thermodynamic continuous media equations will subsequently serve us as an example of application of the formalism developed in this work to the study of link between the gravitational field and its sources.

B. Propagation of the Thermodynamic Waves in Isotropic Media.

By means of a particular study of the phenomenon of thermodynamic waves propagation in the case of isotropic continuous media, we propose to set forth certain of the peculiarities of these wave fronts where we had been able to offer only a qualitative aspect in the general case.

26. Propagation of the Thermodynamic Waves in Isotropic Continuous Media.

We refer to the thermodynamic wave equation:

$$v C \,\overline{\det}\,(\overline{M}^{\beta}_{\alpha}) - (\overline{\min}\ \overline{M})^{\beta}_{\alpha} \overline{N}_{\alpha} P^{\beta} = 0, \qquad 22.24$$

where, by virtue of Eq. (24.1) the quantities Eq. (22.20), Eq. (22.21) and Eq. (22.14-c) assume the particular expressions:

$$\overline{M}^{\beta}_{\alpha} = (\rho + p - TS)(v^2 g^{\beta}_{\alpha} + v_0^2 n_{\alpha} n^{\beta}), \qquad 26.1\text{-a}$$

$$\overline{N}_{\alpha} = -(v q_{\alpha} + L n_{\alpha}), \qquad 26.1\text{-b}$$

$$P^{\alpha} = v(L - TS) n^{\alpha} + 2 n_{\beta} q^{\beta} n^{\alpha} + 2 v^2 q^{\alpha}, \qquad 26.1\text{-c}$$

bearing in mind the relation drawn from Eq. (24.2,3):

$$\lambda = -p + (\rho + p - L)\partial p / \partial \rho, \qquad 26.2$$

and where we have noted:

$$v_0^2 = (\rho + p - L)/(\rho + p - TS)\partial p / \partial \rho. \qquad 26.3$$

We immediately see that when the medium varies according to a succession of states of thermodynamic equilibrium, the wave equation (22.24) is also confirmed, which means that in said case there is no more thermodynamic wave propagation.

If we discard this situation, we can write the thermodynamic wave equation with the following form, after eliminating the superfluous factor $(\rho + p - TS)v^4$ [34]:

$$\begin{aligned} C(\rho + p - TS)v(v^2 - v_0^2) - L(L - TS)v + 2L n_{\alpha} q^{\alpha}(v^2 - 1) \\ + (L - TS) n_{\alpha} q^{\alpha} v^2 + 2 q_{\alpha} q^{\alpha} v(v^2 - v_0^2) + 2(n_{\alpha} q^{\alpha}) v(1 - v_0^2) = 0. \end{aligned} \qquad 26.4$$

[34] Cf. Appendix II

This equation is of the third degree in v, where as the general equation is of the seventh degree: we immediately interpret this fact by saying that the isotropy causes the disappearance of the transverse wave fronts, a known result in the hydrodynamic theory of perfect fluids. The equation Eq. (26.4) must therefore enable us to set forth two elastic wave propagation velocities, the difference between these two waves being due to the existence of the privileged direction of the heat flow vector $\vec{q}$ within the medium, and a thermal wave propagation velocity.

In order to set forth these characteristics and to give an order of magnitude of the propagation velocities of these different types of waves, let us place ourselves in a position close to the thermodynamic equilibrium, whereby we are justified in disregarding the quadratic terms in $|\vec{q}|$ in Eq. (26.4). Bearing this hypothesis in mind, we readily obtain the desired values of the three velocities of thermodynamic wave propagation in the isotropic media in the first order of approximation (which we indicate with the symbol ~):

$$v \sim \pm v'_0 - (L(3 - 2/v'^2_0) - TS)/2C(\rho + p - TS).n^\alpha q_\alpha. \qquad 26.5\text{-a}$$

$$v \sim - (2L/v'^2_0)/C(\rho + p - TS).n^\alpha q_\alpha \qquad 26.5\text{-b}$$

where:

$$v'^2_0 = v^2_0 + L(L - TS)/C(\rho + p - TS) \qquad 26.6$$

Eq. (26.5-a) actually shows the separation of the elastic wave front into two distinct wave fronts, due to the existence of the heat flow, and indicates the correction to be made in the velocity $v_{00} = \pm(\partial p/\partial \rho)^{1/2}$ of the hydrodynamic theory of perfect fluids; let us remark that this separation does not take place, at least in the order of approximation adopted, when the heat flow propagates transversally, i.e., according to a direction situated in the two-plane orthogonal space to the direction of wave propagation $\vec{n}$ Eq. (26.5-b) indicates the value of the thermal wave front propagation velocity; this type of wave disappears during a transverse propagation of the heat flow within the medium.

Remark: This result eliminates one of the great difficulties presented by Pham Mau Quan's theory, a difficulty which is probably due to the fact that this author had been constrained, during the study of the Cauchy problem, to introduce a superfluous variable Z = T; this author, in effect, revealed the existence of a second type of thermal waves propagating at extremely high velocity inversely proportional to $an^{\alpha}q_{\alpha}$, this velocity becoming notably infinite in the case of adiabatic motions of the medium under consideration; such a result is in absolute contradiction with relativistic ideas, and it is therefore gratifying to note that, according to the hypotheses adopted in this work, this type of thermal waves does not exist.

C. The Stefan-Boltzmann World

In closing this work we shall seek to characterize the structure of the gravitational field created by a continuous medium which forms part of a particular class of isotropic thermodynamic continuous media defined above.

27. Position of the Problem

Let us consider the particular class of isotropic thermodynamic media described with the aid of equations (25.7); applying the preceding general considerations, we propose to study the structure of the gravitational field created by such a source.

We are limiting the scope of this study by presenting the following hypotheses:

(i) The state of the medium is adiabatic: It is therefore described with the aid of the pulse rate-energy:

$$T^{\alpha\beta} = \sqrt{-g}\,aT^{n}(u^{\alpha}u^{\beta} - 1/(n-1)\bar{g}^{\alpha\beta}). \qquad 27.1$$

By reason of the finite character of the hydrodynamic wave propagation velocity the only admissible values of n are found in the domain:

$$n \geqslant 2. \qquad 27.2$$

(ii) The world lines form a congruence normal to the triplanar field of space directions:

$$\Omega\alpha\beta = 0. \qquad 27.3$$

(iii) This congruence is without distortion:

$$\sigma_{\alpha\beta} = E_{\alpha\beta} - 1/3\,\bar{g}_{\alpha\beta} = 0 \qquad (\sigma^{\alpha}_{\alpha} = 0). \tag{27.4}$$

a. The writing of the gravitational field equations related to such a source calls for the decomposition of the curvature tensor of V_4 in terms of space and time:

$$P(R_{\alpha\beta\gamma\delta}) = \bar{R}_{\alpha\beta\gamma\delta} + 2\Omega_{\alpha\beta}\Omega_{\gamma\delta} - 2\Omega_{\alpha[\gamma}\Omega_{\delta]\beta} + 2E_{\alpha[\gamma}E_{\delta]\beta}. \tag{27.5-a}$$

$$P(u^{\delta}R_{\alpha\beta\gamma\delta}) = 2\bar{\nabla}_{[\beta}(E_{\alpha]\gamma} + \Omega_{\alpha]\gamma}) - 2\dot{u}_{\gamma}\Omega_{\alpha\beta}. \tag{27.5-b}$$

$$u^{\gamma}u^{\delta}R_{\alpha\gamma\beta\delta} = -\bar{\nabla}_{(\alpha}\dot{u}_{\beta)} + \dot{u}_{\alpha}\dot{u}_{\beta} + \mathcal{L}E_{\alpha\beta} - (E^{\gamma}_{\alpha} + \Omega^{\gamma}_{\alpha})(E_{\beta\gamma} + \Omega_{\beta\gamma}), \tag{27.5-c}$$

where $\bar{R}_{\alpha\beta\gamma\delta}$ is a tensor generalizing the space curvature tensor, the latter being defined only in the case of integrable space sections[35].

The desired equations are written:

$$[\bar{R}_{\alpha\beta} + \bar{\nabla}_{(\alpha}\dot{u}_{\beta)} - \dot{u}_{\alpha}\dot{u}_{\beta}]^{+} = 0, \tag{27.6-a}$$

$$\bar{R} - 2/3\,\theta^{2} = -2aT^{n}, \tag{27.6-b}$$

---

[35] $\bar{R}_{\alpha\beta\gamma\delta}$ has the following properties: 1. It is a space tensor. 2. It has the symmetry properties of a Riemann variation of curvature tensor. 3. When Σ is an integrable space section ($\Omega_{\alpha\beta} = 0$), it is reduced to the curvature tensor Σ. Finally, when the motion of Σ is rigid ($\mathcal{L}\bar{g}_{\alpha\beta} = 0$), $\bar{R}_{\alpha\beta\gamma\delta}$ is invariant during the motion of Σ ($\mathcal{L}\bar{R}_{\alpha\beta\gamma\delta} = 0$).

Note added to the galley proof: The tensor $\bar{R}_{\alpha\beta\gamma\delta}$ has also been set forth by G. FERRARESE, Rend. Mat. (1,2), 24 (1965), p. 57.

$$\bar{\partial}_\alpha \theta = 0, \tag{27.6-c}$$

$$\bar{\nabla}_\alpha \dot{u}^\alpha - \ddot{u}_\alpha \dot{u}^\alpha - \dot{\theta} - 1/3\theta^2 = (n+2)/2(n-1)a\mathrm{T}^n, \tag{27.6-d}$$

the mark [ ]+ designating the traceless portion of the expression situated in brackets, and the mark $\cdot$ the operator $(u^\alpha \nabla_\alpha)$.

b. The field equations Eq. (27.6) must be compatible with the four equations of conservation of $T^{\alpha\beta}$:

$$\theta = -(n-1)\dot{\mathrm{T}}/\mathrm{T}, \tag{27.7-a}$$

$$\dot{u}_\alpha = \bar{\partial}_\alpha \mathrm{T}/\mathrm{T}. \tag{27.7-b}$$

c. Let us consider the decomposition of the metric of V_4 into a sum of squares:

$$ds^2 = -\sum_{i=1}^{3}(\theta^i)^2 + (\theta^0)^2 = \overline{ds}^2 + (u_\alpha dx^\alpha)^2. \tag{27.8}$$

The equations Eq. (27.3) and Eq. (27.7) are the necessary and sufficient conditions of integrability of the Pfaff form $\theta^\circ = u_\alpha dx^\alpha$; it follows that the 3 planes Σ are of the integrable type and may be provided with a Riemann variation structure defined by the metric $\overline{ds}^2$, whose curvature tnesor is the tensor $\bar{R}_{\alpha\beta\gamma\delta}$ introduced above[36]. The form of the conservation equations (27.7) enables us to determine the choice of the metric Eq. (27.8) of the type[36]:

$$ds^2 = -\mathrm{T}^{-2(n-1)/3}\widetilde{ds}^2 + \mathrm{T}^{-2}dt^2. \tag{27.9}$$

28. Structure of the Gravitational Field.

We propose to show that the structure of the field equations introduce the following consequences:

[36]Cf. M. Trümper (1962).

A. The space Σ provided with the metric $\overline{ds}^2$ is isotropic.

B. The Riemann variation V_4 is in agreement with Euclidian theory.

C. The world lines describe geodesic lines of $V_4$1.

That being the case, we shall show that the group of field equations is reduced to a single differential equation which we shall study in a particular case.

a. In order to establish (A) it is necessary to separate two preliminary results.

(i) From the general formula:

$$\mathcal{L}\overline{\nabla}_\alpha \dot{u}_\beta = \overline{\nabla}\mathcal{L}_\alpha \dot{u}_\beta - \dot{u}_\alpha \mathcal{L}\dot{u}_\beta + 2\dot{u}_\alpha(E^\gamma_\beta + \Omega^\gamma_\beta)\dot{u}_\gamma - \dot{u}_\gamma P(\mathcal{L}\Gamma^\gamma_{\alpha\beta}) \qquad 28.1$$

and by virtue of the hypotheses Eq. (27.3-4), of Eq. (27.7) and of Eq. (12.5-a) we obtain first of all:

$$\mathcal{L}(\overline{\nabla}_{(\alpha}\dot{u}_{\beta)} - \dot{u}_\alpha \dot{u}_\beta) = 1/(n-1)(\overline{\nabla}_{(\alpha}\dot{u}_{\beta)} - \dot{u}_\alpha\dot{u}_\beta) + 2(n-4)/(n-1)\theta\dot{u}_\alpha\dot{u}_\beta - 1/3\overline{g}_{\alpha\beta}\dot{u}_\alpha\dot{u}_\beta. \qquad 28.2$$

(ii) Moreover, the variation of the space curvature tensor:

$$\overline{R}_{\alpha\beta\gamma\delta} = -(\overline{g}_{\alpha\gamma}\overline{R}_{\beta\delta} + \overline{g}_{\beta\gamma}\overline{R}_{\alpha\delta} - \overline{g}_{\alpha\delta}\overline{R}_{\beta\gamma} - \overline{g}_{\beta\gamma}\overline{R}_{\alpha\delta}) + \overline{g}_{\alpha[\gamma}\overline{g}_{\delta]\beta}\overline{R} \qquad 28.3$$

is written in the particular case under consideration[37]:

$$\mathcal{L}\overline{R}_{\alpha\beta\gamma\delta} = 1/3\theta\overline{R}_{\alpha\beta\gamma\delta} - 1/3\overline{g}_{\alpha[\gamma}\overline{g}_{\delta]\beta}\theta(-2\overline{R} + 4/3\theta^2 + 4/3\dot{\theta} - (n-2)/(n-1)aT^n). \qquad 28.4$$

We deduce:

$$\mathcal{L}\overline{R}_{\alpha\beta} = -1/3\theta\overline{R}_{\alpha\beta} + 2/9\overline{g}_{\alpha\beta}\theta(\theta^2 + 2\dot{\theta} + 3/(n-1)aT^n). \qquad 28.5$$

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[37]In the general case, the variation $\overline{R}_{\alpha\beta\gamma\delta}$ is written:

$$\begin{aligned}\mathcal{L}\overline{R}_{\alpha\beta\gamma\delta} = {} & E_{\lambda[\gamma}\overline{R}^\lambda{}_{\delta]\alpha\beta} + E_{\lambda[\alpha}\overline{R}^\lambda{}_{\beta]\gamma\delta} \\ & + \overline{\nabla}_{(\delta}\overline{\nabla}_{\beta)}E_{\alpha\gamma} + \overline{\nabla}_{(\gamma}\overline{\nabla}_{\alpha)}E_{\beta\delta} - \overline{\nabla}_{(\delta}\overline{\nabla}_{\alpha)}E_{\beta\gamma} - \overline{\nabla}_{(\gamma}\overline{\nabla}_{\beta)}E_{\alpha\delta} \\ & + 2\overline{\nabla}_{[\beta}(E_{\alpha][\delta}\dot{u}_{\gamma]}) + 2\overline{\nabla}_{[\gamma}(E_{\delta][\alpha}\dot{u}_{\beta]}) \\ & + 2\overline{\nabla}_{[\beta}(E_{\alpha][\delta})\dot{u}_{\gamma]} + 2\overline{\nabla}_{[\gamma}(E_{\delta][\alpha})\dot{u}_{\beta]} \\ & + 4\dot{u}_{[\alpha}E_{\beta][\delta}\dot{u}_{\gamma]} + 2E_{\lambda[\delta}E_{\gamma][\beta}\Omega_{\alpha]}{}^\lambda + 2E_{\lambda[\beta}E_{\alpha][\delta}\Omega_{\gamma]}{}^\lambda.\end{aligned}$$

(iii) With the aids of the results Eq. (28.2-5) we obtain by variation of the first five field equations Eq. (27.6-a):

$$[4(n-4)/(n-1)\dot{u}_\alpha \dot{u}_\beta + (n+2)/3\overline{R}_{\alpha\beta}]^+ = 0. \qquad 28.6$$

By once again changing Eq. (28.6), and by comparison with the initial equation, we obtain the group:

$$(n+2)(n+5)[\overline{R}_{\alpha\beta}]^+ = 0, \qquad 28.7\text{-a}$$

$$(n-4)(n+5)[\dot{u}_\alpha \dot{u}_\beta]^+ = 0. \qquad 28.7\text{-b}$$

Bearing in mind the fundamental hypothesis Eq. (27.2), the equations Eq. (28.7-a) bring in:

$$[\overline{R}_{\alpha\beta}]^+ = 0, \qquad 28.8$$

which proves that space Σ is isotropic.

b. The Euclidian character of V_4 is a result equivalent to the preceding result Eq. (28.8), since according to the adopted hypotheses the Weyl tensor of V_4 assumes the form:

$$C_{\alpha\beta\gamma\delta} = -(\overline{g}_{\alpha\gamma}[\overline{R}_{\beta\delta}]^+ + \overline{g}_{\beta\delta}[\overline{R}_{\alpha\gamma}]^+ - \overline{g}_{\alpha\delta}[\overline{R}_{\beta\gamma}]^+ - \overline{g}_{\beta\gamma}[\overline{R}_{\alpha\delta}]^+) + 4u_{[\alpha}[\overline{R}_{\beta][\delta}]^+ u_{\gamma]} = 0. \qquad 28.9$$

We deduce that the curvature tensor of V_4 is of the type:

$$R_{\alpha\beta\gamma\delta} = 2/3aT^n(\overline{g}_{\alpha[\gamma}\overline{g}_{\delta]\beta} - (n+2)/(n-1)u_{[\alpha}\overline{g}_{\beta][\gamma}u_{\delta]}). \qquad 28.10$$

c. Let us introduce explicitly the space metric $\tilde{ds}^2$ according to $\overline{ds}^2$, defined by Eq. (27.9); by virtue of Eq. (27.7-a) the structure of space $\tilde{\Sigma}$ provided with the metric $\tilde{ds}^2$ is invariant due to the infinitesimal transformation generated by the vector field $\vec{u}$:

$$\mathfrak{L}\tilde{d}s^2 = 0, \tag{28.11}$$

and we therefore have:

$$\mathfrak{L}\tilde{R}_{\alpha\beta} = 0. \tag{28.12}$$

By virtue of a general conformity formula, Ricci's tensor $\tilde{R}_{\alpha\beta}$ of $\tilde{\Sigma}$ assumes the expression:

$$\begin{aligned}\tilde{R}_{\alpha\beta} = \overline{R}_{\alpha\beta} - (n-1)/3\,\overline{\nabla}_{(\alpha}\dot{u}_{\beta)} + (n-1)^2/9\,\dot{u}_\alpha\dot{u}_\beta \\ - \overline{g}_{\alpha\beta}((n-1)/3\,\overline{\nabla}_\gamma\dot{u}^\gamma + (n-1)^2/9\,\dot{u}_\gamma\dot{u}^\gamma).\end{aligned} \tag{28.13}$$

Bearing in mind the equations (27.6-a) and (28.7), it appears from Eq. (28.13) that $\tilde{\Sigma}$ is isotropic:

$$[\tilde{R}_{\alpha\beta}]^+ = 0. \tag{28.14}$$

Let us take the tracer line of the two members of Eq. (28.13); we arrive at:

$$-T^{2(n-1)/3}\tilde{R} = \overline{R} - 4/3(n-1)(\overline{\nabla}_\alpha\dot{u}^\alpha - \dot{u}_\alpha\dot{u}^\alpha) - 2/9(n-1)(n+5)\dot{u}_\alpha\dot{u}^\alpha. \tag{28.15}$$

Let us consider the variation of (28.15), bearing in mind Eq. (28.12) and using Eq. (28.2); the equation obtained is not compatible with the field equations (27.6-b,d) unless:

$$(n-1)(n+5)\dot{u}_\alpha\dot{u}^\alpha = 0, \tag{28.16}$$

i.e., bearing in mind the fundamental hypothesis Eq. (27.2) and the fact that $\dot{u}_\alpha$ is rigorously oriented in space, on condition that:

$$\dot{u}_\alpha = 0, \tag{28.17}$$

which actually proves that the world lines are geodesic lines of V_4; this result could have been immediately attained by using the Bianchi identities related to $\bar{R}_{\alpha\beta\gamma\delta}$, bearing in mind Eq. (28.8) and the field equation (27.6-b), but said process would have concealed the significance of Eq. (28.17) which is, in effect, the compatibility condition of the equations (27.6-b) and (27.6-d); Eq. (27.6-b) is therefore a first integral of Eq. (27.6-b), and the determination of the gravitational field is reduced to the integration of the single field equation Eq. (27.6-b).

From Eq. (28.17) we deduce:

$$\bar{\partial}_a T = 0, \qquad 28.18$$

which is absolutely compatible with the initial hypothesis of the adiabatic state of the medium, bearing in mind the form Eq. (18.9) of the relativistic equation of heat conduction.

d. In relation to the metric Eq. (27.9), the only remaining field equation is written:

$$T^{2(n-1)/3}\tilde{R} + 2/3(n-1)^2(dT/dt)^2 = 2aT^n, \qquad 28.19$$

where $\tilde{R}$ is a constant, by virtue of Eq. (28.12) and the Bianchi identities related to $\tilde{R}_{\alpha\beta}$. The solution of Eq. (28.19), ensured under the conditions:

$$\tilde{R} < 2aT^{(n+2)/3} \qquad 28.20$$

introduces an ellyptical-type integral.

In the particular case of n = 4, the medium under consideration is analogous to a black body described by the famous Stefan-Boltzmann radiation law-(1884) and the V_4 variation agrees with a V'_4 variation homeomorphous to the product V'_4 x R of a three-dimensional isotropic spaceduct the real straight line:

$$ds^2 = T^{-2}(-\tilde{ds}^2 + dt^2); \qquad 28.21$$

the temperature field T of the medium is characterized, according to the symbol $\tilde{R}$ by the functions:

$$T = -\sqrt{3/a}\sqrt{\tilde{R}/6}\,\mathrm{Sin}^{-1}\sqrt{\tilde{R}/6}\,(t - t_0) \qquad \tilde{R} > 0 \qquad 28.22\text{-a}$$

$$T = \sqrt{3/a}\sqrt{-\tilde{R}/6}\,\mathrm{Sh}^{-1}\sqrt{-\tilde{R}/6}\,(t - t_0) \qquad \tilde{R} < 0 \qquad 22.22\text{-b}$$

$$T = \sqrt{3/a}\,(t - t_0)^{-1}. \qquad \tilde{R} = 0 \qquad 22.22\text{-c}$$

Let us summarize these results:

Proposition. Let us consider a material distribution described with the aid of the particular class of isotropic thermodynamic media:

$$\rho = \sqrt{-g}aT^n, \qquad p = 1/(n-1)\sqrt{-g}aT^n \qquad (n > 2),$$

a distribution assumed to be adiabatic, generated by a congruence of world lines normal to the Σ space sections, and without distortion.

The gravitational field created by such a source is characterized by the following properties:

(i) The space sections Σ are isotropic.

(ii) The temperature field is homogeneous in each section of space.

(iii) The world lines are geodesic lines of V_4.

(iv) V_4 is of the Euclidian conformity type.

The metric revealing this structure is of the:

$$ds^2 = -T^{-2(n-1)/2}\tilde{ds}^2 + T^{-2}dt^2$$

where $\tilde{ds}^2$ designates the metric of an invariant structure isotropic space through infinitesimal transformation generated by the vector field $\vec{u}$, and T is determined by the integral:

$$\int T^{-(n-1)/2}(T^{(n+2)/2} - \tilde{R}/2a)^{-1/2}dT = 1/(n-1)(3a)^{1/2}\int dt.$$

Conclusion

The object of this work is to establish a study formalism of continuous media in general relativity; it is totally inspired by the need to substitute the classical description of the states of a material medium the description of the variation of states of the medium. This conception enables us effectively to overcome the difficulty linked to the fact that the definition of an absolute scale for the measurement of conditions of state is not compatible with the principles of general relativity.

In an approach to the problem presented, I have first considered the case of the elastic continuous media. After showing that the structure of such media is represented by a normal type pulse rate-energy, I formulated the group of six linking equations describing the variations of state of the medium; the second member of these equations must satisfy three equations of variation of conservation, the general form of which I have established; by joining with these nine equations the ten equations of variations of field, the four equations of conservation of the pulse rate-energy, and the nine equations of variations of deformations and vortices of the medium, I have been able to set forth a group of 32 equations which should enable us to present the problem of elasticity in general relativity in a manner adapted to the point of view of the variation of states.

I then made an effort to expand this diagram to a thermodynamic framework. A preliminary analysis of the postulates of relativistic thermodynamics has enabled me to forward sound arguments in support of a tensorial interpretation of the concept to total entropy of a material system, the previously assumed vector definition not being adapted to relativistic ideas. This concept necessarily leads to a formulation of the second principle which must introduce four relations describing the behavior of the rate of change vector of the total entropy density, by analogy with the formulation of the first principle, constituted by the four pulse-energy density conservation equations, but which is the rigorous consequence of the field equations. Such a formulation, which must enable us to define the pulse-energy density unequivocally, is linked to the interpretation of the spatial portion of the total entropy rate of change vector, and at this time it remains an open problem. I have therefore limited myself in this work to a special formulation of the two principles, bringing into play, on the one hand, a general postulate of unification of mass and energy, and, on the other, the postulate of the rigorously non-negative character of the time portion of the total entropy rates of change. Within the framework of this axiom, I have formulated the linking equations of the thermodynamics continuous media, generalizing the linking equations of the elastic media; this study has incidentally enabled us to supply a justification of Fourier's relativistic law, as an exact transposition of the classical law.

With the aid of a continuity hypothesis related to the rate of change of total entropy, a hypothesis which clarifies one of the aspects of the second principle, I have set forth the classes of characteristic variations of the problem presented, and this has enabled me to set forth the general form of the thermodynamic wave front equation; the latter shows the existence of seven wave fronts constituted, on the one hand, by the six elastic wave fronts derived from the unfolding of each of the three known wave fronts, separation due to the existence of the privileged direction of the heat flow, and, on the other, by the thermal wave front, the appearance of which is directly linked to the heat flow propagation.

Applying this formalism, I have made a special study of the isotropic thermodynamic continuous media after introducing the concept of isotropy from a peculiar property of wave propagation, I have set forth the structure properties of these media. I have illustrated the preceding general study with a study of the problem of linking the gravitation field with its sources, when the latter are described by a special class of solutions to the linking equations proposed in the isotropic case.

A preliminary classification of the linking equations of the thermodynamic continuous media, in relation with each of the physical situations under consideration, is still necessary before we deal with the fundamental problem of linking the gravitational field to its sources; the solution of this program is linked to the establishment of a satisfactory formulation of the second principle of relativistic thermodynamics, the foundations of which we have been able to outline here.

This work owes its origin to Mr. S. Kichenassamy, whose friendly assistance, and whose advice and criticism inspired by a serious demand for the understanding of the physical phenomena, have been a basic contribution to its development; I express my profound appreciation to him.

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APPENDIX I

CONCERNING THE DETERMINATION OF THE STRUCTURE OF $T^{\alpha\beta}$ (Chap. II)

A. We use the successive identities:

$$\delta/\delta g_{\alpha\beta}(\partial s/\partial y^0) = 1/2\,\partial s/\partial y^0 u^\alpha u^\beta. \qquad \text{A.1}$$

$$\delta/\delta g_{\alpha\beta} u^\gamma = -1/2\, u^\alpha u^\beta u^\gamma. \qquad \text{A.2}$$

$$\delta/\delta \bar{g}_{\alpha\beta} \bar{g}^\delta_\gamma = -u^\delta u^{(\alpha} \bar{g}^{\beta)}_\gamma. \qquad \text{A.3}$$

$$\delta/\delta g_{\alpha\beta} \bar{g}_{\gamma\delta} = \bar{g}^{(\alpha}_\gamma \bar{g}^{\beta)}_\delta. \qquad \text{A.4}$$

to establish:

$$\delta/\delta g^{\alpha\beta} \bar{x}^\gamma{}_{.\mu} = -u^\gamma u^{(\alpha} \bar{x}^{\beta)}{}_{.\mu} \qquad \text{A.5}$$

$$\delta/\delta g_{\alpha\beta}\, \text{dét}\,(\bar{x}^\alpha{}_{.\mu}) = -1/2\, u^\alpha u^\beta\, \text{dét}\,(\bar{x}^\alpha{}_{.\mu}). \qquad \text{A.6}$$

B. From the group of .he two identities:

$$x^\beta{}_{.\mu}\delta/\delta x^\alpha{}_{.\mu}(\partial s/\partial y^0) = \partial s/\partial y^0 u_\alpha u^\beta. \qquad \text{B.1}$$

$$x^\beta{}_{.\mu}\delta/\delta x^\alpha{}_{.\mu} u^\gamma = \bar{g}^\gamma_\alpha u^\beta. \qquad \text{B.2}$$

we first deduce:

$$x^\beta{}_{.\mu}\delta/\delta x^\alpha{}_{.\mu} \bar{x}^\gamma{}_{.\rho} = \bar{g}^\gamma_\alpha \bar{x}^\beta{}_{.\rho} - \bar{g}_{\alpha\delta} \bar{x}^\delta{}_{.\rho} u^\gamma u^\beta. \qquad \text{B.3}$$

$$x^\beta{}_{.\mu}\delta/\delta x^\alpha{}_{.\mu}\, \text{dét}\,(\bar{x}^\alpha{}_{.\mu}) = \bar{g}^\beta_\alpha\, \text{dét}\,(\bar{x}^\alpha{}_{.\mu}). \qquad \text{B.4}$$

then:

$$x^\beta{}_{.\rho}\delta/\delta x^\alpha{}_{.\rho} \gamma_{\mu\nu} = 2\bar{g}_{\alpha\gamma} \bar{x}^\gamma{}_{.(\mu} \bar{x}^\beta{}_{.\nu)}. \qquad \text{B.5}$$

$$x^\beta{}_{.\rho}\delta/\delta x^\alpha{}_{.\rho} \gamma = \bar{g}^\beta_\alpha \gamma. \qquad \text{B.6}$$

APPENDIX II

CONCERNING THE EQUATION OF THERMODYNAMIC WAVE FRONTS (Chap. IV)

A. The calculation of the determinant and of the space minor of a space tensor $\overline{M}_\alpha^\beta$ introduces the identities:

$$1/4!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \delta^{[\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho}\delta^{\delta]}_{\sigma}. \qquad \text{A.1}$$

$$1/3!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} u_\alpha u^\mu = \bar{g}^{[\beta}_{\nu}\bar{g}^{\gamma}_{\rho}\bar{g}^{\delta]}_{\sigma}. \qquad \text{A.2}$$

$$1/2!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\delta} u_\alpha u^\mu = \bar{g}^{[\beta}_{\nu}\bar{g}^{\gamma]}_{\rho}. \qquad \text{A.3}$$

$$1/2!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\gamma\delta} u_\alpha u^\mu = \bar{g}^{\beta}_{\nu}. \qquad \text{A.4}$$

which introduce:

$$\overline{\det}\,(\overline{M}_\alpha^\beta) = 1/6(\overline{M}_\alpha^\alpha)^3 - 1/2\overline{M}_\alpha^\alpha\overline{M}_\beta^\gamma\overline{M}_\gamma^\beta + 1/3\overline{M}_\alpha^\beta\overline{M}_\beta^\gamma\overline{M}_\gamma^\alpha. \qquad \text{A.5}$$

$$(\overline{\min}\,\overline{M})_\alpha^\beta = \overline{M}_\alpha^\gamma\overline{M}_\gamma^\beta - \overline{M}_\gamma^\gamma\overline{M}_\alpha^\beta + 1/2((\overline{M}_\gamma^\gamma)^2 - \overline{M}_\gamma^\delta\overline{M}_\delta^\gamma)\bar{g}_\alpha^\beta. \qquad \text{A.6}$$

B. In the isotropic case, the calculation of the wave equation introduces the identities:

$$1/2!\, \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} u_\alpha u^\mu n_\delta n^\sigma = -\hat{g}^{[\beta}_{\nu}\hat{g}^{\gamma]}_{\rho}, \qquad \hat{g}_\alpha^\beta = \bar{g}_\alpha^\beta + n_\alpha n^\beta. \qquad \text{B.1}$$

$$\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\delta} u_\alpha u^\mu n_\gamma n^\rho = -\hat{g}^{\beta}_{\nu}. \qquad \text{B.2}$$

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