KINETIC THEORY FOR THE ATTENUATION

OF WAVES IN PLASMAS

By Hang-Sheng Tuan

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Department of Electrical Sciences
State University of New York at Stony Brook
Stony Brook, New York 11790

Electronics Research Center
Cambridge, Massachusetts
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Abstract

Collission damping and Landau-type damping are two major loss mechanisms known for electromagnetic waves propagating in plasmas. When an accurate estimation of the attenuation of waves is desired, a simultaneous solution to plasma kinetic equation and Maxwell equations is required but very difficulty to obtain. In this paper, a simplified set of kinetic equations are introduced to replace the original so that simultaneous solutions for problems of wave propagation may be obtained without un-surmountable mathematic difficulty. Special cases of transverse and longitudinal propagations are analysed by using the system of equations to insure that both loss mechanisms are predicted by the new equations.

I. Introduction

It is the purpose of the present work to investigate the interaction of electromagnetic wave with plasma through direct application of Maxwell equations and plasma kinetic equation. Emphasis of the analysis is placed on finding a simplified set of kinetic equations which may be used with Maxwell equations for the treatment of propagation problem in such a way that two basic loss mechanisms, collission and Landau-type dampings, are properly accounted in the theory. Interest in this analysis arises from occasions where answers to basis question in propagation problems as to the attenuation of electromagnetic wave are sought. Information on propagation loss is needed especially for establishing long range radio communication link such as the one in interplanetary space. In order to see clearly the problem, two types of approach normally used are summarized in the following.

Conventionally, electromagnetic waves propagating in an ionized gas is treated by using macroscopic hydrodynamic equations and Maxwell's equations. A great many investigations on the characteristics of wave propagation in the earth ionosphere are based on this approach. [1,2] In a weakly ionized gas, loss is primarily due to collissions between changed particles and neutral particles. Changed particles serve as energy transfer carriers which draw energy from propagating waves and transfer it to neutral particles through successive collissions. To account for the attenuation of field quantities, an average collission frequency ν is introduced into the formulation. In the end, parameters concerning the loss are put in terms of ν which is not known and has to be determined from theories elsewhere. In addition to this deficiency, it is well known that Landan-type damping is not predicted. Thus, hydrodynamic treatment is only phenomelogical as far as loss is concerned and does not furnish a satisfactory way to calculate the attenuation of various waves.

The other approach is the use of kinetic theory. This is to seek solutions of Maxwell equations and kinetic equation of plasma which determines the distribution function of a test particle (also see discussion in section II). The general solution of these coupled integral - differential equations is still not known. However, Landau[3] solved an one-dimensional initial-value problem and obtain the damping constant for a longitudinal wave without introducing the collission mechanism into the theory. This is the famous Landau damping which is understood due to the collective behavior of changed particles in plasma. It is clear that, in addition to collission damping, Landau-type damping is another important loss mechanism for wave propagation in plasma. However, it is not clear (at least to author) that any effort has been made to treat propagation problem of electromagnetic wave in plasma with both collission loss and Landau-type damping built into the theory. This is not because two types of losses could not occur simultaneously, as for a moderate ionized gas, both loss mechanisms would be equally important. The difficulty is mathematical. It is often very difficult to find solution of Maxwell equations and plasma kinetic equation with both collissional and collective behavior properly accounted (which usually result an integral-differential kinetic equation). As a first step forward to overcome this difficulty, it is proposed in the present work to replace the original kinetic equation (see section 2) by a simplified set of differential equations which nevertheless retain the essential features of the original, i.e., with both loss mechanisms in the theory. Hence the major steps concerned in the present work one first to derive the set of equations, then to show they do possess the desired properties by actually obtaining solutions together with Maxwell equations and giving both damping parameters explicitly. After that, the basic set of equations may be used in confidence to treat further radiation and propagation problems.

II. Basic Equations

In recent years, many attempts have been made to the formulation of a kinetic equation for plasmas which will, hopefully, describe its irreversible approach to equilibrium. The common starting point of these theories is the B-B-G-K-Y hierarchy of equations [4,5] derived from Lionville Theorem in Statistic Mechanics. For purpose of describing the nonequilibrium process in plasma, an approximation must be made to break up the chain of equations. There are as many possible ways of achieving this goal as the number of possible theories. An excellent book which summarizes all these approaches is given by Wu.[4] It seems at the present time that there is no way of saying which theory is more successful than others. This is because the kinetic equations derived from various approaches are very complicated and not many details have been worked out yet.

In the present work, the modified Vlasov equation is used as the kinetic equation of plasma for the following reason. Since the basic feature of a plasma is characterized by the long range coulomb interactions among changed particles which results the collective behavior in addition to normal collission phenomena with neutral particles. This is precisely the point where Boltzmann equation fails. As a first approximation (rigorously two particle distribution function should be used) to account for the collective phenomena, a charged test particle is assumed to interact with a smeared out distribution of charge outside the Debye sphere. The equation which is capable to describe this effect is just the Vlasov equation. However, it is known that Vlasov equation does not lead to Maxwell Boltzmann distribution at equilibrium. Thus modifications are needed. It is believed that multiple coulomb collissions and collissions with neutral particles inside the Debye sphere are important for plasma to reach final equilibrium. To account for these processes, a modified Boltzmann collission integral is added at the right-hand side of Vlasov equation. Scattering crossections between charged particles involved in the term should be evaluated by cutting off the impact parameter at Debye length. The approximation so used can not be proved rigorously, same is true for the Boltzmann collission integral of neutral gases. Nevertheless, we will take it as our initial assumption. With this in mind, consider a plasma which consists of electrons, ions and neutral particles. The overall density is assumed not too high and the temperature is moderate such that classical approximation is valid. Quantum and relativistic effects can thus be ignored. The distribution function of electron and electromagnetic fields satisfy the following equations.

$$\frac{\partial}{\partial t} f_e + (\vec{v}.\nabla)\, f_e + (\frac{\vec{F}}{m} \cdot \nabla_v)\, f_e = \frac{\partial fe}{\partial t}\Big|_c \tag{1}$$

$$\vec{F} = -e\, [\vec{E} + \vec{v}\, x \vec{B}] \simeq -\, e\vec{E} \quad \text{(No static magnetic field)} \tag{2}$$

$$\nabla\, x\, \vec{E} = -\, \frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla\, x\, \vec{B} = \mu o [\frac{\partial \vec{D}}{\partial t} + \vec{J}_p + \vec{J}_o] \tag{4}$$

$$\nabla \cdot \vec{D} = \rho_p + \rho_o \tag{5}$$

$$\frac{\partial fe}{\partial t}\Big|_c = \sum_{s=e,i,n} \int [fe(\vec{v}') \, f_s\, (\vec{v}_s') - fe(\vec{v})\, f_s\, (\vec{v}_s)]\sigma_{es}\, (\Omega)\, |\vec{v} - \vec{v}_s| d\Omega \cdot d^3 v_s \tag{6}$$

$$\vec{J}_p = e \int \vec{v}_i\, f_i\, d^3 v_i - e \int \vec{v}\, fe\, d^3 v \tag{7}$$

$$\rho_p = e \int f_i d^3 v_i - e \int f_e\, d^3 v \tag{8}$$

where fs = distribution function of the sth species

$\vec{v}$, m, -e = electron velocity, mass and change

$\vec{v}_s$ = velocity of s particle and $\vec{v}_e \equiv \vec{v}$

$\vec{J}_o$, ρ_o = externally applied electric current and charge

σ_{es} (Ω) = differential scattering crossection between electron and s particle

Ω = solid angle in configuration space

$d^3 v \equiv dv_x\, dv_y\, dv_z$

The remaining task is to look for a simplified set of equations replacing (1) such that solutions to problems of electromagnetic waves may be obtained within reasonable mathematical difficulty. This is sought in next section.

III The Approximate Kinetic Equations

For purpose of deriving a simplified set of differential equations to replace (1) such that simultaneous solutions of (1) - (8) may be obtained, the following procedure is used. The plasma is assumed not far from equilibrium so that a series expansion is permissible from equilibrium solution. Thus, we must restrict ourself to small signal approximation. Since the distribution fe is a function of position, velocity and time, we choose to expand only with respect to velocity in the following manner,

$$f_e(\vec{r},\vec{v},t) = f_o(\vec{r},u,t) + \sum_{\ell=1}^{\infty}\sum_{m=o}^{\ell} fem(\vec{r},u,t)\, P_\ell^m(\cos\Theta)\cos m\Phi + fem'(\vec{r},u,t)\, P_\ell^m(\cos\Theta)\,\mathrm{Sim}\, m\Phi \qquad (9)$$

where $|\vec{v}| = u$ magnitude of velocity vector

$$P_\ell^m(x) = \frac{1}{2^\ell \ell!}(1-x^2)^{\frac{m}{2}} \frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

Θ, Φ = polar angles in spherical coordinates

The essential point of this expansion is to write out explicitly the anisotropic part of velocity distribution due to the interaction of plasma and external perturbation. With the fact that equilibrium distribution is isotropic in velocity space and the assumption that plasma is not far from equilibrium, we shall expect the expansion coefficients in (9) to have fast decreasing magnitudes with respect to increasing ℓ. The next step is to obtain differential equations of those coefficient functions in (9). To this end, substitute (9) into (1), multiply it by $P_\ell^m(\cos\Theta)\cos m\phi$ or $P_\ell^m(\cos\Theta)\sin m\phi$ successively with $\ell = 0,1,2, \ldots$ and integrate (1) through 4π solid angle in velocity space, an infinite set of differential equations concerning f_o, $f_{\ell m}$, $f'_{\ell m}$ can be obtained. The complete calculation is tedious but tractable. As a first approximation, only terms with $\ell=1$ are retained. Equation (9) becomes

$$\begin{aligned} f_e(\vec{r},\vec{v},t) &\approx f_o(r,u,t) + f_{11}\sin\Theta\cos\phi + f'_{11}\sin\Theta\sin\phi \\ &\quad + f_{10}\cos\Theta \\ &\equiv f_o(\vec{r},u,t) + \frac{\vec{v}\cdot\vec{f}'(\vec{r},u,t)}{u} \end{aligned} \tag{10}$$

where

$$(f_{11}, f'_{11}, f_{10}) \equiv (f'_x, f'_y, f'_z)$$

Substitute (10) into (1) and make use of the relation $u^2 = \vec{v}\cdot\vec{v}$, obtaining

$$\frac{\partial f_o}{\partial t} + \frac{\vec{v}}{u}\cdot\frac{\partial \vec{f}'}{\partial t} + (\vec{v}\cdot\nabla) f_o + \frac{\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{f}'}{u}$$

$$- \frac{e}{m} \vec{E} \cdot \left[\frac{\vec{v}}{u} \frac{\partial}{\partial u} f_o + \frac{\vec{f}'}{u} + \frac{\vec{\vec{W}}}{u} \cdot \frac{\partial}{\partial u} \left(\frac{\vec{f}'}{u} \right) \right] = \left. \frac{\partial f_e}{\partial t} \right|_c \quad -- \quad (11)$$

Integrate (11) over 4π solid angles as mentioned, giving

$$\frac{\delta f_o}{\partial t} + \frac{u}{3} (\nabla \cdot \vec{f}') - \frac{e}{m} \vec{E} \cdot \left[\frac{1}{3u^2} \frac{\partial}{\partial u} (u^2 \vec{f}') \right] = RH_1 \quad (12)$$

$$\frac{\delta \vec{f}'}{\partial t} + u \nabla f_o - \frac{e}{m} \vec{E} \left(\frac{\delta f_o}{\partial u} \right) = RH_2 \quad (13)$$

where $RH_1 = \frac{1}{4\pi} \int_{\Omega_v} \left. \frac{\partial f_e}{\partial t} \right|_c d\Omega_v$ (14a)

$$RH_2 = \frac{3}{4\pi u} \int_{\Omega_v} \vec{v} \left. \frac{\delta f_e}{\partial t} \right|_c d\Omega_v \quad (14b)$$

Ω_v = solid angle in velocity space

To complete equations (12) and (13), right hand terms RH_1 and RH_2 must be determined. It is seen from (6) that $\left. \frac{\partial f_e}{\partial t} \right|_c$ involves distributions f_i and f_n. In order to avoid solving f_i and f_n simultaneously with f_e, special collission model may be used. There are many such collission models to account for e-e (electron-electron), e-i and e-n interactions. In general, e-e interaction is negligible in comparison with e-i and e-n interactions, as the immediate neighborhood of a test electron is most likely to be occupied by ions and neutral particles. The following is a list of models which will result, of course, different degree of approximation.

A1. Electron - Neutral Particle Elastic Collission (e-n)

(i) neutral particles are stationary before and after collissions

(ii) neutral particles are stationary before collission and recoil afterwards.

(iii) neutral particles have Maxwell and Boltzmann distribution.

A2. Electron - Ion Elastic Collission (e-i)

(i), (ii) and (iii) are the same as in A1 with ions replacing neutral particles.

For purpose of simplifying the calculation and exposing the physical content of the problem, let us use the simplest set of approximations A1 - (i) and A2 - (i) to calculate RH_1 and RH_2. Thus, we assume the distribution functions f_i and f_n having the forms

$$f_i = N_i \, \delta(\vec{V}_i) \quad , \quad f_n = N_n \, \delta(\vec{V}_n) \tag{15}$$

where $\delta(\vec{v})$ is the three-dimension Dirac delta function.

Substitute (15) into (6) and integrate out the velocity part, each of the collission term becomes

$$\left.\frac{\delta f_e}{\partial t}\right|_{e-n} = N_n \int_\Omega \left[f_e (\vec{v}') - f_e (\vec{v}) \right] \sigma_{en} (\Omega) |\vec{v}| d\Omega \tag{16a}$$

$$-\left.\frac{\delta f_e}{\partial t}\right|_{e-i} = N_i \int_\Omega \left[(f_e(\vec{v}') - f_e(\vec{v})\right] \sigma_{ei}(\Omega)\,|\vec{v}|\,d\Omega \tag{16b}$$

With the help of (10), it is seen that

$$f(\vec{v}') - f(\vec{v}) = \frac{1}{u}(\vec{v}' - \vec{v})\cdot\vec{f}' \tag{17}$$

Substitute (16a,b) and (17) into (14a,b), RH_1 and RH_2 can be obtained as

$$RH_1 = \frac{1}{4\pi}\int_{\Omega_v} \left.\frac{\delta f_e}{\partial t}\right|\,d\Omega_v = o \tag{18a}$$

$$RH_2 = \frac{3}{4\pi u}\int_{\Omega_v}\int_\Omega \frac{1}{u}\vec{v}\,(\vec{v}' - \vec{v})\cdot\vec{f}'\,[N_n\,\sigma_{en}(\Omega) + N_i\,\sigma_{ei}(\Omega)]\,|\vec{v}|\,d\Omega\,d\Omega_v$$

$$= -\nu\vec{f}' \tag{18b}$$

where

$$\nu = \nu_{en} + \nu_{ei}$$

$$\nu_{en} = 2\pi N_n \int_0^\pi u\,(1 - \cos\chi)\,\sigma_{en}(\chi)\,\mathrm{Sin}\,\chi d\chi \tag{18c}$$

$$\nu_{ei} = 2\pi N_i \int_{\chi_o}^\pi u\,(1 - \cos\chi)\,\sigma_{ei}(\chi)\,\mathrm{Sin}\,\chi d\chi$$

$$= \frac{e^4}{8\pi\varepsilon_o^{\,2}\sqrt{m}}\,\frac{N_i}{(3KT)^{3/2}}\,\ln(1+\Lambda^2) \tag{18d}$$

$$\Lambda = 12\pi\left(\frac{\sqrt{\varepsilon_o}}{e}\right)^3\sqrt{\frac{(KT)^3}{N_i}} \text{ and } \mathrm{Cot}\,\frac{\chi_o}{2} = \Lambda$$

K = Boltzmann constant

T = Temperature of electron in Kelvin

In the process of obtaining (18d), the electron impact parameter is cut off at the Debye length to avoid infinite contribution to total scattering crossection due to very small angle deflections. Since these small angle deflections are from interactions over distances greater than the Debye length which have been taken into account by the approximation of smeared out change distribution. Equations (12), (13), (18a) and (18b) are the approximate kinetic equations to be used in the next section.

IV Collission Damping and Landau-Type Damping

The simplified kinetic equations of plasma which have been derived with assumptions A1-(i) and A2-(ii) in the preceding section are rewritten here for easy reference and discussion.

$$\frac{\delta f_o}{\partial t} + \frac{u}{3}\,(\nabla\cdot\vec{f}') - \frac{e}{m}\,\vec{E}\cdot\left[\frac{1}{3u^2}\,\frac{\delta}{\delta u}\,(u^2\vec{f}')\right] = o \qquad (19a)$$

$$\frac{\delta\vec{f}'}{\delta t} + u\,\nabla\,f_o - \frac{e}{m}\,\vec{E}\,\left(\frac{\delta f_o}{\delta u}\right) = -\nu\vec{f}' \qquad (19b)$$

The remaining part of the problem is to show that both loss mechanisms are contained in (19a) and (19b). For this purpose, simultaneous solutions of (19a,b) and Maxwell equations (3) - (5) are sought in the following two simple cases.

Case 1. Transverse Wave Propagation

Consider a plane electromagnetic wave polarized in the χ direction and propagates in a boundless plasma along the z direction of a right-handed rectangular coordinate system. The source terms $\vec{J}_o$ and ρ_o in (4) and (5) are assumed to be infinite away from the region under consideration. Thus the electric field may be written by

$$\vec{E} = \hat{x} E_o e^{i(\omega t - kz)} \tag{20}$$

where k is the wave number to be determined by a simultaneous solution of (20), (19a,b), (3)-(5), (7), (8) and (10). Since the electric field is directed in the χ direction, electrons in plasma are forced to move up and down in the same direction. A χ component perturbation (f_x') to the electron distribution function is thus expected. With the help of (7) and (19b), it can be shown that

$$f_x' \doteq \frac{e}{m} \frac{E_o}{\sqrt{\nu^2+\omega^2}} \left(\frac{\partial f_o}{\partial u}\right) e^{i(\omega t - \theta_1 - kz)} \tag{21a}$$

$$\vec{J}_p = \hat{x} J_x = \hat{x}(-e) \int \vec{v}\left(f_o + \frac{\vec{v}\cdot\vec{f}'}{u}\right) d^3v = \hat{x}\left(-\frac{4\pi e}{3}\right) \int_o^\infty u^3 f_x' \, du \tag{21b}$$

$$\rho_p \doteq o \tag{21c}$$

where $\tan \theta_1 = \omega/\nu$

Equation (21a) is obtained with the approximation $\nabla f_o \simeq o$ which decouples (19a) and (19b). This can be justified by the following argument. If the non-linear term in (19a) is neglected, then ∇f_o is proportional to $\nabla(\nabla\cdot\vec{f}')$ which is very small, if not zero. Since we only concern with small signal theory, the justification is clear. Furthermore, $\vec{J}_p$ and ρ_p

satisfies $\nabla\cdot\vec{J}_p + i\omega\rho_p = o$, with the help of (21b) it can be shown that (21c) is valid. From equations (3) and (4), $\vec{H}$ only has a single component H_y. E_x and H_y satisfy

$$-\frac{\partial H_y}{\partial z} = i\omega\varepsilon_o E_x + J_x \text{ and } \frac{\partial E_x}{oz} = -i\omega\mu_o H_y \tag{22}$$

A simultaneous solution of (22), (21a), (21b) and (20) gives a final expression of k

$$k^2 = \omega^2\mu o\varepsilon s + \frac{4\Pi e^2}{3m}(\omega\mu_o)\int_o^\infty \frac{u^3}{\sqrt{\nu^2+\omega^2}}\left(\frac{\partial f_o}{\partial u}\right)(\text{Sin }\theta_1 + i\cos\theta_1)\,du \tag{23}$$

Define the propagation constant β and the attenuation constant α so that $k \equiv \beta - i\alpha$. With the help of (23), it follows

$$\beta^2-\alpha^2 = \omega^2\mu_o\varepsilon_o + \frac{4\Pi e^2}{3m}(\omega\mu_o)\int_o^\infty \frac{u^3}{\sqrt{\nu^2+\omega^2}}\left(\frac{\partial f_o}{\partial u}\right)\text{Sin }\theta_1\,du \tag{24a}$$

$$\alpha\beta = -\frac{2\Pi e^2}{3m}(\omega\mu_o)\int_o^\infty \frac{u^3}{\sqrt{\nu^2+\omega^2}}\cdot\frac{\partial f_o}{\partial u}\cdot\cos\theta_1\cdot du \tag{24b}$$

With a reasonable assumption of the zero-th order distribution f_o, (24a) and (24b) can be used to solve for α and β which determine completely the properties of propagating plane wave. It is seen from (24b) that α is proportional to cos θ_1. If the collission frequency ν approaches zero, cos θ_1, becomes zero and α vanishes. It is clear that the nature of this attenuation is due to collission damping. Theoretically, f_o should be obtained from (19a). Since we did not furnish the relaxation mechanism by using assumptions A1-(i), A2-(i) and no collissions among electrons, thus (19a) can not determined explicitly the form of f_o. Nevertheless, (24a,b) still furnish the way to calculate the collission damping. If one does not satisfy with the present form of (19a) and seeks for improvement, less restricted assumptions such as A1-(iii) and A2-(iii) may be used. In this way, (19a) will determine explicitly f_o. Details of this calculation is omitted here. Similar expressions for β and α may be obtained with the concept of equivalent conductivity which was calculated by many authors.[6-8] However, in the present analysis, β and α are obtained in a direct application of kinetic equations and Maxwell equations.

Case 2. Longitudinal Wave Propagation

Again, consider an electromagnetic wave which is both propagating and polarized along the z direction in a boundless plasma. Source terms $\vec{J}_o$ and ρ_o are infinite away. The electric field takes the form

$$\vec{E} = \hat{z}\, E_o\, e^{i(\omega t - kz)} \tag{25}$$

Using the same argument as in Case 1, the perturbed electron distribution function has only a component f_z'. With the help of (7) and (8), it follows that

$$\vec{J}_p = \hat{z}\, J_z = \hat{z}\left(-\frac{4\pi e}{3}\right)\int_o^\infty u^3 f_z'\, du \tag{26a}$$

$$\rho_p = e\left[N_i - \int f_o\,(\vec{r},u,t)\, d^3v\right] \tag{26b}$$

Since $\vec{J}_p$ and ρ_p satisfy equation of continuity and $\frac{\partial J_z}{\partial z}$ is not zero, as a result, ρ_p does not vanish. Thus, ∇f_o can no longer be approximated by zero as in the previous case. To account for the non-vanishing ρ_p, the zero-th order distribution function is expanded into the following form

$$f_o(\vec{r},u,t) = f_o(u)\ [1 + \lambda e^{i(\omega t-kz)}] \tag{27}$$

where λ is a small parameter. Neglecting the nonlinear term in (19a) and with $\frac{\partial}{\partial t} = i\omega$ and $\frac{\partial}{\partial z} = -ik$, equations (19a) and (19b) become

$$i\omega\lambda f_o(u)\ e^{i(\omega t-kz)} + \frac{u}{3}\frac{\partial}{\partial z} f_z' = o \tag{28a}$$

$$(i\omega+\nu)f_z' - iku\lambda f_o(u)e^{i(\omega t-kz)} - \frac{e}{m} E_z \left(\frac{\partial f_o}{\partial u}\right) = o \tag{28b}$$

With the help of (3), (4), (5) and (27), it can be shown that $\vec{H} \equiv o$ and E_z satisfies

$$-i\varepsilon_o kE_z = e\left[N_i - \int f_o(u)\ (1 + e^{i\omega t-kz})\ d^3v\right]$$

$$= 4\Pi e\lambda e^{i(\omega t-kz)} \int_o^\infty f_o(u)\ u^2 du \tag{29}$$

Eliminating λ, E_z and f_z' from (28a), (28b) and (29), an equation for k^2 is obtained, giving

$$1 = \frac{4\Pi e^2}{m\varepsilon_o} \int_o^\infty \frac{u^3}{3i\omega(i\omega+\nu)+k^2u^2}\left[\frac{\partial}{\partial u} f_o(u)\right] du \tag{30}$$

With a suitable assumption of $f_o(u)$, equation (30) can be evaluated to give a final equation for k^2. To this end, let us assume $f_o(u)$ has the

Maxwellian distribution,

$$f_o(u) = n \left(\frac{m}{2\Pi\theta}\right)^{3/2} e^{-\frac{mu^2}{2\theta}} \qquad (31)$$

and $\theta = KT$

where K and T are Boltzmann constant and the absolute temperature of electrons respectively. Substituting (31) into (30), it follows

$$k^2 = -\frac{4}{\sqrt{\Pi}} \left(\frac{m}{\theta}\right) \omega_p^2 \int_0^\infty \frac{x^4}{x^2-C^2} e^{-x^2} dx$$

$$= -\frac{4}{\sqrt{\Pi}} \left(\frac{m}{\theta}\right) \omega_p^2 \left[\frac{1}{4}\sqrt{\Pi} + \frac{1}{2}\sqrt{\Pi}\, C^2 + C^4 I \right] \qquad (31a)$$

$$I = \int_0^\infty \frac{1}{x^2-C^2} e^{-x^2} dx \qquad (31b)$$

where

$$C = \frac{1}{k}\sqrt{\frac{3m}{2\theta}} \left(\omega^2 - i\omega\nu\right)^{\frac{1}{2}} \quad , \quad \omega_p^2 = \frac{ne^2}{m\varepsilon_o} \qquad (31c)$$

It may be shown that (31b) can be put in terms of error function, equation (31a) thus becomes

$$k^2 \approx \omega_p^2 \left(\frac{m}{\theta}\right) \left[\frac{2\sqrt{\Pi}}{i} C^3 e^{-c^2} \mathrm{erf}(ic) - 2C^2 - 1 \right] \qquad (32a)$$

$$\mathrm{erf}\,(ic) = \frac{2}{\sqrt{\Pi}} \int_0^{ic} e^{-s^2} ds \qquad (32b)$$

$$k \equiv \beta - i\alpha \qquad (32c)$$

The properties of a propagating longitudinal wave is completely determined by equation (32a). In general, two types of damping are contained in this equation. It is not hard to see that collissions are involved in determining β and α. Hence, the only thing left is to prove that (32a) also contains Landau-type damping. For this purpose, we shall calculate β and α from (32a) and then compare to well-known result [9] for the limiting case of $\nu \to 0$ and $|k| \to 0$. This special case corresponds to the so-called collissionless plasma near resonance in which Landau-type damping is the only loss mechanism for longitudinal waves. To this end, let us expand the error function for large $|c|$ (small $|k|$),

$$\text{erf}\ (ic) \approx 1 - \frac{e^{c^2}}{ic\sqrt{\pi}} \left[1 + \frac{1}{2c^2} + \frac{3}{4c^4} + \frac{15}{8c^6} + \cdots \right] \tag{33}$$

Substituting (32c) and (33) into (32a), it follows

$$\beta^2 - \alpha^2 - 2i\alpha\beta \approx \frac{2}{i}\sqrt{\pi}c^3\ e^{-c^2} \left(\frac{m}{\theta}\right) \omega_p^{\ 2} + \left(\frac{m}{\theta}\right) \omega_p^{\ 2} \left(\frac{3}{2} + \frac{15}{4c^2}\right) \frac{1}{c^2} \tag{34}$$

Equating the real and imaginary parts on both sides of eq. (34), β and α may be proved to be given by

$$\omega^2 \approx \omega_p^{\ 2} + \frac{5}{2} \left(\frac{\theta}{m}\right) \beta^2 \frac{\omega_p^{\ 2}}{\omega^2} \tag{35a}$$

and $$\alpha \approx \sqrt{\pi} \left(\frac{3}{2}\right)^{3/2} \left(\frac{m}{\theta}\right)^{5/2} \frac{\omega_p^{\ 2}\omega^3}{\beta^4}\ e^{-\frac{3}{2}\left(\frac{m}{\theta}\right)\frac{\omega^2}{\beta^2}} \tag{35b}$$

Equation (35a) is exactly the dispersion relation found by Vlašov[9], Bohm, Gross and Pines [10,11] except the factor of 5/3 which is 3/1 in their works. This difference is due to the fact that a three-dimensional problem is considered in the present treatment instead of one dimension. Equation (35b) is also of the form found by Landau [3] except some coefficient differences. This furnishes the proof that (32a) is the correct general dispersion relation which contains Landau-type damping as well as collission damping. A numerical calculation of β and α as a function of ω and ν from (32a) is straightforward. For briefness, it is not included in this paper.

V Conclusion

Two major loss mechanisms for electromagnetic wave propagation in plasmas are known for years. Numerous treatments either to calculate collission damping or Landau-type damping may be found in literatures. However, a general approach to the problem is still lacking. The purpose of the present work is to study how to formulate a relative simple theory such that both loss mechanisms are included. Since the coupled plasma kinetic equation and Maxwell equations are very difficult to solve. We thus derive a simplified set of equations to replace the original kinetic equation. Solutions for two special cases of wave propagation are obtained to insure that both loss mechanisms are built into the theory. Procedures to improve the set of equations are also given. The new set of kinetic equations may be used together with Maxwell equations for treatments of propagation problems when an accurate estimation of the attenuation of waves is desired.