XVI. Communications Systems Research: Sequential Decoding

TELECOMMUNICATIONS DIVISION

A. Performance of *Pioneer*-Type Sequential Decoding Communications Systems With Noisy Oscillators, *J. A. Heller*

1. Introduction

Convolutional coding with sequential decoding will be used as an engineering experiment (Ref. 1) aboard the *Pioneer D* to be launched in August 1968. The information rates to be used are 512, 256, 64 and 16 bits/s. As the spacecraft moves away from the earth, the bit rate will be lowered whenever the overall error probability rises above 10^{-3}. This will continue until the lowest rate (16 bits/s) is used.

The convolutional code that will be used is a rate 1/2 systematic code with constraint length 25. Data will be encoded into blocks of 210 information bits, after which a 14-bit fixed sequence will be inserted in the coder for the purpose of decoder resynchronization. Incoming data at the receiver will be quantized into eight levels prior to inputting it to the decoder.

Pioneer D will operate at a maximum rate of 512 bits/s. A sequential decoder, being designed and built at JPL (SPS 37-50, Vol. II, pp. 71–78), will be available for experimentation on a portion of the *Pioneer D* mission. At rate 1/2, it will have the capability of performing one computation per microsecond and a memory capable of storing about 10^4 branches. This decoder will be so fast, compared to the *Pioneer D* bit rate, that it will be possible to operate at an energy-per-bit $E_b = (E_b)_{min}$ with a negligible ($<10^{-8}$) erasure probability (Ref. 2).

2. Decoder Error Probability

Theoretically, it has been shown that decoder error probability decreases exponentially with constraint length independent of decoder speed and memory size for rates less than capacity (Ref. 3 and SPS 37-50, Vol. III, pp. 241–248). However, probability of decoder memory overflow or block erasure depends on these two factors and on the (energy-per-bit)-to-noise ratio, E_b/N_0. Since the average decoder computation per bit decoded is unbounded for rates above a certain rate, R_{comp}, sequential decoding is limited to operate at rates less than R_{comp}. That is, there is a minimum E_b defined by

$$(E_b)_{min} \triangleq \frac{E}{R_{comp}} \frac{\text{energy/code symbol}}{\text{bits/code symbol}} \tag{1}$$

that must be used in order to induce a stable computational behavior in the decoder. Depending on the decoder speed and memory size, it is usually necessary to operate with an E_b somewhat above $(E_b)_{\min}$ to meet a specified erasure probability (Ref. 2).

3. Noisy Phase Reference

The phase reference used for demodulating the incoming data from *Pioneer* is derived from a phase-locked loop that tracks the unmodulated carrier component of the signal. The received signal is essentially composed of the carrier with power P_C and the phase-shift-keyed (PSK) data with power P_D. The total power is $P_T = P_C + P_D$ and the modulation index, m^2, is the fraction of power in the carrier ($m^2 = P_D/P_T$). Since the *Pioneer* communication system operates at low rates, P_D/N_0 is small. In order to have sufficient carrier signal-to-noise ratio (SNR) to develop a reliable phase reference, the optimum m^2 is typically measured in tenths rather than hundredths as in high-rate systems. Even at these high modulation indices, the effects of an inaccurate phase reference make the required E_b much larger than that needed in a perfectly coherent system.

In addition to the phase uncertainty caused by tracking a carrier in the presence of thermal noise, the spacecraft oscillator itself is noisy to some degree. Four oscillators with differing degrees of noisiness will be considered. Figure 1 shows the rms phase error incurred when the output of these oscillators is tracked by a phase-locked loop as a function of the bandwidth of the tracking loop. The four oscillator characteristics are generated by letting $n = 1$–4 (Fig. 1).

4. System Performance in the Presence of a Noisy Phase Reference

It has been shown (SPS 37-48, Vol. III, pp. 181–187, and Ref. 4) that for an infinite bandwidth unquantized channel with constant but unknown phase, the use of a decision-directed method of obtaining a phase reference from the information signal itself results in an R_{comp} for sequential decoding given by

$$R_{\text{comp}} = \max_{0<\eta<1} \frac{E}{N_0 \ln 2} \left(\frac{\eta}{1+\eta} - \frac{1}{2P_T \tau / N_0} \ln \frac{1}{1-\eta^2} \right) \tag{2}$$

in bits per code symbol, where P_T is the power in the received signal, τ is the time over which the phase reference is formed, and E/N_0 is the SNR per code symbol. In the case at hand, the phase reference is derived from a separate carrier—not from the data signal. In can be argued that the measurement of phase by means independent of the data will improve the performance slightly; however, Eq. (2) will still be approximately true with P_C in place of P_T; W_L, the loop bandwidth, in place

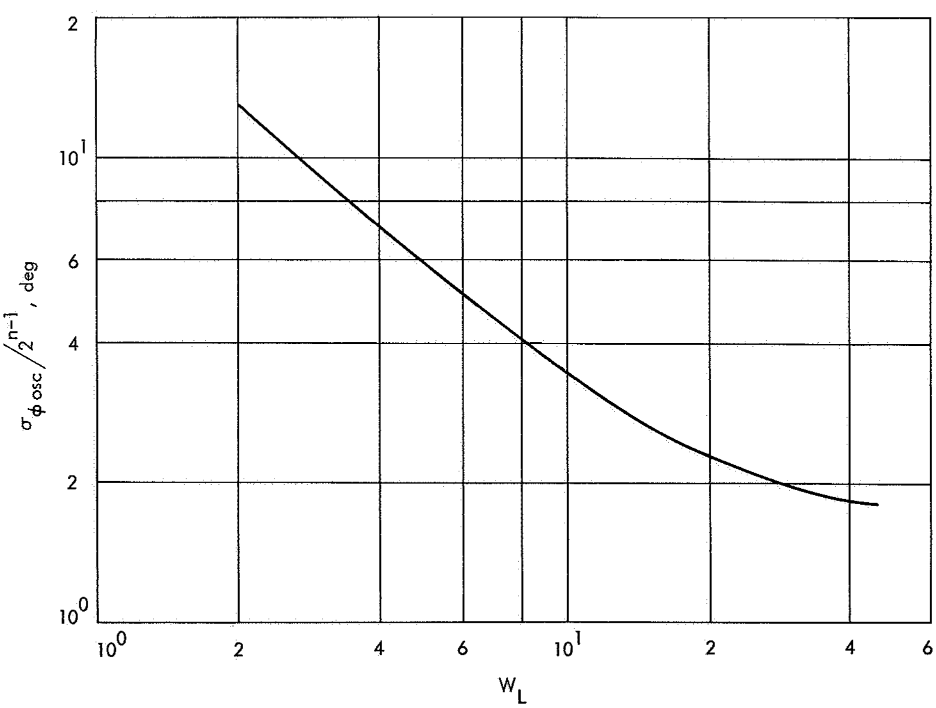

Fig. 1. Rms phase error vs W_L for four oscillators

of $1/\tau$; and E_D, the energy per code symbol in the data signal, in place of E:

$$R_{\text{comp}} \cong \max_{0<\eta<1} \frac{E_D}{N_0 \ln 2} \left(\frac{\eta}{1+\eta} - \frac{1}{2P_C/N_0 W_L} \ln \frac{1}{1-\eta^2} \right) \tag{3}$$

As it stands, this result holds only for a constant phase channel and the effects of a noisy oscillator have not yet been taken into account. When a phase-locked loop is operating in its linear region, the phase-error distribution due to thermal noise is approximately gaussian with variance (Ref. 5):

$$\sigma_\phi^2 = \frac{1}{2P_C/N_0 W_L} \tag{4}$$

Thus, Eq. (3) can be rewritten

$$R_{\text{comp}} \cong \max_{0<\eta<1} \frac{E_D}{N_0 \ln 2} \left(\frac{\eta}{1+\eta} - \sigma_\phi^2 \ln \frac{1}{1-\eta^2} \right) \tag{4}$$

Now, however, any phase jitter due to a noisy oscillator is independent of jitter due to thermal noise. Hence, the total phase-error variance when the oscillator is noisy will be

$$\sigma_\phi^2 = \frac{1}{2P_C/N_0 W_L} + \sigma_{\phi\,\text{osc}}^2 (n, W_L) \tag{5}$$

If we accept the fact that the effect on R_{comp} of a given phase-error variance due to oscillator jitter is the same as that due to a like variance due to thermal noise (a justifiable assumption when σ_ϕ^2 is small), then Eq. (5) may be substituted directly into Eq. (4).

It is now desired to put Eq. (4) into a form that shows its dependence on E_b/N_0; the rate in bits per second, R; and m^2:

$$\frac{2P_C}{N_0 W_L} = \frac{2m^2 P_T}{N_0 W_L} = \frac{2m^2 P_T T_b R}{N_0 W_L} = 2m^2 \frac{E_b}{N_0} \frac{R}{W_L} \tag{6}$$

where $R = 1/T_b$ and $E_b \triangleq P_T T_b / N_0$. Using the fact that $E_D = (1 - m^2) E$, and combining Eqs. (4) and (6), Eq. (1) becomes

$$\frac{E_b}{N_0} \cong \min_{0<\eta<1} \left[\frac{1-m^2}{\ln 2} \left(\frac{\eta}{1+\eta} - \sigma_\phi^2 \ln \frac{1}{1-\eta^2} \right) \right]^{-1} \tag{7}$$

where σ_ϕ^2 is given by Eqs. (5) and (6). Since σ_ϕ^2 is a function of E_b/N_0, Eq. (7) may be solved explicitly for E_b/N_0:

$$\frac{E_b}{N_0} \cong \min_{0\leq\eta<1} \left(\frac{\dfrac{\ln 2}{1-m^2} + \dfrac{W_L}{2m^2 R} \ln \dfrac{1}{1-\eta^2}}{\dfrac{\eta}{1+\eta} - \sigma_{\phi\,\text{osc}}^2 \ln \dfrac{1}{1+\eta^2}} \right) \tag{8}$$

where $\sigma_{\phi\,\text{osc}}$ depends on n and W_L (Fig. 1).

The expression for R_{comp} in Eq. (2) was obtained for an unquantized, infinite bandwidth channel; thus, Eq. (8) holds only for that channel. It has been shown (Ref. 6) that, in going from a coherent, infinite bandwidth (zero rate), unquantized channel to a rate 1/2, 3-bit quantized channel, an additional 1.25 dB in E_b/N_0 are required. Equation (8) has been numerically minimized with respect to η and m^2. Figure 2 shows the E_b/N_0 resulting from this minimization for W_L of 3, 6, 12 and 24 Hz as a function of R. In each graph, there is one curve for each noisy oscillator ($n = 1$ to 4) and an additional curve for a perfect oscillator ($\sigma_{\phi\,\text{osc}} = 0$) that indicates the performance when phase jitter is due to thermal noise alone. Figures 2a–2d include the additional 1.25 dB due to rate 1/2 and 3-bit receiver quantization. These figures also include the value of m^2 that optimizes Eq. (8) versus R.

5. Conclusions and Interpretations

The steps leading up to Eq. (8) depended heavily on the assumption that the linear model of the tracking loop was appropriate. For this reason, it is meaningless to extrapolate the results to very low rates where the phase-error variance of Eqs. (5) and (6) is so large as to clearly indicate non-linear loop operation. To reflect this, the curves in Fig. 2 are truncated at that rate at which $\sigma_\phi = 0.5$ rad. This, for instance, completely eliminates the curves for two oscillators in Fig. 2.

At low-rate operation in a *Pioneer*-type system (16 bits/s), the required E_b/N_0 is several dB larger than that required for a perfectly coherent system. The E_b/N_0 needed for a perfectly coherent system can be obtained from Eq. (8) by letting $\sigma_{\phi\,\text{osc}}^2 \to 0$, $W_L \to 0$, $m \to 0$ and $\eta \to 1$. Thus, for this case, $E_b/N_0 \to 2 \ln 2$. Adding the 1.25 dB for rate 1/2 and 3-bit quantization yields

$$\left(\frac{E_b}{N_0} \right)_{\text{coherent}} = 2.65\,\text{dB} \tag{9}$$

The curves for the perfect oscillators in Fig. 2 asymptotically approach this value at high rates.

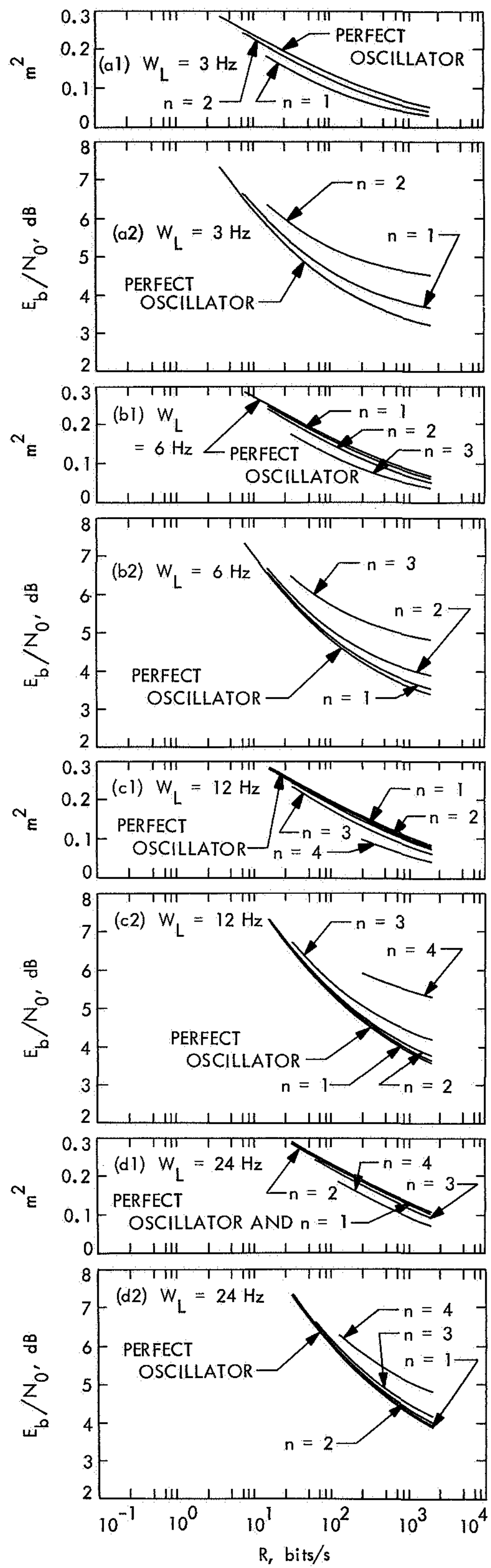

Fig. 2. Minimized E_b/N_0 as a function of R

There is an optimum tracking W_L for any given spacecraft oscillator and rate. This is true because thermal noise considerations dictate the use of the narrowest possible loop while a wide-band loop is best able to track the phase fluctuations of a noisy oscillator. The net result is that at high rates, where the phase-error variance due to thermal noise is small even with a low m^2, noisy-oscillator effects dominate. At low rates, the opposite is true.

The relative looseness of the arguments leading to the E_b/N_0 curves indicates that equipment design should not be based solely on them. It is reasonable to assume, however, that within the linear region of phase-locked loop operation, the curves should be accurate. On this assumption, a comparison with results for uncoded PSK systems (Ref. 6) indicates that, even at rates on the order of 16 bits/s, the sequential decoding system offers about a 2 dB advantage.

References

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