N69-15756
NASA CR-99142

Some one-dimensional problems in a theory of mechanically and thermally interacting continuous media*

by

Charles J. Martin

Department of Mathematics and
Center for Applied Mathematics
Michigan State University
East Lansing, Michigan

December 1, 1968

*The results communicated in this paper were obtained in the course of an investigation conducted under the research grant NGR 23-004-041 of Michigan State University with the N.A.S.A. in Washington, D.C.

Abstract

Using a linearized theory for the mixture of elastic solid and viscous fluid, two one-dimensional boundary value problems are posed and studied.

The first problem is that of a stress-free halfspace to the face of which a transient temperature is applied. The second differs from the first in that its face is constrained rigidly against motion.

With the aid of the Laplace transform, solutions of these problems are given analytically and graphically when the temperature on the boundary is (a) a delta function, (b) a step function and (c) a ramp load.

A comparison of results using this theory and the results for the corresponding thermoelastic theory is made.

step rise applied at $t = 0$.

In contrast with the quasi-static theory in which the normal (to the boundary) stress is identically zero, all references showed that this stress is actually non-zero and exhibits a discontinuity at $t = x$, the spatial coordinate. For the step rise loading, [6] showed the stress discontinuity to be a jump of constant magnitude while [9], which incorporated the mechanical coupling term in the heat conduction equation (ignored in [6]), showed the jump's magnitude decreased as it was propagated.

In [8], the effect of $t_0 > 0$ was to replace the jump discontinuity by a discontinuity in the slope of the stress at $t = x$ and at $t = x + t_0$.

In spite of these contrasts, the conclusions of Muki and Breuer [9] were that except for a very short time interval after application of the thermal loading, and except for a very thin layer near the boundary, the use of the quasi-static field equations in transient thermoelastic problems gave a reasonably valid estimation of the stress field, at least for metals.

In this paper we shall consider the effects of inertia on a mixture of linear elastic solid and viscous fluid in a half-space to which a point load in temperature is applied. Once we have found the kinematic and stress fields due to the point load we shall then consider the ramp and step loadings. This problem will correspond to the uncoupled transient thermoelastic problem of [8] and in a subsequent paper we shall attempt to assess the mechanical coupling effects.

In section 2 we present the salient features of the linearized theory of Green and Steel [3] and Steel [4] for a mixture of isotropic linear elastic solid and viscous fluid. From this, we present in section 3 a nondimensional set of boundary value problems which are to be studied.

Steel [4] terms a parameter occuring in the diffusive resistance vector a measure of diffusive force and we use this parameter as the basis for a perturbation of the equations of section 3. This procedure comprises section 4. The undisturbed equations (zero diffusive force) and the first order theory are solved explicitly in section 5 with the aid of the Laplace transform. After inversion, we discuss the stress and kinematic fields for the ramp and step loadings in section 6.

Application of mixture theories such as [2] to [4] are found in a variety of problems most notable of which are the problems dealing with flows in porous media such as soils and oil bearing rock layers. A similar phennomenon occurs in the ablation of heat shields of re-entering spacecraft in which the rapid temperature rise causes gas bubbles to flow out of the material.

2. Linearized theory for an isotropic elastic solid and a viscous fluid.

The theory presented below is essentially that of Green and Steel [3] for a mixture of elastic solid and a viscous fluid. We assume that the mixture is chemically inert and that at each point of the spatial region occupied by the mixture and at each instant of time, there are material points of each constituent. Thus at each spatial point we define kinematic and mechanical quantities for each constituent and, in addition, we define mechanical and thermodynamic quantities appropriate to the total aggregate.

Since we are concerned only with a linearized theory we shall refer all quantities to a fixed rectangular cartesian coordinate system $x = (x_1,x_2,x_3)$ and thus take the

Lagrangian coordinates as applicable. The assumptions inherent in this linearization process are that the material points of the solid component are displaced infinitesimally from the initial equilibrium state and that the components of the fluid velocity vector are equally small.

We define at time t component densities ρ_1, ρ_2 respectively, solid and fluid. Kinematic quantities associated with the solid are a displacement vector $\vec{w} = (w_i)$, a velocity vector $\vec{u} = (u_i) = (\partial w_i/\partial t)$, a strain tensor $\tilde{e} = (e_{ij})$, a rate of deformation tensor $\tilde{d} = (d_{ij})$, and a vorticity tensor $\tilde{\Gamma} = (\Gamma_{ij})$.* Similarly for the fluid component we have a velocity vector $\vec{v} = (v_i)$, a rate of deformation tensor $\tilde{f} = (f_{ij})$ and a vorticity tensor $\tilde{\Lambda} = (\Lambda_{ij})$. Mechanical quantities associated with the solid are a partial stress tensor $\tilde{\sigma} = (\sigma_{ij})$ and a body force vector $\vec{F} = (F_i)$ while for the fluid we shall denote these quantities by $\tilde{\pi} = (\pi_{ij})$ and $\vec{G} = (G_i)$, respectively. An additional vector accounting for the interaction of the components will be denoted by $\vec{\omega} = (\omega_i)$ and is termed the diffusive resistance.

Green and Steel [3] introduce the thermodynamic quantities, as applied to the total aggregate, by temperature T, specific entropy S, specific Helmholtz free energy A, and the heat flux $\vec{q} = (q_i)$

In the remainder of this section we list but do not re-derive the kinematic relations, the continuity equations and the field equations for each mixture component.

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*All subscripts run over values 1,2,3, and, when repeated indicate a sum on the index over 1,2,3.

Additionally, we pose the constitutive equations*, the energy equation and the entropy-production inequality as given in [1].

For the solid constituent, the velocity-displacement and strain-displacement relations are

$$u_i = \frac{\partial w_i}{\partial t}\ , \ e_{ij} = \frac{1}{2}\,(w_{i,j} + w_{j,i}) \tag{2.1}$$

and the conservation of mass requires that

$$\rho_1 = \bar{\rho}_1(1 - e_{kk}) \tag{2.2}$$

where $\bar{\rho}_1$ is the uniform initial value of the mass density of the solid component.

Rate of deformation and vorticity relations for the solid and fluid components are

$$d_{ij} = \frac{1}{2}\,(u_{i,j}+u_{j,i}),\ \Gamma_{ij} - \frac{1}{2}\,(u_{i,j}-u_{j,i}) \qquad (2.3a)$$

$$f_{ij} = \frac{1}{2}\,(v_{i,j}+v_{j,i}),\ \Lambda_{ij} = \frac{1}{2}\,(v_{i,j}-v_{j,i})\,. \qquad (2.3a)$$

To these we also add:

<u>continuity equation for the fluid</u>

$$\frac{\partial \rho_2}{\partial t} + \bar{\rho}_2\ f_{kk} = 0 \tag{2.4}$$

<u>equations of motion of mixture</u>

$$\sigma_{ij,i} - \omega_j + \bar{\rho}_1 F_j = \bar{\rho}_1\,\frac{\partial u_j}{\partial t} \qquad (2.5a)$$

$$\pi_{ij,i} + \omega_j + \bar{\rho}_2 G_j = \bar{\rho}_2\,\frac{\partial v_j}{\partial t} \qquad (2.5b)$$

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*We use constitutive equations as presented in [4].

<u>energy equation</u>

$$-\bar{\rho}\left(\bar{T}\,\frac{\partial S}{\partial t} + \bar{S}\,\frac{\partial T}{\partial t} + \frac{\partial A}{\partial t}\right) + \bar{\omega}_k(u_k - v_k)$$

$$+\ \bar{\sigma}_{(ij)}d_{ij}+\bar{\pi}_{(ij)}f_{ij}+\bar{\sigma}_{[ij]}(\Gamma_{ji}-\Lambda_{ji})-q_{k,k} = 0 \tag{2.6}$$

where $\bar{\rho} = \bar{\rho}_1 + \bar{\rho}_2$, $\bar{\rho}$ being the total density. Barred quantities refer to values taken at equilibrium. Symbols $f_{(ij)}$ and $f_{[ij]}$ refer to symmetric and skew symmetric components of f_{ij} and partial derivatives with respect to x_i are indicated by $,i$ as a subscript.

The constitutive equations to be used are given by

$$\bar{\rho}A = \bar{\rho}\bar{A} + \alpha_1 e_{kk} + \alpha_2(\rho_2-\bar{\rho}_2) + \alpha_3(T-\bar{T}), \tag{2.7}$$

$$\bar{\rho}S = -[\alpha_3+\alpha_9 e_{kk}+\alpha_{10}(\rho_2-\bar{\rho}_2) + \alpha_7(T-\bar{T})], \tag{2.8}$$

$$q_i = -\,kT_i - K'(u_i-v_i), \tag{2.9}$$

$$\omega_i = -\frac{\bar{\rho}_2}{\bar{\rho}}\,\alpha_1 e_{kk,i} + \frac{\bar{\rho}_1}{\bar{\rho}}\,\alpha_2\rho_{2,i} + \alpha(u_i-v_i)$$

$$+\ a''\epsilon_{ipq}(\Gamma_{pq}-\Lambda_{pq}), \tag{2.10}$$

$$\sigma_{ij} = \Big[\alpha_1 - \Big\{\frac{\bar{\rho}_1}{\bar{\rho}}\,\alpha_1-\alpha_4\Big\}e_{kk} + \Big\{\frac{1}{\bar{\rho}}\,\alpha_1+\alpha_8\Big\}(\rho_2-\bar{\rho}_2)$$

$$+\ \alpha_9(T-\bar{T})\Big]\delta_{ij} + 2(\alpha_1+\alpha_5)e_{ij}$$

$$+\ D\epsilon_{ijk}(u_k-v_k) - D''(\Gamma_{ij}-\Lambda_{ij}), \tag{2.11}$$

$$\pi_{ij} = \Big[-\bar{\rho}_2\alpha_2+\bar{\rho}_2\,\Big\{\frac{\bar{\rho}_1}{\bar{\rho}}\,\alpha_2-\alpha_8\Big\}e_{kk} - \Big\{\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}}\,\alpha_2+\bar{\rho}_2\alpha_6\Big\}(\rho_2-\bar{\rho}_2) +$$

$$- \bar{\rho}_2\alpha_{10}(T-\bar{T}) + \lambda f_{kk}\Big]\delta_{ij} + 2\mu f_{ij}$$

$$- D\epsilon_{ijk}(u_k-v_k) + D''(\Gamma_{ij}-\Lambda_{ij}). \tag{2.12}$$

In these relations all coefficients represent material constants, ϵ_{ijk} is the alternating tensor and bars refer the appropriate quantities to equilibrium. To identify the symmetric and skew-symmetric partial stress components as used in (2.6) with the expressions (2.11), (2.12) we state that the last two terms of (2.11), (2.12) are skew-symmetric.

The material constants, by virtue of the entropy-production inequality, are required to satisfy

$$\mu \geq 0,\ 2\mu+3\lambda \geq 0,\ \alpha \geq 0,\ D'' \geq 0,\ (a''-D)^2 \leq 4\alpha D''$$

$$k \geq 0,\ (K')^2 \leq 4\alpha k\bar{T}\ . \tag{2.13}$$

In addition, by definition of state of equilibrium, we have that

$$\bar{\sigma}_{ij} = \alpha_1\delta_{ij},\ \bar{\pi}_{ij} = -\bar{\rho}_2\alpha_2\delta_{ij},\ \bar{\omega}_i = 0. \tag{2.14}$$

Before considering specific problems we note that the energy equation (2.6), upon substituting (2.7), (2.8), and (2.14), and using (2.1) to (2.4), becomes

$$\alpha_7\,\frac{\partial T}{\partial t} + \alpha_9 d_{kk} - \bar{\rho}_2\alpha_{10}f_{kk} = \frac{1}{\bar{T}}\,q_{k,k}. \tag{2.15}$$

To close out this section we shall quote the uniqueness theorem presented by Atkin, Chadwick and Steel [1].

<u>Uniqueness Theorem</u>. Let B be a bounded regular region

of three-dimensional Euclidean space occupied by a mixture of the type just presented in equations (2.1) to (2.15). By ∂B we denote the boundary of B and by B_0 the interior region of B. By regions $\beta(\beta^\circ)$ we shall mean all pairs (P,t), such that for $t \geq 0$, P is a material point in $B(B_0)$. In addition, we take $\bar{n}$ to be a unit outward normal on ∂B and we use ∂B_1, ∂B_2 as arbitrary subsets of ∂B with complements $\partial \bar{B}_1$, $\partial \bar{B}_2$, respectively.

Let, in addition to (2.13), the α_i, $1 \leq i \leq 8$, satisfy

$$\alpha_1 + \alpha_5 \geq 0, \ \frac{2}{\bar{\rho}}\alpha_2 + \alpha_6 \geq 0, \ \alpha_7 \leq 0$$

$$\alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}} - \frac{1}{3}\right) + \alpha_4 + \frac{2}{3}\alpha_5 \geq 0, \qquad (2.16).$$

$$\left(\frac{1}{\bar{\rho}}\alpha_1 - \frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2 + \alpha_8\right)^2 \leq \left[\alpha_1\left(\frac{2\bar{\rho}_2}{\bar{\rho}} - \frac{1}{3}\right) + \alpha_4 + \frac{2}{3}\alpha_5\right]\left(\frac{2}{\bar{\rho}}\alpha_2 + \alpha_6\right).$$

Then there exists at most one set of functions v_i, ρ_2 of class C^1, and w_i, T of class C^2 on β which satisfy (2.4), (2.5) and (2.15) on β°, (2.1), (2.2) and (2.3) on β and the following initial conditions on B at $t = 0$:

$$w_i = \hat{w}_i, \ u_i = \hat{u}_i, \ v_i = \hat{v}_i, \ \rho_2 = \bar{\rho}_2 + \hat{\rho}_2, \ T = \bar{T} + \hat{T}. \qquad (2.17)$$

In addition these functions satisfy the boundary conditions for $t > 0$:

$$u_i - v_i = \hat{R}_i, \ (\sigma_{ij} + \pi_{ij})n_i = \hat{\sum_j} \ \text{on} \ \partial B_1 \qquad (2.17a)$$

$$u_i = \hat{U}_i, \quad v_i = \hat{V}_i \qquad \text{on} \quad \partial\bar{B}_1 \tag{2.17b}$$

$$T = \bar{T} + \hat{\Theta} \qquad \text{on} \quad \partial\bar{B}_1 \tag{2.18a}$$

$$q_k n_k = \hat{F} \qquad \text{on} \quad \partial\bar{B}_2 \; . \tag{2.18b}$$

The quantities wearing carets are known functions and the quantities $\bar{\rho}_1$, $\bar{\rho}_2$ and $\bar{T}$ are positive constants. Of the choices presented in (2.17), (2.18) either ∂B_1, ∂B_2 or their complements may be empty or the entire boundary.

For the one-dimensional applications to be considered in the remainder of this paper we shall study the pair (2.17a), (2.18a) as one problem called problem A, while by problem B we shall mean (2.17b) together with (2.18a).

Finally we note that although uniqueness is guaranteed by the above theorem, our application is to a half-space and we therefore require that regularity conditions at infinity be imposed upon our stress, displacement and velocity fields.

3. Statement of the problem.

Departing from the indicial notation, let us identify cartesian coordinates as (x,y,z) and consider our body as occupying the region $x \geq 0$. We assume that the body is subjected to a time-dependent temperature field of the form

$$\Theta = \Theta(x,t) \tag{3.1}$$

and is constrained to uniaxial motion so that the displacement vector of the elastic solid has components

$$\vec{w} = [w(x,t),\, 0,\, 0] \tag{3.2}$$

while the fluid velocity vector becomes

$$v = [v(x,t), 0, 0]. \tag{3.3}$$

The temperature field Θ of (3.1) is related to the temperature $T(x,t)$ by

$$\Theta(x,t) = \frac{T(x,t) - \bar{T}}{\bar{T}} . \tag{3.4}$$

Substitution of (3.1) to (3.3) into equations (2.1) to (2.4) shows that we may write

$$\left.\begin{aligned} e_x &= \frac{\partial w}{\partial x}, \quad f_x = \frac{\partial v}{\partial x}, \\ d_x &= \frac{\partial}{\partial t} e_x = \frac{\partial^2 w}{\partial t \partial x}, \end{aligned}\right\} \tag{3.5}$$

$$\rho_1(x,t) = \bar{\rho}_2[1 - \frac{\partial w}{\partial x}] , \tag{3.6}$$

$$\frac{\partial \rho_2}{\partial t}(x,t) + \bar{\rho}_2 \frac{\partial v}{\partial x} = 0 \tag{3.7}$$

as the only non-vanishing kinematic relations. In addition we have that all other strain, rate of deformation and vorticity components are identically zero.

The constitutive relations (2.10) to (2.12) take the form

$$\left.\begin{aligned} \omega_x &= \frac{\bar{\rho}_2 \alpha_1}{\bar{\rho}} \frac{\partial^2 w}{\partial x^2} + \frac{\bar{\rho}_1 \alpha_2}{\bar{\rho}} \frac{\partial \rho_2}{\partial x} + \alpha\left(\frac{\partial w}{\partial t} - v\right), \\ \omega_y &= \omega_z = 0 , \end{aligned}\right\} \tag{3.8}$$

$$\sigma_x(x,t) = [2(\alpha_1+\alpha_5) - \left(\frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} - \alpha_4\right)]\frac{\partial w}{\partial x}$$

$$+ \alpha_9\bar{T}\Theta + \left(\frac{\alpha_1}{\bar{\rho}} + \alpha_8\right)(\rho_2-\bar{\rho}_2)+\alpha_1, \tag{3.9}$$

$$\sigma_y(x,t) = \sigma_z(x,t) = -\left(\frac{\bar{\rho}_1\alpha_1}{\bar{\rho}} - \alpha_4\right)\frac{\partial w}{\partial x}$$

$$+\alpha_9\bar{T}\Theta+\left(\frac{\alpha_1}{\bar{\rho}} + \alpha_8\right)(\rho_2-\bar{\rho}_2) + \alpha_1, \tag{3.10}$$

$$\tau_{xy} = \tau_{xz} = \tau_{yz} = 0, \tag{3.11}$$

$$\pi_x(x,t) = (2\mu+\lambda)\frac{\partial v}{\partial x} + \bar{\rho}_2\left(\frac{\bar{\rho}_1\alpha_2}{\bar{\rho}} - \alpha_8\right)\frac{\partial w}{\partial x}$$

$$- \bar{\rho}_2\alpha_{10}\bar{T}\Theta-\left[\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}}\alpha_2+\bar{\rho}_2\alpha_6\right](\rho_2-\bar{\rho}_2) -\bar{\rho}_2\alpha_2, \tag{3.12}$$

$$\pi_y(x,t) = \pi_z(x,t) = \lambda\frac{\partial v}{\partial x} +\bar{\rho}_2\left(\frac{\bar{\rho}_1}{\bar{\rho}}\alpha_2-\alpha_8\right)\frac{\partial w}{\partial x}$$

$$- \bar{\rho}_2\alpha_{10}\bar{T}\Theta-\left[\frac{\bar{\rho}+\bar{\rho}_2}{\bar{\rho}}\alpha_2+\bar{\rho}_2\alpha_6\right](\rho_2-\bar{\rho}_2) -\bar{\rho}_2\alpha_2, \tag{3.13}$$

$$\pi_{xy} = \pi_{xz} = \pi_{yz} = 0. \tag{3.14}$$

The equations of motion (2.5) and the heat equation (2.15), in view of the constraints (3.1) to (3.3), become

$$\frac{\partial \sigma_x}{\partial x} - \omega_x = \bar{\rho}_1 \frac{\partial^2 w}{\partial t^2} \tag{3.15a}$$

$$\frac{\partial \pi_x}{\partial x} + \omega_x = \bar{\rho}_2 \frac{\partial v}{\partial t} \tag{3.15b}$$

$$\alpha_7 \bar{T} \frac{\partial \Theta}{\partial t} + k \frac{\partial^2 \Theta}{\partial x^2} + \left(\frac{\alpha_9 \bar{T} + K'}{\bar{T}}\right) \frac{\partial^2 w}{\partial x \partial t} - \left(\frac{\bar{\rho}_2 \alpha_{10} \bar{T} + K'}{\bar{T}}\right) \frac{\partial v}{\partial x} = 0 \tag{3.16}$$

provided (3.4) and (2.9) are used.

To complete our statement of the initial-boundary value problem we prescribe that for $t \leq 0$

$$w(x,t) = \frac{\partial w}{\partial t}(x,t) = v(x,0) = 0 \tag{3.17}$$

$$\rho_2 = \bar{\rho}_2, \ \Theta(x,t) = 0.$$

In addition, we require that on the boundary $x = 0$, either:

(<u>problem A</u>) $\quad \Theta(0,t) = f(t),$ (3.18a)

$$\sigma_x(0,t) + \pi_x(0,t) = 0, \tag{3.18b}$$

$$\frac{\partial w(0,t)}{\partial t} - v(0,t) = 0, \tag{3.18c}$$

or: (<u>problem B</u>)

$$\Theta(0,t) = f(t), \tag{3.19a}$$

$$\frac{\partial w}{\partial t}(0,t) = 0, \tag{3.19b}$$

$$v(0,t) = 0, \tag{3.19c}$$

while as $x \rightarrow \infty$, we stipulate that

$\Theta(x,t)$, $w(x,t)$, $\rho_2(x,t)$, $v(x,t)$, $\sigma_x(x,t)$, $\pi_x(x,t)$, $\sigma_y(x,t)$ and $\pi_y(x,t) \rightarrow 0$. (3.20)

At this point, we find it expedient to introduce dimensionless variables. To do so, we note that Steel [4] defines

$$\beta_1 = \frac{\alpha_1}{\bar{\rho}} + \alpha_8, \; \beta_2 = \alpha_4 - \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}}, \; \beta_3 = \alpha_1+\alpha_5 \tag{3.21}$$

$$\gamma_1 = \bar{\rho}_2\alpha_6 + \left(\frac{1}{\bar{\rho}_2} + \frac{1}{\bar{\rho}}\right)\alpha_1, \; \gamma_2 = -\bar{\rho}_2\alpha_8 + \frac{\bar{\rho}_1\alpha_1}{\bar{\rho}}$$

wherein he uses the relation

$$\alpha_1 = \bar{\rho}_2\alpha_2. \tag{3.22}$$

A direct substitution of (3.21) and (3.22) into (3.9) yields

$$\sigma_x(x,t) = (\beta_2+2\beta_3)\frac{\partial w}{\partial x} + \alpha_9\bar{T}\Theta + \beta_1(\rho_2-\bar{\rho}_2)+\alpha_1 \tag{3.23}$$

which, if the material were elastic, would lead us to expect $\beta_2+2\beta_3$ to play the role of the Lamé constants $(\lambda+2\mu)$ while $\alpha_9\bar{T}$ would play the role of $(2\mu+3\lambda)\alpha$ where α is the coefficient of linear thermal expansion of an elastic material. Keeping this in mind we choose a velocity

$$c_1^2 = \frac{\beta_2 + 2\beta_3}{\bar{\rho}_1} \tag{3.24}$$

which would be the irrotional velocity of sound if the material were elastic. Since by (2.16), $\alpha_7 \leq 0$, we define

$$\omega^2 = \frac{-k}{\alpha_7 \bar{T}} . \tag{3.25}$$

By a dimensional analysis we have that (3.24) is a velocity while (3.25) has dimensions of length squared per unit time. Thus, if we take

$$a = \frac{\omega^2}{c_1} , \quad t_0 = \frac{\omega^2}{c_1^2} \tag{3.26}$$

then a dimensionless x-coordinate and time are given by

$$\xi = \frac{x}{a} = \frac{c_1 x}{\omega^2} , \quad \tau = \frac{t}{t_0} = \frac{c_1^2 t}{\omega^2} . \tag{3.27}$$

Proceeding further, we introduce non-dimensional partial stresses, solid displacement, fluid velocity, densities and diffusive force

$$\left.\begin{aligned}
&\hat{\sigma}_x = \frac{\sigma_x}{\beta_2+2\beta_3} , \quad \hat{w} = \frac{w}{a} , \quad \hat{\sigma}_0 = \frac{\alpha_1}{\beta_2+2\beta_3} , \quad \hat{\sigma}_y = \frac{\sigma_y}{\beta_2+2\beta_3} , \\
&\hat{\pi}_x = \frac{\pi_x}{\beta_2+2\beta_3} , \quad \hat{v} = \frac{v t_0}{a} , \quad \hat{\pi}_y = \frac{\pi_y}{\beta_2+2\beta_3} , \\
&\eta_2 = \frac{\rho_2-\bar{\rho}_2}{\bar{\rho}_2} , \quad \eta_1 = \frac{\rho_1-\bar{\rho}_1}{\bar{\rho}_1} , \\
&\hat{\omega} = \frac{a\omega_x}{\beta_2+2\beta_3} .
\end{aligned}\right\} \tag{3.28}$$

In addition, the following quantities are conveniently grouped:

$$s^2 = \frac{2\mu+\lambda}{t_0(\beta_2+2\beta_3)} ,$$

$$d_1 = \frac{\alpha_9\bar{T}}{\beta_2+2\beta_3} , \quad d_2 = \frac{\bar{\rho}_2\alpha_{10}\bar{T}}{\beta_2+2\beta_3} ,$$

$$f = \frac{\bar{\rho}_2}{\bar{\rho}} ,$$

$$\delta_1 = \frac{\gamma_2}{\beta_2+2\beta_3} - f\hat{\sigma}_0 = \frac{-\bar{\rho}_2}{\beta_2+2\beta_3}\left[\beta_1 - \frac{\bar{\rho}_1\alpha_2}{\bar{\rho}}\right], \tag{3.29}$$

$$\delta_2 = \frac{f\alpha_2}{c_1^2} - \frac{\bar{\rho}_2\gamma_1}{\bar{\rho}_1 c_1^2} ,$$

$$\epsilon_1 = \frac{\alpha_9\bar{T} + K'}{\alpha_7\bar{T}^2} , \quad \epsilon_2 = \frac{\bar{\rho}_2\alpha_{10}\bar{T} + K'}{\alpha_7\bar{T}^2} .$$

Incorporating all of these changes leads us the following <u>summary</u>:

<u>constitutive equations</u>

$$\hat{\sigma}_x(\xi,\tau) = \hat{\sigma}_0 + \frac{\partial\hat{w}(\xi,\tau)}{\partial\xi} + d_1\Theta(\xi,\tau) + [(1-f)\hat{\sigma}_0 - \delta_1]\eta_2(\xi,\tau), \tag{3.30a}$$

$$\hat{\pi}_x(\xi,\tau) = -\hat{\sigma}_0 + s^2\frac{\partial\hat{v}}{\partial\xi}(\xi,\tau) + (\delta_1 + f\hat{\sigma}_0)\frac{\partial\hat{w}}{\partial\xi}(\xi,\tau)$$

$$-d_2\Theta(\xi,\tau) + [\delta_2 - (1-f)\hat{\sigma}_0]\eta_2(\xi,\tau) \tag{3.30b}$$

$$\hat{\omega}(\xi,\tau) = -f\hat{\sigma}_0 \frac{\partial^2\hat{w}}{\partial\xi^2}(\xi,\tau) + (1-f)\hat{\sigma}_0 \frac{\partial\eta_2}{\partial\xi}(\xi,\tau) + \frac{\alpha t_0}{\bar{\rho}_1}\left[\frac{\partial\hat{w}}{\partial\tau}(\xi,\tau)\right.$$

$$\left. - \hat{v}(\xi,\tau)\right], \tag{3.30c}$$

$$\hat{\sigma}_y(\xi,\tau) = \frac{\beta_2}{\beta_2+2\beta_3}\frac{\partial\hat{w}}{\partial\xi}(\xi,\tau) + d_1\theta(\xi,\tau) + [(1-f)\hat{\sigma}_0 - \delta_1]\eta_2(\xi,\tau) + \hat{\sigma}_0, \tag{3.30d}$$

$$\hat{\sigma}_z(\xi,\tau) = \hat{\sigma}_y(\xi,\tau), \tag{3.30e}$$

$$\hat{\pi}_y(\xi,\tau) = \hat{\pi}_z(\xi,\tau) = \frac{\lambda}{t_0(\beta_2+2\beta_3)}\frac{\partial\hat{v}}{\partial\xi} + (f\hat{\sigma}_0+\delta_1)\frac{\partial\hat{w}}{\partial\xi}(\xi,\tau)$$

$$- d_2\theta(\xi,\tau) + [\delta_2 - (1-f)\hat{\sigma}_0]\eta_2(\xi,\tau) - \hat{\sigma}_0. \tag{3.30f}$$

<u>equations of motion</u>

$$[1+f\hat{\sigma}_0]\frac{\partial^2\hat{w}}{\partial\xi^2} - \frac{\alpha t_0}{\bar{\rho}_1}\frac{\partial\hat{w}}{\partial\tau} + \frac{\alpha t_0}{\bar{\rho}_1}\hat{v} + d_1\frac{\partial\theta}{\partial\xi} - \delta_1\frac{\partial\eta}{\partial\xi} = \frac{\partial^2\hat{w}}{\partial\tau^2}, \tag{3.31}$$

$$\delta_1\frac{\partial^2\hat{w}}{\partial\xi^2} + \frac{\alpha t_0}{\bar{\rho}_1}\frac{\partial\hat{w}}{\partial\tau} + s^2\frac{\partial^2\hat{v}}{\partial\xi^2} - \frac{\alpha t_0}{\bar{\rho}_1}\hat{v} - d_2\frac{\partial\theta}{\partial\xi} + \delta_2\frac{\partial\eta}{\partial\xi} = \frac{\bar{\rho}_2}{\bar{\rho}_1}\frac{\partial\hat{v}}{\partial\tau}, \tag{3.32}$$

$$\frac{\partial^2\theta}{\partial\xi^2} - \frac{\partial\theta}{\partial\tau} - \epsilon_1\frac{\partial^2\hat{w}}{\partial\xi\partial\tau} + \epsilon_2\frac{\partial\hat{v}}{\partial\xi} = 0, \tag{3.33}$$

$$\frac{\partial\eta_2}{\partial\tau} + \frac{\partial\hat{v}}{\partial\xi} = 0. \tag{3.34}$$

<u>initial conditions</u>

For $\tau \leq 0$, we take

$$\hat{w}(\xi,\tau) = \frac{\partial \hat{w}}{\partial \tau}(\xi,\tau) = \hat{v}(\xi,\tau) = \eta_2(\xi,\tau) = \Theta(\xi,\tau) = 0. \quad (3.35)$$

<u>boundary conditions</u>

at $\xi = 0$:

either

$$\text{(A)} \quad \begin{cases} \Theta(0,\tau) = \hat{f}(\tau) = \dfrac{f(\tau)}{\bar{T}} \\[2ex] \hat{\sigma}_x(0,\tau) + \hat{\pi}_x(0,\tau) = 0 \\[2ex] \dfrac{\partial \hat{w}}{\partial \tau}(0,\tau) - \hat{v}(0,\tau) = 0, \end{cases} \quad (3.36)$$

or

$$\text{(B)} \quad \begin{cases} \Theta(0,\tau) = \hat{f}(\tau) = \dfrac{f(\tau)}{\bar{T}} \\[2ex] \dfrac{\partial \hat{w}}{\partial \tau}(0,\tau) = 0 \\[2ex] \hat{v}(0,\tau) = 0; \end{cases} \quad (3.37)$$

as $\xi \to \infty$, $\Theta(\xi,\tau)$, $\hat{w}(\xi,\tau)$, $\hat{v}(\xi,\tau)$, $\eta_2(\xi,\tau)$, $\hat{\sigma}_x(\xi,\tau)$,

and $\hat{\pi}_x(\xi,\tau) \to 0$. (3.38)

We now introduce the approximations to be followed in the remainder of this paper and call the resulting equations the full <u>uncoupled theory</u>. In subsequent reports, equations (3.30) to (3.38) will be studied as they stand but for a first approximation we postulate that

(a) the stress components depend upon the fluid density only through the initial porosity coefficient f,

(b) the fluid stress components depend upon the solid strain only through the initial porosity coefficient f,

(c) in the heat conduction equation (3.33), the mechanical coupling terms, $\epsilon_1 \frac{\partial^2 \hat{w}}{\partial \xi \partial \tau}$, $\epsilon_2 \frac{\partial \hat{v}}{\partial \xi}$ can be ignored,

(d) the diffusive force parameter $\frac{\alpha t_o}{\bar{\rho}_1}$ is small, i.e. $\frac{\alpha t_o}{\bar{\rho}_1} << 1$.

Mathematically, the approximations (a) and (b) let us neglect the terms $\delta_1 \eta_2(\xi,\tau)$, $\delta_2 \eta_2(\xi,\tau)$ and $\delta_1 \frac{\partial w}{\partial \xi}(\xi,\tau)$ in (3.30a) to (3.30f). Approximation (c) enables us to determine the temperature $\Theta(\xi,\tau)$ independent of the solid and fluid components and we may therefore treat $\Theta(\xi,\tau)$ as a known function when solving (3.31) and (3.32).

4. Perturbation in small diffusive force.

Assume that for $\frac{\alpha t_o}{\bar{\rho}_1} << 1$,

$$
\begin{aligned}
\hat{w}(\xi,\tau) &= \sum_{o} \left(\frac{\alpha t_o}{\bar{\rho}_1}\right)^k w_k(\xi,\tau) \\
\hat{v}(\xi,\tau) &= \sum_{o} \left(\frac{\alpha t_o}{\bar{\rho}_1}\right)^k v_k(\xi,\tau)
\end{aligned}
\tag{4.1}
$$

and incorporate these expansions into equations (3.30) to (3.37) along with the assumptions (a) to (c) of the last section. Then

$$\hat{\sigma}_x(\xi,\tau) = \sigma_{x0}(\xi,\tau) + \frac{\alpha t_0}{\bar{\rho}_1}\sigma_{x1}(\xi,\tau)+\ldots,$$

$$\hat{\pi}_x(\xi,\tau) = \pi_{x0}(\xi,\tau) + \frac{\alpha t_0}{\bar{\rho}_1}\pi_{x1}(\xi,\tau)+\ldots, \tag{4.2}$$

$$\hat{\omega}(\xi,\tau) = \omega_0(\xi,\tau) + \frac{\alpha t_0}{\bar{\rho}_1}\omega_1(\xi,\tau)+\ldots,$$

provided we define

$$\begin{aligned}\sigma_{x0}(\xi,\tau) &= \hat{\sigma}_0 + \frac{\partial w_0}{\partial \xi} + d_1\Theta_0 + (1-f)\hat{\sigma}_0\eta_{20},\\ \sigma_{x1}(\xi,\tau) &= \frac{\partial w_1}{\partial \xi} + d_1\Theta_1 + (1-f)\hat{\sigma}_0\eta_{21},\end{aligned} \tag{4.3}$$

$$\begin{aligned}\pi_{x0}(\xi,\tau) &= -\hat{\sigma}_0 + s^2\frac{\partial v_0}{\partial \xi} + f\hat{\sigma}_0\frac{\partial w_0}{\partial \xi} - d_2\Theta_0 - (1-f)\hat{\sigma}_0\eta_{20},\\ \pi_{x1}(\xi,\tau) &= s^2\frac{\partial v_1}{\partial \xi} + f\hat{\sigma}_0\frac{\partial w_1}{\partial \xi} - d_2\Theta_1 - (1-f)\hat{\sigma}_0\eta_{21},\end{aligned} \tag{4.4}$$

$$\begin{aligned}\omega_0(\xi,\tau) &= -f\hat{\sigma}_0\frac{\partial^2 w_0}{\partial \xi^2} + (1-f)\hat{\sigma}_0\frac{\partial \eta_{20}}{\partial \xi},\\ \omega_1(\xi,\tau) &= -f\hat{\sigma}_0\frac{\partial^2 w_1}{\partial \xi^2} + (1-f)\hat{\sigma}_0\frac{\partial \eta_{21}}{\partial \xi} + \frac{\partial w_0}{\partial \tau} - v_0.\end{aligned} \tag{4.5}$$

The quantities w_0, v_0, η_{20} and Θ_0 must satisfy

$$\left.\begin{aligned} &\gamma \frac{\partial^2 w_0}{\partial \xi^2} - \frac{\partial^2 w_0}{\partial \tau^2} + d_1 \frac{\partial \Theta_0}{\partial \xi} = 0, \\ &s^4 \frac{\partial^2 v_0}{\partial \xi^2} - \beta^2 \frac{\partial v_0}{\partial \tau} - d_2 s^2 \frac{\partial \Theta_0}{\partial \xi} = 0, \end{aligned}\right\} \tag{4.6}$$

$$\frac{\partial^2 \Theta_0}{\partial \xi^2} - \frac{\partial \Theta_0}{\partial \tau} = 0, \tag{4.7}$$

$$\frac{\partial \eta_{20}}{\partial \tau} + \frac{\partial v_0}{\partial \xi} = 0, \tag{4.8}$$

while w_1, v_1, η_{21} and Θ_1 are solutions of

$$\left.\begin{aligned} &\gamma \frac{\partial^2 w_1}{\partial \xi^2} - \frac{\partial^2 w_1}{\partial \tau^2} + d_1 \frac{\partial \Theta_1}{\partial \xi} = \frac{\partial w_0}{\partial \tau} - v_0, \\ &s^4 \frac{\partial^2 v_1}{\partial \xi^2} - \beta^2 \frac{\partial v_1}{\partial \tau} - d_2 s^2 \frac{\partial \Theta_1}{\partial \xi} = -s^2\left[\frac{\partial w_0}{\partial \tau} - v_0\right], \end{aligned}\right\} \tag{4.9}$$

$$\frac{\partial^2 \Theta_1}{\partial \xi^2} - \frac{\partial \Theta_1}{\partial \tau} = 0, \tag{4.10}$$

$$\frac{\partial \eta_{21}}{\partial \tau} + \frac{\partial v_1}{\partial \xi} = 0. \tag{4.11}$$

For ease of manipulation we have introduced into (4.6) and (4.9) the constants $\gamma = 1 + f\hat{\sigma}_0$ and $\beta^2 = s^2\bar{\rho}_2/\bar{\rho}_1$.

To complete the zero and first order problems we require that the functions w_j, v_j, η_{2j} and Θ_j, j=0,1 satisfy the initial conditions

$$w_j = \frac{\partial w_j}{\partial \tau} = v_j = \eta_{2j} = \Theta_j = 0 \quad \text{for} \quad t \leq 0,\ j=0,1 \qquad (4.12)$$

and one of the following sets of boundary conditions:

<u>either</u>

Problem A

$$\begin{cases} \Theta_0(0,\tau) = \hat{f}(\tau),\ \Theta_1(0,\tau) = 0 & (4.13) \\ \left.\begin{aligned} &\sigma_{xj}(0,\tau) + \pi_{xj}(0,\tau) = 0 \quad j=0,1 \\ &\frac{\partial w_j}{\partial \tau}(0,\tau) - v_j(0,\tau) = 0 \quad j=0,1 \end{aligned}\right\} & (4.14) \end{cases}$$

<u>or</u>

Problem B

$$\begin{cases} \Theta_0(0,\tau) = \hat{f}(\tau),\ \Theta_1(0,\tau) = 0, & (4.13) \\ \frac{\partial w_j}{\partial \tau}(0,\tau) = v_j(0,\tau) = 0,\ j=0,1. & (4.15) \end{cases}$$

In either case, i.e., (4.14) or (4.15), we also ask that as ξ approaches infinity, Θ_j, w_j, v_j, η_{2j}, σ_{xj} and π_{xj}, j=0,1 all approach zero.

The technique employed for the solution of these equations is that of the Laplace transform with respect to time. Hence, we define

$$\bar{A}(p) = L(a(\tau);\ p) = \int_0^\infty e^{-p\tau} a(\tau)\, d\tau \tag{4.16}$$

with the inverse relation

$$a(\tau) = L^{-1}(\bar{A}(p);\ \tau) = \frac{1}{2\pi i} \int_\Gamma e^{p\tau} \bar{A}(p)\, dp \tag{4.17}$$

where Γ is the Bromwich contour in the complex p-plane.

The application of the transform (4.16) to these problems is straight forward. Restricting ourselves to the zeroth order terms we find that solutions of (4.7) and (4.6) which satisfy (4.13) and the regularity conditions as ξ approaches infinity may be written (in the transform plane) as

$$\bar{\Theta}_0(\xi,p) = \bar{f}(p)\, e^{-\xi\sqrt{p}}, \tag{4.18}$$

$$\bar{w}_0(\xi,p) = B_0(p)\, e^{-p\xi/\sqrt{\gamma}} - \frac{d_1 \bar{f}(p)\, e^{-\xi\sqrt{p}}}{\sqrt{p}[p - \gamma]}, \tag{4.19}$$

$$\bar{v}_0(\xi,p) = D_0(p)\, e^{-\frac{\xi\beta}{s^2}\sqrt{p}} - \frac{d_2 s^2 \bar{f}(p)\, e^{-\xi\sqrt{p}}}{\sqrt{p}[s^4 - \beta^2]}, \tag{4.20}$$

provided

$$\bar{f}(p) = \int_0^\infty e^{-p\tau}\, \hat{f}(\tau)\, d\tau. \tag{4.21}$$

If (4.1() and (4.20) are to be solutions of problem A or problem B, then $B_0(p)$ and $D_0(p)$ must be chosen so that (4.14) or (4.15) are satisfied. Thus, if we apply

(4.16) to (4.14) and substitute (4.19) and (4.20) into the transformed equations a complete solution of problem A (in the transform plane) is given by (4.19), (4.20) provided

$$B_0(p) = \frac{d_1(\beta+1)\bar{f}(p)}{(\beta\sqrt{p}+\sqrt{\gamma})(p-\gamma)} - \frac{d_2\bar{f}(p)}{(s^2+\beta)(\sqrt{p}+\frac{\sqrt{\gamma}}{\beta})p}, \quad (4.22)$$

$$D_0(p) = \frac{d_1\sqrt{p}\bar{f}(p)}{(\beta\sqrt{p}+\sqrt{\gamma})(\sqrt{p}+\sqrt{\gamma})} + \frac{d_2\bar{f}(p)[\beta^2\sqrt{p}+s^2\sqrt{\gamma}]}{(\beta\sqrt{p}+\sqrt{\gamma})\sqrt{p}(s^4-\beta^2)}. \quad (4.23)$$

The zeroth order partial stresses and fluid density are then obtained from (4.19), (4.20) by means of (4.3), (4.4) and (4.8), i.e.

$$\bar{\sigma}_{x0}(\xi,p) = \frac{\hat{\sigma}_0}{p} + \frac{\partial\bar{w}_0}{\partial\xi}(\xi,p) + d_1\bar{\Theta}_0(\xi,p) + (1-f)\hat{\sigma}_0\bar{\eta}_{20}(\xi,p),$$

$$\begin{aligned}\bar{\pi}_{x0}(\xi,p) &= -\frac{\hat{\sigma}_0}{p} + s^2\frac{\partial\bar{v}_0}{\partial\xi}(\xi,p) + f\hat{\sigma}_0\frac{\partial\bar{w}_0}{\partial\xi}(\xi,p) \\ &\quad - d_2\bar{\Theta}_0(\xi,p) - (1-f)\hat{\sigma}_0\bar{\eta}_{20}(\xi,p),\end{aligned} \quad (4.24)$$

$$\bar{\eta}_{20}(\xi,p) = -\frac{1}{p}\frac{\partial\bar{v}_0}{\partial\xi}(\xi,p).$$

By the same procedure the solution of problem B is obtained if

$$B_0(p) = \frac{d_1\bar{f}(p)}{\sqrt{p}(p-\gamma)}, \quad D_0(p) = \frac{d_2 s^2\bar{f}(p)}{(s^4-\beta^2)\sqrt{p}}. \quad (4.25)$$

We are now in a position to invert the zeroth order displacement, velocity, partial stresses and fluid density components for both boundary value problems. Before doing so, we anticipate the requirement of transformed zeroth order quantities needed to determine the first order terms in (4.3) to (4.5) and (4.9) to (4.15). That is, to find $\bar{w}_1(\xi,p)$ and $\bar{v}_1(\xi,p)$ we must solve

$$
\begin{aligned}
&(\gamma \frac{\partial^2}{\partial\xi^2} - p^2)\bar{w}_1(\xi,p) = p\bar{w}_0(\xi,p) - \bar{v}_0(\xi,p) - d_1 \frac{\partial\bar{\Theta}_1}{\partial\xi}(\xi,p), \\
&(s^4 \frac{\partial^2}{\partial\xi^2} - \beta^2 p)\bar{v}_1(\xi,p) = -s^2(p\bar{w}_0(\xi,p) - \bar{v}_0(\xi,p) - d_2 \frac{\partial\bar{\Theta}_1}{\partial\xi}(\xi,p)),
\end{aligned}
\tag{4.26}
$$

$$
(\frac{\partial^2}{\partial\xi^2} - p)\bar{\Theta}_1(\xi,p) = 0, \tag{4.27}
$$

$$
p\bar{\eta}_{21}(\xi,p) = -\frac{\partial\bar{v}_1}{\partial\xi}(\xi,p), \tag{4.28}
$$

subject to (4.14) in problem A and (4.15) in problem B, as well as the regularity conditions as $\xi \longrightarrow \infty$.

We first note that by (4.27) and (4.13),

$$
\bar{\Theta}_1(\xi,p) \equiv 0 \tag{4.29}
$$

so that temperature enters our problems explicitly only in the undisturbed term $\bar{\Theta}_0(\xi,p)$ and implicitly through $\bar{v}_0(\xi,p)$ and $\bar{w}_0(\xi,p)$.

Equations (4.26) have solutions that satisfy the regularity conditions as ξ approaches infinity of the form

$$\bar{w}_1(\xi,p) = B_1(p)e^{-\frac{\xi p}{\sqrt{\gamma}}} - \frac{\xi B_0(p)}{2\sqrt{\gamma}} e^{-\frac{\xi p}{\sqrt{\gamma}}} + D_0(p)e^{-\frac{\xi\beta\sqrt{p}}{s^2}}$$

$$+ \frac{\bar{F}_1(p)e^{-\xi\sqrt{p}}}{p^{3/2}(p-\gamma)}, \qquad (4.30)$$

$$\bar{v}_1(\xi,p) = D_1(p)e^{-\frac{\xi\beta\sqrt{p}}{s^2}} - \frac{B_0(p)s^2\gamma e^{-\frac{p\xi}{\sqrt{\gamma}}}}{s^4p-\beta^2\gamma} - \frac{\xi D_0(p)}{2\beta} e^{-\frac{\xi}{s^2}\beta\sqrt{p}}$$

$$+ \frac{s^2\bar{F}_1(p)e^{-\xi\sqrt{p}}}{(s^4-\beta^2)p^{3/2}} \qquad (4.31)$$

in which we have introduced $\bar{F}_1(p)$ defined by

$$\bar{F}_1(p) = \bar{f}(p)\left[\frac{pd_1}{p-\gamma} - \frac{d_2s^2}{s^4-\beta^2}\right]. \qquad (4.32)$$

In these expressions the constants of integration $B_1(p)$ and $D_1(p)$ are to be chosen so that either (4.14), for problem A, or (4.15), for problem B, are satisfied. In so choosing we must use $B_0(p)$ and $D_0(p)$ as given by (4.22) and (4.23) in the A case and (4.25) in the B case.

For problem A, we apply (4.16) to (4.14) and substitute (4.30) and (4.31) directly. After some algebra we achieve for $\bar{w}_1(\xi,p)$ and $\bar{v}_1(\xi,p)$

$$\bar{w}_1(\xi,p) = B_0(p)\Big[-\frac{\xi}{2\sqrt{\gamma}} + \frac{1}{(\beta\sqrt{p}+\sqrt{\gamma})\,(p-\frac{\beta^2\gamma}{s^4})}\,\Big(\sqrt{\gamma} - \beta\sqrt{p} - \frac{2\beta\gamma}{s^2\sqrt{p}}\Big)$$

$$- \frac{s^2(\sqrt{p} + \frac{\beta\sqrt{\gamma}}{s^2})}{2\beta p(\beta\sqrt{p}+\sqrt{\gamma})}\Big]\, e^{-\frac{p\xi}{\sqrt{\gamma}}} + B_0(p)\, e^{-\frac{\xi\beta}{s^2}\sqrt{p}}$$

$$+ \frac{\bar{F}_1(p)}{\beta\sqrt{p}+\sqrt{\gamma}}\Big[\frac{\beta}{p(p-\frac{\beta^2\gamma}{s^4})}\Big(\frac{\gamma}{s^2 p} + 1\Big) - \frac{\gamma+\beta p}{p^2(p-\gamma)}$$

$$+ \frac{s^2(s^2-\beta)}{2\beta p^2(s^2+\beta)}\Big]\, e^{-\frac{p\xi}{\sqrt{\gamma}}} - \frac{\bar{F}_1(p)}{p^{3/2}(p-\frac{\beta^2\gamma}{s^4})}\, e^{-\frac{\xi\beta}{s^2}} + \frac{\bar{F}_1(p)}{p^{3/2}(p-\gamma)}\, e^{-\xi\sqrt{p}}, \tag{4.33}$$

$$\bar{v}_1(\xi,p) = B_0(p)\Big[-\frac{\xi}{2\beta}\sqrt{p} + \frac{1}{(\beta\sqrt{p}+\sqrt{\gamma})\,(p-\frac{\beta^2\gamma}{s^4})}\,\Big(2\sqrt{\gamma}\,p\, -$$

$$\frac{\beta\gamma}{s^2}\sqrt{p} + \frac{\gamma^{3/2}}{s^2}\Big) - \frac{\beta\sqrt{\gamma} + s^2\sqrt{p}}{2\beta(\beta\sqrt{p}+\sqrt{\gamma})}\Big]\, e^{-\frac{\xi\beta}{s^2}\sqrt{p}} - \frac{\gamma B_0(p)}{s^2(p-\frac{\beta^2\gamma}{s^4})}\, e^{-\frac{\xi p}{\sqrt{\gamma}}}$$

$$+ \bar{F}_1(p)\Big[\frac{\xi}{2\beta p} + \frac{1}{p(\beta\sqrt{p}+\sqrt{\gamma})}\Big(\frac{\sqrt{\gamma}}{\sqrt{p}+\sqrt{\gamma}} + \frac{s^2}{2\beta} - \frac{s^4}{s^4-\beta^2}$$

$$- \frac{\sqrt{\gamma}\, s^2}{\sqrt{p}(s^4-\beta^2)} - \frac{\sqrt{\gamma}}{\sqrt{p} + \frac{\beta\sqrt{\gamma}}{s^2}} \Big)\Big] e^{-\frac{\xi\beta}{s^2}\sqrt{p}} + \frac{s^2 \bar{F}_1(p)}{p^{3/2}(s^4-\beta^2)} e^{-\xi\sqrt{p}}, \tag{4.34}$$

wherein $B_0(p)$ is given by (4.22).

Turning now to problem B we find that if

$$B_1(p) = \frac{B_0(p)}{4p} - \frac{D_0(p)}{p(p - \frac{\gamma\beta^2}{s^4})} - \frac{\bar{F}_1(p)}{\sqrt{p}(p-\gamma)}, \tag{4.35}$$

$$D_1(p) = \frac{\gamma B_0(p)}{s^2(p - \frac{\gamma\beta^2}{s^4})} + \frac{s^2 D_0(p)}{4p\beta^2} - \frac{s^2 \bar{F}_1(p)}{\sqrt{p}(s^4-\beta^2)}, \tag{4.36}$$

are used in (4.30) and (4.31), then (4.15) are satisfied provided $B_0(p)$, $D_0(p)$ are as given in (4.25).

Partial stresses and fluid density can be found by substituting the displacement and velocity expressions (4.30), (4.31) into (4.3), (4.4) and (4.11) after inversion has been accomplished.

To conclude this section we make two remarks. The first is the fact that second and higher order terms may be found at will by the methods employed here.

The second remark anticipates the fact that these solutions (4.18) to (4.20) and (4.30) to (4.31) are valid for any imposed temperature distribution $\hat{f}(\tau)$ provided $\bar{f}(p)$ exists. If we take $\bar{f}(p) = 1$, then the field variables $\bar{w}_0$, $\bar{v}_0$, $\bar{w}_1$, $\bar{v}_1$, etc, are those that result from a thermal boundary loading of the form

$$\Theta(0,\tau) = \delta(\tau)$$

and these same quantities can be related to those which result from any other loading, say $\Phi(\tau)$, by means of a convolution integral.

Using this fact, we devote section 5 to the inversion of (4.18) to (4.20) and (4.31), (4.32) when $\bar{f}(p) = 1$.

5. <u>Inversion for the case $\bar{f}(p) = 1$</u>. Inversion is readily accomplished with the aid of tables and in terms of some formulae listed below.

Define

$$L^{-1}\left[\frac{e^{-\xi p}}{p-k_1} \cdot \frac{1}{\sqrt{p}+k_2}\right] \equiv I_1(k_1,\xi,\tau,k_2) \tag{5.1}$$

$$L^{-1}\left[\frac{e^{-\xi p}}{p-k_1} \cdot \frac{1}{(\sqrt{p}+k_2)^2}\right] \equiv -\frac{\partial I_1}{\partial k_2}(k_1,\xi,\tau,k_2) \tag{5.2}$$

$$L^{-1}\left[\frac{1}{p-k_1} \cdot \frac{e^{-\xi\sqrt{p}}}{\sqrt{p}+k_2}\right] \equiv I_3(k_1,\xi,\tau,k_2) \tag{5.3}$$

$$L^{-1}\left[\frac{e^{-k_1\sqrt{p}}}{\sqrt{p}+k_2}\right] \equiv I_4(k_1,k_2,\tau) \tag{5.4}$$

$$L^{-1}\left[\frac{e^{-k_1\sqrt{p}}}{(\sqrt{p}+k_2)^2}\right] \equiv -\frac{\partial I_4}{\partial k_2}(k_1,k_2,\tau) \tag{5.5}$$

where

$$I_1(k_1,\xi,\tau,k_2) = \int_0^{\tau-\xi} e^{k_1(\tau-\xi-r)}\Big[\frac{1}{\sqrt{\pi r}} - k_2 e^{k_2^2 r}\,\mathrm{Erfc}(k_2\sqrt{r})\Big]dr H(\tau-\xi) \tag{5.6}$$

by no. 4, p. 223 and no. 9, p. 242 of [13],

$$I_3(k_1,\xi,\tau,k_2) = \int_0^{\tau} e^{k_1(\tau-r)}\Big[\frac{e^{-\frac{\xi^2}{4r}}}{\sqrt{\pi r}} - k_2 e^{k_2\xi+k_2^2 r}\,\mathrm{Erfc}\Big(\frac{\xi}{2\sqrt{r}} + k_2\sqrt{r}\Big)\Big]dr H(\tau) \tag{5.7}$$

by no. 1, p. 229 and no. 12, p. 246 of [13],

$$I_4(k_1,k_2,\tau) = \frac{e^{-\frac{k_1^2}{4t}}}{\sqrt{\pi\tau}} - k_2 e^{k_1k_2+k_2^2\tau}\,\mathrm{Erfc}\Big(\frac{k_1}{2\sqrt{\tau}} + k_2\sqrt{\tau}\Big), \tag{5.8}$$

by no. 12, p. 246 of [13].

$\mathrm{Erfc}(\lambda)$ is the complimentary error function defined by

$$\mathrm{Erfc}(\lambda) = \frac{2}{\sqrt{\pi}}\int_\lambda^\infty e^{-m^2}dm \tag{5.9}$$

and is related to the error function $\Phi(\lambda)$ by

$$\mathrm{Erfc}(\lambda) = 1 - \Phi(\lambda). \tag{5.10}$$

Some useful properties of these functions (see [14], for example) are

$$\left.\begin{aligned} &\mathrm{Erfc}(\infty) = 0, \ \mathrm{Erfc}(0) = 1, \\ &\Phi(\infty) = 1, \ \Phi(0) = 0, \\ &\Phi(-\lambda) = -\Phi(\lambda), \\ &\mathrm{Erfc}(-\lambda) = 1 + \Phi(\lambda) = 2 - \mathrm{Erfc}(\lambda), \end{aligned}\right\} \tag{5.11}$$

and any others will be listed as required.

We invert the zeroth order terms for problem A first and follow with the first order terms. Problem B is treated separately.

Beginning with (4.18) (for $\bar{f}(p) = 1$), the temperature field becomes, by means of no. 1, p. 245 of [13],

$$\Theta_0(\xi,\tau) = \frac{\xi}{2\sqrt{\pi}\ \tau^{3/2}}\, e^{-\xi^2/4\tau} . \tag{5.12}$$

Equations (4.19) and (4.20) can be inverted directly after (4.22) and (4.23) are used to replace $B_0(p)$ and $D_0(p)$. In terms of (5.1) to (5.5) we have

$$w_0(\xi,\tau) = \frac{d_1(\beta+1)}{\beta} I_1(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta}) - \frac{d_2}{s^2+\beta} I_1(0, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta})$$

$$- d_1 I_3(\gamma,\xi,\tau,0), \tag{5.13}$$

and

$$v_0(\xi,\tau) = \left[\frac{d_1}{\beta(1-\beta)} - \frac{d_2}{s^2+\beta}\right] I_4\left(\frac{\xi\beta}{s^2}, \frac{\sqrt{\gamma}}{\beta}, \tau\right)$$

$$+ \frac{d_1}{\beta-1} I_4\left(\frac{\xi\beta}{s^2}, \sqrt{\gamma}, \tau\right) + \frac{d_2 s^2}{s^4-\beta^2} I_4\left(\frac{\xi\beta}{s^2}, 0, \tau\right)$$

$$- \frac{d_2 s^2}{s^4-\beta^2} I_4(\xi, 0, \tau). \tag{5.14}$$

To calculate the partial stresses we require $\eta_{20}(\xi,\tau)$. Turning to (4.24) we have

$$\bar{\eta}_{20}(\xi,p) = -\frac{1}{p}\frac{\partial \bar{v}_0}{\partial \xi}(\xi,p)$$

or, what is equivalent,

$$\eta_{20}(\xi,\tau) = -\int_0^\tau \frac{\partial v_0}{\partial \xi}(\xi,r)\,dr. \tag{5.15}$$

From (5.14) and (5.3), (5.4) we have

$$\eta_{20}(\xi,\tau) = -\int_0^\tau \left\{\left[\frac{d_1}{s^2(1-\beta)} - \frac{d_2\beta}{s^2(s^2+\beta)}\right]\frac{\partial I_4}{\partial k_1}\left(k_1, \frac{\sqrt{\gamma}}{\beta}, r\right)\right.$$

$$+ \frac{d_1\beta}{s^2(\beta-1)}\frac{\partial I_4}{\partial k_1}(k_1, \sqrt{\gamma}, r) + \frac{d_2\beta}{s^4-\beta^2}\frac{\partial I_4}{\partial k_1}(k_1, 0, r)$$

$$\left. - \frac{d_2 s^2}{s^4-\beta^2}\frac{\partial I_4}{\partial \xi}(\xi, 0, r)\right\} dr \tag{5.16}$$

wherein $k_1 = \xi\beta/s^2$. Interchanging $\partial/\partial k_1$ and integration, and using

$$\int_0^\tau I_4(k_1,k_2,r)\,dr = I_3(0,k_1,\tau,k_2)$$

we arrive at

$$\eta_{20}(\xi,\tau) = -\frac{\partial}{\partial\zeta}\Big[\Big(\frac{d_1}{s^2(1-\beta)} - \frac{d_2\beta}{s^2(s^2+\beta)}\Big) I_3(0,\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta})$$

$$+ \frac{d_1\beta}{s^2(\beta-1)} I_3(0,\zeta,\tau,\sqrt{\gamma}) + \frac{d_2\beta}{s^4-\beta^2} I_3(0,\zeta,\tau,0)\Big]\Big/_{\zeta = \frac{\xi\beta}{s^2}}$$

$$+ \frac{d_2 s^2}{s^4-\beta^2}\frac{\partial I_3}{\partial\xi}(0,\xi,\tau,0) \tag{5.17}$$

as the final form for η_{20}. We note that the integrand in (5.16) is $\partial v_0(\xi,r)/\partial\xi$.

To complete the zero order terms we return to (4.3), (4.4) and note that all quantities needed to describe $\sigma_{x0}(\xi,\tau)$ and $\pi_{x0}(\xi,\tau)$ are known.

The first order terms are easily inverted although the algebra is longer. Without further delay, the inversion of (4.33) with the aid of (4.22) gives

$$w_1(\xi,\tau) = \Big(a_{11} - \frac{d_1(\beta+1)\xi}{2\beta\sqrt{\gamma}}\Big) I_1(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta})$$

$$+ \Big(a_{12} + \frac{d_2\xi}{2\sqrt{\gamma}(s^2+\beta)}\Big) I_1(0, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta})$$

$$+ a_{13}I_1(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + a_{14}I_1(\frac{\beta^2\gamma}{s^4}, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + a_{15}I_1(\frac{\beta^2\gamma}{s^4}, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta}$$

$$+ a_{16}I_1(0, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + a_{17}\frac{\partial I_1}{\partial k_1}(k_1, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta})\Big/_{k_1=\gamma}$$

$$+ a_{18}\frac{\partial I_1}{\partial_k 1}(k_1, \frac{\xi}{\sqrt{\gamma}}, \tau, \frac{\sqrt{\gamma}}{\beta})\Big/_{k_1=0}$$

$$+ a_{19}\frac{\partial I_1}{\partial k_2}\left(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, k_2\right)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}} + a_{20}\frac{\partial I_1}{\partial k_2}\left(\frac{\beta^2\gamma}{s^4}, \frac{\xi}{\sqrt{\gamma}}, \tau, k_2\right)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+ a_{21}\frac{\partial I_1}{\partial k_2}\left(0, \frac{\xi}{\sqrt{\gamma}}, \tau, k_2\right)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+ a_{22}\frac{\partial^2 I_1}{\partial k_1 \partial k_2}\left(k_1, \frac{\xi}{\sqrt{\gamma}}, \tau, k_2\right)\Big/_{\substack{k_1=0\\ k_2=\frac{\sqrt{\gamma}}{\beta}}} + a_{23} I_3\left(\frac{\gamma\beta^2}{s^4}, \frac{\xi\beta}{s^2}, \tau, 0\right)$$

$$+ a_{24} I_3\left(\frac{\gamma\beta^2}{s^4}, \frac{\xi\beta}{s^2}, \tau, \frac{\sqrt{\gamma}}{\beta}\right)$$

$$+ a_{25} I_3\left(\frac{\gamma\beta^2}{s^4}, \frac{\xi\beta}{s^2}, \tau, \sqrt{\gamma}\right) + a_{26} I_3\left(0, \frac{\xi\beta}{s^2}, \tau, \frac{\sqrt{\gamma}}{\beta}\right) + a_{27} I_3\left(0, \frac{\xi\beta}{s^2}, \tau, 0\right)$$

$$+ a_{28}\frac{\partial I_3}{\partial k_1}(k_1, \xi, \tau, 0)\Big/_{k_1=\gamma} + a_{29}[I_3(\gamma, \xi, \tau, 0) - I_3(0, \xi, \tau, 0)]. \quad (5.18)$$

Coefficients a_{11} to a_{29} are listed below.

$$a_{11} = \frac{d_1}{\gamma}\left[\frac{s^2}{s^4-\beta^2}(s^2+1+3\beta(\beta+1)) + \frac{s^2(\beta+1)}{2\beta} + \frac{1}{\beta} + \frac{s^2(s^2-\beta)}{2\beta(s^2+\beta)}\right] + \frac{d_2 s^2(\beta+1)}{\gamma\beta(s^4-\beta^2)},$$

$$a_{12} = \frac{d_1(s^2-\beta)(s^2+2\beta)}{2\beta^2\gamma(s^2+\beta)} + \frac{d_2 s^2}{\gamma(s^2+\beta)}\left[2 - \frac{1}{2\beta} + \frac{s^4(s^2+\beta^2)}{\beta^4(s^2-\beta)} - \frac{(\beta+1)}{\beta(s^2-\beta)}\right],$$

$$a_{13} = -\frac{d_1 s^2(\beta+1)(s^4+5\beta^2)}{2\beta\gamma(s^4-\beta^2)}, \quad a_{14} = \frac{\beta^2+2s^4}{s^2\gamma(s^2+\beta)}\left[\frac{d_1\beta(\beta+1)}{s^2-\beta} + d_2\right],$$

$$a_{15} = -\frac{d_1}{\gamma(s^4-\beta^2)}\left[\frac{s^4(s^2+\beta^2)}{\beta^2} + \frac{\beta(\beta+1)(\beta^2+2s^4)}{s^2}\right] - \frac{d_2}{\gamma(s^2+\beta)}\left[\frac{\beta^2+2s^4}{s^2} + \frac{s^6(s^2+\beta^2)}{\beta^4(s^2-\beta)}\right],$$

$$a_{16} = \frac{d_2 s^2(1-4\beta)}{2\beta\gamma(s^2+\beta)} \;, \; a_{17} = -\frac{d_1(\beta+1)}{\beta} \;, \; a_{18} = -\frac{d_2 s^2}{\beta(s^2+\beta)}\left[\frac{1}{s^2-\beta} + \frac{s^2}{2\beta(s^2+\beta)}\right] ,$$

$$a_{19} = -\frac{d_1(\beta+1)}{\beta^2\sqrt{\gamma}}\left[\frac{s^2(s^2+3\beta^2)}{s^4-\beta^2} + \frac{s^2-1}{2}\right] ,$$

$$a_{20} = +\frac{1}{\beta^2\sqrt{\gamma}(s^2+\beta)}\left(\frac{d_1(\beta+1)}{s^2-\beta} + \frac{d_2}{\beta}\right)\left(s^4+2s^2\beta^2+\frac{\beta^4}{s^2}\right) ,$$

$$a_{21} = -\frac{d_1(\beta+1)}{2\beta^2\sqrt{\gamma}} - \frac{d_2 s^2}{\beta\sqrt{\gamma}(s^2+\beta)}\left[\frac{s^2}{\beta^2} + 2\beta^2 - \frac{1}{2} + \frac{s^2\sqrt{\gamma}}{\beta(s^2-\beta)}\right] ,$$

$$a_{22} = -\frac{d_2\sqrt{\gamma}}{2\beta(s^2+\beta)} \;, \; a_{23} = \frac{d_1}{\gamma} + \frac{s^6 d_2}{\gamma\beta^2(s^4-\beta^2)} \;, \; a_{24} = \frac{d_1\beta}{\gamma(1-\beta)} - \frac{s^4 d_2}{\gamma\beta^2(s^2+\beta)} ,$$

$$a_{25} = \frac{d_1}{\gamma(\beta-1)} \;, \; a_{26} = \frac{s^4 d_2}{\gamma\beta^2(s^2+\beta)} \;, \; a_{27} = -\frac{s^6 d_2}{\gamma\beta^2(s^4-\beta^2)} ,$$

$$a_{28} = d_1, \; a_{29} = -\frac{d_2 s^2}{\gamma(s^4-\beta^2)} \;. \qquad (5.19)$$

Proceeding in the same manner with the inversion of (4.34) we have

$$v_1(\xi,\tau) = b_1 I_4(\frac{\xi\beta}{s^2}, -\sqrt{\gamma}, \tau) + (b_2 + \frac{d_1\xi}{2\beta\sqrt{\gamma}(\beta-1)}) I_4(\frac{\xi\beta}{s^2}, \sqrt{\gamma}, \tau)$$

$$+ \left[b_3 - \frac{\xi}{2\sqrt{\gamma}}\left(\frac{d_1}{\beta(\beta-1)} + \frac{d_2}{s^2+\beta}\right)\right] I_4\left(\frac{\xi\beta}{s^2}, \frac{\sqrt{\gamma}}{\beta}, \tau\right)$$

$$+ (b_4 + \frac{d_2\xi}{2\sqrt{\gamma}(s^2+\beta)}) I_4(\frac{\xi\beta}{s^2}, 0, \tau) + b_5 \frac{\partial I_4}{\partial k_2}(\frac{\xi\beta}{s^2}, k_2, \tau)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+ \frac{d_2 s^2\xi}{2\beta(s^4-\beta^2)} \frac{\partial I_4}{\partial k_2}(\frac{\xi\beta}{s^2}, k_2, \tau)\Big/_{k_2=0} + b_6 \frac{\partial I_3}{\partial k_2}(\gamma, \frac{\xi\beta}{s^2}, \tau, k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+ b_7 \frac{\partial I_3}{\partial k_2}(\frac{\beta^2\gamma}{s^4}, \frac{\xi\beta}{s^2}, \tau, k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}} + b_8 \frac{\partial I_3}{\partial k_2}(0, \frac{\xi\beta}{s^2}, \tau, k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+ b_9 I_3(\frac{\beta^2\gamma}{s^4}, \frac{\xi\beta}{s^2}, \tau, -\sqrt{\gamma}) + b_{10} I_3(\frac{\beta^2\gamma}{s^4}, \frac{\xi\beta}{s^2}, \tau, \sqrt{\gamma})$$

$$+ b_{11} I_3(\frac{\beta^2\gamma}{s^4}, \frac{\xi\beta}{s^2}, \tau, \frac{\sqrt{\gamma}}{\beta}) + b_{12} I_3(\frac{\beta^2\gamma}{s^4}, \frac{\xi\beta}{s^2}, \tau, 0)$$

$$+ b_{13} I_3(\gamma, \frac{\xi\beta}{s^2}, \tau, \frac{\sqrt{\gamma}}{\beta})$$

$$+ b_{14} I_3(\gamma, \frac{\xi\beta}{s^2}, \tau, 0) + b_{15} I_3(0, \frac{\xi\beta}{s^2}, \tau, \frac{\sqrt{\gamma}}{\beta}) + b_{16} I_3(\gamma, \frac{\xi\beta}{s^2}, \tau, \sqrt{\gamma})$$

$$+b_{17}I_3\left(0,\frac{\xi\beta}{s^2},\tau,0\right)+b_{18}I_1\left(\frac{\beta^2\gamma}{s^4},\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{19}I_1\left(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta}\right)$$

$$+b_{20}I_1\left(0,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{21}I_3(\gamma,\xi,\tau,0)-b_{17}I_3(0,\xi,\tau,0), \quad (5.20)$$

with the b_k defined by

$$b_1=-\frac{d_1 s^2}{4\beta\gamma(\beta+1)}, \quad b_2=-\frac{d_1 s^2(\beta+1)}{2\beta\gamma(1-\beta)^2}, \quad b_3=\frac{d_1 s^2(\beta^2+1)}{2\beta\gamma(\beta-1)^2(\beta+1)}-\frac{d_2 s^2}{2\gamma(s^2+\beta)},$$

$$b_4=\frac{d_2 s^2}{2\gamma(s^2+\beta)}, \quad b_5=\frac{s^2}{2\beta\sqrt{\gamma}}\left[\frac{d_1}{\beta(\beta-1)}+\frac{d_2}{s^2+\beta}\right],$$

$$b_6=\frac{d_1(\beta+1)\sqrt{\gamma}}{2\beta^2}\left[1-\frac{2s^2(2s^2+1)}{s^4-\beta^2}\right],$$

$$b_7=d_1\sqrt{\gamma}\left[\frac{(\beta+1)(s^2+2\beta^2)}{\beta^2(s^4-\beta^2)}+\frac{1}{s^2(\beta-1)}\right]+\frac{d_2\sqrt{\gamma}}{s^2+\beta}\left[\frac{s^2}{\beta^2}+\frac{2}{\beta}+\frac{\beta}{s^2}\right],$$

$$b_8=-\frac{d_2\sqrt{\gamma}(2s^2+\beta^2)}{2\beta^3(s^2+\beta)}, \quad b_9=-\frac{d_1\beta}{2s^2(\beta+1)}, \quad b_{10}=-\frac{d_1\beta(\beta+1)}{2s^2(1-\beta)^2},$$

$$b_{11}=d_1\left[\frac{\beta(\beta^2+1)}{s^2(\beta+1)(\beta-1)^2}-\frac{s^2+\beta^2}{s^4-\beta^2}\right]-\frac{d_2}{s^2+\beta}\left[\frac{\beta^2}{s^2}+\frac{s^2+\gamma}{s^2-\beta}\right]$$

$$b_{12}=\frac{d_1\beta^2}{s^4-\beta^2}+\frac{d_2}{s^2+\beta}\left[\frac{s^2}{s^2-\beta}+\frac{\beta^2}{s^2}\right],$$

$$b_{13}=d_1\left[\frac{s^2}{2\beta^2}+\frac{1}{\beta-1}+\frac{s^4(\beta-1)+2s^2\beta}{\beta(s^4-\beta^2)}\right]+\frac{d_2 s^2(\beta-1)}{\beta(s^4-\beta^2)} ,$$

$$b_{14}= -\frac{s^2}{s^4-\beta^2}\,(d_1(s^2+1)+d_2) ,\quad b_{15}=\frac{d_2}{s^4-\beta^2}\,(\gamma-\frac{s^4}{2\beta^2}-\frac{s^2}{s^2+\beta}) ,$$

$$b_{16}=\frac{d_1}{1-\beta} ,\quad b_{17}=\frac{d_2 s^4}{(s^4-\beta^2)^2} ,\quad b_{18}=\frac{s^2}{\beta(s^2+\beta)}\left(\frac{d_1(\beta+1)}{s^2-\beta}+\frac{d_2}{\beta}\right),$$

$$b_{19}= -\frac{d_1 s^2(\beta+1)}{\beta(s^4-\beta^2)} ,\quad b_{20}= -\frac{d_2 s^2}{\beta^2(s^2+\beta)} ,\quad b_{21}=\frac{d_1 s^2}{s^4-\beta^2} . \qquad (5.21)$$

To compute the partial stresses σ_{x1}, π_{x1}, and the fluid density η_{21} we require both $\partial w_1/\partial\xi$ and $\partial v_1/\partial\xi$. By (5.18) and (5.20) these quantities are

$$\frac{\partial w_1}{\partial\xi}(\xi,\tau) = -\frac{d_1(\beta+1)}{2\beta\sqrt{\gamma}}\,I_1(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta})+\left(\frac{a_{11}}{\sqrt{\gamma}}-\frac{d_1(\beta+1)\xi}{2\beta\gamma}\right)\frac{\partial I_1}{\partial\zeta}(\gamma,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$+\frac{d_2}{2\sqrt{\gamma}(s^2+\beta)}\,I_1(0,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta})+\left(\frac{a_{12}}{\sqrt{\gamma}}+\frac{d_2\xi}{2\gamma(s^2+\beta)}\right)\frac{\partial I_1}{\partial\zeta}(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$+\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial\zeta}\Big[a_{13}I_1(\gamma,\zeta,\tau,0)+a_{14}I_1(\frac{\beta^2\gamma}{s^4},\zeta,\tau,0)+a_{15}I_1(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})$$

$$+a_{16}I_1(0,\zeta,\tau,0)+a_{17}\frac{\partial I_1}{\partial k_1}(k_1,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})\Big/_{k_1=\gamma}+a_{18}\frac{\partial I_1}{\partial k_1}(k_1,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})\Big/_{k_1=0}$$

$$+a_{19}\frac{\partial I_1}{\partial k_2}(\gamma,\zeta,\tau,k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}+a_{20}\frac{\partial I_1}{\partial k_2}(\frac{\beta^2\gamma}{s^4},\zeta,\tau,k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+a_{21}\frac{\partial I_1}{\partial k_2}(0,\zeta,\tau,k_2)\Big/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+a_{22}\frac{\partial^2 I_1}{\partial k_1\partial k_2}(k_1,\zeta,\tau,k_2)\Big/_{\substack{k_1=0\\ k_2=\frac{\sqrt{\gamma}}{\beta}}}\Big]\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$+\frac{\beta}{s^2}\frac{\partial}{\partial\zeta}\Big[a_{23}I_3(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0)+a_{24}I_3(\frac{\gamma\beta^2}{s^4},\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})+a_{25}I_3(\frac{\gamma\beta^2}{s^4},\zeta,\tau,\sqrt{\gamma})$$

$$+a_{26}I_3(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta})+a_{27}I_3(0,\zeta,\tau,0)\Big]\Big/_{\zeta=\frac{\xi\beta}{s^2}}$$

$$+a_{28}\frac{\partial^2 I_3}{\partial\xi\partial k_1}(k_1,\xi,\tau,0)\Big/_{k_1=\gamma}+a_{29}\frac{\partial}{\partial\xi}[I_3(\gamma,\xi,\tau,0)-I_3(0,\xi,\tau,0)], \quad (5.22)$$

and

$$\frac{\partial v_1}{\partial \xi}(\xi,\tau) = \frac{d_1}{2\beta\sqrt{\gamma}(\beta-1)} I_4\left(\frac{\xi\beta}{s^2}, \sqrt{\gamma}, \tau\right) - \frac{1}{2\sqrt{\gamma}}\Bigg(\frac{d_1}{\beta(\beta-1)}$$

$$+ \frac{d_2}{s^2+\beta}\Bigg) I_4\left(\frac{\xi\beta}{s^2}, \frac{\sqrt{\gamma}}{\beta}, \tau\right)$$

$$+ \frac{d_2}{2\sqrt{\gamma}(s^2+\beta)} I_4\left(\frac{\xi\beta}{s^2}, 0, \tau\right) + \frac{d_2 s^2}{2\beta(s^4-\beta^2)} \frac{\partial I_4}{\partial k_2}\left(\frac{\xi\beta}{s^2}, k_2, \tau\right)\Bigg/_{k_2=0}$$

$$+ \frac{\beta}{s^2}\frac{\partial}{\partial \zeta}\Bigg[b_1 I_4(\zeta, -\sqrt{\gamma}, \tau) + \left(b_2 + \frac{d_1\xi}{2\beta\sqrt{\gamma}(\beta-1)}\right) I_4(\zeta, \sqrt{\gamma}, \tau)$$

$$+\left(b_3 - \frac{\xi}{2\sqrt{\gamma}}\left(\frac{d_1}{\beta(\beta-1)} + \frac{d_2}{s^2+\beta}\right)\right) I_4\left(\zeta, \frac{\sqrt{\gamma}}{\beta}, \tau\right) + \Bigg(b_4$$

$$+ \frac{d_2\xi}{2\sqrt{\gamma}(s^2+\beta)}\Bigg) I_4(\zeta, 0, \tau)$$

$$+b_5 \frac{\partial I_4}{\partial k_2}(\zeta, k_2, \tau)\Bigg/_{k_2=\frac{\sqrt{\gamma}}{\beta}} + \frac{d_2 s^2 \xi}{2\beta(s^4-\beta^2)} \frac{\partial I_4}{\partial k_2}(\zeta, k_2, \tau)\Bigg/_{k_2=0}$$

$$+b_6 \frac{\partial I_3}{\partial k_2}(\gamma, \zeta, \tau, k_2)\Bigg/_{k_2=\frac{\sqrt{\gamma}}{\beta}} + b_7 \frac{\partial I_3}{\partial k_2}\left(\frac{\beta^2\gamma}{s^4}, \zeta, \tau, k_2\right)\Bigg/_{k_2=\frac{\sqrt{\gamma}}{\beta}}$$

$$+b_8 \frac{\partial I_3}{\partial k_2}(0, \zeta, \tau, k_2)\Bigg/_{k_2=\frac{\sqrt{\gamma}}{\beta}} + b_9 I_3\left(\frac{\beta^2\gamma}{s^4}, \zeta, \tau, -\sqrt{\gamma}\right)$$

$$+b_{10}I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\sqrt{\gamma}\right)+b_{11}I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{12}I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,0\right)$$

$$+b_{13}I_3\left(\gamma,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{14}I_3(\gamma,\zeta,\tau,0)+b_{15}I_3\left(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)$$

$$+b_{16}I_3(\gamma,\zeta,\tau,\sqrt{\gamma}) + b_{17}I_3(0,\zeta,\tau,0)\Big]\Big/_{\zeta=\frac{\xi\beta}{s^2}}$$

$$+\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial\zeta}\left[b_{18}I_1\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{19}I_1\left(\gamma,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)+b_{20}I_1\left(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta}\right)\right]\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$+b_{21}\frac{\partial I_3}{\partial\xi}(\gamma,\xi,\tau,0) - b_{17}\frac{\partial I_3}{\partial\xi}(0,\xi,\tau,0). \qquad (5.23)$$

The next step in the process is to use (5.23) in (4.11), or its equivalent,

$$\eta_{21}(\xi,\tau) = -\int_0^\tau \frac{\partial v_1}{\partial\xi}(\xi,r)\,dr. \qquad (5.24)$$

The integration of (5.23) is accomplished directly if we note that differentiation with respect to ξ, k_1, or k_2 may be interchanged with the time integral, and, if we use

$$\int_0^\tau I_1(k_1,\xi,r,k_2)\,dr = \begin{cases} \frac{1}{k_1}\left[I_1(k_1,\xi,\tau,k_2)-I_1(0,\xi,\tau,k_2)\right] & \text{for } k_1\neq 0, \\ \frac{\partial I_1}{\partial k_1}(k_1,\xi,\tau,k_2)\Big/_{k_1=0}; & \end{cases}$$

$$\int_0^\tau I_3(k_1,\xi,r,k_2)\,dr = \begin{cases} \dfrac{1}{k_1}\left[I_3(k_1,\xi,\tau,k_2)-I_3(0,\xi,\tau,k_2)\right] & \text{for } k_1\neq 0 \\ \dfrac{\partial I_3}{\partial k_1}(k_1,\xi,\tau,k_2)\Big/_{k_1=0}\,; & \end{cases}$$

$$\int_0^\tau I_4(k_1,k_2,r)\,dr = I_3(0,k_1,\tau,k_2)\,. \qquad (5.25)$$

Thus,

$$\eta_{21}(\xi,\tau) = -\frac{d_1}{2\beta\sqrt{\gamma}(\beta-1)}\,I_3(0,\frac{\xi\beta}{s^2},\tau,\sqrt{\gamma}) + \frac{1}{2\sqrt{\gamma}}\,(\frac{d_1}{\beta(\beta-1)}$$

$$+\frac{d_2}{s^2+\beta})\,I_3(0,\frac{\xi\beta}{s^2},\tau,\frac{\sqrt{\gamma}}{\beta})$$

$$-\frac{d_2}{2\sqrt{\gamma}(s^2+\beta)}\,I_3(0,\frac{\xi\beta}{s^2},\tau,0) - \frac{d_2 s^2}{2\beta(s^4-\beta^2)}\,\frac{\partial I_3}{\partial k_2}(0,\frac{\xi\beta}{s^2},\tau,k_2)\Big/_{k_2=0}$$

$$-\frac{\beta}{s^2}\frac{\partial}{\partial\zeta}\Big[(b_1-\frac{b_9 s^4}{\beta^2\gamma})\,I_3(0,\zeta,\tau,-\sqrt{\gamma}) + (\frac{d_1\xi}{2\beta\sqrt{\gamma}(\beta-1)} + b_2$$

$$-\frac{b_{10}s^4}{\beta^2\gamma} - \frac{b_{16}}{\gamma})\,I_3(0,\zeta,\tau,\sqrt{\gamma})$$

$$+\left(b_3 - \frac{\xi}{2\sqrt{\gamma}}\left(\frac{d_1}{\beta(\beta-1)} + \frac{d_2}{s^2+\beta}\right) - \frac{b_{11}s^4}{\beta^2\gamma} - \frac{b_{13}}{\gamma}\right) I_3(0,\zeta,\tau,\sqrt{\tfrac{\gamma}{\beta}})$$

$$+\left(b_4 + \frac{d_2\xi}{2\sqrt{\gamma}(s^2+\beta)} - \frac{b_{12}s^4}{\beta^2\gamma} - \frac{b_{14}}{\gamma}\right) I_3(0,\zeta,\tau,0) + \left(b_5 - \frac{b_6}{\gamma}\right.$$

$$\left. - \frac{b_7 s^4}{\beta^2\gamma}\right) \frac{\partial I_3}{\partial k_2}(0,\zeta,\tau,k_2)\Big/_{k_2=\sqrt{\frac{\gamma}{\beta}}}$$

$$+ \frac{d_2 s^2 \xi}{2\beta(s^4-\beta^2)} \frac{\partial I_3}{\partial k_2}(0,\zeta,\tau,k_2)\Big/_{k_2=0} + \frac{\partial}{\partial k_2}\left\{\frac{b_6}{\gamma} I_3(\gamma,\zeta,\tau,k_2)\right.$$

$$\left. + \frac{b_7 s^4}{\beta^2\gamma} I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,k_2\right)\right\}\Big/_{k_2=\sqrt{\frac{\gamma}{\beta}}}$$

$$+b_8 \frac{\partial^2 I_3}{\partial k_2 \partial k_1}(k_1,\zeta,\tau,k_2)\Big/_{\substack{k_1=0\\ k_2=\sqrt{\frac{\gamma}{\beta}}}} + \frac{\partial}{\partial k_1}\left\{b_{15} I_3(k_1,\zeta,\tau,\sqrt{\tfrac{\gamma}{\beta}})\right.$$

$$\left. + b_{17} I_3(k_1,\zeta,\tau,0)\right\}\Big/_{k_1=0}$$

$$+ \frac{s^4}{\beta^2\gamma}\left\{b_9 I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,-\sqrt{\gamma}\right) + b_{10} I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\sqrt{\gamma}\right) + b_{11} I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,\sqrt{\tfrac{\gamma}{\beta}}\right)\right.$$

$$\left. + b_{12} I_3\left(\frac{\beta^2\gamma}{s^4},\zeta,\tau,0\right)\right\}$$

$$+ \frac{1}{\gamma}\left\{b_{13}I_3(\gamma,\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta}) + b_{14}I_3(\gamma,\zeta,\tau,0) + b_{16}I_3(\gamma,\zeta,\tau,+\sqrt{\gamma})\right\}\Big]\Big/_{\zeta=\frac{\xi\beta}{s^2}}$$

$$- \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial\zeta}\Big[\frac{s^4 b_{18}}{\beta^2\gamma} I_1(\tfrac{\beta^2\gamma}{s^4},\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta}) + \frac{b_{19}}{\gamma} I_1(\gamma,\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta})$$

$$+ b_{20}\frac{\partial}{\partial k_1} I_1(k_1,\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta})\Big/_{k_1=0}$$

$$-(\frac{s^4 b_{18}}{\beta^2\gamma} + \frac{b_{19}}{\gamma}) I_1(0,\zeta,\tau,\tfrac{\sqrt{\gamma}}{\beta})\Big]\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}} - \frac{b_{21}}{\gamma}\frac{\partial}{\partial\xi}\,(I_3(\gamma,\xi,\tau,0)$$

$$- I_3(0,\xi,\tau,0))$$

$$+ b_{17}\frac{\partial^2 I_3}{\partial k_1\partial\xi}(k_1,\xi,\tau,0)\Big/_{k_1=0}\,. \tag{5.26}$$

The partial stresses π_{x1}, σ_{x1} are given by

$$\sigma_{x1}(\xi,\tau) = \frac{\partial w_1}{\partial\xi}(\xi,\tau) + (1-f)\hat{\sigma}_0\,\eta_{21}(\xi,\tau), \tag{5.27}$$

$$\pi_{x1}(\xi,\tau) = f\hat{\sigma}_0\frac{\partial w_1}{\partial\xi}(\xi,\tau) + s^2\frac{\partial v_1}{\partial\xi}(\xi,\tau) - (1-f)\hat{\sigma}_0\,\eta_{21}(\xi,\tau)$$

into which (5.22), (5.23) and (5.26) can be substituted to give the stresses directly.

This completes the inversion of the deformation fields for the case of a delta function thermal loading and boundary conditions of problem A. The inversion is complete in the sense that these quantities are stated explicitly as functions of the integrals, I_1, I_3 and I_4 and these in turn are elementary integrals expressible in terms of error function.

By an anologous procedure, we invert the zero and first order terms associated with problem B (for $\bar{f}(p)=1$). Return to (4.19), (4.20) and use (4.25) in them. Then inverting directly gives

$$w_0(\xi,\tau) = d_1(I_1(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,0) - I_3(\gamma,\xi,\tau,0)), \qquad (5.28)$$

and

$$v_0(\xi,\tau) = \frac{d_2 s^2}{s^4-\beta^2}\,(I_4(\frac{\xi\beta}{s^2},0,\tau) - I_4(\xi,0,\tau)). \qquad (5.29)$$

Fluid density $\eta_{20}(\xi,\tau)$ is found from (5.15) and (5.29) to be

$$\eta_{20}(\xi,\tau) = -\int_0^\tau \frac{d_2 s^2}{s^4-\beta^2}\left[\frac{\beta}{s^2}\frac{\partial I_4}{\partial\zeta}(\zeta,0,r)\Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_4}{\partial\xi}(\xi,0,r)\right]dr$$

or,

$$\eta_{20}(\xi,\tau) = -\frac{d_2 s^2}{s^4-\beta^2}\left[\frac{\beta}{s^2}\frac{\partial I_3}{\partial\zeta}(0.\zeta,\tau,0)\Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_3}{\partial\xi}(0,\xi,\tau,0)\right]. \qquad (5.30)$$

As in problem A,

$$\sigma_{x0}(\xi,\tau)-\hat{\sigma}_0 = d_1\left[\frac{1}{\sqrt{\gamma}}\frac{\partial I_1}{\partial\zeta}(\gamma,\zeta,\tau,0)\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}} - \frac{\partial I_3}{\partial\xi}(\gamma,\xi,\tau,0)\right]$$

$$+ \frac{d_1\xi}{2\sqrt{\pi}\tau^{3/2}}\exp\left(-\frac{\xi^2}{4\tau}\right) +$$

$$- \frac{(1-f)\hat{\sigma}_0 d_2 s^2}{s^4-\beta^2}\left[\frac{\beta}{s^2}\frac{\partial I_3}{\partial \zeta}(0,\zeta,\tau,0)\Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_3}{\partial \xi}(0,\xi,\tau,0)\right]; \tag{5.31}$$

$$\pi_{x0}(\xi,\tau)+\hat{\sigma}_0 = \frac{d_2 s^4}{s^4-\beta^2}\left[\frac{\beta}{s^2}\frac{\partial I_4}{\partial \zeta}(\zeta,0,\tau)\Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_4}{\partial \xi}(\xi,0,\tau)\right]$$

$$+f\hat{\sigma}_0 d_1\left[\frac{1}{\sqrt{\gamma}}\frac{\partial I_1}{\partial \zeta}(\gamma,\zeta,\tau,0)\Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}} - \frac{\partial I_3}{\partial \xi}(\gamma,\xi,\tau,0)\right]$$

$$- \frac{d_2 \xi}{2\sqrt{\pi}\tau^{3/2}} e^{-\xi^2/4\tau}$$

$$+(1-f)\frac{\hat{\sigma}_0 d_2 s^2}{s^4-\beta^2}\left[\frac{\beta}{s^2}\frac{\partial I_3}{\partial \zeta}(0,\zeta,\tau,0)\Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_3}{\partial \xi}(0,\xi,\tau,0)\right]. \tag{5.32}$$

Equations (4.30) to (4.32), (4.35), (4.36) and (4.25) will yield

$$w_1(\xi,\tau) = -d_1\left[\frac{\xi}{2\sqrt{\gamma}} I_1(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,0)+ \frac{\partial}{\partial k_1}\left(I_1(k_1,\frac{\xi}{\sqrt{\gamma}},\tau,0)\right.\right.$$

$$\left.\left. + I_3(k_1,\xi,\tau,0)\right)\Big/_{k_1=0}\right]$$

$$+ \frac{d_2 s^2}{\gamma(s^4-\beta^2)}\left[I_1(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,0) - I_3(\gamma,\xi,\tau,0) - I_1(0,\frac{\xi}{\sqrt{\gamma}},\tau,0) + I_3(0,\xi,\tau,0)\right.$$

$$- \frac{s^4}{\beta^2} I_1(\frac{\gamma\beta^2}{s^4}, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + \frac{s^4}{\beta^2} I_1(0, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) \Big]$$

$$+ \frac{d_2 s^6}{\gamma\beta^2(s^4-\beta^2)} [I_3(\frac{\gamma\beta^2}{s^4}, \frac{\xi\beta}{s^2}, \tau, 0) - I_3(0, \frac{\xi\beta}{s^2}, \tau, 0)]; \tag{5.33}$$

$$v_1(\xi, \tau) = \frac{d_1 s^2}{s^4-\beta^2} [I_1(\frac{\gamma\beta^2}{s^4}, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) - I_3(\frac{\gamma\beta^2}{s^4}, \frac{\xi\beta}{s^2}, \tau, 0)$$

$$- I_1(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + I_3(\gamma, \xi, \tau, 0)]$$

$$+ \frac{d_2 s^2}{s^4-\beta^2} \Big[\frac{s^2}{s^4-\beta^2} \{I_3(0, \frac{\xi\beta}{s^2}, \tau, 0) - I_3(0, \xi, \tau, 0)\}$$

$$+ \frac{\xi \partial I_4}{\partial\beta\partial k_2} (\frac{\xi\beta}{s^2}, k_2, \tau) \Big/_{k_2=0} \Big]. \tag{5.34}$$

Returning to (4.3), (4.4) to compute the stresses we again require $\partial w_1/\partial\xi$, $\partial v_1/\partial\xi$ and η_{21}. By (5.33) and (5.34) the first two are

$$\frac{\partial w_1}{\partial\xi} = - d_1\Big[\frac{1}{2\sqrt{\gamma}} I_1(\gamma, \frac{\xi}{\sqrt{\gamma}}, \tau, 0) + \frac{\xi}{2\gamma} \frac{\partial I_1}{\partial\zeta} (\gamma, \zeta, \tau, 0) \Big/_{\zeta = \frac{\xi}{\sqrt{\gamma}}}$$

$$+ \frac{1}{\sqrt{\gamma}} \frac{\partial^2}{\partial\zeta\partial k_1} I_1(k_1, \zeta, \tau, 0) \Big/_{\substack{k_1=0 \\ \zeta=\xi/\sqrt{\gamma}}} + \frac{\partial^2 I_3}{\partial\xi\partial k_1} (k_1, \xi, \tau, 0) \Big/_{k_1=0} \Big]$$

$$+ \frac{d_2 s^2}{\gamma(s^4-\beta^2)} \left[\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial\zeta} \left\{ I_1(\gamma,\zeta,\tau,0) - I_1(0,\zeta,\tau,0) - \frac{s^4}{\beta^2} I_1(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0) \right.\right.$$

$$\left.\left. + \frac{s^4}{\beta^2} I_1(0,\zeta,\tau,0) \right\} \Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}} + \frac{\partial}{\partial\xi} \{ I_3(0,\xi,\tau,0) - I_3(\gamma,\xi,\tau,0) \} \right]; \quad (5.35)$$

$$\frac{\partial v_1}{\partial\xi}(\xi,\tau) = \frac{d_1 s^2}{s^4-\beta^2} \left[\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial\zeta} \left\{ I_1(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0) - I_1(\gamma,\zeta,\tau,0) \right\} \Big/_{\zeta=\frac{\xi}{\sqrt{\gamma}}} \right.$$

$$\left. + \frac{\partial I_3}{\partial\xi}(\gamma,\xi,\tau,0) - \frac{\beta}{s^2} \frac{\partial}{\partial\zeta} I_3(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0) \Big/_{\zeta=\frac{\xi\beta}{s^2}} \right]$$

$$+ \frac{d_2 s^2}{s^4-\beta^2} \left[\frac{s^2}{s^4-\beta^2} \left\{ \frac{\beta}{s^2} \frac{\partial}{\partial\zeta} I_3(0,\zeta,\tau,0) \Big/_{\zeta=\frac{\xi\beta}{s^2}} - \frac{\partial I_3}{\partial\xi}(0,\xi,\tau,0) \right\} \right.$$

$$\left. + \frac{1}{2\beta} \frac{\partial I_4}{\partial k_2}(\frac{\xi\beta}{s^2},k_2,\tau) \Big/_{k_2=0} + \frac{\xi}{2s^2} \frac{\partial^2 I_4}{\partial k_1 \partial k_2}(k_1,k_2,\tau) \Big/_{\substack{k_1=\frac{\xi\beta}{s^2} \\ k_2=0}} \right].$$

$$(5.36)$$

We find $\eta_{21}(\xi,\tau)$ by

$$\eta_{21}(\xi,\tau) = -\int_0^\tau \frac{\partial v_1}{\partial\xi}(\xi,r)\,dr$$

and (5.25) to be

$$\eta_{21}(\xi,\tau) = -\frac{d_1 s^2}{s^4-\beta^2}\Bigg[\frac{s^4}{\gamma^{3/2}\beta^2}\frac{\partial}{\partial\zeta}\left\{I_1\left(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0\right)-I_1(0,\zeta,\tau,0)\right\}\Bigg/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$-\frac{1}{\gamma^{3/2}}\frac{\partial}{\partial\zeta}\{I_1(\gamma,\zeta,\tau,0) - I_1(0,\zeta,\tau,0)\}\Bigg/_{\zeta=\frac{\xi}{\sqrt{\gamma}}}$$

$$+\frac{1}{\gamma}\frac{\partial}{\partial\xi}\{I_3(\gamma,\xi,\tau,0) - I_3(0,\xi,\tau,0)\}$$

$$-\frac{s^2}{\gamma\beta}\frac{\partial}{\partial\zeta}\left\{I_3\left(\frac{\gamma\beta^2}{s^4},\zeta,\tau,0\right) - I_3(0,\zeta,\tau,0)\right\}\Bigg/_{\zeta=\frac{\xi\beta}{s^2}}\Bigg]$$

$$-\frac{d_2 s^2}{s^4-\beta^2}\Bigg[\frac{\beta}{s^4-\beta^2}\frac{\partial^2 I_3}{\partial\zeta\partial k_1}(k_1,\zeta,\tau,0)\Bigg/_{\substack{k_1=0\\ \zeta=\frac{\xi\beta}{s^2}}}$$

$$-\frac{s^2}{s^4-\beta^2}\frac{\partial^2 I_3}{\partial\xi\partial k_1}(k_1,\xi,\tau,0)\Bigg/_{k_1=0}$$

$$+\frac{1}{2\beta}\frac{\partial}{\partial k_2}I_3\left(0,\frac{\xi\beta}{s^2},\tau,k_2\right)\Bigg/_{k_2=0}+\frac{\xi}{\partial s^2}\frac{\partial^2}{\partial\zeta\partial k_2}I_3(0,\zeta,\tau,k_2)\Bigg/_{\substack{\zeta=\frac{\xi\beta}{s^2}\\ k_2=0}}\Bigg]. \tag{5.37}$$

Substitution of these functions into (4.3), (4.4) completes the zero and first order solutions of problem B.

6. Examples of temperature loads.

As an application of the results obtained so far let us consider in some detail the stresses, displacements and velocities which result when the boundary temperature $\hat{f}(\tau)$ is the ramp loading

$$\hat{f}(\tau) = \frac{\tau}{\tau_0} H(\tau_0 - \tau) + H(\tau - \tau_0) \tag{6.1}$$

where τ_0 is a known "loading time".

The relation between the field variables of section 5, e.g., $w_0(\xi,\tau)$, found for $\hat{f}(\tau) = \delta(\tau)$ and the response of the same function due to (6.1) is the convolution integral

$$w_0^*(\xi,\tau) = \int_0^\tau \hat{f}(\tau - s)\, w_0(\xi,s)\, ds \tag{6.2}$$

where, for purpose of illustration, here $w_0^*(\xi,\tau)$ is the response due to (6.1) and $w_0(\xi,s)$ is the response of the last section.

Since all of the stresses, displacements, etc., of section 5 are expressed in terms of the integrals I_1, I_3 and I_4, as defined in (5.1) to (5.8), and their partial derivatives, integrals of the type (6.2) require us to compute

$$\int_0^\tau \hat{f}(\tau - t')\{I_1(k_1,\xi,t',k_2),\ I_3(k_1,\xi,t',k_2),\ I_4(k_1,k_2,t')\}dt',$$

in order to obtain the stress and displacement fields due to the loading (6.1).

In general then, for a thermal boundary loading $\hat{f}(\tau)$

we define integrals

$$L_\alpha(k_1,\xi,\tau,k_2) = \int_0^\tau \hat{f}(\tau-t')I_\alpha(k_1,\xi,t',k_2)\,dt', \quad \alpha = 1,3 \quad (6.3)$$

$$L_4(k_1,k_2,\tau) = \int_0^\tau \hat{f}(\tau-t')I_4(k_1,k_2,t')\,dt'$$

and obtain the response such as (6.2) by simply replacing the I_α's by the corresponding L_α and their partial derivatives.

For the loading (6.1), in particular, we follow the scheme just mentioned and define

$$L_\alpha(k_1,\xi,\tau,k_2,\tau_0) = L^{-1}\left\{\frac{1-e^{-p\tau_0}}{p^2\tau_0}\cdot \bar{I}_\alpha(k_1,\xi,p,k_2);\ p\right\}\alpha = 1,3, \quad (6.4)$$

and

$$L_4(k_1,k_2,\tau,\tau_0) = L^{-1}\left\{\frac{1-e^{-p\tau_0}}{p^2\tau_0}\cdot \bar{I}_4(k_1,k_2,p);p\right\} \quad (6.5)$$

where we have used (4.16) on (6.1), (5.1), (5.3) and (5.4).

Before proceeding we note that in (6.4) partial differentiation of $\bar{I}_\alpha$ with respect to k_1, ξ or k_2 may be interchanged with the integration so that for example

$$L^{-1}\left\{\frac{1-e^{-p\tau_0}}{p^2\tau_0}\cdot \frac{\partial\bar{I}_\alpha}{\partial k_1}(k_1,\xi,p,k_2);\ p\right\} = \frac{\partial}{\partial k_1}L_\alpha(k_1,\xi,\tau,k_2,\tau_0). \quad (6.6)$$

In addition to (6.4), (6.5) and (6.6), we record here additional relations that are useful in evaluating the solutions of section 5.

From the convolution (6.4) we have

$$L_1(k_1,\xi,\tau,k_2,\tau_0)= k_1^{-2}\tau_0^{-1}[I_1(k_1,\xi,\tau,k_2)+I_1(0,\xi+\tau_0,\tau,k_2)$$

$$- I_1(k_1,\xi+\tau_0,\tau,k_2)-I_1(0,\xi,\tau,k_2)]$$

$$+k_1^{-1}\tau_0^{-1}\left[\frac{\partial}{\partial k_1}\{I_1(k_1,\xi+\tau_0,\tau,k_2)-I_1(k_1,\xi,\tau,k_2)\}\Big/_{k_1=0}\right],$$

$$\text{for } k_1,\tau_0 \text{ nonzero;} \quad (6.7)$$

$$L_1(0,\xi,\tau,k_2,\tau_0)= -\frac{1}{2\tau_0}\frac{\partial^2}{\partial k_1^2}[I_1(k_1,\xi+\tau_0,\tau,k_2)-I_1(k_1,\xi,\tau,k_2)]\Big/_{k_1=0}$$

$$\text{for } \tau_0 \text{ nonzero} \quad (6.8)$$

$$L_1(k_1,\xi,\tau,k_2,0)=k_1^{-1}[I_1(k_1,\xi,\tau,k_2)-I_1(0,\xi,\tau,k_2)] \text{ for } k_1 \text{ nonzero;} \quad (6.9)$$

$$L_1(0,\xi,\tau,k_2,0) = \frac{\partial I_1}{\partial k_1}(k_1,\xi,\tau,k_2)\Big/_{k_1=0}. \quad (6.10)$$

Formulae for $L_3(k_1,\xi,\tau,k_2,\tau_0)$ and the special cases when k_1 or τ_0 vanish are of the same form as for L_1 if in (6.7) to (6.10) we replace I_1 by I_3 and the arguments $\xi+\tau_0,\tau$ by $\xi,\tau-\tau_0$, respectively, in those integrals I_1 containing $\xi+\tau_0,\tau$.

Before discussing L_4, we note that the integral $I_1(k_1,\xi,\tau,k_2)$, as defined by (5.6), carries the restriction that it is non-zero only if $\tau > \xi$. Similarly, by (5.7), $I_3(k_1,\xi,\tau,k_2)$ is zero if $\tau < 0$. Since, in determining L_3 as explained above, integrals of the type $I_3(k_1,\xi,\tau-\tau_O)$ will be encountered, it must be remembered that such integrals, for non-zero values, require $\tau > \tau_O$.

From (6.5) follows

$$L_4(k_1,k_2,\tau,\tau_O) = \frac{1}{\tau_O}\frac{\partial}{\partial\zeta}\left[I_3(\zeta,k_1,\tau,k_2)-I_3(\zeta,k_1,\tau-\tau_O,k_2)\right]\Big/_{\zeta=0} \tag{6.11}$$

for τ_O = nonzero;

$$L_4(k_1,k_2,\tau,0) = I_3(0,k_1,\tau,k_2)\,. \tag{6.12}$$

Returning now to the expression (6.2) when (6.1) is used we have that the response $w_O(\xi,\tau,\tau_O)$ due to (6.1) is of the form

$$w_O(\xi,\tau,\tau_O) = \frac{d_1(\beta+1)}{\beta} L_1(\gamma,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta},\tau_O) - \frac{d_2}{s^2+\beta} L_1(0,\frac{\xi}{\sqrt{\gamma}},\tau,\frac{\sqrt{\gamma}}{\beta},\tau_O)$$

$$- d_1L_3(\gamma,\xi,\tau,0,\tau_O) \tag{6.13}$$

if the $w_O(\xi,\tau)$ in (6.2) is the displacement given in (5.13).

To evaluate (6.13) we use (6.7) to (6.10) for L_1 and L_3. This in turn requires evaluation of I_1, I_3, and, in other expressions similar to $w_O(\xi,\tau,\tau_O)$, I_4 as well as their partial derivatives. Equations (5.6), (5.7) can be evaluated in a straightforward way and are expressible in terms of the error function and its complement. We do so now.

By (5.6) we have, after an integration by parts,

$$I_1(k_1,\xi,\tau,k_2) = -\frac{\sqrt{k_1}}{k_2^2-k_1} e^{k_1(\tau-\xi)} \Phi(\sqrt{k_1(\tau-\xi)})$$

$$+ \frac{k_2}{k_2^2-k_1} e^{k_1(\tau-\xi)} - \frac{k_2}{k_2^2-k_1} e^{k_2^2(\tau-\xi)} \mathrm{Erfc}(k_2\sqrt{\tau-\xi})$$

if $\tau > \xi$ k_1, k_2 not both zero,

$$= 0 \quad \text{if} \quad \tau \leq \xi; \tag{6.14}$$

$$I_1(0,\xi,\tau,k_2) = k_2^{-1}[1-e^{k_2^2(\tau-\xi)} \mathrm{Erfc}(k_2\sqrt{\tau-\xi})] \text{ if } \tau > \xi,\ k_2 \neq 0$$

$$= 0 \quad \text{if} \quad \tau \leq \xi; \tag{6.15}$$

$$I_1(0,\xi,\tau,0) = \frac{2}{\sqrt{\pi}}\sqrt{\tau-\xi} \quad \text{when} \quad \tau > \xi$$

$$= 0 \quad \text{when} \quad \tau \leq \xi; \tag{6.16}$$

$$I_3(k_1,\xi,\tau,k_2) = \frac{1}{2(k_2+\sqrt{k_1})} e^{k_1(\tau-\frac{\xi}{\sqrt{k_1}})} \mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}} - \sqrt{k_1\tau}) H(\tau)$$

$$+ \frac{1}{2(k_2-\sqrt{k_1})} e^{k_1(\tau+\frac{\xi}{\sqrt{k_1}})} \mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}} + \sqrt{k_1\tau}) H(\tau)$$

$$- \frac{k_2}{k_2^2-k_1} e^{k_2^2\tau+k_2\xi} \mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}} + k_2\sqrt{\tau}) H(\tau) \text{ if } k_1, k_2$$

are not both zero; (6.17)

$$I_3(0,\xi,\tau,0) = [-\xi \,\mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}})+2\sqrt{\frac{\tau}{\pi}}\, e^{-\xi^2/4\tau}]H(\tau). \qquad (6.18)$$

Additional formulae to (6.14), (6.15), etc. are required, such as their first and second derivatives with respect to k_1, k_2, ξ. As required by need we shall compute them at that point.

A program has been written for the CDC 3600 digital computer to determine as functions of spatial variable, ξ, and time, τ, the values of $w_0, w_1, v_0, v_1, \eta_{20}, \eta_{21}, \sigma_{x0}, \sigma_{x1}, \pi_{x0}, \pi_{x1}$ and temperature Θ_0 under the applied temperature load $\hat{f}(\tau)$ specified by (6.1) for τ_0 zero and nonzero. Both problems A and B have been studied.

To program a function such as $w_0(\xi,\tau,\tau_0)$ given by (6.13) we required subroutines for I_1, I_3, I_4 and their partial derivatives with respect to their arguments. Such functions in turn required accurate values of $\Phi(\lambda)$ and $\mathrm{Erfc}(\lambda)$. For this purpose we used (see [14]) the power series

$$\Phi(\lambda) = \frac{2}{\sqrt{\pi}} e^{-\lambda^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \ldots (2n+1)} \lambda^{2n+1} \text{ if } \lambda < 3 \qquad (6.19)$$

and the asymptotic expansion for $\mathrm{Erfc}(\lambda)$

$$\sqrt{\pi}\, \lambda e^{\lambda^2} \mathrm{Erfc}(\lambda) = 1+\sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \ldots (2m-1)}{(2\lambda^2)^m} \text{ when } \lambda > 3. \quad (6.20)$$

By checking with the tables [15] we were able to maintain at least 10 place accuracy for all values of the argument.

Input data required consisted of the thermal expansion coefficients d_1, d_2, the initial stress $\hat{\sigma}_0$, the porosity $f = \bar{\rho}_2/\bar{\rho}$, the initial densities $\bar{\rho}_1$, $\bar{\rho}_2$, the parameter s^2, the diffusive force parameter $\alpha t_0/\bar{\rho}_1$ and the loading time for the boundary temperature, τ_0.

To check the validity and accuracy of our solutions we considered the case when

$$\hat{\sigma}_0 = 0, \ \bar{\rho}_2 = 0, \ s^2 = 0, \ d_1 = -1 \tag{6.21}$$

which, when used in (3.29) to (3.35) and the boundary conditions for problem A, i.e., (3.36), reduces the exact problem specified there to the transient uncoupled ($\epsilon_1=\epsilon_2=0$) thermoelastic problems of [6] and [8] if $\hat{f}(\tau)$ is given by (6.1).

If (6.21) is used, the zeroth order solution is the complete solution and in fact

$$\frac{\partial w_0}{\partial \tau} = v_0 \tag{6.22}$$

in the region.

The results of [6] show that $\sigma_{x0}(\xi,\tau)$, plotted versus τ for ξ fixed, exhibits a jump discontinuity at $\tau = \xi$. By [8] this same stress becomes continuous at $\tau = \xi$ and at $\tau = \xi + \tau_0$ for $\tau_0 > 0$ but has discontinuous slope at these points.

It is natural to expect that our results should approach those of [6] and [8] if one were to choose d_2, α, s^2 and $\bar{\rho}_2$ near zero. To illustrate this we took as one set of fluid material properties

$$d_2 = 0, \ \alpha = 0, \ s = 2, \ \bar{\rho}_2 = 1.1 \times 10^{-5}. \tag{6.23}$$

A second set of parameters were chosen so that $\bar{\rho}_2$ would be of the same order as $\bar{\rho}_1$.

Thus

$$d_2 = 0, \ \alpha = 0, \ s = 2, \ \bar{\rho}_2 = 1. \tag{6.24}$$

For the sets (6.23), (6.24) the solid material parameters were taken to approximate the thermoelastic values:

$$d_1 = -1, \ \bar{\rho}_1 = 2, \ \hat{\sigma}_0 = 10^{-4}. \tag{6.25}$$

Before discussing and comparing our results with those of [6] and [8] we begin by showing a plot, in figure 1, of $\Theta_0(\xi,\tau,\tau_0)$ at $\xi = 1$ and for $\tau_0 = 0$, 1/2, 1 and 2. The temperature Θ_0 is the same for problems A and B and, since it is independent of material properties, is the same distribution used in [6] and [8].

Turning now to the stress $\sigma_{x0}(\xi,\tau) - \hat{\sigma}_0$ for problem A we show, in figure 2, the stress at $\xi = 1$ for the parameter values (6.23) and (6.25). The effect of a non-zero $\bar{\rho}_2$ is to eliminate the jump discontinuity found in [6] for the step loading $\tau_0 = 0$. In its place we have a slope discontinuity at $\tau = \xi/\sqrt{\gamma}$. The shock front is no longer plane to the boundary but is rounded as one approaches from the right.

The slope of $\sigma_{x0}(\xi,\tau) - \hat{\sigma}_0$ at $\tau = \xi^+/\sqrt{\gamma}$ changes continuously from a finite value to infinity whereas the slope from the left, i.e., as $\tau \to \xi^-/\sqrt{\gamma}$, is finite. The overall character is, as expected, like that found in [6] with the exceptions noted.

When $\tau_0 > 0$, the slope discontinuities which occured in [8] at $\tau = \xi$ and $\tau = \xi + \tau_0$ disappear in our case and $\sigma_{x0}(\xi,\tau) - \hat{\sigma}_0$ has a continuous slope at all points.

In figure 3 we plot the same stress using the properties given in (6.24), (6.25). Here the shock has dissipated even further for $\tau_0 = 0$ and when $\tau_0 > 0$ the stress is clearly continuous with continuous derivatives.

To confrim these results analytically we returned to (4.3) and substituted for w_O, Θ_O and η_{2O} the expressions obtained when (6.1) is the loading. Hence,

$$\sigma_{xO}(\xi,\tau,\tau_O)-\hat{\sigma}_O=\frac{\partial w_O}{\partial\xi}(\xi,\tau,\tau_O)+d_1\Theta_O(\xi,\tau,\tau_O)+(1-f)\hat{\sigma}_O\eta_{2O}(\xi,\tau,\tau_O) \quad (6.26)$$

where by (4.18), (5.3), (5.17), (6.1), (6.7) to (6.10), and (6.13)

$$\Theta_O(\xi,\tau,\tau_O) = \frac{1}{\tau_O}\left[(\tau+\frac{\xi^2}{2})\,\mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}})-\xi\sqrt{\frac{\tau}{\pi}}\,e^{-\xi^2/4\tau}\right]H(\tau)$$

$$+\frac{1}{\tau_O}\left[\frac{\xi}{\sqrt{\pi}}\sqrt{\tau-\tau_O}\;e^{-\xi^2/4(\tau-\tau_O)}-(\frac{\xi^2}{2}+\tau-\tau_O)\,\mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau-\tau_O}})\right]$$

$$H(\tau-\tau_O) \quad (6.27)$$

if $\tau_O \neq 0$;

$$\Theta_O(\xi,\tau,0) = \mathrm{Erfc}(\frac{\xi}{2\sqrt{\tau}})H(\tau); \quad (6.28)$$

$$\frac{\partial w_O}{\partial\xi}(\xi,\tau,\tau_O) = \frac{d_1(\beta+1)}{\beta\sqrt{\gamma}}\frac{\partial}{\partial\zeta}L_1(\gamma,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta},\tau_O)\Big/_{\zeta=\xi/\sqrt{\gamma}}$$

$$-\frac{d_2}{\sqrt{\gamma}(s^2+\beta)}\frac{\partial}{\partial\zeta}L_1(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta},\tau_O)\Big/_{\zeta=\xi/\sqrt{\gamma}}$$

$$-d_1\frac{\partial L_3}{\partial\xi}(\gamma,\xi,\tau,0,\tau_O); \quad (6.29)$$

$$\eta_{20}(\xi,\tau,\tau_0) = -\frac{\partial}{\partial\zeta}\Big[\Big(\frac{d_1}{s^2(1-\beta)} - \frac{d_2\beta}{s^2(s^2+\beta)}\Big)L_3(0,\zeta,\tau,\frac{\sqrt{\gamma}}{\beta},\tau_0)$$

$$+\frac{d_1\beta}{s^2(\beta-1)}L_3(0,\zeta,\tau,\sqrt{\gamma},\tau_0) + \frac{d_2\beta}{s^4-\beta^2}L_3(0,\zeta,\tau,0,\tau_0)\Big]\Big/_{\zeta=\frac{\xi\beta}{s^2}}$$

$$+\frac{d_2 s^2}{s^4-\beta^2}\frac{\partial L_3}{\partial\varsigma}(0,\xi,\tau,0,\tau_0). \qquad (6.30)$$

A long but straightforward computation indicates that when $\tau_0 = 0$, both $\theta_0(\xi,\tau,0)$ and $\eta_{20}(\xi,\tau,0)$ are continuous and differentiable at $\tau = \xi/\sqrt{\gamma}$. The discontinuity in $\partial/\partial\tau[\sigma_{x0}(\xi,\tau,0)-\hat{\sigma}_0]$ therefore comes from $\partial^2 w_0/\partial\tau\partial\xi$. If one expands (6.29) for $\tau_0 = 0$ we have that

$$\frac{\partial w_0}{\partial\xi}(\varsigma,\tau,0) = \frac{d_1}{2\gamma}\Big[e^{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}\,\mathrm{Erfc}\Big(\frac{\xi}{2\sqrt{\tau}}-\sqrt{\gamma\tau}\Big) + e^{\gamma(\tau+\frac{\xi}{\sqrt{\gamma}})}\,\mathrm{Erfc}\Big(\frac{\xi}{2\sqrt{\tau}}+\sqrt{\gamma\tau}\Big)$$

$$-2\,\mathrm{Erfc}\Big(\frac{\xi}{2\sqrt{\tau}}\Big)\Big]H(\tau)$$

$$+\frac{d_1}{1-\beta}\Big[\frac{\beta}{\gamma}\,e^{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}\,\Phi\Big(\sqrt{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}\Big) - \frac{1}{\gamma}\,e^{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}$$

$$+\frac{1}{\gamma}\,e^{\frac{\gamma}{\beta^2}(\tau-\frac{\xi}{\sqrt{\gamma}})}\,\mathrm{Erfc}\Big(\frac{1}{\beta}\sqrt{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}\Big)\Big]H\Big(\tau-\frac{\xi}{\sqrt{\gamma}}\Big)$$

$$+\frac{d_2\beta}{\gamma(s^2+\beta)}\Big[1-e^{\frac{\gamma}{\beta^2}(\tau-\frac{\xi}{\sqrt{\gamma}})}\,\mathrm{Erfc}\Big(\frac{1}{\beta}\sqrt{\gamma(\tau-\frac{\xi}{\sqrt{\gamma}})}\Big)\Big]H\Big(\tau-\frac{\xi}{\sqrt{\gamma}}\Big) \qquad (6.31)$$

and it is readily confirmed that (6.31) is continuous at $\tau = \xi/\sqrt{\gamma}$ provided $\beta \neq 0$. By the expression $W_I(\xi,\xi/\sqrt{\gamma})$

we shall mean

$$\lim_{\tau \to \frac{\xi^-}{\sqrt{\gamma}}} \frac{\partial}{\partial \tau}\left(\frac{\partial w_0}{\partial \xi}\right) = W_I(\xi, \frac{\xi^-}{\sqrt{\gamma}}). \qquad (6.32)$$

Thus differentiating (6.31) with respect to τ and taking the limit as $\tau \to \xi^+/\sqrt{\gamma}$ we have

$$\lim_{\tau \to \frac{\xi^+}{\sqrt{\gamma}}} \frac{\partial}{\partial \tau}\left(\frac{\partial w_0}{\partial \xi}\right) = W_I(\xi, \frac{\xi}{\sqrt{\gamma}}) + \frac{1}{\beta}\left[\frac{d_1(\beta+1)}{\beta} - \frac{d_2}{s^2+\beta}\right]$$

$$+ \lim_{\epsilon \to 0^+}\left\{\frac{1}{\sqrt{\pi\gamma}}\left(\frac{d_2}{s^2+\beta} - \frac{d_1(\beta+1)}{\beta}\right) \cdot \frac{1}{\sqrt{\epsilon}}\right\} \qquad (6.33)$$

which exhibits the discontinuity in slope at $\tau = \xi/\sqrt{\gamma}$ unless

$$\frac{1}{\sqrt{\pi\gamma}}\left(\frac{d_2}{s^2+\beta} - \frac{d_1(\beta+1)}{\beta}\right) = 0. \qquad (6.34)$$

For the parameter sets (6.23), (6.24) and (6.25), (6.34) is nonzero so that a discontinuity in slope at $\tau = \xi/\sqrt{\gamma}$ is found.

Figure 4 is a plot of the stress $\sigma_{x0}(\xi,\tau) - \hat{\sigma}_0$ for the parameters given in (6.25) and d_2, s and β chosen to satisfy (6.34): i.e.,

$$d_2 = -5-3\sqrt{2}, \ \bar{\rho}_2 = 1, \ s = 2, \qquad (6.35)$$

$$d_1 = -1, \ \bar{\rho}_1 = 2, \ \hat{\sigma}_0 = 0.1.$$

As expected, all discontinuities have disappeared and the form of the response for different τ_0 is the same.

The next set of graphs, figures 5 to 10, are the solid strain, $\partial w_0/\partial \xi$, and the fluid rate of deformation component, $\partial v_0/\partial \xi$, plotted versus τ at $\xi = 1$ and for the three parameter sets (6.23) to (6.25) and (6.35). As already discussed the discontinuity in stress slope for $\tau_0 = 0$ is due to the strain component, and this discontinuity dies away when (6.35) is used.

Figure 11 is a plot of the fluid density $\eta_{20}(\xi,\tau,\tau_0)$ for the parameters (6.23) and (6.25), while figure 12 is the same function if (6.24) replaces (6.23). The effect of the thermal loading on the density is to decrease the value from its initial equilibrium value with time so that the fluid is less dense near the boundary than it was initially.

If one next turns to the parameter set (6.35), then figure 13 shows that the fluid density, after initially decreasing for $\tau_0 = 0$ rapidly increases with τ.

We examine the fluid partial stress $\pi_{x0}(\xi,\tau,\tau_0)+\hat{\sigma}_0$ given in figures 14 to 16. Figure 14 is the graph obtained when $\bar{\rho}_2$ is almost zero, i.e. using (6.23) and (6.25). Although the stress exhibits the discontinuous slope in this case at points $\tau = \tau_0$, its magnitude is much smaller than the corresponding solid partial stress shown in figure 2.

When (6.24) replaces (6.23), the stress increases appreciably in magnitude, and, with the exception of the discontinuity in its slope at $\tau = \xi/\sqrt{\gamma}$ for $\tau_0 = 0$, figure 15, the stress is continuous and differentiable with respect to time.

For a solid fluid mixture with properties satisfying (6.35), the fluid partial stress becomes the dominant stress component and, like the solid stress, is continuous and differentiable everywhere.

Finally we examine the total stress at $\xi = 1$ versus τ for $\tau_0 = 0$, .5, 1 and 2. Figures 16 to 18 illustrate the behavior of $\sigma_{x0}(\xi,\tau) + \pi_{x0}(\xi,\tau)$ for the three parametric sets chosen in this study.

Let us now proceed to a discussion of the zero order terms found for problem B under the ramp loading (6.1). As mentioned earlier, the thermal distribution of the uncoupled theory is the same for problems A and B even though these problems are physically distinct.

The boundary conditions of problem B state that the half space, $\xi \geq 0$, is initially at rest and is constrained at the face $\xi = 0$ so that there is no displacement and no velocity, fluid or solid, allowed at the boundary. At time $\tau = 0$ the face of the half-space is heated according to (6.1) and problem B is the study of the reaction of the half-space to this load.

The defect of the zero order theory is immediately apparent in problem B. The displacement and fluid velocity fields, given in (4.19), (4.20) are physically independent of each other, i.e., $w_0(\xi,\tau,\tau_0)$ depends only upon the solid material properties while $v_0(\xi,\tau,\tau_0)$ is characterized by fluid properties alone. The zero order theory for this problem, as in the corresponding theory for problem A, should tend to the thermoelastic theory as $\bar{\rho}_2$, s^2, α and d_2 approach zero. This will at least give us a basis for comparison.

Accordingly, we have plots of $\partial w_0/\partial\xi$ and σ_{x0} versus τ for the ramp loading (6.1) and for $\xi = 1$. Figure 19 is the thermoelastic strain and figure 20 the thermoelastic stress obtained when $\bar{\rho}_2 = s^2 = \alpha = d_2 = 0$ and when $d_1 = -1$, $\gamma = 1$. Unlike the elastic discontinuities found for the stress-free half-space problem of [6] and the ramp problem of [8], our strain and stress are continuous at $\tau = \xi$, and at $\tau = \xi + \tau_0$. In fact, the slope discontinuity in both $\partial w_0/\partial\xi$ and σ_{x0} at $\tau = \xi$ for $\tau_0 = 0$ is of the type found for problem A. Again unlike problem A, however, the strain $\partial w_0/\partial\xi$ does not change with $\bar{\rho}_2$ or s^2 or d_2 so that the result shown in figure 19 is the same regardless of the fluid properties.

In problem A we were able to obtain a criterion whereby the slope discontinuity in the solid partial stress would vanish. This was expressed by equation (6.34). No such equation can be found for problem B so that the response shown in figure 20 is typical for all values of $\bar{\rho}_2$, s^2 and d_2. This again points out the defects of the zero order theory for the mixture of solid and fluid.

It is reasonable to expect that when the diffusive for a parameter is nonzero the first order terms will yield a relation between material properties such as that found for problem A. This is to be considered in a subsequent publication.

7. Acknowledgment. The author would like to thank Mr. Yong Lee of Michigan State University who programmed the solutions for numerical evaluation on the CDC-3600 digital computer at Michigan State University. The author would also like to thank N.A.S.A. for their financial support through the research grant no. NGR 23-004-041.

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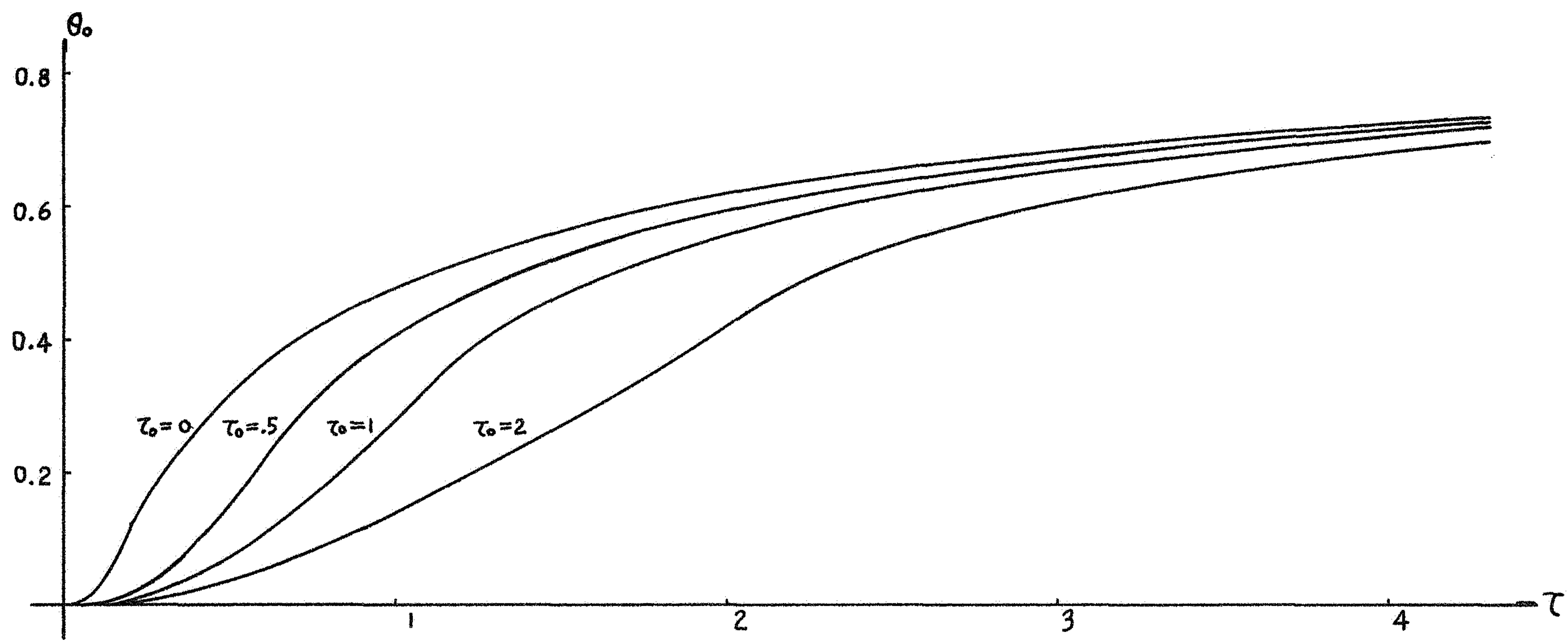

Figure 1. Plot of $\Theta_o(\xi, \tau, \tau_o)$ at $\xi = 1$ for $\tau_o = 0$, 1/2, 1 and 2.

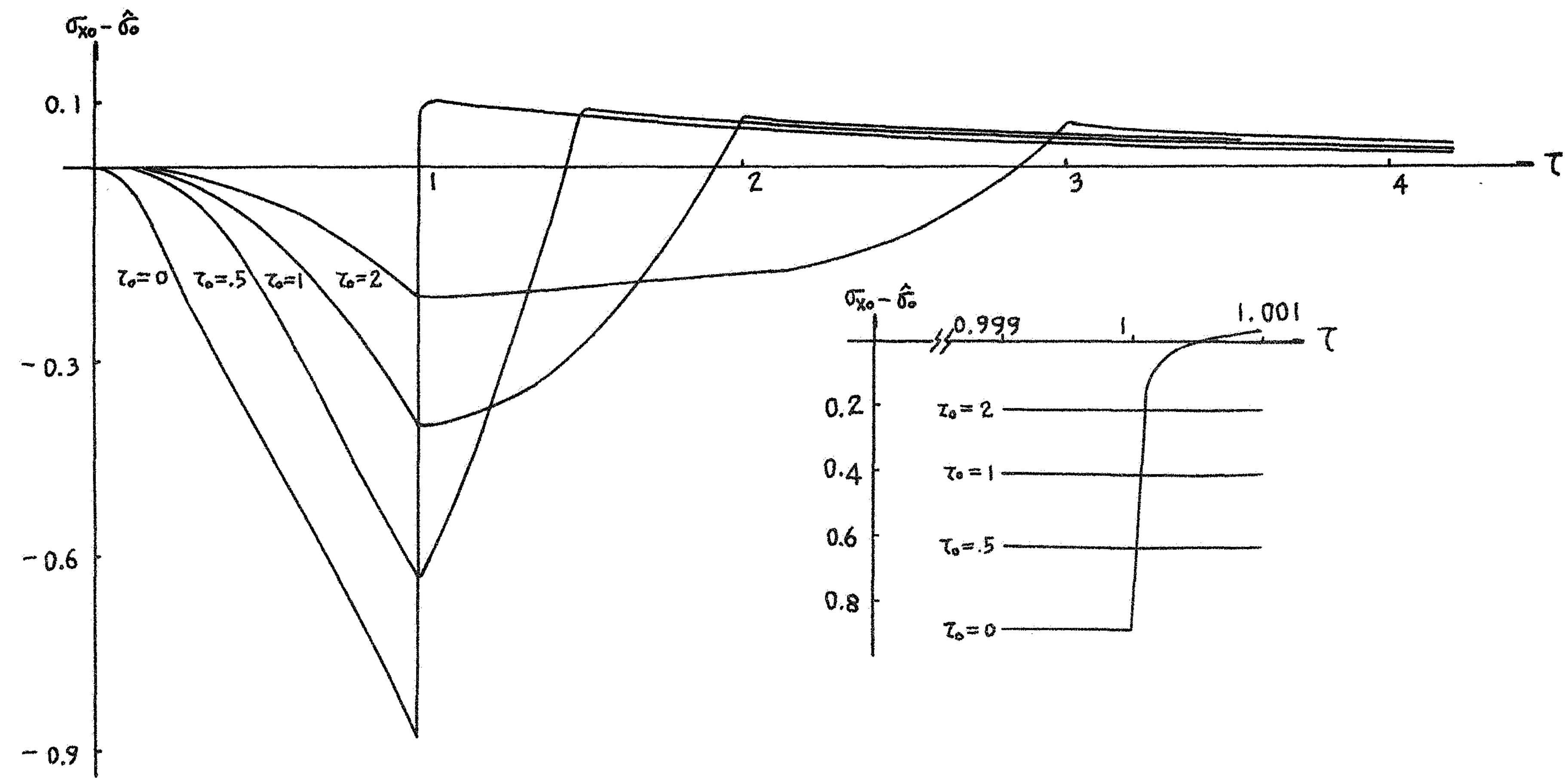

Figure 2. Plot of $\sigma_{xo}(\xi, \tau, \tau_o) - \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.23), (6.25).

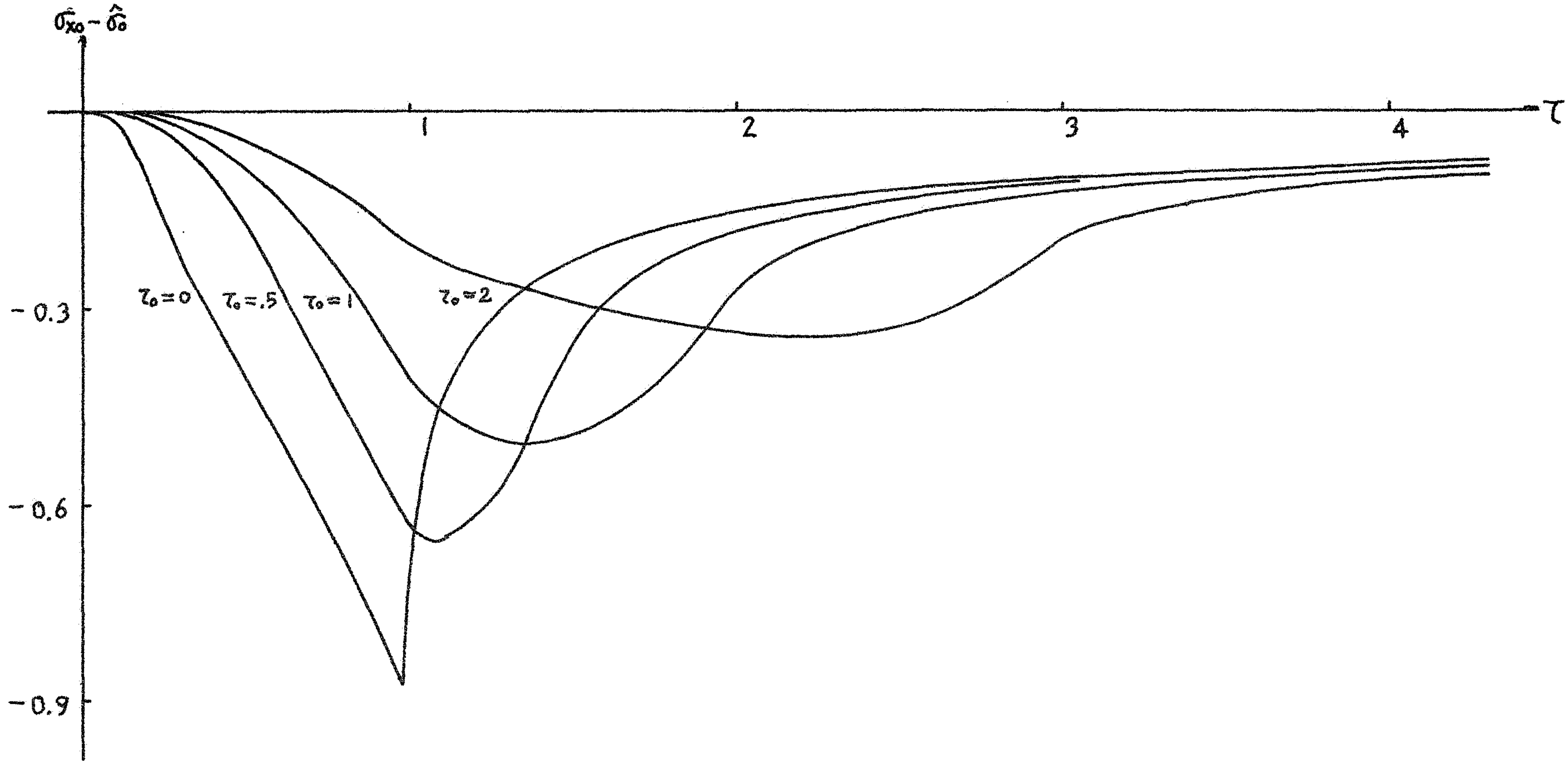

Figure 3. Plot of $\sigma_{xo}(\xi,\tau,\tau_o) - \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.24), (6.25).

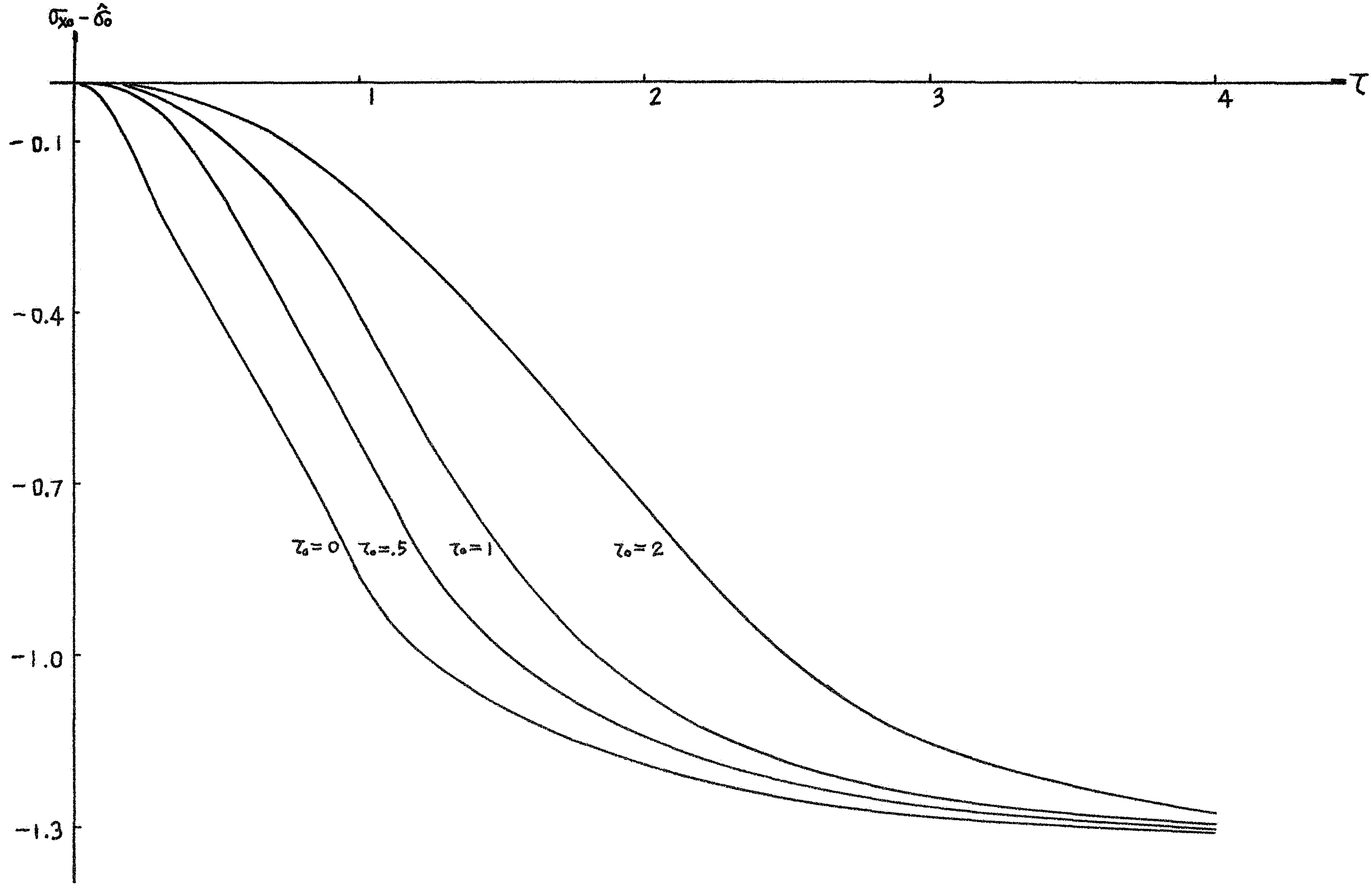

Figure 4. Plot of $\sigma_{xo}(\xi, \tau, \tau_o) - \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

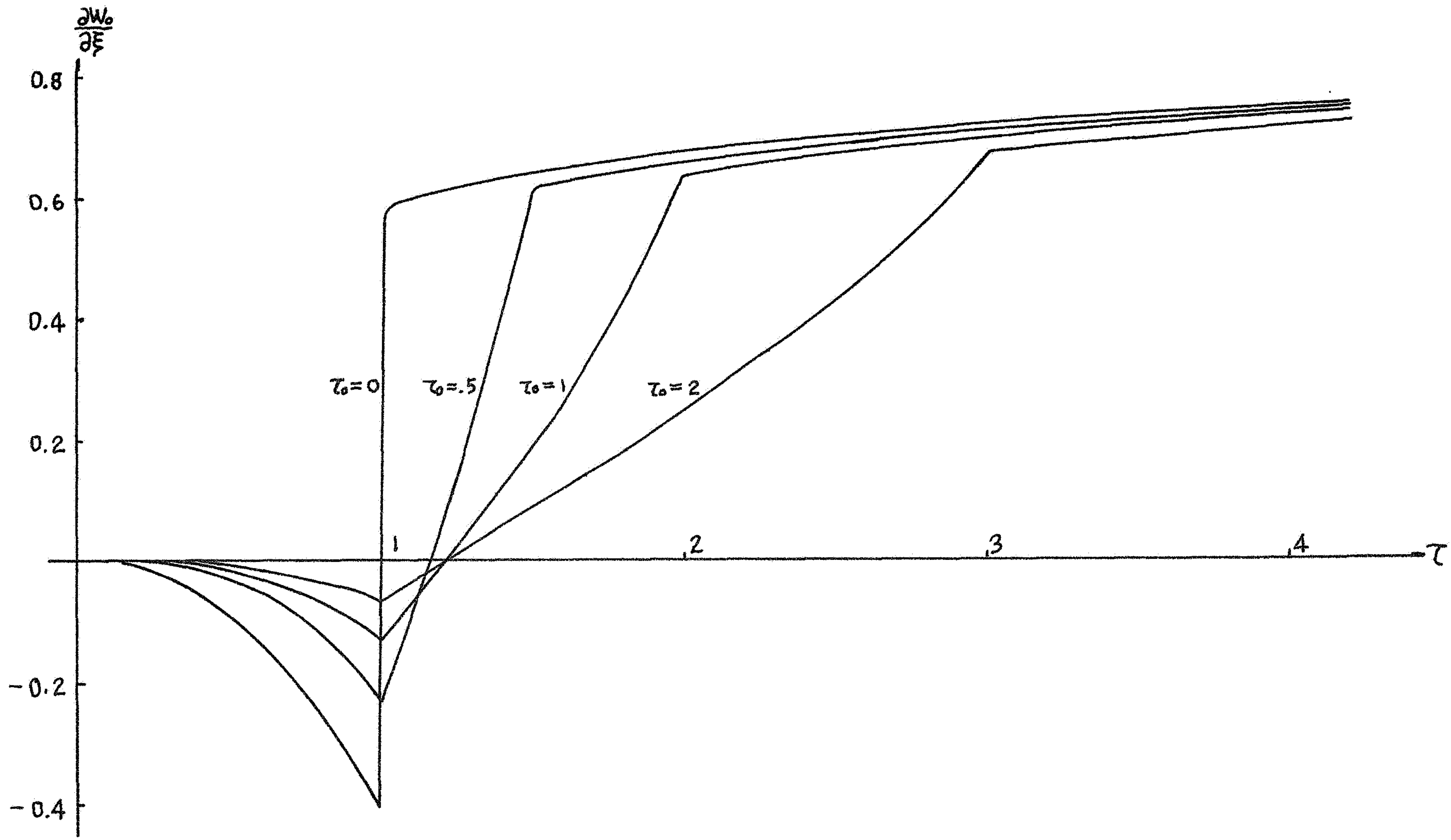

Figure 5. Plot of $\partial w_o(\xi,\tau,\tau_o)/\partial\xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.23) and (6.25).

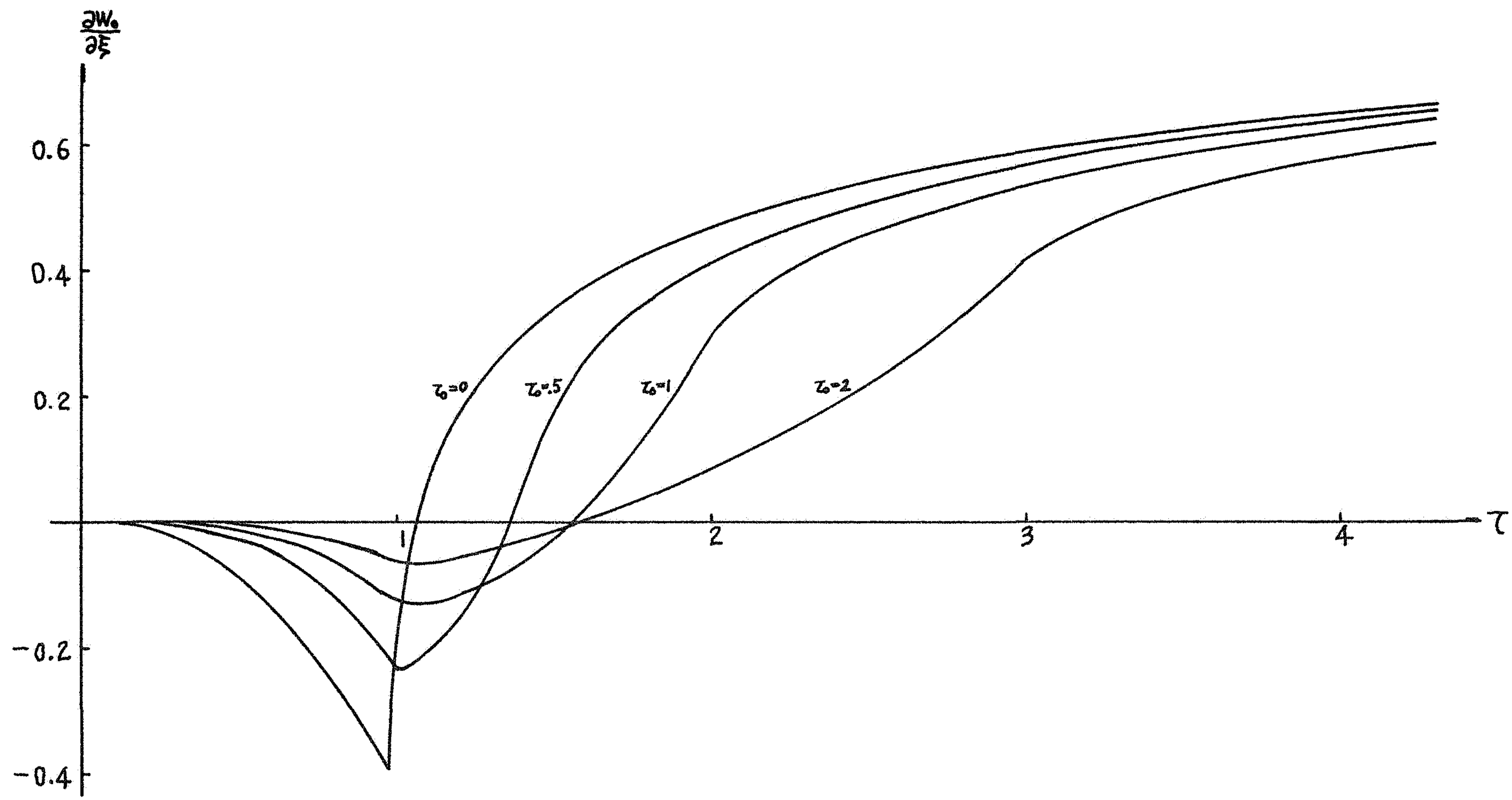

Figure 6. Plot of $\partial w_o(\xi,\tau,\tau_o)/\partial\xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.24) and (6.25).

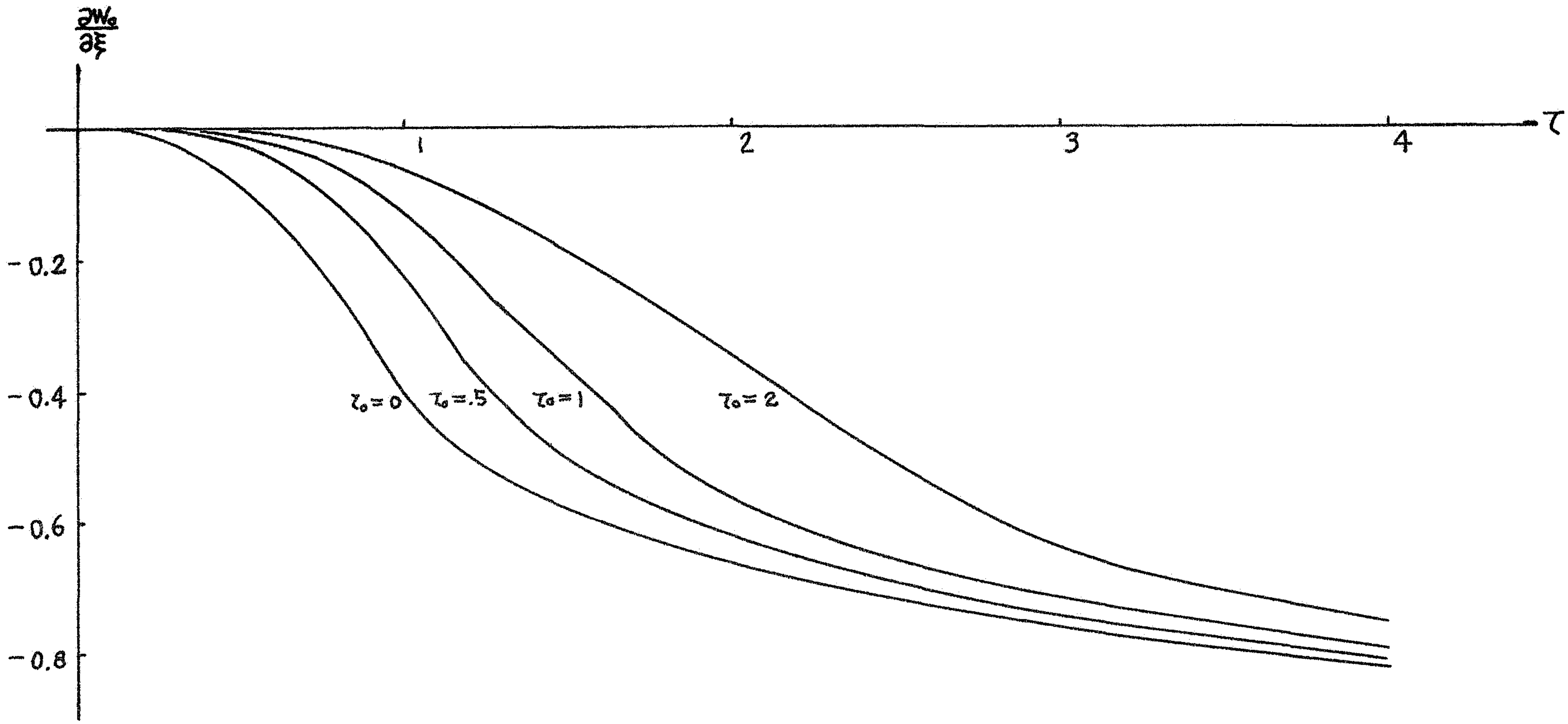

Figure 7. Plot of $\partial w_o(\xi,\tau,\tau_o)/\partial\xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

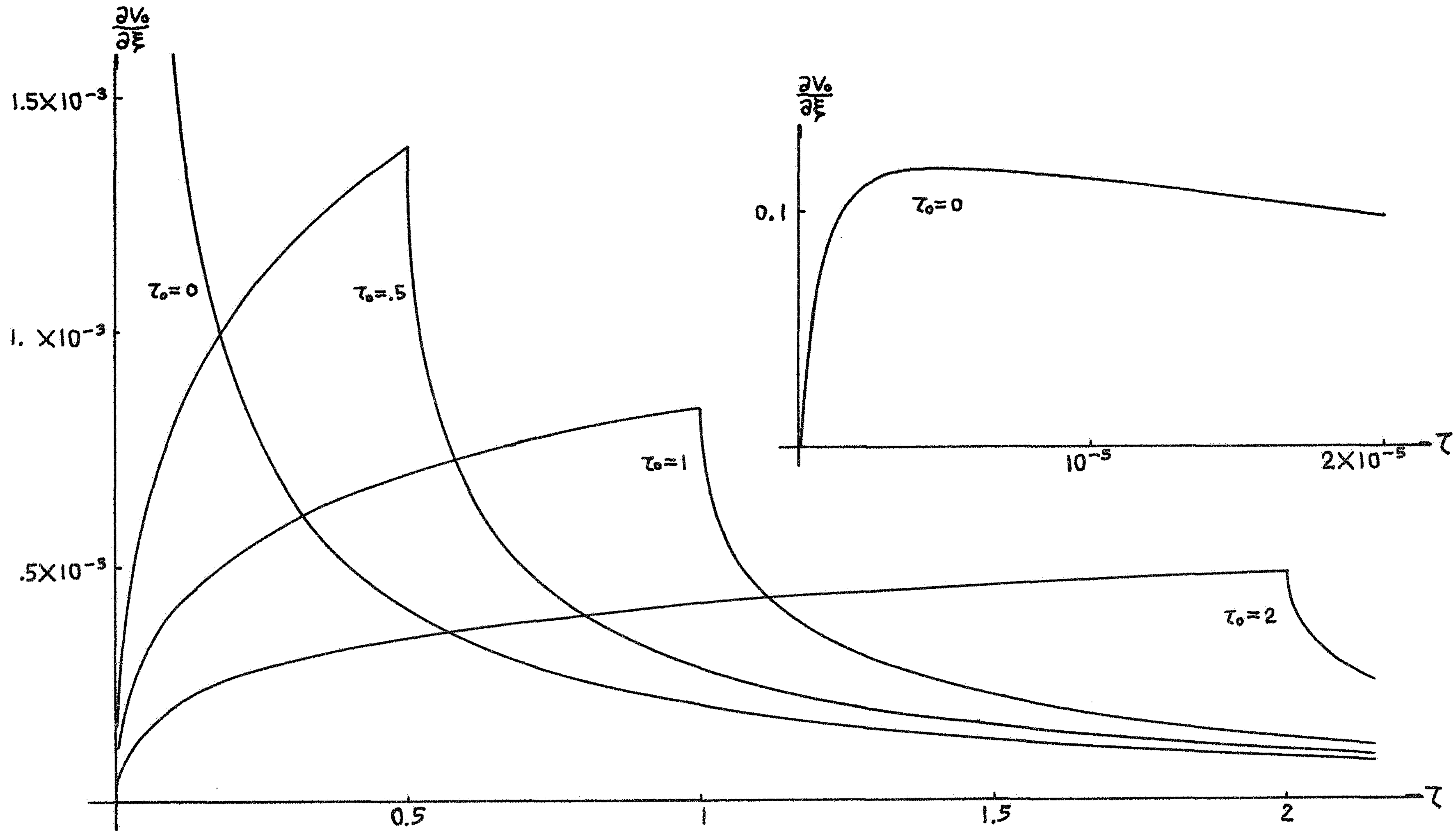

Figure 8. Plot of $\partial v_o(\xi,\tau,\tau_o)/\partial\xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.23) and (6.25).

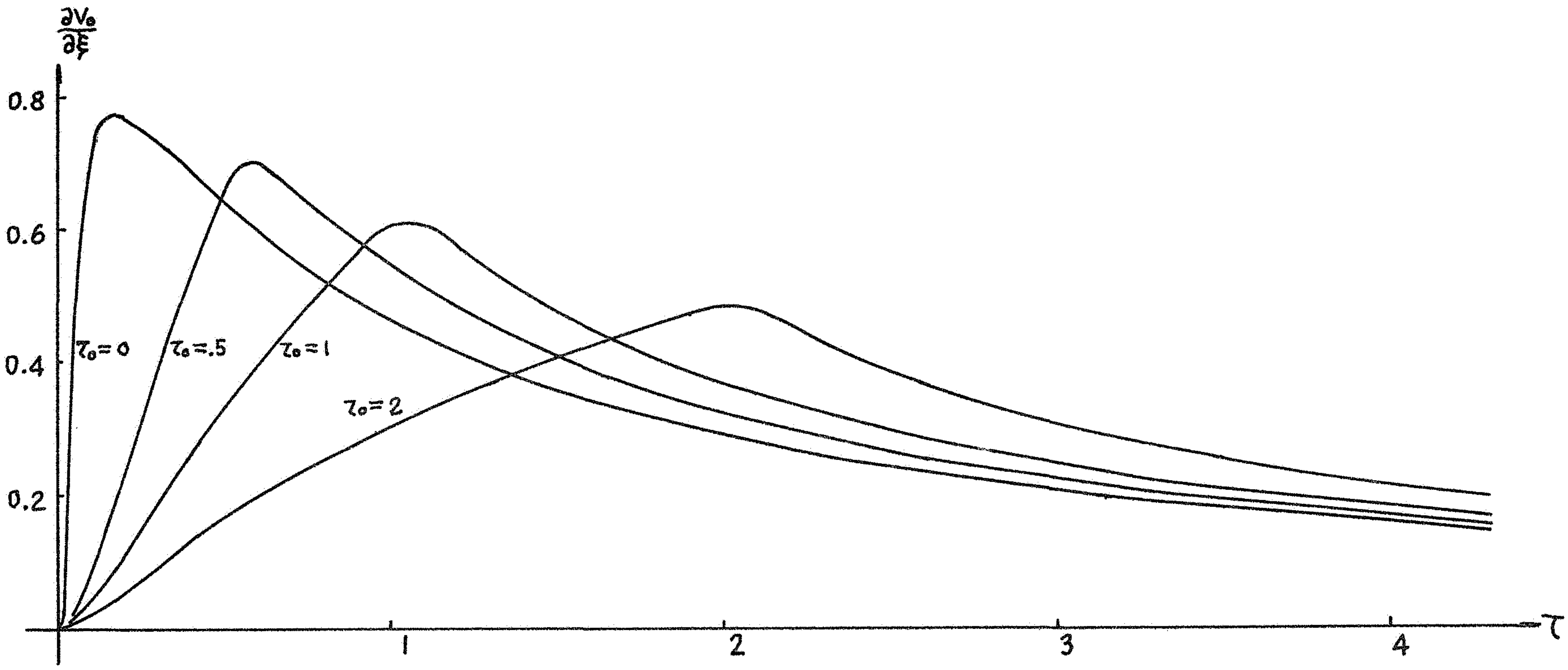

Figure 9. Plot of $\partial v_o(\xi, \tau, \tau_o)/\partial \xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.24) and (6.25).

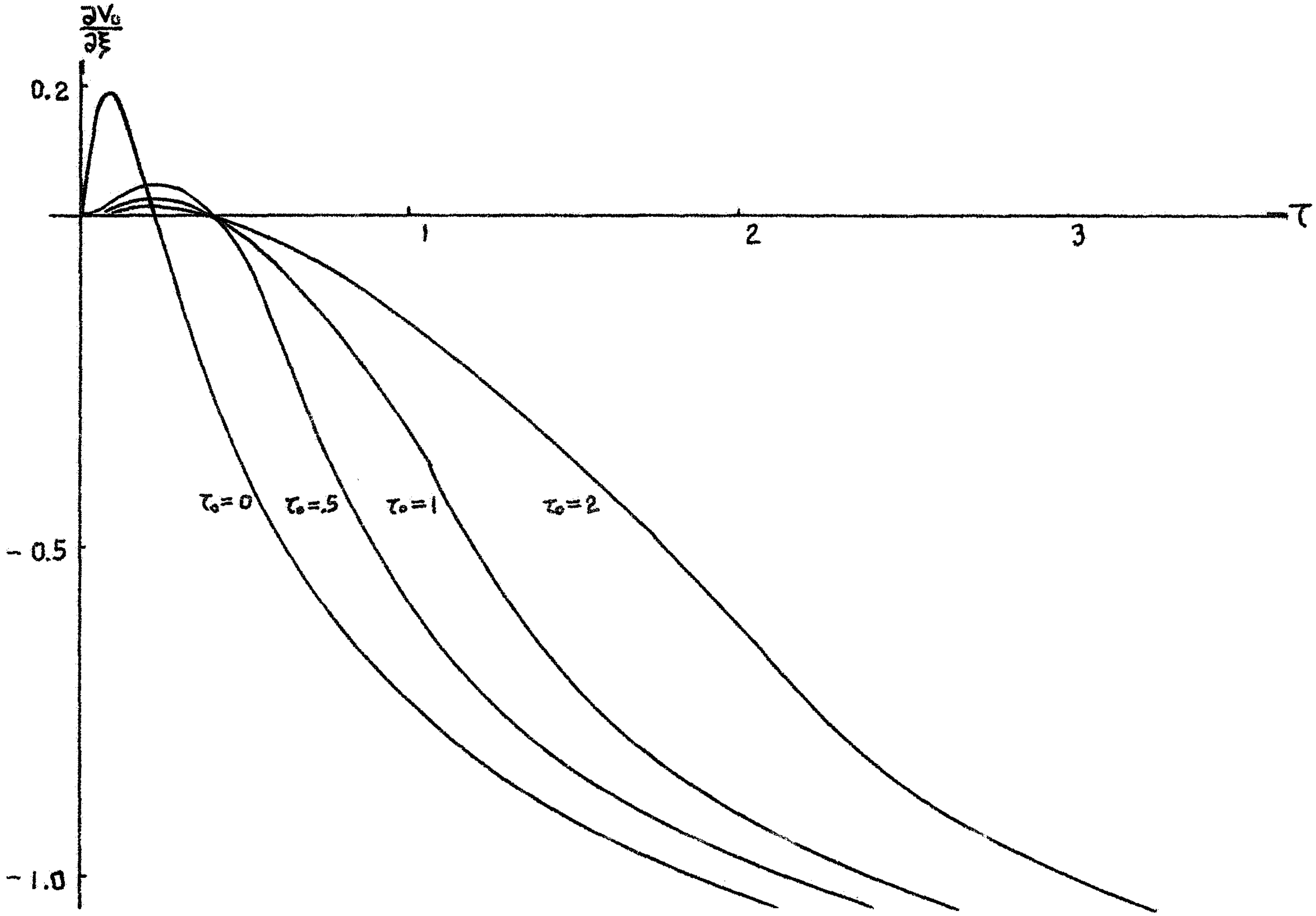

Figure 10. Plot of $\partial v_o(\xi,\tau,\tau_o)/\partial\xi$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

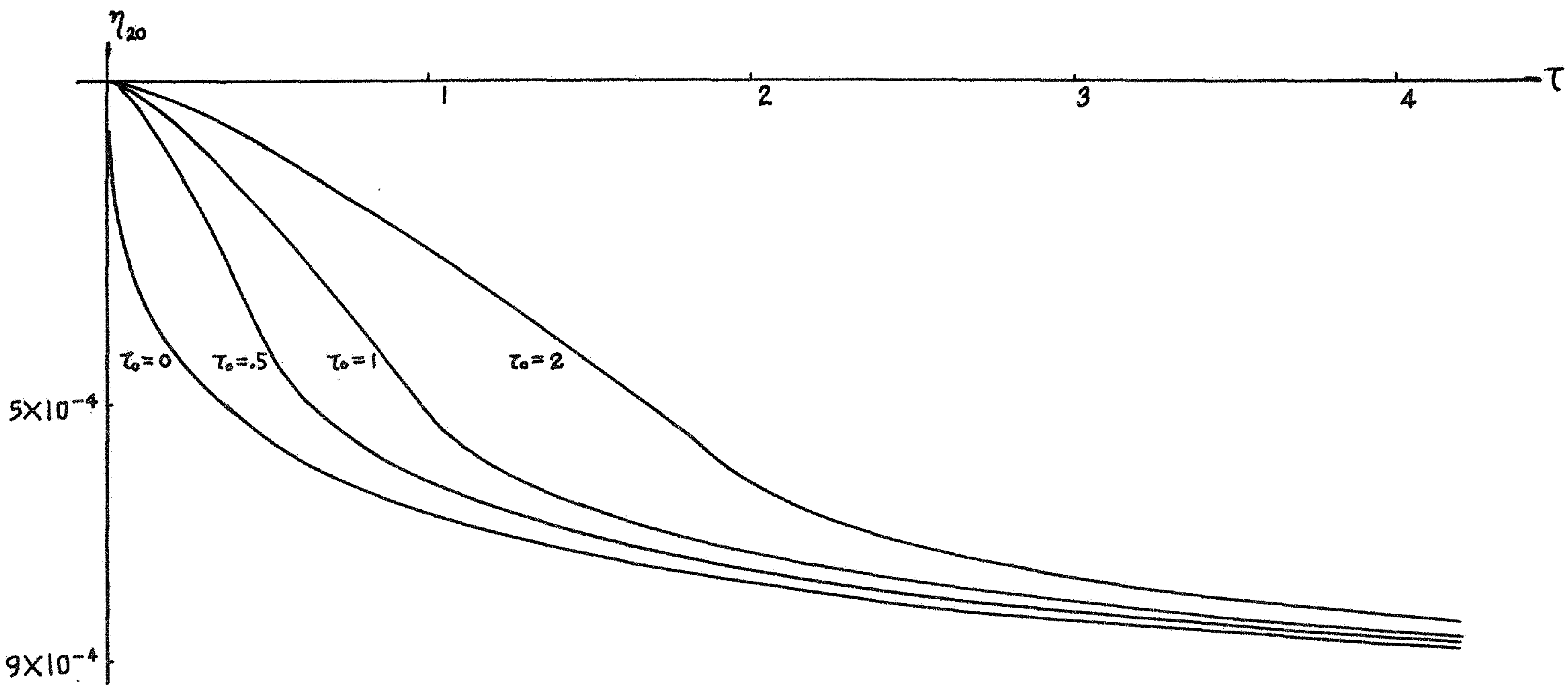

Figure 11. Plot of $\eta_{20}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.23) and (6.25).

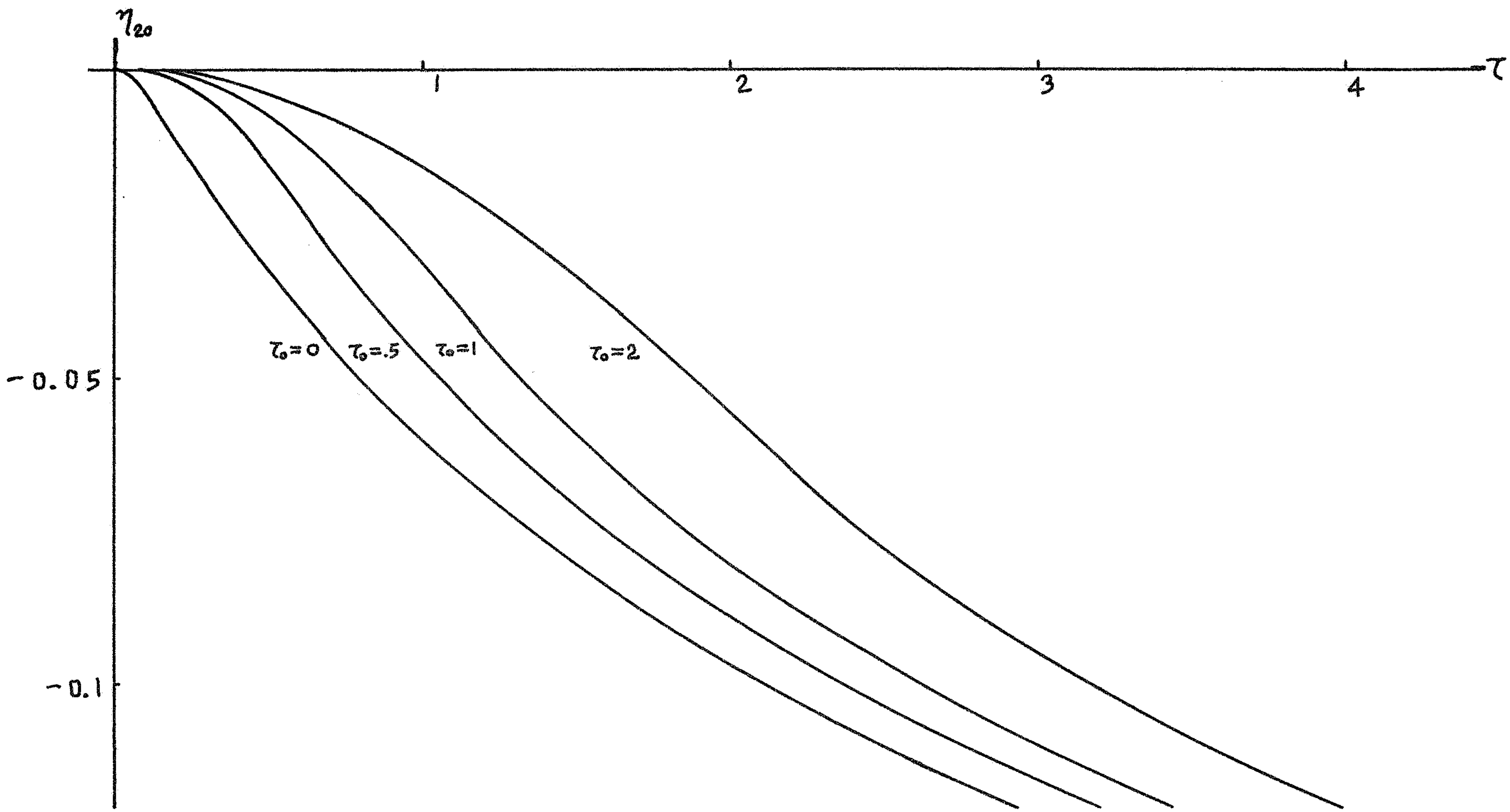

Figure 12. Plot of $\eta_{20}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.24) and (6.25).

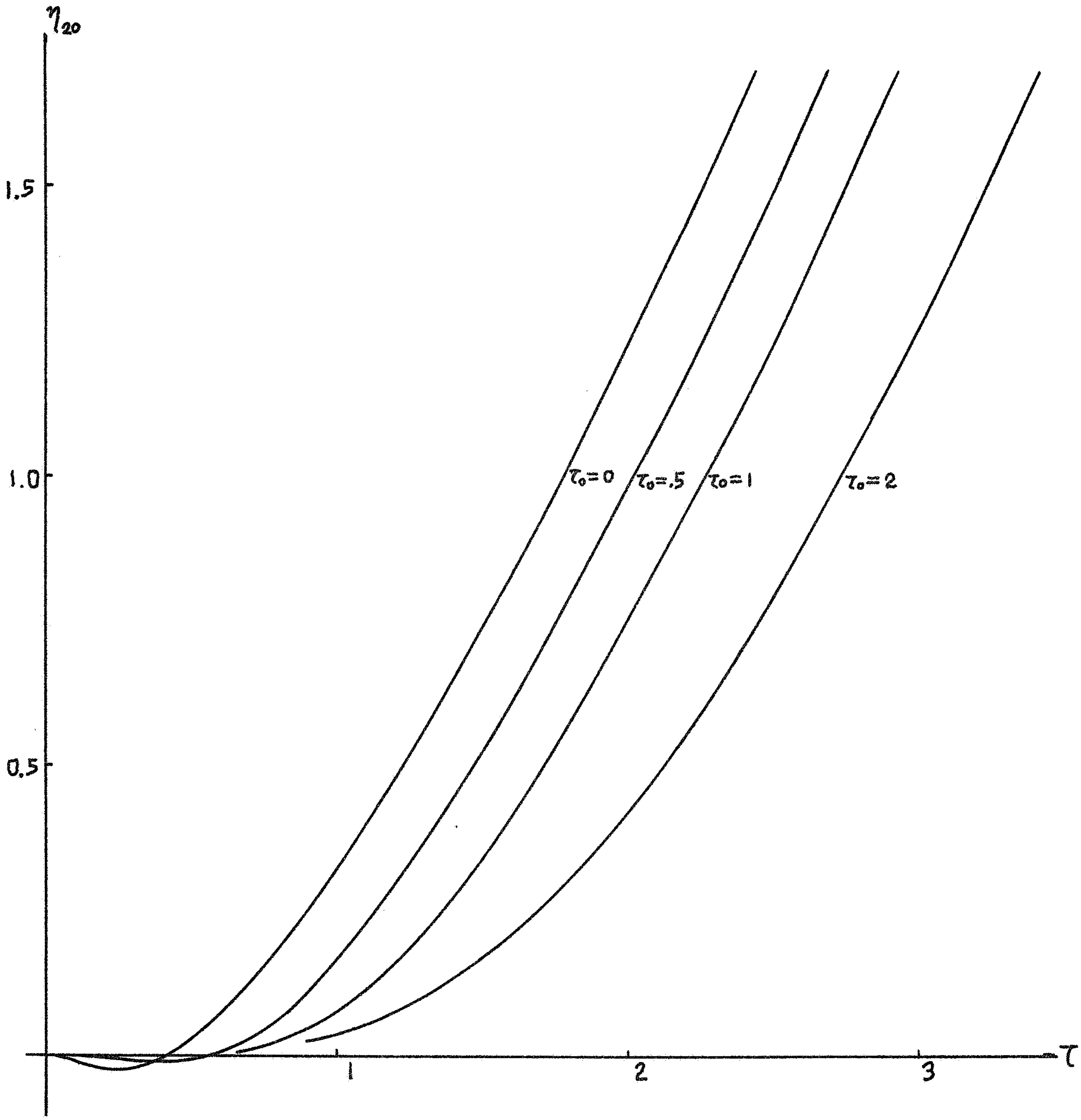

Figure 13. Plot of $\eta_{20}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

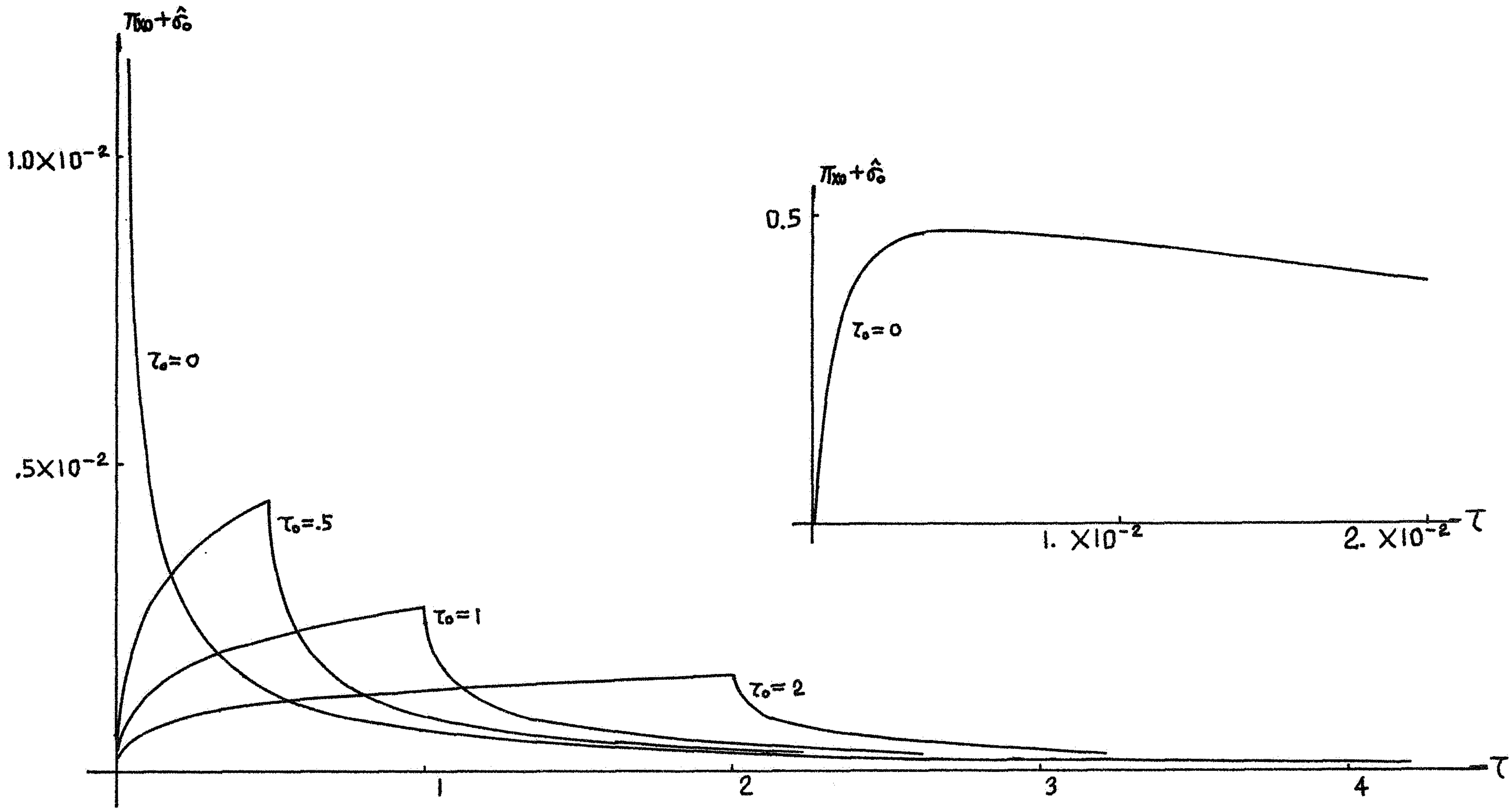

Figure 14. Plot of $\pi_{xo}(\xi,\tau,\tau_o) + \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.23) and (6.25).

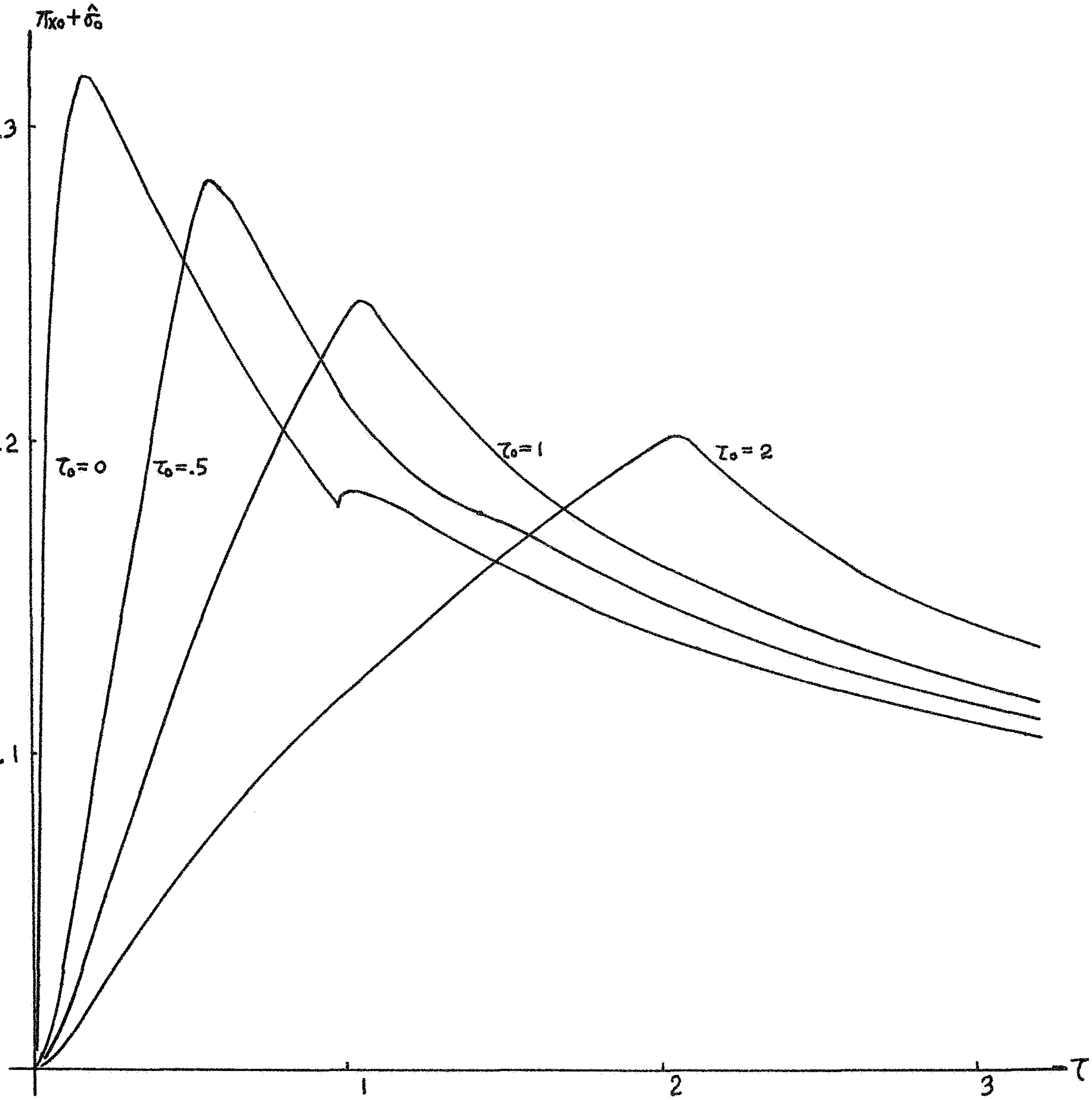

Figure 15. Plot of $\pi_{xo}(\xi,\tau,\tau_o) + \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.24) and (6.25).

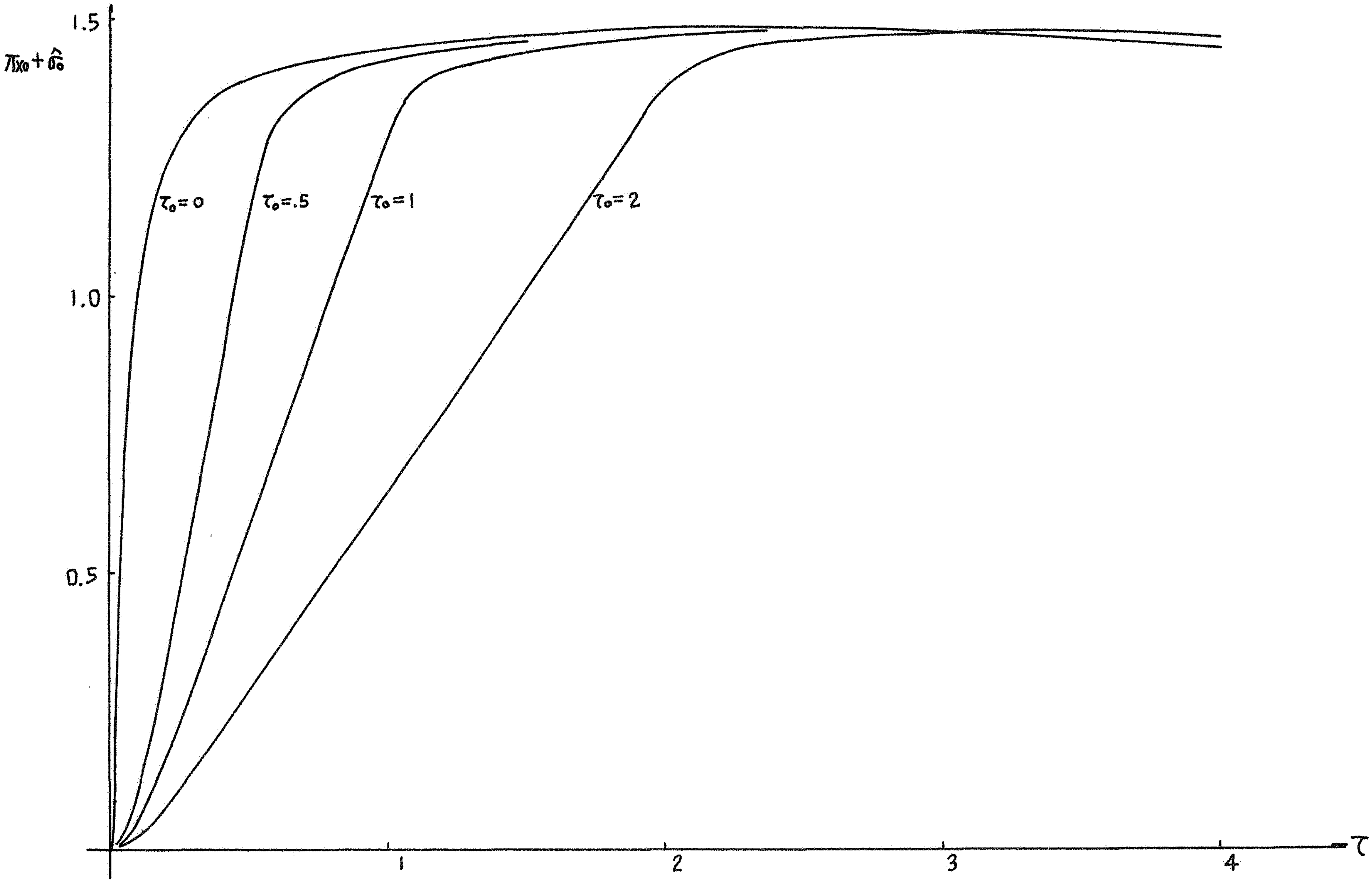

Figure 16. Plot of $\pi_{xo}(\xi,\tau,\tau_o) + \hat{\sigma}_o$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

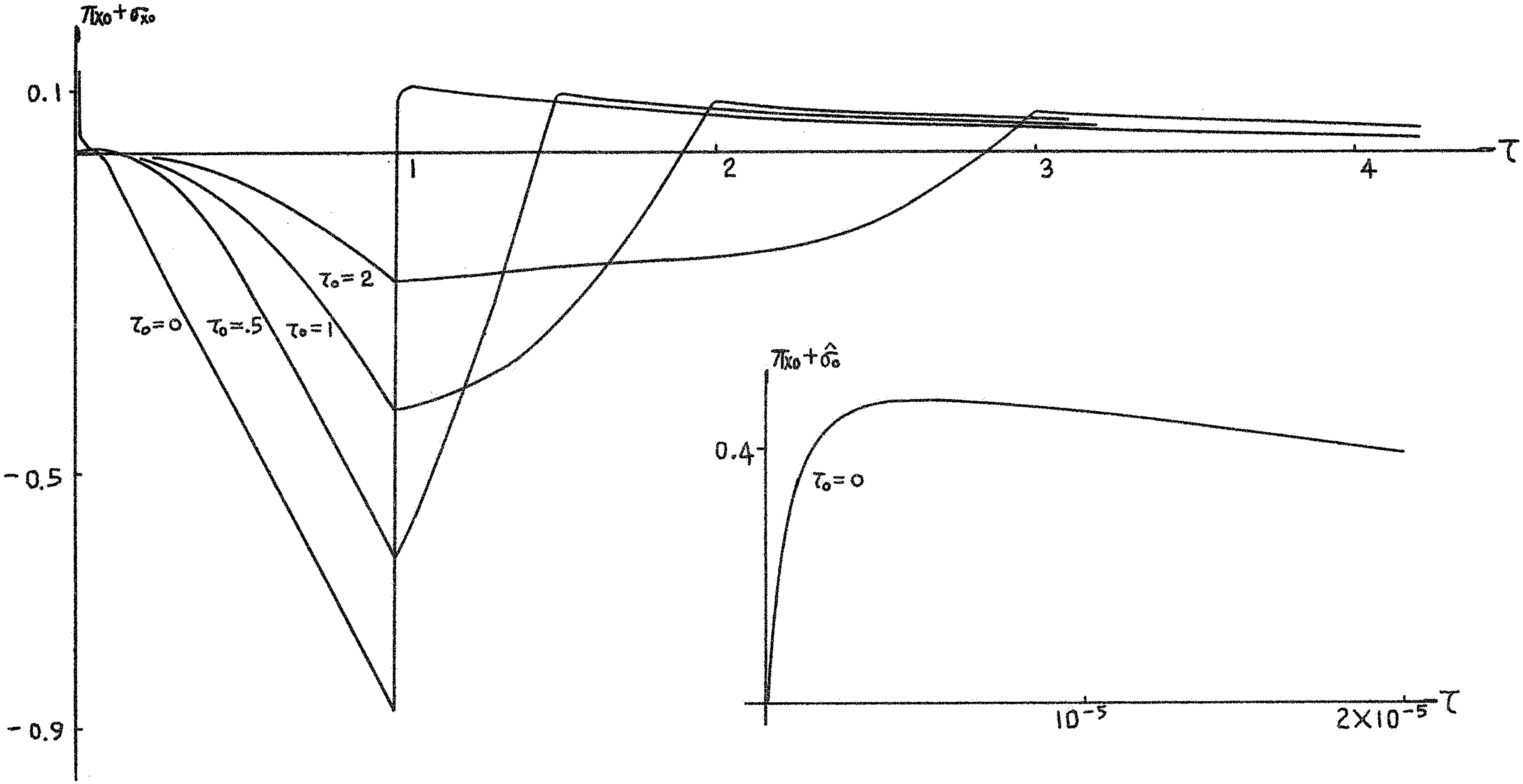

Figure 17. Plot of $\pi_{xo}(\xi,\tau,\tau_o) + \sigma_{xo}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.23) and (6.25).

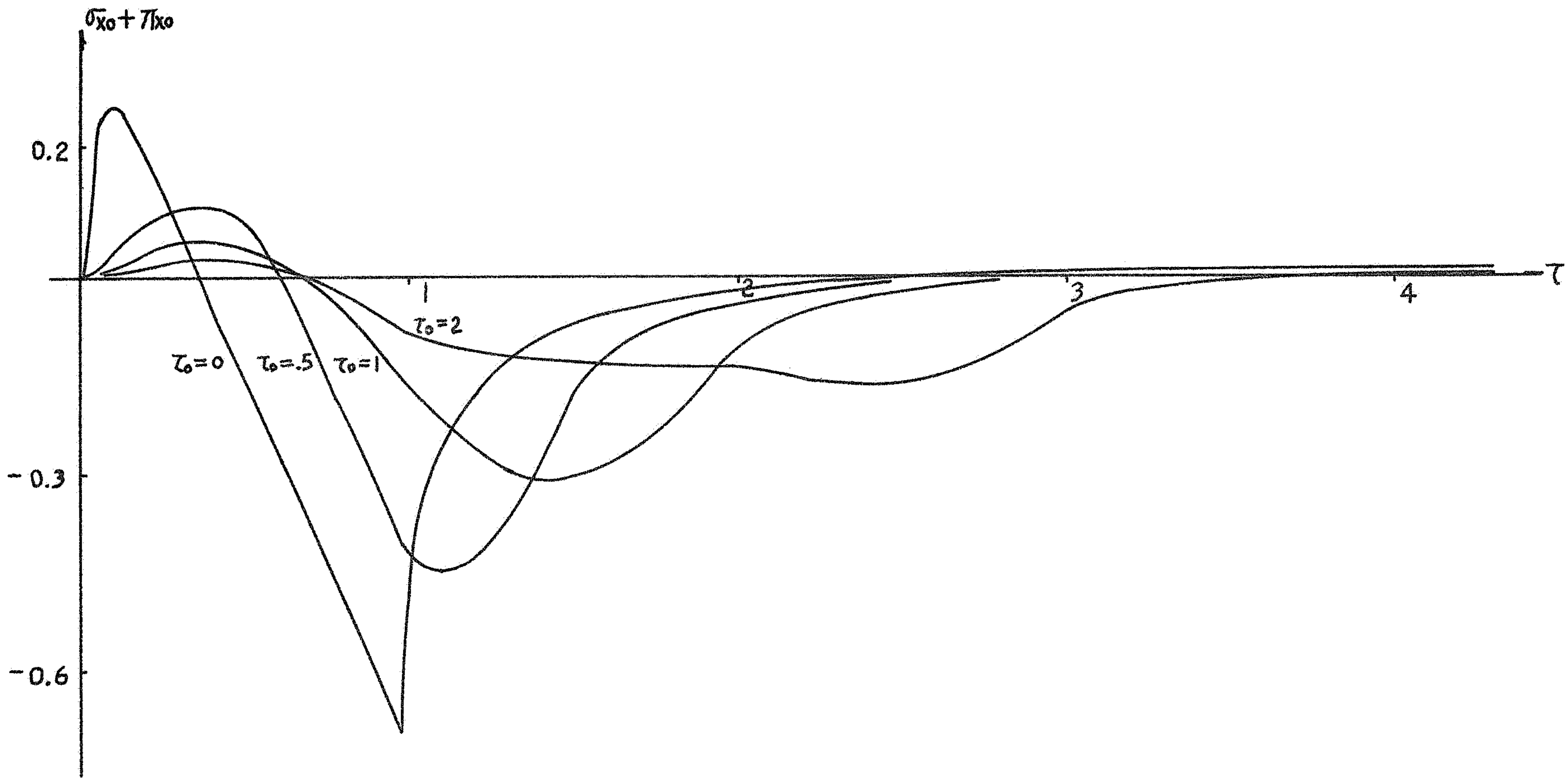

Figure 18. Plot of $\sigma_{xo}(\xi,\tau,\tau_o) + \pi_{xo}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.24) and (6.25).

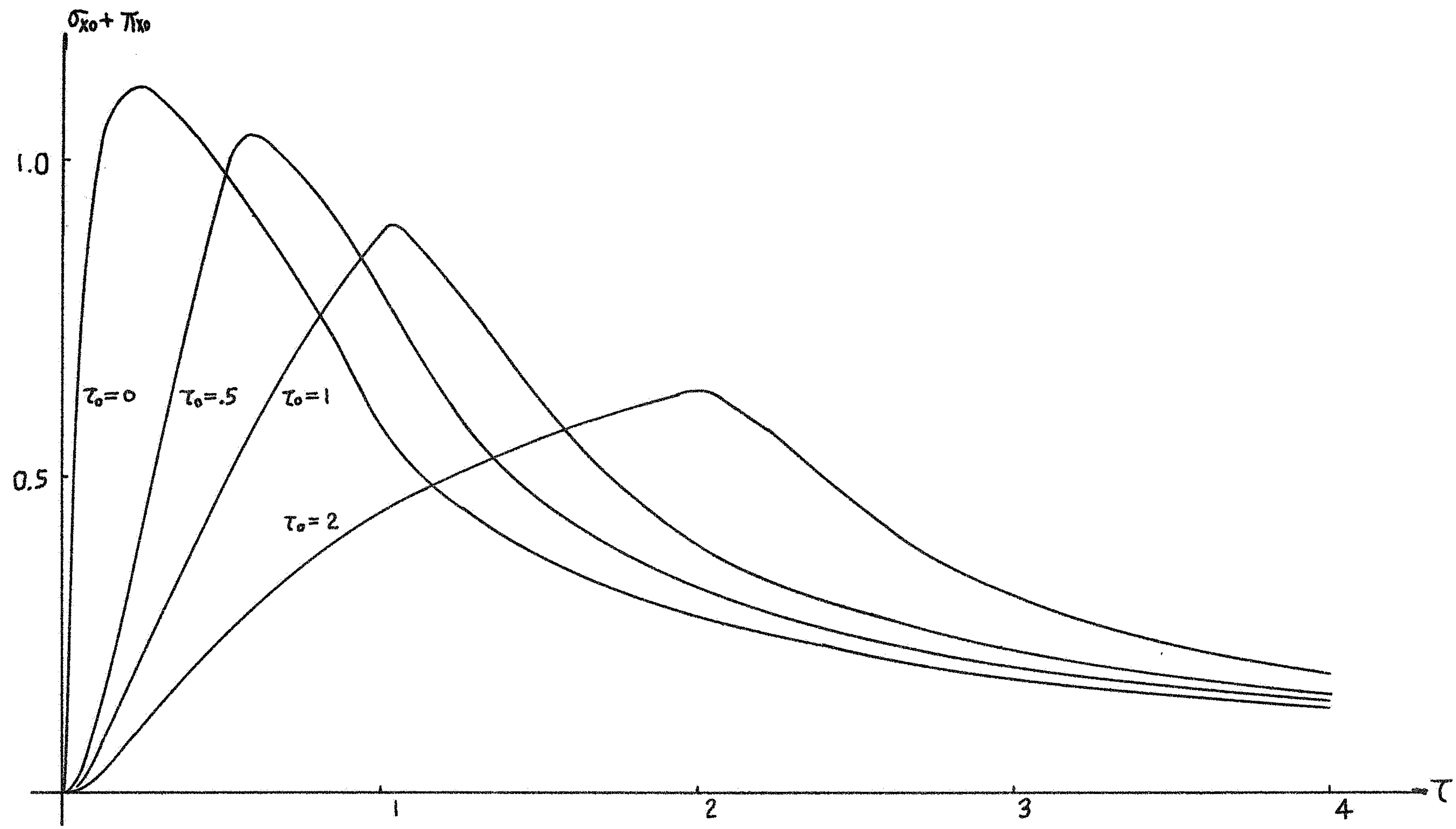

Figure 19. Plot of $\sigma_{xo}(\xi,\tau,\tau_o) + \pi_{xo}(\xi,\tau,\tau_o)$ at $\xi = 1$ for various τ_o and properties satisfying (6.35).

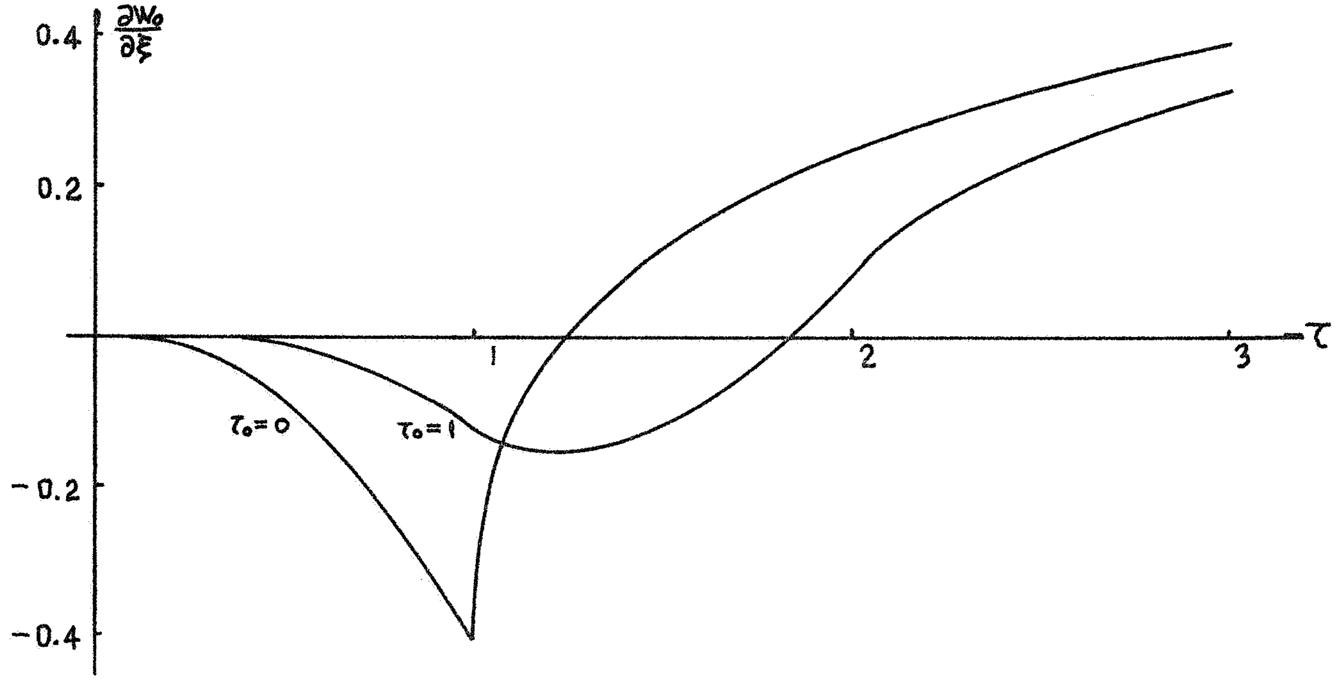

Figure 20. Thermoelastic strain $\partial w_o/\partial \xi$ at $\xi = 1$ for various τ_o when $d_1 = -1$, $\gamma = 1$.

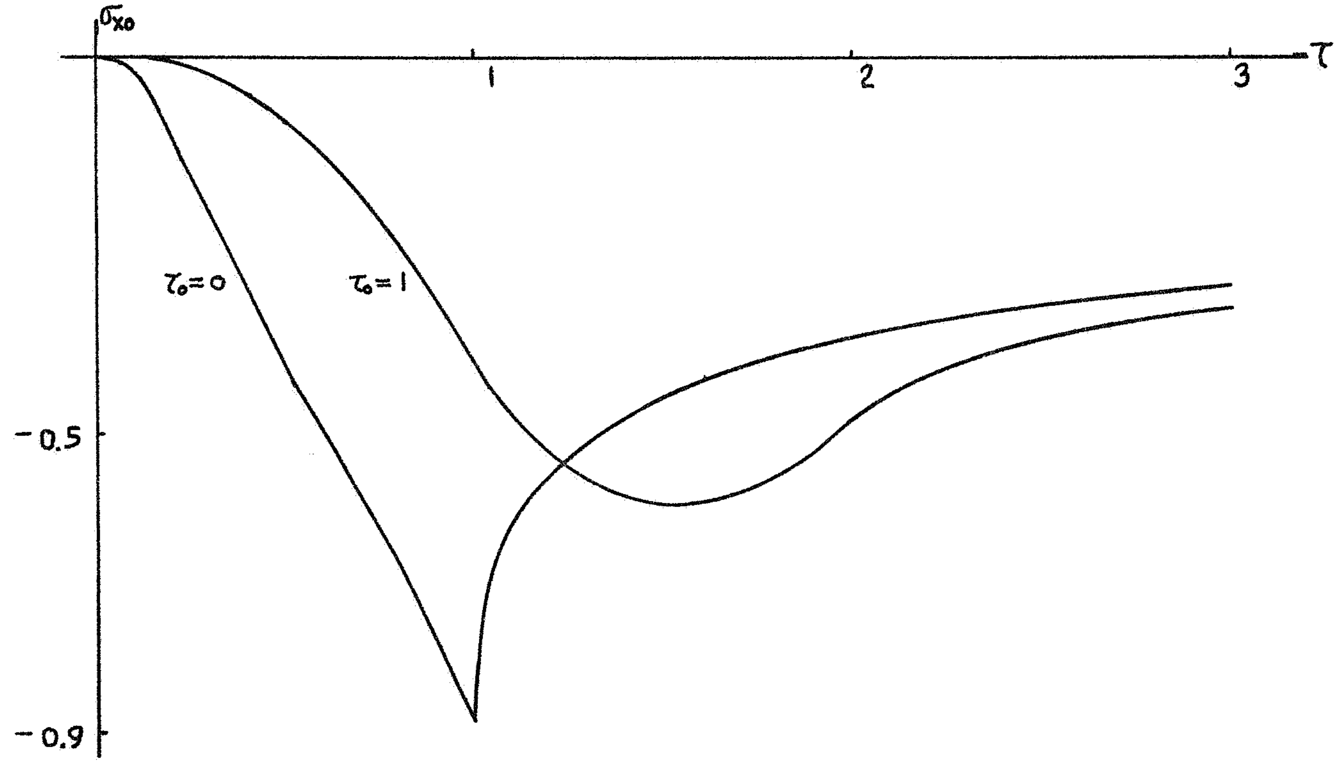

Figure 21. Elastic stress at $\xi = 1$ for various τ_o when $d_1 = -1$, $\gamma = 1$.