XXI. Communications Systems Research: Information Processing

TELECOMMUNICATIONS DIVISION

A. Digital Filtering of Random Sequences,

G. Jennings

1. Introduction

This article reports recently established results concerning the output of a digital filter when the input is a random sequence. The results are an improvement on those reported in SPS 37-48, Vol. III, pp. 213–220. The stability of the numerical method discussed therein is established under weaker conditions for recurrences of general length. A conjecture is used that has been proved in special cases.

In SPS 37-48, Vol. III, we considered filtering a random sequence $\{x_1, x_2, \cdots\}$ to form another sequence $\{y_1, y_2, \cdots\}$ by a linear recurrence of the form

$$y_n = \sum_{j=1}^{K} a_j y_{n-j} + x_n \tag{1}$$

When the values of $\{x_n\}$ are arbitrary real numbers, Eq. (1) can be solved only approximately. The sequence that is actually found satisfies

$$y_n = \sum_{j=1}^{K} a_j y_{n-j} + x_n + \delta_n \tag{2}$$

where δ_n is the error involved in evaluating the right side of Eq. (1).

We consider the type of error that occurs if we try to solve Eq. (1) on a digital computer. The details of the approximation procedure were specified in the previous article. Recall that

$$P = \{0, \pm 2\delta, \pm 4\delta, \cdots, 2k_o\delta = \pm \mathrm{M}\}$$

is the set of possible values of each y_i. We had established bounds for the mean square of the difference between the solutions of Eqs. (1) and (2) under the condition that

$$\sum | a_j | < 1$$

This condition is weakened for recurrences of length 2 to the condition that the equation

$$x^2 - a_1 x - a_2 = 0$$

has only roots of modulus less than one. A lemma, proved for $K = 2$, is conjectured to hold for any K. Assuming this, the same generalization can be made for $K > 2$.

We denote the solution of Eq. (1) by $\tilde{y}_1, \tilde{y}_2, \cdots$, rewriting that equation as

$$\tilde{y}_n = \sum_{j=1}^{K} a_j \tilde{y}_{n-j} + x_n \tag{3}$$

We want to compare the solution of Eq. (2) with the solution of Eq. (3) when the same initial values are used for each sequence:

$$y_n = \widetilde{y}_n = 0, \qquad n = 0,1, \cdots, K-1$$

Moreover, we shall require that each of the roots of the polynomial

$$t^k - \sum_{j=1}^{K} a_j t^{K-j} \tag{4}$$

be less than one in modulus.

2. The Conjecture

It is convenient to reformulate the problem as a matrix equation. We note the identity

$$\mathbf{Y}_n = A\mathbf{Y}_{n-1} + \mathbf{X}_n + \mathbf{D}_n \tag{5}$$

where

$$\mathbf{Y}_n = \begin{pmatrix} \mathbf{y}_{n-k+1} \\ \vdots \\ \mathbf{y}_n \end{pmatrix}, \qquad \mathbf{X}_n = \begin{pmatrix} 0 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}, \qquad \mathbf{D}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta_n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 0 & 0 & 1 \\ \mathbf{a}_j & \mathbf{a}_{j-1} & & \cdot & \cdot & \cdot & \mathbf{a}_1 \end{pmatrix}$$

Equation (5) provides a formulation of the problem which is equivalent to Eq. (2) in an obvious way. A minimal polynomial of A is Eq. (4), and therefore, the spectral radius of A, $\rho(A)$, is strictly less than one because of the restrictions on the roots of Eq. (4). Hence, by a fundamental theorem of spectral theory, there exists a norm on R^K, depending on ϵ, such that the induced matrix norm on A is less than $\rho(A) + \epsilon$, where ϵ may be chosen to be any positive number. This follows from the fact that a matrix is similar to a matrix of the form

$$\begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_n \end{pmatrix}$$

where each Λ_i is a block of the form

$$\begin{pmatrix} \lambda & \epsilon & 0 & & & \\ 0 & \lambda & \epsilon & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & 0 & \lambda & \epsilon \\ & & & & 0 & \lambda \end{pmatrix}$$

where ϵ is an arbitrary real number (Ref. 1).

We denote such a norm by $\|\cdot\|_\epsilon$. We shall require in the following that $\rho(A) + \epsilon$ be less than one.

For such an ϵ, it immediately follows that A carries the set

$$B_{\eta,\epsilon} = \{\mathbf{y} \in R^K : \|\mathbf{y}\|_\epsilon \leq \eta\}$$

into itself, where η is taken to be any positive number. As any two norms on a finite dimensional Euclidean space are equivalent (Ref. 2), there exists an $\eta > 0$, given $\epsilon > 0$, such that $B_{\eta,\epsilon}$ is contained in the unit hypercube

$$H_1^K = \left\{ \mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_K \end{pmatrix} : |\mathbf{y}_i| \leq 1, i = 1, \cdots, K \right\}$$

We assume that such an η and ϵ are chosen and fixed.

We shall now formulate the conjecture. We view the rounding process as being composed of two parts written symbolically as R_M and R_δ. R_M applied to a vector in R^K changes any component of that vector that exceeds M in absolute value to M, if that component is positive, to $-M$, otherwise; recall that M is positive. R_M leaves unchanged any component less than M in absolute value. R_δ rounds each component of a vector to that element of P to which it is closest in absolute value. With these conventions the calculation of the right-hand side of Eq. (4) can be viewed as applying A to $\mathbf{Y}_{n-1}$, adding $\mathbf{X}_n$, and then applying R_M followed by R_δ. That is,

$$\mathbf{Y}_n = R_\delta R_M (A\mathbf{Y}_{n-1} + \mathbf{X}_n)$$

We then have the following conjecture:

Conjecture. For a matrix A,

$$A = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & & 0 & 1 \\ a_K & a_{K-1} & & & & \cdot & a_1 \end{pmatrix}$$

where the polynomial

$$t^K - \sum_{j=1}^{K} a_j t^{K-j}$$

has only roots of modulus less than one, there exists a positive integer N, such that $(R_1 A)^N \mathbf{Y}_0$ is contained in $B_{\eta/3,\epsilon}$, if $\mathbf{Y}_0$ is any vector contained in the unit hypercube. (N may depend on A, η, and ϵ but must be independent of $\mathbf{Y}_0$.) ϵ is chosen so that $\rho(A) + \epsilon < 1$. η is fixed so that $B_{\eta,\epsilon}$ is contained in H_1^K.

3. Estimates Using the Conjecture

We now proceed to derive estimates for $\int (y_i - \tilde{y}_i)^2 \, d\mu$ with the conjecture as an hypothesis. Let μ denote the probability measure that is induced when the x_i's are independent random variables with a gaussian distribution. (We do not try for the best possible estimates.) Assuming that $M = 1$, we apply the conjecture to deduce that $(R_1 A)^N \mathbf{Y}_{i-N-1}$ is contained in $B_{\eta/3,\epsilon}$ regardless of the value of $\mathbf{Y}_{i-N-1}$, since $\mathbf{Y}_{i-N-1}$ must lie in the unit hypercube. With

$$\hat{\mathbf{Y}}_n = R_1 \, (A \hat{\mathbf{Y}}_{n-1} + \mathbf{X}_n)$$

neglecting to apply R_δ, the process of determining the $\hat{\mathbf{Y}}_K$'s is a continuous process of the $\mathbf{X}_K$'s. Therefore, if each x_j is restricted to be sufficiently small for $j = i - N, \cdots, i - 1$, $\hat{\mathbf{Y}}_{i-1}$ must be contanied in $B_{2\eta/3,\epsilon}$ where $\hat{\mathbf{Y}}_i$ has been determined from the equation

$$\hat{\mathbf{Y}}_K = R_1 \, (A \hat{\mathbf{Y}}_{K-1} + \mathbf{X}_K)$$

Let $a > 0$ be such a bound on the absolute value of the x_j's, $j = i - N, \cdots, i - 1$. We can increase the scale of allowed values of $\hat{\mathbf{Y}}_i$ by a factor of M, rounding to the hypercube each of whose edges is the length $2M$ and requiring

$$|x_j| \leq aM, \qquad \text{for } j = i - N + 1, \cdots, i - 1$$

to obtain a similar result. More precisely, define

$$C_i^M = \{\{x_j\}_{j=K}^{\infty} \,||\, x_j \,| \leq aM, \quad \text{for } j = i - N - 1, \cdots, i - 1\}$$

Then for $\{x_j\}$ in C_i^M, $\hat{Y}_{i-1}$ $\{x_j\}$ is contained in $B_{(2\eta/3)\,(M,\epsilon)}$. In this instance $\hat{\mathbf{Y}}_K$ satisfies the recursion

$$\hat{\mathbf{Y}}_K = R_M(A\hat{\mathbf{Y}}_{K-1} + \mathbf{X}_K)$$

Noting that

$$\mathbf{Y}_K = R_\delta R_M (A\mathbf{Y}_{K-1} + \mathbf{X}_K)$$

we can deduce that

$$\mathbf{Y}_K - \hat{\mathbf{Y}}_K = A(\mathbf{Y}_{K-1} - \hat{\mathbf{Y}}_{K-1}) + E_K$$

where each component of $\mathbf{E}_K$ is less than δ in magnitude. Noting that $\mathbf{Y}_K$ and $\hat{\mathbf{Y}}_K$ have the same value at the index $i - N - 1$, and that the solution of a linear recurrence depends continuously on the inhomogeneous term, we can deduce that, for $M \geq M_0 > 0$, there exists a δ_0 depending on M_0 but independent of M such that $\delta \leq \delta_0$, $M \geq M_0$ implies that $\mathbf{Y}_{i-1}(\{x_j\})$ is contained in $B_{\eta M,\epsilon}$, for $\{x_j\}$ in C_i^M. We thus obtain the condition that $A\mathbf{Y}_{i-1}$ is contained in $B_{(\rho(A)+\epsilon)\eta M,\epsilon}$, and therefore,under the additional constraint that the modulus of x_i be less than $(1 - \rho(A) - \epsilon)M$, the absolute value of δ_i must be less than δ. Let

$$S = C_i^M \cap \{\{x_j\} \,||\, x_i \,| \leq (1 - \rho(A) - \epsilon)M\}$$

and let S^C be its complement, and integrate with respect to probability measure:

$$\int \delta_i^2 d\mu \leq \int_C \delta_i^2 d\mu + \int_{S^C} \delta_i^2 d\mu$$

$$\leq \delta^2 + \int_{S^C} \delta_i^2 d\mu$$

The integrand of the last integral can be bounded using

$$|\delta_i| \leq \left(1 + \sum_j |a_j|\right) M + |x_i|$$

It is a straightforward estimate that the integral of the right-hand side of the above inequality over the complement of S goes to zero as δ and M go to zero and ∞, respectively. The proof follows techniques used in SPS 37-48, Vol. III. We have, therefore, the following result:

Lemma

$$\int \delta_i^2 d\mu$$

goes to zero as δ goes to zero and M goes to infinity, uniformly in i.

Similarly, using techniques from SPS 37-48, Vol. III, we obtain:

Theorem 1

$$\int (y_i - \tilde{y}_i)^2 \, d\mu$$

goes to zero, uniformly in i, as δ and M approach zero and infinity, respectively.

4. The Conjecture for $K = 2$

We now proceed to prove the conjecture for recurrences of length 2 subject to the condition that the roots of Eq. (4) are less than one in modulus. First, we establish the notation that $\bar{\lambda}$ is the complex conjugate of λ, which is a complex number. If

$$\mathbf{v} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

and

$$\mathbf{w} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{pmatrix}$$

are column vectors over the complex numbers, we define the inner product of the two $(\mathbf{v},\mathbf{w})$ as

$$v_1\bar{w}_1 + v_2\bar{w}_2$$

Let λ be a root of Eq. (4); then a calculation verifies that $\binom{1}{\lambda}$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ a_2 & a_1 \end{pmatrix}$$

with eigenvalue λ.

$$\mathbf{v}_1 = \frac{1}{(1 + |\lambda|^2)^{1/2}} \cdot \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$$

and

$$\mathbf{v}_2 = \frac{1}{(1 + |\lambda|^2)^{1/2}} \cdot \begin{pmatrix} -\bar{\lambda} \\ 1 \end{pmatrix}$$

form an orthonormal basis for the two-dimensional vector space of column vectors over the complex numbers. As $\mathbf{v}_1$ is an eigenvector of A, the matrix of A in the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is upper triangular. Hence, $(A\mathbf{v}_2, \mathbf{v}_2)$ is an eigenvalue of A and must be less than one in magnitude.

Let

$$\mathbf{w} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

be a vector with real entries each of which is less than one in absolute value. As $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis, $\mathbf{w}$ may be written in the form

$$\mathbf{w} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$$

A computation yields that

$$\alpha_2 = (\mathbf{w}, \mathbf{v}_2) = (-\lambda \mathbf{y}_1 + \mathbf{y}_2)/(1 + |\lambda|^2)^{1/2}$$

Recall that y_1 and y_2 are real. The square of the modulus of α_2 is

$$[\mathrm{Re}\,\lambda y_1 + y_2) + (\mathrm{Im}\,\lambda)^2 y_1^2]/(1 + |\lambda|^2)$$

Multiplying by $1 + |\lambda|^2$ and calculating the partial derivative with respect to y_2 we achieve

$$\frac{1}{2}\frac{\partial}{\partial y_2}(1 + |\lambda|^2)\,|\alpha_2|^2 + \mathrm{Re}\,\lambda y_1 + y_2$$

We note two easily derived inequalities when $|y_1| \leq 1$:

$$\left.\begin{aligned} &\mathrm{Re}\,\lambda y_1 + y_2 \geq y_2 - |\lambda| \\ \text{and}\qquad & \\ &\mathrm{Re}\,\lambda y_1 + y_2 \leq y_2 + |\lambda| \end{aligned}\right\} \tag{6}$$

Recalling Eq. (5), specialized to the case at hand, we see that the image of a vector $\mathbf{w}$ in the unit square has first component less than or equal to one. If the second component, y_2, is greater than or equal to one, rounding will decrease the second component of the vector. Selecting the first of the two inequalities in Eq. (6), we see

$$1/2\frac{\partial}{\partial y_2}(1 + |\lambda|^2)\,|\alpha_2|^2 \geq 1 - |\lambda| \geq 0$$

from which we obtain

$$\frac{\partial}{\partial y_2} | \alpha_2 |^2 \geq 0$$

when $y_2 \geq 1$. In this case, the effect of rounding is to decrease the modulus of α_2. Similarly, if y_2 is less than -1, rounding increases y_2. The second inequality of Eq. (6) yields

$$\frac{\partial}{\partial y_2} | \alpha_2 |^2 \leq (-1 + | \lambda |) \frac{2}{1 + | \lambda |^2} \leq 0$$

when $y_2 \leq -1$. It follows that the modulus of α_2 is also decreased in this case.

We thus obtain the condition that the effect of applying $R_1 A$ to a vector in the unit cube

$$\alpha_1^{(0)} \mathbf{v}_1 + \alpha_2^{(0)} \mathbf{v}_2$$

produces another vector in the unit cube

$$\alpha_1^{(1)} \mathbf{v}_1 + \alpha_2^{(1)} \mathbf{v}_2$$

where

$$| \alpha_2^{(1)} | \leq \rho(A) | \alpha_2^{(0)} |$$

We see that successive application of $R_1 A$ yields a vector in the unit cube

$$\alpha_1^{(n)} \mathbf{v}_1 + \alpha_2^{(n)} \mathbf{v}_2$$

with

$$| \alpha_2^{(n)} | \leq \rho(A)^n | \alpha_2 |$$

and

$$\alpha_1^{(n)} = \lambda \alpha_1^{(n-1)} + \gamma \alpha_2^{(n-1)}$$

where $\gamma = (A\mathbf{v}_2, \mathbf{v}_1)$. This last equation defines $x_1^{(n)}$ as the solution of a nonhomogeneous linear recurrence.

The solution for $\gamma_1^{(n)}$ can be written in the closed form

$$\alpha_1^{(n+j)} = \lambda^j \alpha_1^{(n)} + \gamma \sum_{s=0}^{j} \lambda^s \alpha_2^{(n+j-s-1)}$$

If

$$| \alpha_2^{(r)} | \leq b \text{ for } r = n - 1$$

we obtain

$$| \alpha_1^{(n+j)} | \leq | \lambda |^j | \alpha_1^{(n)} | + \frac{| \gamma | b}{1 - \rho(A)^2}$$

As b may be taken arbitrarily small, $\alpha_1^{(n+j)}$ may be made arbitrarily small. As

$$| \alpha_2^{(n+j)} | \leq \rho(A)^n | \alpha_2^{(0)} |$$

$\alpha_2^{(n)}$ goes to zero, and hence, $(R_1 A)^n$ shrinks the unit square uniformly to a point. Hence, the conjecture is established for $K = 2$.

References

1. John, F., *Notes on Ordinary Differential Equations*, p. 101, Courant Institute of Mathematical Sciences, New York, N.Y., 1964–1965.
2. Taylor, A. E., *Introduction to Functional Analysis*, p. 95, John Wiley & Sons, Inc., New York, N.Y., 1958.

B. Maximum Likelihood Symbol Synchronization for Binary Systems With Coherent Subcarrier-Symbol Rate, *W. J. Hurd*

1. Introduction

This article considers the symbol synchronization problem for binary systems in which the subcarrier frequency and the symbol rate are coherent; i.e., they are derived from the same frequency reference. In such systems, with the additional assumption that the subcarrier phase is known at the receiver, there are only a finite number of possible phases for the symbol clock. Typically there are an integral number, e.g., N, of subcarrier half cycles in each symbol time, so that the symbol phase can occur at N positions. Normally the subcarrier phase is tracked by a phase-locked loop.

The analysis and results are also applicable to systems in which there are an infinite number of possible phases for the subcarrier clock. In these cases, however, one must assume a finite number of possible phases, and accept phase errors smaller than the difference between the assumed phase position candidates. There is, however, the additional requirement that the symbol repetition rate be known exactly, a requirement that is automatically satisfied when the subcarrier is tracked and the symbol timing is derived from the same clock.

The basic symbol synchronization problem is to find the optimum decision rule for estimation of the correct symbol timing based on observations of the received noisy data for a fixed length of time. Other related problems are to evaluate the performance of the optimum and near-optimum decision rules as functions of the symbol signal-to-noise ratio (SNR) and the observation time, and to find the required observation times to achieve given error probabilities at a given SNR. Stiffler[1] has derived the maximum likelihood decision rule for general (n-ary) amplitude-modulation (AM) systems, of which binary AM and biphase modulation systems are special cases. Here we present a different derivation for the binary case, and a more concise expression for the final result. We also examine two methods that approximate the maximum likelihood rule, one at high SNRs and one at low SNRs, and present numerical results for these approximations. The two approximations were suggested by Stiffler, but numerical results were not given, although numerical comparison of the two methods has been given by Stiffler for a problem that somewhat resembles the synchronization problem (SPS 37-29, Vol. 14, pp. 285–290).

2. Derivation of Maximum Likelihood Rule

Suppose the received signal waveform $r(t) = s(t) + n(t)$ is observed for $0 \leq t \leq (M+1)T$ sec, where $s(t)$ is the signal, $n(t)$ is white gaussian noise with two-sided spectral density $N_0/2$, T is the duration of one symbol, and $M+1$ is the number of symbol times observed. The signal $s(t)$ is constant at either $+A$ or $-A$ over each symbol time, so that the symbol energy is A^2T, but it is not known at which points in time the symbols start. The N possible candidates for the starting time of the first full symbol observed are $0, T/N, 2T/N, \cdots, (N-1)T/N$, and if the actual starting time is kT/N, successive symbols start $kT/N, (kT/N) + T, (kT/N) + 2T, \cdots$. We denote the first M symbols by the vector $\mathbf{A} = (A_1, A_2, \cdots, A_M)$. The probability that each symbol is $+A$ or $-A$ is one half, and each symbol is independent of all others.

To make a maximum likelihood decision as to correct symbol timing, we must compute the *a posteriori* probability of each candidate, given that $r(t)$ is received, and choose the candidate for which this probability is maximized. The first step is to convert the problem from one involving random functions to a finite dimensional vector problem. The signal component of any possible received waveform can then be expressed as

$$s(t) = \sum_{i=1}^{N(M+1)} s_i \, \phi_i(t) \tag{1}$$

where the $\phi_i(t)$ are the orthonormal functions

$$\phi_i(t) = \begin{cases} (N/T)^{1/2}, & \text{for } (i-1)T/N \leq t < iT/N \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

and

$$s_i = \int_0^{(M+1)T} s(t) \, \phi_i(t) \, dt = \pm A(T/N)^{1/2} \tag{3}$$

The vector $\mathbf{s} = (s_1, s_2, \cdots, s_{N(M+1)})$ completely defines $s(t)$. Similarly, we define the noise vector

$$\mathbf{n} = (n_1, n_2, \cdots, n_{N(M+1)}) \tag{4}$$

where

$$n_i = \int_0^{(M+1)T} n(t) \, \phi_i(t) dt \tag{5}$$

The noise components n_i are independent zero mean gaussian random variables with variances $N_0/2$.

We also define the received signal plus noise vector

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{6}$$

With this notation, it can be shown (Ref. 1) that the vector $\mathbf{r}$ is a sufficient statistic, and contains all of the data that is relevant to determining the received signal waveform; i.e., the *a posteriori* probability of the transmitted signal vector conditioned on $r(t)$ is the same as that conditioned on $\mathbf{r}$.

$$p(\mathbf{s} \mid r(t)) = p(\mathbf{s} \mid \mathbf{r}) \tag{7}$$

The *a posteriori* probability that the correct timing occurs at position k (i.e., at $t = kT/N$), is

$$p(k \mid r(t)) = p(k \mid \mathbf{r}) = \frac{p(\mathbf{r} \mid k) \, p(k)}{p(\mathbf{r})} \tag{8}$$

[1]Stiffler, J. J., *The Synchronization in Communication Systems*, to be published by Prentiss Hall Pub. Co., Englewood Cliffs, N.J., in 1969.

Since $p(\mathbf{r})$ is not a function of k, and since $p(k) = 1/N$ for all k, the best estimate of k can be made by computing $p(\mathbf{r} \mid k)$ for each k and choosing the maximum.

It is now convenient to neglect the data for $t < kT/N$ and for $t \geq MT + kT/N$, i.e., to neglect r_i for $i \leq k$ and for $i > MN + k$, $k = 0, 1, \cdots, N-1$. This will result in negligible degradation for reasonably large M, and the more exact result can easily be obtained if desired. Since the n_i and the A_m are all independent, and the signal components s_i are all the same for $mN - N + k < i \leq mN + k$, the probability density of $\mathbf{r}$, conditioned on phase position k, is

$$p(\mathbf{r} \mid k) = \prod_{m=1}^{M} p(r_{mN-N+k+1}, \cdots, r_{mN+k} \mid k) \tag{9}$$

Conditioned on A_m, the r_i for $mN - N + k < i \leq mN + k$ are independent. Furthermore, the s_i are all $+A(T/N)^{1/2}$ or all $-A(T/N)^{1/2}$ with equal probability, so

$$p(r_{mN-N+k+1}, \cdots, r_{nM+k} \mid k) = \frac{1}{2} \prod_{i=mN-N+k+1}^{mN+k} p(r_i \mid k, s_i = +A(T/N)^{1/2}) + \frac{1}{2} \prod_{i=mN-N+k+1}^{mN+k} p(r_i \mid k, s_i = -A(T/N)^{1/2}) \tag{10}$$

Using the gaussian densities with means $\pm A(T/N)^{1/2}$ and variances $N_0/2$ for the conditional densities in Eq. (10), substituting into Eq. (9), and simplifying, we get

$$p(\mathbf{r}|k) \approx (\pi N_0)^{-NM/2} \exp\left\{-(MA^2T/N_0) - N_0^{-1} \sum_{i=k+1}^{NM+k} r_i^2\right\} \prod_{m=1}^{M} \cosh\left(2A(T/N)^{1/2} N_0^{-1} \sum_{i=mN-N+k+1}^{mN+k} r_i\right) \tag{11}$$

But, neglecting end effects,

$$\sum_{i=k+1}^{NM+k} r_i^2$$

is approximately the same for all k, so the exponential terms in Eq. (11) can be dropped.

Finally, the maximum likelihood decision rule, neglecting only negligible end effects, is to choose the k which maximizes the function

$$L(k) = \prod_{m=1}^{M} \cosh\left(2A(T/N)^{1/2} N_0^{-1} \sum_{i=mN-N+k+1}^{mN+k} r_i\right) \tag{12}$$

which, using the defining relation for the r_i, can be written as

$$L(k) = \prod_{m=1}^{M} \cosh\left(2AN_0^{-1} \int_{(m-1)T+kT/N}^{mT+kT/N} r(t)\, dt\right) \tag{13}$$

3. Approximation Methods

The maximum likelihood rule is impractical to use because of the product of the hyperbolic cosines, and because the expression depends on knowledge of the signal amplitude A and the noise spectral density. However, $L(k)$ can be approximated using one expression for high SNRs and another for low SNRs.

Suppose we define

$$x_m(k) = \int_{(m-1)T+kT/N}^{mT+kT/N} r(t)\, dt \tag{14}$$

Then, conditioned on the mth symbol, the mean and variance of the argument $2AN_0^{-1} x_m(k)$ of the hyperbolic cosine are

$$E\left\{2AN_0^{-1} x_m(k) \mid A_m = \pm A\right\} = \pm 2A^2T/N_0 \tag{15}$$

$$\mathrm{var}\left\{2AN_0^{-1} x_m(k) \mid A_m = \pm A\right\} = 2A^2T/N_0 \tag{16}$$

Since the SNR is equal to the square of the conditional mean divided by the conditional variance, the quantity $2A^2T/N_0$ is the SNR. Hence, on the average, the argument of the hyperbolic cosine is small for low SNRs and large for high SNRs.

a. Squaring method. For low SNRs, the product of hyperbolic cosines can be expanded into a product of Taylor series, and all but the first terms can be dropped.

$$\prod_{m=1}^{M} \cosh\,(x_m(k)) \approx 1 + \sum_{m=1}^{M} (2AN_0^{-1}\, x_m(k))^2 \qquad (17)$$

In this case, synchronization is determined by measuring

$$L_S(k) = \sum_{m=1}^{M} x_m^2\,(k) \qquad (18)$$

for each k and choosing the largest. This is called the *squaring method.*

b. Absolute value method. For high SNRs, the hyperbolic cosine is approximately exponential, so

$$\prod_{m=1}^{M} \cosh\,(2AN_0^{-1}\, x_m(k)) \approx \prod_{m=1}^{M} \frac{1}{2} \exp\,(\,|\,2AN_0^{-1}\, x_m(k)\,|\,) \qquad (19)$$

It suffices to measure

$$L_A(k) = \sum_{m=1}^{M} |\, x_m(k)\,| \qquad (20)$$

for each k and to choose the largest. This is called the *absolute value method.*

c. Performance evaluation. By the central limit theorem, the $L_S(k)$, $1 \leq k \leq N-1$, and the $L_A(k)$, $1 \leq k \leq N-1$, are approximately jointly gaussian for sufficiently large M. Let us assume that the $k = 0$ position is the actual transition point, and define the normalized variables

$$M^{1/2} A_k = \frac{L_A(0) - L_A(k)}{(\mathrm{var}\,\{L_A(0) - L_A(k)\})^{1/2}} \qquad (21)$$

and

$$M^{1/2} S_k = \frac{L_S(0) - L_S(k)}{(\mathrm{var}\,\{L_S(0) - L_S(k)\})^{1/2}}$$

The normalization is chosen so that the statistics of A_k and S_k are independent of M. A correct decision is made whenever all of the A_k (or S_k) are greater than zero for $k = 1, 2, \cdots, N-1$.

Since $M^{1/2}A_k$ and $M^{1/2}S_k$ are linear combinations of approximately gaussian random variables, they are also approximately gaussian. Furthermore, their variances are unity by the normalization in Eq. (22), so

$$\Pr\{A_k < 0\} = (2\pi)^{-1/2} \int_{-\infty}^{0} \exp\left\{-\frac{1}{2}\,(x - M^{1/2}\, E\{A_k\})^2\right\} dx$$

$$= \frac{1}{2}\,\mathrm{erfc}\,(2^{-1/2}\, M^{1/2}\, E\{A_k\}) \qquad (23)$$

and

$$\Pr\{S_k < 0\} = \frac{1}{2}\,\mathrm{erfc}\,(2^{-1/2}\;\, M^{1/2}\, E\{S_k\}) \qquad (24)$$

where $\mathrm{erfc}\,(a) = 1 - \mathrm{erf}\,(a)$ is the complementary error function and

$$\mathrm{erf}\,(a) = 2\pi^{-1/2} \int_0^a \exp\,\{-x^2\}\, dx \qquad (25)$$

For the absolute value method, for example, the probability of error is at least as great as the probability that $A_1 \leq 0$ and by a "union bound" does not exceed

$$\sum_{k=1}^{N-1} \Pr\,\{A_k < 0\}$$

Hence for the absolute value method, the error probability is bounded below by

$$P_E \geq \frac{1}{2}\,\mathrm{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_1\}\right) \qquad (26)$$

and above by

$$P_E \leq \frac{1}{2} \sum_{k=1}^{N-1} \mathrm{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_k\}\right) \qquad (27)$$

Similar expressions for the squaring method are obtained by substituting $E\,\{S_k\}$ for $E\,\{A_k\}$ throughout. Note that by symmetry $E\,\{A_k\} = E\,\{A_{N-k}\}$ so that the k and $N - k$ terms in Eq. (27) are equal.

d. Numerical results. The expected values of A_k and S_k are derived in the following *Subsection 4.* The results are shown as a function of symbol SNR, $R = 2A^2T/N_0$, in Fig. 1 for fractional timing errors $k/N = 1/2$, $1/4$, $1/8$, $\cdots$, $1/1024$. As expected, these curves show that the squaring method is better at low symbol signal-to-noise ratios and that the absolute value method is better at higher SNRs. At SNRs of interest for coded systems, around $R = 1$, the two methods are approximately equally good. The squaring method may be preferred because

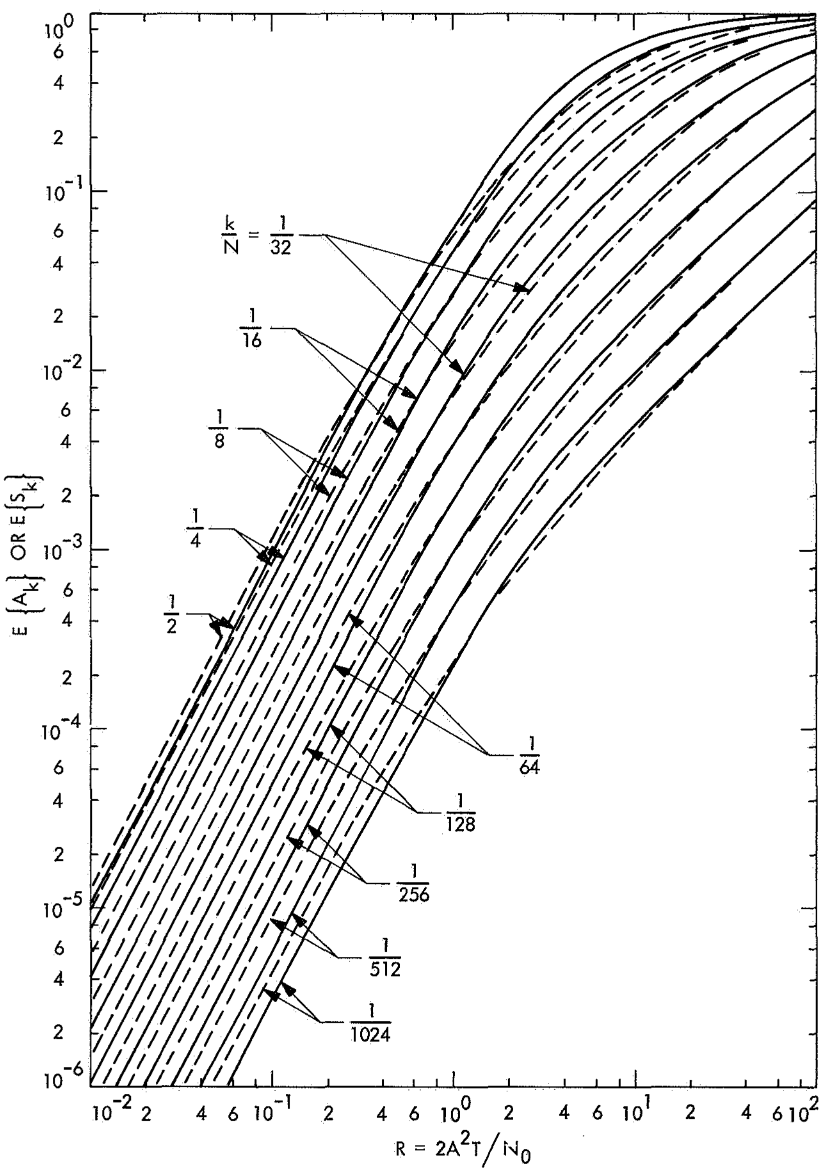

Fig. 1. A_k and S_k as functions of symbol SNR

it is about 1.5 dB better for low SNRs, and the absolute-value method is not much better at higher SNRs. On the other hand, the absolute-value method may be easier to implement in some situations.

e. Simplified upper bound on P_E. The results shown in Fig. 1 can be used to simplify the expression for the upper bound on the probability of incorrect synchronization. The basic idea is that synchronizing incorrectly at position $k = 1$ or $k = N - 1$ is much more likely than at any other position, so that all terms in Eq. (27) can be dropped except the $k - 1$ and $k = N - 1$ terms.

To reduce the lower bound of Eq. (26) on probability of synchronization error to a reasonably low level, we must choose M large enough so that

$$0.5 \operatorname{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_1\}\right)$$

for the absolute value method, is small. Since $E\{A_2\}$ for fixed N is typically 1.5 to 2 times $E\{A_1\}$ for that N,

$$\operatorname{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_2\}\right)$$

will be negligible compared to

$$\operatorname{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_1\}\right)$$

whenever the latter is small, because erfc(z) decreases approximately as $\exp\{-z^2\}/z$. Similarly, all terms in Eq. (27) will be small except the $k - 1$ and $k = N - 1$ terms, which are equal by symmetry, so that the upper bound becomes approximately

$$P_E \lesssim \operatorname{erfc}\left(\left(\frac{M}{2}\right)^{1/2} E\{A_1\}\right) \tag{28}$$

for the absolute value method, and similarly for the squaring method.

4. Calculations for Approximate Methods

The mean values of A_k and S_k can be expressed in terms of the relevant statistics for the $x_m(k)$, given that $k = 0$ is the correct position. Normalizing to unit noise energy per symbol at the detector (integrator, matched filter) output, we define

$$\sigma^2 = \frac{N_0 T}{2} = 1 = \text{noise energy per symbol}$$

$$\mu = |AT|$$

$$R = \frac{2A^2T}{N_0} = \frac{\mu^2}{\sigma^2} = \mu^2$$

$$= \text{SNR at detector output}$$

$$\rho_k = \frac{N-k}{N} = \text{fraction of } m\text{th symbol in assumed position of the } m\text{th symbol for given } k$$

$$\alpha_k^2 = \rho_k^2 + (1-\rho_k)^2$$

$$\beta_k = \rho_k(1-\rho_k)$$

and

$$\gamma_k = 2\rho_k - 1$$

With this notation, we can write $x_m(k)$ as

$$x_m(k) = \int_{(m-1)T+kT/N}^{mT+kT/N} n(t)\,dt + \rho_k A_m T + (1-\rho_k) A_{m+1} T \tag{29}$$

a. Squaring method. To calculate the statistics of the $L_S(k)$, we note that $x_m(k)$ and $x_n(k)$ are independent for $|m - n| > 1$, and that $x_m(0)$ and $x_n(k)$ are independent except for $n = m$ and $n = m - 1$. In other cases, no symbol affects both $x_m(k)$ and $x_n(k)$ or both $x_m(0)$ and $x_n(k)$, and the noise components are independent because the noise is white. Hence

$$E\{L_S(k)\} = ME\{x_m^2(k)\} \tag{30}$$

$$\begin{aligned} E\{L_S^2(k)\} &= E \sum_{m=1}^{M} \sum_{n=1}^{M} x_m^2(k)\, x_n^2(k) \\ &= ME\{x_m^4(k)\} + 2(M-1)\, E\{x_m^2(k)\, x_{m+1}^2(k)\} \\ &\quad + (M^2 - 3M + 2)\, E^2\{x_m^2(k)\} \end{aligned} \tag{31}$$

$$\begin{aligned} E\{L_S(0)L_S(k)\} &= E \sum_{m=1}^{M} \sum_{n=1}^{M} x_m^2(0) x_n^2(k) \\ &= ME\{x_m^2(0) x_m^2(k)\} \\ &\quad + (M-1)\, E\{x_m^2(0)\, x_{m-1}^2(k)\} \\ &\quad + (M^2 - 2M + 1)\, E\{x_m^2(0)\, x_m^2(k)\} \end{aligned} \tag{32}$$

The expectations in the above equations are obtained by using Eq. (29), expanding, and taking expectations term by term:

$$E\{x_m^2(k)\} = 1 + \alpha_k^2 R \tag{33}$$

$$E\{x_m^4(k)\} = 3 + 6\alpha_k^2 R + (\alpha_k^4 + 4\beta_k^2) R^2 \tag{34}$$

$$E\{x_m^2(k)\, x_{m+1}^2(k)\} = 1 + 2\alpha_k^2 R + \alpha_k^4 R^2 = E^2\{x_m^2(k)\} \tag{35}$$

$$E\{x_m^2(0) x_m^2(k)\} = 1 + 2\rho_k^2 + (2 - 2\rho_k + 6\rho_k^2)R + \alpha_k^2 R^2 \tag{36}$$

$$\begin{aligned} E\{x_m^2(0) x_{m-1}^2(k)\} &= 3 - 4\rho_k + 2\rho_k^2 \\ &\quad + (6 - 10\rho_k + 6\rho_k^2)R + \alpha_k^2 R^2 \end{aligned} \tag{37}$$

Substituting Eqs. (33) to (37) into Eqs. (30) to (32), combining terms, and assuming $M \gg 1$ so that terms not depending on M are negligible, the mean and variance of $L_S(0) - L_S(k)$ are

$$E\{L_S(0) - L_S(k)\} = 2M\,\beta_k\, R \tag{38}$$

and

$$\operatorname{var}\{L_S(0) - L_S(k)\} = 8M\,\beta_k \left(1 + R + \frac{\beta_k R^2}{2}\right) \tag{39}$$

As expected, these expressions are linear in M and symmetric in k and $N - k$. Finally, the mean of S_k is

$$E\{S_k\} = R\left(\frac{\beta_k}{2(1+R)+\frac{\beta_k^2}{2}}\right)^{1/2} \tag{40}$$

b. Absolute value method. Following the same procedure as above but replacing squares by absolute values, we get

$$\frac{1}{M} E\{L_A(0) - L_A(k)\} = E\{|x_m(0)|\} - E\{|x_m(k)|\} \tag{41}$$

and

$$\begin{aligned}\frac{1}{M}\operatorname{var}\{L_A(0) - L_A(k)\} = {} & E\{x_m^2(0)\} - E^2\{|x_m(0)|\} \\ & + E\{x_m^2(k)\} \\ & + 2E\{|x_m(k)x_{m+1}(k)|\} \\ & - 3E^2\{|x_m(k)|\} \\ & - 2E\{|x_m(0)x_m(k)|\} \\ & - 2E\{|x_m(0)x_{m-1}(k)|\} \\ & + 4E\{|x_m(0)|\}\, E\{|x_m(k)|\}\end{aligned} \tag{42}$$

The absolute moments in Eq. (42) must now be evaluated. Conditioned on the received symbols, $x_m(0)$ and $x_n(k)$ are jointly gaussian, and, by symmetry, we can always assume that the first symbol affecting the desired statistic is equal to $+A$. Hence, for one variate,

$$E\{|x_m(k)|\} = \frac{1}{2}E\left\{|x_m(k)|\,|\, A_m = A, A_{m+1} = A\right\} + \frac{1}{2}E\left\{|x_m(k)|\, |\, A_m = A, A_{m+1} = -A\right\} \tag{43}$$

Since

$$E\{x_m(k)\,|\,A_m = A, A_{m+1} = \pm A\} = (1 - (1-\rho_k) \pm (1-\rho_k))\mu$$

and

$$\operatorname{var}\{x_m(k)\,|\,A_m, A_{m+1}\} = \sigma^2 = 1$$

$$\begin{aligned}E\{|x_m(k)|\,\big|A_m = A, A_{m+1} = \pm A\} &= (2\pi)^{-1/2}\left[\int_0^{\infty} - \int_{-\infty}^{0}\right] x \exp\left\{-\frac{1}{2}\left(x - [1-(1-\rho_k)\pm(1-\rho_k)]\mu\right)^2\right\}dx \\ &= [1-(1-\rho_k)\pm(1-\rho_k)]\,\mu \operatorname{erf}\left(2^{-1/2}[1-(1-\rho_k)\pm(1-\rho_k)]\,\mu\right) \\ &\quad + 2(2\pi)^{-1/2}\exp\left\{-\frac{[1-(1-p_k)\pm(1-p_k)]^2\mu^2}{2}\right\}\end{aligned} \tag{44}$$

and

$$E\{|x_m(k)|\} = \frac{\mu}{2}\operatorname{erf}(2^{-1/2}\mu) + \frac{\gamma_k\mu}{2}\operatorname{erf}(2^{-1/2}\gamma_k\mu) + (2\pi)^{-1/2}\exp\left\{-\frac{\mu^2}{2}\right\} + (2\pi)^{-1/2}\exp\left\{-\frac{\gamma_k^2\mu^2}{2}\right\} \tag{45}$$

To evaluate the joint absolute moments, we first define the function g by

$$g(m_x,m_y,\rho) = E\{|xy|\} \tag{46}$$

where x and y are jointly gaussian with means m_x and m_y, unit variances, and covariance ρ. Then

$$E\{|x_m(k)x_{m+1}(k)|\} = \frac{1}{4}[g(\mu,\mu,0) + g(\mu,\gamma_k\mu,0) + g(\gamma_k\mu, -\gamma_k\mu,0) + g(\gamma_k\mu, -\mu,0)] \tag{47}$$

where the first two arguments of g reflect the four possible combinations of A_{m+1} and A_{m+2}. Similarly

$$E\{|x_m(0)x_m(k)|\} = \frac{1}{2}[g(\mu,\mu,\rho_k) + g(\mu,\gamma_k\mu\,\rho_k)] \tag{48}$$

and

$$E\{|x_m(0)x_{m-1}(k)|\} = \frac{1}{2}[g(\mu,\mu,1-\rho_k) + g(-\gamma_k\mu,\mu,1-\rho_k)] \tag{49}$$

The function g is

$$g(m_x, m_y,\rho) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy|\exp\left\{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T\mathbf{R}^{-1}(\mathbf{x}-\mathbf{m})\right\}dx\,dy \tag{50}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{51}$$

$$\mathbf{m} = \begin{pmatrix} m_x \\ m_y \end{pmatrix} \tag{52}$$

and $\mathbf{R}$ is the covariance matrix of $\mathbf{x}$. We now perform a transformation to polar coordinates, letting

$$\mathbf{B} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{53}$$

so

$$\mathbf{x} = r\,\mathbf{B} \tag{54}$$

Then g becomes

$$g(m_x,m_y,\rho) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\int_0^{2\pi}d\theta\int_0^{\infty}dr\,r^3\,|\cos\theta\sin\theta|\,x\exp\left\{-\frac{1}{2}(r\,\mathbf{B}-\mathbf{m})^T\mathbf{R}^{-1}(r\mathbf{B}-\mathbf{m})\right\} \tag{55}$$

The infinite integral can now be integrated in closed form yielding

$$g(m_x,m_y,\rho) = \frac{1}{2\pi(1-\rho^2)^{1/2}}\int_0^{2\pi}d\theta\,|\cos\theta\sin\theta|\exp\{-(a_3 - a_2^2/a_1)/2\}$$
$$\times\,[(2a_1 + a_2^2)\,a_1^{-3}\exp\{-a_2^2/(2a_1)\} + \pi^{1/2}\,2^{-1/2}\,(a_2^3 + 3a_1a_2)\,a_1^{-7/2}\,\mathrm{erfc}\,(-a_2(2a_1)^{1/2})] \tag{56}$$

where

$$a_1 = \mathbf{B}^T \mathbf{R}^{-1} \mathbf{B} \tag{57}$$

$$a_2 = \mathbf{B}^T \mathbf{R}^{-1} \mathbf{m} \tag{58}$$

and

$$a_3 = \mathbf{m}^T \mathbf{R}^{-1} \mathbf{m} \tag{59}$$

Equation (56) was integrated numerically to obtain the curves in Fig. 1.

5. Summary

Although the maximum likelihood method for symbol synchronization derived in *Subsection 2* is impractical to implement, it is closely approximated by the squaring method at low SNRs and by the absolute value method at high SNRs. The probability that synchronization does not occur at exactly the correct place is bounded by

$$\frac{1}{2}\,\text{erfc}\left((M/2)^{1/2}\,E\{A_1\}\right) \leq P_E \lesssim \text{erfc}\left((M/2)^{1/2}\,E\{A_1\}\right) \tag{60}$$

for the absolute value, and by

$$\frac{1}{2}\,\text{erfc}\left((M/2)^{1/2}\,E\{S_1\}\right) \leq P_E \lesssim \text{erfc}\left((M/2)^{1/2}\,E\{S_1\}\right) \tag{61}$$

for the squaring method. In these expressions, M is the number of symbols used in the estimation, and A_1 and S_1 are given as a function of SNR in Fig. 1. The parameter N in Fig. 1 is the number of places at which synchronization might occur, which is typically the number of subcarrier half cycles in one symbol time.

Reference

1. Wozencraft, J. M., and Jacobs, I. M., *Principles of Communication Engineering,* Chap. 4, John Wiley and Sons, New York, 1965.