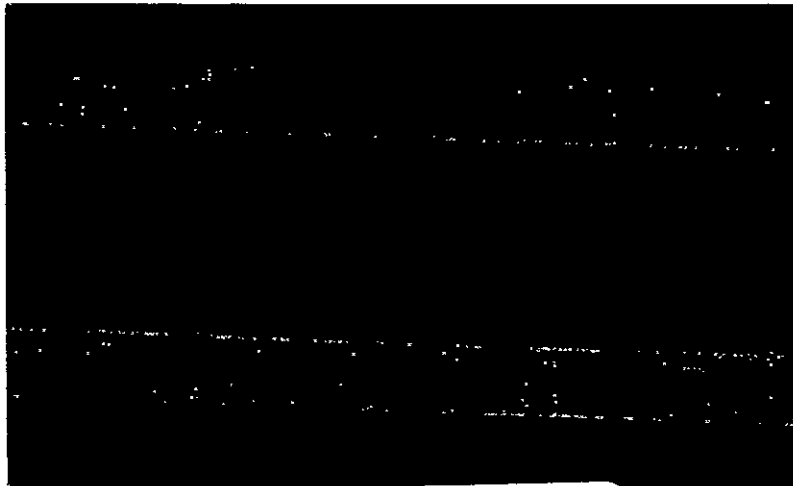


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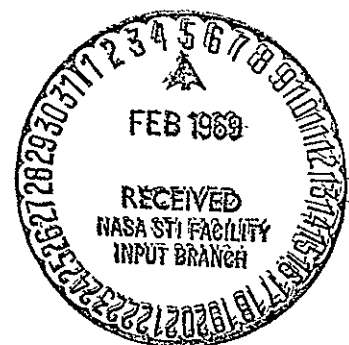
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MINIMAL TIME CONTROL OF LINEAR
SYSTEMS WITH CONTROL AMPLITUDE
AND RATE SATURATIONS

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December, 1968

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TABLE OF CONTENTS

	Page
LIST OF TABLES	v
LIST OF FIGURES	vii
ABSTRACT	ix
CHAPTER I INTRODUCTION	1
1.1 Motivation	1
1.2 Review of the Literature	2
1.2.1 Minimal Time Open-Loop Operation	2
1.2.2 Sub-Optimal Closed-Loop Operation	6
1.2.3 Recoverability	8
1.3 Organization of the Thesis	9
CHAPTER II PROBLEM FORMULATION	11
2.1 System Definition	11
2.2 Open-Loop Solution	12
2.3 Set of Recoverability	15
2.4 Sub-Optimal Closed-Loop Solution	16
2.5 Recoverability	17
CHAPTER III OPEN-LOOP SOLUTION	18
3.1 Introduction	18
3.2 Boundary Points on the Set of Recoverability	18
3.3 Extremal Controls	19
3.4 Decomposition of the Problem	20
3.5 Necessary Conditions	22
3.6 Modified Adjoint Solution	25
3.7 Construction of the Extremal Controls	37
3.8 Extremal Control Tables	50
3.9 Computer Program	63
3.10 Open-Loop Minimal Time Control	66
3.11 An Example-Ranger Block III Attitude Control System	67
3.12 General Remarks	77
CHAPTER IV A SUB-OPTIMAL CLOSED-LOOP SOLUTION	78
4.1 Introduction	78
4.2 Introduction of a Sub-Optimal Method of Solution	79
4.3 Isochrones and the Set of Recoverability	84

	Page
4.4 Partitioning the Set M	86
4.5 Construction of the Isochrones	91
4.6 Approximation of the Isochrones	96
4.7 Computation of the Matrices $\bar{C}_k$ and $\bar{C}_k$	101
4.8 Obtaining the Approximation Functions $V(x,T)$ and $\bar{V}(x,T)$	105
4.9 On-Line Operation	112
4.10 Consideration of the Motivating Criteria	117
4.11 Illustrative Example	118
4.12 Application of Step (1)	122
4.13 Application of Step (2)	130
4.14 Application of Step (3)	133
4.15 Sub-Optimal Controller Performance for System (4.40)	151
4.16 Sub-Optimal Controller Performance for System (3.78)	165
4.17 Computation of the Parameter T	173
CHAPTER V RECOVERABILITY	176
5.1 Introduction	176
5.2 Complete Recoverability and the Adjoint Equations	177
5.3 A Necessary Condition for Complete Recoverability	182
5.4 A Sufficient Condition for Complete Recoverability	184
5.5 Examples	192
5.6 Comparison with Complete Recoverability in the Bounded Amplitude Case	194
5.7 Construction of the Set of Recoverable Initial Conditions	202
CHAPTER VI CONCLUSIONS AND RECOMMENDATIONS	204
6.1 Minimal Time Open-Loop Solution	204
6.2 Sub-Optimal Closed-Loop Solution	206
6.3 Recoverability	207
BIBLIOGRAPHY	209
APPENDIX A PROPERTIES OF THE SET OF RECOVERABILITY	212
APPENDIX B APPLICATION OF THE MAXIMUM PRINCIPLE	214
B.1 Regularity Condition	215
B.2 Necessary Conditions	217
B.3 Jump Conditions	220
B.4 Transversality Condition	221
APPENDIX C COMPUTER PROGRAM LISTING	222
APPENDIX D NUMERICAL RESULTS FOR STEP(1) AND STEP (2)	237

LIST OF TABLES

Table	Page
3.1 Parameter Definitions	53
3.2 Initial Classifications	54
3.3 Final Classifications	57
3.4 Final Classifications	58
3.5 Final Classifications	59
3.6 Values of the Parameters	61
4.1 8-th Order Power Series Approximations [5.9,18.5]	143
4.2 8-th Order Power Series Approximations [1.3,5.9)	146
4.3 Log Approximations [1.3,5.9)	149
4.4 Representative Initial Conditions for System (4.40)	153
4.5 Minimum Values of σ_a and σ_b	157
4.6 Evaluation of Sub-Optimal Performance for System (4.40).	159
4.7 Corresponding Initial Conditions for System (3.78)	166
4.8 Minimum Value of $ \theta $	168
4.9 Evaluation of Sub-Optimal Performance for System (3.78).	170
4.10 Calculation of the Value of the Parameter T	175
Appendix	
Table	
D.1 Numerical Values of the Angles	238
D.2 Computed Values of the Elements $\tilde{c}_{11} - \tilde{c}_{15}$	239
D.3 Computed Values of the Elements $\tilde{c}_{22} - \tilde{c}_{33}$	241

Appendix
Table

	Page
D.4 Computed Values of the Elements $\tilde{c}_{34} - \tilde{c}_{55}$	243
D.5 Computed Values of the Elements $\bar{c}_{11} - \bar{c}_{15}$	245
D.6 Computed Values of the Elements $\bar{c}_{22} - \bar{c}_{33}$	247
D.7 Computed Values of the Elements $\bar{c}_{34} - \bar{c}_{55}$	249

LIST OF FIGURES

Figure	Page
3.1 Function $f(t)$	39
3.2 Case (1) with Case (4)	47
3.3 Case (2) with Case (5)	48
3.4 Case (3) with Case (5)	49
3.5 Constructed $\bar{w}(t)$	73
3.6 Minimal Time Responses $\theta^*(t)$, $\ddot{\theta}^*(t)$, and $\dddot{\theta}^*(t)$	75
3.7 Minimal Time Torque ($q^*(t)$) Pitch Deviation ($\theta^*(t)$) and Response ($\theta(t)$) to Present Controller	76
4.1 Comparison of the Computed Values of $\tilde{c}_{15}$ and $\bar{c}_{15}$	132
4.2 Initial Power Series Approximation for Element $\bar{c}_{33}$...	135
4.3 Initial Log Approximation for Element $\bar{c}_{33}$	138
4.4 Power Series Approximation for Element $\tilde{c}_{11}$ (5.9-18.5) .	142
4.5 Final Dual Approximation for Element $\bar{c}_{45}$	145
4.6 Comparison of Log and Power Series Approximations for element $\tilde{c}_{13}$ (1.3-5.9)	147
4.7 Final Dual Approximation for Element $\tilde{c}_{24}$	150
4.8 Comparison of $\bar{x}_5(t)$ and $x_5^*(t)$ for Initial Condition x^0	160
4.9 Comparison of $\bar{x}_1(t)$ and $x_1^*(t)$ for Initial Condition x^0	161
4.10 Comparison of $\bar{x}_2(t)$ and $x_2^*(t)$ for Initial Condition x^0	162
4.11 Comparison of $\bar{x}_3(t)$ and $x_3^*(t)$ for Initial Condition x^0	163

Figure	Page
4.12 Comparison of $\bar{x}_4(t)$ and $x_4^*(t)$ for Initial Condition $\bar{x}$	164
4.13 Comparison of $\bar{\theta}(\tau)$, $\theta^*(\tau)$, and $\theta(\tau)$ for Initial Condition $\bar{\theta}$	171
4.14 Comparison of $d\theta^*/d\tau$ and $d\bar{\theta}/d\tau$ for Initial Condition $\bar{\theta}$	172

ABSTRACT

This study considers three aspects of the minimal time control problem for a class of linear systems. The class of systems is characterized by constraints on both the control amplitude and rate.

The first problem that is considered is the minimal time open-loop control of linear time-varying processes. Based on the Maximum Principle, necessary conditions are developed for a specific modified adjoint solution. Two additional conditions that are associated with this modified adjoint solution are also derived. These conditions are then utilized to develop a systematic procedure of constructing the extremal controls. The results are presented in six tables. These tables consist of groups of inequalities which are used to determine the unknown parameters of the extremal controls. These inequalities are also incorporated into a digital computer program. The program is used to obtain the extremal controls for an illustrative physical example. Finally, a method of obtaining the minimal time open-loop control from the extremal controls is given.

The second problem that is considered is the development of a sub-optimal closed-loop solution. A method of constructing a sub-optimal controller which is feasible for practical high order systems, is established. This method is based on obtaining a parameterized

approximation to the isochrones of the given system. This approximation is then used to determine the sub-optimal controller. An application of the method to a fourth order practical system, is given as an illustration.

The last aspect to be considered, involves the problem of recoverability. A necessary and a sufficient condition are derived. These conditions provide a test criterion that determines whether the entire state space is recoverable. Applications of the conditions are illustrated by examples.

CHAPTER I

INTRODUCTION

1.1 Motivation

Minimal time control of linear systems with control amplitude saturations has been studied by various authors, including Bellman, Glicksberg and Gross [1], Pontryagin [2] and Neustadt [3]. They have shown that the minimal time control is in the familiar "bang-bang" form. In many problems it is desirable to allow for controls whose rates are also finite, due to inertial or other factors. One example of this type of problem occurs in the Ranger Block III Attitude Control System [4]. During midcourse maneuver it is desirable to operate the autopilot in a minimal time fashion, in order to avoid the accumulation of vehicle position errors. The autopilot is under the control of a jet-vane actuator, which provides an output torque that is bounded both in amplitude and rate. Thus, the minimal time problem has the additional constraint of a control rate saturation. Another example is the minimal time control of an aerodynamically unstable booster, where the actuator has both position and rate limits [5]. As in the previous example, this additional rate saturation introduces a finite control rate constraint to the minimal time problem.

In general, the addition of a finite rate constraint on the controls forms a new class of minimal time problems, viz., one in which "bang-bang" operation is no longer allowable. This is because of the non-admissibility of discontinuous control inputs.

The purpose of this study is to analyze carefully three aspects of this class of problems, viz., minimal time open-loop operation, sub-optimal closed-loop operation and the general problem of recoverability.

1.2 Review of the Literature

In this study, for analysis purposes, the minimal time problem with control amplitude and rate saturations will be reformulated as a bounded state variable problem (Section 2.2). Therefore in this section, a discussion of the published results concerning the bounded state variable problem will also be included. The three previously mentioned aspects of the problem are considered separately.

1.2.1 Minimal Time Open-Loop Operation

The general open-loop bounded state variable control problem has been investigated by, among others, Berkovitz [9], Gamkrelidze [2] and Dreyfus [10]. Each author used different methods of approach to obtain the necessary conditions for the optimal open-loop control. Berkovitz used the calculus of variations approach, Gamkrelidze modified the Maximum Principle of Pontryagin and Dreyfus applied the method of Dynamic Programming. In later work [11, 12], the equivalence between the variously derived necessary conditions, was shown.

For the particular application to the open-loop minimal time control problem with control amplitude and rate saturations, these necessary conditions are difficult to apply in a systematic manner. The major problem is that the necessary conditions merely provide guidelines and offer very little direct indication as to the method of construction of the minimal time controls. As an example, the Maximum

Principle [2] indicates the conditions that must be satisfied, when the state variables are both on and off their respective bounds. However the total number of times that a given state variable is on its bound during the control interval, is not directly apparent from the necessary conditions. Also, the corresponding entry and exit time instants are difficult to obtain.

More recently, Luh, et. al. [13] developed sufficiency conditions for the open-loop minimal time bounded state variable problem. The authors showed that the necessary conditions were also sufficient and proved uniqueness for systems satisfying certain normality criteria. Also, a "backing out" procedure for the calculation of the minimal time trajectories, was suggested. As in the previous work, the difficulty is that the procedure is not systematic. It requires a great deal of insight and labor to solve each specific problem.

Many computational techniques have also been developed for finding the optimal solution to the general open-loop bounded state variable problem. Ho and Brontani [14], Denham [15] and others [16, 17] are but just a few examples of the many authors who have done work in this area. One type of technique involves first replacing the hard constraints by penalty functions. Then this new unconstrained problem is solved by using gradient methods. A major drawback of this type of technique is that one is no longer solving the original optimal problem. Also, the obtained solutions depend on the particular penalty functions chosen. Another type of technique involves the direct application of the necessary conditions. In this technique, a certain set of initial quantities are guessed. The trajectory is then found by numerical integration. Next, the necessary conditions are tested. If they are

not all satisfied, the initial guess is corrected and the procedure is repeated. The basic drawback of this technique is that it becomes extremely complex, in cases where the state variable bounds are achieved two or more times during the control interval. Another drawback, common to all of the computational techniques, is that these techniques all require a good deal of computer time.

The specific open-loop minimal time problem with control amplitude and rate saturations, was investigated by Chang [18], and Schmaedeke and Russell [19]. Chang developed a set of necessary conditions for the minimal time control of linear autonomous systems. He also established the uniqueness of the minimal time controls, for systems that satisfy the normality criterion of LaSalle [6]. Chang's results were based on the particular properties of the "region of accessible states." That is, the set of states that can be reached in a finite time, starting from a given initial condition. Later, Schmaedeke and Russell extended Chang's results to linear time variable systems. They approached the problem from the point of view of extremal controls, i.e., the controls corresponding to points on the boundary of the region of accessible states; or by their definition, the set of attainability. The authors utilized the fact that every minimal time control is also an extremal control. On this basis, the authors obtained a set of necessary conditions for the extremal controls. Neither of these two papers determine a systematic procedure for the computation of the minimal time controls. That is the essential weakness as far as actual solution of the problem is concerned.

Luh, et. al. [8, 21] have also investigated the problem from the point of view of extremal controls. Their results were obtained for

linear autonomous systems. The authors found the complete analytical solution for two problems, a second order unstable system and a second order oscillatory system with two control inputs. Also a set of necessary conditions were derived, based on results from the Maximum Principle and the calculus of variations. Their derivation was extremely complicated due to the particular approach taken by the authors. Also, as in the previous papers, no general systematic technique was established.

Harvey [20] has outlined a number of possible computational schemes, such as linear and nonlinear programming, for obtaining the extremal controls. All of his schemes require an excessive amount of computer time. This is partially due to the fact that the number of times the minimal time control enters and exits from its amplitude bound is an unknown number. This number must be guessed at the start of each computer run. Also, the computational schemes become extremely unwieldy and complex as the order of the system and the number of control inputs become large.

In summary, the major drawback of all the previous work, is that no systematic method exists for constructing open-loop minimal time controls.

In this study, a general non-iterative computational scheme will be developed. This scheme computes the extremal controls for linear time variable systems with control amplitude and rate saturations. Also, a method for determining the minimal time control from the extremal controls will be given.

1.2.2 Sub-Optimal Closed-Loop Operation

For linear systems with control or altitude saturations, the minimal-time closed-loop solution has most often been approached from the point of view of obtaining the switching surface. This was the method of approach taken in references [2], [12] and [33]. The authors obtained the analytical expression of the switching surface for some simple systems of less than fourth order. However if the order of the system is greater than three, then it is generally impossible to obtain a closed form expression for the switching surface, therefore for practical high order systems, some type of approximation procedure must be used. This results in a sub-optimal closed-loop controller.

Russell [36] used a computational procedure to obtain an approximate expression for the switching surface of a third order system associated with a high performance aircraft. His technique was based on first computing the numerical values of a set of points on the switching surface. Then by using a least mean squares fit, these numerical values were approximated by a high order polynomial. In this manner, the author was able to approximate a small portion of the switching surface by a polynomial function. Smith [34] used the same general procedure to obtain an approximate expression for the switching surface of a third order integrator (triple-integrator).

The basic drawback of the switching surface approach for high order practical systems is that the resulting sub-optimal controller operation requires a large amount of on-line computation time. This occurs, because in order to assign the correct control for any system state, the controller must first determine where the system state is located in relation to the switching surface. As an example, in the third order

system considered by Russell [36], the controller must determine in which of six possible regions the system state is located.

An alternate type of sub-optimal approach has been suggested by Kalman and Bertalan [35]. This approach consists of first finding a Lyapunov function for the unforced system. Then at each system state, the control is chosen so as to minimize the time rate of change of the Lyapunov function. The basic drawback of this approach is that the resulting sub-optimal controller is dependent on the particular choice of the Lyapunov function. However, there is no general procedure whereby this function can be selected so as to provide good sub-optimal controller performance. Therefore, it is difficult to determine which particular form of the Lyapunov function to select for a given system.

There are very few published results on the minimal time closed-loop control of systems with both amplitude and rate saturations. Friedland [5] found an analytical expression of the switching surface for a specific second order unstable booster. However, the expression was extremely complex because of the additional rate constraint. Therefore, even for that simple system the author used an approximation to the switching surface in order to implement the controller.

In general, the addition of a rate constraint on the control makes it more difficult to apply the sub-optimal methods that were outlined earlier in this section. Therefore from the discussion of these methods, the following conclusion can be drawn concerning the closed-loop control of systems with both control amplitude and rate saturations. "There is no general method that can be applied to practical high order systems to obtain a feasible sub-optimal controller". This study considers the development of such a general method.

1.2.3 Recoverability

For a given system, an initial condition is said to be recoverable if it can be driven to a specified terminal state in finite time, using admissible controls. The admissible controls are those that satisfy the given constraints on the system. The recoverability problem involves the investigation of the properties of the set of recoverable initial conditions.

The recoverability problem for linear processes with control amplitude saturations has been investigated by many authors, including LaSalle [6] and LeMay [7]. Their basic result is a necessary and sufficient condition for complete recoverability. That is, this condition indicates whether the set of all possible initial conditions is recoverable.

Very few results have been published on the subject of recoverability for linear systems with both control amplitude and rate saturations. Friedland [5], and Ish and Harvey [8] studied the recoverability problem for an unstable booster with actuator position and rate limits. For that specific unstable second order system, the authors succeeded in finding the maximal region of recoverable initial conditions. Friedland's method involved the use of the Maximum Principle of Pontryagin. He first obtained the adjoint solution, and from this was able to find a parametric expression for the switching surface in terms of the last switching time and the final time. Then, by allowing the final time to grow without bound, the author obtained the maximal set of initial conditions that could be brought to the terminal state in finite time. The major drawback of this method is that one must find an analytical expression for the switching surface. From the discussion in the preceding section, this is only possible for simple low order systems.

Therefore, the method cannot be applied in general. Luh and Harvey obtained identical results by a different procedure. Their method was based on first finding analytical expressions for all the extremal controls as a function of the final time. Then, by allowing the final time to approach infinity, they obtained the limits of the maximal region of recoverable initial conditions. The problem with this method is that it is impossible to obtain the analytical expressions of all the extremal controls, for systems above second order.

The basic drawback of the above two methods is that they can only be applied to a few simple low order systems. Therefore, it would be useful to obtain conditions that could determine the recoverability properties of general linear systems with control amplitude and rate saturations. The development of these conditions is considered in one aspect of this study.

1.3 Organisation of the Thesis

The above three aspects of the problem are formulated mathematically in Chapter II. Chapter III is devoted to the minimal time open-loop solution. This solution is based on obtaining a systematic procedure for constructing the extremal controls. The results are presented in the form of tables which are incorporated into a digital computer program.

In Chapter IV, a technique is established for obtaining a sub-optimal closed-loop solution that is applicable to high order practical systems. This technique is illustrated by a practical fourth order example. The results are then compared to the known minimal time open-loop solutions, obtained from Chapter III.

The recoverability problem is discussed in Chapter V. A necessary

and a sufficient condition are derived, which provide a test criterion that determines whether the set of all possible initial conditions is recoverable. Applications of the conditions are illustrated by examples.

The thesis is concluded in Chapter VI with a discussion of the findings, and with suggestions for future research.

CHAPTER II

PROBLEM FORMULATION

2.1 System Definition

The class of linear time-varying processes to be investigated are described by the system of differential equations,

$$\dot{y}(t) = E(t)y(t) + F(t)u(t) + g(t) \quad (2.1)$$

on $t \in [t_0, t_a]$. The n -dimensional vector y describes the state of the system, u is an m -dimensional vector, $m \leq n$, representing the control input, g is an n -dimensional bounded vector and $E(t)$ and $F(t)$ are bounded and continuous matrices of dimension n by n and n by m respectively. It is assumed that the system is completely controllable, for general initial time t_0 , in the sense defined by LaSalle [6]. Let G and Ω be two restraint sets in R^m given by

$$G = \{u \mid |u_i| \leq 1, i = 1, 2, \dots, m\} \quad (2.2)$$

and

$$\Omega = \{\dot{u} \mid |\dot{u}_i| \leq h_i, h_i > 0, i = 1, 2, \dots, m\} \quad (2.3)$$

The normalised restraint set G describes the amplitude bound and the restraint set Ω describes the rate bound on the control vector u . The control vector u of system (2.1) is said to be admissible if:

- (a) $u(t)$ has a piecewise continuous derivative,
- (b) $u(t) \in G$ and $\dot{u}(t) \in \Omega$ for all $t \in [t_0, t_a]$ and

(c) $u(T) = 0$.

The symbol T and the justification of the condition (c) are as follows:

The general problem can be described as follows. At some time $t = t_0$, the system defined by equation (2.1) is disturbed in some manner from its rest or steady state position, $y = \tilde{y}$. The problem is to select an admissible control that drives the system from its disturbed position at $t = t_0$, back to its steady state position in the shortest possible time. For the remainder of this study the steady state position will be taken to be the origin, i.e., $\tilde{y} = 0$. There is no loss of generality since the system is linear and therefore, by a proper translation of the coordinate system of the state vector, $\tilde{y}$ can be taken to be a zero vector.

From a closed-loop point of view, it is desirable that the control input be zero whenever system (2.1) is at steady state, $\tilde{y} = 0$. If this is not the case, then because of the continuity requirement on admissible controls, the control actuator must be disengaged after each control interval. Since Chapter IV is based on the open-loop solution developed in Chapter III, in order to avoid a repetitive and very similar derivation, both problems are formulated with that additional condition on the admissible controls, i.e., $u = 0$ whenever $y = 0$.

In the following sections the three basic aspects of the problem are formulated.

2.2 Open-Loop Solution

Suppose at time $t = t_0$, system (2.1) is at its disturbed initial condition $y(t_0) = y_0 \neq 0$. The open-loop problem is to find an admissible control $u(t)$ on $t \in [t_0, T]$, $T \leq t_n$, that drives the system to

$y(T) = 0$, the steady state position, and such that T is minimal. From condition (c) of Section 2.1, the terminal condition requirement on the admissible controls is that $u(T) = 0$. Up to this point, no mention has been made of $u(t_0)$, the initial value of the control variable. Because of the continuity requirement on admissible controls, condition (a) of Section 2.1, the minimal time open-loop control is dependent on the choice of the initial control value, $u(t_0)$. Since the control is open-loop it is reasonable to assume that the control operations are sequential, i.e., the system is driven from the given initial condition to the origin (steady state value $y = 0$) before another disturbance occurs. For this reason, it is assumed that $u(t_0) = u(T) = 0$. However, in order to facilitate the adaptation of the results of the open-loop problem to the sub-optimal closed-loop solution, the problem will be solved by first allowing $u(t_0)$ to be a parameter and then by selecting the particular minimal time control with $u(t_0) = 0$. Essentially the problem has been imbedded in a larger class of problems.

In order to apply the results of Pontryagin's Maximum Principle [2], the problem is reformulated as a bounded state variable process. Let

$$x_{n+1}(t) = u_1(t) \quad (2.4)$$

and

$$v_i(t) = u_i(t), \quad i = 1, 2, \dots, m. \quad (2.5)$$

Then system (2.1) can be augmented as

$$\dot{x}(t) = A(t)x(t) + Bv(t) + c(t) \quad (2.6)$$

with the initial conditions

$$x(t_0) = x_0 = \begin{bmatrix} y_0 \\ u_0 \end{bmatrix}, \quad (2.7)$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \\ x_{n+1}(t) \\ \vdots \\ x_{n+m}(t) \end{bmatrix} = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \\ u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} E(t) & F(t) \\ 0_1 & 0_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix}, \quad c(t) = \begin{bmatrix} g(t) \\ 0_4 \end{bmatrix}$$

and 0_1 is an n by n zero matrix, 0_2 is an m by m zero matrix, 0_3 is an n by n zero matrix, 0_4 is an n -dimensional zero vector and I_3 is an n by n identity matrix. System (2.6) is in the bounded state variable formulation since now $x(t) \in \hat{G} = \{x \mid |x_{n+i}| \leq 1, i = 1, 2, \dots, m\}$ for all $t \in [t_0, t_a]$. The class of admissible controls in this formulation may be written in terms of the new control variable v as

$$\Lambda(t) = \{v \mid v \text{ is piecewise continuous, } v(s) \in \Omega \quad (2.8)$$

for all $s \in [t_0, \sigma]$, $x(\sigma) \in \hat{G}$ for all $\sigma \in [t_0, t]$,

$$x_{n+i}(t) = 0, i = 1, 2, \dots, m\} ,$$

where the last m -components of the vector x , as defined in equation (2.4), represent the m -components of the original control vector u . The problem in the new formulation is to choose an admissible control $v(t)$ on $t \in [t_0, T]$ that steers system (2.6) from the initial condition $x(t_0) = x_0 = \begin{bmatrix} y_0 \\ u_0 \end{bmatrix}$ to the steady state condition $x(T) = \begin{bmatrix} y(T) \\ u(T) \end{bmatrix} = 0$, such that T is minimal. It should be noted that the terminal condition requirement $u(T) = 0$ is automatically satisfied. This is due to the structure of the class of admissible controls defined in equation (2.8).

This problem is discussed in Chapter III. In the following section the definition of the set of recoverability is established. It will be applied in the later analyses.

2.3 Set of Recoverability

From the variation of parameters formula, the solution to system (2.6) may be written as

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^t \phi(t, s)[Bv(s) + c(s)]ds, \quad v \in A(t) \quad (2.9)$$

where $\phi(t, t_0)$ is a fundamental solution matrix of the homogeneous differential equations corresponding to system (2.6) with $\phi(t_0, t_0) = I$, the identity matrix. The set of recoverability, $K(t)$, is defined as the set of initial conditions at time t_0 that can be brought to the origin at time t , using admissible controls. By setting $x(t) = 0$ and by solving for $x_0 = x(t_0)$ in equation (2.9), the following expression for the set of recoverability is obtained, i.e.,

$$K(t) = \left\{ x \mid x = - \int_{t_0}^t \phi(t_0, s)[Bv(s) + c(s)]ds, \quad v \in A(t) \right\} \quad (2.10)$$

It can be shown that the set of recoverability, $K(t)$, is compact, convex

and continuous in t . These properties are shown in Appendix A.

2.4 Sub-Optimal Closed-Loop Solution

The class of linear systems to be considered in the development of a sub-optimal closed-loop solution is described by the autonomous version of system (2.1). That is, $E(t) = E$ and $F(t) = F$, are constant matrices. Also, the vector $g(t)$ is taken to be zero. It is further assumed that the matrices E and F satisfy the normality criterion given by LaSalle [6]. The reformulated bounded state variable problem is then described by system (2.6) with $A(t) = A$, a constant matrix, and $e(t)$ a zero vector. Thus the system under investigation can be described by,

$$\dot{x}(t) = Ax(t) + Bv(t), \quad x \in \hat{G} \quad (2.11)$$

The minimal time closed-loop problem is to find an admissible feedback controller $v^*(x)$, that steers every non-zero state $x \in \hat{G}$ of system (2.11), to the origin in minimum time. With reference to equation (2.8), it is seen that the set of admissible feedback controllers is given by

$$J = \{v(x) | v(x) \text{ is piecewise continuous, } v(x) \in \Omega \text{ with } x \in \hat{G}\} \quad (2.12)$$

As discussed in Chapter I, the above minimal time closed-loop control problem has not been solved. An alternate approach is to find a sub-optimal closed-loop solution. A method for obtaining such a solution is developed in Chapter IV.

2.5 Recoverability

The solutions of the minimal time open-loop control problem and the related sub-optimal closed-loop problem are both based on the assumption that a given initial condition $x(t_0) = x_0$ of system (2.6) can be driven to the origin in a finite amount of time, using admissible controls. The problem of determining the conditions under which the above assumption is valid is formulated below.

An initial condition $x(t_0) = x_0$ of system (2.6) is said to be recoverable if there exists an admissible control $v(t)$ on $t \in [t_0, T]$ that can drive system (2.6) from $x(t_0) = x_0$ to $x(T) = 0$, where T is finite. From the discussion of the set of recoverability, $K(t)$, in Section 2.3, it is clear that an initial condition $x(t_0) = x_0$ is recoverable if and only if $x_0 \in K(t)$ for some $t \geq t_0$. Let $S(t_0)$ denote the set of all recoverable initial conditions. Then, from the preceding discussion,

$$S(t_0) = \bigcup_{t \geq t_0} K(t) \subset \hat{G} \quad (2.13)$$

where the set $\hat{G}$ was defined in Section 2.2. With the aid of the above definitions, the problem of recoverability is to determine the conditions under which $S(t_0) = \hat{G}$, i.e., every given initial condition can be steered to the origin in finite time with admissible controls. This is called the problem of determining complete recoverability.

A related problem occurs when it is found that for a particular system $S(t_0) \neq \hat{G}$. In this case it is desirable to know the extent of the set of recoverable initial conditions. The problem in this case is to determine a technique that establishes the extent of the set $S(t_0)$.

Both of the above problems are investigated in Chapter V.

CHAPTER III

OPEN-LOOP SOLUTION

3.1 Introduction

In this chapter a subclass of the admissible controls, called extremal controls, will be introduced. The solution for the minimal time open-loop problem will be developed, based on these extremal controls.

3.2 Boundary Points of the Set of Recoverability

Consider the set of recoverability, $K(t)$, defined in equation (2.10). Also, consider the initial condition $x(t_0) = x_0 \neq 0$ as a point in the augmented $(n+m)$ - dimensional state variable space. If a minimal time control exists for this particular initial condition, then as t increases from t_0 , there is a first time instant, say $t = T$, at which the point $x_0 \in K(t)$, the set of recoverability. By virtue of the convexity and continuity of the set of recoverability, it can be shown [22] that $x_0 \in \partial K(T)$, the boundary of $K(T)$. Therefore, a corresponding minimal time control for x_0 , i.e., a control that steers system (2.6) from $x(t_0) = x_0$ to $x(T) = 0$, is a control that corresponds to a boundary point of the set of recoverability at the time $t = T$. In general, then, the only candidates for minimal time controls are those that can steer an initial condition $x_0 \in \partial K(T)$, for some $T > t_0$, to $x(T) = 0$.

In the following section, extremal controls will be defined and shown to be the only candidates for minimal time controls.

3.3 Extremal Controls

An admissible control $v(t)$ on $t \in [t_0, T]$ is said to be an extremal control $\bar{v}(t)$ if there exists a nontrivial solution $Y(t) = [Y_1(t), \dots, Y_{n+m}(t)]$ of the adjoint system $\dot{Y}(t) = -Y(t)A(t)$ such that

$$\int_{t_0}^T Y(s)B\bar{v}(s)ds = \max_{v \in A(T)} \int_{t_0}^T Y(s)Bv(s)ds \quad (3.1)$$

The following theorem establishes extremal controls as the only candidates for minimal time controls.

Theorem 3.1

An admissible control $\bar{v}(t)$ on $t \in [t_0, T]$ is an extremal control if and only if the corresponding initial condition $\bar{x}_0 = -\int_{t_0}^T \phi(t, s)[B\bar{v}(s) + c(s)]ds$ is on the boundary of the set of recoverability $K(T)$.

Proof

Suppose $\bar{x}_0 \in \partial K(T)$ for some time $T > t_0$ and that $\bar{x}_0$ corresponds to some admissible control $\bar{v} \in A(T)$. Let $\eta = [\eta_1, \eta_2, \dots, \eta_{n+m}]$ be an outward normal to $K(T)$ at $\bar{x}_0$. Since the set $K(T)$ is convex, [Appendix A], then

$$\eta[\bar{x}_0 - x] \geq 0 \quad (3.2)$$

for all initial conditions $x \in K(T)$. By utilizing the definition of the set of recoverability, equation (2.10), the above equation may be written as

$$-\eta \int_{t_0}^T \phi(t_0, s)E[\bar{v}(s) - v(s)]ds \geq 0, \quad v \in A(T) \quad (3.3)$$

Now let $\Psi(t_0) = -\eta$. Then this particular solution of the adjoint system is given by

$$\Psi(s) = -\eta \phi(t_0, s) \quad (3.4)$$

Hence, by equation (3.4), equation (3.3) and equation (3.1) are equivalent and therefore $\bar{v}(t)$ on $t \in [t_0, T]$ is an extremal control.

Now assume $\bar{v}(t)$ is an extremal control on $t \in [t_0, T]$ and that $\bar{x}_0$ is the corresponding initial condition. Let $-\eta = \Psi(t_0)$. Then equation (3.3) is equivalent to equation (3.1). As before, equation (3.3) is equivalent to equation (3.2). Since the set $K(T)$ is convex, equation (3.2) implies that $\bar{x}_0 \in \partial K(T)$ and that η is an outward normal to the set of recoverability at the point $\bar{x}_0$.

From Theorem 3.1 and the discussion in Section 3.2, it may be concluded that a minimal time control is always an extremal control. The conditions under which an extremal control is also a minimal time control will be discussed in Section 3.10.

In the next section, the problem of obtaining the extremal controls will be simplified by decomposing it into a number of second order problems.

3.4 Decomposition of the Problem

The definition of an extremal control is given by equation (3.1).

Since

$$\int_{t_0}^T \Psi(s) B v(s) ds = \int_{t_0}^T \sum_{i=1}^m \Psi_{n+1}(s) v_i(s) ds = \sum_{i=1}^m \int_{t_0}^T \Psi_{n+1}(s) v_i(s) ds \quad (3.5)$$

and since, by definition of the set of admissible controls, equation (2.8), each component of $v(s)$ can be chosen independently of all other components, an extremal control $\bar{v}(s)$ can be defined by

$$\int_{t_0}^T Y(s) B \bar{v}(s) ds = \sum_{i=1}^m \text{Max}_{v \in A(T)} \int_{t_0}^T Y_{n+1}(s) v_i(s) ds. \quad (3.6)$$

Therefore, an extremal control $\bar{v}(t)$ on $t \in [t_0, T]$ may be found by solving equation (3.6) for each value of i , $i=1, 2, \dots, m$. In other words, the solution of the extremal control problem consists of solving the same type of maximization problem m -times, where in each case $Y_{n+1}(t)$ is the $(n+1)^{\text{th}}$ component of the adjoint solution and its form can be obtained for each choice of the initial value of $Y(t)$, i.e., $Y(t_0)$.

Since the subscript i is not involved in the general solution, it will be omitted in the further discussion. Also, in order to simplify the notation and the application of the necessary conditions in the next section, let the following scalar functions (for general i , $i=1, 2, \dots, m$), be defined by

$$f(t) = Y_{n+1}(T + t_0 - t), \quad (3.7a)$$

$$w(t) = v_1(T + t_0 - t), \quad (3.7b)$$

$$\xi_1(t) = -x_{n+1}(T + t_0 - t) = -u_1(T + t_0 - t) \quad (3.7c)$$

and

$$k = h_1, \quad (3.7d)$$

where h_1 is the amplitude bound on the particular component of control v_1 . From equation (3.7c), $\xi_1(t_0) = -u_1(T)$. Therefore, by definition of

the admissible controls, equation (2.8), $\xi_1(t_0) = 0$. Thus the transformation, equation (3.7c) has replaced a terminal condition with an initial condition on the admissible controls. By applying the transformations defined in equation (3.7) to the 1th member of the summation in equation (3.6), the general optimization problem can be formulated in terms of the second order vector $\xi = \begin{bmatrix} \xi_0 \\ \xi_1 \end{bmatrix}$ as follows. Given the system of differential equations

$$\begin{cases} \dot{\xi}_0 = -f(t)w(t) \\ \dot{\xi}_1 = w(t) \end{cases} \quad (3.8)$$

with the initial conditions $\xi(t_0) = 0$. The basic problem is to determine a scalar control $w(t)$ on $t \in [t_0, T]$, T fixed, such that for all $t \in [t_0, T]$,

$$w(t) \in \tilde{\Omega} = \{w \mid |w| \leq k\} , \quad (3.9a)$$

$$\xi_1(t) \in \tilde{G} = \{\xi_1 \mid |\xi_1| \leq 1\} , \quad (3.9b)$$

$\xi_1(t)$ is not identically zero and such that $\xi_0(T)$ is minimal. The problem defined above is the basic building block in the solution of the extremal controls. For a given choice of T and $\forall(t_0)$, each component of the extremal control may be found by solving the above problem n -times.

In the next section a set of necessary conditions for the above problem will be derived with the aid of the Maximum Principle [2].

3.5 Necessary Conditions

It is assumed that the time interval $[t_0, T]$ can be partitioned into subintervals according to whether $\xi_1(t)$ is on or off its bound, $|\xi_1| = 1$, in the following manner. Let

$$t_0 = \tau_0 < \tau_1 \leq \tau_2 < \tau_3 \leq \dots \tau_{2N-1} \leq \tau_{2N} \leq T, \quad (3.10a)$$

where $0 \leq N < \infty$, such that if $N > 0$, then for $j = 0, 1, \dots, N-1$,

$$|\xi_1(t)| < 1 \quad \text{on } t \in (\tau_{2j}, \tau_{2j+1}), \quad (3.10b)$$

$$|\xi_1(t)| = 1 \quad \text{on } t \in [\tau_{2j+1}, \tau_{2j+2}] \quad (3.10c)$$

and if $\tau_{2N} < T$, then

$$|\xi_1(t)| < 1 \quad \text{on } t \in (\tau_{2N}, T) \quad (3.10d)$$

or if $\tau_{2N} = T$, then

$$\tau_{2N-1} < \tau_{2N} = T. \quad (3.10e)$$

If $N = 0$, then

$$|\xi_1(t)| < 1 \quad \text{on } t \in [\tau_{2N}, T] = [t_0, T]. \quad (3.10f)$$

It is shown in Appendix B.1 that every solution $\xi(t)$ of system (3.8) satisfies the regularity condition given in Reference 2. Therefore, the Maximum Principle is applicable. From Appendix B.2, if $\bar{w}(t)$ is the extremal control on $t \in [t_0, T]$, i.e., the control that minimizes $\xi_0(T)$, then there exists a row vector function $\lambda(t) = [\lambda_1(t), \lambda_2(t)]$ such that on the intervals (τ_{2j}, τ_{2j+1}) , $j = 0, 1, \dots, N-1$, for $N > 0$; and on the interval (τ_{2N}, T) for $N \geq 0$:

$$\bar{w}(t) = k \operatorname{sgn} [f(t) + \lambda_1(t)], \quad (3.11a)$$

where

$$\text{sgn}(\alpha) = \begin{cases} 1, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha = 0 \\ -1, & \text{if } \alpha < 0 \end{cases}; \quad (3.11b)$$

$$f(t) + \lambda_1(t) \neq 0 \quad (3.12)$$

on any finite subinterval;

$$\lambda_1(t) = \gamma \quad (3.13)$$

where γ is constant on each interval and

$$\lambda_2(t) = -k|f(t) + \lambda_1(t)| \quad (3.14)$$

Furthermore on the intervals $(\tau_{2j+1}, \tau_{2j+2})$, $j=0,1,\dots,N-1$, for $N > 0$:

$$\bar{w}(t) = 0; \quad (3.15)$$

$$f(t) \text{sgn} [\bar{E}_1(t)] \leq 0; \quad (3.16)$$

$$\lambda_2(t) = 0 \quad (3.17)$$

and

$$\lambda_1(t) = \gamma, \quad (3.18)$$

where, as before, γ is constant on each interval.

The above necessary conditions, equations (3.11) through (3.18), are difficult to apply in their present form. This is due to the fact that a systematic procedure for their application is lacking. In the following section these necessary conditions will be combined with the "jump conditions", which are discussed in Appendix B.3, to establish a set of systematic and simplified necessary conditions for a modified

adjoint solution, $\rho(t)$. It will be shown that this particular solution is continuous on $[t_0, T)$ and that the extremal control $\bar{w}(t) = k \operatorname{sgn}[\rho(t)]$ on $t \in [t_0, T]$.

3.6 Modified Adjoint Solution

The following proposition establishes values for $\lambda_1(t)$ in terms of the known function $f(t)$ on $t \in [t_0, T]$.

Proposition 3.1

If $H > 0$, then $\gamma = -f(\tau_{2j+1})$ on $t \in (\tau_{2j}, \tau_{2j+1})$, $j=0,1,\dots,N-1$, where the function $f(t)$ is defined in Section 3.4.

Proof

Consider the open interval (τ_{2j}, τ_{2j+1}) for some j , $j=0,1,\dots,N-1$, and suppose that $\tau_{2j+1} < \tau_{2j+2}$. From the "jump conditions", Appendix B.3, $\lambda_2(t)$ is continuous on the whole interval (t_0, T) . Therefore, since $\tau_{2j+1} \in (t_0, T)$ for all $j=0,1,\dots,N-1$, then

$$\lambda_2(\tau_{2j+1}^-) = \lambda_2(\tau_{2j+1}^+) \quad (3.19)$$

where $\lambda_2(\alpha^-) = \lim_{\epsilon \rightarrow 0} \lambda_2(\alpha - \epsilon)$, $\epsilon > 0$ and $\lambda_2(\alpha^+)$ is similarly defined. From equation (3.17), $\lambda_2(t) = 0$ for $t \in (\tau_{2j+1}, \tau_{2j+2})$ and from equation (3.14) $\lambda_2(t) = -k|f(t) + \lambda_1(t)|$ for $t \in (\tau_{2j}, \tau_{2j+1})$. Combining these results with equation (3.19) yields $|f(\tau_{2j+1}) + \lambda_1(\tau_{2j+1})| = 0$. However, from equation (3.18) $\lambda_1(t)$ is constant on (τ_{2j}, τ_{2j+1}) . Therefore,

$$\lambda_1(t) = \gamma = -f(\tau_{2j+1}) \text{ on } (\tau_{2j}, \tau_{2j+1}) \quad (3.20)$$

Suppose instead, $\tau_{2j+1} = \tau_{2j+2}$, i.e., $|\bar{E}_1(t)| = 1$ only at the time instant

$t = \tau_{2j+1} = \tau_{2j+2}$. For this case the "jump conditions", as shown in Appendix B.3 yield,

$$[\lambda_1(\tau_{2j+1}^+) - \lambda_1(\tau_{2j+1}^-)] \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \geq 0 \quad (3.21)$$

In this case $|\bar{E}_1(\tau_{2j+1})| = |\bar{E}_1(\tau_{2j+2})| = 1$ and by definition $|\bar{E}_1(t)| < 1$ for $t \in (\tau_{2j}, \tau_{2j+1})$ and $t \in (\tau_{2j+2}, \tau_{2j+3})$, $j=0,1,\dots,N-2$, and also for $t \in (\tau_{2N}, T)$. Therefore, since $\bar{w}(t) = d\bar{E}_1/dt$, then

$$\bar{w}(\tau_{2j+1}^+) \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \leq 0 \quad (3.22)$$

and

$$\bar{w}(\tau_{2j+1}^-) \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \geq 0 \quad (3.23)$$

Substituting for $\bar{w}(t)$ from equation (3.11a) and noting that $k > 0$ and $k \operatorname{sgn} [f(t) + \lambda_1(t)] \geq 0$ imply that $f(t) + \lambda_1(t) \geq 0$, we obtain

$$[f(\tau_{2j+1}) + \lambda_1(\tau_{2j+1}^+)] \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \leq 0 \quad (3.24)$$

and

$$[f(\tau_{2j+1}) + \lambda_1(\tau_{2j+1}^-)] \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \geq 0 \quad (3.25)$$

Subtracting equation (3.25) from equation (3.24) yields

$$[\lambda_1(\tau_{2j+1}^+) - \lambda_1(\tau_{2j+1}^-)] \operatorname{sgn} [\bar{E}_1(\tau_{2j+1})] \leq 0 \quad (3.26)$$

By comparing equation (3.26) with equation (3.21) and by noting that $|\bar{E}_1(\tau_{2j+1})| = 1 \neq 0$, we conclude that $\lambda_1(\tau_{2j+1}^+) = \lambda_1(\tau_{2j+1}^-)$. Consequently by equations (3.24) and (3.25), $\lambda_1(\tau_{2j+1}^+) = \lambda_1(\tau_{2j+1}^-) = -f(\tau_{2j+1})$. Again, since $\lambda_1(t) = \gamma$ is constant on (τ_{2j}, τ_{2j+1}) , the result is identical to equation (3.20). Finally, since the choice of j ,

$j=0,1,\dots,N-1$, was arbitrary, equation (3.20) is true for all $j=0,1,\dots,N-1$.

In a similar manner it can be shown that if $N > 0$, then on the intervals (τ_{2j}, τ_{2j+1}) , $j=1,2,\dots,N-1$, and on the interval (τ_{2N}, T) ,

$$\gamma = -f(\tau_{2j}) \quad (3.27)$$

This result can be combined with Proposition 3.1 to yield the following relation. If $N > 0$ then for $j=1,2,\dots,N-1$,

$$f(\tau_{2j}) = f(\tau_{2j+1}) \quad (3.28)$$

Equation (3.28) is important in that it relates the unknown parameters τ_j , $j=1,2,\dots,N-1$, to the known function $f(t)$. With these preliminary results as background, it is now possible to define the previously mentioned modified adjoint solution as follows. For the case $N > 0$, the modified adjoint solution is defined by

$$p(t) = \begin{cases} f(t) - f(\tau_{2j+1}) & , \tau_{2j} \leq t < \tau_{2j+1}, \\ 0 & , \tau_{2j+1} \leq t < \tau_{2j+2}, \\ & , j = 0,1,\dots,N-1, \\ 0 & , \text{if } t = \tau_{2N} < T, \\ f(t) + p(T^-) - f(T) & , \text{if } \tau_{2N} < t < T, \\ f(T) & , \text{if } t = T, \end{cases} \quad (3.29)$$

where

$$p(T^-) = f(T) - f(\tau_{2N}), \quad \text{if } \tau_{2N} < T \quad (3.30)$$

and

$$\rho(T^-) = 0, \quad \text{if } \tau_{2N} = T. \quad (3.31)$$

The relationship between the modified adjoint solution $\rho(t)$ and the extremal control $\bar{w}(t)$ is established by the next proposition.

Proposition 3.2

If $N > 0$, then the extremal control $\bar{w}(t)$ on $t \in [t_0, T]$ can be expressed by $\bar{w}(t) = k \operatorname{sgn} [\rho(t)]$, where $\rho(t)$ is given by equation (3.29) and $\rho(T^-)$ by equation (3.30) if $\tau_{2N} < T$, or by equation (3.31) if $\tau_{2N} = T$. Furthermore, $\rho(t)$ is continuous on $t \in [t_0, T)$.

Proof

From Proposition 3.1, $\lambda_1(t) = \gamma = -f(\tau_{2j+1})$ on the intervals (τ_{2j}, τ_{2j+1}) for $j=0, 1, \dots, N-1$. Therefore on these intervals equation (3.29) is equivalent to equation (3.11a) of the necessary conditions. Also, on the intervals $(\tau_{2j+1}, \tau_{2j+2})$ for $j=0, 1, \dots, N-1$, equation (3.29) is identical to equation (3.15) of the necessary conditions. If $\tau_{2N} < T$, then on the interval (τ_{2N}, T) the extremal control is given by equation (3.11a). On this interval $\lambda_1(t) = \gamma = -f(\tau_{2N})$, by equation (3.27). Also by equation (3.30), $\rho(T^-) = f(T) - f(\tau_{2N})$. Therefore once again equation (3.29) is equivalent to equation (3.11a). The continuity of $\rho(t)$ on $t \in [t_0, T)$ is established by noting that $f(\tau_{2j}) = f(\tau_{2j+1})$ for $j=1, 2, \dots, N-1$, from equation (3.28). Also, if $\tau_{2N} < T$, then $\rho(\tau_{2N}) = 0$ and therefore $\rho(t)$ is also continuous at $t = \tau_{2N}$.

In a similar manner, a modified adjoint solution $\rho(t)$, continuous on $t \in [t_0, T)$ can be constructed for the case $N = 0$. In this case define the function $\rho(t)$ on $t \in [t_0, T]$ as

$$\rho(t) = \begin{cases} f(t) + \rho(T^-) - f(T), & \text{if } t_0 \leq t < T, \\ f(T) & \text{if } t = T \end{cases} \quad (3.32)$$

where

$$\rho(T^-) = f(T) - c \quad (3.33)$$

and c is some constant. In this case $\tau_{2n} = t_0$ and from equation (3.11a), $\bar{w}(t) = k \operatorname{sgn} [f(t) + \lambda_1(t)]$ on $t \in (t_0, T)$. However from equation (3.13) $\lambda_1(t) = \gamma$ where γ is a constant on this interval. Therefore equation (3.32) is equivalent to equation (3.11a). The continuity of $\rho(t)$ on $t \in [t_0, T]$ is established by noting that $f(t)$, related to the $(n+1)$ -th component of the adjoint solution, is a continuous function and therefore $\rho(t)$, by equation (3.32) is continuous on $[t_0, T]$.

Up to this point it has been shown that the modified adjoint solution $\rho(t)$, given by equations (3.29) or (3.32) is equivalent to the necessary conditions given by equations (3.11) through (3.15) and equations (3.17) and (3.18). Equations (3.16) and (3.28) are two necessary conditions that still must be satisfied. Another condition that the modified adjoint solution must satisfy is established in Theorem 3.2.

Theorem 3.2

The modified adjoint solution $\rho(t)$ may have a discontinuity at $t = T$, the terminal time, but this can only happen if $|\bar{\xi}_1(T)| = 1$. Furthermore, the following terminal time condition must always be satisfied.

$$[f(T) - \rho(T^-)] \operatorname{sgn} [\bar{\xi}_1(T)] \geq 0, \quad (3.34)$$

where $\bar{E}_1(t)$ is the response to the control $\bar{w}(t) = k \operatorname{sgn} [\rho(t)]$.

Proof

(a) Suppose $|\bar{E}_1(T)| < 1$. Then from the Transversality Condition, Appendix B.4,

$$\lambda_1(T) = 0 \quad (3.35)$$

If $N > 0$ and $\tau_{2N} < T$, then by equation (3.27), $\gamma = -f(\tau_{2N})$ on (τ_{2N}, T) . Since $|\bar{E}_1(T)| < 1$ and by definition $|\bar{E}_1(t)| < 1$ on $t \in (\tau_{2N}, T)$, then $\lambda_1(t) = \lambda_1(T) = 0$ on $t \in (\tau_{2N}, T]$ by equation (3.13). Thus $f(\tau_{2N}) = 0$ and from equation (3.30), $\rho(T^-) = f(T)$. Substituting for $\rho(T^-)$ in equation (3.29) yields the desired result on the continuity of $\rho(t)$ at $t = T$. Similarly replacing $\rho(T^-)$ with $f(T)$ in equation (3.34) shows that this equation is satisfied. If $N = 0$ and $|\bar{E}_1(T)| < 1$, then $c = 0$ by similar reasoning. From equation (3.33), $\rho(T^-) = f(T)$ and therefore by equation (3.32), $\rho(t)$ is continuous at $t = T$. Again equation (3.34) is also satisfied. Thus the modified adjoint solution is continuous at $t = T$ if $|\bar{E}_1(T)| < 1$. Also equation (3.34) is satisfied. Therefore a discontinuity can occur at the terminal time only if $|\bar{E}_1(T)| = 1$. Notice that the case $N > 0$ and $\tau_{2N} = T$, was not included in the previous proof. The reason is that it is not applicable since in this case, by definition, $|\bar{E}_1(T)| = 1$.

(b) Now suppose that $|\bar{E}_1(T)| = 1$. Since $\bar{w}(t)$ is the extremal control for system (3.8), then by definition

$$-\int_{t_0}^T f(t)[\bar{w}(t) - \tilde{w}(t)]dt \leq 0 \quad (3.36)$$

where $\tilde{w}$ is any other admissible control for system (3.8), i.e., one

that satisfies the constraint equation (3.9). Let $\tilde{\xi}(t)$ be the response to $\tilde{w}(t)$ and, as before, $\bar{\xi}(t)$ correspond to the extremal control $\bar{w}(t)$.

With these preliminaries completed, the proof will now be carried out for the two possible values of $p(T^-)$.

(1) $p(T^-) = 0$. This case occurs for $N > 0$, $\tau_{2N} = T$ and

$$\tau_{2N-1} < \tau_{2N}$$

By substitution, equation (3.34) reduces to

$$f(T) \operatorname{sgn} [\bar{\xi}_1(T)] \geq 0 \quad (3.37)$$

where, as previously defined, $\bar{\xi}_1(t)$ is the response to the extremal control $\bar{w}(t) = k \operatorname{sgn} [p(t)]$. Suppose, if possible, equation (3.37) is not satisfied, i.e., $f(T) \operatorname{sgn} [\bar{\xi}_1(T)] < 0$. Then since $f(t)$ is continuous on the whole time interval $[t_0, T]$, there exists a time instant t_1 ,

$\tau_{2N-1} \leq t_1 < T$, such that

$$f(t) \operatorname{sgn} [\bar{\xi}_1(T)] < 0, \quad t \in [t_1, T] \quad (3.38)$$

The above equation may be rewritten as

$$\operatorname{sgn} [f(t)] = -\operatorname{sgn} [\bar{\xi}_1(T)] \quad \text{on } t \in [t_1, T] \quad (3.39)$$

Let the time instant t^* be defined by

$$t^* = \max [t_1, T-2/k] \quad (3.40)$$

and the control $\tilde{w}(t)$ on $t \in [t_0, T]$ be defined by

$$\tilde{w}(t) = \begin{cases} \bar{w}(t), & \text{if } t_0 \leq t \leq t^* \\ k \operatorname{sgn} [f(t)], & \text{if } t^* < t \leq T \end{cases} \quad (3.41)$$

Since the extremal control $\bar{w}(t) = k \operatorname{sgn} [p(t)]$ is certainly admissible, by the above definition, $\tilde{w}(t)$ and its corresponding response $\tilde{\xi}_1(t)$ satisfy the constraint equation (3.9) on $t \in [t_0, t^*]$. Also, since, by equation (3.41), $|\tilde{w}(t)| \leq k$ on $t \in (t^*, T]$, equation (3.9a) is satisfied for all $t \in [t_0, T]$. In order to show that equation (3.9b) is also satisfied on the whole time interval, consider system (3.8). The response $\tilde{\xi}_1(t)$ to the control $\tilde{w}(t)$ on $t \in [t^*, T]$ is given by

$$|\tilde{\xi}_1(t)| = |\tilde{\xi}_1(t^*) + \int_{t^*}^t k \operatorname{sgn} [f(t)] dt| \quad (3.42)$$

Since, by definition, $|\bar{\xi}_1(t)| = 1$ on $t \in [t_{2N-1}, T]$, then by the above definitions of the time t^* and the control $\tilde{w}(t)$,

$$\tilde{\xi}_1(t^*) = \bar{\xi}_1(t^*) = \bar{\xi}_1(T) = \operatorname{sgn} [\bar{\xi}_1(T)] \quad (3.43)$$

Then with the aid of equation (3.39) and the above result, equation (3.42) may be written as

$$|\tilde{\xi}_1(t)| = |\operatorname{sgn} [\bar{\xi}_1(T)] (1 - k(t - t^*))|, \quad t \in [t^*, T] \quad (3.44)$$

However by definition of the time t^* in equation (3.40), $T - t^* \leq 2/k$. Therefore

$$|\tilde{\xi}_1(t)| \leq 1, \quad t \in [t^*, T] \quad (3.45)$$

The above equation implies that the constraint equation (3.9b) is satisfied by $\tilde{\xi}_1(t)$, the response to $\tilde{w}(t)$ for $t \in [t^*, T]$. Since we have now shown that equation (3.9) is satisfied for all $t \in [t_0, T]$, then $\tilde{w}(t)$ is an admissible control. Now substituting for $\tilde{w}(t)$ in equation (3.36) and noting that by definition, $\bar{w}(t) = 0$ on

$t \in [\tau_{2N-1}, T]$, we obtain

$$-\int_{t^*}^T f(t) [-k \operatorname{sgn} [f(t)]] dt = \int_{t^*}^T k |f(t)| dt \leq 0 \quad (3.46)$$

Since $k > 0$ and from equation (3.36), $f(t) \neq 0$ on $t \in [t_1, T] \supset [t^*, T]$, then equation (3.46) is a contradiction. Therefore for this given case, $\rho(T^-) = 0$, equation (3.37) and correspondingly equation (3.34) must be satisfied.

(2) $\rho(T^-) = f(T) - \alpha$, where $\alpha = f(\tau_{2N})$ if $N > 0$ and $\alpha = c$ if $N = 0$.

This case occurs if $N > 0$ and $\tau_{2N} < T$ or if $N = 0$, where c is some constant.

In the above case, by definition, $|\bar{g}_1(t)| < 1$ on $t \in (\tau_{2N}, T)$ where $\tau_{2N} = t_0$ if $N = 0$. Equation (3.34) reduces to

$$\alpha \operatorname{sgn} [\bar{g}_1(T)] \geq 0 \quad (3.47)$$

From equations (3.29) and (3.32) the extremal control $\bar{w}(t) = k \operatorname{sgn} [\rho(t)]$ on the interval (τ_{2N}, T) is given by

$$\bar{w}(t) = k \operatorname{sgn} [f(t) - \alpha] \quad (3.48)$$

Let the time instant t^* be defined by $\tau_{2N} \leq t^* < T$, and such that

$$f(t^*) = \alpha \quad (3.49a)$$

and

$$[f(t) - \alpha]^2 > 0, \quad t \in (t^*, T) \quad (3.49b)$$

Certainly the time t^* exists if $\alpha = f(\tau_{2N})$ or if $\alpha = c$ and $f(t) = c$ for at least one time instant $t \in [t_1, T]$. If $f(t) \neq c$ for all $t \in [t_1, T]$

$t \in [t_0, T)$ then there exists no time t^* satisfying the above definition. In this special case, from equation (3.48),

$$\bar{w}(t) = k \operatorname{sgn} [f(t) - c] \neq 0, \quad t \in [t_0, T) \quad (3.50)$$

The corresponding response $\bar{\xi}_1(T)$ is, from system (3.8), given by

$$|\bar{\xi}_1(T)| = \left| \int_{t_0}^T k \operatorname{sgn} [f(t) - c] dt \right| \quad (3.51)$$

However we are considering the case $|\bar{\xi}_1(T)| = 1$, and since $f(t) \neq c$ on $t \in [t_0, T)$, then equation (3.51) implies that

$$T - t_0 = 1/k \quad (3.52)$$

Since $\xi_1(t_0) = 0$, $|w(t)| \leq k$ and $|\xi_1(t)| \leq 1$ on $t \in [t_0, T]$, then this case can be viewed as one having no bound on the response $\xi_1(t)$, $t \in [t_0, T]$. Thus the Transversality Condition [Appendix B.4] is applicable and from the first part of this theorem, it has been shown that $\alpha = c = 0$. Thus equation (3.47) and correspondingly equation (3.34) are satisfied. Returning to the definition of t^* in equation (3.49), it is seen that the time instant t^* exists for all the remaining cases. Now suppose, if possible, $\alpha \operatorname{sgn} [\bar{\xi}_1(T)] < 0$, i.e., equation (3.47) is not satisfied for these remaining cases. From the definition of the time instant t^* , and since $f(t)$ is continuous on the whole time interval, there exists another time instant t_1 , $t^* < t_1 \leq T$, such that

$$f(t) \operatorname{sgn} [\bar{\xi}_1(T)] < 0, \quad t \in [t^*, t_1] \quad (3.53)$$

Equation (3.53) may be written as

$$\operatorname{sgn} [f(t)] = -\operatorname{sgn} [\bar{\xi}_1(T)], \quad t \in [t^*, t_1] \quad (3.54)$$

Let the control $\tilde{w}(t)$ on $t \in [t_0, T]$ be defined by

$$\tilde{w}(t) = \begin{cases} \bar{w}(t) & , \text{ if } t_0 \leq t \leq t^* \\ 0 & , \text{ if } t^* < t \leq t_1 \\ \bar{w}(t) & , \text{ if } t_1 < t \leq T \end{cases} \quad (3.55)$$

where $\bar{w}(t)$ is the extremal control given by $\bar{w}(t) = k \operatorname{sgn} [p(t)]$ on $t \in [t_0, T]$. Since $\bar{w}(t)$ is certainly admissible on the whole time interval, then $\tilde{w}(t)$ and its corresponding response $\tilde{\xi}_1(t)$ satisfy the constraint equation (3.9) for $t \in [t_0, t^*]$. Also, by definition, $|\tilde{w}(t)| \leq k$ for all $t \in [t_0, T]$, and therefore constraint equation (3.9a) is satisfied on the whole time interval. In order to show that $\tilde{\xi}_1(t)$ satisfies constraint equation (3.9b) on the entire time interval, consider equations (3.48) and (3.49). From these two equations and from system (3.8), the response $\tilde{\xi}_1(t)$ to the extremal control $\tilde{w}(t)$ on $t \in [t^*, T]$ is given by

$$|\tilde{\xi}_1(t)| = |\tilde{\xi}_1(t^*) + (t-t^*) k \operatorname{sgn} [f(t)-\alpha]| \quad (3.56)$$

In a similar manner the response $\tilde{\xi}_1(t)$ to the control $\tilde{w}(t)$ on $t \in [t_1, T]$ is given by

$$|\tilde{\xi}_1(t)| = |\tilde{\xi}_1(t_1) + (t-t_1) k \operatorname{sgn} [f(t)-\alpha]| \quad (3.57)$$

However, from equation (3.49) and the definition of the time instant t_1 , equation (3.53), we know that $t_1 > t^*$ and that $\tilde{\xi}_1(t_1) = \tilde{\xi}_1(t^*) = \tilde{\xi}_1(t^*)$. Thus since $|\tilde{\xi}_1(t)| \leq 1$ on $t \in [t^*, T]$, then $|\tilde{\xi}_1(t)| \leq 1$ on $t \in [t^*, T]$. The conclusion is that $|\tilde{\xi}_1(t)| \leq 1$ for all $t \in [t_0, T]$ and therefore $\tilde{w}(t)$ is admissible. Substituting for $\tilde{w}(t)$ in equation

(3.36) yields

$$- \int_{t^*}^{t_1} f(t) \bar{w}(t) dt \leq 0 \quad (3.58)$$

Now, from system (3.8), $\bar{E}_1(T)$ is given by

$$|\bar{E}_1(T)| = |\bar{E}_1(t^*) + \int_{t^*}^T \bar{w}(t) dt| \quad (3.59)$$

where $\bar{w}(t) = k \operatorname{sgn} [f(t) - \alpha]$, by equation (3.48). However, the case under consideration is such that $|\bar{E}_1(T)| = 1$. Also, from equation (3.49), $f(t) \neq \alpha$ on $t \in (t^*, T)$ which implies that $\bar{w}(t)$ is of one sign on $t \in (t^*, T)$. Therefore, in order to satisfy equation (3.59),

$$\bar{w}(t) = k \operatorname{sgn} [\bar{E}_1(T)], \quad t \in (t^*, T) \quad (3.60)$$

Also, from equation (3.54), on $t \in [t^*, t_1]$

$$f(t) = |f(t)| \operatorname{sgn} [f(t)] = -|f(t)| \operatorname{sgn} [\bar{E}_1(T)] \quad (3.61)$$

Substituting equations (3.60) and (3.61) into equation (3.58) and noting that $t_1 \leq T$, yields

$$k \int_{t^*}^{t_1} |f(t)| (\operatorname{sgn} [\bar{E}_1(T)])^2 dt \leq 0 \quad (3.62)$$

However $|\bar{E}_1(T)| = 1$, $k > 0$ and from equation (3.54) $f(t) \neq 0$ on $t \in [t^*, t_1]$. Therefore equation (3.62) is a contradiction which implies that equation (3.47) and correspondingly equation (3.34) must be satisfied.

Theorem 3.2 provides a necessary condition for the modified adjoint solution at the terminal time. This condition, equation (3.34), along with the previously derived condition, equation (3.28), and equation (3.16) all facilitate the actual construction of the extremal controls.

In the next section, the procedure of construction will be developed. The basic results of this section are summarized below.

If $\bar{w}(t)$ is the extremal control on $t \in [t_0, T]$ for system (3.8), then there exists a finite number of time-parameters defined in equation (3.10) and a modified adjoint solution $p(t)$ such that $\bar{w}(t) = k \operatorname{sgn} [p(t)]$, where

(A) $p(t)$ is given by either equation (3.29) or (3.32) with $p(T^-)$ given by equations (3.30), (3.31), or (3.33). The function $p(t)$ is continuous on $t \in [t_0, T)$ and may have a discontinuity at $t = T$, but this can only happen if $|\bar{g}_1(T)| = 1$.

(B) $\dot{f}(t) \operatorname{sgn} [\bar{g}_1(t)] \leq 0$ on $t \in (\tau_{2j+1}, \tau_{2j+2})$, $j=0, 1, \dots, N-1$, for $N > 0$.

(C) $f(\tau_{2i}) = f(\tau_{2i+1})$, $i=1, 2, \dots, N-1$, for $N > 1$.

(D) $[f(T) - p(T^-)] \operatorname{sgn} [\bar{g}_1(T)] \geq 0$.

3.7 Construction of the Extremal Controls

In this section, the following will be shown:

For a given fixed terminal time T and unit vector η , an extremal control $\bar{w}(t)$, $t \in [t_0, T]$, for system (2.6) can be constructed from system (3.8) and conditions (A) through (D) of Section 3.6.

Since $y_{n+1}(t)$ is the $(n+1)$ -th component of the solution of the adjoint system $\dot{y}(t) = -y(t)A(t)$, it can be computed on $t \in [t_0, T]$ for each choice of $\eta = -y(t_0)$ and T . Then for each i , $i=1, 2, \dots, m$, the set of transformations defined in equation (3.7) can be applied to form the function $f(t)$ in system (3.8). The corresponding i -th component of the extremal control, denoted by $\bar{w}$ in system (3.8), then can be found by applying conditions (A) through (D). The only unknown quantities are

the time parameters, τ_j , $j=1,2,\dots,2N$, N , $\rho(T^-)$ and the constant c . Once these unknowns are found, then the modified adjoint solution $\rho(t)$ is completely specified. The corresponding component of the extremal control $\bar{w}(t)$ on $t \in [t_0, T]$ is then obtained as k times the signum of $\rho(t)$, where $\rho(t)$ is given by equation (3.29) if $N > 0$, or by equation (3.32) if $N = 0$. This process is repeated until all m -components of the extremal control have been found. In summary, the basic procedure of constructing the extremal controls consists of finding the unknown parameters for an arbitrary function $f(t)$, where the function $f(t)$ is continuous and its form is known.

To best illustrate the procedure, the construction of the extremal control for a particular class of functions with one extremum and one zero crossing will be carried out in detail below. Let $f(t)$ (Figure 3.1) on $t \in [t_0, T]$ be such that $t_0 < t_1 < t_2 < t_3 < T$, with the time instants t_i , $i=1,2,3$ that describe $f(t)$ defined by

$$\begin{cases} \dot{f}(t) > 0, & \text{if } t_0 \leq t < t_1 \\ \dot{f}(t) = 0, & \text{if } t = t_1 \\ \dot{f}(t) < 0, & \text{if } t_1 < t \leq T, \end{cases} \quad (3.63a)$$

$$f(t_0) = f(t_2) \quad (3.63b)$$

and

$$\begin{cases} f(t) > 0, & \text{if } t_0 \leq t < t_3 \\ f(t) = 0, & \text{if } t = t_3 \\ f(t) < 0, & \text{if } t_3 < t \leq T \end{cases} \quad (3.63c)$$

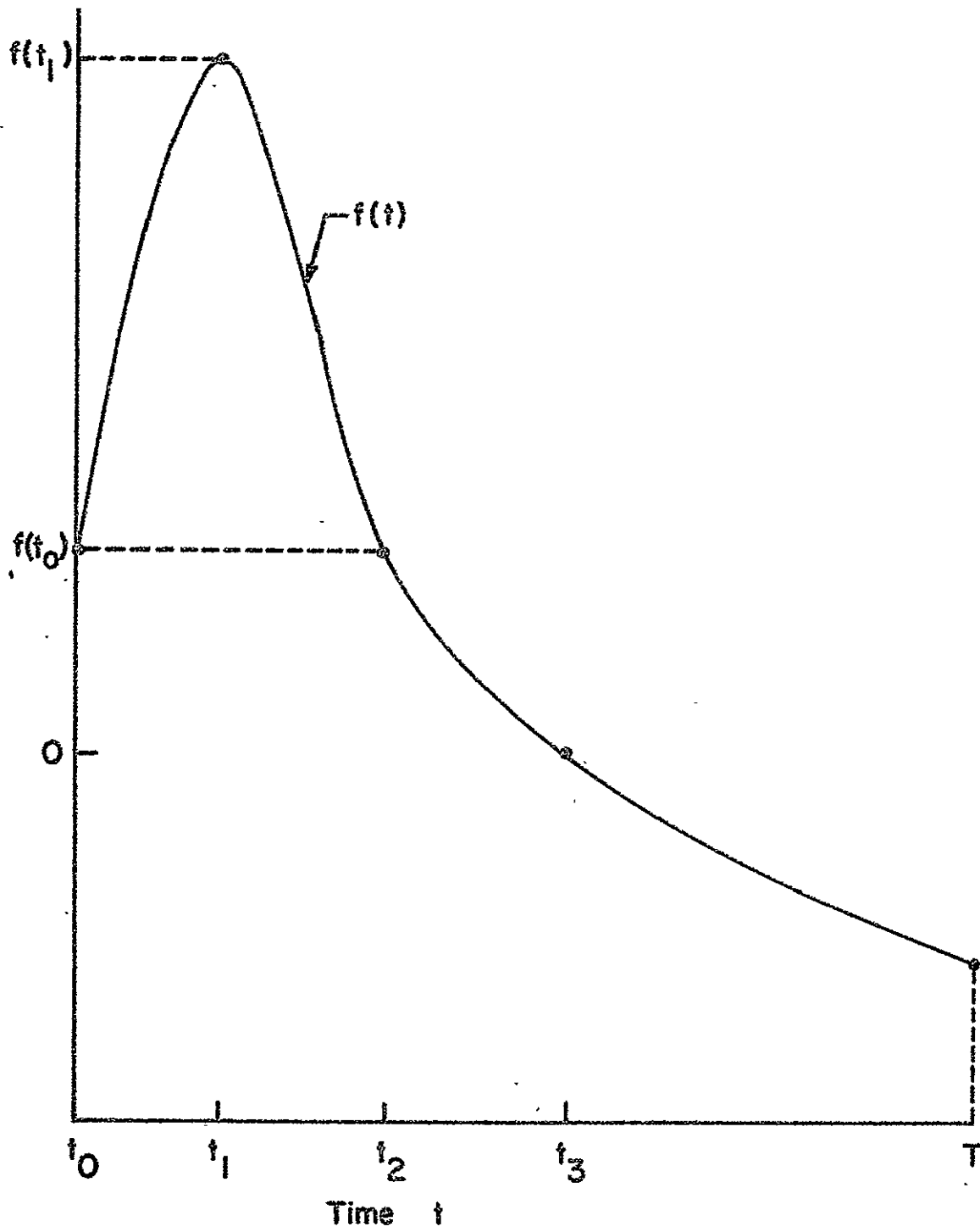


FIGURE 3.1. FUNCTION $f(t)$

Also suppose that

$$t_3 - t_0 > 3/k \quad (3.63d)$$

where $k > 0$ is the magnitude bound on the control w .

To begin with, we have from system (3.8) that for any two time instants t_α, t_β ,

$$\bar{E}_1(t_\beta) = \int_{t_\alpha}^{t_\beta} \bar{w}(t) dt + \bar{E}_1(t_\alpha) \quad (3.64a)$$

and also the initial value of $\bar{E}_1(t)$ is

$$\bar{E}_1(t_0) = 0 \quad (3.64b)$$

The first step is to decide whether $N = 0$, or $N > 0$, i.e., whether equation (3.32) or equation (3.29) is the applicable case. Certainly, from a pictorial point of view the case $N = 0$, i.e., $|\bar{E}_1(t)| < 1$ on $t \in [t_0, T]$, is quite unlikely to occur because of the form of the function $f(t)$ and equation (3.63d) which implies that $T - t_0 > 3/k$. The fact that the time length involved is certainly large enough for the bound, $|\bar{E}_1(t)| = 1$, to be reached at least once makes the case $N > 0$ seem more intuitively obvious. However, for the purposes of illustration, suppose that $N = 0$. It will now be shown that this case cannot occur because it violates either the constraint equation (3.9b) or one of the necessary conditions (A) through (D) of Section 3.6.

If $N = 0$, then from condition (A), the modified adjoint solution $p(t)$ is given by equation (3.32) with $p(T^-)$ given by equation (3.33). From these equations, we obtain

$$-\bar{w}(t) = k \operatorname{sgn} [f(t) - c], \quad t \in [t_0, T] \quad (3.65)$$

where c is some real number. First suppose that $c = 0$. Then equation (3.65) reduces to $\bar{w}(t) = k \operatorname{sgn} [f(t)]$ on $t \in [t_0, T]$. From equation (3.63e), $f(t) > 0$ on $t \in [t_0, t_3)$ and therefore $\bar{w}(t) = k > 0$ on $t \in [t_0, t_3)$. The corresponding response $\bar{\xi}_1$ at time t_3 is given by equation (3.64) as

$$\bar{\xi}_1(t_3) = \int_{t_0}^{t_3} \bar{w}(t) dt = k(t_3 - t_0) \quad (3.66)$$

However, by equation (3.63d), $\bar{\xi}_1(t_3) > 1$ which violates the constraint equation (3.9b). Thus the case $c = 0$ cannot occur. Now consider the case $c \neq 0$. From condition (D)

$$c \operatorname{sgn} [\bar{\xi}_1(T)] \geq 0 \quad (3.67)$$

Also, from condition (A), $\rho(t)$ can be discontinuous at $t = T$ only if $|\bar{\xi}_1(T)| = 1$. Since $c \neq 0$, it is seen that, from equation (3.32), a discontinuity does occur at $t = T$ and therefore $|\bar{\xi}_1(T)| = 1$ in this case. First, assume that $c < 0$. Then $f(t) - c > f(t)$ on $t \in [t_0, T]$ and correspondingly $\bar{w}(t) = k \operatorname{sgn} [f(t) - c] = k$ on $t \in [t_0, t_3)$. This is equivalent to the case $c = 0$ and therefore $\bar{\xi}_1(t_3) > 1$ which is a constraint violation. Instead, suppose that $c > 0$. In this case $f(t) - c < f(t) < 0$ on $t \in [t_3, T]$. The corresponding extremal control $\bar{w}(t) = -k$ on this time interval. From equation (3.64e), $\bar{\xi}_1(T)$ is given by

$$\bar{\xi}_1(T) = \bar{\xi}_1(t_3) + \int_{t_3}^T \bar{w}(t) dt = \bar{\xi}_1(t_3) - k(T - t_3) \quad (3.68)$$

Since $|\bar{\xi}_1(T)| = 1$, then by equation (3.68)

$$-\bar{\xi}_1(T) = -1 \quad (3.69)$$

However, it has been assumed that $c > 0$, and this fact combined with equation (3.69) violates equation (3.67) which is necessary condition (D). Therefore, this case is also not allowed. All the possible values for the real number c have now been shown to violate either the constraint equation (3.9) or in particular, condition (D). Therefore, the case $N = 0$ cannot occur as was originally predicted by pictorial reasoning.

Now consider the case $N > 0$. From equation (3.29),

$$\bar{w}(t) = k \operatorname{sgn} [f(t) - f(\tau_1)], \quad t \in [t_0, \tau_1] \quad (3.70)$$

and, by definition,

$$1 = |\bar{E}_1(\tau_1)| = \left| \int_{t_0}^{\tau_1} \bar{w}(t) dt \right|, \quad (3.71)$$

where the time instant τ_1 is to be determined. For convenience in the following analysis, the function $f(t)$ is further divided into five subcases, viz: (1) $t_2 - t_0 \leq 1/k$, (2) $t_2 - t_0 > 1/k$, $f(1/k) > f(3/k)$, (3) $t_2 - t_0 > 1/k$, $f(1/k) \leq f(3/k)$, (4) $T - t_3 < 2/k$ and (5) $T - t_3 \geq 2/k$. The first three cases are concerned with the characteristics of the function $f(t)$ at the beginning and middle portions of the time interval. The last two cases are concerned with the characteristics of the function $f(t)$ at the end of the time interval. Each of the cases (1) through (3) can be coupled with either a case (4) or case (5) ending. For the purposes of analysis, each case is considered separately below.

(1) $t_2 - t_0 \leq 1/k$. Since τ_1 is the first time at which $|\bar{E}_1(t)| = 1$, and since $\bar{E}_1(t_0) = 0$, then certainly $\tau_1 - t_0 \geq 1/k$.

Therefore, in this case $\tau_1 \geq t_2$. For any choice of $\tau_1 \in [t_2, t_3]$, the extremal control $\bar{w}(t)$, given by equation (3.70), becomes, $\bar{w}(t) = k$. Substituting this value of $\bar{w}(t)$ in equation (3.71) yields the relation

$$\tau_1 = t_0 + 1/k \quad (3.72)$$

Note that this value for τ_1 is acceptable, since by equation (3.63d), $t_3 - t_0 > 3/k$, and therefore $\tau_1 \in [t_2, t_3]$ as was originally assumed.

(2) $t_2 - t_0 > 1/k$, $f(1/k) > f(3/k)$. In this case it will be shown that $t_1 < \tau_1 < t_2$. Suppose, if possible, that $\tau_1 \leq t_1$. Then by equation (3.70) and equation (3.63a), $\bar{w}(t) = -k$ on $t \in [t_0, \tau_1]$. Therefore by equation (3.71), $\bar{\xi}_1(\tau_1) = -1$ and $\tau_1 = 1/k + t_0$. Now consider the parameter τ_2 . On the time interval $[\tau_1, \tau_2]$, $\bar{\xi}_1(t) = -1$. From condition (B) and equation (3.63a), $\tau_2 \leq t_1$ and therefore $f(\tau_2) > 0$. First suppose that $N = 1$, i.e., τ_2 is the last parameter. Since $\tau_2 < T$, then $\rho(T^-) = f(T) - f(\tau_2)$, from equation (3.30). Also, since $f(\tau_2) > 0$, then equation (3.29) implies that $\rho(t)$ is discontinuous at $t = T$. Therefore, from condition (A), $|\bar{\xi}_1(T)| = 1$. The extremal control on $t \in (\tau_2, T]$ is given by equation (3.29) as $\bar{w}(t) = k \operatorname{sgn}[f(t) - f(\tau_2)]$. This implies that $\bar{w}(t) = -k$ for $t \in [t_w, T]$, where $t_w \geq t_1$ is the time instant at which $f(\tau_2) = f(t_w)$. Thus from equation (3.64a), $\bar{\xi}_1(T) = \bar{\xi}_1(t_w) - k(T - t_w)$. Since $\bar{\xi}_1(t)$ at the terminal time is on its bound, then $\bar{\xi}_1(T) = -1$. However $f(\tau_2) > 0$ which violates condition (D) and therefore the case $N = 1$ cannot occur. On the other hand, suppose $N > 1$. In this case, there must be a parameter τ_3 such that $f(\tau_2) = f(\tau_3)$ and $\bar{\xi}_1(\tau_3) = 1$, from condition (C). Since $\bar{\xi}_1(\tau_0) = -1$, and $\tau_2 \leq t_1$, then $\tau_3 = \tau_2 + 2/k$. However, in this

case $f(1/k) > f(3/k)$. Also, it has been established that if $\tau_1 \leq t_1$, then $\tau_1 = 1/k + t_0$. Therefore, no such time instant τ_3 exists. In summary, if $\tau_1 \leq t_1$, then a contradiction of one of the necessary conditions occurs in the choice of τ_2 . It is fairly simple to show that the case $\tau_1 \geq t_2$ also cannot occur. For all choices of $\tau_1 \geq t_2$, $\bar{w}(t) = k$ on $t \in [t_0, \tau_1]$, by equations (3.63) and (3.70). Since in this case $t_2 - t_0 > 1/k$, then by equation (3.64) $\bar{S}_1(t_2) > 1$, which is a violation of the constraint equation (3.9b). It can now be concluded that $t_1 < \tau_1 < t_2$. The actual value of τ_1 may be found by applying the fact that $|\bar{S}_1(\tau_1)| = 1$. This yields the relation

$$\tau_1 = 1/k + (2\sigma - t_0) \quad (3.73)$$

where $\sigma < t_1$ and $f(\sigma) = f(\tau_1)$.

(3) $t_2 - t_0 > 1/k, f(1/k) \leq f(3/k)$. The calculation of the parameter τ_1 for this case is quite similar to case (2). The only difference is that $f(1/k) \leq f(3/k)$. This fact changes the results in that now a parameter τ_3 , satisfying condition (C), does exist. The results for this case are as follows.

$$\begin{cases} \tau_1 = 1/k \\ \tau_3 = \tau_2 + 2/k \end{cases} \quad (3.74)$$

where $f(\tau_2) = f(\tau_3)$ and $\tau_2 < t_1 < \tau_3 < t_2$.

Up to this point values have been determined for the parameter τ_1 and also for τ_2 and τ_3 in case (3). Now, assume that $N = 1$ in cases (1) and (2) and that $N = 2$ in case (3). Under this assumption, $\bar{S}_1(\tau_{2N-1}) = 1$ for all three cases and it remains only to determine the

value of the parameter τ_{2N} . It can be shown that the above assumption is valid. The basic reasoning is similar to that used in case (2) and will therefore be omitted. Returning to the determination of the parameter τ_{2N} , it is first necessary to decide whether $\tau_{2N} = T$ or $\tau_{2N} < T$. In other words whether or not $\bar{E}_1(t)$ remains on or leaves its bound, $\bar{E}_1(t) = 1$, on $t \in (\tau_{2N-1}, T)$. From condition (A), $\rho(T^-) = 0$ if $\tau_{2N} = T$, and in this case $\bar{E}_1(t)$ would remain on its bound for the remaining time interval. However, at the terminal time, $f(T)$ and $\bar{E}_1(T)$ would be of opposite sign and hence condition (D) would be violated. Thus, for this function $f(t)$, the only alternative is that $\tau_{2N} < T$ and $\bar{E}_1(t)$ does leave its bound at $t = \tau_{2N}$, $\tau_{2N} \in (\tau_{2N-1}, T)$. In other words the value of $\bar{w}(t)$ changes from 0 to $-k$ at $t = \tau_{2N}$, where τ_{2N} is still an unknown parameter. For $\tau_{2N} < T$, the value of $\rho(T^-)$ is given by equation (30) as $\rho(T^-) = f(T) - f(\tau_{2N})$. The two possible previously stated subcases, viz, (4) $T - t_3 < 2/k$ and (5) $T - t_3 \geq 2/k$, will now be considered.

(4) $T - t_3 < 2/k$. First assume that $|\bar{E}_1(T)| < 1$. Then from condition (A), $\rho(t)$ must be continuous at the terminal time and therefore $f(\tau_{2N}) = 0$, by equation (3.29). Thus by definition

$$\tau_{2N} = t_3 \quad (3.75)$$

where t_3 is the zero crossing of $f(t)$ on $t \in (t_0, T)$. Note that if it had been assumed that $|\bar{E}_1(T)| = 1$, then since $\bar{E}_1(\tau_{2N-1}) = \bar{E}_1(\tau_{2N}) = 1$, the only possible choice would be that $\bar{E}_1(T) = -1$. However, in order for this to occur, it is required that $\tau_{2N} = T - 2/k < t_3$ which implies $f(\tau_{2N}) > 0$. This is a violation of condition (D) since $f(\tau_{2N})$ and $\bar{E}_1(T)$ are of opposite sign.

(5) $T - t_3 \geq 2/k$. In this case the previous choice of $\tau_{2N} = t_3$ is no longer valid, for now $\bar{g}_1(T) = 1 - k(T - t_3) \leq -1$, which contradicts the original assumption that $|\bar{g}_1(T)| < 1$. Therefore for this case $\bar{g}_1(T) = -1$, and the corresponding value of τ_{2N} is given by

$$\tau_{2N} = T - 2/k \quad (3.76)$$

Note that condition (D) is now satisfied because $\tau_{2N} \geq t_3$ and therefore $f(\tau_{2N}) \leq 0$.

The results of this example, equations (3.72) through (3.76) are illustrated in Figures 3.2 through 3.4. Figure 3.2 shows case (1), i.e., $t_2 - t_0 \leq 1/k$, coupled with a case (4) ending, i.e., $T - t_3 < 2/k$. Figure 3.3 shows case (2), i.e., $t_2 - t_0 > 1/k$, $f(1/k) > f(3/k)$, with a case (5) ending, i.e., $T - t_3 \geq 2/k$. Figure 3.4 shows case (3), i.e., $t_2 - t_0 > 1/k$, $f(1/k) \leq f(3/k)$, also with a case (5) ending.

As a simple numerical application of the results, consider the function $f(t) = .5 + \sin t$ on $t \in [0, 3\pi/2]$. The parameters t_i , $i=1,2,3$ describing this function are, by a simple calculation, given by $t_1 = \pi/2$, $t_2 = \pi$, and $t_3 = 7\pi/6$. For this application, let $k = 6/\pi$. Then, since $t_2 - t_0 > \pi/6$ and $f(\pi/6) < f(\pi/2)$, case (3) is applicable. From equation (3.74), $\tau_1 = \pi/6$, $\tau_2 = \pi/3$ and $\tau_3 = 2\pi/3$. Finally, since $T - t_3 = \pi/3 = 2/k$, from equation (3.76), $\tau_4 = 7\pi/6$. The modified adjoint solution on $t \in [0, 3\pi/2]$ is now completely determined. From $\bar{w}(t) = k \operatorname{sgn} [\rho(t)]$ we obtain

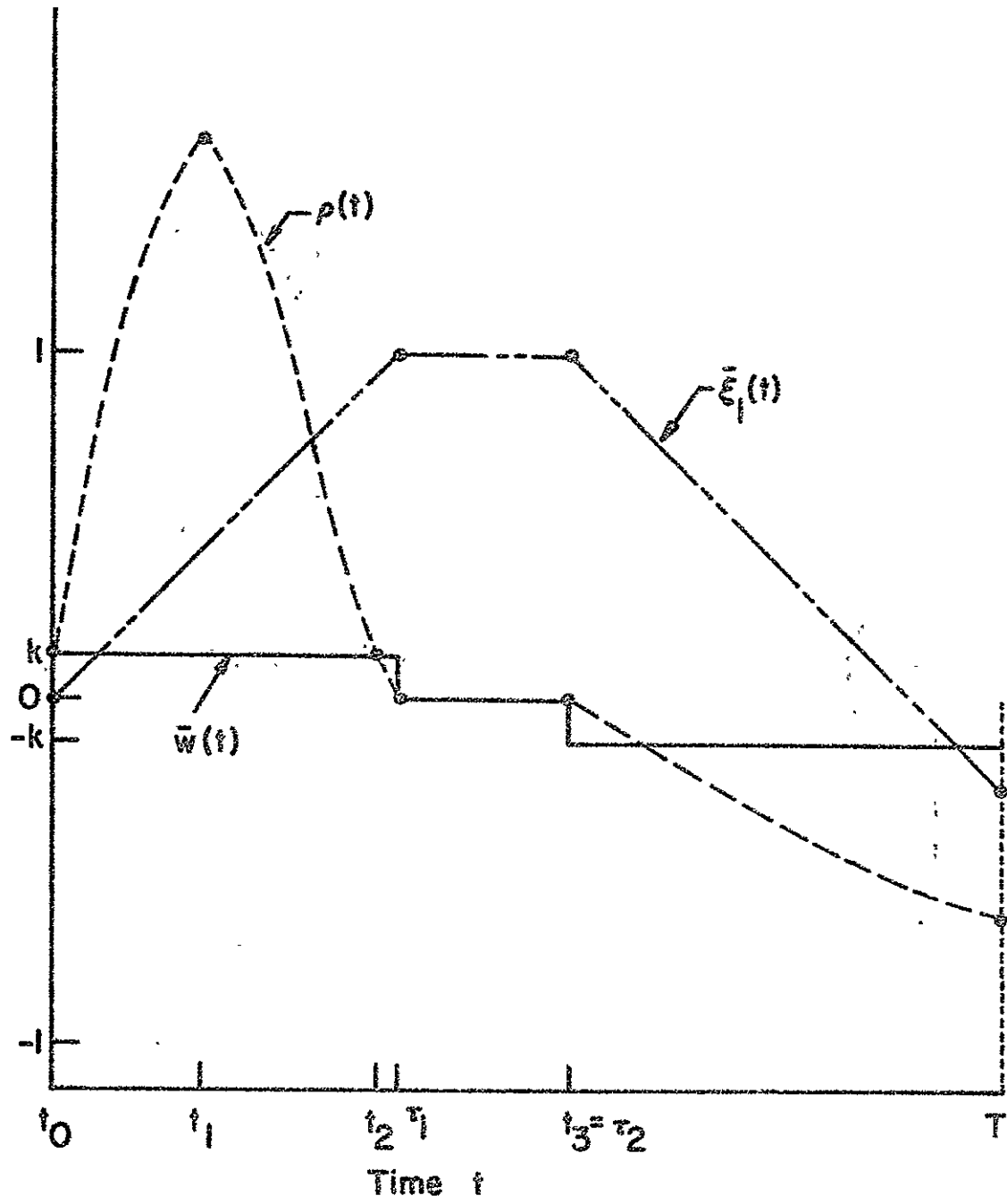


FIGURE 3.2. CASE (1) WITH CASE (4)

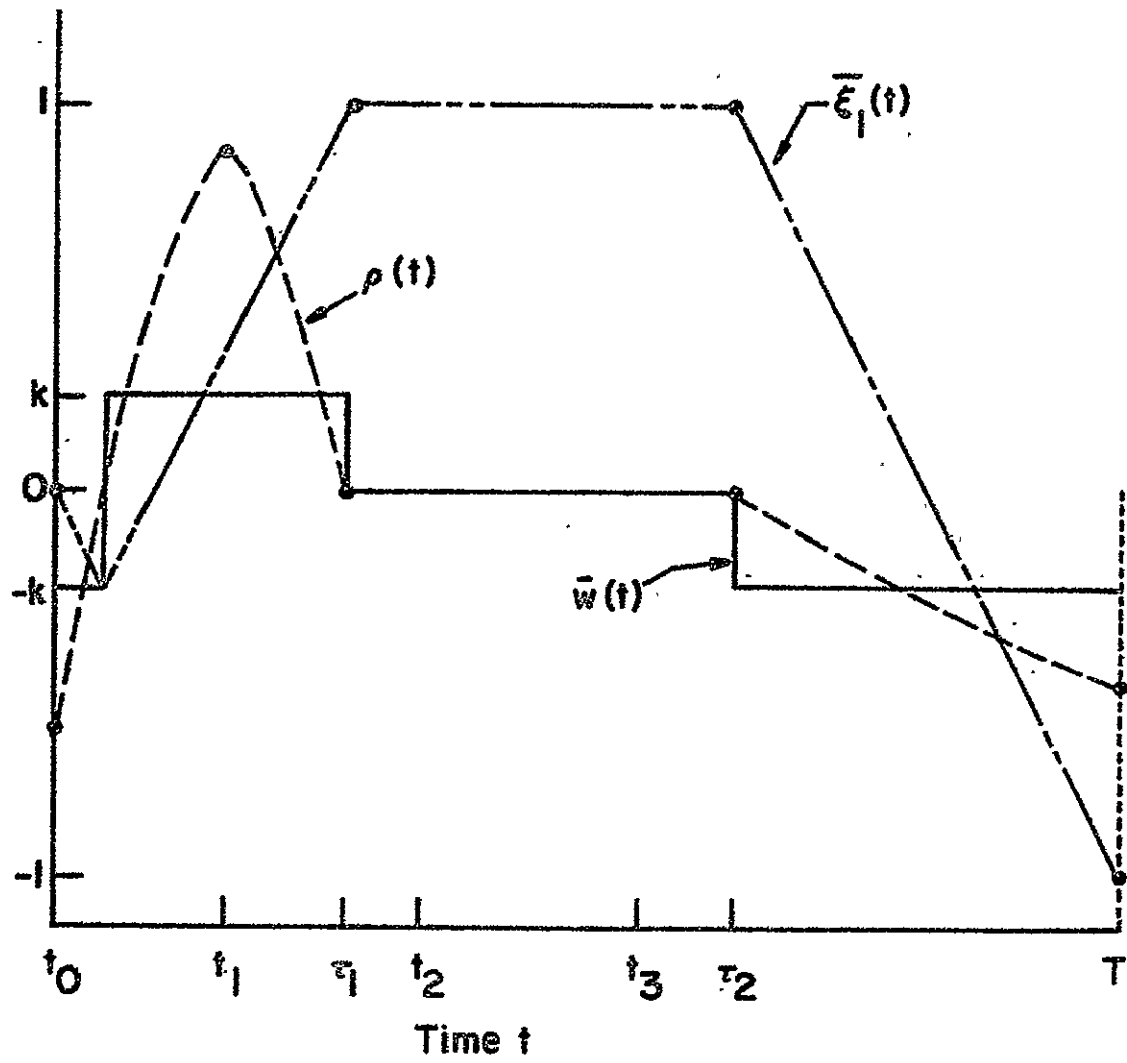


FIGURE 3.3 . CASE (2) WITH CASE (5)

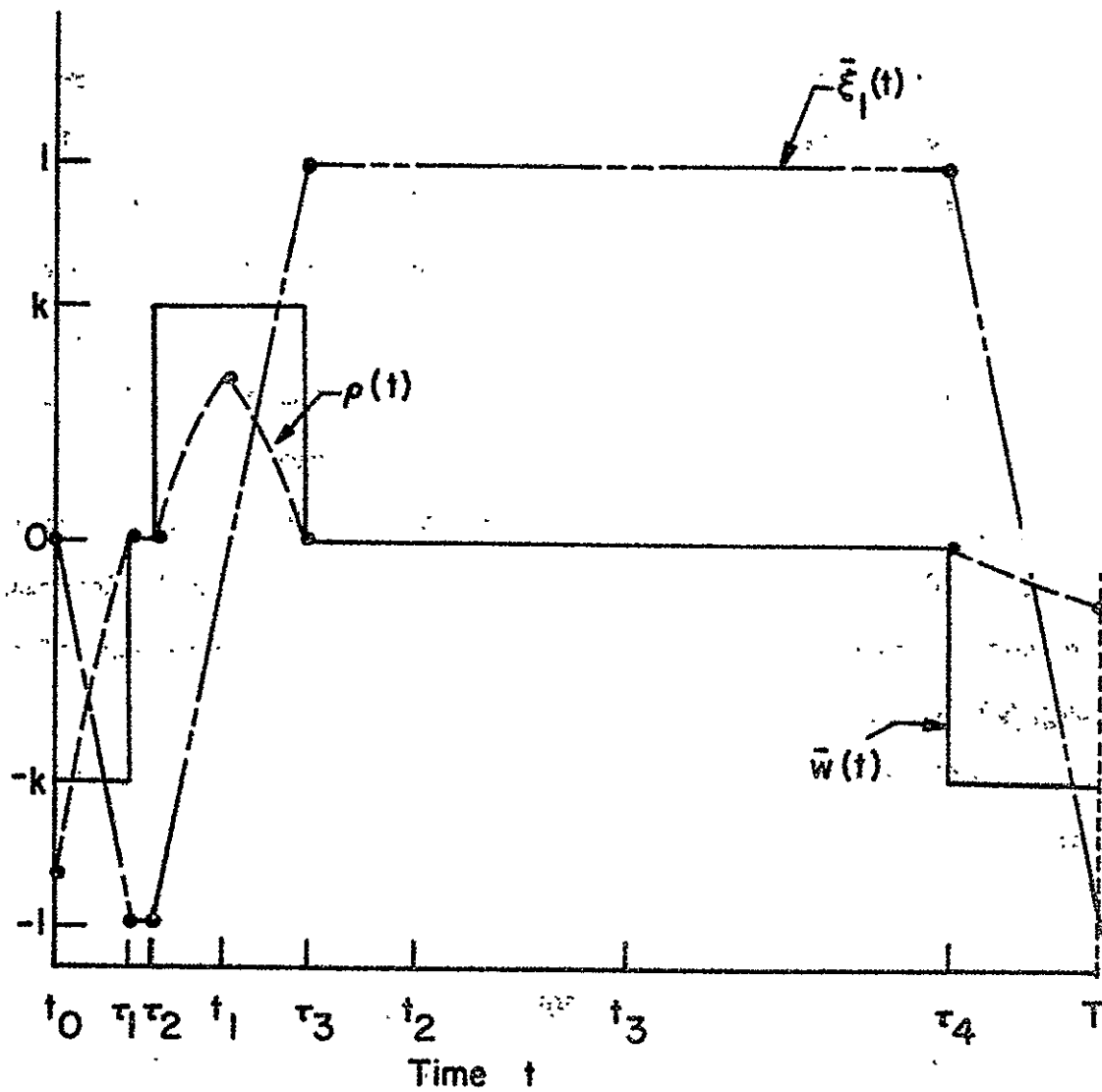


FIGURE 3.4 . CASE (3) WITH CASE (5)

$$\bar{w}(t) = \begin{cases} -6/\pi, & \text{if } 0 \leq t < \pi/6 \\ 0, & \text{if } \pi/6 \leq t < \pi/3 \\ 6/\pi, & \text{if } \pi/3 \leq t < 2\pi/3 \\ 0, & \text{if } 2\pi/3 \leq t \leq 7\pi/6 \\ -6/\pi, & \text{if } 7\pi/6 < t \leq 3\pi/2 \end{cases} \quad (3.77)$$

In the above example, the procedure of constructing an extremal control for a given class of functions $f(t)$, was illustrated. The general procedure is based on the repeated application of the steps analyzed above, to other classes of functions $f(t)$. Since the governing conditions are necessary, it is theoretically required to compute all the possible outcomes first, and then eliminate those that violate conditions (A) through (D). This was the procedure followed in the example. In actual practice the calculation of the unknown parameters of the modified adjoint solution is somewhat easier, in that pictorial reasoning and previous experience can be used to eliminate many of the possible outcomes.

In the next section, the results of this construction procedure are developed and presented in the form of Tables 3.1 through 3.6. The explanation of these tables is also given.

3.8 Extremal Control Tables

In order to arrange the construction procedure in a systematic manner, it is desirable to group the possible types of functions $f(t)$ into a number of mutually exclusive classes based on their distinctive characteristics, i.e., those that determine specific extremal controls. Referring to the illustrative example, some of the dominant factors that determine the parameters τ_i are as follows:

- (a) number of extrema of $f(t)$ - this is related to the number of times $\bar{g}_1(t)$ enters upon and exits from its bound -1 ,
- (b) number of zero crossings of $f(t)$ - this affects the initial and terminal τ parameters,
- (c) the total time interval involved and the time intervals between parameters describing the shape of $f(t)$ and the value of the magnitude bound k on $w(t)$ (e.g., in the illustrative example, the relation between $t_2 - t_0$ and $1/k$ plays an important role in the determination of the unknown parameters) - these determine whether or not $\bar{g}_1(t)$ enters upon its bound, stays on the bound if entered and exits from the bound,
- (d) relative magnitudes of $f(t)$ at time instants related to the magnitude bound k (e.g., the relation between $f(1/k)$ and $f(3/k)$ in the illustrative example) - these also affect the initial and terminal τ parameters.

As an example, consider cases (2) and (4) of the illustrative example in Section 3.7. The distinctive parameters are listed below.

- (i) one extremum,
- (ii) $|f(t_1) - f(t_0)| < |f(t_1) - f(T)|$,
- (iii) $t_2 - t_0 > 1/k$, where $f(t_2) = f(t_0)$
- (iv) $f(1/k) > f(3/k)$
- (v) one zero crossing with $t_3 - t_0 > 3/k$, where $f(t_3) = 0$,
- (vi) $T - t_3 < 2/k$

The results for this case were that $N = 2$ with τ_1 given by equation (3.73) and τ_2 by equation (3.75). Any function $f(t)$ on $t \in [t_0, T]$ which satisfies specifications (i) through (vi) has the same results

for N , τ_1 and τ_2 . This of course only illustrates the fact that the number of switches are the same. The exact numerical values of the switching times of course depend on the particular function $f(t)$. Thus the extremal control $\bar{w}(t)$ may be different, depending on the numerical values of τ_1 and τ_2 .

The above describes the basic approach for the selection of the dominant parameters which divide the functions $f(t)$ into mutually exclusive classes. We again have been relying on the fact that for a fixed T and a fixed $\eta = -v(t_0)$, the form of the function $f(t)$ can be computed. In order to generalize the notation to correspond with that in the tables, consider the following definitions for a given function $f(t)$ on $t \in [t_0, T]$.

$$(1) \quad t_0 < Z_1 < Z_2 < \dots < Z_L < T,$$

such that $f(Z_j) = 0$, $j=1,2,\dots,L$ (L = number of zero crossings);

$$(2) \quad t_0 = B_0 < B_1 < \dots < B_p < B_{p+1} = T,$$

such that $f(B_j)$, $j=1,2,\dots,p$, are the extrema values of $f(t)$ (p = number of extrema); and

$$(3) \quad Y_j = f(B_j), \quad j=0,1,\dots,p+1.$$

Finally, let $t_0 = 0$. There is no loss of generality in setting $t_0 = 0$, since the computed time instants at which $f(t)$ is zero or an extremum can be re-labelled.

With these definitions, it is found that 12 parameters are required to describe the various classes of functions $f(t)$ on $t \in [0, T]$. These 12 parameters are defined in Table 3.1. With the aid of these 12 parameters and the number of extrema of $f(t)$ on $t \in (0, T)$, the various forms of $f(t)$ can be grouped into 15 mutually exclusive classes. These classes are defined in Table 3.2. It is assumed that the various

Table 3.1. Parameter Definitions

<u>Parameter</u>	<u>Definition</u>	<u>Parameter</u>	<u>Definition</u>
α_1	$B_1 < \alpha_1 < B_2, f(0)=f(\alpha_1)$	α_6	$B_{p-1} < \alpha_6 < B_p, f(\alpha_6)=f(T)$
α_2	$\begin{cases} B_0 < (\alpha_2 - 1/k)/2 < B_1 < \alpha_2 < B_2 \\ f(\alpha_2/2 - 1/2k) = f(\alpha_2) \end{cases}$	γ_i	$\begin{cases} B_{i-1} < \gamma_i - 2/k < B_i < \gamma_i < B_{i+1} \\ f(\gamma_i - 2/k) = f(\gamma_i), i=1, \dots, p \end{cases}$
α_3	$\begin{cases} B_0 < \alpha_3 - (T/2 - 1/2k) < B_1 < \alpha_3 < B_2 \\ f(\alpha_3 - T/2 + 1/2k) = f(\alpha_3) \end{cases}$	c_1	$c_1 = Y_1 - Y_0 - Y_1 - Y_2 $
α_4	$\begin{cases} B_{p-1} < \alpha_4 < B_p < (\alpha_4 + T)/2 < B_{p+1} \\ f(\alpha_4) = f(\alpha_4/2 + T/2) \end{cases}$	c_2	$c_2 = Y_1 - f(1/k) - Y_1 - f(3/k) $
α_5	$\begin{cases} B_{p-2} < \alpha_5 < B_{p-1} < (\alpha_5 + T)/2 < B_p \\ f(\alpha_5) = f(\alpha_5/2 + T/2) \end{cases}$	c_3	$c_3 = Y_2 - Y_3 - Y_2 - f(1/k) $
		c_4	$c_4 = Y_2 - Y_3 - Y_2 - f(\alpha_2) $
		c_5	$c_5 = Y_p - Y_{p+1} - Y_p - f(\gamma_{p-1}) $

Table 3.2. Initial Classifications

Class No.	(p-) No. of Extrema	Description	Parameters	Refer to Table No.	When Use Table 6, Choose Sub- case (if Any)
1	0	Any Condition	$t_1=0, t_2=1/k, t_3=0, t_4=(T+1/k)/4$	3	a
2	1	$C_2 \leq 0, \alpha_2 \leq 1/k$	$t_1=\alpha_2, t_2=1/k, t_3=0, t_4=(T+1/k)/4$	3	a
3	1	$\begin{cases} C_1 \leq 0, 1/k < \alpha_1 \leq 3/k; \text{ or} \\ \alpha_2 > 3/k, C_2 \leq 0, C_3 \leq 0 \end{cases}$	$\begin{cases} t_1=(t_2+1/k)/2, t_2=\alpha_2, t_3=0, \\ 1/k < t_4 < t_2 < B_2, 1(1/k) < t_4 \end{cases}$	3	b
4	1	$C_1 \leq 0, \alpha_1 > 3/k, C_2 > 0$	$t_1=(t_2+1/k)/2, t_2=\alpha_1, t_3=0, t_4=T+1/k$	3	c
5	1	$\begin{cases} C_1 > 0, \alpha_1 \leq 1/k, \text{ with } \alpha_2 > 3/k; \\ \text{or with } \alpha_2 > 3/k, C_2 \leq 0 \end{cases}$	$\begin{cases} t_1=(t_2+1/k)/2, t_2=\alpha_2, t_3=\alpha_1, \\ 1/k < t_4 < t_2 < B_2, 1(1/k) < t_4 \end{cases}$	3	b
6	1	$\begin{cases} C_1 > 0, \alpha_1 \leq 1/k, T > 3/k; \\ \text{and } C_2 > 0 \end{cases}$	$t_1=(t_2+1/k)/2, t_2=\alpha_1, t_3=0, t_4=T+1/k$	3	c
7	1	$C_1 > 0, \alpha_2 > 1/k$	$\begin{cases} t_1=\alpha_1 \text{ (or } t_1=0 \text{ if } L=0), t_2=1/k, \\ t_3=\alpha_2, t_4=0 \end{cases}$	4	a
8	2	$\alpha_1 \leq 1/k, C_3 \leq 0$	$t_1=\alpha_1, t_2=1/k, t_3=\alpha_2, t_4=B_2$	4	a
9	2	$\alpha_1 \leq 1/k, C_3 > 0$	$t_1=1/k, t_2=1/k, t_3=1/k, t_4=B_2$	4	a
10	2	$\begin{cases} C_1 < 0, 1/k < \alpha_1 \leq 3/k, C_4 \leq 0 \\ \text{or } C_1 < 0, \alpha_1 \geq 3/k, C_2 \leq 0, C_4 \leq 0 \\ \text{or } C_1 \geq 0, C_2 \leq 0, C_4 \leq 0 \end{cases}$	$t_1=(t_2+1/k)/2, t_2=\alpha_1, t_3=\alpha_2, t_4=B_2$	4	b
11	2	$\begin{cases} C_1 < 0, 1/k < \alpha_1 \leq 3/k, C_4 > 0 \\ \text{or } C_1 < 0, \alpha_1 \geq 3/k, C_2 < 0, C_4 > 0 \\ \text{or } C_1 \geq 0, C_2 < 0, C_4 > 0 \end{cases}$	$t_1=1/k, t_2=1/k, t_3=1/k, t_4=B_2$	4	b
12	2	$\begin{cases} C_1 < 0, \alpha_1 > 3/k, C_2 > 0; \text{ or} \\ C_1 \geq 0, C_2 > 0 \end{cases}$	$t_1=1/k, r=2$	5	No Sub- case.
13	≥ 3	$C_1 < 0, \alpha_1 \leq 1/k$	$\begin{cases} t_1=1/k, \\ t_{2J+2}=\alpha_{J+2}-2/k, J=0, 1, \dots, p-4 \\ t_{2J+3}=\alpha_{J+2}, r=2p-4 \end{cases}$	5	No Sub- case
14	≥ 3	$\begin{cases} C_1 < 0, 1/k < \alpha_1 \leq 3/k; \text{ or} \\ C_1 < 0, \alpha_1 > 3/k, C_2 \leq 0; \text{ or} \\ C_1 \geq 0, C_2 \leq 0 \end{cases}$	$\begin{cases} t_1=\alpha_2; \\ t_{2J+2}=\alpha_{J+2}-2/k, J=0, 1, \dots, p-4 \\ t_{2J+3}=\alpha_{J+2}, r=2p-4 \end{cases}$	5	No Sub- case
15	≥ 3	$\begin{cases} C_1 < 0, \alpha_1 > 3/k, C_2 > 0; \text{ or} \\ C_1 \geq 0, C_2 > 0 \end{cases}$	$\begin{cases} t_1=1/k, \\ t_{2J+2}=\alpha_{J+2}-2/k, J=0, 1, \dots, p-3 \\ t_{2J+3}=\alpha_{J+2}, r=2p-2 \end{cases}$	5	No Sub- case.

functions $f(t)$ satisfy the following conditions.

- (a) If $f(t) \in$ class 8, then $C_1 < 0$.
- (b) If $f(t) \in$ class 9, then $C_1 < 0$ and $\gamma_2 \geq 3/k$.
- (c) If $f(t) \in$ class 11, then $\gamma_2 \geq \alpha_2 + 2/k$.
- (d) If $f(t) \in$ class 12, then $\gamma_2 > \gamma_1 + 2/k$ if $C_5 > 0$.
- (e) If $f(t) \in$ class 13 through class 15, then
 - (e-1) $\gamma_p \geq \gamma_{p-1} + 2/k$ if $C_5 > 0$,
 - (e-2) $\gamma_{i+1} \geq \gamma_i + 2/k$ for $i=2,3,\dots,p-2$, if $p \geq 4$ and
 - (e-3) if p is set equal to 2, t_1 is set equal to B_3 and $f(t)$ on $t \in [0, t_1]$ belongs to one of the classes 8, 9, 11, or 12, then the corresponding condition listed in (a) through (d) must be satisfied.

Conditions listed in (a) through (e) are constraints on the general shape of $f(t)$ in relation to the magnitude of k . These constraints are satisfied for the class of problems that has a sufficiently large k , so that the number of times that $\bar{E}_1(t)$ reaches its bound is essentially proportional to the number of extrema of $f(t)$.

The procedure to be followed in classifying the functions $f(t)$ that satisfy the conditions listed in (a) through (e) is best illustrated with the aid of an example. Consider the example in Section 3.7 described by specifications (i) through (vi). In terms of the notations and definitions of Table 3.1, specifications (i) through (vi) become

- (i) $p = 1$,
- (ii) $C_1 = |Y_1 - Y_0| - |Y_1 - Y_2| < 0$,
- (iii) $\alpha_1 > 1/k$, where $B_1 < \alpha_1 < B_2$ and $f(0) = f(\alpha_1)$,
- (iv) $C_2 = |Y_1 - f(1/k)| - |Y_1 - f(3/k)| < 0$,
- (v) $L = 1$, $Z_L > 3/k$,

$$(vi) \ 3/k < T < Z_L + 2/k$$

Reference to Table 3.2, shows that specifications (i) through (iv) coincide with the description of class 3. However, specification (v), describing the number of zero crossings and the value of each zero crossing, has not yet been used. Also, specification (vi), describing the value of the terminal time is still to be applied. Since both of these specifications are dominant factors in the determination of the extremal control $\bar{w}(t)$, obviously further classification is necessary. This further classification is provided by Tables 3.3, 3.4 and 3.5 which consider the remaining specifications. The reason for the use of the additional tables is that it is desirable, for the sake of clarity, to economize on each individual table. In order to insure the correct correspondance between each class of functions $f(t)$ and Tables 3.3 through 3.5, cross-references are necessary. These are provided by the last three columns of Table 3.2. For the example under consideration, the parameters t_i , $i=1,2,3,4$, are given by (see class 3) Table 3.2. They are $t_2 = \alpha_2$, where by the definition of the parameter α_2 in Table 3.1, it is equivalent to the parameter τ_1 defined by equation (3.73), $t_1 = (t_2 + 1/k)/2$, $t_3 = 0$ and t_4 is defined by $1/k < B_1 < t_4 < B_2$, $f(1/k) = f(t_4)$. Also, Table 3.2 indicates that Table 3.3 should be used with "sub-case b".

In Table 3.3, the number of zero crossings and the numerical value of each zero crossing is now applied to find the correct row. From specification (v), $L = 1$ and $Z_L > 3/k$. The corresponding column is found from the value of T , which in this example, from specification (vi), is given by $3/k < T < Z_L + 2/k$. With reference to Table 3.3,

Table 3.3. Final Classifications

		Note: (M) = Case No. M				
T Condition		0 < T < 1/k	1/k < T < 2t ₃ + 1/k		2t ₃ + 1/k < T < 3/k	T > 3/k
			1/k < T < 2Z _L + 1/k	2Z _L + 1/k < T < 3/k		
0	$(-1)^P Y_2 < (-1)^P Y_1 $	(1)	(2)			(4)
1	Class 6					
2	Class 5	(1)	(2)			(4)
1	C ₁ ≤ 0, Z _L ≤ 1/k	(1)	(3)			(6)
1	C ₁ ≤ 0, Z _L > 1/k	(1)	(1) if Z ₁ ≤ 1/k, Z ₂ - Z ₁ > (T - 1/k) / 2 (3) if Z ₁ ≤ 1/k, Z ₂ - Z ₁ ≤ (T - 1/k) / 2; or if Z ₁ > 1/k, t ₄ ≤ (T + 1/k) / 2 (7a) if Z ₁ > 1/k, t ₄ > (T + 1/k) / 2, Z ₂ > (T + Z ₁) / 2 (8a) if Z ₁ > 1/k, t ₄ > (T + 1/k) / 2, Z ₂ ≤ (T + Z ₁) / 2	(5)	(6)	
≥ 2	Z _L ≥ t ₂					
≥ 2	Z _L < t ₂	(1)	(1) if Z ₁ ≤ 1/k, Z ₂ - Z ₁ > (T - 1/k) / 2 (3) if Z ₁ ≤ 1/k, Z ₂ - Z ₁ ≤ (T - 1/k) / 2; or if Z ₁ > 1/k, t ₄ ≤ (T + 1/k) / 2 (7a) if Z ₁ > 1/k, t ₄ > (T + 1/k) / 2, Z ₂ > (T + Z ₁) / 2 (8a) if Z ₁ > 1/k, t ₄ > (T + 1/k) / 2, Z ₂ ≤ (T + Z ₁) / 2	(2)	(6)	
0	$(-1)^P Y_2 > (-1)^P Y_1 $	(1)	(3) if B ₂ ≤ 1/k; or if B ₂ > 1/k, t ₄ ≤ (T + 1/k) / 2 (8a) if B ₂ > 1/k, t ₄ > (T + 1/k) / 2	(2)	(6)	
T Condition		0 < T < 1/k	1/k < T < 2t ₁ + 1/k		2t ₁ + 1/k < T < t ₂ + 2/k	T > t ₂ + 2/k
			1/k < T < Z _L + 2/k			

Table 3.4. Final Classifications

NOTE: (M) = Case No. M

L \ T	Condition	T			
		$0 < T < 2t_1 + 1/k$	$2t_1 + 1/k < T < t_2 + 2/k$	$t_2 + 2/k < T < t_3 + 2/k$	$T > t_3 + 2/k$
0	$(-1)^p y_2 < (-1)^p y_1 $	(3)	(2)	(6)	(9)
≥ 2	Classes 8, 9, 11 & $Z_L < B_2$	(1) if $Z_L - Z_{L-1} > (T-1/k)/2$			
≥ 2	Class 10 & $Z_L < \alpha_1$	(5) if $Z_L - Z_{L-1} \leq (T-1/k)/2$	(2)	(6)	(9)
1	Classes 7, 8, 9 & $Z_L \leq 1/k$	(1)	(1) if $Z_L > (T-1/k)/2$ (2) if $Z_L \leq (T-1/k)/2$	(6)	(9)
1	Classes 7, 8 & $Z_L > 1/k$	(5)		(6)	(9)
1	Class 10 & $Z_L > \alpha_1$				
≥ 2	Class 10 & $\alpha_1 \leq Z_L \leq t_4$				
≥ 2	$Z_L \geq \gamma_p$ & $T > t_3 + 2/k$; or $Z_L \geq t_4$ & $T \leq t_3 + 2/k$	(7) if $Z_L > (T+Z_{L-1})/2$		(10) → (11)	(11)
1	Class 9 & $Z_L > 1/k$	(8) if $Z_L \leq (T+Z_{L-1})/2$			
1	Class 11 & $Z_L > \alpha_1$				
≥ 2	$t_4 \leq Z_L < \gamma_p$ & $T > t_3 + 2/k$; or $Z_L \geq t_4$ & $T \leq t_3 + 2/k$	(7) if $Z_L > (T+Z_{L-1})/2$; (8) if $Z_L \leq (T+Z_{L-1})/2$		(11)	
0	$(-1)^p y_2 > (-1)^p y_1 $	(8)		(11)	
1	Classes 10, 11 & $Z_L < \alpha_1$				
L	Condition	$0 < T < Z_L + 2/k$			$T > Z_L + 2/k$
		$0 < T < \gamma_p + 2/k$			$T > \gamma_p + 2/k$

Table 3.5. Final Classifications

NOTE: (M) = Case No. M

Condition \ T		0 < T < t ₁ +2/k		t ₁ +2/k < T < t ₂ +2/k		T > t ₂ +2/k	
		0 < T < Z _L +2/k			Z _L +2/k < T < t ₂ +2/k		
0	y _{p-1} < y _p	(15)		(13)		(12)	
≠	B _j < Z _L < B _{j+1} , p-j=odd no., j=0,1,...,p-1, if c ₅ >0 or j=0,1,...,p-2, if c ₅ ≤0						
≠	c ₅ <0, y _{p-1} >Z _L >B _{p-1}	(14) if Z _L >(T+Z _{L-1})/2 (15) if Z _L ≤(T+Z _{L-1})/2		(13)		(12)	
≠	c ₅ <0, y _{p-1} <Z _L <B _p	(16)		(13)		(12)	
≠	c ₅ <0, Z _L >y _p , T>t ₂ +2/k; or c ₅ <0, Z _L >B _p , T≤t ₂ +2/k	(20) if Z _L >(T+Z _{L-1})/2 (19) if Z _L ≤(T+Z _{L-1})/2				(17)	(18)
≠	c ₅ >0, Z _L >y _p						
≠	B _p <Z _L <y _p , T>t ₂ +2/k; or Z _L >B _p , T≤t ₂ +2/k	(20) if Z _L >(T+Z _{L-1})/2 (19) if Z _L ≤(T+Z _{L-1})/2				(18)	
0	y _{p-1} > y _p	(19)				(18)	
≠	B _j < Z _L < B _{j+1} , p-j=even no., j=0,1,...,p-2						
Condition \ T	T	0 < T < Z _L + 2/k				T > Z _L +2/k	
		0 < T < y _p + 2/k				T > y _p + 2/k	

the applicable case for this example is labeled as Case 5. Notice that in this particular example, the values of the parameters t_i , $i=1,2,3,4$, were not needed in the classification of Table 3.3. The values of the unknown parameters are now found by referring to Case 5 in Table 3.6.

In Table 3.6, Case 5 has three subcases. The proper choice for this example is "subcase b", which was obtained from Table 3.2. The results are that $N = 1$, $\tau_1 = \alpha_2$, $\tau_2 = Z_L$ and $\rho(T^-) = f(T)$. These results are equivalent to those derived in the illustrative example (see equations (3.73) and (3.75)).

The preceding paragraphs give a brief description of how the various forms of $f(t)$ on $t \in [0, T]$ are classified and their corresponding τ_j 's are grouped and classified. The construction of this set of six tables involves tedious but straightforward analysis of the various forms of $f(t)$ on $t \in [0, T]$. That is, conditions (A) through (D) of Section 3.6 are applied to all 15 mutually exclusive classes and their associated sub-classes to obtain the values of the unknown parameters N , τ_j , $j=1,2,\dots,2N$ and $\rho(T^-)$. The results are then included in the appropriate table, i.e., one of the Tables 3.3 through 3.6.

The general step by step procedure to use the Tables is as follows:

- (I) Locate the specific class of $f(t)$ from Table 3.2 by noting the number of extrema of $f(t)$ and by calculating the values of the appropriate parameters.
- (II) Once the class is specified, Table 3.2 gives the pertinent case-table, i.e., Table 3.3, 3.4 or 3.5, and subcase a, b, c, or none, along with the values of the parameters to be used with that specific table.
- (III) On that particular case-table locate the case number from the

Table 3.6. Values of the Parameters

Case No.	Sub-case	Values Of τ 's	N	$\rho(\tau^-)$
1	None	$\tau_0 = 0$	0	$f(T)$
2	None	$\tau_0 = 0$	0	$f(T) - f(T/2 - 1/2k)$
3	None	$\tau_0 = 0$	0	$F(T) - f(\alpha_3)$
4	a	$\tau_1 = 1/k, \tau_2 = T$	1	0
4	b	$\tau_1 = \alpha_2, \tau_2 = T$	1	0
4	c	$\tau_1 = 1/k, \tau_2 = \gamma_1 - 2/k, \tau_3 = \gamma_1, \tau_4 = T$	2	0
5	a	$\tau_1 = 1/k, \tau_2 = Z_L$	1	$f(T)$
5	b	$\tau_1 = \alpha_2, \tau_2 = Z_L$	1	$f(T)$
5	c	$\tau_1 = 1/k, \tau_2 = \gamma_1 - 2/k, \tau_3 = \gamma_1, \tau_4 = Z_L$	2	$f(T)$
6	a	$\tau_1 = 1/k, \tau_2 = T - 2/k$	1	$f(T) - f(T - 2/k)$
6	b	$\tau_1 = \alpha_2, \tau_2 = T - 2/k$	1	$f(T) - f(T - 2/k)$
6	c	$\tau_1 = 1/k, \tau_2 = \gamma_1 - 2/k, \tau_3 = \gamma_1, \tau_4 = T - 2/k$	2	$f(T) - f(T - 2/k)$
7	a	$\tau_1 = 1/k, \tau_2 = Z_{L-1}$	1	$f(T)$
7	b	$\tau_1 = \alpha_2, \tau_2 = Z_{L-1}$	1	$f(T)$
8	a	$\tau_1 = 1/k, \tau_2 = \alpha_4$	1	$f(T) - f(\alpha_4)$
8	b	$\tau_1 = \alpha_2, \tau_2 = \alpha_4$	1	$f(T) - f(\alpha_4)$
9	a	$\tau_1 = 1/k, \tau_2 = \gamma_p - 2/k, \tau_3 = \gamma_p, \tau_4 = T$	2	0
9	b	$\tau_1 = \alpha_2, \tau_2 = \gamma_2 - 2/k, \tau_3 = \gamma_2, \tau_4 = T$	2	0
10	a	$\tau_1 = 1/k, \tau_2 = \gamma_p - 2/k, \tau_3 = \gamma_p, \tau_4 = Z_L$	2	$f(T)$
10	b	$\tau_1 = \alpha_2, \tau_2 = \gamma_2 - 2/k, \tau_3 = \gamma_2, \tau_4 = Z_L$	2	$f(T)$
11	a	$\tau_1 = 1/k, \tau_2 = \gamma_p - 2/k, \tau_3 = \gamma_p, \tau_4 = T - 2/k$	2	$f(T) - f(T - 2/k)$
11	b	$\tau_1 = \alpha_2, \tau_2 = \gamma_2 - 2/k, \tau_3 = \gamma_2, \tau_4 = T - 2/k$	2	$f(T) - f(T - 2/k)$
12	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = \gamma_p - 2/k, \tau_{r+3} = \gamma_p, \tau_{r+4} = T$	$(r+4)/2$	0
13	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = T - 2/k$	$(r+2)/2$	$f(T) - f(T - 2/k)$
14	None	$\tau_r = Z_{L-1}$	$r/2$	$f(T)$
15	None	$\tau_r = \alpha_5$	$r/2$	$f(T) - f(\alpha_5)$
16	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = Z_L$	$(r+2)/2$	$f(T)$
17	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = \gamma_p - 2/k, \tau_{r+3} = \gamma_p, \tau_{r+4} = Z_L$	$(r+4)/2$	$f(T)$
18	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = \gamma_p - 2/k, \tau_{r+3} = \gamma_p, \tau_{r+4} = T - 2/k$	$(r+4)/2$	$f(T) - f(T - 2/k)$
19	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = \alpha_4$	$(r+2)/2$	$f(T) - f(\alpha_4)$
20	None	$\tau_r = \gamma_{p-1} - 2/k, \tau_{r+1} = \gamma_{p-1}, \tau_{r+2} = Z_{L-1}$	$(r+2)/2$	$f(T)$

value of T and the values of the specified parameters. If T is such that two cases are applicable, then use either case. Finally, for each case number use Table 3.6 to obtain the values of the unknown parameters $N, \tau_j, j=1,2,\dots,2N$ and $\rho(T^-)$.

Note 1 If the case number of Table 3.6 has more than one set of parameters, use Table 3.2 to locate the specific subcase.

Note 2 If classes 12 through 15 are applicable, then Table 3.2 also gives the value of $r, r < 2N$, and $\tau_j, j=1,2,\dots,r$. The remaining parameters are obtained as explained previously.

In the preceding paragraphs, the procedure of constructing an extremal control for a fixed T and a given $\eta = -\dot{V}(t_0)$ was presented. The results were given in the form of six tables. The procedure of using the tables to obtain the extremal control for a given function $f(t)$ was also explained. The major difficulties with the results in the form of tables are fairly obvious. First of all the general procedure of using the tables is both complex and time-consuming. Also, the parameters in Table 3.1 are in some cases implicitly defined, and therefore their calculation may not be a simple task. Finally, in many cases the function $f(t)$, which is related to the $(n+1)$ -th component of the adjoint solution, may only be known numerically which essentially makes the use of the tables impractical. In order to overcome the above mentioned difficulties, the complete process of selecting the extremal control for any given linear system has been incorporated into a digital computer program in Fortran IV. This program is discussed in the following section.

3.9 Computer Program

[see Appendix C for listing]

The basic operation of the program is closely allied to the procedural steps (I) through (III) describing the use of the tables, which was outlined in Section 3.8. The user must supply the original system matrices $E(t)$ and $F(t)$ as shown in equation (2.1), the initial and terminal values of time, the initial value of the adjoint solution, i.e., $\Psi(t_0) = -\eta$ and the value of the rate bounds $h_1, h_1 > 0$, $i=1,2,\dots,m$. The program then calculates the adjoint solution $\Psi(t)$ on $t \in [t_0, T]$ by integrating the adjoint system equations $\dot{\Psi}(t) = -\Psi(t)A(t)$, with $\Psi(t_0) = -\eta$. If, as in some relatively simple systems, the adjoint solution is known analytically, the above integration may be bypassed.

The program now operates on the $(n+1)$ -th component of the adjoint solution to form the function $f(t)$ of system (3.8), along with the corresponding value of $k = h_1$. The program then follows the following steps in sequence.

(1) The extrema of $f(t)$, their corresponding time instants, and the zero crossings are determined on the interval $t \in [t_0, T]$. This process is accomplished by Subroutine PSICLC, shown in Appendix C. This subroutine operates by basically searching the time interval $[t_0, T]$, while monitoring the stored values of the function $f(t)$ for zero value times and for maximum and minimum times. When such a time is sensed, the subroutine locates the actual value of time by a linear interpolation scheme based on the stored values of the function $f(t)$. The results of this step are the parameters $p, L, B_i, i=1,2,\dots,p, Y_i, i=1,2,\dots,p$, and $Z_i, i=1,2,\dots,L$, which were defined in Section 3.8.

(2) The parameters of Table 3.1 that are needed in the classification of the function $f(t)$ are then found by three subroutines. The parameter α_1 is found by Subroutine TALC, the parameter α_6 is determined by Subroutine TBLC and the parameters α_2 and γ_{p-1} by Subroutine TLIC. Since all of the above parameters are defined in an implicit manner (Table 3.1), the above three subroutines all operate by first choosing an initial value for the unknown parameter, and then by checking the implicit relation for that particular parameter. For example, the implicit relation for the parameter α_1 , from Table 3.1, is given by $f(0)-f(\alpha_1) = 0$. If the implicit relation is not satisfied, then the parameter value is adjusted in a direction determined by the sign of the implicit relation. This process is repeated until the implicit relation changes sign, and then the actual value of the parameter is found by linear interpolation on the stored values of the function $f(t)$. The parameters C_i , $i=1,2,\dots,5$, (Table 3.1) are not implicitly defined and are therefore found directly from their definitions. Note that it is not necessary to compute all of the above parameters for each function $f(t)$. In fact, depending on the particular $f(t)$, some of the above parameters may not even be defined. The determining factors are the number of extrema, p , of $f(t)$ and the values of particular parameters. As an example, if $p = 0$, then from Table 3.2, no parameter calculation is necessary in the classification process. If $p = 1$ and if α_1 is calculated and it is found that $\alpha_1 \leq 1/k$ and also C_1 is found to be such that $C_1 \leq 0$, then the class is fully specified as class 2 of Table 3.1. Again, in this case no further parameter calculations are necessary for classification. The governing logic which underlies this classification process is performed in Subroutine EXTR, if $p \leq 1$ or in Subroutine EXTRP,

if $p > 1$.

(3) The function $f(t)$ is further classified into an appropriate "case" by a series of inequality tests based on the value of T , the number and values of the zero crossings, and the values of some of the parameters listed in Table 3.1. These inequalities are given in Tables 3.3 through 3.5. This operation is carried out in Subroutine EXTR or EXTRP, depending on the number of extrema of $f(t)$.

(4) Once the proper case for $f(t)$ on $t \in [t_0, T]$ is established, all the corresponding parameters listed in Table 3.6 are determined with the aid of Subroutines TALC, TBLC and TILC. The operation of these subroutines has already been explained.

(5) Based on the information obtained in Step 4, $p(t)$ on $t \in [t_0, T]$ is computed from equation (3.29) or equation (3.32).

(6) Finally $\bar{w}(t) = k \operatorname{sgn} [p(t)]$ is calculated on $t \in [t_0, T]$. The corresponding i -th component of the extremal control, $\bar{v}_i$ is found from the transformation equation (3.7).

Steps (1) through (6) are repeated for each $\bar{v}_{n+1}(t)$, $i=1,2,\dots,m$, until all m -components of the extremal control are found.

The program described above enables one to systematically obtain the extremal controls for a given linear system with amplitude and rate constraints. For a given interval of time $[t_0, T]$ and a given initial condition on the adjoint solution, $\eta = -\bar{Y}(t_0)$, the corresponding extremal control $\bar{v}(t)$ on $t \in [t_0, T]$ is obtained quite rapidly from the computer program because of its basically non-iterative nature. The actual time involved of course depends on the particular time interval under investigation and the particular form of the components of the adjoint solution $\bar{Y}(t)$. As a very rough figure, it has been found that

the typical computer time required for Steps (1) through (6) is about .02 sec(CDC-6500) for a function $f(t)$ with about twelve extrema on the given time interval. A function $f(t)$ having fewer extrema would require correspondingly shorter computer time. The reason is that more switches occur in the extremal control for a function with many extrema than does in the case of a function with few extrema. Thus there are correspondingly fewer unknowns, τ_j , $j=1,2,\dots,2N$, in the latter case and the computation time is thereby reduced. It has also been found that the effect of the length of the time interval $[t_0, T]$ on the computation time, can be considerably lessened. This is accomplished by time-scaling the problem within the program itself through the use of a built-in scale factor option. Of course there is a limit to the allowed contraction of the real time interval $[t_0, T]$. As the time scale is contracted, accuracy is lost in the determination of the parameters in Table 3.1 and also in the time instants of the extrema and zero crossings of the function $f(t)$.

In general, the program is able to determine a large number of extremal controls in a sufficiently short amount of time. This property is essential to the practical development of the sub-optimal closed-loop method established in Chapter IV. Also, the above property is necessary for the calculation of the open-loop minimal time solution. This will be discussed in the following section.

3.10 Open-Loop Minimal Time Control

The method of constructing the extremal controls has been shown in the preceding sections. Additional effort is necessary when the minimal time open-loop control for a given initial condition is desired. The

discussion in Sections 3.2 and 3.3 has shown that the only candidates for minimal time controls are extremal controls. The method of determining the open-loop minimal time control from these extremal controls is described as follows. Suppose the initial condition of system (2.6) is given by $x(t_0) = x_0$. For the given x_0 , estimate the time length required to drive x_0 to $x(T) = 0$, i.e., the origin. Now choose $\eta = -\dot{V}(t_0)$ and construct the extremal control $\bar{v}(t)$ on $t \in [t_0, T]$. With the aid of the expression for the set of recoverability, equation (2.10), compute the initial condition $\bar{x}_0$ that corresponds to the calculated extremal control $\bar{v}$. If this computed $\bar{x}_0$, with a minimum possible estimated time T , is equal to the given value of the initial condition, $x(t_0) = x_0$, then the constructed extremal control $\bar{v}(t)$ is indeed the open-loop minimal time control $v^*(t)$ on $t \in [t_0, T]$. Otherwise, change the value of η and/or the value of T and then repeat the procedure.

Although the above procedure may require many runs in order to determine the correct value for η and T , the process is not overly time consuming. The reason is that the computer program is able to calculate each extremal control in a very short amount of time. The general procedure of finding the minimal time control will be illustrated by the example in the next section.

3.11 An Example-Ranger Block III Attitude Control System

The Ranger Block III Attitude Control System [4] uses jet-vanes to position the vehicle so that the thrust vector of the midcourse motor is pointed through the vehicle's center of gravity. The equations of motion for the pitch mode can be approximated by [4, p. 46]

$$s^2 \theta \left\{ 109 \left[\frac{s^2}{(1.037)(18.1)^2} + 1 \right] \right\} = q \left[\frac{s^2}{(18.1)^2} + 1 \right], \quad (3.78)$$

where θ = deviation from nominal pitch angle in radians

and q = applied torque to the pitch axis in ft - lbs.

Because of the physical constraints on the jet-vanes, the magnitude and the rate of the torque are bounded by

$$|q| \leq 6.69 \text{ ft - lbs.}, \text{ and} \quad (3.79)$$

$$|\dot{q}| \leq 484 \frac{\text{ft - lbs.}}{\text{sec.}} \quad (3.80)$$

The bound on q given in equation (3.79) is introduced by the mechanical stop with a recommended 85% safety factor [4, p. 66, $7.87 \times 85\%$], and the bound on $\dot{q}$ in equation (3.80) is taken from the maximum slope of the $q(t)$ vs. t curve [4, p. 68]. Since the vehicle accumulates errors as long as the pitch angle deviates from its nominal position, it is desirable to nullify an initial given deviation in the least possible time. Also, since the torque inputs must satisfy the constraints, equations (3.79) and (3.80), the basic problem can be formulated as a minimal time problem with control (torque) amplitude and rate saturations. Thus, for a given initial pitch angle deviation $\theta(0) = \theta_0$, the problem is to find the open-loop control torque $q(t)$ on $t \in [0, T]$ that drives system (3.78) to $\theta(T) = 0$, where T is minimal. Note that in the above formulation the initial time was taken to be zero, i.e., $t_0 = 0$. There is no loss of generality, since system (3.78) is autonomous. It is also assumed, as discussed in Chapter II, that $q(T) = 0$ and that $q(0) = 0$. The terminal time condition has been added in order

to avoid the necessity of disengaging the controller after each control period. The initial condition on the torque is based on the physical reasoning that the control periods are sequential, so that $q(0)=q(T)=0$.

In order to apply the extremal control construction method developed earlier, system (3.78) along with the constraints, equations (3.79) and (3.80), must be reformulated as a bounded state variable problem in the form given by system (2.6), i.e.,

$$\dot{x}(t) = A(t)x(t) + Bv(t) + c(t) \quad (3.81)$$

This is accomplished by using the "direct programming" method which yields the 5-th order system of differential equations in the form given by equation (3.81) with

$$A(t)=A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -339.73 & 0 & 0.0636 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, c(t)=c=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (3.82)$$

with the constraints

$$|x_5| = |u| = |q/6.69| \leq 1 \quad (3.83)$$

and

$$|\dot{x}_5| = |\dot{v}| \leq 484/6.69 = 72.35, \quad (3.84)$$

The transformation is given by $\theta = 327.6x_1 + x_3$, $\dot{\theta} = 327.6x_2 + x_4$, $\ddot{\theta} = -12.1x_3 + 0.0636x_5$, $\ddot{\theta} = -12.1x_4 + 0.0636v$, $q = 6.69x_5$ and $\dot{q} = 6.69v$.

In the above system of equations, the vector c is a zero vector. This occurs, because in this particular physical problem the terminal condition is already at the origin and therefore no transformation of the x -coordinate system is necessary. The minimal time control problem in terms of the transformed system is to choose a control $v(t)$ on $t \in [0, T]$ that drives system (3.81) from $x(0) = x_0$ (x_0 is related to θ_0 through the transformation equations) to $x(T) = 0$, where T is minimal. It is also required that the constraints, equations (3.83) and (3.84), be satisfied and that $x_5(0) = u(0) = 0$.

The problem will now be solved by using the method developed in this chapter. For a given initial condition $\theta(0) = \theta_0$ and $q(0) = 0$, the required time T to drive θ_0 to $\theta(T) = 0$ can be estimated. For this fixed T and a fixed $\eta = -\Psi(0)$, the adjoint solution $\Psi(t)$ on $t \in [0, T]$ for system (3.81) can be found from the equation $\dot{\Psi}(t) = -\Psi(t)A$. The modified adjoint solution $p(t)$ is then found by the method developed in this chapter. The extremal control $\bar{v}(t)$ on $t \in [0, T]$ is obtained from $\bar{v}(t) = 72.35 \operatorname{sgn} [p(t)]$ through the transformation equation (3.7c). Finally, with the aid of the definition of the set of recoverability, equation (2.10), the initial condition $\bar{x}_0$, which corresponds to the extremal control $\bar{v}(t)$, can be calculated. The corresponding value of $\bar{\theta}_0$ and $\bar{q}(0)$ is then determined by the transformation equations. If $\bar{\theta}_0 = \theta_0$ and $\bar{q}(0) = q(0) = 0$, then $\bar{v}(t)$ is the minimal time control for the initial condition $\bar{x}_0$ and correspondingly $q^*(t) = 6.69\bar{x}_5(t)$ is the minimal time torque control for the initial condition $\theta(0) = \theta_0$ with $q^*(0) = 0$. If the initial conditions are not equal, then the process is repeated with a different $\eta = -\Psi(0)$ and/or T . The above procedure is best illustrated by a specific initial condition, which is described

below.

Consider the initial condition $\theta(0) = 21.17$ milli-radians, $\dot{\theta}(0) = -33.02$ m-rad/sec, $\ddot{\theta}(0) = 5.31$ m-rad/sec² and $\ddot{\ddot{\theta}}(0) = -2.86$ rad/sec³. Of course the initial condition on the torque is $q(0) = 0$. For an initial rough estimation of the required time length, neglect the higher order terms in system (3.78). Then we obtain $109\ddot{\theta} \approx q$, or $\ddot{\theta} \approx q/109 = 0.03065$, where an average value of $q = 6.69/2$ has been used for the control torque. Thus, the approximate time duration needed to drive θ from 21.17 m-rad to zero m-rad is given by $\sqrt{2\theta/\ddot{\theta}} = \sqrt{2(0.02117)/(0.03065)} = 1.38$ seconds. This gross estimate is employed as a guide in choosing the value of T .

For the purposes of illustration the analytical solution of the adjoint system will be established for the example. In actual practice, the computer program could integrate the adjoint equations with $\Psi(0) = -\eta$ to find the numerical values of $\Psi(t)$ on $t \in [0, T]$. Since, in this example $m = 1$, i.e., scalar control, there is only one component of the adjoint solution to consider, viz, $\Psi_{n+1}(t) = \Psi_5(t)$. In terms of the initial condition $\Psi(0) = -\eta$, the analytical expression for $\Psi_5(t)$ for $t \geq 0$ is found to be given by

$$\begin{aligned} \Psi_5(t) = & -\eta_1 \gamma \alpha^{-4} [(\alpha t)^2/2 - 1 + \cos \alpha t] + \eta_2 \gamma \alpha^{-3} [\alpha t - \sin \alpha t] \\ & - \eta_3 \gamma \alpha^{-2} [1 - \cos \alpha t] + \eta_4 \gamma \alpha^{-1} [\sin \alpha t] - \eta_5 \end{aligned} \quad (3.85)$$

where $\alpha = 18.43$ and $\gamma = 0.0636$. Let

$$\begin{aligned} \eta = & -\cos \phi_4 [\cos \phi_1 \cos \phi_2 \cos \phi_3, \sin \phi_1 \cos \phi_2 \cos \phi_3, \sin \phi_2 \cos \phi_3, \\ & \sin \phi_3, \tan \phi_4] \end{aligned} \quad (3.86)$$

where $-\pi < \phi_1 \leq \pi$ and $|\phi_j| \leq \pi/2$, $j=2,3,4$. Then by choosing the proper values of ϕ_k , $k=1,2,3,4$, every unit vector in R^5 can be represented by equation (3.86).

For a particular choice of η , let $\phi_1 = 2.043$ radians, $\phi_2 = 1.359 \times 10^{-2}$ rad., $\phi_3 = -1.987 \times 10^{-3}$ rad. and $\phi_4 = -1.739 \times 10^{-3}$ rad. Then by equation (3.86),

$$\eta = [0.95569, -0.29406, -0.01359, 0.00199, 0.00174] \quad (3.87)$$

The numerical expression for $\Psi_5(t)$ is found by substituting the value for η in equation (3.87) into the expression for $\Psi_5(t)$, equation (3.85). Then, by using the transformation equation (3.76), $f(t)$ is given by

$$\begin{aligned} f(t) = 7.836 \times 10^{-3} \{ & -0.06485 [(\alpha(T-t)/2)^2 - 1 + \cos\alpha(T-t)] \\ & -0.3678 [\alpha(T-t) - \sin\alpha(T-t)] + 0.31338 [1 - \cos\alpha(T-t)] \\ & + 0.84443 [\sin\alpha(T-t)] - 0.22191 \} \end{aligned} \quad (3.88)$$

where the values of the parameters α and γ are given in equation (3.85). Let $T = 0.97115$ seconds. With these choices for η and T , the computer program yields the following values for the parameters τ_i , $i=1,2,\dots,2N$, and N in the modified adjoint solution $\rho(t)$: $N = 6$, $\tau_1 = 0.01382$ (seconds), $\tau_2 = 0.24584$, $\tau_3 = 0.27348$, $\tau_4 = 0.27813$, $\tau_5 = 0.30577$, $\tau_6 = 0.55506$, $\tau_7 = 0.5827$, $\tau_8 = 0.65142$, $\tau_9 = 0.67906$, $\tau_{10} = 0.87626$, $\tau_{11} = 0.9039$, and $\tau_{12} = 0.95732$. Since $\tau_{12} = \tau_{2N} < T$, then in this case, by equation (3.30), $\rho(T^-) = f(T) - f(\tau_{12})$. The corresponding functions $\bar{w}(t) = 72.35 \operatorname{sgn}[\rho(t)]$ and $\bar{\xi}_1(t)$ have been determined by the computer program and are plotted along with the function $f(t)$ in

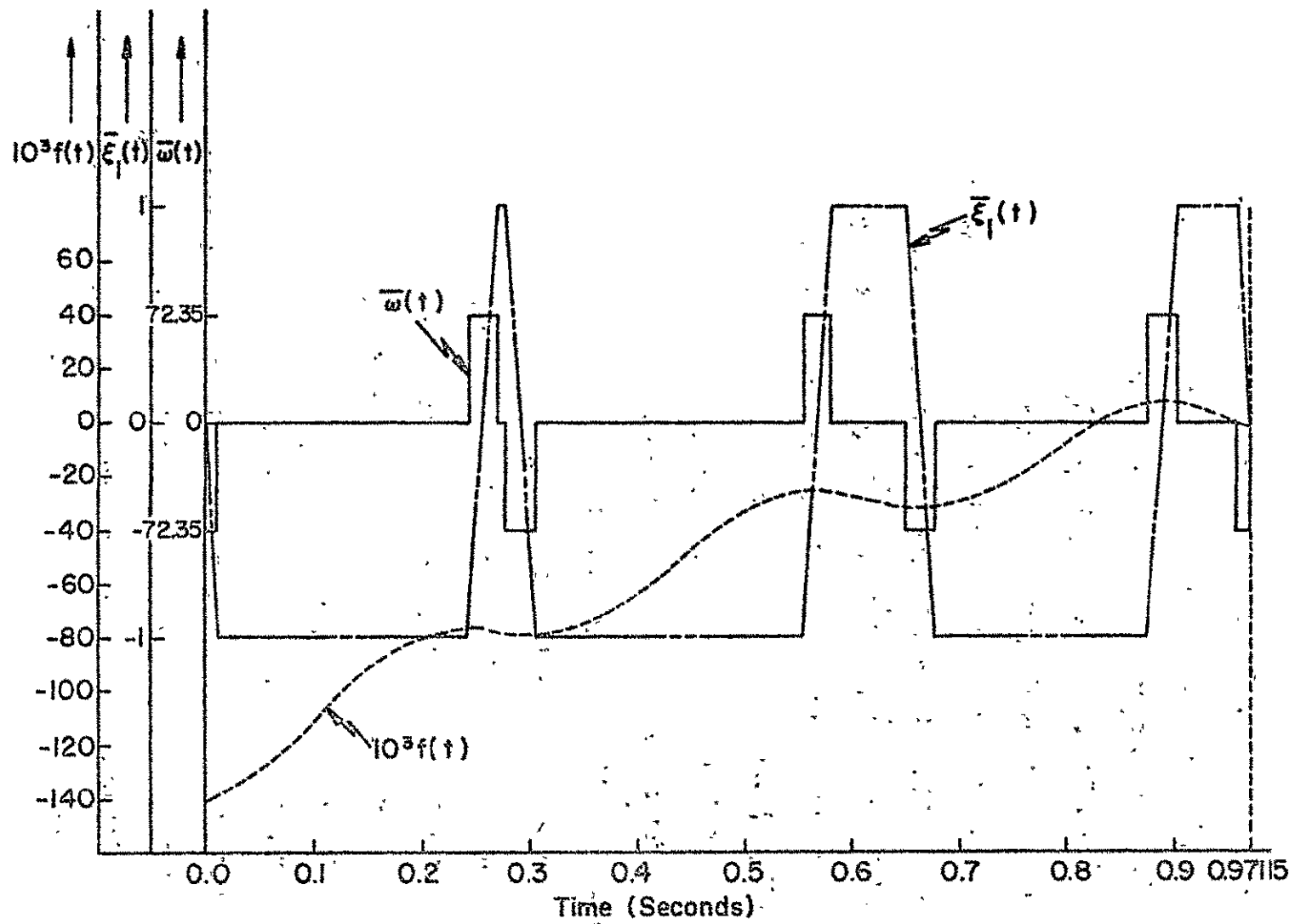


FIGURE 3.5. CONSTRUCTED $\bar{\omega}(t)$.

Figure 3.5. From the function $\bar{w}(t)$, the resulting numerical expression for the extremal control $\bar{v}(t)$ on $t \in [0, 0.97115]$ has also been determined. Finally, the numerical values of the initial conditions $\bar{q}(0)$ and $\bar{\theta}_0$ have been found for this extremal control. The obtained results indicate that $\bar{\theta}_0 = \theta_0$ and $\bar{q}(0) = q(0) = 0$, i.e., the initial conditions are equivalent. Therefore $\bar{q}(t) = 6.69x_5(t)$ is the minimal time control torque input for the given set of initial conditions, and $T = 0.97115$ is the minimal time. The results of this problem are shown in Figures 3.6 and 3.7. Figure 3.6 shows the minimal time responses of the derivatives, i.e., $\dot{\theta}^*(t)$, $\ddot{\theta}^*(t)$ and $\dddot{\theta}^*(t)$. Note that $\dddot{\theta}^*(t)$ is piecewise continuous since it is directly related to the rate of change of control. In Figure 3.7 the minimal time pitch deviation response, i.e., $\theta^*(t)$, is plotted along with the minimal time control torque $q^*(t) = \bar{q}(t)$. Also, for comparison purposes, the response using the proportional plus derivative feedback control system [4, p. 70] presently in use on the Ranger (for $\theta(0) = 0.7$ m-rad), is also plotted in Figure 3.7. From this figure it is seen that the open-loop minimal time control computed via the method presented in this chapter achieves the same control goal in approximately one-eighth of the time required by the presently used controller.

The above numerical example provides typical values for the relative efficiency of the open-loop optimal control and the presently used feedback controller. The minimal time controls for other choices of initial conditions have also been found and the results are of the same order.

In the following section, this chapter will be concluded with some general remarks.

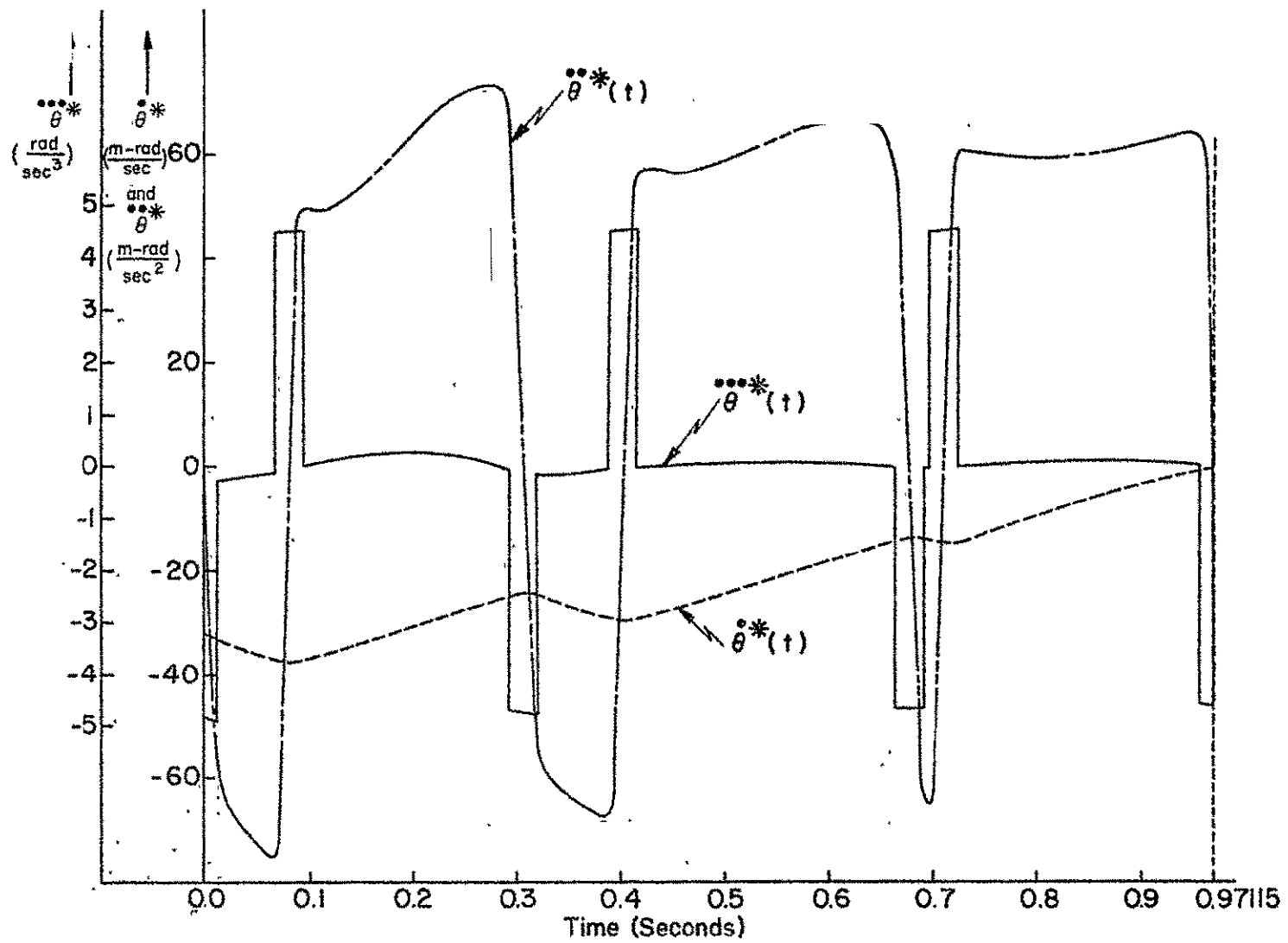


FIGURE 3.6. MINIMAL TIME RESPONSES $\dot{\theta}^*(t)$, $\ddot{\theta}^*(t)$, AND $\theta^*(t)$.

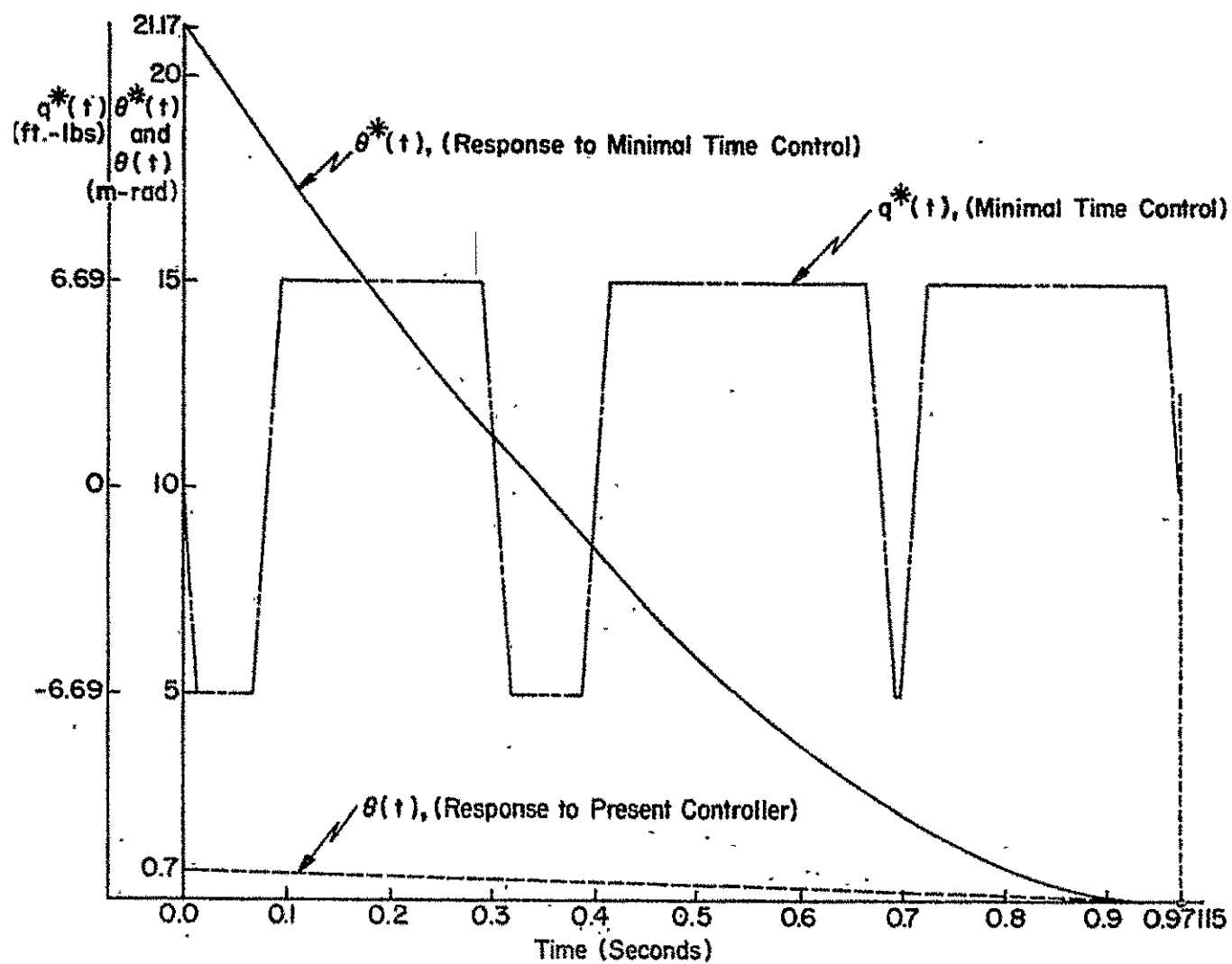


FIGURE 3.7. MINIMAL TIME TORQUE ($q^*(t)$) PITCH DEVIATION ($\theta^*(t)$) AND RESPONSE ($\theta(t)$) TO PRESENT CONTROLLER.

3.12 General Remarks

The calculation of the extremal controls for some specific systems has been carried out in detail, (for example see [21, 23, 24]). In Reference 23, the construction of the extremal controls, for a second order oscillator with the control amplitude bounded by unity and the control rate bounded by $2.5/\pi$, is discussed. In [24] the results are extended to general rate bounds. In [21], the same system is considered with two control inputs. Reference 24 also considers a second order unstable booster. The extremal controls are constructed for this problem and the results are shown to be equivalent to those derived by a different method.

Since all of the above results are special cases of the results presented in this chapter, they will be omitted in this study.

It should be also pointed out that the general formulation of the open-loop problem may be altered somewhat, without changing the basic results of this chapter. For example, in [26] the problem is formulated with the initial control zero and the terminal control value free. The basic construction procedure, however, is identical to that developed in this chapter. The major difference is that the extremal controls can now be obtained directly from the tables without the transformations, equations (3.7a) through (3.7d).

In the following chapter, a sub-optimal closed loop solution will be developed. This solution is based on the results of this chapter.

CHAPTER IV

A SUB-OPTIMAL CLOSED-LOOP SOLUTION

4.1 Introduction

In this chapter a method will be developed for finding a sub-optimal closed-loop solution for the class of systems described by equation (2.11). The method will be based on the results of the minimal time open-loop solution of the previous chapter.

The term sub-optimal can essentially be applied to any solution which is not optimal. Therefore in order to motivate the development of a sub-optimal closed-loop solution for system (2.11), the following set of criteria is established.

(1) The sub-optimal solution should be related in some manner to the optimal solution. In other words, there should be a direct line of reasoning that leads from the unsolved minimal time closed-loop problem of Section 2.4, to the sub-optimal solution.

(2) The sub-optimal method of solution should be relatively straightforward in application. The basic method should be general, so that it can be applied to a class of systems. This eliminates any methods that are only applicable to specific systems.

(3) The resulting controller configuration should be simple enough so that it may actually be implemented. The basic objective is an engineering solution which is realizable. Ideally, the loss of optimal performance should be balanced by a lower order of complexity of the sub-optimal solution.

(4) The sub-optimal method of solution should be applicable to high order practical systems. Again, this is necessary from the point of view of engineering applications.

These four criteria serve as a guide in the development of the sub-optimal solution. In the next section a sub-optimal closed-loop method of solution, based on approximating the isochrones of system (2.11), is introduced. The various aspects of the method are investigated in detail in the later sections.

4.2 Introduction of a Sub-Optimal Method of Solution

The isochrone $I(T)$, $T \geq 0$, of system (2.11), is defined to be the set of state vectors x that can be driven to the origin in a time T , using minimal time controls. Suppose that a scalar function $W(x)$ could be found that describes the isochrones $I(T)$, for all times $T \geq 0$. That is, the state $\tilde{x}$ belongs to the isochrone $I(\tilde{T})$, if and only if $W(\tilde{x}) = \tilde{T}$. Then the isochrones could be described by,

$$I(T) = \{x | W(x) = T, \quad T \geq 0\} \quad (4.1)$$

Note that the function $W(x)$ is equivalent to the minimum return function of the method of Dynamic Programming [27]. From the definition of the function $W(x)$, the minimal time closed-loop controller could be determined by the following method: At all state vectors x where the derivative dW/dt exists, choose the control that minimizes dW/dt , subject to the constraints of equation (2.12).

As discussed in Chapter I, the above problem is not solved. However it does provide the motivation for a sub-optimal approach, which is based on the following assumption. System (2.11) is the $(n+m)$ -dimensional

reformulated version of an original linear autonomous n -dimensional system. The original n -dimensional state vector y is subject to disturbances, which cause the state vector y to be non-zero. It is assumed that for the systems under consideration, the disturbed state vectors y will always belong to some prescribed bounded set $\tilde{M}$. That is, it is only required to control a bounded subset of the entire n -dimensional state space. This assumption will be valid in most practical systems since the disturbances will normally be bounded. Also, in these systems the components of the state vectors usually represent physical variables (e.g., angles, velocities, etc.), and are therefore necessarily bounded in value.

With the above assumption satisfied, the related $(n+m)$ -dimensional state vectors x of the reformulated system (2.11) will always belong to some bounded maximal set $M \subset \hat{G}$. Then the sub-optimal controller need only operate on the state vectors x that belong to this set M .

The basic method of obtaining a sub-optimal closed-loop solution is established in the following seven steps. These steps are presented here for the purpose of introducing the sub-optimal method. A detailed discussion of the actual procedure is given in the later sections.

(1) The maximal set M is partitioned by constructing a sequence of isochrones $I(T_k)$, $k = 1, 2, \dots, N$, such that the set,

$$L(T_1, T_N) = \bigcup_{T_1 \leq T \leq T_N} I(T), \quad (4.2)$$

essentially covers the entire maximal set M . This is done to insure that the later approximations will be valid for all of the state vectors $x \in M$.

(2) Each isochrone $I(T_k)$, $k = 1, 2, \dots, N$, is then approximated by the equation

$$V(x, T_k) = 1/2[x'C_k x] = T_k, \quad k = 1, 2, \dots, N, \quad (4.3)$$

where $C_k = C_k(T_k)$ is a positive definite $(n+m)$ by $(n+m)$ symmetric matrix whose elements are bounded for all $T_k \geq 0$, and x' denotes the transpose of x .

(3) The corresponding elements $c_{ij}(T_k)$ of each matrix C_k , $k = 1, 2, \dots, N$, are then approximated by continuous functions. The resulting approximation of the isochrones is given by,

$$V(x, T) = 1/2[x'C(T)x] = T \quad (4.4)$$

Steps (1) through (3) are done off-line for each given system. Since equation (4.4) is an approximation of the isochrones of the set M , then from equation (4.1), the function $V(x, T)$ approximates the previously defined scalar function $W(x)$, on the set M . Thus the problem of minimizing the control time is equivalent to the problem of minimizing $\dot{V}(x, T)$. This is accomplished by steps (4) and (5), which are both done on-line.

(4) At the beginning of the control interval, the value of the parameter T is found by iteratively solving the nonlinear equation (4.4).

(5) Once the value of T is known, the sub-optimal closed-loop controller $\bar{v}(x)$ is chosen to minimize $\dot{V}(x, T)$, i.e.,

$$\begin{aligned} \dot{V}(x, T) \Big|_{v=\bar{v}(x)} &= \min_{v \in J} [\dot{V}(x, T)] = \min_{v \in J} [(vV)x] \\ &= (vV)Ax + \min_{v \in J} [(vV)Bv] \end{aligned} \quad (4.5)$$

where $vV = [\partial V/\partial x_1, \partial V/\partial x_2, \dots, \partial V/\partial x_{n+m}]$ and the set J , defined by

equation (2.12), is the set of all admissible controllers. From the form of the matrix B of system (2.11), equation (4.5) becomes,

$$\dot{V}(x, T) \Big|_{v=\bar{v}(x)} = (vV)Ax + \min_{v \in J} \sum_{i=1}^m (\partial V / \partial x_{n+1}) v_i \quad (4.6)$$

Then from the definitions of the sets J and $\hat{G}$, the resulting sub-optimal controller is given by the following expression.

For each i, $i = 1, 2, \dots, m$,

$$\bar{v}_i(x) = -\text{sgn}[\partial V / \partial x_{n+1}], \text{ if } |x_{n+1}| < 1; \quad (4.7a)$$

or if $|x_{n+1}| = 1$, then

$$\bar{v}_i(x) = \begin{cases} 0 & , \text{ if } x_{n+1} \text{sgn}[\partial V / \partial x_{n+1}] \geq 0, \\ -\text{sgn}[\partial V / \partial x_{n+1}], & \text{ if } x_{n+1} \text{sgn}[\partial V / \partial x_{n+1}] < 0. \end{cases} \quad (4.7b)$$

(6) Since the value of the parameter T must be found on-line, it is not feasible to continually solve equation (4.4) for T at every time instant of the control interval. An alternate method may be obtained by noting that the parameter T represents the minimal time to the origin, for the particular state x. Suppose that the value of $T = \tilde{T}$ is found from Step (4), at some real time $t = \tilde{t}$. Since the sub-optimal controller, given by equation (4.7), approximates the performance of the minimal time controller, then the equation,

$$T = \tilde{T} + (t - \tilde{t}), \quad T \geq T_1, \quad (4.8)$$

serves as a simpler method for approximating the value of the parameter T. Of course equation (4.8) is not exact, since the sub-optimal

controller is not acting optimally and since it is possible for another disturbance to occur during the control interval. Also, equation (4.8) should only be used if the resulting parameter T is not less than T_1 . Thus it may be required to repeat Step (4) a number of times on each control interval. A criterion for deciding when to recalculate the parameter T can be derived as follows. Suppose the particular choice of the function $V(x, T)$ is a reasonable approximation to the isochrone $I(T)$, which corresponds to the state vector x . Then the resulting sub-optimal controller $\bar{v}(x)$ should satisfy the inequality,

$$\left. \frac{dV}{dt} \right|_{\bar{v} = \bar{v}(x)} \leq 0 \quad (4.9)$$

Relation (4.9) implies that the minimal time to the origin is decreasing. Thus if relation (4.9) is violated at any time t , a new value of the parameter T is required.

(7) A final aspect of the sub-optimal controller is the Termination Criterion. Since the controller is not optimal, it cannot be expected to drive the state x of system (2.11) to the origin. Thus a Termination Criterion must be established. This Termination Criterion will depend on the particular problem under consideration. When the Termination Criterion is satisfied, the controller is disengaged until a new disturbance occurs.

The basic sub-optimal method of solution is summarized as follows. For a particular system in the bounded state variable formulation given by equation (2.11), the maximal set M of the state vectors to be controlled should be specified beforehand. Also, the desired Termination Criterion should be specified. Then the isochrones of system (2.11)

are approximated off-line by a parameterized function $V(x, T)$, by applying Steps (1) through (3). The on-line operation proceeds according to Steps (4) and (5), while the inequality of Step (6) is monitored. The parameter T is recalculated whenever relation (4.9) is not met, or whenever the value of T from equation (4.8) becomes less than T_1 . This procedure continues until the pre-assigned Termination Criterion is satisfied. At this point the controller is disengaged until another disturbance occurs or until the system no longer satisfies the Termination Criterion.

In the following sections, the various aspects of this sub-optimal method are analyzed in detail.

4.3 Isochrones and the Set of Recoverability

In this section an important relationship between the isochrones $I(T)$, $T \geq 0$, and the set of recoverability $K(T)$, is established. This relationship will form a basic part of the procedure for constructing the isochrones of system (2.11).

For the class of linear autonomous systems described by equation (2.11), Chang [18] has shown that the extremal controls are the unique solution to the open-loop minimal time problem. That is, if the extremal control $\bar{v}(t)$ steers the state vector $\bar{x} \neq 0$ to the origin in T seconds, then there exists no other admissible control that can steer $\bar{x}$ to the origin in a time $\tilde{T} < T$. Also, the extremal control $\bar{v}(t)$ is the only control that can steer the state vector $\bar{x} \neq 0$ to the origin in the time T . Therefore, in this case extremal controls are also minimal time controls.

In Chapter III, from Theorem 3.1, it has been shown that state

vectors on the boundary of the set of recoverability correspond to the extremal controls. Then from the definition of the isochrones $I(T)$, $T \geq 0$, in Section 4.2, it is seen that the boundary of the set of recoverability at time T is equivalent to $I(T)$. From equation (2.10), the set of recoverability $K(T)$, $T \geq t$, of system (2.11) is given by,

$$K(T) = \left\{ x \mid x = - \int_t^T \phi(t, s) B v(s) ds, v \in \Lambda(T) \right\} \quad (4.10)$$

Notice that the set $K(T)$ is independent of the choice of the time t , since system (2.11) is autonomous. Therefore, there is no loss of generality in taking $t = 0$, in equation (4.10). From the above discussion,

$$I(T) = \partial K(T) = \left\{ x \mid x = - \int_0^T \phi(0, s) B \bar{v}(s) ds, \bar{v} \in \Lambda(T) \right\}, \quad (4.11)$$

where $\bar{v}(s)$ on $s \in [0, T]$ denotes an extremal control.

From equation (4.11), the boundary of the set of recoverability, $\partial K(T)$, can be constructed by computing the extremal controls $\bar{v}(t)$ on $t \in [0, T]$. Thus the problem of constructing the isochrone $I(T_k)$ is based on finding the extremal controls of system (2.11) on the time interval $[0, T_k]$. In Chapter III, a method for constructing the extremal controls of system (2.6) has been established. Since system (2.11) is a special sub-class of system (2.6), this method can also be used to construct the extremal controls of system (2.11). The basic procedure is developed in the following sections. This procedure is based, in part, on the various properties of the set of recoverability $K(T)$. From Appendix A, the set of recoverability $K(T)$ is compact, convex and continuous in T . It is shown in Chapter V that the set of

recoverability for system (2.11) is also non-decreasing, i.e., $K(T_a) \subset K(T_b)$ for $0 \leq T_a \leq T_b$. Also, from the definition of the class of admissible controls, equation (2.8), if the control $v \in A(T)$, then the control $-v \in A(T)$. Therefore the set of recoverability is symmetric with respect to the origin. That is, if the state vector $x \in K(T)$, then the state vector $-x \in K(T)$.

Since the set of recoverability is non-decreasing, then from equations (4.10) and (4.11),

$$K(T) = \bigcup_{0 \leq t \leq T} \dot{I}(t) = L(0, T), \quad (4.12)$$

where the set $L(T_1, T_N)$ was defined in equation (4.2).

The first step of the sub-optimal method involves the partitioning of the maximal set M . This is discussed in the following section.

4.4 Partitioning the Set M

Step (1) of the sub-optimal method requires that a sequence of isochrones $I(T_k)$, $k = 1, 2, \dots, N$, be chosen such that the later approximations will be valid for all the state vectors $x \in M$. Thus, the above sequence should be chosen in such a manner as to be a partition of the set M . In this section a procedure for choosing the value of the integer N and the value of the times T_k , $k = 1, 2, \dots, N$, will be established.

Since the set of recoverability for system (2.11) is non-decreasing, then from equation (4.12) it is reasonable to order the times T_k , $k = 1, 2, \dots, N$, as follows.

$$0 \leq T_1 < T_2 < \dots < T_N \leq \tilde{T}, \quad (4.13)$$

where the time instant $\tilde{T}$ is defined to be any time instant that satisfies the relation

$$M \subset K(\tilde{T}) \quad (4.14)$$

It is assumed that a time $\tilde{T}$ satisfying relation (4.14) does indeed exist. Since the set of recoverability of system (2.11) is non-decreasing, then as discussed in Chapter V, relation (4.14) implies that every state vector $x \in M$ is recoverable. A detailed discussion of the problem of recoverability is given in Chapter V.

First, consider the choice of the value of the maximum time T_N . The time $\tilde{T}$, defined in equation (4.14), serves as an upper bound on the value of T_N . It is desirable that the sequence of isochrones $I(T_k)$, $k = 1, 2, \dots, N$, be chosen in such a manner as to cover the maximal set M . Therefore a reasonable choice for the time T_N is given by,

$$T_N = \inf [\tilde{T}] \quad (4.15)$$

Note that as in all aspects of a sub-optimal method where an engineering solution is sought, the above equation (4.15) need only serve as a guide to the selection of the time T_N . The procedure for finding the time T_N is as follows.

First, choose a nominal value of T , say $T = T_a$. Then choose an initial value of the adjoint vector $V(0)$. The corresponding extremal control on the time interval $[0, T_a]$ is then found with the aid of the computer program developed in Chapter III. The corresponding state vector x on the boundary of the set of recoverability ($x \in \partial K(T_a) = I(T_a)$), is obtained from equation (4.11). This process is repeated for other choices of the initial value of the adjoint vector. Note that by the

symmetry property of the set of recoverability, each extremal control yields two points on the boundary of $K(T_a)$.

The convexity property of the set of recoverability can then be applied to these computed boundary points to obtain a rough estimate as to the size of the set $K(T_a)$. This is compared with the given maximal set M . If it is found that the set $K(T_a)$ is too small to cover the set M , then a larger value of time is needed. If on the other hand, it is found that the set $K(T_a)$ is too large, then a smaller value of time should be chosen. Note that the continuity of the set $K(T)$ with respect to T can be used as an aid in choosing these new values of T .

The above process is repeated until a value of T is found such that the set of recoverability approximately covers the set M . This value of T then becomes the choice for T_N .

Once the value of T_N is known, it is necessary to choose the value of the integer N and the values of the times T_k , $k = 1, 2, \dots, N-1$. By actual practice, it was found that the best method of choosing these values is as follows. After choosing the value of the time T_N , construct the isochrone $I(T_N)$. Then find the corresponding approximation $V(x, T_N) = 1/2[x'C_N x] = T_N$ of Step (2) (Section 4.2). The method for obtaining the above approximation is developed in the later sections. Now choose the time $T_{N-1} = T_N - \Delta T$, where ΔT is some small nominal positive value of time. Repeat the above process to obtain the approximation of $I(T_{N-1})$, i.e., $V(x, T_{N-1}) = 1/2[x'C_{N-1} x] = T_{N-1}$. The corresponding elements c_{ij} of the matrices C_N and C_{N-1} are then compared term by term. The magnitude difference between the pairs of elements is dependent on the respective shapes of the isochrones $I(T_N)$ and $I(T_{N-1})$. Hence it is related to the value of the time ΔT . Note that the elements

of the two approximations can be made to be as close as desired by decreasing the value of the choice of ΔT . This is because the set of recoverability is continuous and therefore, by equation (4.11),

$\lim_{\Delta T \rightarrow 0} I(T_{N-1}) = I(T_N)$. The actual choice of the value of the time ΔT is dependent on the following factors.

Since the elements of the matrices will later be approximated by continuous functions in Step (3) of Section 4.2, it is desirable to have many closely spaced numerical values of these elements. Thus the time ΔT should be chosen small enough to satisfy this criterion. However, at the times at which the shape of the isochrones is changing very rapidly, the time ΔT will become quite small. In this case the construction of a large number of isochrones is required in order to cover the entire maximal set H . Since each isochrone is constructed by using the computer program developed in Chapter III, this means that a corresponding large amount of off-line computer time will be needed in order to construct these isochrones. There is obviously an engineering trade-off involved. Thus the choice of the time ΔT will in general depend on the trade-off between the accuracy of the approximation desired and the amount of off-line computer time that is available for the given problem.

Once the value of ΔT is established, the remaining times are chosen by the sequential relation $T_{k-1} = T_k - \Delta T$, $k = N-1, N-2, \dots, 2$. At each value of time T_k , the isochrone $I(T_k)$ is constructed and the corresponding approximation $1/2[x'C_k x] = T_k$ is obtained. Thus Step (2) of Section 4.2 is actually carried out as part of the process of Step (1). That is, the time T_{k-1} is selected according to the results of the approximation of Step (2) at the time instant T_k . Note that the value of ΔT

may be altered at any step of the interval if it is found that the elements of the approximation matrices C_k begin to change very rapidly.

The final choice to be made is the value of the smallest parameter T_1 . It has been found from experience that the elements of the approximation matrix tend to become unbounded as T approaches zero. This is due to the fact that the state vectors $x \in K(T)$ take on extremely small values as T approaches zero. Thus for small values of the time T_1 , the approximation $1/2[x'C_1x] = T_1$ becomes very difficult to apply accurately, since both sides of the approximation are approaching zero. Therefore, there is a lower positive limit to the choice of the time T_1 which depends on each individual system. The sub-optimal controller will therefore not be able to drive the state vector x to the origin. As previously mentioned, this problem can be overcome by using a Termination Criterion. However, in some systems it is necessary to drive the state vector to the origin. For these systems a dual controller scheme can be used. The sub-optimal controller is used to drive the state vectors x to within the set of recoverability $K(T_1)$. As previously mentioned, the state vectors $x \in K(T_1)$ have small values. For these small values a linear feedback controller can be employed. The dual controller would then consist of the sub-optimal controller for the state vectors $x \notin K(T_1)$ and a linear controller for the state vectors $x \in K(T_1)$.

In this section, the basic procedure for choosing the times T_k , $k = 1, 2, \dots, N$, has been established. It is based on first finding the choice of the maximum time T_N . Then the remaining time instants are chosen in descending order by computing the approximation of Step(2), and by observing the rate of change of the elements of the matrix C_k .

In Section 4.2, a method of constructing the isochrone $I(T)$, for any value of $T \geq 0$, has been indicated. This method is examined in detail in the next section.

4.5 Construction of the Isochrones

The method of constructing the isochrone $I(T)$, outlined in Section 4.2, is as follows. The extremal controls $\bar{v}(t)$ on $t \in [0, T]$ are first found by the procedure developed in Chapter III. Then the state vectors x on the isochrone $I(T)$ are found by applying equation (4.11).

From the definition of an extremal control, equation (3.1), it is seen that the extremal controls $\bar{v}(t)$ on $t \in [0, T]$ are functions of the initial choice of the unit adjoint vector $\Psi(0)$. For some simple low order systems, an analytical expression can be obtained for the extremal controls $\bar{v}(t)$ on $t \in [0, T]$ as a function of the choice of $\Psi(0)$ (e.g., [21], [28]). In this case, the values of all the state vectors x on the isochrone $I(T)$ may be obtained from equation (4.11). However, in more complex higher order systems, it is not feasible to find analytical expressions for the extremal controls. For these systems, the computer program developed in Chapter III must be applied. This computer program can calculate the extremal control $\bar{v}(t)$ on $t \in [0, T]$, for each given choice of the unit vector $\Psi(0)$. It is therefore only possible to find the extremal controls for a finite discrete set of choices of $\Psi(0)$. Correspondingly, it is only possible to find a discrete set of state vectors x on the isochrone $I(T)$.

The sub-optimal method should be applicable to general systems described by equation (2.11). Therefore in most systems, it will be necessary to select some finite discrete subset of all the possible

values of $\Psi(0)$ in order to numerically construct the isochrone $I(T)$. A general set of engineering criteria for selecting the values of $\Psi(0)$ to be used in the construction process, is developed below.

As indicated in the example of Section 3.11, the complete set of all possible (nm) -dimensional unit vectors $\Psi(0)$ can be put into a one-to-one correspondence with a set of angles ϕ_i , $i = 1, 2, \dots, nm-1$. Since the set of recoverability is symmetric with respect to the origin, it is only necessary for these angles to lie in the ranges, $0 \leq \phi_1 < \pi$, and $|\phi_i| \leq \pi/2$, $i = 2, 3, \dots, nm-1$. If the reformulated system (2.11) is less than fourth order ($nm < 4$), it is feasible to partition the entire range of each of the angles ϕ_i , $i = 1, 2, \dots, nm-1$. In this manner a uniform sample of closely spaced values of the vector $\Psi(0)$ can be used to construct each isochrone $I(T)$. However, for systems of order four or higher this method becomes impractical, since any reasonably closely spaced partition of the angles ϕ_i , $i = 1, 2, \dots, nm-1$ ($nm-1 \geq 3$), now yields an extremely large number of values of $\Psi(0)$. The computer program developed in Chapter III operates by calculating the extremal control, component by component. Therefore for each value of $\Psi(0)$, it requires m -operations to calculate the m -dimensional extremal control $\bar{v}(t)$ on $t \in [0, T]$. Thus the number of values of $\Psi(0)$ that can be used in the construction process is limited by the amount of available off-line computer time. In order to keep this necessary off-line computation time at a reasonable value for systems of order four or higher, a subset of the partitioned values of $\Psi(0)$ must be chosen. A set of general engineering criteria for choosing this subset is established as follows.

A general criterion for choosing the values of $\Psi(0)$ can be obtained

by analyzing the basic purpose of the construction process. The basic purpose is to obtain numerical values of the state vectors x on the isochrones $I(T_k)$, $k = 1, 2, \dots, N$. These obtained numerical values will be used in the approximation scheme of Step (2) of Section 4.2. The approximation of each of the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, in Step (2), is accomplished by a least mean squares fit. Therefore, in order to insure that the approximation is reasonable, it is necessary to obtain a wide scattering of the numerical values of the state vectors x on each of the isochrones. From equation (4.11) it is seen that for a given system, the values of the state vectors x on the isochrone $I(T)$ are solely determined by the extremal controls $\bar{v}(t)$ on $t \in [0, T]$. Thus, in order to insure that a wide scattering of values of the state vectors are obtained, the following general criterion involving the selection of the initial values of $\Psi(0)$, should be applied:

"The initial values of the adjoint vector $\Psi(0)$ should be chosen so as to yield a wide variety of extremal controls".

It has been shown in Chapter III that each component $\bar{v}_j(t)$, $j = 1, 2, \dots, m$, of the extremal control $\bar{v}(t)$ on $t \in [0, T]$ is dependent on certain distinctive characteristics of the corresponding component $\Psi_{n+j}(t)$ of the adjoint solution, on the same time interval. These characteristics are stated in Section 3.8. In order to insure a wide variety of extremal controls, it is therefore necessary to choose the initial values of $\Psi(0)$ so that a wide variety of these distinctive characteristics are present in the adjoint solutions. Three of these distinctive characteristics are discussed below.

(1) One of the most important characteristics in determining the component $\bar{v}_j(t)$ of the extremal control is the number of extrema of the

function $\Psi_{n+j}(t)$ on $t \in (0, T)$. This is evident from Chapter III, where it has been shown that the number of extrema of the function $\Psi_{n+j}(t)$ on $t \in (0, T)$, determines the number of times the boundary of the set $\hat{G}$ is reached on the same time interval. Of course the maximum number of extrema of the adjoint solution $\Psi(t)$ is dependent on the particular system under consideration. However, in all cases it is important to select the values of $\Psi(0)$ such that each component $\Psi_{n+j}(t)$, $j = 1, 2, \dots, m$, of the adjoint solution has various numbers of extrema on the time interval $(0, T)$.

(ii) Another consideration in the selection of the set of values for $\Psi(0)$ is the relative time instants of the extrema of $\Psi_{n+j}(t)$, $j = 1, 2, \dots, m$, on the time interval $(0, T)$. This determines the sub-intervals of time during which the corresponding component $\bar{v}_j(t)$ of the extremal control will be zero.

(iii) A third characteristic to be considered is the value of the last zero crossing of each function $f(t) = \Psi_{n+j}(T-t)$, on the time interval $[0, T]$. As indicated by necessary condition (D) of Section 3.6, the last zero crossing determines whether or not the component of the extremal control $\bar{v}_j(t)$ is initially on its bound, $|\bar{v}_j| = h_j$.

As indicated in Section 3.8, the above three characteristics are by no means the only dominant factors in the determination of the extremal controls. However, by experience, it has been found that a wide variety of extremal controls are obtained whenever a corresponding wide variety of the above three characteristics are present.

In order to apply the above characteristics in the selection of the choices of $\Psi(0)$, some preliminary analysis must be done on the adjoint solution $\Psi(t)$ on $t \in [0, T]$. This analysis is most easily accomplished by

finding the analytical expression for the adjoint solution $\bar{V}(t)$ as a function of the initial condition $\bar{V}(0)$. Then the form of the adjoint solution $\bar{V}(t)$ on $t \in [0, T]$ can be readily observed as a function of the choice of $\bar{V}(0)$. In this manner, a reasonable set of values for $\bar{V}(0)$ can be selected so that a wide variety of the above three characteristics are present. For systems in which the analytical expression of the adjoint solution is too complex for simple analysis, a numerical method can be used to observe the above three characteristics as the choice of $\bar{V}(0)$ is varied. For example, Subroutine PSICIC of the computer program developed in Chapter III can be used to obtain the number of extrema, etc., of any numerically known scalar function. Therefore, this program could also be used in cases where the adjoint solution can only be found by a numerical integration of the adjoint equations, $\dot{\bar{V}}(t) = -\bar{V}(t)A$.

In summary, the procedure of constructing the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, of Step (1) of Section 4.2, consists of the following three alternative steps.

(1a) For simple low order systems, obtain the extremal controls $\bar{V}(t)$ on $[0, T_k]$ as an analytic function of the choice of $\bar{V}(0)$. Then use equation (4.11) to find all of the state vectors x on the isochrone $I(T_k)$.

(1b) For more complex systems of order $n + m < 4$, use the partitioning method discussed in this section to obtain a set of choices for $\bar{V}(0)$. Next, use the computer program developed in Chapter III to obtain the extremal controls $\bar{V}(t)$ on $t \in [0, T_k]$ for these choices of $\bar{V}(0)$. Then proceed as in Step (1a).

(1c) For general systems of order $n + m \geq 4$, choose a subset of

the partitioned values of $V(0)$ by observing the three characteristics discussed in this section. This requires a preliminary analysis of the adjoint solution. Then proceed according to Step (1b).

In all cases, the choice of the set of values for $V(0)$ remains fixed throughout the construction procedure. The remaining part of the procedure is repeated for each value of T_k , $k = 1, 2, \dots, N$.

Steps (1a) and (1b) are both fairly straightforward and require no further explanation. However the application of Step (1c) is relatively complex. In Section 4.12, the example presented to illustrate the application of the sub-optimal method to practical systems, also illustrates the use of Step (1c).

In the next section, the isochrone approximations of Step (2) of Section (4.2) will be discussed in detail.

4.6 Approximation of the Isochrones

In Step (2) of Section 4.2, the numerical values of the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, are to be approximated by functions of the form $V(x, T_k) = 1/2[x'C_k x] = T_k$, where $C_k = C_k(T_k)$ is a positive definite symmetric matrix. The positive definite property of the matrix C_k implies that each approximation $V(x, T_k) = T_k$ of the isochrone $I(T_k)$, represents the equation of an $(n+m)$ -dimensional ellipsoid. The reason for choosing this particular form for the function $V(x, T_k)$ is as follows. From equation (4.11), the isochrone $I(T_k)$ represents the boundary of the set of recoverability $K(T_k)$. The set of recoverability is convex and symmetric with respect to the origin. These two properties of the set of recoverability indicate that the above form for the function $V(x, T_k)$ can be expected to yield a reasonable approximation to the

isochrones $I(T_k)$, $k = 1, 2, \dots, N$.

Of course, the form of the function $V(x, T_k)$ is by no means limited to terms of second order. Certainly, terms of even orders of four or higher could be added to the form of $V(x, T_k)$. This would improve the accuracy of each isochrone approximation. However this introduces a corresponding increase in the complexity of the sub-optimal method. This increase of complexity is due to the considerable number of additional coefficients that must be found in systems of higher order. As an example, consider a system of order $n = 5$. In this case each 5 by 5 symmetric matrix C_k has 15 unknown elements to be calculated. If all terms of order four are added to the form of the function $V(x, T_k)$, then the values of an additional 35 coefficients must be computed. This requires a considerable increase of the off-line computation time. Also, the on-line storage requirements are much larger, since it now requires 50 rather than 15 continuous functions to describe the final approximation function $V(x, T)$. Thus there is a trade-off between the accuracy of the approximations and the complexity of the sub-optimal method. It has been found from experience that a reasonable compromise between these two factors can be obtained by using the above second order form for the functions $V(x, T_k)$, $k = 1, 2, \dots, N$.

The approximation of the numerical values of the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, is accomplished by a least mean squares fit. In order to establish this approximation, a basic criterion that determines the type of fit must be chosen. There are two factors to be considered in the choice of such a criterion. These two factors are discussed below.

- (1) For a given state vector x , the value of the parameter T represents the time required to drive the state vector x to the origin,

using minimal time controls. That is, if the state vector $\bar{x}$ belongs to the isochrone $I(\bar{T})$, then the value of the parameter T is equal to $\bar{T}$. In order to apply the sub-optimal method, the value of T must be found on-line at each given state vector x . For most of the control interval, the value of the parameter T can be obtained from equation (4.8). However, as discussed in Section 4.2, there are times during the control interval at which the value of T must be found by iteratively solving equation (4.4), i.e., $V(x, T) = T$. Consider a given state vector $\bar{x}$ which belongs to the isochrone $I(\bar{T})$. For this given state vector, equation (4.4) becomes $V(\bar{x}, T) = T$. Since the value of the parameter T obtained from equation (4.4) should be equal to $\bar{T}$, it is necessary that $V(\bar{x}, \bar{T}) \approx \bar{T}$. Thus in general, at every state vector x it is desirable that the value of the function $V(x, T)$ closely approximates the correct value of the parameter T .

(2) Once the value of the parameter T is obtained for the given state vector x , the sub-optimal closed loop controller $\bar{v}(x)$ is found by applying equation (4.7). By equation (4.5), the sub-optimal controller $\bar{v}(x)$ is determined by the direction of the row vector VV . Similarly, from the discussion of Section 4.2, the minimal time controller at a given state vector x is determined by the direction of VW . Therefore in order to insure that the sub-optimal controller closely approximates the minimal time controller, it is necessary for the direction of the vector VV to closely approximate the direction of the vector VW at every state vector x . In order to accomplish this, consider the definition of the function $W(x)$. From equation (4.1), the isochrones $I(T)$, $T \geq 0$, are described by the expressions $W(x) = T$. Therefore the above condition may be satisfied by requiring that the $(n+m)$ -dimensional ellipsoids

$V(x, T) = T$, $T \geq 0$, closely approximate the numerical values of the isochrones $I(T)$.

The above factors establish two conditions that should be satisfied by the approximation function $V(x, T)$ to insure proper operation of the sub-optimal controller. Since the function $V(x, T)$ is obtained from the set of approximations $V(x, T_k)$, $k = 1, 2, \dots, N$, then these same two conditions should form the criteria for the least mean squares fit of Step (2) of Section 4.2. With reference to the above two conditions, the criteria for the least mean squares fit are as follows.

Criterion A

At each value of T_k , $k = 1, 2, \dots, N$, choose the elements of the matrix C_k so that $V(x, T_k) \approx T_k$ at all the computed values of the state vectors x belonging to the isochrone $I(T_k)$.

Criterion B

At each value of T_k , $k = 1, 2, \dots, N$, choose the elements of the matrix C_k so that the $(n+m)$ -dimensional ellipsoid $V(x, T_k) = T_k$, provides a close approximation to the isochrone $I(T_k)$.

If the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, could be exactly described by $(n+m)$ -dimensional ellipsoids, then only one set of functions would be needed to satisfy both of the above criteria. In this case, each $(n+m)$ -dimensional ellipsoid $V(x, T_k) = T_k$ which exactly describes the isochrone $I(T_k)$, would also exactly satisfy Criterion A, i.e., $V(x, T_k) = T_k$ at all the state vectors $x \in I(T_k)$. However in general, the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, cannot be described exactly by $(n+m)$ -dimensional ellipsoids. Therefore in order to satisfy both of the above criteria, two sets of approximation functions are generally needed.

These two functions are defined as follows. Let

$$\bar{V}(x, T_k) = 1/2[x^T \bar{C}_k x], \quad k = 1, 2, \dots, N, \quad (4.16)$$

where the elements $\bar{c}_{ij}$ of each $(n \times n)$ by $(n \times n)$ positive definite symmetric matrix $\bar{C}_k$ are to be chosen according to Criterion A. Also, let

$$\tilde{V}(x, T_k) = 1/2[x^T \tilde{C}_k x], \quad k = 1, 2, \dots, N, \quad (4.17)$$

where the elements $\tilde{c}_{ij}$ of each $(n \times n)$ by $(n \times n)$ positive definite symmetric matrix $\tilde{C}_k$ are to be chosen according to Criterion B.

The equations governing the calculation of the elements of the matrices $\bar{C}_k$ and $\tilde{C}_k$, $k = 1, 2, \dots, N$, are established in the next section. Once the functions $\bar{V}(x, T_k)$ and $\tilde{V}(x, T_k)$, $k = 1, 2, \dots, N$, are obtained, the procedure of Step (3) of Section 4.2 is then applied. This yields two approximation functions, viz, $\bar{V}(x, T)$ and $\tilde{V}(x, T)$. Since the function $\bar{V}(x, T)$ has been obtained according to Criterion A, it approximates the correct value of the parameter T at each state vector x . Therefore the approximation function $\bar{V}(x, T)$ should be used in place of the function $V(x, T)$ in equation (4.4). The approximation function $\tilde{V}(x, T)$ has been obtained according to Criterion B. Therefore at each value of T , the equation $\tilde{V}(x, T) = T$, represents an $(n \times n)$ -dimensional ellipsoid approximation to the isochrone $I(T)$. From the discussion of Factor (2), this implies that the approximation function $\tilde{V}(x, T)$ should be employed in the calculation of the sub-optimal controller $\bar{v}(x)$. Hence, the function $\tilde{V}(x, T)$ should be used in place of $V(x, T)$ in equation (4.7). Finally, from Section 4.2, equation (4.9) has been developed based on the assumption that the isochrone $I(T)$ is closely approximated by the function $V(x, T)$. Thus, the approximation function

$\tilde{V}(x, T)$ should also be used in place of $V(x, T)$ in equation (4.9).

In summary, the sub-optimal procedure is basically still the same as the one described in Section 4.2. The only change is that now two different approximation functions, viz, $\bar{V}(x, T)$ and $\tilde{V}(x, T)$ are to be used in the sub-optimal procedure. The approximation function $\bar{V}(x, T)$ replaces $V(x, T)$ in equation (4.4) and the approximation function $\tilde{V}(x, T)$ replaces $V(x, T)$ in equations (4.7) and (4.9). Also, note that the procedure given in Section 4.4 for finding the values of the parameters T_k , $k = 1, 2, \dots, N$, must be slightly altered. It is now required to observe the changes in the elements of both the matrices $\bar{C}_k$ and $\tilde{C}_k$ as the time interval ΔT is varied.

4.7 Computation of the Matrices $\bar{C}_k$ and $\tilde{C}_k$

In this section a procedure will be established for computing the elements of the matrices $\bar{C}_k$ and $\tilde{C}_k$, $k = 1, 2, \dots, N$, of equations (4.16) and (4.17). This procedure is based on the application of the two criteria of the preceding section, to form the governing equations of two least mean squares fits.

First, consider the computation of the elements $\bar{c}_{ij}$ of the matrix $\bar{C}_k$. From equation (4.16), these elements are to be chosen according to Criterion A of Section 4.6. With reference to Criterion A, the governing equation of a least mean squares fit that satisfies this criterion is as follows. At each value of T_k , $k = 1, 2, \dots, N$, choose the elements $\bar{c}_{ij}$ of the matrix $\bar{C}_k$ to minimize $\bar{P}$, where

$$\bar{P} = \sum_{\ell=1}^{M_k} [\bar{V}(x^\ell, T_k) - T_k]^2. \quad (4.18)$$

In the above equation, the $(n+m)$ -dimensional state vectors x^l , $l = 1, 2, \dots, M_k$, represent the computed values of the state vectors x belonging to the isochrone $I(T_k)$. The positive integer M_k in equation (4.18) represents the total number of computed state vectors x^l on the isochrone $I(T_k)$.

Now, consider the computation of the elements $\tilde{c}_{ij}$ of the matrix $\tilde{C}_k$. From equation (4.17), these elements are to be chosen according to Criterion B of Section 4.6. In order to satisfy Criterion B, a measure of the closeness of the fit of the $(n+m)$ -dimensional ellipsoid $\tilde{V}(x, T_k) = T_k$ to the isochrone $I(T_k)$, must be chosen. Such a measure is established below. Let

$$r_l = \left[\sum_{i=1}^{n+m} (x_i^l)^2 \right]^{1/2} \quad (4.19)$$

represent the radial distance of each computed state vector x^l , $l = 1, 2, \dots, M_k$, on the isochrone $I(T_k)$. Also, let the unit vector α^l represent the normalized form of each vector x^l , i.e.,

$$\alpha_i^l = x_i^l / r_l, \quad i = 1, 2, \dots, n+m. \quad (4.20)$$

Consider an arbitrary computed state vector x^l belonging to the isochrone $I(T_k)$. From equation (4.20), the direction of this state vector is given by the unit vector α^l . Let the square of the inverse radial distance of the state vector x^l be denoted by d_l . Then from equation (4.19),

$$d_l = 1/(r_l)^2 \quad (4.21)$$

Now consider the vector $\tilde{x}^l$ on the $(n+m)$ -dimensional ellipsoid $\tilde{V}(x, T_k) = T_k$, which is in the same direction as the computed state vector x^l . Let

the radial distance of this vector be denoted by $\tilde{r}_k$ and let the square of the inverse radial distance be denoted by $\tilde{d}_k$, i.e., $\tilde{d}_k = 1/(\tilde{r}_k)^2$. Since $\tilde{V}(x, T_k)/T_k = 1$ and since the direction of the vector $\tilde{x}_k$ is given by the unit vector α^k , then from equations (4.17) and (4.20),

$$\tilde{d}_k = \left[(\tilde{x}^k)' \tilde{C}_k (\tilde{x}^k) \right] / 2(\tilde{r}_k)^2 T_k = \left[(\alpha^k)' \tilde{C}_k (\alpha^k) \right] / 2T_k = \tilde{V}(\alpha^k, T_k) / T_k \quad (4.22)$$

The difference between the square of the inverse radial distances d_k and $\tilde{d}_k$, can serve as a measure of the closeness of the vector $\tilde{x}^k$ on the $(n+m)$ -dimensional ellipsoid $\tilde{V}(x, T_k) = T_k$, to the computed state vector x^k on the isochrone $I(T_k)$.

With the aid of equations (4.21) and (4.22), the governing equation for a least mean squares fit that satisfies Criterion B of Section 4.6, is as follows. At each value of T_k , $k = 1, 2, \dots, N$, choose the elements of the matrix $\tilde{C}_k$ to minimize $\tilde{P}$, where

$$\tilde{P} = \sum_{k=1}^{M_k} \left[\tilde{V}(\alpha^k, T_k) - T_k/r_k^2 \right]^2, \quad (4.23)$$

and the positive integer M_k was defined in equation (4.18). Notice that in equation (4.23), the measure of the closeness of the fit has been multiplied by T_k . Since T_k is a constant at each value of k , $k = 1, 2, \dots, N$, this multiplication does not affect the resulting $(n+m)$ -dimensional ellipsoid approximations. In fact, all of these approximations could have been normalized by letting $T_k = 1$, in equation (4.23).

In this case the resulting approximations would be of the form,

$\tilde{V}(x, T_k) = 1$. However, it has been found that the numerical values of the elements $\tilde{c}_{ij}$ of the matrices $\tilde{C}_k$, $k = 1, 2, \dots, N$, are easier to

approximate in Step (3) of Section 4.2, if the actual value of T_k is used in equation (4.23).

Equations (4.18) and (4.23) are the governing equations for the two least mean squares fits. The procedure for finding the two sets of elements that minimize equations (4.18) and (4.23) is well known (for example, [29]), and is therefore omitted here. For each approximation, the results of this procedure is a set of linear equations involving the elements of the corresponding matrix. Since the $(n+m)$ by $(n+m)$ matrices $\bar{C}_k$ and $\tilde{C}_k$ are symmetric, then the total number of elements of each matrix that can be chosen independently is given by

$$R = (n+m)(n+m+1)/2 \quad (4.24)$$

Therefore, in order to obtain the values of the elements of each matrix, it is necessary to solve a set of R linear equations. This procedure is repeated at each value of T_k , $k = 1, 2, \dots, N$.

The matrices $\bar{C}_k$ and $\tilde{C}_k$, $k = 1, 2, \dots, N$, are defined to be positive definite in equations (4.16) and (4.17). It should be pointed out that these matrices are not constrained to be positive definite as part of the approximation procedure. The positive definite property of these matrices should occur as a result of the properties of the computed state vectors. This factor was discussed at the beginning of Section 4.6. The positive definite property of the matrices $\bar{C}_k$ and $\tilde{C}_k$, can also be used as an indication of the quality of the two approximations. This is accomplished in the following manner. At each value of T_k , $k = 1, 2, \dots, N$, the obtained matrices $\bar{C}_k$ and $\tilde{C}_k$ are both tested to see if they are positive definite. This can easily be done by applying Sylvester's Theorem [30]. If at some value of T_k , say $T_k = \tilde{T}$, it is

found that at least one of the matrices is not positive definite, then this indicates that the corresponding approximation is not reasonable. In order to improve the approximation, additional values of the state vectors x on the isochrone $I(T)$ must be computed. In some cases it may even be necessary to alter the initial choices of the adjoint vector $\psi(0)$, in order to insure that both matrices are positive definite.

The next section discusses the approximation described in Step (3) of Section 4.2.

4.8 Obtaining the Approximation Functions $\bar{V}(x,T)$ and $\bar{\psi}(x,T)$

The numerical values of the elements of the matrices $\bar{C}_k$ and $\bar{\psi}_k$ are found by applying equations (4.18) and (4.23) at each value of T_k , $k = 1, 2, \dots, N$. The next step (Step (3) of Section 4.2) of the sub-optimal method, is to approximate the numerical values of each of these elements by continuous functions of the parameter T . From equation (4.24), there are a total of R independent elements in each matrix. Therefore a total of $2R$ approximations are necessary. The general procedure for obtaining these $2R$ approximations is discussed below. For convenience in the following discussion, let the elements to be approximated be denoted by d_i , $i = 1, 2, \dots, 2R$. Also, let the corresponding continuous approximations of these elements be denoted by $d_i^*(T)$, $i = 1, 2, \dots, 2R$.

The first aspect of the approximation procedure is the selection of the form of each continuous function $d_i^*(T)$, $i = 1, 2, \dots, 2R$. There are many possible choices for the form of these functions, e.g. polynomials, sinusoids, etc. For each element d_i , the form of the function $d_i^*(T)$ is dependent on the following three factors.

- (1) Each element approximation is to be accomplished by a least

mean squares fit. Therefore the form chosen for the function $d_i^*(T)$ must be applicable to this method of approximation. That is, it must be possible to obtain the coefficients of the function $d_i^*(T)$, by applying the method of least mean squares.

(ii) For each choice of the form of the function $d_i^*(T)$, a least mean squares fit yields a particular approximation to the computed numerical values of the element d_i . In the final analysis, the quality of this approximation is determined by the operation of the sub-optimal controller. In order to insure the proper operation of the sub-optimal controller, it is desirable for each function $d_i^*(T)$ to closely approximate all of the computed values of the element d_i . Therefore a general criterion for choosing the form of the function $d_i^*(T)$ is as follows.

"The form of each function $d_i^*(T)$, $i = 1, 2, \dots, 2R$, should be selected so that the resulting least mean squares fit yields a reasonably accurate approximation to all the computed values of the corresponding element d_i ."

This criterion indicates that the form of each approximation function is dependent on the computed values of the particular element under consideration. As an example, suppose it is found that the computed values of a particular element vary in a nearly linear manner, when plotted on a semi-log scale. This implies that an exponential form should be chosen for the corresponding approximation function.

(iii) The operation of the sub-optimal controller requires that the element approximation functions $d_i^*(T)$, $i = 1, 2, \dots, 2R$, be evaluated on-line. For general high order systems, this requires the use of an on-line digital computer. Therefore, for these systems the forms of the functions $d_i^*(T)$, $i = 1, 2, \dots, 2R$, are constrained by the on-line

storage limitations. Also since the computer is operating in real time, it is necessary for all computer operations to occur at a much faster rate than the rate of change of the state vector. This limits the complexity of the forms of the element approximation functions.

The above discussion indicates that the choice of the form of each element approximation function $d_i^*(T)$, $i = 1, 2, \dots, 2R$, is dependent on the particular system under consideration. However in order to develop a general approximation procedure, some form must be chosen for these functions. In the absence of any specific system information, the following form is suggested.

Let each function $d_i^*(T)$, $i = 1, 2, \dots, 2R$, be represented by a power series form. That is,

$$d_i^*(T) = \sum_{j=0}^{Q_i} \alpha_{ij} T^j, \quad i = 1, 2, \dots, 2R, \quad (4.25)$$

where α_{ij} , $j = 0, 1, \dots, Q_i$, represents the coefficients of each power series, and the positive integer Q_i represents the order of the power series. The next step of the approximation procedure involves the choice of the value of each integer Q_i , and the evaluation of each set of coefficients α_{ij} , $j = 0, 1, \dots, Q_i$, by the method of least mean squares. This is accomplished in the following manner.

Let the governing equations for the least mean squares fits be given as follows. For each element d_i , $i = 1, 2, \dots, 2R$, choose the coefficients of the power series approximation $d_i^*(T)$ given in equation (4.25), to minimize P_i , where

$$P_i = \sum_{k=1}^N \left(d_i^*(T_k) - d_{ik} \right)^2 \quad (4.26)$$

and d_{ik} , $k = 1, 2, \dots, N$, denotes the computed values of the element d_i . In order to find the set of coefficients that minimizes equation (4.26) for each i , $i = 1, 2, \dots, 2R$, the value of the corresponding integer Q_i in equation (4.25) must be specified. The procedure for specifying the value of each integer Q_i is based on the discussions in Factors (ii) and (iii) of this section. From the discussion in Factor (ii), it is desirable for each function $d_i^*(T)$ to closely approximate all of the computed values of the element d_i . From equation (4.26), a measure of the accuracy of each approximation is provided by the value of the mean squared error P_i . Therefore the value of each integer Q_i should be chosen so that the resulting mean squared error P_i is small. Also, from the discussion in Factor (iii), it is desirable from the point of view of implementation, for the actual value of each integer Q_i to be as small as possible. The procedure for choosing the value of the integer Q_i to satisfy both of these criteria is given below.

Let the maximum allowable value of all the mean squared errors be denoted by the positive number Δ . That is, it is required that

$$P_i \leq \Delta, \quad i = 1, 2, \dots, 2R. \quad (4.27)$$

Initially, choose some nominal value for Δ . The actual choice of the value for Δ is dependent on the particular system under consideration. This is discussed in a later part of the section. Then for each element d_i , select an initial value of the integer Q_i . For this chosen value of the integer Q_i , the set of coefficients α_{ij} , $j = 0, 1, \dots, Q_i$, that minimizes equation (4.26), may be found by applying the method of least mean squares [29]. Once this set of coefficients has been determined, the value of P_i is evaluated by equation (4.26). This value of P_i is

measure of the accuracy of the least mean fit for the given value of the integer Q_i . If $P_i > \Delta$, then a larger value of the integer Q_i is needed. On the other hand, if it is found that $P_i < \Delta$, then a smaller value of the integer Q_i should be chosen. The procedure is repeated until the smallest value of the integer Q_i that can satisfy relation (4.27) is found. All of the values of the integers Q_i , $i = 1, 2, \dots, 2R$, are established in the same manner. Once the values of these integers are specified, then all of the coefficients of the element approximation functions $d_i^*(T)$, $i = 1, 2, \dots, 2R$, are also determined.

In the above discussion, the procedure for obtaining a set of element approximation functions for a given value of Δ , has been established. In general, for each choice of the value of Δ , this procedure yields a different set of element approximation functions. Therefore a criterion for choosing the value of Δ is needed. Such a criterion can be developed by considering the effect of varying the value of Δ . Suppose that a set of element approximation functions has been obtained for some nominal value of Δ , say $\Delta = \Delta_1$. Now, suppose that the value of Δ is decreased by choosing $\Delta = \Delta_2 < \Delta_1$. In general, this will yield a new set of element approximation functions. From relation (4.27), this new set of functions will more accurately approximate the computed values of the elements. This will result in an improvement of the performance of the sub-optimal controller. However this will also necessitate the use of more terms of the power series in equation (4.25). This is because it will now require larger values of the integers Q_i , $i = 1, 2, \dots, 2R$, to satisfy relation (4.27). Once again there is an engineering trade-off between the performance and the complexity of the sub-optimal controller. Therefore, for each given system, the value of

should be chosen according to the following general criterion.

"The value of Δ should be chosen so as to yield a reasonable compromise between the performance and the complexity of the sub-optimal controller".

For a given system, it may be necessary to try a few different values of Δ , until this criterion is satisfied.

In the above procedure for finding the set of element approximation functions, each element is to be approximated by one least mean squares fit. That is, from equation (4.26), the function $d_1^*(T)$ is to be used as an approximation to the computed values of the corresponding element i_1 , on the entire interval $T \in [T_1, T_N]$. In some cases, it may be found that all the numerical values of a particular element cannot be reasonably approximated by a single function. As an example, in Section 4.4 it has been shown that the computed values of some of the elements tend to become unbounded as the value of T approaches zero. On the other hand, it has been found that the computed values of the elements tend to become smaller as the value of T increases. Therefore, in some cases the values of a particular element may vary over an extremely large range. This is dependent on both the particular system under consideration and the choices of the values for the parameters T_1 and T_N . In these problems, the much larger computed values of the element may dominate the least mean squares fit. The resulting element approximation function would then poorly characterize the smaller values of that particular element. This leads to a degeneration in the performance of the sub-optimal controller at the corresponding larger values of the parameter T . Also, in the case of a power series approximation, it may not be feasible to merely increase the order of the approximation to obtain good performance. This is because it would then be

necessary to use an extremely high order approximation. As discussed in Factor (iii) of this section, a high order approximation is not desirable since it requires large on-line computation times for each evaluation. In order to overcome this problem, it is generally necessary to employ more than one approximation function for such elements.

The use of more than one function to approximate the values of a particular element does not alter the sub-optimal method to any large extent. In fact it introduces an additional flexibility into the method, since a different form can be selected for each function. As an example, consider the case of using two functions to approximate all the values of the element d_i . Let these functions be denoted by $\bar{d}_i(T)$ and $\tilde{d}_i(T)$. Also, let T_m represent some intermediate value of the parameter T , where $T_m \in (T_1, T_N)$. The parameter T_m should be chosen so that the intervals $[T_1, T_m)$ and $[T_m, T_N]$ each contain element values of approximately the same range. Then the function $\bar{d}_i(T)$ could be used to approximate the values of d_i on $[T_1, T_m)$, while the function $\tilde{d}_i(T)$ is used to approximate the remaining values of d_i on $[T_m, T_N]$. If a power-series form is chosen for both $\bar{d}_i(T)$ and $\tilde{d}_i(T)$, then the procedure outlined in this section could still be used to find the orders and the coefficients of these functions. The only difference is that now two least mean squares fits are required, and therefore P_i in relation (4.27), now represents the total mean squared error.

For the above example, the final element approximation function is given by,

$$d_i^*(T) = \begin{cases} \bar{d}_i(T) & , \text{ if } T_1 \leq T < T_m \\ \tilde{d}_i(T) & , \text{ if } T_m \leq T \leq T_N \end{cases} \quad (4.28)$$

This dual element approximation function is evaluated by the on-line digital computer. The only differences in operation are that now two sets of coefficients must be stored to represent $d_i^*(T)$ on the entire interval of $T \in [T_1, T_2]$. Also, an additional instruction must be included to insure that the correct approximation function is evaluated at each given value of $T \in [T_1, T_2]$.

In this section, the method for obtaining the approximations described in Step (3) of Section 4.2, has been discussed. Once these $2R$ approximations are completed, then the final approximation functions $\bar{V}(x, T)$ and $\tilde{V}(x, T)$ are determined. With reference to equations (4.16) and (4.17), the functions $\bar{V}(x, T)$ and $\tilde{V}(x, T)$ are given by,

$$\bar{V}(x, T) = 1/2[x' \bar{C}(T)x] \quad (4.29)$$

and

$$\tilde{V}(x, T) = 1/2[x' \tilde{C}(T)x] , \quad (4.30)$$

where each element of the matrices $\bar{C}(T)$ and $\tilde{C}(T)$ is represented by one of the approximations $d_i^*(T)$, $i = 1, 2, \dots, 2R$, of this section.

This completes the off-line portion of the sub-optimal method, described by Steps (1) through (3) of Section 4.2. In the next section, the various aspects of the on-line operation described by Steps (4) through (7) of Section 4.2, are discussed in detail.

4.9 On-Line Operation

For general high order systems, the sub-optimal control operation is most easily accomplished with the use of an on-line digital computer. The sub-optimal controller then operates in a discrete fashion by

sampling the values of the state vector at intervals of time Δt . The actual value of Δt is dependent on the speed of operation of the digital computer and the rate of change of the state vector of the particular system under consideration.

Consider a general sampling interval Δt . At the beginning of this interval, the value of the state vector x is obtained. Then there are three basic operations that must be performed by the on-line digital computer. It is assumed that the operational computer time is short enough, so that the value of the state vector does not change significantly during this time period. The three basic computer operations are as follows.

(I) Calculating the Value of the Parameter T

As discussed in Section 4.2, at most sampling instants during the control interval the value of the parameter T is obtained from equation (4.8). In terms of the sampling interval Δt , equation (4.8) becomes,

$$T = \tilde{T} - \Delta t, \quad T \geq T_1, \quad (4.31)$$

where $\tilde{T}$ denotes the value of the parameter T at the previous sampling instant. Equation (4.31) is to be used as long as the value of $(T - T_1)$ is non-negative, and relation (4.9) is satisfied. From the discussion in Section 4.7, the function $\tilde{V}(x, T)$ replaces $V(x, T)$ in relation (4.9). Therefore, this relation becomes,

$$\left[\frac{d\tilde{V}}{dt} \right] \Big|_{v = \bar{v}(x)} < 0 \quad (4.32)$$

In order to check whether relation (4.32) is satisfied, the value of the sub-optimal controller $\bar{v}(x)$ must be obtained. Therefore at each

sampling instant, the following sequence of steps occurs. Initially, the value of the parameter T is found from equation (4.31). This value of T is first tested to insure that, $(T-T_1) \geq 0$. If $(T-T_1)$ is non-negative, then it is used in the calculation of the value of the sub-optimal controller $\bar{v}(x)$. This is discussed in Operation II. Once $\bar{v}(x)$ is obtained, then relation (4.32) is evaluated. If this relation is satisfied, then the obtained value of $\bar{v}(x)$ is used for the sampling interval Δt .

If the value of the parameter T obtained from equation (4.31) is not positive or if relation (4.32) is not satisfied, then the value of the parameter T must be re-calculated by a different procedure. Also, at the beginning of the control interval, equation (4.31) does not apply, since the value of T is of course unknown. Therefore at these sampling instants, the nonlinear equation (4.4) must be used to obtain the value of T . From Section 4.7, the function $\bar{V}(x, T)$ replaces $V(x, T)$ in equation (4.4). Therefore this equation becomes,

$$\bar{V}(x, T) = 1/2[x' \bar{C}(T)x] = T \quad (4.33)$$

For a given value of the state vector x , there are many iterative techniques for obtaining a solution to the above nonlinear equation. Since equation (4.33) is to be solved on-line, the particular technique selected must operate rapidly. One technique which has been used with success is Mueller's Iteration Method [31]. For this method, the convergence is quadratic in most cases. In order to use the method for a given state vector x , two values, T_a and T_b of the parameter T must be found, such that

$$T_1 \leq T_a \leq T_b \leq T_N \quad (4.34)$$

and

$$\mathcal{F}(T_a)\mathcal{F}(T_b) \leq 0, \quad (4.35)$$

where $\mathcal{F}(T)$ denotes the function $[\bar{V}(x, T) - T]$, evaluated at the given value of x . The procedure for finding T_a and T_b is based on selecting discrete values of the parameter T , on the interval $[T_1, T_2]$. These discrete values are selected according to the equation,

$$T = T_1 + (j-1)\delta T, \quad j = 1, 2, \dots, T \leq T_N, \quad (4.36)$$

where the value of the time interval δT depends on the given system. Initially, the value of T_1 is selected and then the sign of $\mathcal{F}(T_1)$ is evaluated and stored. At each succeeding step, the sign of $\mathcal{F}(T)$ is evaluated and relation (4.35) is checked with $T_a = T_1$ and $T_b = T$. The procedure continues until relation (4.35) is satisfied. Suppose this occurs at the $(k\text{-th})$ step. Then T_a and T_b are given by,

$$T_a = T_1 + (k-2)\delta T \quad (4.37a)$$

and

$$T_b = T_a + \delta T. \quad (4.37b)$$

Notice that the largest possible value has been chosen for T_a in equation (4.37a). This is done in order to increase the speed of operation of the subsequent method.

When the values of T_a and T_b are supplied, the method rapidly obtains a value of the parameter T , say $T = \hat{T}$, such that $T_a \leq \hat{T} \leq T_b$.

and

$$|\mathcal{F}(\hat{T})| \leq \epsilon. \quad (4.38)$$

In the above equation, the positive number ϵ denotes the prescribed error bound. From the definition of the function $\mathcal{F}(T)$, it is seen that $T = \hat{T}$ is an approximate solution to equation (4.33). The accuracy of this solution is dependent on the value of the error bound ϵ . As the value of ϵ decreases, the accuracy of the method increases. However, the required operation time of the method also increases. Therefore the value of ϵ should be chosen so as to obtain a compromise between the accuracy and the speed of operation of the method. This value is dependent on the particular system under consideration.

It should be pointed out that equation (4.33) may have multiple roots. Therefore relation (4.32) should be tested at the obtained value of $\hat{T}$. If this relation is not satisfied, then another root of equation (4.33) must be found. This is accomplished by continuing the selection procedure of equation (4.36), until new values are obtained for T_a and T_b . The method is then repeated for these new values.

In summary, at most sampling instants the value of the parameter T is obtained from equation (4.31). However, at the beginning of the control interval and possibly at certain sampling instants during the control interval, the value of the parameter T is obtained by iteratively solving equation (4.33).

(II) Obtaining the Sub-Optimal Control

Once the value of the parameter T is calculated, then the sub-optimal control is obtained from equation (4.7). As discussed in

Section 4.7, the function $\bar{V}(x, T)$ should be used in place of $V(x, T)$ in this equation.

(III) Monitoring the Termination Criterion

The Termination Criterion will usually be an inequality involving some of the components of the state vector x . This prescribed criterion is checked at each sampling instant. When the Termination Criterion is satisfied, the sub-optimal controller is disengaged.

In this section, the on-line operation of the sub-optimal controller has been discussed. This completes the description of the sub-optimal method. In the next section, this method of solution will be compared with the standards established by the four motivating criteria of Section 4.1.

4.10 Consideration of the Motivating Criteria

Criterion 1 of Section 4.1 states that the sub-optimal solution should be related to the optimal solution. From the discussion in Section 4.2, it is seen that the proposed sub-optimal solution is based on the approximation of the optimal function $W(x)$. Therefore, this criterion is indeed satisfied.

The second criterion of Section 4.1 states that the method of solution should be applicable to general systems. In the preceding sections, the procedure for applying the sub-optimal method of Section 4.2, has been discussed. The results of these sections are a set of sequential procedural steps that can indeed be applied to general systems.

Throughout the entire development of the sub-optimal method of Section 4.2, it has been shown that there is an engineering trade-off

between the performance and the complexity of the sub-optimal controller. In other words, the quality of the performance of the sub-optimal controller is directly related to the corresponding level of allowed complexity. Also, the implementation of the sub-optimal controller for general systems has been discussed in Section 4.9. It has been shown that the implementation is accomplished with the use of an on-line digital computer. These results indicate that the sub-optimal controller does satisfy Criterion 3 of Section 4.1.

The last criterion of Section 4.1 concerns the application of the sub-optimal method to high order practical systems. In the next section, such a system is presented as an illustrative example.

4.11 Illustrative Example

The Ranger Block III spacecraft control problem presented in Section 3.11, will also be used to illustrate the application of the sub-optimal method to practical systems. This will facilitate the comparison of the resulting sub-optimal closed-loop solution to the minimal time open-loop solution of Chapter III.

With reference to Section 3.11, the system to be controlled is represented by equation (3.78) along with the constraints, equations (3.79) and (3.80). In these equations, θ denotes the pitch angle deviation and q denotes the input control torque. From the discussion in Section 3.11, the closed-loop minimal time problem can be stated as follows. Find an admissible feedback controller that can drive every non-zero value of the pitch deviation θ to the origin, in minimum time. In this problem, a feedback controller is admissible if the corresponding control torque q always satisfies the magnitude and rate constraints

given by equations (3.79) and (3.80).

The sub-optimal method developed in this chapter, will be used to obtain an admissible sub-optimal feedback controller for the above system. In order to apply this method, the system must be reformulated as a bounded state variable problem in the form given by system (2.11). This can be accomplished by the "direct programming" method used in Section 3.11. However, in order to simplify the presentation, in this section the reformulated problem will operate on a different time scale. For convenience, let t denote the time scale of the reformulated bounded state variable system and let τ denote the time scale of the original system (3.78). The relationship between the two time scales is given by,

$$t = (18.43)\tau. \quad (4.39)$$

With the aid of equation (4.39), system (3.78) along with the constraints, equations (3.79) and (3.80), can be written as the following reformulated 5-th order system.

$$dx/dt = Ax + Bv, \quad (4.40)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and with the constraints,

$$|x_5| = |u| = |q/6.69| \leq 1 \quad (4.41a)$$

and

$$|dx_5/dt| = |v| \leq 3.925 \quad (4.41b)$$

The transformation between system (4.40) and system (3.78) is given by equation (4.39) along with the equations, $\theta = 1.874 \times 10^{-4}(0.974x_1 + x_3)$, $d\theta/d\tau = 3.4 \times 10^{-3}(0.992x_2 + 1.018x_4)$, $d^2\theta/d\tau^2 = 61.4 \times 10^{-3}(-0.037x_3 + 1.037x_5)$, $d^3\theta/d\tau^3 = 1.13(-0.037x_4 + 1.037v)$, $q = 6.69x_5$ and $dq/d\tau = 123.3v$.

The basic procedure to be followed in the remaining sections, is now summarized. First, the sub-optimal method of this chapter will be applied to system (4.40) to obtain the controller $\bar{v}(x)$ of equation (4.7). Next, this sub-optimal closed-loop solution will be compared to the minimal time open-loop solution $v^*(t)$, for a representative set of initial conditions. This is done in order to illustrate some of the general results of the sub-optimal method, as applied to system (4.40). Notice that the above comparison is not difficult to carry out, since the open-loop minimal time controls are already computed as part of the sub-optimal procedure. After this comparison is completed, the sub-optimal closed-loop solution for the practical system (3.78), will be obtained. This is accomplished by applying the above transformation relations. Then, the performance of the sub-optimal controller for system (3.78) will be evaluated. The procedure of evaluation is similar to the one used for system (4.40). Finally, the sub-optimal feedback controller for system (3.78) will be compared to the presently

used proportional plus derivative feedback controller on the Ranger.

The first consideration of the sub-optimal procedure is the specification of the bounded maximal set $M \subset \hat{G}$. As discussed in Section 4.2, the size of this set is determined by the particular properties of the given system. The only available data from Reference 4, is that the initial pitch displacements of system (3.78) are in the order 5 m-rad [4, p.93]. This data was obtained from the analog simulation of the pitch axis midcourse maneuver response. There is no information available concerning the maximum disturbed values of the derivatives of the pitch displacement, i.e., $d\theta/dr$, $d^2\theta/dr^2$, etc. Because of the absence of this information, the maximal set M cannot be completely specified. Therefore, for this illustrative example, an alternate approach must be used for specifying the set M , viz., selecting the value of the parameter T_N . From Section 4.4, the value of the parameter T_N determines the largest set of recoverability, $K(T_N)$. Once $K(T_N)$ is specified, the corresponding maximum disturbed values of the pitch deviation θ are also determined, by applying the transformation relations to the state vectors $x \in K(T_N)$. For this problem, the numerical value of T_N was selected to be,

$$T_N = 18.5 \text{ seconds.} \quad (4.42)$$

From equation (4.39), this value of T_N corresponds to approximately 1 second on the time scale τ , of system (3.78). Also, the corresponding maximum pitch deviations were found to be in the order of 25 m-rad. These pitch deviations are approximately five times as large as the given expected values obtained from Reference 4. Therefore, it is reasonable to assume that the value of T_N given by equation (4.42), is

sufficiently large.

With the above value of T_N specified, Step (1) of Section 4.2 may now be applied to obtain the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, of system (4.40). This is discussed in the next section.

4.12 Application of Step (1)

In order to construct the isochrones $I(T_k)$, $k = 1, 2, \dots, N$, the values of the initial adjoint vectors $\Psi(0)$, must be selected for system (4.40). Since this system is of order $n+m=5$, then as discussed in Section 4.5, a discrete subset of the partitioned values of the initial adjoint vector $\Psi(0)$ must be chosen. These values are to be chosen so as to insure that a wide variety of each of the three distinctive characteristics established in Section 4.5, are present in the adjoint solutions. These characteristics are (i) the number of extrema of each component $\Psi_{n+j}(t)$, $j = 1, 2, \dots, m$, on the time interval $(0, T)$, (ii) the relative positions of these extrema, and (iii) the value of the last zero crossing of each function $f(t) = \Psi_{n+j}(T-t)$, $j = 1, 2, \dots, m$. Following the procedure outlined in Step (1c) of Section 4.5, a preliminary analysis of the adjoint solution of system (4.40) is presented below.

Since system (4.40) has a scalar control input, then $m = 1$, and therefore only the component $\Psi_5(t)$ of the adjoint solution need be analyzed. The procedure for finding $\Psi_5(t)$ is similar to that used in Section 3.11. The resulting equation for the function $\Psi_5(t)$ of system (4.40) is given by,

$$\begin{aligned} \Psi_5(t) = & \mu_1(t^2/2 - 1 + \cos t) - \mu_2(t - \sin t) + \mu_3(1 - \cos t) \\ & - \mu_4 \sin t + \mu_5 \end{aligned} \quad (4.43)$$

where $\Psi(0) = \mu = [\mu_1 \mu_2 \mu_3 \mu_4 \mu_5]$. Equation (4.43) describes the function $\Psi_5(t)$ for a general choice of the initial adjoint vector $\Psi(0) = \mu$. The problem is, how to select the values of μ_i , $i = 1, 2, \dots, 5$, so as to provide a wide variety of the above three distinctive characteristics? In order to obtain a better understanding of how each component μ_i , $i = 1, 2, \dots, 5$, affects the form of the function $\Psi_5(t)$, define

$$\sigma = \sqrt{(\mu_1 - \mu_3)^2 + (\mu_2 - \mu_4)^2}, \quad (4.44a)$$

$$\beta = \tan^{-1} \left(\frac{\mu_2 - \mu_4}{\mu_1 - \mu_3} \right) \quad (4.44b)$$

and

$$\gamma = (\mu_3 + \mu_5 - \mu_1). \quad (4.44c)$$

With the aid of the above three definitions, equation (4.43) can be written as,

$$\Psi_5(t) = \sigma \cos(t - \beta) + \mu_1 t^2 / 2 - \mu_2 t + \gamma \quad (4.45)$$

First, consider characteristics (i) and (ii), which involve the number and relative positions of the extrema of $\Psi_5(t)$ on $t \in (0, T)$. The equation determining the interior extrema of $\Psi_5(t)$ on any interval $t \in (0, T)$, is given by

$$d\Psi_5(t)/dt = 0 = \sigma \sin(t - \beta) - \mu_1 t + \mu_2. \quad (4.46)$$

Notice that the value of μ_5 only affects the determination of γ . Therefore, since equation (4.46) does not involve γ , then characteristics (i) and (ii) are independent of the choice of μ_5 . Also, note

that equation (4.46) is not affected, if each value of μ_i , $i=1, \dots, 4$, is multiplied by a constant. Therefore, there is no loss of generality in choosing μ_i , $i = 1, \dots, 4$, to satisfy the equation,

$$\left(\sum_{i=1}^4 \mu_i^2 \right)^{1/2} = 1. \quad (4.47)$$

In order to most easily observe the variation of characteristics (i) and (ii), the analytical solution to equation (4.46) as a function of σ , β , μ_1 and μ_2 , should be obtained. In general, it is not possible to analytically solve equation (4.46). However, an analytical solution is possible in the special case of $\sigma = 0$. From equation (4.44a), this case occurs when

$$\mu_1 = \mu_3 \quad \text{and} \quad \mu_2 = \mu_4 \quad (4.48)$$

If $\sigma = 0$, then equation (4.46) yields,

$$t^* = \mu_2 / \mu_1, \quad \mu_1 \neq 0, \quad (4.49)$$

where t^* denotes the time instant of the extremum of $\Psi_5(t)$. Therefore if μ_i , $i = 1, \dots, 4$, are chosen to satisfy equation (4.48), then the resulting function $\Psi_5(t)$ will have exactly one extremum. Furthermore, the value of the time instant of this extremum is determined by the value of the ratio μ_2/μ_1 .

The above analysis indicates that for this special case of one extremum, the ratios μ_2/μ_1 should be chosen so as to provide a wide variety of characteristic (ii). For each interval of time $(0, T)$, this can be accomplished by first choosing the values of $t^* \in (0, T)$ at equally spaced intervals. Then, for each value of t^* , the corresponding values

of μ_i , $i = 1, \dots, 4$ are obtained by combining the results of equations (4.47) through (4.49). This yields,

$$\mu_1 = \mu_3 = [2(1 + (t^*)^2)]^{-1/2} \quad (4.50a)$$

and

$$\mu_2 = \mu_4 = t^* \mu_1. \quad (4.50b)$$

For system (4.40), the values of $t^* \in (0, T)$ were chosen at intervals of 1 second. Also, the value $t^* = -1$ was chosen. This corresponds to the case of no extrema of $\Psi_5(t)$ on $t \in (0, T)$.

By choosing μ_i , $i = 1, \dots, 4$, according to the above procedure, a wide variety of characteristic (ii) will be included in the resulting functions $\Psi_5(t)$. This completes the analysis for the functions $\Psi_5(t)$ which have at most one extremum on $t \in (0, T)$. The remaining cases ($\sigma \neq 0$) of $\Psi_5(t)$ with two or more extrema on $t \in (0, T)$, are more difficult to analyze. In these cases, a numerical procedure must be used to observe the variation of characteristics (i) and (ii) with respect to the choices of μ_i , $i = 1, \dots, 4$. This analysis is most easily accomplished by first expressing μ_i , $i = 1, \dots, 4$, in the following form. Let,

$$\begin{cases} \mu_1 = \cos\phi_1 \cos\phi_2 \cos\phi_3, \\ \mu_2 = \sin\phi_1 \cos\phi_2 \cos\phi_3, \\ \mu_3 = \sin\phi_2 \cos\phi_3, \\ \mu_4 = \sin\phi_3, \end{cases} \quad (4.51)$$

where $0 \leq \phi_1 < \pi$ and $|\phi_i| \leq \pi/2$, $i = 2, 3$. Then each set of values μ_i , $i = 1, \dots, 4$, that determine a different solution to equation (4.46), can be represented by equation (4.51). Therefore, it is only necessary

to analyze the solutions to equation (4.46), as a function of the choices for the angles ϕ_i , $i = 1, 2, 3$. The numerical procedure that was followed, is outlined below.

First, the angles ϕ_i , $i = 1, 2, 3$, were partitioned at intervals of 10 degrees. Then for each choice of ϕ_i , $i = 1, 2, 3$, the extrema of $\Psi_5(t)$ on the time interval $(0, T_N) = (0, 18.5)$, were found by solving equation (4.46). This was accomplished by using Subroutine PSICLC [Appendix C]. Next, the partitioned values of the angles were classified, according to the number of extrema of the corresponding functions $\Psi_5(t)$ on $(0, T_N)$. It was found that a total number of 11 classifications were required (two extrema through twelve extrema). For each of these classifications, a subset of the angles was chosen so as to provide a wide variety of characteristic (ii). In other words, the angles were chosen so that the extrema of the corresponding set of functions $\Psi_5(t)$, occurred at a wide variety of time instants on the time interval $(0, T_N)$. This was accomplished by observing the actual numerical values of the time instants of the extrema for each classification. Then, the cases which had relatively similar numerical values were eliminated.

In the above manner, a set of values for ϕ_i , $i = 1, 2, 3$, consisting of the combined subsets of each classification, was obtained. Since this set of values includes members from each classification, then a wide variety of characteristic (i), is also present in the resulting functions $\Psi_5(t)$. Therefore, the above procedure yields a set of values for ϕ_i , $i = 1, 2, 3$, that provide a wide variety of both characteristics (i) and (ii). The actual result of this numerical procedure was a set of 229 values for the angles ϕ_i , $i = 1, 2, 3$. These values are presented in Table D.1.

Once the values of μ_i , $i = 1, \dots, 4$, are selected, it remains only to choose the values of μ_5 . These values can be selected by considering characteristic (iii). This characteristic involves the value of the last zero crossing of the function $f(t) = \bar{Y}_5(T-t)$, on the time interval $[0, T]$. Since μ_5 only appears as an additive constant in equation (4.43), then for each given choice of μ_i , $i = 1, \dots, 4$, the value of the last zero crossing of the function $f(t)$ is dependent upon the value of μ_5 . As an example, consider some arbitrary choice for the values of μ_i , $i = 1, \dots, 4$. Suppose that it is desirable for the last zero crossing of $f(t)$ to occur at the time instant $t = T - \epsilon$, where ϵ is some small number, $\epsilon < T$. First, assume that $f(t)$ does not have an extrema on $t \in (T - \epsilon, T)$. Then the value of μ_5 should be given by

$$\mu_5 = -\bar{f}(T - \epsilon) \quad (4.52)$$

where $\bar{f}(t)$ denotes the function $f(t)$, with μ_5 identically zero. Of course, if $f(t)$ has at least one extremum on $t \in (T - \epsilon, T)$, then it may not be possible for the last zero crossing to occur at the time instant, $t = T - \epsilon$.

The above example indicates that the particular choices of μ_5 are dependent on the numerical values of the function $\bar{f}(t)$. Another factor which must be considered in choosing values for μ_5 , is indicated by the discussion of illustrative cases (4) and (5) in Section 3.7. From this discussion, it is seen that the value of the last zero crossing of $f(t)$ is a distinctive characteristic, only if it occurs on the time interval $[T - 2/k, T]$. Here, k denotes the value of the rate constraint, i.e., $k = 3.925$. Therefore, in order to satisfy this factor and also to provide a wide variety of characteristic (iii), the values of μ_5

were chosen in the following manner.

For each choice of μ_i , $i = 1, \dots, 4$, five values of μ_5 were selected. These values are given by,

$$\mu_5^j = -\bar{f}(T-jk/2) \quad , \quad j = 0, 1, \dots, 4, \quad (4.53)$$

where μ_5^j denotes the j -th value of μ_5 . As indicated by the above example, these choices for μ_5 should generally yield a set of five different values for the last zero crossings of $f(t)$. Furthermore, these crossings should generally occur on the time interval $[T-2/k, T]$.

In summary, for system (4.40) the initial values of the adjoint vector $\Psi(0) = \mu$, were chosen according to the following procedure. The first set of values of μ_i , $i = 1, \dots, 4$, were obtained from equation (4.51), for the given numerical values of ϕ_i , $i = 1, 2, 3$, in Table D.1. Also, for each given time interval $[0, T]$, additional values of μ_i , $i = 1, \dots, 4$, were obtained from equation (4.50), for the chosen numerical values of t^* . Finally, for each obtained set of values for μ_i , $i = 1, \dots, 4$, the corresponding five values of μ_5 , were obtained from equation (4.53).

With the values of $\Psi(0) = \mu$ determined, the basic isochrone construction procedure is now established for system (4.40). At each value of the partitioning time instant T_k , $k = 1, 2, \dots, N$, this procedure is as follows. For each of the above values of $\Psi(0)$, the corresponding extremal control on the time interval $[0, T_k]$ is obtained, by using the computer program developed in Chapter III. Then the state vector x on the isochrone $I(T_k)$ is obtained from equation (4.11). This procedure is repeated for each value of $\Psi(0)$. There are approximately 1250 different choices for the initial adjoint vector $\Psi(0)$ (250×5).

Therefore, the result of this procedure is a set of about 1250 computed state vectors on each isochrone $I(T_k)$.

The above procedure is identical for each choice of T_k , $k = 1, 2, \dots, N$. The only remaining problem is the specification of the values of T_k , $k = 1, 2, \dots, N$. The value of T_N has already been specified in Section 4.11 to be 18.5 seconds. The remaining values are obtained by following the procedure outlined in Section 4.4. For system (4.40), it was found that the shape of the isochrones changed rather slowly, for values of T_k in the interval, $5.9 \leq T_k \leq 18.5$. This was reflected in the relatively small changes of the approximation matrices $\bar{C}_k$ and $\tilde{C}_k$, as the value of T_k was decreased from 18.5 seconds. Therefore, from 5.9 seconds through 18.5 seconds, the values of the parameter T_k were taken at intervals of 0.2 seconds, i.e., from Section 4.5, $\Delta T = 0.2$ seconds. The spacing interval ΔT was then reduced to 0.1 seconds. It was found that the value $\Delta T = 0.1$ was sufficient, until T_k was decreased to 2.1 seconds. At this point, some of the element values of the matrices $\bar{C}_k$ and $\tilde{C}_k$ began to increase quite rapidly. Therefore, for values of $T_k < 2.1$ seconds, the value of ΔT was decreased to 0.05 seconds.

The value of the smallest parameter T_1 was chosen to be 1.3 seconds. It was found that any further decrease in the value of T_1 resulted in excessively high element values for the corresponding matrices $\bar{C}_1$ and $\tilde{C}_1$. This aspect of the method was discussed in Section 4.4. From equation (4.39), the value of 1.3 seconds for T_1 corresponds to 0.0705 seconds on the time scale τ , of system (3.78). Also, the corresponding pitch deviations are in the order of 0.1 m-rad. Therefore from the discussion in Section 4.11, the sub-optimal

controller for system (3.78) should operate for pitch deviations from approximately 0.1 m-rad through 25 m-rad. As discussed in Section 4.4, a dual controller scheme could be used to drive the pitch deviations of less than 0.1 m-rad, to the origin.

Altogether, a total of 118 time instants T_k , were chosen for system (4.40) (i.e., $N=118$). At each of these time instants, the corresponding isochrone was constructed by computing the values of approximately 1250 state vectors. This was accomplished by using the computer program of Chapter III.

The second step of the sub-optimal method can now be applied. This step involves the calculation of the elements of the matrices $\bar{C}_k$ and $\tilde{C}_k$ for $k = 1, 2, \dots, 118$. The results of this calculation are presented in the next section. Notice that these matrices are actually obtained as part of the procedure for choosing the values of T_k , $k = 1, 2, \dots, N$. However, in order to simplify the presentation of the results, the actual calculation of $\bar{C}_k$ and $\tilde{C}_k$ is discussed in a separate section.

4.13 Application of Step (2)

At each value of T_k , $k = 1, 2, \dots, 118$, the numerical values of the elements $\tilde{c}_{ij}$ and $\bar{c}_{ij}$, are obtained by applying the two least mean squares fits described in Section 4.7. The governing equations for these two least mean squares fits are given by equations (4.18) and (4.23).

Since system (4.40) is of 5-th order, it requires a total of 15 independent element values to describe each 5 by 5 symmetric matrix $\bar{C}_k$ and $\tilde{C}_k$. Therefore, as discussed in Section 4.7, the resulting

values of the 15 independent elements of each matrix, are obtained by solving a set of 15 linear equations.

The above procedure was carried out at all of the 118 specified values of the time instants T_k . The resulting numerical values of the independent elements are presented in Tables D.2 through D.7. Each of these tables gives the computed values of five of the elements, along with the corresponding specified value of T_k , $1.3 \leq T_k \leq 18.5$. There are a total of 30 elements with 118 numerical values per element, presented in these tables.

There are two general features that can be observed, concerning the computed numerical values of the elements. These features are discussed below.

(a) First, notice that in most cases, the computed values of each element $\tilde{c}_{ij}$, $i = 1, \dots, 5$, $j = 1, \dots, 5$, are reasonably close to the values of the corresponding element $\bar{c}_{ij}$. This is illustrated in Figure 4.1, for the computed values of the elements $\tilde{c}_{15}$ and $\tilde{c}_{15}$.

(b) In Figure 4.1, the log of the element values is plotted vs. the parameter T . This points out another interesting feature about the computed element values, viz, their wide range of variation. This feature is most pronounced for the values of the element $\tilde{c}_{11}$. From Table D.2, it is seen that the element $\tilde{c}_{11}$ has values ranging from 2.5×10^5 at $T_1 = 1.3$ seconds, to 6.08×10^{-3} at $T_{118} = 18.5$ seconds. With reference to Tables D.2 through D.7, it can be concluded that almost all of the elements attain a wide range of values on the interval, $1.3 \leq T \leq 18.5$. The only exceptions are the elements $\tilde{c}_{45}$, $\bar{c}_{45}$, $\tilde{c}_{55}$ and $\bar{c}_{55}$.

The above two features will be discussed in more detail in the

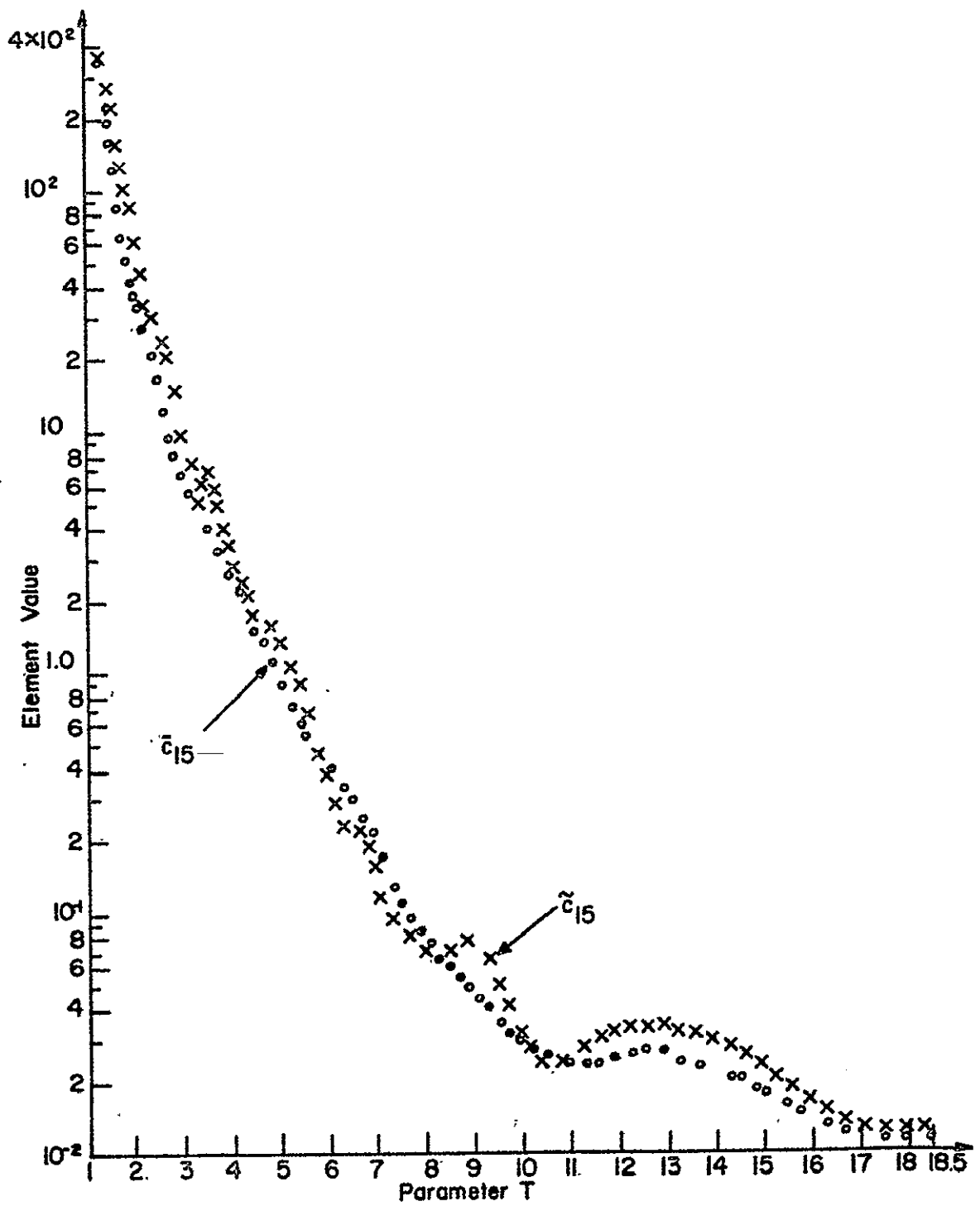


FIGURE 4.1. COMPARISON OF THE COMPUTED VALUES OF $\tilde{c}_{15}$ AND $\bar{c}_{15}$

following section. At this point, there are two general remarks that should be added. First, a total of 2 3/4 hours of off-line computation time was used to reach this point of the sub-optimal method. It should also be pointed out that the obtained matrices $\bar{C}_k$ and $\tilde{C}_k$, $k = 1, 2, \dots, 118$, were all tested and were all found to be positive definite. From the discussion in Section 4.7, this is an indication that all of the corresponding approximations $\bar{V}(x, T_k)$ and $\tilde{V}(x, T_k)$, are reasonable.

The next section will discuss the application of Step (3) of Section 4.2, to obtain the approximations for the computed values of the elements.

4.14 Application of Step (3)

The element approximation functions are obtained, by following the general procedure outlined in Section 4.7. For system (4.40), there are a total of 30 sets of element values to be approximated. In the preceding section, two general features of these computed values have been established. These features will prove useful in the development of the element approximations.

Feature (a) of Section 4.13 indicates that the values of each element $\bar{c}_{ij}$, are reasonably close to the values of the corresponding element $\tilde{c}_{ij}$. Therefore, the form of the approximation function selected for each element $\bar{c}_{ij}$, should also be used for the corresponding element $\tilde{c}_{ij}$.

Feature (b) of Section 4.13 indicates that almost all of the elements attain a wide range of values. As discussed in Section 4.7, this implies that for these elements it may be extremely difficult to

approximate the entire set of element values, with only one approximation function.

In order to illustrate the application of the procedure outlined in Section 4.7 to a specific system, the actual sequence of steps that was followed for system (4.40) is described below, in some detail.

Initially, Feature (b) of Section 4.13 was disregarded, and the general power series approximation scheme of Section 4.7 was followed. Each power series was selected to be of 8-th order, i.e., from equation (4.25), $Q_1 = 8$, $i = 1, 2, \dots, 30$. The least mean squares fit governed by equation (4.26) was then applied, to obtain the values of the coefficients of each power series. The resulting approximation for the element $\bar{c}_{33}$ is presented in Figure 4.2. From this figure, it is obvious that the 8-th order power series is not a reasonable approximation to all of the computed numerical values of the element. The approximation function follows the variation of the computed values, only on the interval $1.3 \leq T \leq 2.2$. For larger values of the parameter T , the function begins to oscillate about zero. Notice that the negative excursions are not shown in Figure 4.2. This is because the scale on the ordinate axis is logarithmic. The approximation function actually oscillates between the values of about ± 200 .

The general characteristics exhibited by the 8-th order power series approximation for element $\bar{c}_{33}$, were found to be the same for almost all of the other element approximations. That is, each approximation would follow the variation of the computed values of the corresponding element, only for values of the parameter T in some small neighborhood of $T_1 = 1.3$. As the value of T increased, the approximations would begin to oscillate about zero. The only exceptions were the approximation

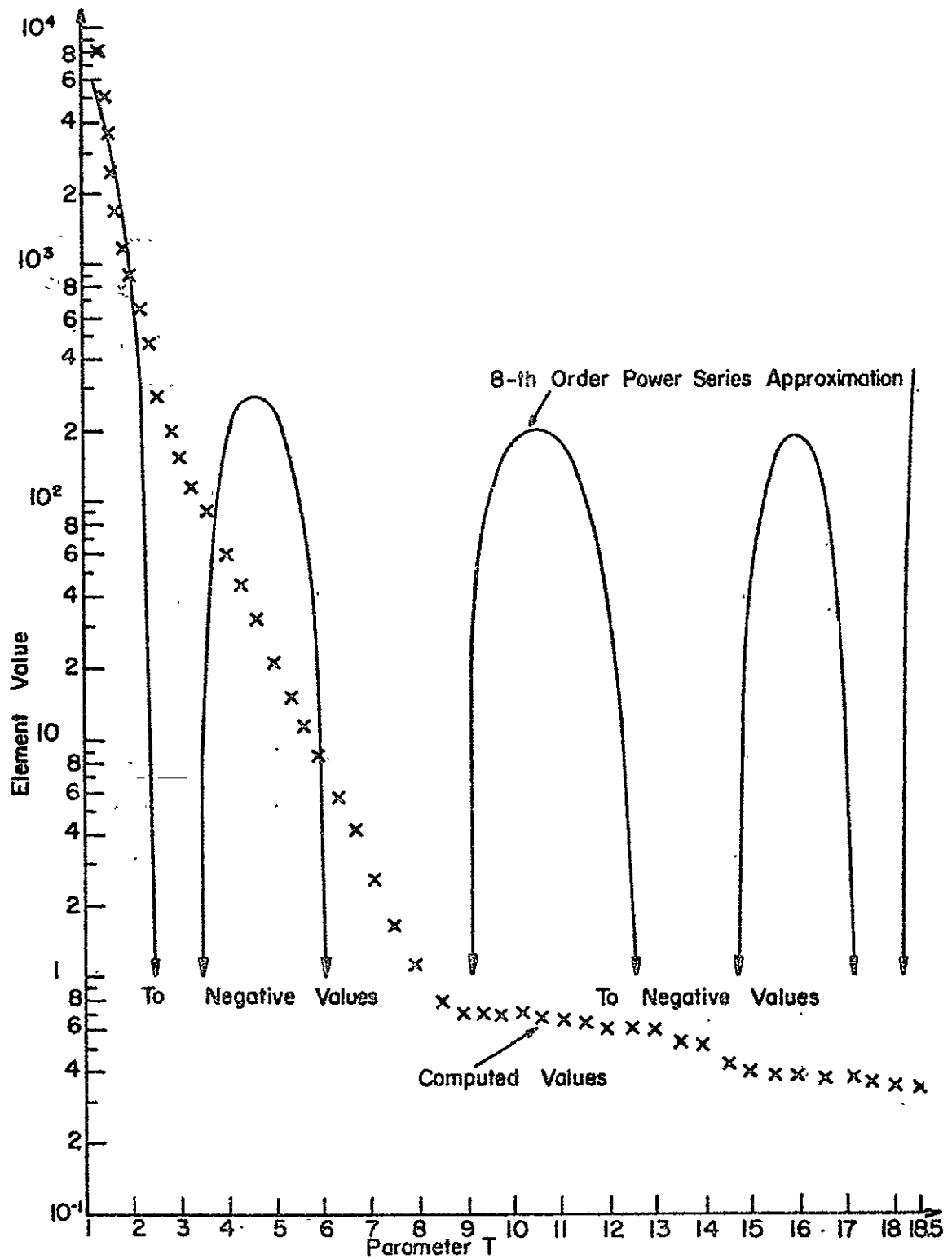


FIGURE 4.2. INITIAL POWER SERIES APPROXIMATION FOR ELEMENT c_{33} .

functions for the elements $\tilde{c}_{45}$, $\bar{c}_{45}$, $\tilde{c}_{55}$ and $\bar{c}_{55}$. For these four elements, the 8-th order power series approximations were reasonably accurate over the entire range of the parameter T , i.e., $1.3 \leq T \leq 18.5$.

The above discussion indicates that 26 of the 30 elements cannot be reasonably approximated by an 8-th order power series. Therefore, the accuracy of the approximations for these 26 elements must be improved. In the following discussion three possible methods for improving the accuracy of the 26 element approximations, are described. These three methods were established for general systems, in Section 4.7.

One possible method of approach is to increase the order of each power series. This method was applied to the computed element values of system (4.40). However, it was found that the accuracy of the approximations were not significantly improved, even when the order of each power series was raised to 16.

Another possible method of approach is to choose the form of each approximation function according to the particular characteristics of the corresponding element values. This method of approach was also applied to the computed element values of system (4.40). It was found that the 26 sets of element values all exhibited a certain degree of linearity, when plotted on logarithmic axes. Therefore, it seems reasonable to select the form of each approximation function, so that the function appears as a straight line on a log-log plot. With reference to the notation used in Section 4.7, this implies that the 26 element approximation functions should be of the form,

$$d_i^*(T) = \beta_i T^{\gamma_i}, \quad i = 1, 2, \dots, 26. \quad (4.54)$$

For the functions $d_i^*(T)$, $i = 1, 2, \dots, 26$, defined in equation (4.54), the values of β_i and γ_i can be obtained by the general least mean squares procedure of Section 4.7. However in this case, the least mean squares fits must be applied to the logarithms of the computed element values. With reference to the notation used in equation (4.26), the governing equation for the least mean squares fits for the above form of approximation function, is as follows. For each element d_i , $i = 1, 2, \dots, 26$, choose the values of β_i and γ_i to minimize LP_i , where,

$$LP_i = \sum_{k=1}^{118} \left[\log(d_i^*(T_k)) - \log d_{ik} \right]^2 \quad (4.55)$$

From equation (4.54), the above equation can be written as,

$$LP_i = \sum_{k=1}^{118} \left[\log \beta_i + \gamma_i \log T_k - \log d_{ik} \right]^2. \quad (4.56)$$

The function obtained by using the governing equation (4.56) will be called a log approximation. This is done in order to distinguish it from the power series approximation obtained by using the governing equation (4.26).

The least mean squares fit governed by equation (4.56), was applied to the computed element values of system (4.40). The resulting log approximation for the element $\bar{c}_{33}$, is shown by the log-log plot in Figure 4.3. From this figure, it is seen that the log approximation follows the variation of the computed values over a parameter interval of about, $1.9 \leq T \leq 5.5$. For values of T greater than about 5.5, the approximation degenerates. This occurs because the computed values of the element $\bar{c}_{33}$, are now no longer varying in a near linear fashion. However, observe that the variation of the computed values is nearly

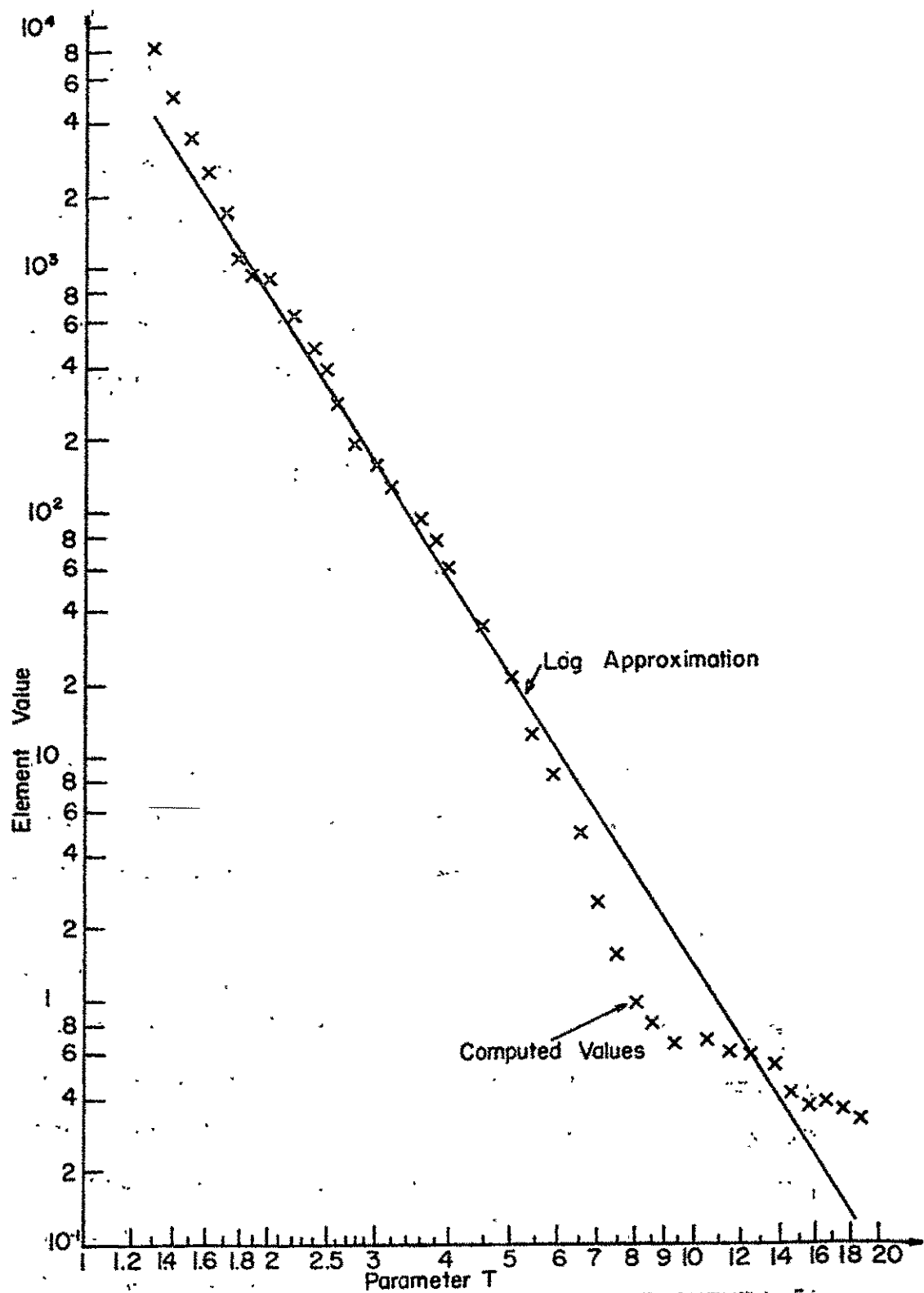


FIGURE 4.3 . INITIAL LOG APPROXIMATION FOR ELEMENT c_{33}

linear from $T=1.3$ seconds to $T=5.5$ seconds. This indicates that a log approximation to the computed values belonging to $1.3 \leq T \leq 5.5$, should be reasonably accurate over this parameter interval.

These general characteristics of the log approximation for the element $\bar{c}_{33}$ were also observed for the other 25 elements. The computed values varied in a near linear fashion, for values of the parameter T of less than about 6 seconds. The corresponding log approximations were found to be fairly reasonable over this parameter interval. However for values of T greater than 6 seconds, the log approximations were no longer able to follow the variation of the computed element values.

The basic drawback of the two preceding methods, is that they both involve the use of only a single approximation function for each set of 118 computed element values. The difficulty is that the 26 elements under discussion have a wide range of computed values over the entire parameter interval (Feature (b) of Section 4.13). Therefore by the nature of the least mean squares fit, the much larger computed values tend to dominate the resulting approximation. This was best illustrated in Figure 4.2. Observe that the 8-th order power series approximation in Figure 4.2 is only reasonable for the initial larger computed values. The same result is illustrated by Figure 4.3. Again, the approximation favors the larger computed element values. Notice that this feature is less apparent in Figure 4.3. This is because the range of the computed values is essentially compressed, when the least mean squares fit governed by equation (4.56) is used.

The above discussion indicates a third possible method of approach, viz., the use of multiple approximation functions for each element. This method was applied to system (4.40). However, it was decided to fix the

number of approximation functions to be two per element, i.e., a dual approximation. This was done in order to limit the complexity of the resulting sub-optimal controller. It was also decided to use the dual approximation approach for all 30 elements. Notice that in the earlier part of this section, it was stated that the computed values of the elements $\bar{c}_{45}$, $\bar{c}_{45}$, $\bar{c}_{55}$ and $\bar{c}_{55}$ could be approximated by a single 8-th order power series. However, it was felt that the computed values of these four elements could be approximated much more accurately, if a dual approximation scheme were used.

The general procedure for obtaining a dual approximation is described in Section 4.7. The first step of this procedure is the choice of some intermediate value of the parameter T . This intermediate value is denoted by T_m . The parameter T_m divides the entire parameter interval into two disjoint intervals, viz., $1.3 \leq T < T_m$ and $T_m \leq T \leq 18.5$. From Section 4.7, the value of T_m should be chosen so that the computed element values on each interval are in the same general range. Obviously, a different value of T_m could be chosen for each of the elements. However for system (4.40), the value of T_m was taken to be fixed, for all of the elements. Again, this was done in order to limit the complexity of the resulting sub-optimal controller operation.

For the computed element values of system (4.40), the parameter T_m was taken to be,

$$T_m = 5.9 \text{ seconds} \quad (4.57)$$

From Tables D.2 through D.7, it is seen that for $T \geq 5.9$ seconds, the corresponding 30 sets of element values lie in the range of from about 10^{-3} to 10. Also, for $T < 5.9$ seconds, the element values are all generally greater than 1. However, the maximum range of these values is dependent on the particular element. It should be pointed out that for $T < 5.9$ seconds, some

of the elements still have values over an extremely large range. However as indicated in the previous discussion, on this parameter interval, these computed values can be reasonably approximated by using functions in the form given by equation (4.54).

Once the value of T_m is selected, the least mean squares procedure of Section 4.7 is again applied to system (4.40). However, in this case there is an increased flexibility. This is because the form of the two approximation functions for each element need not be identical. In fact, the two approximations can be considered separately. This is the procedure followed in the succeeding discussion.

First, consider the element approximations on the interval, $5.9 \leq T \leq 18.5$. On this interval, the computed element values of system (4.40) are all less than ten. It was found that every element could be accurately approximated on this interval, by using an 8-th order power series form. The resulting least mean squares approximation for the element α_{11} , is presented in Figure 4.4. Observe that the approximation function closely follows the computed values of the element, over the entire interval, $5.9 \leq T \leq 18.5$. The general results of this 8-th order power series approximation on the interval $T \in [5.9, 18.5]$, are presented in Table 4.1. For each of the 30 elements, the numerical values of the nine coefficients that determine the corresponding 8-th order power series approximation are given. These coefficients correspond to α_{1j} , $j = 0, 1, \dots, 8$, defined in equation (4.25).

This completes the approximation procedure for the parameter interval, $5.9 \leq T \leq 18.5$. Now, consider the approximation of the computed element values on the interval, $1.3 \leq T < 5.9$. Initially, the procedure followed for the previous approximation interval was also used on $T \in [1.3, 5.9)$. That is, an 8-th order power series form was chosen for all of the element approximation functions. It was found that the

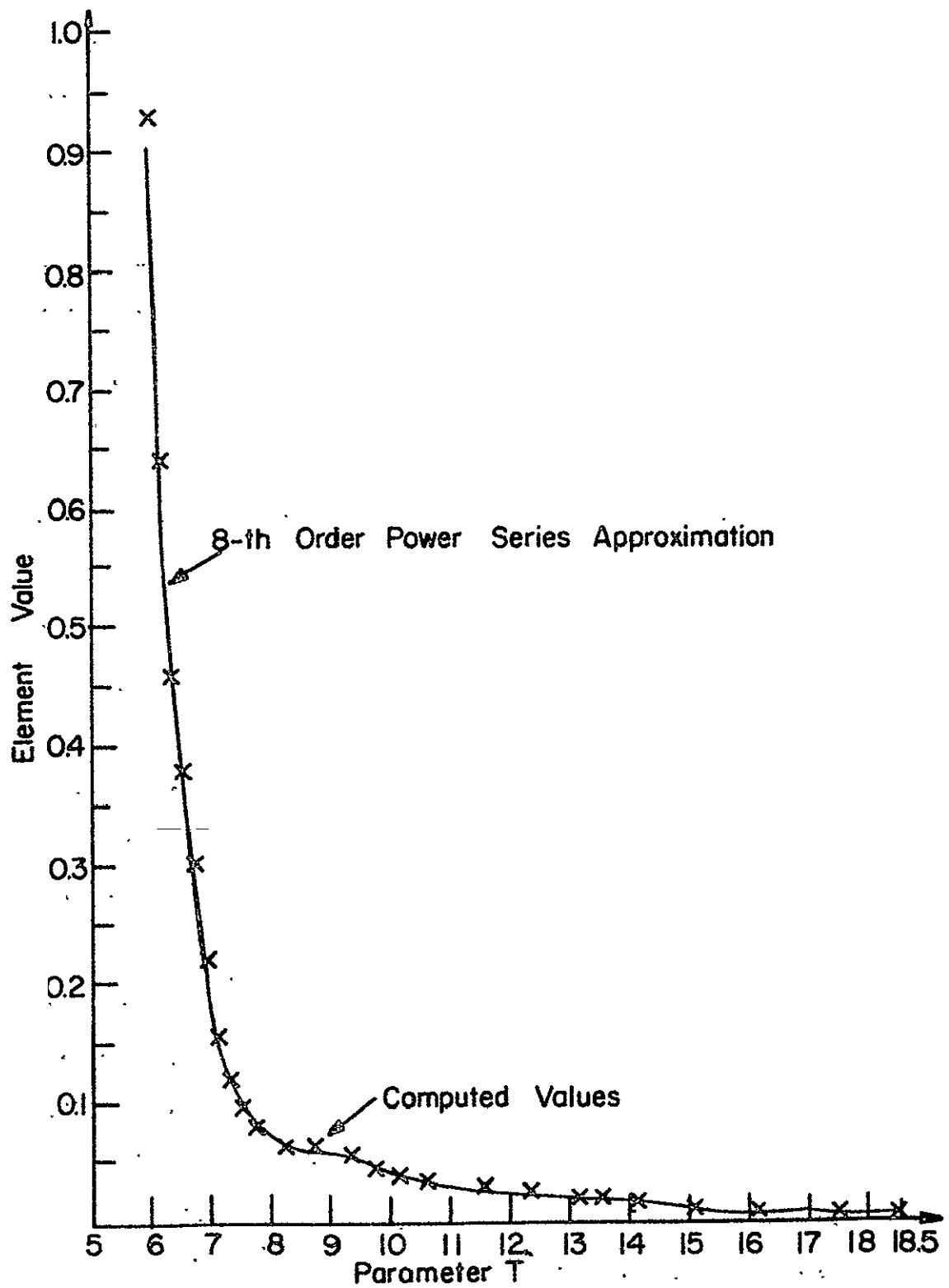


FIGURE 4.4. POWER SERIES APPROXIMATION FOR ELEMENT $\tilde{c}_{11}$ (5.9-18.5).

Table 4.1. 8-th Order Power Series Approximations [5.9,18.5]

Dependent	Value of the Coefficients								
	0-Order	1-Order	2-Order	3-Order	4-Order	5-Order	6-Order	7-Order	8-Order
c_{11}	0.25167E 03	-0.16613E 03	0.47338E 02	-0.76017E 01	0.75250E 00	-0.47034E-01	0.18134E-02	-0.39452E-04	0.37098E-06
c_{12}	0.55593E 03	-0.36055E 03	0.10096E 03	-0.15933E 02	0.15500E 01	-0.95195E-01	0.36056E-02	-0.77033E-04	0.71105E-06
c_{23}	0.59531E 03	-0.39644E 03	0.11478E 03	-0.18836E 02	0.19134E 01	-0.12303E 00	0.49855E-02	-0.10949E-03	0.10600E-05
c_{31}	0.22416E 03	-0.13297E 03	0.33570E 02	-0.46645E 01	0.38487E 00	-0.18907E-01	0.51719E-03	-0.63756E-05	0.11905E-07
c_{15}	0.51204E 02	-0.31217E 02	0.81116E 01	-0.11626E 01	0.99540E-01	-0.51343E-02	0.15232E-03	-0.22360E-05	0.10242E-07
c_{22}	0.14029E 04	-0.91985E 03	0.26125E 03	-0.41910E 02	0.41507E 01	-0.25973E 00	0.10025E-01	-0.21816E-03	0.20495E-05
c_{23}	0.14861E 04	-0.99591E 03	0.29083E 03	-0.46208E 02	0.49499E 01	-0.32176E 00	0.17913E-01	-0.29225E-03	0.28550E-05
c_{21}	0.54008E 03	-0.31534E 03	0.77997E 02	-0.10485E 02	0.81753E 00	-0.35959E-01	0.74680E-03	-0.11321E-05	-0.14198E-06
c_{25}	0.13646E 03	-0.83658E 02	0.21861E 02	-0.31440E 01	0.26898E 00	-0.13764E-01	0.39716E-03	-0.54028E-05	0.17504E-07
c_{33}	0.10892E 04	-0.69303E 03	0.19335E 03	-0.30823E 02	0.30642E 01	-0.19405E 00	0.76275E-02	-0.16984E-03	0.16380E-05
c_{14}	0.42146E 03	-0.26434E 03	0.74593E 02	-0.12240E 02	0.12630E 01	-0.83102E-01	0.33801E-02	-0.77358E-04	0.76101E-06
c_{31}	0.11374E 03	-0.76178E 02	0.22685E 02	-0.38754E 01	0.41139E 00	-0.27598E-01	0.11379E-02	-0.26309E-04	0.26103E-06
c_{14}	0.18134E 03	-0.77270E 02	0.59860E 01	0.17618E 01	-0.48995E 00	0.53435E-01	-0.30579E-02	0.90635E-04	-0.10991E-05
c_{15}	0.82576E 02	-0.47219E 02	0.10969E 02	-0.12514E 01	0.60670E-01	-0.98462E-03	-0.25309E-03	0.10806E-04	-0.15658E-06
c_{15}	0.72319E 02	-0.30373E 02	0.25059E 01	-0.77626E 00	-0.20999E 00	0.22316E-01	-0.17416E-02	0.39729E-04	-0.42028E-06
c_{11}	0.15571E 03	-0.98564E 02	0.27035E 02	-0.41938E 01	0.40231E 00	-0.24438E-01	0.91802E-03	-0.19502E-04	0.17940E-06
c_{12}	0.29330E 03	-0.17255E 03	0.43778E 02	-0.62489E 01	0.54847E 00	-0.30286E-01	0.10263E-02	-0.19400E-04	0.15821E-06
c_{13}	0.28258E 03	-0.16614E 03	0.42447E 02	-0.61568E 01	0.55461E 00	-0.31761E-01	0.11285E-02	-0.22724E-04	0.19836E-06
c_{14}	-0.19538E 02	0.40394E 02	-0.18514E 02	0.40049E 01	-0.49207E 00	0.36445E-01	-0.16166E-02	0.39644E-04	-0.41396E-06
c_{15}	-0.13716E 02	0.15094E 02	-0.58462E 01	0.11682E 01	-0.13707E 00	0.98538E-02	-0.42809E-03	0.10339E-04	-0.10671E-06
c_{23}	0.35150E 03	-0.14255E 03	0.17584E 02	0.55437E 00	-0.35173E 00	0.40316E-01	-0.21894E-02	0.60581E-04	-0.68474E-06
c_{22}	0.32637E 03	-0.17760E 03	0.14521E 02	0.73593E 00	-0.33968E 00	0.36010E-01	-0.18994E-02	0.51434E-04	-0.57146E-06
c_{24}	-0.52835E 03	0.45728E 03	-0.15814E 03	0.29521E 02	-0.33053E 01	0.22937E 00	-0.96884E-02	0.22863E-03	-0.23140E-05
c_{23}	-0.13647E 03	0.11324E 03	-0.38190E 02	0.70102E 01	-0.77555E 00	0.53339E-01	-0.22374E-02	0.52506E-04	-0.52897E-06
c_{33}	0.52608E 02	0.80224E 02	-0.52086E 02	0.12580E 02	-0.15221E 01	0.12281E 00	-0.54924E-02	0.13475E-03	-0.14014E-05
c_{34}	-0.66621E 03	0.54617E 03	-0.18162E 03	0.32790E 02	-0.35618E 01	0.24030E 00	-0.98860E-02	0.22762E-03	-0.22517E-05
c_{35}	-0.16803E 03	0.13884E 03	-0.43739E 02	0.78064E 01	-0.84134E 00	0.56443E-01	-0.23126E-02	0.53086E-04	-0.52398E-06
c_{41}	-0.86852E 03	0.66531E 03	-0.21191E 03	0.37258E 02	-0.39875E 01	0.26731E 00	-0.10995E-01	0.25423E-03	-0.25332E-05
c_{45}	-0.19482E 03	0.14784E 03	-0.46770E 02	0.81816E 01	-0.87231E 00	0.58809E-01	-0.23931E-02	0.55234E-04	-0.54956E-06
c_{55}	0.19976E 02	-0.56708E 01	-0.85966E 00	0.65078E 00	-0.12635E 00	0.12403E-01	-0.67272E-03	0.19226E-04	-0.22644E-06

resulting least mean squares approximations accurately described the computed values for ten of the thirty elements. The approximation results for one of these ten elements is shown in Figure 4.5. In this figure, the final dual approximation for element $\bar{c}_{45}$ is presented. The dual approximation consists of two different 8-th order power series approximations, labelled as 1 and 2 in Figure 4.5. As previously indicated, the coefficients that determine Power Series 2 for the element $\bar{c}_{45}$, are found in Table 4.1. The coefficients that describe Power Series 1 for the element $\bar{c}_{45}$, are presented in Table 4.2. This table shows the 10 elements for which two 8-th order power series approximations are used. For each of these ten elements, the values of the coefficients that determine the 8-th order power series approximation on $TE[1.3, 5.9)$, are also presented. Tables 4.1 and 4.2 completely specify the final dual approximation for 10 of the 30 elements.

For the remaining 20 elements, it was found that the power series form for the approximation function yielded poor results on $TE[1.3, 5.9)$. This is illustrated by the log-log plot in Figure 4.6, for the element $\bar{c}_{13}$. This figure shows the least mean squares approximation that was obtained for an 8-th order power series form (broken line). Observe that the characteristics of this approximation are similar to those obtained from the single 8-th order power series approximations. The problem is that the computed values of the 20 remaining elements still cover an extremely wide range, even when the interval is reduced to, $1.3 \leq T < 5.9$. Therefore, the larger computed values still tend to dominate the least mean squares fit.

For these 20 elements it was found that the log approximations obtained from equation (4.56), reasonably described the element values

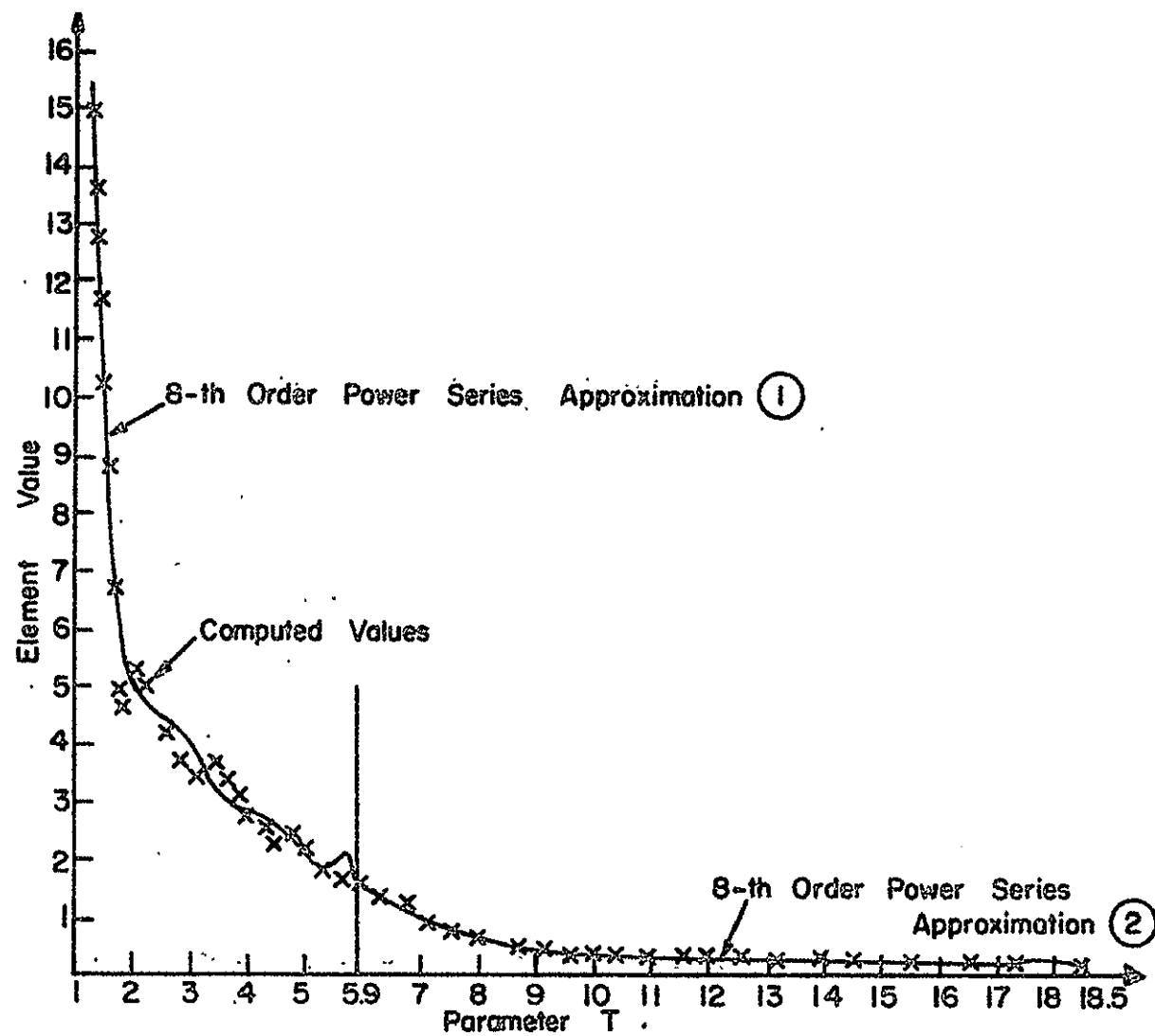


FIGURE 4.5. FINAL DUAL APPROXIMATION FOR ELEMENT $\bar{c}_{45}$.

Table 4.2. 8-th Order Power Series Approximations [1,3,5,9)

<u>Element</u>	<u>Value of the Coefficients</u>								
	<u>0-ORDER</u>	<u>1-ORDER</u>	<u>2-ORDER</u>	<u>3-ORDER</u>	<u>4-ORDER</u>	<u>5-ORDER</u>	<u>6-ORDER</u>	<u>7-ORDER</u>	<u>8-ORDER</u>
c_{34}	0.22711E 05	-0.45168E 05	0.39521E 05	-0.19578E 05	0.59595E 04	-0.11380E 04	0.13795E 03	-0.86804E 01	0.24242E 00
c_{35}	-0.74125E 03	0.30196E 04	-0.40894E 04	0.28208E 04	-0.11316E 04	0.27540E 03	-0.40119E 02	0.32204E 01	-0.10961E 00
c_{44}	0.19312E 04	-0.30772E 04	0.20720E 04	-0.76812E 03	0.16590E 03	-0.21253E 02	0.16701E 01	-0.79246E -01	0.23819E -02
c_{45}	-0.99174E 02	0.40565E 03	-0.52576E 03	0.34331E 03	-0.12870E 03	0.28835E 02	-0.38862E 01	0.27218E 00	-0.91022E -02
c_{55}	0.14660E 03	-0.42474E 03	0.53127E 03	-0.36959E 03	0.15590E 03	-0.40733E 02	0.64305E 01	-0.56077E 00	0.20700E -01
c_{34}	0.54910E 05	-0.12755E 06	0.12804E 06	-0.71990E 05	0.24746E 05	-0.53248E 04	0.70994E 03	-0.51661E 02	0.16342E 01
c_{35}	0.17704E 04	-0.34494E 04	0.28513E 04	-0.12734E 04	0.32754E 03	-0.47315E 02	0.32438E 01	-0.23695E -01	-0.59189E -02
c_{44}	0.17156E 04	-0.14849E 05	0.14189E 05	-0.75828E 04	0.24745E 04	-0.50501E 03	0.63803E 02	-0.43995E 01	0.13175E 00
c_{45}	0.16740E 03	-0.18460E 03	-0.16263E 02	0.12407E 03	-0.82443E 02	0.26309E 02	-0.45868E 01	0.42028E 00	-0.15870E -01
c_{55}	-0.92259E 01	0.65087E 02	-0.11223E 03	0.91492E 02	-0.41695E 02	0.11251E 02	-0.17879E 01	0.15475E 00	-0.56300E -02

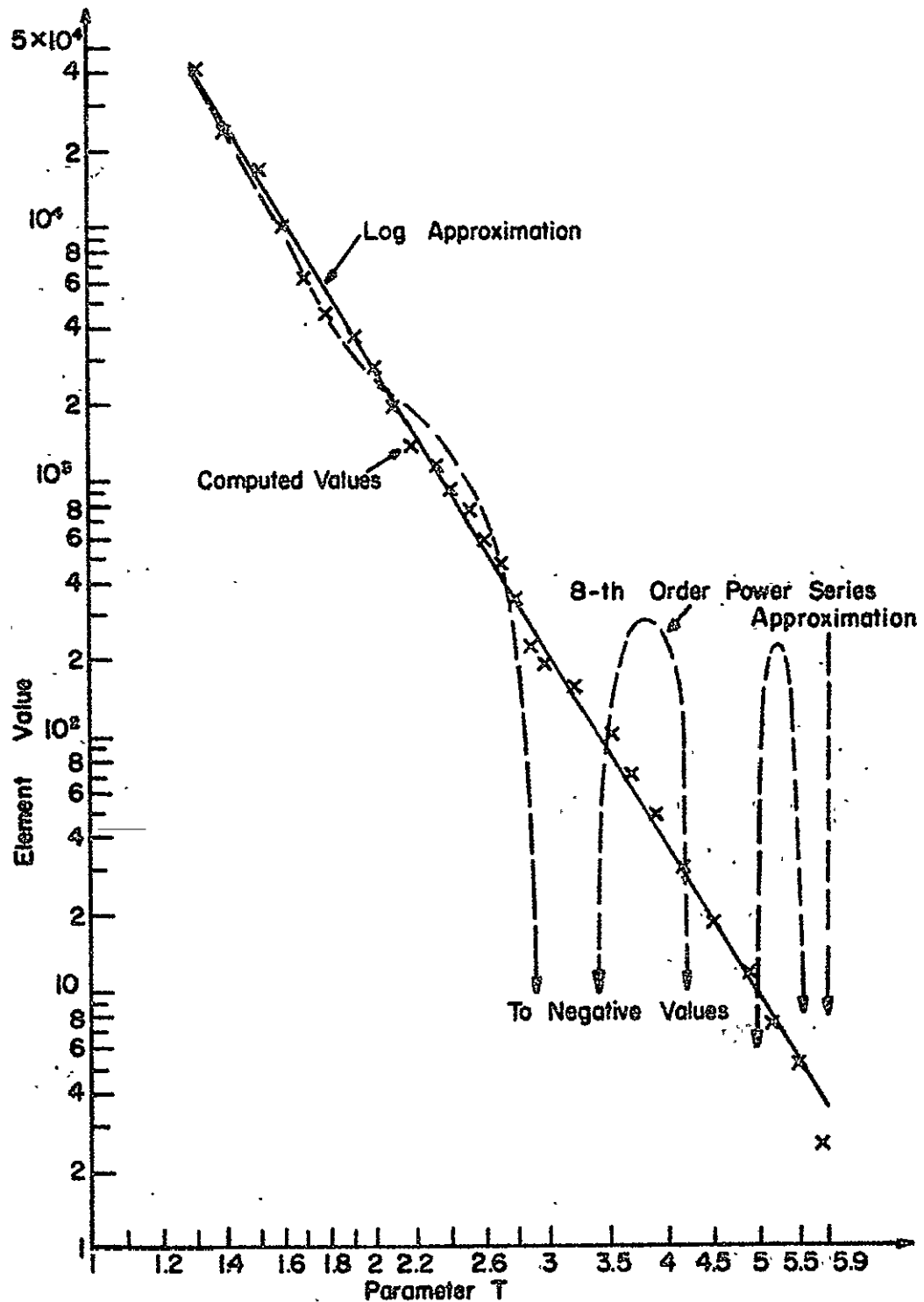


FIGURE 4.6. COMPARISON OF LOG AND POWER SERIES APPROXIMATIONS FOR ELEMENT $\bar{\alpha}_{13}$ (1.3-5.9).

on $TE[1.3, 5.9)$. The resulting least mean squares log approximation for the element $\tilde{c}_{13}$ is also shown in Figure 4.6 (solid line). Observe the comparison between the log and power series approximations for this element.

The numerical values that determine the log approximations are given in Table 4.3. For each of the 20 elements, this table gives the numerical values of the coefficient β_i and the exponent γ_i , of the approximation function defined in equation (4.54). For the elements belonging to Table 4.3, the dual approximation consists of a log approximation for $1.3 \leq T < 5.9$, and an 8-th order power series approximation for $5.9 \leq T \leq 18.5$. As an illustration, the dual approximation for the element $\tilde{c}_{24}$ is presented on the semi-log plot of Figure 4.7.

In general, the numerical values of the dual approximations are obtained from Table 4.1 for $5.9 \leq T \leq 18.5$, and from either Table 4.2 or 4.3 for $1.3 \leq T < 5.9$. Observe that the corresponding elements $\tilde{c}_{ij}$ and $\tilde{c}_{ij}$ are always located on same tables. This result was indicated by Feature (a) of Section 4.13.

The 30 dual element approximations obtained for system (4.40), specify the two final approximations functions, $\bar{V}(x, T)$ and $\tilde{V}(x, T)$. This completes the off-line portion of the sub-optimal procedure. A total of about 3 hours of off-line computer time (CDC-6500) was required to accomplish Steps (1) through (3) of the sub-optimal method.

At this point, various aspects of the sub-optimal controller performance were investigated. The general results of this investigation are presented in the following three sections.

Table 4.3. Log Approximations (1.3,5.9)

Element	Value of the Coefficient and Exponent	
	Coefficient (β)	Exponent (γ)
$\tilde{c}_{11}$	0.177572E 07	-0.803793E 01
$\tilde{c}_{12}$	0.884285E 06	-0.705427E 01
$\tilde{c}_{13}$	0.187544E 06	-0.611762E 01
$\tilde{c}_{14}$	0.187352E 05	-0.500135E 01
$\tilde{c}_{15}$	0.120043E 04	-0.431095E 01
$\tilde{c}_{22}$	0.448144E 06	-0.606286E 01
$\tilde{c}_{23}$	0.100850E 06	-0.517087E 01
$\tilde{c}_{24}$	0.102864E 05	-0.402982E 01
$\tilde{c}_{25}$	0.689473E 03	-0.335554E 01
$\tilde{c}_{33}$	0.231119E 05	-0.425627E 01
$\bar{c}_{11}$	0.142564E 07	-0.799542E 01
$\bar{c}_{12}$	0.693681E 06	-0.698567E 01
$\bar{c}_{13}$	0.145731E 06	-0.605805E 01
$\bar{c}_{14}$	0.136625E 05	-0.491872E 01
$\bar{c}_{15}$	0.752279E 03	-0.413126E 01
$\bar{c}_{22}$	0.345429E 06	-0.597221E 01
$\bar{c}_{23}$	0.770948E 05	-0.508518E 01
$\bar{c}_{24}$	0.742141E 04	-0.392599E 01
$\bar{c}_{25}$	0.426979E 03	-0.314930E 01
$\bar{c}_{33}$	0.177048E 05	-0.418051E 01

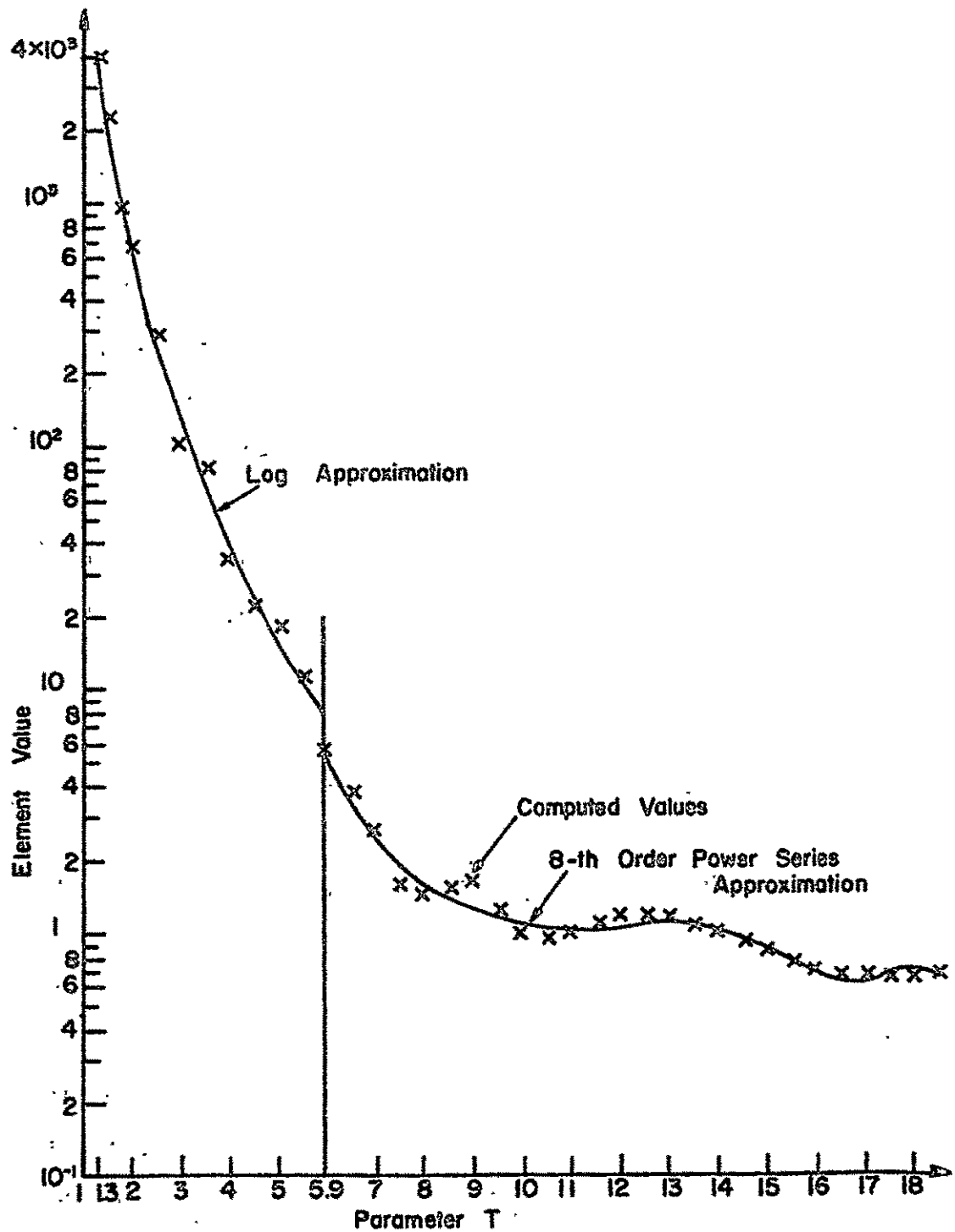


FIGURE 4.7. FINAL DUAL APPROXIMATION FOR ELEMENT z_{24} .

4.15 Sub-Optimal Controller Performance for System (4.40)

The investigation of the sub-optimal controller performance was accomplished with the aid of a digital computer program. The program basically operates in the following manner.

At the beginning of each run, the value of the initial disturbed state vector x of system (4.40) must be provided. The program then performs the following sequence of steps.

(i) For the given value of the state vector x , the various computations outlined in Operations I and II of Section 4.9 are carried out. These computations involve the evaluation of the two approximation functions $\bar{V}(x,T)$ and $\tilde{V}(x,T)$, which were obtained in the preceding section. The result of these computations is a value for the sub-optimal control $\bar{v}(x)$.

(ii) For this value of control, the differential equations that describe system (4.40) are then integrated over a time interval of $\Delta t = 0.01$ seconds. This yields a new value for the state vector x of system (4.40).

(iii) Steps (i) and (ii) are repeated, until some specified Termination Criterion is satisfied.

This program was used for a representative set of initial conditions, which was selected from the computed values of the state vector x on the isochrones $I(T_k)$, $k = 1, 2, \dots, 118$. By choosing the initial conditions in this manner, the later comparison between the sub-optimal controller and the minimal time open-loop control, is facilitated. This is seen by noting that the value of every computed state vector x was obtained, by first computing the corresponding extremal control. In Section 4.3 it has been established for system (4.40), that every

extremal control is also a minimal time control. Therefore for the computed state vectors, the open-loop minimal time controls have already been found as part of Step (1) of Section 4.12.

From all of the computed state vectors, a set of 10 representative initial conditions was chosen. The values of these initial conditions x^1 , $i = 1, 2, \dots, 10$, are given in columns 7 through 11 of Table 4.4.

The reason that these particular values were chosen is discussed below.

It was felt that the representative set of initial conditions should correspond to a variety of extremal controls. Also, each initial condition should correspond to a different isochrone.

It can be shown from the values in the first five columns of Table 4.4, that this set of initial conditions does possess the above two properties. First consider columns 1 through 4 of Table 4.4. For each initial condition x^1 , $i = 1, 2, \dots, 10$, there are a corresponding set of values for μ_5 and either μ_1 , $i = 1, \dots, 4$, or the angles ϕ_i , $i = 1, 2, 3$. As discussed in Section 4.12, these values determine the initial adjoint vector $\Psi(0) = \mu$. For initial conditions x^1 and x^8 , the values of μ_1 , $i = 1, \dots, 4$, have been chosen to satisfy equation (4.50). For the other eight initial conditions, $\Psi(0) = \mu$ is obtained from equation (4.51), for the given values of ϕ_i , $i = 1, 2, 3$. These values of ϕ_i , $i = 1, 2, 3$, have been selected from Table D.1 (p.238). In all ten cases, the values in the first four columns were chosen so that the resulting extremal controls would be of various different types. This satisfies the first property.

Now consider the values of T_k in column 5 of Table 4.4. For each initial condition, the corresponding isochrone is given by the value of T_k . For example, the initial condition x^2 belongs to the isochrone

Table 4.4. Representative Initial Conditions for System (4.40)

Reference					Initial Condition Number	Value of the Components				
ϕ_1	ϕ_2	ϕ_3	Value Of μ_j	T_k		x_1	x_2	x_3	x_4	$x_5 = u$
degrees				Seconds						
$\mu_1 = \mu_3 = 0.55$ $\mu_2 = \mu_4 = 0.44$			-0.067	4.4	x^1	7.6657	- 4.9944	0.8062	2.2651	-0.3331
60	-50	-10	-0.014	7.5	x^2	-19.488	- 0.5331	3.3983	1.3900	0.6478
120	50	10	0.015	8.7	x^3	34.579	- 6.6230	-0.3441	0.9736	-0.3225
70	10	0	-1.	9.5	x^4	-28.749	- 0.3224	1.1664	1.8406	1.0000
30	-60	70	0.247	10.5	x^5	-43.262	9.2838	1.3467	-3.7647	0.0000
100	70	-20	-0.162	13.5	x^6	59.096	- 8.3158	-5.3952	2.9592	0.3332
90	0	-30	-0.195	15.5	x^7	50.192	-13.892	1.2597	6.9151	0.6683
$\mu_1 = \mu_3 = 0.5$ $\mu_2 = \mu_4 = 0.5$			0.092	16.7	x^8	135.68	-17.331	1.6400	0.8153	0.3333
100	40	-60	-0.228	17.9	x^9	119.59	-16.997	-2.3373	6.8277	0.0000
70	-70	-40	-1.	18.5	x^{10}	-98.246	0.2580	6.7675	5.0501	1.0000

I(7.5). Notice that the values of T_k have been chosen so that the initial conditions x^1 , $i = 1, 2, \dots, 10$, all belong to different isochrones. This satisfies the second property.

In order to use the sub-optimal controller, some Termination Criterion must be specified. This criterion determines when the sub-optimal controller is to be disengaged. For practical systems, this criterion may be specified by considering the actual physical significance of the components of the state vector. For example, in system (3.78) it is desirable to drive the pitch deviation θ to zero as quickly as possible. Therefore, for this system, a reasonable Termination Criterion would be, $|\theta| \leq 0.1$ m-rad. That is, the sub-optimal controller is disengaged whenever this inequality is satisfied. Since system (4.40) is really the reformulated version of system (3.78), the above criterion could also be related to the state vector x . However in order to generalize the results for system (4.40), in this section no physical significance will be attached to the state vectors. The physical aspects of the problem will be considered in the following section.

There are many possible criteria that could be specified for the system. Two of these criteria are established as follows. Let the scalar functions $\sigma_a(x)$ and $\sigma_b(x)$ be defined by,

$$\sigma_a(x) = \frac{(x_1^2 + x_2^2)^{1/2}}{(x_1(0)^2 + x_2(0)^2)^{1/2}} \quad (4.58a)$$

and

$$\sigma_b(x) = \frac{\left(\sum_{i=1}^4 x_i^2\right)^{1/2}}{\left(\sum_{i=1}^4 x_i^2(0)\right)^{1/2}} \quad (4.58b)$$

where $x_i(0)$ denotes the initial value of the component x_i , $i = 1, \dots, 4$. The function $\sigma_a(x)$ is a measure of the distance of the components x_1 and x_2 from the origin at any time t , over the initial distance. The function $\sigma_b(x)$ is similarly defined. Notice that the component x_5 of the state vector is not included in these two definitions. This is because x_5 actually represents the control input for the original system.

Either function $\sigma_a(x)$ or $\sigma_b(x)$ can be used to form the Termination Criterion. For example, consider the Termination Criterion,

$$\sigma_a(x) \leq \beta, \quad (4.59)$$

where β is some number which satisfies, $0 < \beta < 1$. This criterion implies that the sub-optimal controller is to be applied, until the distance of the components x_1 and x_2 from the origin is reduced to some fraction of the original distance. A similar criterion could be defined for the function $\sigma_b(x)$.

For system (4.40), two Termination Criteria were selected. They are denoted by Cx_a and Cx_b . The criterion Cx_a is given by,

$$\sigma_a \leq 7\% (=0.07), \quad (4.60)$$

while Cx_b is given by,

$$\sigma_b \leq 9\% (=0.09). \quad (4.61)$$

For each of the above two Termination Criteria, the performance of the sub-optimal controller was observed for the ten representative initial conditions given in Table 4.4. This was accomplished by using the digital computer program that was described at the beginning of this section. However, in order to distinctly illustrate the various aspects of the

sub-optimal method, the following modification was made in the procedure.

From Operation I of Section 4.9, it is seen that the value of the parameter T is usually found by using equation (4.31). However, at certain time instants on the control interval, the value of T is computed by iteratively solving the nonlinear equation (4.33). In this section and also in Section 4.16, the value of the parameter T was obtained exclusively from equation (4.31). The calculation of the value of T from equation (4.33) will be discussed in Section 4.17.

Notice that with the above modification, equation (4.31) can be rewritten as,

$$T = \tilde{T} - t, \quad T \geq T_1 = 1.3, \quad (4.62)$$

where $\tilde{T}$ represents the initial value of the parameter T . In order to use equation (4.62), the value of $\tilde{T}$ must be provided. Since $\tilde{T}$ specifies the isochrone corresponding to each initial state vector x , then for the ten initial conditions in Table 4.4 the value of $\tilde{T}$ is given by the corresponding value of T_k .

The first set of results obtained for the initial conditions x^1 , $i = 1, 2, \dots, 10$, are presented in Table 4.5. These results were obtained by applying the sub-optimal controller for the entire range of validity of equation (4.62). That is, from $t = 0$ seconds to $t = \tilde{T} - 1.3$ seconds. Table 4.5 shows the minimum values of the scalar functions $\sigma_a(x)$ and $\sigma_b(x)$ on this control interval, along with their corresponding times of occurrence. Also, for purposes of reference, the actual minimal time to the origin is given. Observe that for initial conditions x^2 through x^9 , the sub-optimal controller was able to reduce the values of σ_a and σ_b below the specified limits of relations (4.60) and (4.61). This

Table 4.5. Minimum Values of σ_a and σ_b

Initial Condition Number	Minimal Time to Origin	Minimum Value of σ_a	Time of Occurrence	Minimum Value of σ_b	Time of Occurrence
	Seconds	percent	Seconds	percent	Seconds
x^1	4.4	9.33	3.02	12.52	3.10
x^2	7.5	4.90	6.20	5.87	6.20
x^3	8.7	6.92	6.90	8.63	7.40
x^4	9.5	6.75	7.66	7.69	7.86
x^5	10.5	1.28	9.15	4.87	9.20
x^6	13.5	2.75	10.82	3.87	11.12
x^7	15.5	5.90	13.44	8.98	11.34
x^8	16.7	2.37	13.85	2.86	13.95
x^9	17.9	0.69	16.22	1.84	16.60
x^{10}	18.5	0.41	17.08	2.96	17.20

indicates that the two Termination Criteria Cx_a and Cx_b , can be achieved on this control interval. For initial condition x^1 , σ_a and σ_b were reduced to .0933 and .1252, respectively. In this case, it is evident that neither Termination Criterion can be achieved on the given control interval. Therefore, for this initial condition, the value of the parameter T would have to be recalculated by using equation (4.33). This is discussed in Section 4.17.

The next set of results are presented in Table 4.6. This table compares the performance of the sub-optimal controller to the minimal time open-loop control. For each initial condition x^i , $i = 1, 2, \dots, 10$, the table compares the time required by the sub-optimal controller to satisfy each criterion, to the corresponding time required by the minimal time control. The basic conclusion that can be drawn from the given values, is that both the sub-optimal and minimal time controls take about the same amount of time to achieve the Termination Criteria. This indicates that the sub-optimal controller response $\bar{x}(t)$, is fairly close to the minimal time response $x^*(t)$. Notice that the single exception occurs for the initial condition x^1 . As discussed previously, in this case the two Termination Criteria can only be achieved by recalculating the value of the parameter T from equation (4.33).

A comparison between the sub-optimal response $\bar{x}(t)$ and the minimal time response $x^*(t)$ for the initial condition x^7 , is illustrated in Figures 4.8 through 4.12. The comparison is made from $t = 0$, to $t = 11.84$ seconds. From Table 4.6, this is the time required by the sub-optimal controller to satisfy Termination Criterion Cx_a , for the given initial condition x^7 .

Figure 4.8 shows the sub-optimal response $\bar{x}_5(t)$ along with the

Table 4.6. Evaluation of Sub-Optimal Performance for System (4.40)

Initial Condition Number	Time (seconds) Required to Satisfy the Termination Criterion Cx_a		Time (seconds) Required to Satisfy the Termination Criterion Cx_b	
	Sub-Optimal Closed-Loop Control	Minimal Time Open-Loop Control	Sub-Optimal Closed-Loop Control	Minimal Time Open-loop Control
x^1	-----	2.68	-----	3.41
x^2	5.48	5.30	5.55	5.58
x^3	6.66	6.02	6.55	5.99
x^4	7.36	7.00	7.14	7.07
x^5	7.76	7.60	7.94	7.47
x^6	9.57	9.82	9.49	9.60
x^7	11.84	12.27	11.33	12.04
x^8	11.82	12.21	11.35	11.74
x^9	14.00	13.66	13.74	13.21
x^{10}	14.98	14.57	14.73	14.13

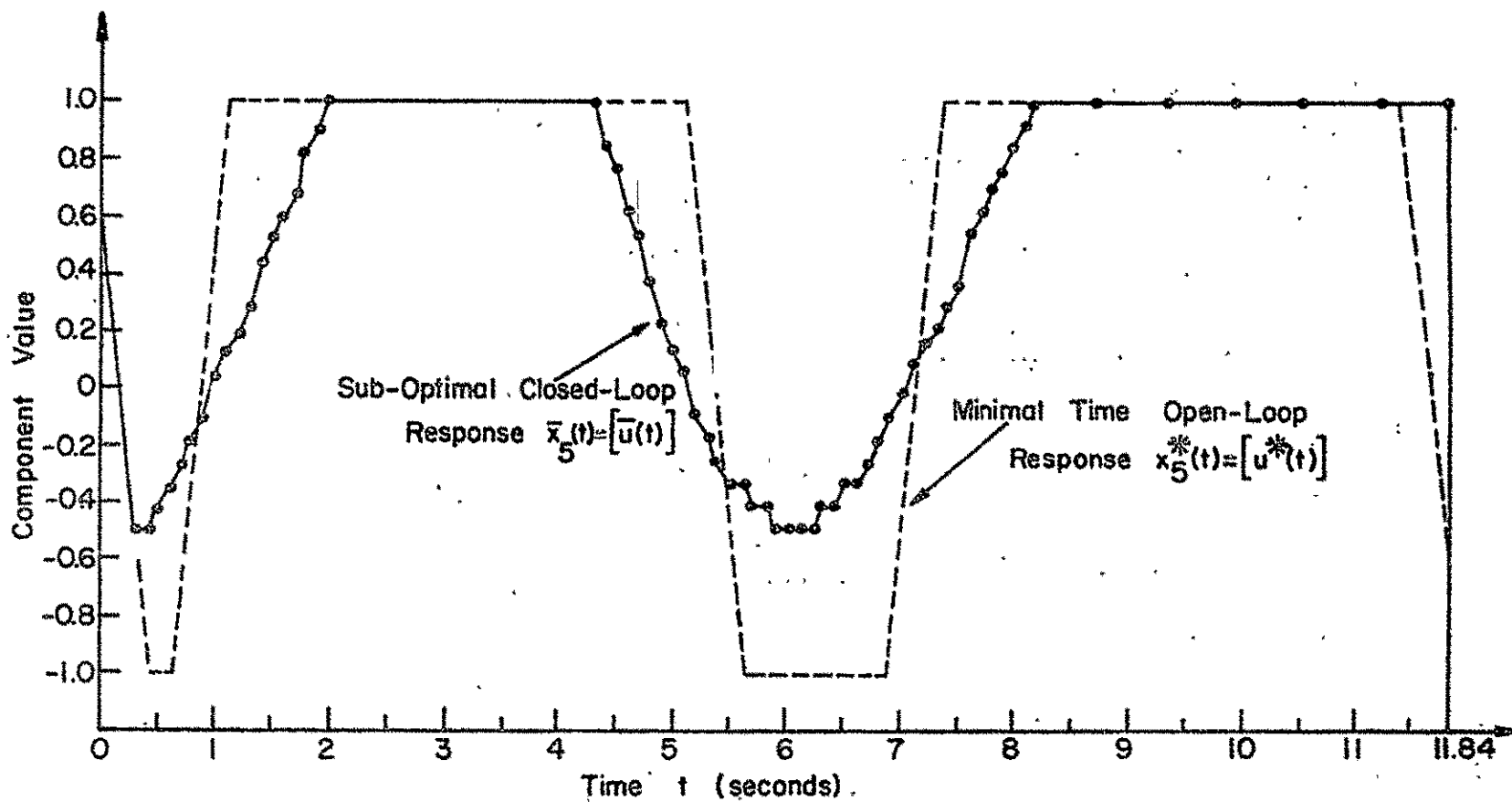


FIGURE 4.8 . COMPARISON OF $\bar{x}_5(t)$ AND $x_5^*(t)$ FOR INITIAL CONDITION x^7

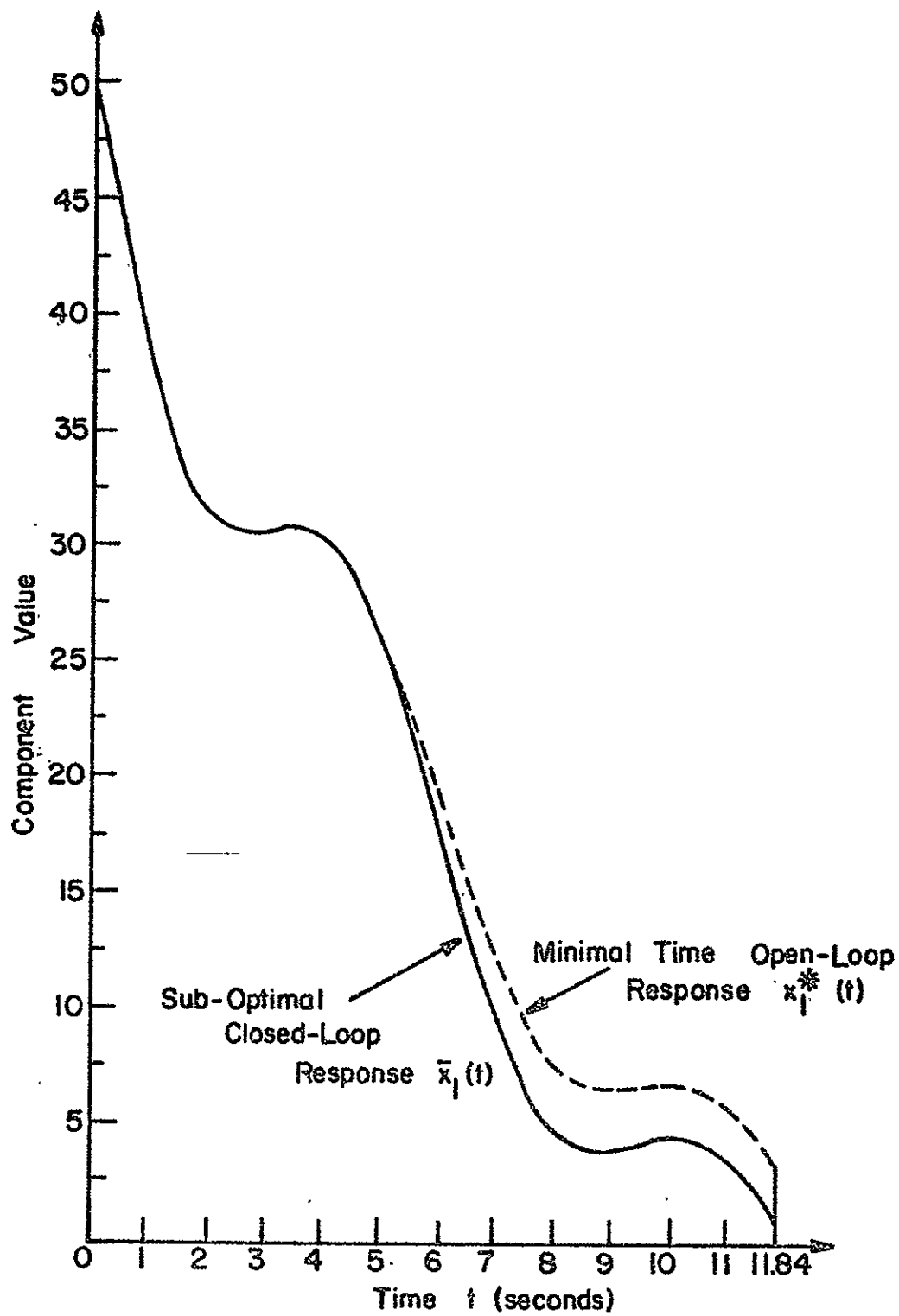


FIGURE 4.9 COMPARISON OF $\bar{x}_1(t)$ AND $x_1^*(t)$ FOR INITIAL CONDITION x^T .

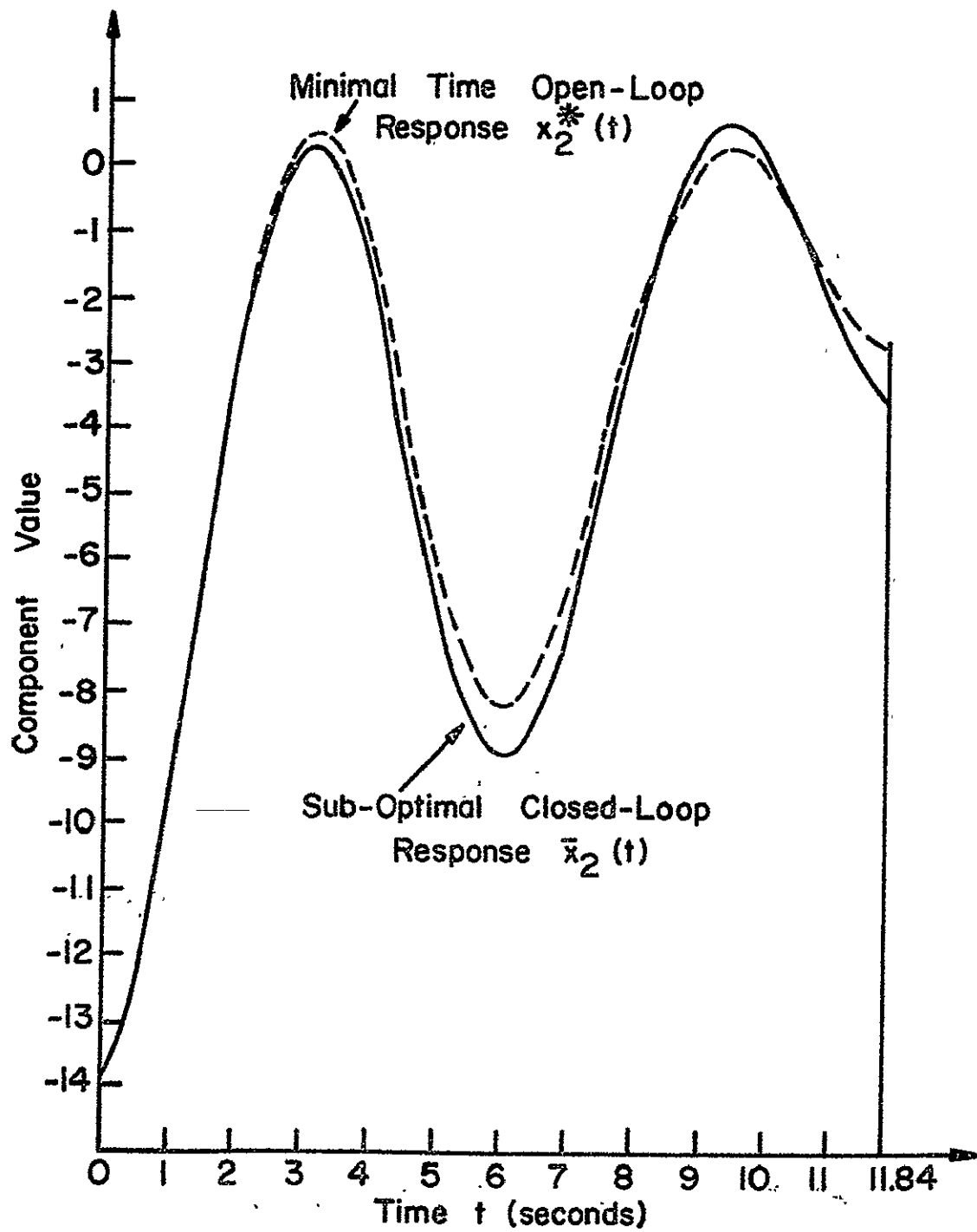


FIGURE 4.10 . COMPARISON OF $\bar{x}_2(t)$ AND $x_2^*(t)$ FOR INITIAL CONDITION x^7

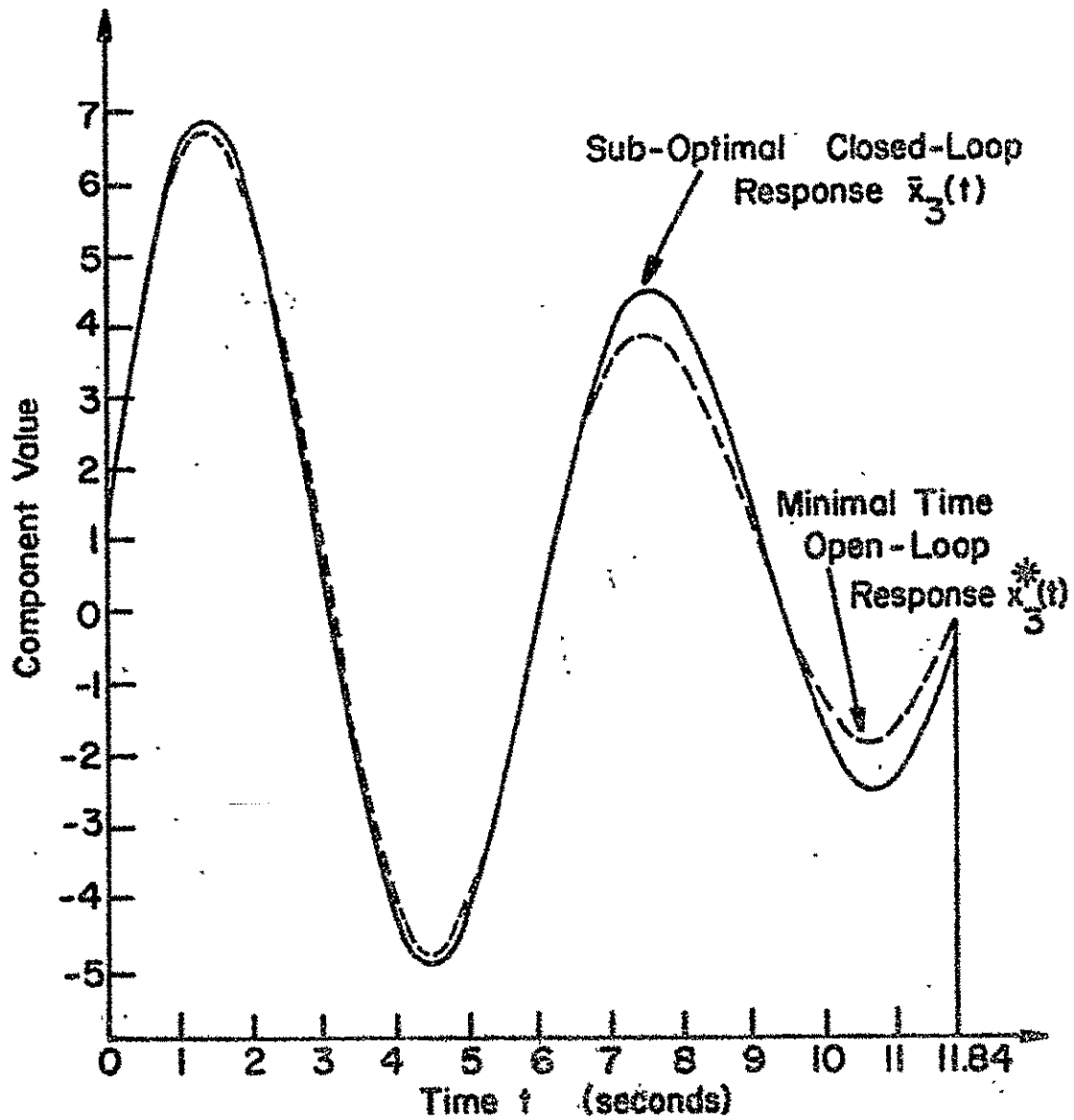


FIGURE 4.11 . COMPARISON OF $\bar{x}_3(t)$ AND $x_3^*(t)$ FOR INITIAL CONDITION x^7 .

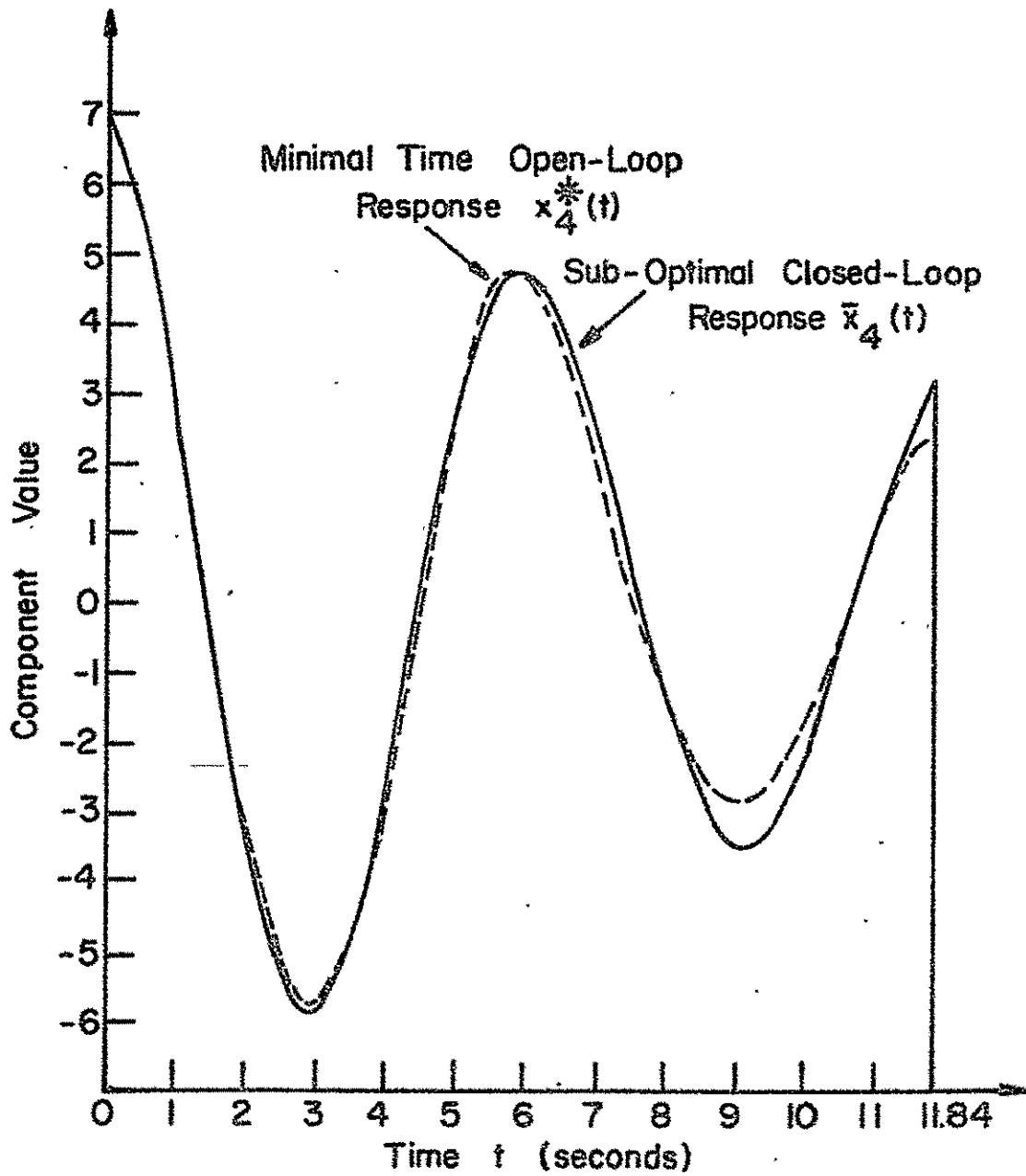


FIGURE 4.12 . COMPARISON OF $\bar{x}_4(t)$ AND $x_4^*(t)$ FOR INITIAL CONDITION x^7

minimal time response $x_5^*(t)$. Note that this component of the state vector actually represents the original control input. From this figure, it is seen that $\bar{x}_5(t)$ is identical to $x_5^*(t)$ for $0 \leq t \leq 0.3$ seconds. For the remainder of the time interval, $\bar{x}_5(t)$ follows the general variation of $x_5^*(t)$. It should be mentioned that the values used to plot the sub-optimal response $\bar{x}_5(t)$, were taken at intervals of 0.1 seconds. The actual sub-optimal controller sampling interval was 0.01 seconds. Therefore, each straight line segment of $\bar{x}_5(t)$ represents the average motion of $\bar{x}_5(t)$ over that particular 0.1 second interval.

Figures 4.9 through 4.12 show the comparisons between $\bar{x}_i(t)$ and $x_i^*(t)$, $i = 1, \dots, 4$. These figures illustrate the closeness of the sub-optimal closed-loop response to the minimal time open-loop response.

This completes the sub-optimal performance evaluation for system (4.40). In the next section, a similar evaluation for system (3.78) will be discussed.

4.16 Sub-Optimal Controller Performance for System (3.78)

The program described in Section 4.15, was also used to observe the performance of the sub-optimal controller for system (3.78). This was accomplished by including the computation of the transformation relations of Section 4.11, as part of the program operation. That is, at each value of the state vector x of system (4.40), the corresponding values of τ , θ , $d\theta/d\tau$, etc., of system (3.78) were computed.

The representative set of initial conditions for system (3.78) are given in Table 4.7. Each initial condition θ^1 corresponds to the initial condition x^1 in Table 4.4. In column 1 of Table 4.7, the minimal

Table 4.7. Corresponding Initial Conditions for System (3.78)

Minimal time to origin: (seconds)	Initial Condition Number	θ	$d\theta/dt$	$d^2\theta/dt^2$	$d^3\theta/dt^3$	q (torque)
		m-rad	$\frac{\text{m-rad}}{\text{sec}}$	$\frac{\text{m-rad}}{\text{sec}^2}$	$\frac{\text{m-rad}}{\text{sec}^3}$	ft.-lbs.
0.2387	θ^1	1.5359	- 8.8091	-23.032	- 94.812	-2.2285
0.4069	θ^2	-2.8860	3.0242	33.535	- 58.176	4.3337
0.4720	θ^3	6.1826	-18.692	-19.747	- 40.753	-2.1577
0.5154	θ^4	-4.9752	5.2821	60.998	- 77.043	6.6900
0.5697	θ^5	-7.5635	17.914	- 3.0578	157.58	0.0000
0.7324	θ^6	9.6656	-17.472	33.460	-123.86	2.2292
0.8409	θ^7	9.3037	-22.382	39.677	-289.45	4.4711
0.9060	θ^8	24.820	-54.896	17.483	- 34.126	2.2301
0.9712	θ^9	21.167	-33.021	5.3078	-285.79	0.0000
1.004	θ^{10}	-16.4812	18.2978	48.2785	-211.38	6.6900

Table 4.8. Minimum Value of $|\theta|$

Initial Condition	Minimal Time to Origin	Minimum Value of $ \theta $	Corresponding Value $ d\theta/dr $	Time of Occurrence
Number	Seconds	m-rad	m-rad/sec	Seconds
θ^1	0.2387	0.0035	5.829	0.164
θ^2	0.4069	0.0068	4.973	0.337
θ^3	0.4720	0.0069	7.090	0.374
θ^4	0.5154	0.0068	7.273	0.416
θ^5	0.5697	0.0032	9.211	0.497
θ^6	0.7324	0.0064	10.943	0.587
θ^7	0.8409	0.0687	0.013	0.654
θ^8	0.9060	0.0100	10.199	0.751
θ^9	0.9712	0.0182	10.315	0.880
θ^{10}	1.004	0.0127	11.677	0.927

times to the origin for these 10 initial conditions are given. These times are related to the values of T_k in Table 4.4, through the transformation given by equation (4.38).

In Section 4.15, it was indicated that a physically motivated Termination Criterion for system (3.78) is given by,

$$|\theta| \leq 0.1 \text{ m-rad} . \quad (4.63)$$

For convenience, this criterion is denoted by $C\theta_a$. Termination Criterion $C\theta_a$, does not include any of the derivatives of θ . However the sub-optimal controller is to be disengaged, whenever this criterion is satisfied. Therefore it is reasonable to require that the rate of change of the pitch deviation be small, whenever the controller is disengaged. In order to satisfy this requirement, the following Termination Criterion was also used for system (3.78).

$$|\theta| \leq 0.1 \text{ m-rad} \quad \text{and} \quad |d\theta/dt| \leq 10 \frac{\text{m-rad}}{\text{sec}} \quad (4.64)$$

This Termination Criterion is denoted by $C\theta_b$.

The sub-optimal controller performance for system (3.78) was observed, with respect to each of the above two Termination Criteria. The first set of results is presented in Table 4.8. These results were obtained, by allowing the sub-optimal controller to operate over the entire range of equation (4.62). This was the procedure used to obtain the results in Table 4.5. In Table 4.8, the minimum value of $|\theta|$ on each control interval is given, along with the value of time at which the minimum occurred. The corresponding value of $|d\theta/dt|$ at this value of time, is also presented. Observe that the sub-optimal controller was able to drive the pitch deviation to very small values, for all of the

10 representative initial conditions. However for some initial conditions, the corresponding value of the rate of change of θ was not within the specifications of relation (4.64). This indicates that for some of the initial conditions, it may not be possible to achieve Termination Criterion CO_b on the given control interval.

The next set of results for system (3.78) are presented in Table 4.9. This table compares the results under sub-optimal closed-loop control, to those obtained under open-loop minimal time control. This comparison is with respect to the time required to satisfy each of the two Termination Criteria CO_a and CO_b . Also, for Criterion CO_a , Table 4.9 gives the corresponding value of $|d\theta/dr|$. Observe that the minimal time control requires a larger amount of time to satisfy Criterion CO_a , than does the sub-optimal control. However, the corresponding values of $|d\theta/dr|$ are generally smaller under minimal time control. Also, observe that for initial conditions θ^6 and θ^{10} , the Termination Criterion CO_b was not satisfied under sub-optimal control. In these two cases, a new value of the parameter T would have to be found from equation (4.33).

The general conclusion that can be drawn from Table 4.9, is that in most cases the sub-optimal response $\bar{\theta}(\tau)$ is close to the minimal time response $\theta^*(\tau)$. This is implied by the fact that Criterion CO_a is achieved at about the same time instant, for both types of control.

The above general conclusion is illustrated in Figure 4.13, for the initial condition θ^9 . In this figure, the minimal time response $\theta^*(\tau)$ and the sub-optimal response $\bar{\theta}(\tau)$ are compared over the time interval $\tau \in [0, 0.886]$. From Table 4.9, 0.886 seconds is the time required by the sub-optimal controller to satisfy Criterion CO_b , for the given initial condition θ^9 . From Figure 4.13, it is observed that the

Table 4.9. Evaluation of Sub-Optimal Performance for System (3.78)

Initial Condition Number	Time (seconds) Re- quired to Satisfy the Termination Criterion $C\theta_a$		Corresponding Values of $ d\theta/dr $ ($\frac{m-rad}{sec}$)		Time (seconds) Re- quired to Satisfy the Termination Criterion $C\theta_b$	
	Sub- Optimal Control	Minimal Time Control	Sub- Optimal Control	Minimal Time Control	Sub- Optimal Control	Minimal Time Control
θ^1	0.148	0.172	6.752	3.494	0.148	0.172
θ^2	0.319	0.344	6.032	3.568	0.321	0.344
θ^3	0.360	0.407	7.646	3.696	0.360	0.407
θ^4	0.404	0.458	8.035	3.110	0.404	0.458
θ^5	0.486	0.508	9.894	9.378	0.486	0.508
θ^6	0.578	0.641	11.52	3.825	-----	0.641
θ^7	0.622	0.775	2.005	3.690	0.622	0.775
θ^8	0.743	0.841	10.74	3.444	0.754	0.841
θ^9	0.868	0.907	11.08	3.446	0.886	0.907
θ^{10}	0.917	0.947	12.26	3.495	-----	0.947

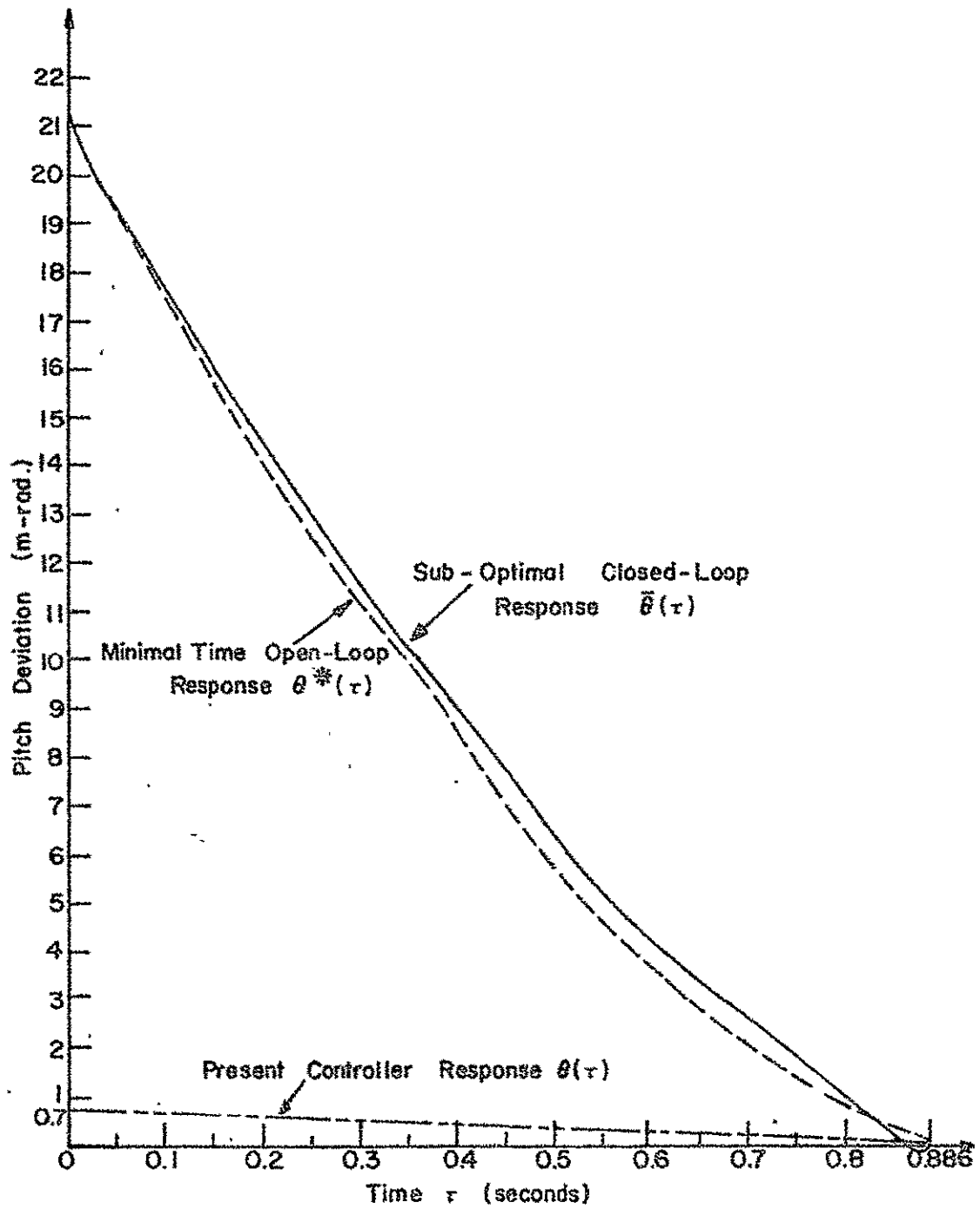


FIGURE 4.13. COMPARISON OF $\bar{\theta}(\tau)$, $\theta^*(\tau)$, AND $\theta(\tau)$, FOR INITIAL CONDITION θ^0 .

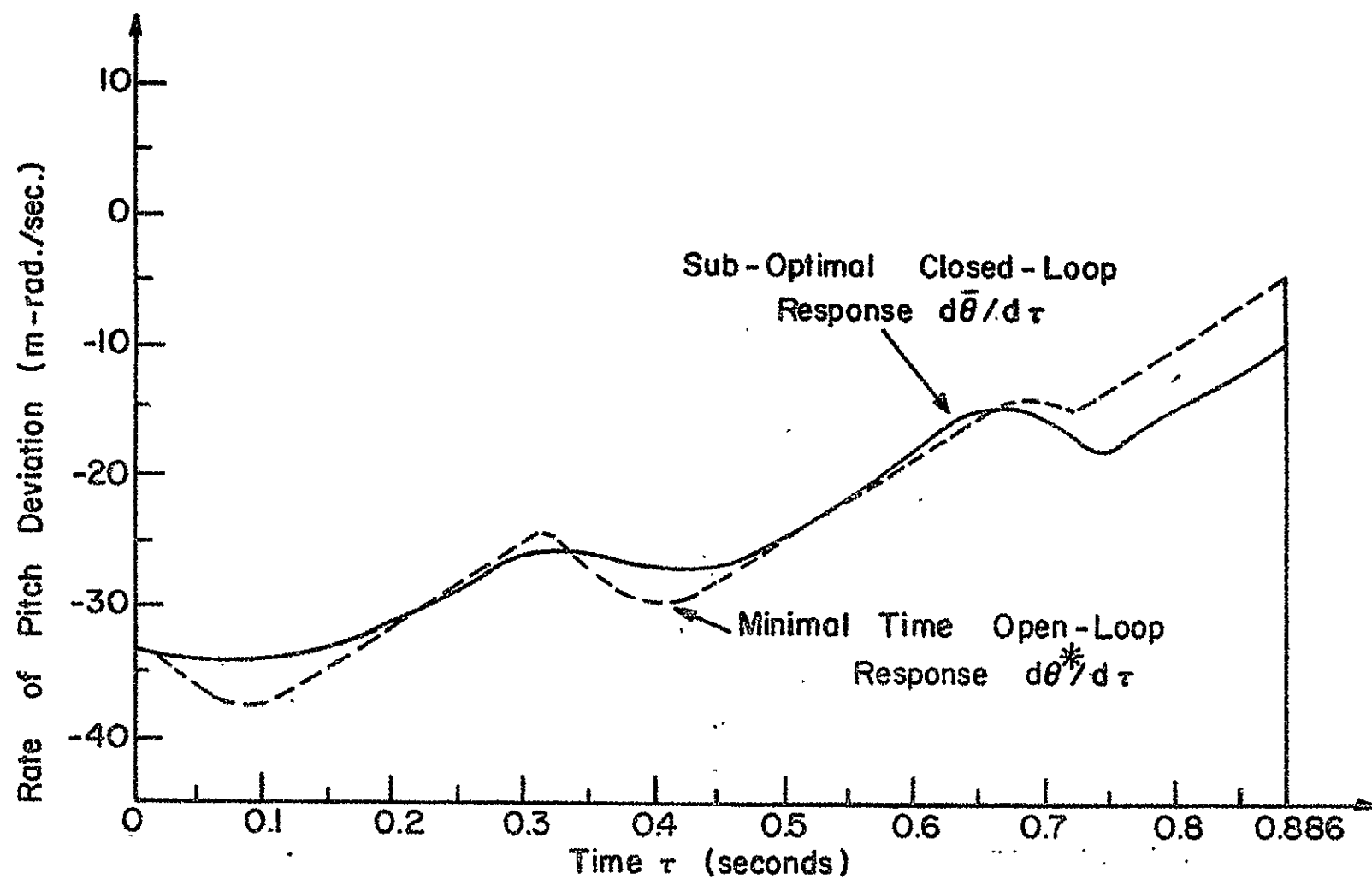


FIGURE 4.14. COMPARISON OF $d\theta^*/d\tau$ AND $d\bar{\theta}/d\tau$ FOR INITIAL CONDITION θ^0 .

responses $\bar{\theta}(\tau)$ and $\theta^*(\tau)$ are close over the entire time interval. The response $\theta(\tau)$ to the present proportional plus derivative controller used on the Ranger, is also shown in this figure (for $\theta(0)=0.7$ m-rad). Observe that the sub-optimal controller can drive a pitch deviation of about $\theta = 21$ m-rad, to $|\theta| \leq 0.1$ m-rad in a time interval of 0.886 seconds. In this same time interval, the present controller can generally only reduce the pitch deviation by about 1 m-rad [4].

As a final illustration, $d\bar{\theta}/d\tau$ and $d\theta^*/d\tau$ are compared in Figure 4.14. From this figure, it is seen that for $\tau \geq 0.67$ seconds, the minimal time response $d\theta^*/d\tau$ is closer to zero than the sub-optimal response $d\bar{\theta}/d\tau$. This characteristic was indicated by the results in Table 4.9. Also notice that the absolute value of the sub-optimal response $d\bar{\theta}/d\tau$, becomes less than $10 \frac{\text{m-rad}}{\text{sec}}$ at 0.886 seconds. It is at this time instant that the Termination Criterion $C\theta_b$, defined by relation (4.64), is satisfied.

The following section discusses the computation of the parameter T from equation (4.33).

4.17 Computation of the Parameter T

A general technique for computing the value of the parameter T from equation (4.33) was described in Section 4.9. This method was applied to system (4.40), at each initial condition x^i , $i = 1, 2, \dots, 10$, of Table 4.4.

For system (4.40), the parameter interval, $1.3 \leq T \leq 18.5$, was sampled at steps of 1 second. This specifies the value of δT in equation (4.36). At each initial condition, the parameter samples were taken according to equation (4.36), until the two sampling instants T_a

and T_b that satisfied relation (4.35), were found. These two values were then used as the input parameters to Mueller's Iteration Method [31]. For system (4.40), the value of ϵ in equation (4.38) was taken to be, $\epsilon = 10^{-3}$.

The results of this method are presented in Table 4.10. For each of the 10 initial conditions, Table 4.10 gives the computed value of the parameter T ($T = \hat{T}$), along with the actual value. The actual value of T was obtained from Table 4.4 ($T = T_k$). The value of $|F(T)|$ evaluated at $T = \hat{T}$, is also presented in Table 4.10. Observe that these values of $|F(\hat{T})|$ satisfy relation (4.38), for the given choice of ϵ . The results indicate that the computed value of the parameter is generally less than 1 second away from the actual value of the parameter.

Since the above operations are normally performed on-line, the total amount of computer time must be small. It was found that for each initial condition, it required about 0.15 seconds of computer time (CDC-6500), to find the value of $\hat{T}$. This does not include the time used in obtaining the parameter values T_a and T_b . For this 5-th order system, it takes about 0.02 seconds of computation time to determine the value of the function $F(T)$ at any given value of T . Therefore the total time needed to find the parameter values T_a and T_b , is dependent on the number of parameter samples that are evaluated.

This completes the investigation of the sub-optimal method of solution. In the following chapter, the two recoverability problems formulated in Section 2.5 will be investigated.

Table 4.10. Calculation of the Value of the Parameter T

Initial Condition Number	Input Value T_a	Input Value T_b	Computed Value of the parameter $\hat{T}$	Actual Value of the parameter	Value of
	seconds	seconds	seconds	seconds	$ E(\hat{T}) $
x^1	4.3	5.3	4.759	4.4	4.46×10^{-7}
x^2	6.3	7.3	7.182	7.5	5.75×10^{-7}
x^3	8.3	9.3	9.034	8.7	1.87×10^{-6}
x^4	9.3	10.3	10.066	9.5	1.27×10^{-4}
x^5	10.3	11.3	10.371	10.5	1.06×10^{-4}
x^6	12.3	13.3	12.376	13.5	1.84×10^{-4}
x^7	14.3	15.3	15.220	15.5	1.51×10^{-7}
x^8	17.3	18.3	17.760	16.7	1.9×10^{-4}
x^9	17.3	18.3	17.728	17.9	1.9×10^{-5}
x^{10}	17.3	18.3	17.840	18.5	5.78×10^{-4}

CHAPTER V

RECOVERABILITY

5.1 Introduction

In this chapter, two theorems which provide a necessary and a sufficient condition for complete recoverability, i.e., $S(t_0) = \hat{G}$, will be determined for system (2.6). These two theorems will be developed, based on a series of lemmas. A technique for determining the set of recoverable initial conditions in the case $S(t_0) \neq \hat{G}$, will also be discussed in Section 5.7.

In the following sections it will be assumed that the vector g , given in system (2.1) is a zero vector. In this case, by virtue of the transformations defined for system (2.6), the vector c is also a zero vector. The corresponding expression for the set of recoverability, from equation (2.10), becomes

$$K(t) = \left\{ x \mid x = - \int_{t_0}^t \phi(t_0, s) B v(s) ds, v \in \Lambda(t) \right\} \quad (5.1)$$

The above modification of system (2.1) is made in order to simplify the analysis in this chapter.

It is shown in Appendix A that the set of recoverability, $K(t)$, is compact, convex and continuous in t . With the set $K(t)$ defined by equation (5.1), it can be shown that the set $K(t)$ is also non-decreasing, i.e.,

$$K(t_1) \subset K(t_2), \text{ whenever } t_0 \leq t_1 \leq t_2 \quad (5.2)$$

In order to establish this property, suppose that the initial condition $x(t_0) = \tilde{x} \in K(t_1)$, where $t_1 \geq t_0$. Then, by equation (5.1), there exists a control $\tilde{v} \in \Lambda(t_1)$, such that

$$\tilde{x} = - \int_{t_0}^{t_1} \phi(t_0, s) \tilde{B} \tilde{v}(s) ds \quad (5.3)$$

Now, let the time t_2 be such that $t_2 \geq t_1$ and define the control $\hat{v}(t)$ on $t \in [t_0, t_2]$ by

$$\hat{v}(t) = \begin{cases} \tilde{v}(t) & t_0 \leq t \leq t_1 \\ 0 & t_1 < t \leq t_2 \end{cases} \quad (5.4)$$

Since $\tilde{v} \in \Lambda(t_1)$, then by the definition of the class of admissible controls, equation (2.8), $\hat{v} \in \Lambda(t_2)$. The corresponding initial condition $x(t_0) = \hat{x} \in K(t_2)$ is given by equation (5.1) as

$$\hat{x} = - \int_{t_0}^{t_2} \phi(t_0, s) \hat{B} \hat{v}(s) ds = - \int_{t_0}^{t_1} \phi(t_0, s) \tilde{B} \tilde{v}(s) ds + 0 = \tilde{x} \quad (5.5)$$

Thus the initial condition $\tilde{x} \in K(t_2)$. Since the choices of $\tilde{x}$, t_1 and t_2 ($t_0 \leq t_1 \leq t_2$), were arbitrary, the above is true for all initial conditions $x(t_0) = x \in K(t_1)$. Thus $K(t_1) \subset K(t_2)$, which establishes relation (5.2). This non-decreasing property of $K(t)$ is necessary in establishing the convexity of the set of recoverable initial conditions. This is done in the following section.

5.2 Complete Recoverability and the Adjoint Equations

In this section, the concept of complete recoverability is related to the solutions of the adjoint equation, $\dot{V}(t) = -V(t)A(t)$. This is accomplished by a series of three lemmas.

Lemma 5.1

The set of recoverable initial conditions, $S(t_0)$, is convex.

Proof

Suppose that the initial conditions $\tilde{x}, \hat{x} \in S(t_0)$. Then by definition of the set $S(t_0)$, equation (2.13), $\tilde{x} \in K(t_1)$ and $\hat{x} \in K(t_2)$ for some time instants $t_1, t_2 \geq t_0$. Let the time t_3 be defined by,

$$t_3 = \text{Max} [t_1, t_2] \quad (5.6)$$

By relation (5.2), the set $K(t)$ is non-decreasing and therefore $\tilde{x}, \hat{x} \in K(t_3)$. Since the set $K(t)$ is convex [Appendix A], all initial conditions $x(t_0) = x$ given by,

$$x = \alpha \tilde{x} + (1 - \alpha) \hat{x}, \quad \text{for all } 0 \leq \alpha \leq 1, \quad (5.7)$$

also belong to the set $K(t_3)$. However, by definition, $K(t_3) \subset S(t_0)$ and hence $x \in S(t_0)$. Since the initial conditions $\tilde{x}$ and $\hat{x}$ and the initial times t_1 and t_2 were arbitrary, the above is true for all initial conditions belonging to the set $S(t_0)$. Thus the set $S(t_0)$ is convex.

For convenience, in the following discussion let the $(n + m)$ -dimensional initial conditions $x \in S(t_0)$ and the $(n + m)$ -dimensional adjoint vector $y(t)$ be partitioned by,

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \quad (5.8)$$

and

$$y(t) = [y^1(t) y^2(t)] , \quad (5.9)$$

where

$$x^1 = [x_1, x_2, \dots, x_n]' , \quad (5.10a)$$

$$x^2 = [x_{n+1}, x_{n+2}, \dots, x_{n+m}]' , \quad (5.10b)$$

$$y^1 = [y_1, y_2, \dots, y_n] , \quad (5.11a)$$

$$y^2 = [y_{n+1}, y_{n+2}, \dots, y_{n+m}] . \quad (5.11b)$$

and $(\quad)'$ is the transpose of $(\quad)$. Also, let the $(n+m)$ -dimensional row vector η be similarly partitioned, i.e., $\eta = [\eta^1 \ \eta^2]$, where η^1 represents the first n -components and η^2 represents the last m -components of the vector η .

The next lemma employs the convexity property of the set $S(t_0)$ to express the property of complete recoverability in terms of general vectors η .

Lemma 5.2

A necessary and sufficient condition that $S(t_0) = \hat{G}$ is that for any number $H > 0$ and any vector $\eta = [\eta^1 \ \eta^2]$, with η^1 non-zero, there exists an initial condition $x \in S(t_0)$ such that $\eta x > H$.

Proof

First suppose that $S(t_0) = \hat{G}$ and, if possible, there exists some number $H > 0$ and some vector $\eta = [\eta^1 \ \eta^2]$, with η^1 non-zero, such that $\eta x \leq H$ for all initial conditions $x \in S(t_0)$. Then, by the definitions in equations (5.8) through (5.11),

$$\eta x = \eta^1 x^1 + \eta^2 x^2 \leq H \quad (5.12)$$

However $S(t_0) \subset \hat{G}$, and therefore, from Section 2.2, $|x_{n+1}| \leq 1$, $i=1,2,\dots,m$. Thus,

$$\eta^1 x^1 \leq H + \|\eta^2\| = H^*, \quad (5.13)$$

where $\|\eta^2\| = \sum_{i=1}^m |\eta_{n+1}^i|$. Since equation (5.13) is true for all $x \in S(t_0)$ and since η^1 is non-zero, then the first n -components of every initial condition $x \in S(t_0)$ are contained in the half-space $\eta^1 x^1 \leq H^*$, $x^1 \in \mathbb{R}^n$. However, by the definition of the set $\hat{G}$ in Section 2.2, this implies that $S(t_0) \neq \hat{G}$ which is a contradiction. This proves the necessity portion of the lemma.

Now, suppose that for every number $H > 0$ and every vector η , with η^1 non-zero, there exists an initial condition $x \in S(t_0)$ such that $\eta x > H$. Also assume, if possible, that $S(t_0) \neq \hat{G}$. Then since the set $S(t_0)$ is contained in the set $\hat{G}$, there exists boundary points of $S(t_0)$ contained in $\hat{G}$. From Lemma 5.1, the set $S(t_0)$ is convex. Thus there exists supporting hyperplanes at each of these boundary points such that all of $S(t_0)$ is contained in the half-space generated by each of the hyperplanes. Let the outward normal to each hyperplane be represented by the vector η . Then, $\eta x \leq H$ for some number $H > 0$ and all initial conditions $x \in S(t_0)$. Also, at least one of the vectors η must have a non-zero η^1 . To see this last statement, suppose all outward normals are such that $\eta^1 = 0$. Then for every vector η , the half-space becomes,

$$\eta x = \eta^2 x^2 \leq H \quad \text{for all } x \in S(t_0) \quad (5.14)$$

However relation (5.14) only restricts the last m -components of the

initial conditions, which is equivalent to the definition of the set $\hat{G}$. Thus if all outward normals have zero η^1 , then $S(t_0) = \hat{G}$. Since it has been assumed that $S(t_0) \neq \hat{G}$, then this cannot occur and therefore at least one of the vectors η must have a non-zero η^1 and be such that $\eta x \leq H$, for some number $H > 0$, and all initial conditions $x \in S(t_0)$. This contradicts the original assumption, thereby proving the sufficiency portion of the lemma.

Lemma 5.2 can be re-expressed in terms of $\Psi(t)$, the adjoint vector, in the following manner. If the initial condition $x \in S(t_0)$, then by the definition of the set of recoverable initial conditions, equation (2.13), $x \in K(T)$, the set of recoverability, for some time $T \geq t_0$. Now, let $\Psi(t_0) = -\eta$, where $\eta = [\eta^1 \ \eta^2]$ is any $(n+m)$ -dimensional row vector with η^1 non-zero. Since $\Psi(s) = \Psi(t_0)\phi(t_0, s)$, then from the definition of the set of recoverability, equation (5.1),

$$\eta x = \int_{t_0}^T \Psi(s)Bv(s)ds, \quad v \in \Lambda(T) \quad (5.15)$$

Equation (5.15) may be further specified by noting the form of the matrix B as given in system (2.6). This yields the relation,

$$\eta x = \int_{t_0}^T \Psi^2(s)v(s)ds, \quad v \in \Lambda(T) \quad (5.16)$$

where the m -dimensional vector Ψ^2 was defined in equation (5.11b).

From the preceding discussion, it is seen that Lemma 5.2 can be rewritten as the following lemma.

Lemma 5.3

$S(t_0) = \hat{G}$ if and only if for any number $H > 0$ and any choice $\Psi(t_0) = [\Psi^1(t_0) \ \Psi^2(t_0)]$, with $\Psi^1(t_0)$ non-zero, of the initial value of

the adjoint vector, there exists a time $T > t_0$ and a control $v \in \Lambda(T)$, such that

$$\int_{t_0}^T v^2(s)v(s)ds > H \quad (5.17)$$

A necessary condition that the adjoint solution must satisfy for complete recoverability is developed in the next section. The development is based, in part, on Lemma 5.3.

5.3. A Necessary Condition for Complete Recoverability

System (2.6) is in the bounded state variable formulation since $x \in \hat{G}$, where $\hat{G} \neq R^{n+m}$. That is, the restraint set is not equal to the complete $(n+m)$ -dimensional space. Suppose that system (2.6) is completely recoverable, i.e., $S(t_0) = \hat{G}$. Now consider the same system of equations with the state variable constraint removed. Then, as a direct consequence of the convexity of the set $S(t_0)$, the system must also be completely recoverable in this case, i.e., $S(t_0) = R^{n+m}$. Thus a necessary condition for the complete recoverability of system (2.6) is that the system must be completely recoverable when the state variable constraint is removed.

If the state variable constraint is removed, then from equation (2.8), a control $v(t)$ is admissible if it is piecewise continuous and satisfies $v(t) \in \Omega$ for all $t \geq t_0$. The condition for complete recoverability for this class of admissible controls has been established in Reference 7. From the previous discussion, this condition is a necessary condition for the complete recoverability of system (2.6).

For the sake of completeness, this necessary condition is established in the following theorem. The method of proof is similar to

that used in Reference 7.

Theorem 5.1

If $S(t_0) = \hat{G}$, then for every choice $\Psi(t_0)$, with $\Psi^1(t_0)$ non-zero, of the initial value of the adjoint vector, the following relation must be satisfied,

$$\int_{t_0}^{\infty} \|\Psi^2(s)\| ds = \infty, \quad (5.18)$$

where the norm is defined by,

$$\|\Psi^2(s)\| = \sum_{i=1}^m |\Psi_{n+1}^i(s)| \quad (5.19)$$

Proof

Suppose that $S(t_0) = \hat{G}$ and, if possible, equation (5.18) is not satisfied. Then, there exists a number $H_1 > 0$ and an initial value of the adjoint vector, say $\Psi(t_0) = -\eta$, with η^1 non-zero, such that

$$\int_{t_0}^{\infty} \|\Psi^2(s)\| ds \leq H_1 \quad (5.20)$$

From equation (5.11b), for any time $T > t_0$,

$$\int_{t_0}^T \Psi^2(s) v(s) ds = \int_{t_0}^T \sum_{i=1}^m \Psi_{n+1}^i(s) v_i(s) ds = \sum_{i=1}^m \int_{t_0}^T \Psi_{n+1}^i(s) v_i(s) ds \quad (5.21)$$

By definition of the class of admissible controls, equation (2.8), if $v \in \Lambda(T)$, then $|v_i| \leq h_i$, $h_i > 0$, $i=1,2,\dots,m$. Let $H_2 = \text{Max}[h_1, h_2, \dots, h_m]$. Then from equation (5.21), for all controls $v \in \Lambda(T)$,

$$\int_{t_0}^T \Psi^2(s) v(s) ds \leq \sum_{i=1}^m \int_{t_0}^T |\Psi_{n+1}^i(s)| |v_i(s)| ds \leq H_2 \int_{t_0}^T \|\Psi^2(s)\| ds \quad (5.22)$$

Now, let $v(t_0) = -\eta_0$. Then from equation (5.20)

$$\int_{t_0}^T v^2(s) v(s) ds \leq H_1 H_2 = H, \quad v \in \Lambda(T) \quad (5.23)$$

Since equation (5.23) is true for all choices of $T > t_0$, it is seen from Lemma 5.3 that $S(t_0) \neq \hat{G}$. However, this is a contradiction of the original assumption and therefore equation (5.18) must be satisfied if $S(t_0) = \hat{G}$.

A sufficiency condition for complete recoverability is established in the following section:

5.4 A Sufficient Condition for Complete Recoverability

A sufficiency condition will be developed for the functions $v^2(t) = [v_{n+1}(t), v_{n+2}(t), \dots, v_{n+m}(t)]$ that belong to a certain Class Γ of vector functions. The definition of this class of functions is developed below.

The scalar function $v_{n+j}(t)$, $j=1,2,\dots,m$, defined for all $t \geq t_0$, is said to belong to Class Γ_1 if for any number $H > 0$, there exists a time $T > t_0$ such that,

$$|v_{n+j}(t)| > H, \quad \text{for all } t \geq T \quad (5.24)$$

The scalar function $v_{n+j}(t)$, $j=1,2,\dots,m$, defined for all $t \geq t_0$, is said to belong to Class Γ_2 if it does not belong to Class Γ_1 and if it satisfies the following three relations:

$$(1) \quad |dv_{n+j}(t)/dt| \leq r, \quad r > 0, \quad \text{for all } t \geq t_0 \quad (5.25)$$

i.e., the derivative of $v_{n+j}(t)$ is bounded for all $t \geq t_0$.

$$(2) \Delta t_1 = (t_{1+1} - t_1) \leq \beta, \beta > 0, i=1, 2, \dots, N-1, \quad (5.26)$$

where $t_0 < t_1 < t_2 < \dots < t_N < T$, $N = N(T)$, are defined to be the time instants at which $y_{n+j}(t)$ is at a local maximum or minimum value.

Equation (5.26) implies that the time between any two adjacent extrema is bounded.

(3) For any number $H > 0$, there exists a time $T > t_0$ and a corresponding integer $N = N(T) > 0$, such that

$$\sum_{i=1}^{N-1} (\Delta y_i)^2 > H, \quad (5.27)$$

where $\Delta y_i = |y_{n+j}(t_{i+1}) - y_{n+j}(t_i)|$.

Based on the above definition of Class Γ_2 , it is seen that any periodic function $y_{n+j}(t)$ belongs to this class.

The Class Γ of vector functions $y^2(t)$, to be considered in the development of a sufficiency condition, is defined in terms of Classes Γ_1 and Γ_2 as follows. The function $y^2(t)$, defined for all $t \geq t_0$ is said to belong to the Class Γ of vector functions if at least one component of $y^2(t)$ belongs to either Class Γ_1 or Class Γ_2 .

A sufficiency condition for complete recoverability is stated in the following theorem.

Theorem 5.2

If the vector function $y^2(t)$ of the adjoint vector $y(t)$, defined for all $t \geq t_0$, belongs to Class Γ for every choice $y(t_0)$ with $y^1(t_0)$ non-zero, then $S(t_0) = \hat{G}$.

Proof

The proof is carried out with the aid of the following two lemmas.

Lemma 5.4

If for a given choice of initial condition $\bar{y}(t_0)$ of the adjoint vector, with $\bar{y}^1(t_0)$ non-zero, there exists a component of $\bar{y}^2(t)$ belonging to Class Γ_1 , then for any number $H > 0$ there exists a time $T > t_0$ and a control $\bar{v} \in \Lambda(T)$, such that equation (5.17) is satisfied, i.e.,

$$\int_{t_0}^T \bar{y}^2(s) \bar{v}(s) ds > H \quad (5.28)$$

Proof

Suppose for the given initial condition of $\bar{y}(t_0)$, with $\bar{y}^1(t_0)$ non-zero, that $\bar{y}_{m+j}(t)$, for some j , $j=1,2,\dots,m$, belongs to Class Γ_1 . Then, for any number $H > 0$, there exists a time $\tilde{T} > t_0$ such that

$$|\bar{y}_{m+j}(t)| > H, \quad \text{for all } t \geq \tilde{T}. \quad (5.29)$$

Let the time T be given by

$$T = \tilde{T} + 1/h_j \quad (5.30)$$

and define the control $\bar{v}(t)$ on $t \in [t_0, T]$ by,

$$\bar{v}_i(t) \equiv 0, \quad i \neq j, \quad i=1,2,\dots,m \quad (5.31a)$$

and

$$\bar{v}_j(t) = \begin{cases} 0 & , \text{ if } t_0 \leq t < \tilde{T}, \\ h_j \operatorname{sgn} [\bar{y}_{m+j}(t)] & , \text{ if } \tilde{T} \leq t \leq T, \end{cases} \quad (5.31b)$$

where $h_j > 0$ is the amplitude bound on the (j-th)-component of the admissible controls. From the definition of the class of admissible controls, equation (2.8), it is seen that $\tilde{v}(t)$ on $t \in [t_0, T]$ is an admissible control, i.e., $\tilde{v} \in \Lambda(T)$. For this choice of control, equation (5.21) becomes

$$\int_{t_0}^T \Psi^2(s) \tilde{v}(s) ds = h_j \int_{t_0}^T |\Psi_{n+j}(s)| ds, \quad \tilde{v} \in \Lambda(T) \quad (5.32)$$

However from equations (5.29) and (5.30), equation (5.32) implies that

$$\int_{t_0}^T \Psi^2(s) \tilde{v}(s) ds > h_j H / h_j = H, \quad \tilde{v} \in \Lambda(T) \quad (5.33)$$

Since the above procedure is valid for any choice of the number $H > 0$, then relation (5.33) is equivalent to relation (5.28).

Lemma 5.4 considered the Class Γ_1 . The Class Γ_2 is considered in the following lemma.

Lemma 5.5

Lemma 5.4 is also valid if Class Γ_1 is replaced by Class Γ_2 in the statement of the lemma.

Proof

The proof will be carried out in a similar manner to the proof of Lemma 5.4. That is, for any number $H > 0$, a time $T > t_0$ and an admissible control $\tilde{v}(t)$ on $t \in [t_0, T]$ will be found, such that equation (5.28) is satisfied.

Suppose for the given initial condition of $\Psi(t_0)$ with $\Psi^1(t_0)$ non-zero, that $\Psi_{n+j}(t)$, for some j , $j=1, 2, \dots, m$, belongs to Class Γ_2 . Then, from equation (5.27), for any number $\tilde{H} > 0$, there exists a time $T > t_0$, such that

$$\sum_{i=1}^{N-1} (\Delta \Psi_i)^2 > \tilde{H} \quad (5.34)$$

Since $\tilde{H} > 0$, then from equation (5.34), $N = N(T) \geq 2$. Therefore, by equation (5.26), there must be at least two extrema of $\Psi_{n+j}(t)$ on the time interval (t_0, T) . For convenience, suppose that the first extrema is a local maximum. Then by definition of the time instants t_i , $\Psi_{n+j}(t_i)$, $i=1, 3, \dots$, $i \leq N$, are the maxima of $\Psi_{n+j}(t)$ for $t \in (t_0, T)$. Thus,

$$\Psi_{n+j}(t) \leq \Psi_{n+j}(t_i), \text{ for all } t \in [t_{i-1}, t_{i+1}], \quad i=1, 3, \dots, i \leq N \quad (5.35)$$

Similarly, for the minima of $\Psi_{n+j}(t)$ on $t \in (t_0, T)$,

$$\Psi_{n+j}(t) \geq \Psi_{n+j}(t_i), \text{ for all } t \in [t_{i-1}, t_{i+1}], \quad i=2, 4, \dots, i \leq N \quad (5.36)$$

There is no loss of generality in taking t_1 to be the time instant at which $\Psi_{n+j}(t)$ is at a local maximum value, since the indexing parameter can be redefined in the case that t_1 corresponds to a local minimum of the function $\Psi_{n+j}(t)$.

Let the control $\tilde{v}(t)$ on $t \in [t_1, T]$ be defined by

$$\tilde{v}_i(t) \equiv 0, \quad i \neq j, \quad i=1, 2, \dots, m \quad (5.37)$$

and

$$\tilde{v}_j(t) = \begin{cases} (-1)^{i+1} h_j, & \text{if } t_i \leq t < t_i + \delta_i, \\ 0, & \text{if } t_i + \delta_i \leq t < t_{i+1} - \delta_i, \\ (-1)^i h_j, & \text{if } t_{i+1} - \delta_i \leq t < t_{i+1}, \\ & i = 1, 2, \dots, N-1, \\ 0, & \text{elsewhere,} \end{cases} \quad (5.38)$$

where δ_i , $i=1, 2, \dots, N-1$, is given by,

$$\delta_i = \begin{cases} \Delta y_i / 3r, & \text{if } \Delta y_i \leq 3r/h_j \\ \Delta t_i / h_j \beta, & \text{if } \Delta y_i > 3r/h_j \end{cases} \quad (5.39a)$$

$$\quad (5.39b)$$

and the numbers r and β were defined in equations (5.25) and (5.26).

In order to show that $\tilde{v} \in \Lambda(T)$, consider the definition of the Class Γ_2 . From equations (5.25) and (5.26)

$$\Delta y_i \leq r \Delta t_i \quad (5.40)$$

and

$$\Delta t_i \leq \beta, \quad i=1, 2, \dots, N-1 \quad (5.41)$$

Now, suppose that $\Delta y_i \leq 3r/h_j$. Then δ_i is given by equation (5.39a) and therefore $\delta_i \leq 1/h_j$. Also, from equation (5.40), $\delta_i \leq \Delta t_i/3$. If $\Delta y_i > 3r/h_j$, then δ_i is given by equation (5.39b). From equation (5.41), $\delta_i \leq 1/h_j$. Also, from equation (5.40), in this case $\Delta t_i > 3/h_j$. Since $\Delta t_i \leq \beta$, then $\delta_i < \Delta t_i/3$. Thus in both cases δ_i satisfies the following two constraints, viz.

$$\delta_i < 1/h_j \quad (5.42)$$

and

$$\delta_i \leq \Delta t_i / 3, \quad i = 1, 2, \dots, N-1 \quad (5.43)$$

From the definition of the class of admissible controls, equation (2.8), it is seen that the control $\tilde{v}(t)$ on $t \in [t_0, T]$ given by equations (5.37) and (5.38), with δ_i given by equation (5.39), is indeed admissible, i.e. $\tilde{v} \in A(T)$. The admissibility of the control $\tilde{v}(t)$ is a direct consequence of the fact that the constraints, equations (5.42) and (5.43) are satisfied.

For this choice of admissible control, equation (5.21) becomes

$$\begin{aligned} \int_{t_0}^T v^2(s) \tilde{v}(s) ds = h_j \sum_{i=1}^{N-1} \left\{ (-1)^{i+1} \int_{t_i}^{t_i + \delta_i} v_{n+j}(t) dt \right. \\ \left. + (-1)^i \int_{t_{i+1} - \delta_i}^{t_{i+1}} v_{n+j}(t) dt \right\} \end{aligned} \quad (5.44)$$

Let k be an odd number, $1 \leq k \leq N-1$, and consider the term $i = k$ of the summation in equation (5.44). Since k is odd, then from equations (5.35) and (5.36),

$$\int_{t_k}^{t_k + \delta_k} v_{n+j}(t) dt \geq \delta_k v_{n+j}(t_k + \delta_k) \quad (5.45)$$

and

$$- \int_{t_{k+1} - \delta_k}^{t_{k+1}} v_{n+j}(t) dt \geq -\delta_k v_{n+j}(t_{k+1} - \delta_k). \quad (5.46)$$

However, since the derivative of $v_{n+j}(t)$ is bounded, then

$$v_{n+j}(t_k + \delta_k) \geq v_{n+j}(t_k) - r\delta_k \quad (5.47)$$

and

$$-v_{n+j}(t_{k+1} - \delta_k) \geq -v_{n+j}(t_{k+1}) - r\delta_k \quad (5.48)$$

Let I_k represent the (k-th) term of the summation in equation (5.44). Then, by combining the results of equations (5.45) through (5.48), the following inequality for I_k is obtained.

$$I_k \geq [\Delta y_k - 2r\delta_k] h_j \delta_k, \quad (5.49)$$

where in this case $\Delta y_k = [y_{n+j}(t_k) - y_{n+j}(t_{k+1})]$ since k is odd and therefore equation (5.35) is applicable.

Consider the two possible values of δ_k from equations (5.39a) and (5.39b):

(1) $\Delta y_k \leq 3r/h_j$. In this case $\delta_k = \Delta y_k/3r$. Substituting this value of δ_k into equation (5.49) yields,

$$I_k \geq h_j (\Delta y_k)^2 / 9r \quad (5.50)$$

(2) $\Delta y_k > 3r/h_j$. In this case $\delta_k = \Delta t_k/h_j\beta$. Since $\Delta y_k > 3r/h_j$ and $\delta_k \leq 1/h_j$, then $\delta_k \leq \Delta y_k/3r$. Substituting for δ_k in equation (5.49) yields,

$$I_k \geq (\Delta y_k) \Delta t_k / 3\beta \quad (5.51)$$

However, from equation (5.40), equation (5.51) implies that

$$I_k \geq (\Delta y_k)^2 / 3r\beta \quad (5.52)$$

The number $\beta > 0$ is defined by equation (5.26) as $\Delta t_1 \leq \beta$. Therefore the inequality $\beta \geq 3/h_j$ can always be satisfied for any given problem. With this inequality satisfied, it is seen that equation (5.52) is valid for both choices of δ_k .

If k is an even number, equation (5.52) can be established by similar reasoning. Substituting equation (5.52) into the summation,

equation (5.44), yields,

$$\int_{t_0}^T \mathbf{Y}^2(s) \tilde{\mathbf{v}}(s) ds \geq \sum_{i=1}^{N-1} (\Delta \mathbf{Y}_i)^2 / 3\alpha \quad (5.53)$$

However, since $\mathbf{Y}_{n+j}(t)$ belongs to Class Γ_2 , then from equation (5.34)

$$\int_{t_0}^T \mathbf{Y}^2(s) \tilde{\mathbf{v}}(s) ds > \tilde{H} / 3\alpha \quad (5.54)$$

For any number $H > 0$, let $\tilde{H} = 3\alpha H$. Since the above procedure is valid for any number $\tilde{H} > 0$, then equation (5.54) is equivalent to equation (5.28).

The combined result of Lemmas 5.4 and 5.5 is that equation (5.28), or equivalently equation (5.17), is satisfied for any number $H > 0$, whenever the vector function $\mathbf{Y}^2(t)$ belongs to Class Γ . Thus if $\mathbf{Y}^2(t)$ belongs to Class Γ for all choices of $\mathbf{Y}(t_0)$, with $\mathbf{Y}^1(t_0)$ non-zero, then Theorem 5.2 follows immediately from Lemma 5.3.

Theorems 5.1 and 5.2 are the basic results on complete recoverability. In order to apply these theorems to a given problem, the adjoint solution must be constructed as a function of the initial condition $\mathbf{Y}(t_0)$, with $\mathbf{Y}^1(t_0)$ non-zero. Then the conditions of the theorems can be checked to see if the set of recoverable initial conditions, $S(t_0)$, equals the restraint set $\hat{G}$. This procedure is illustrated by two examples in the next section.

5.5 Examples

An example of a system that is not completely recoverable is given below. It will be shown that this system does not satisfy the necessary condition given by Theorem 5.1.

Consider the following unstable system for $t \geq 0$,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 + u, \end{cases} \quad (5.55)$$

with the control $u(t)$ bounded in both amplitude and rate for all time t . By using the transformation discussed in Chapter II, system (5.55) may be reformulated as a bounded state variable process and written as,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \quad (5.56)$$

where $x' = [x_1 x_2 x_3] = [x_1 x_2 u]$ and $v = \dot{u}$. For this example, $n = 2$, $m = 1$ and therefore $y^2(t) = y_3(t)$. The third component of the adjoint solution is given by

$$y_3(t) = \eta_3 - \eta_1 + \eta_1 \cosh t - \eta_2 \sinh t, \quad (5.57)$$

where $y(0) = [\eta_1 \eta_2 \eta_3]$. Consider ~~was~~ particular choice of $y(0) = [2, 2, 2]$. For this choice, $y^3(0)$ is non-zero and $y_3(t) = 2e^{-t}$. Substituting for $y^2(t) = y_3(t)$ in equation (5.18) yields.

$$\int_0^\infty |y^2(t)| dt = 2 < \infty \quad (5.58)$$

Thus the necessary condition of Theorem 5.1 is violated and therefore $S(t_0) \neq \hat{G}$ in this case.

As an illustration of the sufficiency condition of Theorem 5.2 consider the following oscillatory system with bounded control amplitudes and rates, for $t \geq 0$.

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = -x_1 + u_2 \end{cases} \quad (5.59)$$

The bounded state variable reformulation of system (5.59) is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v, \quad (5.60)$$

where $x' = [x_1 x_2 x_3 x_4] = [x_1 x_2 u_1 u_2]$, $v' = [v_1 v_2] = [\dot{u}_1 \dot{u}_2]$. For this example $n = 2$, $m = 2$ and $y^2(t) = [y_3(t) y_4(t)]$. The third and fourth components of the adjoint solution are given by,

$$\begin{aligned} y_3(t) &= -\eta_1 \sin t - \eta_2 (1 - \cos t) + \eta_3 \\ y_4(t) &= \eta_1 (1 - \cos t) - \eta_2 \sin t + \eta_4 \end{aligned} \quad (5.61)$$

where $y(0) = [\eta_1 \eta_2 \eta_3 \eta_4]$. From equation (5.61), both $y_3(t)$ and $y_4(t)$ are periodic as long as $y^1(0) = [\eta_1 \eta_2]$ is non-zero. Since all periodic functions belong to Class Γ_2 , then $y^2(t)$ always belongs to Class Γ and therefore Theorem 5.2 is satisfied. Thus $S(t_0) = \hat{G}$ and therefore every initial condition is recoverable.

In the following section, the two theorems developed in this chapter will be compared to the necessary and sufficient condition for the complete recoverability of system (2.1) in the case when instantaneous rate of change of control is allowed.

5.6 Comparison with Complete Recoverability in the Bounded Amplitude Case

If the constraint on the rate of change of the admissible controls, equation (2.3), is removed, then the class of admissible control inputs is described as follows: A control $u(t)$ is admissible if it is piecewise continuous and satisfies equation (2.2) for all $t \geq t_0$. In this

case, the complete recoverability of system (2.1) implies that every initial condition $y(t_0) \in R^n$ can be driven to $y(T) = 0$ with admissible controls, such that T is finite. A necessary and sufficient condition for the complete recoverability of linear systems whose control inputs are constrained only in amplitude, has been established in Reference 7. This condition can be applied to system (2.1), with $g(t) = 0$, for the class of admissible control inputs described above. Let $\phi(t)$ represent the n -dimensional vector solution to the adjoint equation $\dot{\phi}(t) = -\phi(t)E(t)$. Then Reference 7 gives the necessary and sufficient condition for complete recoverability as

$$\int_{t_0}^{\infty} \|\phi(t)F(t)\| dt = \infty, \quad (5.62)$$

for all choices of $\phi(t_0) \neq 0$.

The complete recoverability problem considered in this chapter has included the rate constraint on the admissible controls. The reformulated $(n+m)$ -dimensional problem is described by system (2.6), with $c(t)=0$. The results for this problem are Theorems 5.1 and 5.2 which provide a necessary and a sufficient condition for the complete recoverability of system (2.6).

A basic relationship between the two problems can be established by noting the form of the matrix $A(t)$ in system (2.6). Since the $(n+m)$ -dimensional adjoint vector $\Psi(t) = [\Psi^1(t)\Psi^2(t)]$ satisfies the equation $\dot{\Psi}(t) = -\Psi(t)A(t)$, then the following two relations are established; viz

$$\Psi^1(t) = \phi(t), \quad (5.63)$$

where $\dot{y}^1(t_0) = \phi(t_0) \neq 0$, and

$$\dot{y}^2(t) = -\dot{y}^1(t)F(t) = -\phi(t)F(t), \quad (5.64)$$

a comparison of the results for the two problems can now be made with the aid of the relationships established by equations (5.63) and (5.64).

Certainly, every system that is completely recoverable in the case when the rate of change of the admissible controls is constrained should also be recoverable when the rate constraint is removed. Thus if Theorem 5.2 is satisfied, then equation (5.62) should also be satisfied. To see that this is indeed the case, consider equation (5.64). This relation implies that equation (5.62) can be written in terms of $\dot{y}^2(t)$ as,

$$\int_{t_0}^{\infty} \|\dot{y}^2(t)\| dt = \infty \quad (5.65)$$

Now, suppose that Theorem 5.2 is satisfied, i.e., $\dot{y}^2(t)$ belongs to Class Γ for all choices of $\dot{y}(t_0)$, with $\dot{y}^1(t_0) \neq 0$. Then by definition of Class Γ , there exists at least one component of $\dot{y}^2(t)$, say $\dot{y}_{n+j}(t)$, belonging to either Class Γ_1 or Class Γ_2 , for all choices of $\dot{y}(t_0)$, with $\dot{y}^1(t_0) \neq 0$. First, suppose that $\dot{y}_{n+j}(t)$ for some j , $j=1,2,\dots,m$, belongs to Class Γ_1 . Then by equation (5.24), for any number $H > 0$ there exists a time $T > t_0$, such that $|\dot{y}_{n+j}(t)| > H$ for all $t \geq T$. Since,

$$\int_{t_0}^{\infty} \|\dot{y}^2(t)\| dt \geq \int_{t_0}^{\infty} |\dot{y}_{n+j}(t)| dt \geq \lim_{T \rightarrow \infty} |\dot{y}_{n+j}(T) - \dot{y}_{n+j}(t_0)|, \quad (5.66)$$

and since $\dot{y}_{n+j}(t_0)$ is some constant, then equation (5.66) implies that equation (5.65) is satisfied. Now, suppose that some $\dot{y}_{n+j}(t)$,

$j=1,2,\dots,m$, belongs to Class Γ_2 . From equations (5.25) and (5.26),

$$\int_{t_0}^{\infty} \|\dot{y}^2(t)\| dt \geq \int_{t_0}^{\infty} |\dot{y}_{n+j}(t)| dt \geq \sum_{i=1}^{\infty} \Delta y_i \quad (5.67)$$

Suppose, if possible, that the summation in equation (5.67) converges.

Then $\lim_{i \rightarrow \infty} \Delta y_i = 0$, and therefore, there exists an integer $M > 0$ such that,

$$\Delta y_i \leq 1/2, \text{ for all } i > M \quad (5.68)$$

Equation (5.68) implies that

$$(\Delta y_i)^2 < \Delta y_i, \text{ for all } i > M \quad (5.69)$$

Thus, by the Comparison Test [26], since the summation in equation (5.67) converges, then

$$\sum_{i=1}^{\infty} (\Delta y_i)^2 < \infty \quad (5.70)$$

however equation (5.70) contradicts the definition of Class Γ_2 , equation (5.27), and therefore the summation in equation (5.67) must diverge. Thus, by equation (5.67), equation (5.65) is satisfied. Since equation (5.65) is satisfied for all choices of $y(t_0)$, with $\dot{y}^1(t_0)$ non-zero, this implies, from equation (5.64), that equation (5.62) is satisfied.

In the above paragraph it has been shown that if Theorem 5.2 is satisfied, then equation (5.62) is indeed satisfied. On the other hand, it is not necessarily true that if equation (5.62) is satisfied, then Theorem 5.1 will be satisfied. In other words, a system may be completely recoverable with admissible controls that are bounded only in amplitude. However the property of complete recoverability may be lost when constraints on the rate of change of the admissible controls are added.

An example of such a system is presented below.

Consider the following scalar system for $t > 0$.

$$\dot{y}(t) = -y(t) + f(t)u(t), \text{ for } t \geq 0, \quad (5.71)$$

with

$$f(t) = -e^{-t}(-1)^{k-1} \sin k\pi(t-t_{k-1}), \text{ if } t_{k-1} \leq t < t_k, \quad k=1,2,\dots, \quad (5.72)$$

and with the time instants $t_k, k=0,1,2,\dots$, given by,

$$t_0 = 0, \quad t_k = \sum_{n=1}^k 1/n \quad (5.73)$$

From equation (5.73), for any integer $k \geq 1$,

$$t_k = t_{k-1} + 1/k \quad (5.74)$$

First, suppose that the control $u(t)$ is bounded only in amplitude.

Since $n = 1$, the adjoint solution $\phi(t)$ is scalar and is given by

$\phi(t) = \phi(0)e^t$. Substituting the expression for $\phi(t)f(t)$ into equation (5.62) yields,

$$\int_0^\infty |\phi(t)f(t)| dt = \sum_{k=1}^\infty |\phi(0)| \int_{t_{k-1}}^{t_k} |\sin k\pi(t-t_{k-1})| dt \quad (5.75)$$

The integral in the summation of equation (5.75) can be evaluated with the aid of equation (5.74). This yields the equation,

$$\int_0^\infty |\phi(t)f(t)| dt = \sum_{k=1}^\infty 2|\phi(0)|/k\pi = \infty, \quad (5.76)$$

for all choices of $\phi(0) \neq 0$. Thus, by equation (5.76), equation (5.62) is satisfied and therefore system (5.71) is completely recoverable for the case that the control u is only bounded in amplitude.

Now, suppose that a rate constraint on the control $u(t)$ is added, i.e., $|\dot{u}(t)| \leq h$, $h > 0$, for all $t \geq 0$. System (5.71) can be reformulated as a bounded state variable problem and written as

$$\dot{x}(t) = \begin{bmatrix} -1 & f(t) \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t), \quad (5.77)$$

where $x' = [x_1 x_2] = [y u]$ and $v = \dot{u}$. From equation (5.63), $y^1(t) = \phi(t)$, with $y^1(0) = \phi(0) \neq 0$. Since $m=1$, $y^2(t)$ is scalar and, from equation (5.64), is given by,

$$y^2(t) = y^2(0) + \int_0^t -y^1(s)f(s)ds \quad (5.78)$$

Substituting for $y^1(t)f(t)$ in equation (5.78) yields the relation,

$$[y^2(t) - y^2(0)]/y^1(0) = \begin{cases} \int_0^t \sin \pi t dt, & \text{if } 0 \leq t < t_1 \\ \sum_{k=1}^{N-1} \int_{t_{k-1}}^{t_k} (-1)^{k-1} \sin k\pi(t-t_{k-1}) dt \\ + \int_{t_{N-1}}^t (-1)^{N-1} \sin N\pi(t-t_{N-1}) dt \\ , & \text{if } t_{N-1} \leq t < t_N, N > 1. \end{cases} \quad (5.79)$$

With the aid of equation (5.74), equation (5.79) becomes,

$$[y^2(t) - y^2(0)]/y^1(0) = \begin{cases} (1 - \cos \pi t)/\pi, & \text{if } 0 \leq t < t_1 \\ \sum_{k=1}^{N-1} (-1)^{k-1} 2/\pi + (-1)^{N-1} (1 - \cos N\pi(t-t_{N-1}))/N\pi \\ , & \text{if } t_{N-1} \leq t < t_N, N > 1 \end{cases} \quad (5.80)$$

Now, in order to check whether Theorem 5.1 is satisfied, consider equation (5.18). In this case, $\varphi^2(t)$ is a scalar and, from equation (5.73), $t_1 = 1 > 0$. Therefore,

$$\int_0^\infty \|\varphi^2(t)\| dt = \int_0^\infty |\varphi^2(t)| dt = \int_0^{t_1} |\varphi^2(t)| dt + \int_{t_1}^\infty |\varphi^2(t)| dt = I_1 + I_2, \quad (5.81)$$

where $I_1 = \int_0^{t_1} |\varphi^2(t)| dt$ and $I_2 = \int_{t_1}^\infty |\varphi^2(t)| dt$.

From equation (5.80), and since $t_1 = 1$, then

$$I_1 = 1/\pi \int_0^1 |\varphi^2(0) + \varphi^1(0)(1 - \cos \pi t)| dt \leq H_1, \quad H_1 > 0, \quad (5.82)$$

for all choices of $\varphi(0) = [\varphi^1(0) \varphi^2(0)]$, with $\varphi^1(0) \neq 0$. Also, by equation (5.80),

$$I_2 = \sum_{N=2}^\infty \int_{t_{N-1}}^{t_N} |\varphi^2(0) + 2\varphi^1(0)/\pi \sum_{k=1}^{N-1} (-1)^{k-1}/k + [(-1)^{N-1} \varphi^1(0)(1 - \cos N\pi(t - t_{N-1}))]/N\pi| dt \quad (5.83)$$

Let,

$$J_1 = \sum_{N=2}^\infty \int_{t_{N-1}}^{t_N} |\varphi^2(0) + 2\varphi^1(0)/\pi \sum_{k=1}^{N-1} (-1)^{k-1}/k| dt \quad (5.84)$$

and

$$J_2 = \sum_{N=2}^\infty \int_{t_{N-1}}^{t_N} |[(-1)^{N-1} \varphi^1(0)(1 - \cos N\pi(t - t_{N-1}))]/N\pi| dt \quad (5.85)$$

Then, from equation (5.83),

$$I_2 \leq J_1 + J_2 \quad (5.86)$$

For a particular choice of $y(0)$, let $y^1(0)$ be arbitrary, $y^1(0) \neq 0$, and let,

$$y^2(0) = - [2y^1(0) \ln(2)]/\pi \quad (5.87)$$

Since $\sum_{k=1}^{\infty} (-1)^{k-1}/k = \ln(2)$, then from equation (5.87), equation (5.84) becomes

$$J_1 = \sum_{N=2}^{\infty} \int_{t_{N-1}}^{t_N} |2y^1(0)/\pi \sum_{k=N}^{\infty} (-1)^{k-1}/k| dt \quad (5.88)$$

Applying Leibniz's Alternating Series Test [26] to equation (5.88) and noting that $t_N - t_{N-1} = 1/N$, yields,

$$J_1 \leq \sum_{N=2}^{\infty} \int_{t_{N-1}}^{t_N} |2y^1(0)/N\pi| dt \leq \sum_{N=2}^{\infty} 2y^1(0)/N^2\pi \leq H_2 < \infty, \quad (5.89)$$

where H_2 is some number, $H_2 > 0$.

In a similar manner, it can be shown that there exists a number $H_3 > 0$, such that—

$$J_2 \leq H_3. \quad (5.90)$$

Substituting the results of equations (5.82), (5.89) and (5.90) into equations (5.86) and (5.81) yields the result that

$$\int_0^{\infty} \|y^2(t)\| dt = I_1 + I_2 \leq H_1 + H_2 + H_3 = H < \infty, \quad (5.91)$$

for these particular choices of $y(0) = [y^1(0), y^2(0)]$, with $y^1(0) \neq 0$.

Thus, equation (5.18) does not hold and therefore Theorem 5.1 is violated. System (5.71) is then not completely recoverable in the case when the admissible controls are constrained both in amplitude and rate, although it is completely recoverable when the rate constraint is removed.

System (5.71) and the unstable system, equation (5.55), of Section 5.5 are both examples of systems in which $S(t_0) \neq \hat{G}$. In these cases, it would be advantageous to know the actual limits of the set of recoverable initial conditions. In the concluding section of this chapter, a procedure for finding the limits of the set $S(t_0)$, when $S(t_0) \neq \hat{G}$, is established.

5.7. Computation of the Set of Recoverable Initial Conditions

Suppose that it has been established, through Theorem 5.1, that system (2.6) is not completely recoverable, i.e., $S(t_0) \neq \hat{G}$. Then, the limits of the set $S(t_0)$ can be obtained through the calculation of the extremal controls. To see this, consider Theorem 3.1. Theorem 3.1 establishes a correspondence between points $\bar{x}$ on the boundary of the set of recoverability, i.e., $\partial K(T)$, for any time $T \geq t_0$, and the extremal controls $\bar{v}(t)$ on $t \in [t_0, T]$. Thus from equation (5.1)

$$\partial K(t) = \left\{ \bar{x} | \bar{x} = - \int_{t_0}^t \phi(t_0, s) B \bar{v}(s) ds, \bar{v} \in \Lambda(t) \right\} \quad (5.92)$$

Since the set of recoverability is convex and non-decreasing, then from equation (2.13), the range of the set $S(t_0)$ may be found by obtaining the limits of the boundary points of $K(t)$ as t approaches infinity, i.e., $\lim_{t \rightarrow \infty} \partial K(t)$. Thus, from equation (5.92), this requires the calculation of the extremal controls $\bar{v}(s)$ on $s \in [t_0, t]$ as t approaches infinity.

In Reference [24], this procedure has been applied to the unstable system, equation (5.55), of Section 5.5. The reformulated bounded state variable system is given by equation (5.56). All the extremal controls were found analytically, as a function of the interval of time

$[0, t]$. The limits of the set of recoverable initial conditions were then found, with the aid of equation (5.92), by letting the time t approach infinity. The results were found to agree with those obtained by an alternate method [5].

In the following chapter, the thesis is concluded with a discussion of the results and with suggestions for future research.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

The three basic aspects of the problem are considered separately, below.

6.1 Minimal Time Open-Loop Solution

In Chapter III, a method for obtaining the minimal time open-loop control from the extremal controls has been indicated. It has been shown how each component of the extremal control can be expressed as a multiple of the signum of the corresponding component of a modified adjoint solution. With the aid of the Maximum Principle, inequalities have been derived and are presented in the form of tables. These tables determine the unknown parameters of the extremal controls. The results have also been incorporated into a digital computer program.

The most significant result from Chapter III is the set of six tables (Table 3.1 through 3.6). With the aid of these tables, the extremal controls can be constructed for general linear time-varying systems with control amplitude and rate saturations. Therefore, the construction of the extremal controls has been reduced to a "table look-up" operation. The significance of this result is that up to this point there was no systematic procedure for obtaining the extremal controls. Previously, a set of necessary conditions had to be used in order to construct each extremal control. This required a good deal of time-consuming effort.

Another important result is the digital computer program developed in Chapter III. With this program, a great number of extremal controls can be determined in a small amount of time. This is useful from two points of view. First, for a given initial condition, the program facilitates the calculation of the minimal time open-loop control from the extremal controls. This is because it will generally require the computation of a number of extremal controls in order to determine the one that matches the given initial condition. In addition, the program also facilitates the off-line computational procedure developed in Chapter IV. This procedure is based on first obtaining the numerical values of the state vectors on the various isochrones. These state vectors correspond to the extremal controls. Therefore for high order practical systems, it is important that the extremal controls can be obtained easily and quickly.

There are two aspects of the minimal time open-loop problem that should be further investigated. The first aspect is the extension of Tables 3.1 through 3.6 to include some or all of the cases that were eliminated in Section 3.8. The basic method for accomplishing this is still the same, i.e., by applying necessary conditions (A) through (D) of Section 3.6.

The second aspect concerns the procedure whereby the open-loop minimal time control is obtained from the extremal controls. In the procedure outlined in Chapter III, the values of the initial adjoint vector and the terminal time are guessed, for the given initial condition. Then these values are changed, until the resulting initial condition obtained from the extremal control is equivalent to the given initial condition. This can be viewed as a two point boundary value problem.

Therefore, it would be useful to investigate the possibility of obtaining an iterative computational scheme for this problem. This would be an improvement over the present "trial and error" procedure.

6.2 Sub-Optimal Closed-Loop Solution

In Chapter IV, a method for obtaining a sub-optimal closed-loop solution which is feasible for practical high order systems, has been developed. This method is based on approximating a discrete set of the isochrones of the system, over a specified maximal set of initial conditions. The resulting parameterized functions serve as an approximation to the isochrones, for the given maximal set. The sub-optimal closed-loop control is then chosen so as to minimize the time rate of change of these functions, subject to the given control amplitude and rate constraints.

The significance of the sub-optimal method of Chapter IV is that it is directly applicable to practical high order systems. This is because of the following reasons. First, the basic procedural steps of the method are independent of the particular system under consideration. Therefore, the method can be applied to general high order systems. In addition, the resulting sub-optimal controller can be easily implemented by using an on-line digital computer. Also, because of the nature of the method, both the on-line storage requirements and computation times are not prohibitive.

There are various aspects of the sub-optimal method of solution that remain to be investigated. These are enumerated below.

- (1) There is a tendency for some of the elements of the approximation matrices to grow without bound as the parameter T approaches

zero. This causes two problems. First, the succeeding element approximations are difficult to obtain. The second problem is that these high values tend to make a small region about the origin unstable. That is, the sub-optimal response trajectories will not remain near the origin, unless the controller is disengaged. In order to overcome these problems, some refinement of the isochrone approximation procedure is needed for small values of the parameter T . One possibility is to compute a larger number of state vectors on the isochrone. This will improve the resulting isochrone approximation.

(2) The element approximation procedure should be investigated to see if a different type of fitting technique could be used. For example, it may be found that a least mean squares fit based on minimizing the percentage deviation, may yield good sub-optimal controller performance. Also, it would be worthwhile to investigate the possibility of specifying some orthonormal form for the element approximation functions (e.g., Laguerre polynomials).

(3) The sub-optimal method should be applied to more systems, especially those of higher order. Also, the complete set of steps outlined in Operation I of Section 4.9, should be employed. That is, on each control interval the value of the parameter T should be computed from both equations (4.31) and (4.33).

6.3 Recoverability

In Chapter V, a necessary and a sufficient condition for complete recoverability have been derived. These conditions have been developed by a series of lemmas which involve the particular properties of the set of recoverable initial conditions. Also, for systems which are

not completely recoverable, a technique for obtaining the maximum region of recoverable initial conditions has been established.

The two theorems derived in Chapter V, provide a systematic method for investigating the complete recoverability of general linear systems with bounded control amplitudes and rates. For a given system, this involves the testing of the augmented components of the adjoint solution to the reformulated bounded state variable process.

It appears that the necessary condition for complete recoverability is too weak. Essentially, this condition reduces to the necessary and sufficient condition for the complete recoverability of systems whose controls are constrained only in amplitude. In order to improve the necessary condition, information about the form of the extremal controls should be used in the derivation. This is because the extremal controls correspond to points on the boundary of the set of recoverability. Therefore, these controls establish the limits of the set of recoverable initial conditions. The information about the form of the extremal controls can be obtained from necessary conditions (A) through (D) of Section 3.6.

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APPENDIX A

PROPERTIES OF THE SET OF RECOVERABILITY

Consider the set of recoverability given by

$$K(t) = \left\{ x \mid x = \int_{t_0}^t \phi(t, s) [Bv(s) + c(s)] ds, v \in \Lambda(t) \right\} \quad (A.1)$$

where $\phi(t, s)$ is a fundamental solution matrix, and $\Lambda(t)$ is the class of admissible controls. Let the set $L(t)$ be defined as

$$L(t) = \left\{ \xi \mid \xi = \int_{t_0}^t \phi(t, s) [Bv(s) + c(s)] ds, v \in \Lambda(t) \right\} \quad (A.2)$$

Since $v \in \Lambda(t)$, then by definition of the class of admissible controls, $u(t) = 0$, where $du(s)/ds = v(s)$, $s \in [t_0, t]$. Now consider the following transformations, viz.

$$\tilde{w}(s) = v(t + t_0 - s) \quad (A.3)$$

and

$$\zeta(s) = -u(t + t_0 - s), s \in [t_0, t]. \quad (A.4)$$

From equations (A.3) and (A.4), $d\zeta(s)/ds = w(s)$, $s \in [t_0, t]$ and $\zeta(t_0) = -u(t) = 0$. Let the set $Z(t)$ be defined as

$$Z(t) = \left\{ z \mid z = \int_{t_0}^t \phi(t, s) [Bw(s) + c(s)] ds \right\} \quad (A.5)$$

where $w(s)$ on $s \in [t_0, t]$ is given by equation (A.3). It is shown in Reference 19 that the set $Z(t)$, defined in equation (A.5), is compact,

convex and continuous in t . Since the proof of these properties in Reference 19 does not depend on the transformation given by equations (A.3) and (A.4), then the set $L(t)$ is also compact, convex and continuous in t . From equations (A.1) and (A.2), the sets $K(t)$ and $L(t)$ are related by the transformation

$$x = -\beta(t_0, t) \xi \quad (A.6)$$

Since the two sets are related by a linear continuous transformation, then the set $K(t)$ is also compact, convex and continuous in t .

APPENDIX B
APPLICATION OF THE MAXIMUM PRINCIPLE

Consider the second order system of differential equations on $t \in [t_0, T]$,

$$\begin{cases} \dot{\xi}_0 = -f(t)w(t) \\ \dot{\xi}_1 = w(t) \end{cases}, \quad \xi(t_0) = 0, \quad (B.1)$$

where the function $f(t)$ is continuous and $w(t)$ is a piecewise continuous function on $t \in [t_0, T]$. The restraint sets are given by

$$w \in \tilde{\Omega} = \{w \mid |w| \leq k, k > 0\} \quad (B.2)$$

and

$$\xi_1 \in \tilde{G} = \{\xi_1 \mid |\xi_1| \leq 1\} \quad (B.3)$$

Define the functions $g(\xi) = \xi_1^2 - 1$ and $q(w) = w^2 - k^2$. Then the restraint sets defined in equations (B.2) and (B.3) can be written as

$$\tilde{\Omega} = \{w \mid q(w) \leq 0\} \quad (B.4)$$

and

$$\tilde{G} = \{\xi_1 \mid g(\xi_1) \leq 0\} \quad (B.5)$$

Let $\xi_2 = t$. Then $\dot{\xi}_2 = 1$ and $\xi_2(t_0) = t_0$. System (B.1) can now be rewritten, in terms of the augmented variable ξ_2 , as the autonomous

system,

$$\begin{cases} \dot{\xi}_0 = -f(\xi_2)w \\ \dot{\xi}_1 = w \\ \dot{\xi}_2 = 1 \end{cases} \quad (\text{B.6})$$

with the initial conditions $\xi_0(t_0) = \xi_1(t_0) = 0$, $\xi_2(t_0) = t_0$. The optimization problem for system (B.6) is to find the control $\bar{w}(t)$, not identically zero, on $t \in [t_0, T]$ such that $\bar{w} \in \tilde{\Omega}$, $\bar{\xi}_1 \in \tilde{G}$ and $\bar{\xi}_0(T)$ is minimal.

As a first step in the application of the Maximum Principle to the above problem, it must be shown that every response trajectory $\xi(t) \in \partial\tilde{G}$, the boundary of $\tilde{G}$, is a regular trajectory. This is done in the following section.

B.1 Regularity Condition

The regularity condition [2] reduces to the following conditions for the case of scalar control.

A point $\tilde{\xi} \in \partial\tilde{G}$, the boundary of $\tilde{G}$, is said to be regular with respect to a point $\tilde{w} \in \tilde{\Omega}$ if

$$p(\tilde{\xi}, \tilde{w}) = 0, \quad (\text{B.7})$$

$$\partial p(\tilde{\xi}, \tilde{w}) / \partial w \neq 0 \quad (\text{B.8})$$

and

$$\tilde{w} \in \text{Int}(\tilde{\Omega}), \text{ the interior of } \tilde{\Omega}, \quad (\text{B.9})$$

where the function $p(\xi, w)$ is given by

$$p(\xi, w) = dg(\xi)/dt. \quad (B.10)$$

From the definition of the function $g(\xi)$, in this problem,

$p(\xi, w) = 2\tilde{\xi}_1(d\tilde{\xi}_1/dt) = 2\tilde{\xi}_1\tilde{w}$. Substituting for $p(\xi, w)$ into equations (B.7) and (B.8) yields the regularity condition for this problem, i.e.,

$$2\tilde{\xi}_1\tilde{w} = 0, \quad (B.11)$$

$$2\tilde{\xi}_1 \neq 0 \quad (B.12)$$

and repeating equation (B.9),

$$w \in \text{Int } (Q) \quad (B.13)$$

Now suppose that $\xi(t) \in \partial\tilde{G}$ on $t \in [\tau_A, \tau_B]$ where $t_0 \leq \tau_A < \tau_B \leq T$.

Then from system (B.6),

$$|\xi_1(t)| = 1 \quad (B.14)$$

and

$$|w(t)| = 0, \quad t \in [\tau_A, \tau_B]. \quad (B.15)$$

Substituting the results of equations (B.14) and (B.15) into equations (B.11) through (B.13), it is seen that the regularity condition is satisfied for every point $\xi(t) \in \partial\tilde{G}$, $t \in [\tau_A, \tau_B]$. Consequently from Reference 2, every response trajectory $\xi(t) \in \partial\tilde{G}$ of system (B.6) and hence system (B.1), is regular.

The Maximum Principle will now be applied to system (B.6) to find a set of necessary conditions.

B.2 Necessary Conditions

Assume that the response trajectory $\bar{\xi}(t)$ on $t \in [t_0, T]$ of system (B.6), corresponding to the optimal control $\bar{w}(t)$, enters upon and exits from its bound, $|\bar{\xi}_1| = 1$, at most a finite number of times. Define the function $H(\bar{\xi}, \bar{w}, \lambda)$ by

$$H(\bar{\xi}, \bar{w}, \lambda) = \lambda(t) \dot{\bar{\xi}}(t), \quad (\text{B.16})$$

where $\lambda(t) = [-1, \lambda_1(t), \lambda_2(t)]$ is a row vector function. The Maximum Principle [2] states that if $\bar{w}(t)$ is the optimal control on $t \in [t_0, T]$, then on every time interval that $\bar{\xi}(t) \in \text{Int}(\tilde{G})$,

$$H(\bar{\xi}, \bar{w}, \lambda) = \sup_{w \in \Omega} H(\bar{\xi}, w, \lambda) = 0 \quad (\text{B.17})$$

and

$$\dot{\lambda}' = -\partial H(\bar{\xi}, \bar{w}, \lambda) / \partial \bar{\xi}, \quad (\text{B.18})$$

where $(\quad)'$ is the transpose of $(\quad)$.

On the interior of every time interval that $\bar{\xi}(t) \in \partial \tilde{G}$ and consequently $\bar{w}(t) = 0$,

$$H(\bar{\xi}, 0, \lambda) = 0 \quad (\text{B.19})$$

and

$$\dot{\lambda}' = -\partial H(\bar{\xi}, 0, \lambda) / \partial \bar{\xi} + \mu(t) \partial p(\bar{\xi}, 0) / \partial \bar{\xi}, \quad (\text{B.20})$$

where the function $p(\bar{\xi}, w)$ was defined in equation (B.10) and $\mu(t)$ is the solution of the relation

$$\partial H(\bar{\xi}, 0, \lambda) / \partial w = \mu(t) \partial p(\bar{\xi}, 0) / \partial w. \quad (\text{B.21})$$

Also, at all points where the derivative of $\mu(t)$ exists,

$$d\mu/dt \leq 0 \quad (\text{B.22})$$

Substituting for $\xi(t)$ from system (B.6) into equation (B.16) yields the expression for the function $H(\xi, w, \lambda)$ for this problem, i.e.,

$$H(\xi, w, \lambda) = f(\xi_2)w + \lambda_1 w + \lambda_2 \quad (\text{B.23})$$

Now, consider a time interval such that $\bar{\xi}(t) \in \text{Int}(\tilde{G})$. Substituting equation (B.23) into equation (B.17) and noting that $\xi_2 = t$, yields the expression

$$H(\bar{\xi}, \bar{w}, \lambda) = \sup_{w \in \Omega} H(\bar{\xi}, w, \lambda) = k|f(t) + \lambda_1(t)| + \lambda_2(t) = 0 \quad (\text{B.24})$$

From the above equation,

$$\lambda_2(t) = -k|f(t) + \lambda_1(t)| \quad (\text{B.25})$$

and the optimal control $\bar{w}(t)$ is given by

$$\bar{w}(t) = k \operatorname{sgn} [f(t) + \lambda_1(t)], \quad (\text{B.26})$$

where $f(t) + \lambda_1(t) \neq 0$ on any finite sub-interval of time and,

$$\operatorname{sgn} [\alpha] = \begin{cases} 1, & \text{if } \alpha > 0 \\ 0, & \text{if } \alpha = 0 \\ -1, & \text{if } \alpha < 0 \end{cases} \quad (\text{B.27})$$

Also, from equations (B.18) and (B.23),

$$\dot{\lambda}_1(t) = 0, \quad (\text{B.28})$$

i.e., λ_1 is constant on this time interval. Thus, for this problem, equations (B.25) through (B.28) are the necessary conditions for system (B.6) and consequently system (B.1), when $\bar{\xi}_1(t) \in \text{Int}(\tilde{G})$.

Now, consider the interior of a time interval such that $\bar{\xi}(t) \in \partial\tilde{G}$. It has already been established that for this interval of time

$$\bar{w}(t) = 0 \quad (\text{B.29})$$

Substituting the expression for $H(\xi, w, \lambda)$ from equation (B.23) into (B.21) and noting that $g(\xi) = \xi_1^2 - 1$ and that $p(\xi, w) = dg(\xi)/dt$, yields

$$f(\bar{\xi}_2) + \lambda_1 = 2\bar{\xi}_1 \quad (\text{B.30})$$

Since $\bar{\xi}(t) \in \partial\tilde{G}$, then $|\bar{\xi}_1| = 1$ and therefore $\text{sgn} [\bar{\xi}_1] = \bar{\xi}_1$. Thus, from the above equation, since $\bar{\xi}_2 = t$,

$$\mu(t) = [f(t) + \lambda_1(t)] \text{sgn} [\bar{\xi}_1(t)]/2 \quad (\text{B.31})$$

Now, since $v_p(t, v)/v_s = 0$, equation (B.20) reduces to equation (B.18) and therefore, as before, by equation (B.23),

$$\dot{\lambda}_1(t) = 0 \quad (\text{B.32})$$

Also, by equations (B.19) and B.23),

$$\lambda_2(t) = 0 \quad (\text{B.33})$$

Finally, by equation (B.31), it is seen that $\mu(t)$ is always differentiable. Then since $\dot{\lambda}_1(t) = 0$ on this time interval, substituting equation (B.31) into equation (B.22) yields the relation

$$d\mu/dt = \dot{f}(t) \text{sgn} [\bar{\xi}_1(t)] \leq 0 \quad (\text{B.34})$$

Equation (B.29) and equations (B.32) through (B.34) are the necessary conditions for system (B.6) and consequently system (B.1), on the interior of intervals such that $\bar{\xi}_1(t) \in \partial\tilde{G}$.

The following section establishes the "jump conditions" for this problem, i.e., the conditions that the optimal response trajectory $\bar{\xi}(t)$ must satisfy at time instants when $\bar{\xi}_1(t)$ exits from or enters upon its bound, $|\bar{\xi}_1(t)| = 1$.

B.3 Jump Conditions

a time instant $t_1 \in (t_0, T)$ is called a junction time if the response trajectory enters upon or exits from $\partial\tilde{G}$ at $t = t_1$. At a junction time, the following relation must be satisfied [2].

$$\lambda(t_1^+) = \lambda(t_1^-) + \alpha \nabla g(\bar{\xi}(t_1^-)), \quad (B.35)$$

where α is a real number and $t_1^- = \lim_{\epsilon \rightarrow 0} t_1 + \epsilon$, $\epsilon > 0$. The time t_1^- is similarly defined. Also, the real number α must satisfy the relation,

$$\alpha \geq 0, \text{ if } \bar{\xi}(t_1^-), \bar{\xi}(t_1^+) \in \text{Int}(\tilde{G}) \quad (B.36)$$

For this problem, $g(\xi) = \xi_1^2 - 1$ and hence

$$\nabla g(\bar{\xi}) = \begin{bmatrix} 0 \\ 2\bar{\xi}_1 \\ 0 \end{bmatrix} \quad (B.37)$$

Thus from equation (B.35), in this case $\lambda_2(t)$ is continuous on $t \in (t_0, T)$ and

$$\lambda_1(t_1^+) = \lambda_1(t_1^-) + 2\alpha \bar{\xi}_1(t_1) \quad (B.38)$$

For the special case of $\bar{\xi}_1(t_1^-), \bar{\xi}_1(t_1^+) \in \text{Int}(\tilde{G})$, by equation (B.36), $\alpha \geq 0$. Since $|\bar{\xi}_1(t_1)| = 1$, then equation (B.38) becomes,

$$\lambda_1(t_1^+) - \lambda_1(t_1^-) = 2\alpha \operatorname{sgn} [\bar{\xi}_1(t_1)], \alpha \geq 0 \quad (\text{B.39})$$

Multiplying both sides of the above equation by $\operatorname{sgn} [\bar{\xi}_1(t_1)]$ yields,

$$[\lambda_1(t_1^+) - \lambda_1(t_1^-)] \operatorname{sgn} [\bar{\xi}_1(t_1)] \geq 0. \quad (\text{B.40})$$

Equations (B.38) and (B.40) along with the continuity requirement on $\lambda_2(t)$ for $t \in (t_0, T)$ are the jump conditions for system (B.6) and also system (B.1).

In the closing section of this Appendix, the Transversality Condition will be stated for this problem.

B.4 Transversality Condition

The Transversality Condition [2] states that for the unbounded problem the vector $[\lambda_1(t), \lambda_2(t)]$ at the terminal time $t = T$, is orthogonal to every vector $[\theta_1, \theta_2]$ which belongs or is parallel to the terminal manifold. At $t = T$, $\xi_2(T) = T$, T fixed. If $|\bar{\xi}_1(T)| < 1$, then the problem can be considered to be unbounded. The Transversality Condition in this case becomes

$$\lambda_1(T) = 0 \quad (\text{B.41})$$

APPENDIX C

COMPUTER PROGRAM LISTING

This appendix presents a listing of the subroutines discussed in Section 3.9. They are PSICLC, TALC, TELC, TILC, EXTR and EXTRP.

Subroutine PSICLC

```

UBROUTINE PSICLC(IPICK)
COMMON /BLOK1/ T
COMMON /BLOK3/ U
COMMON /BLOK3A/ DUOT
COMMON /BLOK7/ Y(100),Z(101),S(100),L
COMMON /BLOK8/ N
COMMON /BLOK9/ ZST(101),LST
COMMON /BLOK15/ SIGMA
T=1
F(IPICK,GT,1) GO TO 1300
T=1
IGR=SIGMA
Z=0
IGOUT=0
S=0
CALL DPSIDT(S)
SLP1=DUOT
IF(SLP1)100,90,100
IN CASE SLOPE IS ZERO
STEST=S+.01
CALL DPSIDT(STEST)
GO TO 50
100 CALL PSI(S)
YIN=U
BIN=S
N=N+1
IF(N,GT,2) GO TO 150
P1=Y(1)
IF(P1*SLP1,LT,0.) GO TO 125
GO TO 300
125 IF(ABS(P1),LT,.0005) GO TO 300
IZ=1
GO TO 300
IF(IZ,GT,0) GO TO 175
IZ=1
P1=Y(N-1)
GO TO 300
175 IF(YIN-1)*P1)190,180,180
180 IZ=0
GO TO 300
190 IF(ABS(YIN-1),LT,.0005) GO TO 180
P2=YIN-X)
GO TO 600
C STEP OF SIZE SIGMA
300 IF(IGOUT,GT,0) GO TO 1000
S=S+SIGMA
IF(S,GE,T)IGOUT=1
IF(ABS(T-S),LT,.0005) IGOUT=1
IF(IGOUT,EQ,1)S=T
CALL DPSIDT(S)
SLP2=DUOT
IF(SLP1*SLP2)400,350,310
IF(ABS(SLP2),GE,.0001) GO TO 340
IF(S,EQ,T) GO TO 325
C TEST FOR SADDLE POINT
320 STEST=S+.01
CALL DPSIDT(STEST)
IF(SLP1*DUOT)1325,320,340
325 SLP1=SLP2
GO TO 100
340 IF(IZ)300,300,300
350 IF(S,EQ,T) GO TO 355
C TEST FOR SADDLE POINT
352 STEST=S+.01
CALL DPSIDT(STEST)
IF(SLP1*DUOT)355,352,340
355 SLP1=SLP2
GO TO 75
C CALCULATION OF EXTREMA
400 SLP1=SLP2
S=S+.01
CALL DPSIDT(S)
SLP2=DUOT
IF(SLP1*SLP2)450,355,400
450 -ALPHA=SLP2/(SLP2-SLP1)
IF(ABS(ALPHA-.5),GT,.5)WRITE(6,2)ALPHA
DFORMAT(27HERROR IN SUBROUTINE PSICLC 5X,6HALPHA=F10.5,5X,
17HEXTREMA)
S=S+.01*ALPHA
GO TO 100
C CALCULATION OF ZERO CROSSINGS
500 SZ=S
CALL PSI(SZ)
P2=U
IF(P1*P2)620,505,300
505 IF(IGOUT,EQ,1)GO TO 1000
P1=P2
Z(L)=SZ
L=L+1
525 IF(P1*SLP1)300,550,510
550 IZ=0
GO TO 300
600 SZ=S
620 P1=P2
SZ=SZ+.01
CALL PSI(SZ)
P2=U
IF(P1*P2)650,510,620
650 ALPHA=P2/(P2-P1)
IF(ABS(ALPHA-.5),GT,.5) WRITE(6,3)ALPHA
DFORMAT(27HERROR IN SUBROUTINE PSICLC 5X,6HALPHA=F10.5,5X,
1AHZERO)
Z(L)=SZ+.01*ALPHA
GO TO 525
EXIT
1000 IF(ABS(T-BIN-1),LT,.0005) GO TO 1100
IF(ABS(T-S),LT,.0005) GO TO 1050
S=T
GO TO 100
1050 CALL PSI(T)
YIN=U
BIN=T
GO TO 1110
N=N+1
L=L+1
LST=L
IF(L,EQ,0) GO TO 1500
DO 1200 I=1,L
2ST(I)=Z(I)
GO TO 1300
1300 N1=N-1
DO 1400 I=1,N1

```

Subroutine PSIGC (cont'd)

```

      CALL PSI(B(I))
      Y(I)=U
      IF(ABS(Y(I)).LT..0005) GO TO 1400
      CALL PSI(B(I+1))
      Y(I+1)=U
      IF(Y(I)*Y(I+1))1325,1400,1400
1325  IF(ABS(Y(I+1)).LT..0005) GO TO 1400
C     CALCULATION OF ZERO CROSSING
      ALPHA=-Y(I)/(Y(I+1)-Y(I))
      IF(ABS(ALPHA-.5).GT..5)WRITE(6,3)ALPHA
      SZ=B(I)+ALPHA*(B(I+1)-B(I))
      CALL PSI(SZ)
      P2=U
      IF(P2*Y(I+1))1330,1380,1340
1330  DELTA=.01
      GO TO 1350
1340  DELTA=-.01
350   P1=P2
      SZ=SZ+DELTA
      CALL PSI(SZ)
      P2=U
      IF(P1*P2)1360,1380,1350
1360  ALPHA=P2/(P2-P1)
      IF(ABS(ALPHA-.5).GT..5)WRITE(6,3)ALPHA
      Z(L)=SZ-ALPHA*DELTA
      GO TO 1390
1380  Z(L)=SZ
1390  L=L+1
1400  CONTINUE
      CALL PSI(B(N))
      Y(N)=U
      GO TO 1110
1500  RETURN
      END

```

Subroutine TALC

```

SUBROUTINE TALC(Y1,Y2,B2)
COMMON /BLOK3/ U
COMMON /BLOK4/ TA
COMMON /BLOK13/ IERR
COMMON/BLOK15/SIGMA
SIGA=SIGMA
ICOUNT=0
DELTA=ABS(Y2-Y1)
100 S=B2+SIGA
CALL PSI(S)
DELTA2=ABS(Y2-U)
IF(DELTA2-DELTA)200,500,300
200 IF(ICOUNT.EQ.1) GO TO 400
225 DELTA1=DELTA2
S=S+SIGA
IF(S.LE.B2)GO TO 600
GO TO 100
300 IF(ICOUNT.EQ.1) GO TO 225
ICOUNT=1
SIGA=-.01
GO TO 225
400 ALPHA=(DELTA2-DELTA)/(DELTA2-DELTA1)
IF(ABS(ALPHA-.5).GT..5) GO TO 440
GO TO 450
440 WRITE(6,2) ALPHA
2 FORMAT(25H0ERROR IN SUBROUTINE TALC 5X,6HALPHA= F10.5)
IERR=1
450 TA=S-SIGA*ALPHA
GO TO 1000
500 TA=S
GO TO 1000
600 S=B2+.005
CALL PSI(S)
DELTA2=ABS(Y2-U)
TA=B2+(DELTA*.005)/DELTA2
IF(TA-B2.GT..01) GO TO 640
GO TO 1000
640 WRITE(6,1)DELTA
1 FORMAT(25H0ERROR IN SUBROUTINE TALC 5X,6HDELTA= E15.3)
IERR=1
1000 RETURN
END

```

Subroutine TBLC

```

SUBROUTINE TBLC(Y3,Y2,B2)
COMMON /BLOK3/ U
COMMON /BLOK5/ TB
COMMON /BLOK13/ IERR
COMMON/BLOK15/SIGMA
SIGB=SIGMA
ICOUNT=0
DELTA=ABS(Y2-Y3)
100 S=B2-SIGB
CALL PSI(S)
DELTA2=ABS(Y2-U)
IF(DELTA2-DELTA)200,500,300
200 IF(ICOUNT.EQ.1) GO TO 400
225 DELTA1=DELTA2
S=S-SIGB
IF(S.GE.B2)GO TO 600
GO TO 100
300 IF(ICOUNT.EQ.1) GO TO 225
ICOUNT=1
SIGB=-.01
GO TO 225
400 ALPHA=(DELTA2-DELTA)/(DELTA2-DELTA1)
IF(ABS(ALPHA-.5).GT..5) GO TO 440
GO TO 450
440 WRITE(6,2) ALPHA
2 FORMAT(25H0ERROR IN SUBROUTINE TBLC 5X,6HALPHA=F10.5)
IERR=1
450 TB=S+SIGB*ALPHA
GO TO 1000
500 TB=S
GO TO 1000
600 S=B2-.005
CALL PSI(S)
DELTA2=ABS(Y2-U)
TB=B2-(DELTA*.005)/DELTA2
IF(B2-TB.GT..01) GO TO 640
GO TO 1000
640 WRITE(6,1)DELTA
1 FORMAT(25H0ERROR IN SUBROUTINE TBLC 5X,6HDELTA= E15.3)
IERR=1
1000 RETURN
END

```

Subroutine TILC

```

SUBROUTINE TILC(YP,BP,1)
COMMON /BLOK1/ T
COMMON /BLOK3/ U
COMMON /BLOK6/ T1,T2
COMMON /BLOK11/ A,D
COMMON /BLOK13/ IERR
COMMON /BLOK16/ BETA
ICOUNT=0
IF(T-1150.51.52
SIG1=(19.*BETA+.01)/A.
GO TO 80
51 SIG1=BETA
GO TO 80
32 SIG1=.01
80 T2=BP+SIG1
IF(T1.LT.2) GO TO 100
IF(T2.LT.T) GO TO 100
SIG1=SIG1/2.
WRITE(6,31)
3 FORMAT(48H0 ERROR IN SUBROUTINE TILC T2 GREATER THAN T. I= 12)
GO TO 80
100 CALL PSI(T2)
ALP2=U
IF(T1-1)110,120,130
110 T1=(T2-D)/2.
GO TO 200
120 T1=T2-2.*D
GO TO 200
130 IF(1.EQ.3) GO TO 140
T1=T2-(T-D)/2.
GO TO 200
140 T1=2.*T2-T
200 CALL PSI(T1)
ALP1=U
DELTA2=ABS(YP-ALP2)-ABS(YP-ALP1)
IF(DELTA2)300,1000,400
300 IF(ICOUNT.EQ.1) GO TO 500
310 DELTA1=DELTA2
320 T2=T2-SIG1
IF(T1.LT.2) GO TO 350
IF(T2.LT.T) GO TO 350
T2=T2-SIG1
SIG1=SIG1/2.
WRITE(6,31)
GO TO 320
350 IF(T2.GT.BP) GO TO 100
WRITE(6,11)
1 FORMAT(46H0 ERROR IN SUBROUTINE TILC T2 LESS THAN BP. I= 12)
T2=T2-SIG1
SIG1=SIG1/2.
GO TO 320
400 IF(ICOUNT.EQ.1) GO TO 310
ICOUNT=1
SIG1=.01
GO TO 310
500 ALPHA=DELTA2/(DELTA2-DELTA1)
IF(ABS(ALPHA-.5).GT.5) GO TO 540
GO TO 550
540 WRITE(6,2) ALPHA,I
2 FORMAT(25H0ERROR IN SUBROUTINE TILC 5X,6HALPHA= F10.5,5X,2HI= 12)
IERR=1
550 T2=T2-SIG1*ALPHA
IF(I-1)600,625,650
600 T1=(T2-D)/2.
GO TO 1000
625 T1=T2-2.*D
GO TO 1000
650 IF(1.EQ.3) GO TO 675
T1=T2-(T-D)/2.
GO TO 1000
675 T1=2.*T2-T
1000 RETURN
END

```

Subroutine EXTR

```

SUBROUTINE EXTR(I)
LESS THAN TWO EXTREMA
COMMON /BLOK1/ T
COMMON /BLOK3/ U
COMMON /BLOK4/ TA
COMMON /BLOK5/ TB
COMMON /BLOK6/ T1,T2
COMMON /BLOK7/ Y(100),Z(101),B(100),L
COMMON /BLOK8/ N
COMMON /BLOK10/ V(210),TAU(210),KV,KCON
COMMON /BLOK11/ A,D
COMMON /BLOK12/ XJ
GO TO (100,114,115,125,133,259,296,297,300,301,316,322,334,332),I
100 IF(N.EQ.3) GO TO 250
NO EXTREMA (N=2)
IF(L.EQ.1) GO TO 120
NO ZERO CROSSINGS
IF(ABS(Y(21)).GT.ABS(Y(11)))GO TO 108
IF(Y(11).GT.0.) GO TO 104
XJ=-1.
GO TO 105
104 XJ=1.
105 IF(T.GT.0) GO TO 107
CONTROL 1
106 V(1)=XJ*A
TAU(1)=T
KV=1
KCON=1
GO TO 2500
CONTROL 2
107 V(1)=XJ*A
TAU(1)=D
V(2)=0.
TAU(2)=T
KV=2
KCON=2
GO TO 2500
108 IF(Y(2).LT.0.) GO TO 110
XJ=-1.
GO TO 111
110 XJ=1.
111 IF(T.GT.0) GO TO 113
CONTROL 3
112 V(1)=XJ*A
TAU(1)=T
KV=3
KCON=3
GO TO 2500
113 IF(T.GT.3.*D) GO TO 115
CONTROL 4
114 V(1)=XJ*A
TAU(1)=(T-D)/Z.
V(2)=XJ*A
TAU(2)=T
KV=2
KCON=4
GO TO 2500
CONTROL 5
115 V(1)=XJ*A
TAU(1)=D
V(2)=0.

```

```

TAU(2)=T-(2.*D)
V(3)=XJ*A
TAU(3)=T
KV=3
KCON=5
GO TO 2500
C ONE ZERO CROSSING
120 IF(Y(2).LT.Y(11)) GO TO 123
XJ=-1.
GO TO 124
123 XJ=1.
124 IF(T.GT.0) GO TO 126
CONTROL 6
125 V(1)=XJ*A
TAU(1)=Z(1)
V(2)=XJ*A
TAU(2)=T
KV=2
KCON=6
GO TO 2500
126 IF(Z(1).GT.0) GO TO 128
IF(T.GT.0+(2.*Z(1))) GO TO 129
GO TO 125
129 IF(T.GT.13.*D) GO TO 131
GO TO 114
131 GO TO 115
132 IF(T.GT.2*(1)+(2.*D)) GO TO 134
CONTROL 7
133 V(1)=XJ*A
TAU(1)=D
V(2)=0.
TAU(2)=Z(1)
V(3)=XJ*A
TAU(3)=T
KV=3
KCON=7
GO TO 2500
134 GO TO 115
C ONE EXTREMUM (N=3)
250 IF(ABS(Y(21)-Y(11)).GT.ABS(Y(31)-Y(21))) GO TO 300
C ABS(Y(21)-Y(11)) LESS THAN OR EQUAL TO ABS(Y(31)-Y(21))
IF(L-1) 251,271,311
251 IF(ABS(Y(21)).LT.ABS(Y(11))) GO TO 401
C NO ZERO CROSSINGS, ABS(Y(21)) GREATER THAN OR EQUAL TO ABS(Y(11))
IF(Y(1).GT.0.) GO TO 255
XJ=-1.
GO TO 256
255 XJ=1.
256 IF(T.GT.0) GO TO 258
GO TO 106
258 CALL TALL(Y(1),Y(21),B(21))
C CALCULATION OF TA
IF(TA.GT.0) GO TO 261
GO TO 107
261 IF(TA.LE.3.*D) GO TO 266
CALL PSI(D)
ALP1=U
CALL PSI(3.*D)
ALP2=U
IF(ABS(Y(2)-ALP2).LT.ABS(Y(21)-ALP1)) GO TO 268
C CALCULATION OF T1 AND T2, T2=2.*T1+D

```

Subroutine FXXV (cont'd)

```

266 CALL TILC(Y(2),B(2),0)
C CONTROL 8
267 V(1)=XJ*A
TAU(1)=T1
V(2)=XJ*A
TAU(2)=T2
V(3)=0
TAU(3)=T
KV=3
KCON=8
GO TO 2500
C CALCULATION OF T1 AND T2: T2=T1+2.*D
268 CALL TILC(Y(2),B(2),1)
C CONTROL 9
269 V(1)=XJ*A
TAU(1)=0
V(2)=0
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
V(4)=0
TAU(4)=T
KV=4
KCON=9
GO TO 2500
C ONE ZERO CROSSING
271 IF(Y(1).GT.0.) GO TO 274
XJ=-1
GO TO 275
274 XJ=1
275 IF(T.GT.D) GO TO 277
GO TO 125
277 IF(Z(1).GT.D) GO TO 283
IF(T-0.GT.2.*Z(1)) GO TO 280
GO TO 125
280 IF(T.GT.3.*D) GO TO 282
GO TO 114
282 GO TO 115
283 CALL TALC(Y(1),Y(2),B(2))
IF(TA.GT.D) GO TO 288
IF(T.GT.(2.*D+Z(1))) GO TO 287
GO TO 133
287 GO TO 135
288 IF(TA.LE.3.*D) GO TO 294
CALL PSI(D)
ALP1=0
CALL PSI(3.*D)
ALP2=0
IF(ABS(Y(2)-ALP2).LT.ABS(Y(2)-ALP1)) GO TO 298
CALL TILC(Y(2),B(2),0)
IF(T.GT.Z(1)+(2.*D)) GO TO 297
C CONTROL 10
296 V(1)=XJ*A
TAU(1)=T1
V(2)=XJ*A
TAU(2)=T2
V(3)=0
TAU(3)=Z(1)
V(4)=XJ*A
TAU(4)=T
KV=4

```

```

KCON=10
GO TO 2500
C CONTROL 11
297 V(1)=XJ*A
TAU(1)=T1
V(2)=XJ*A
TAU(2)=T2
V(3)=0
TAU(3)=T-(2.*D)
V(4)=XJ*A
TAU(4)=T
KCON=11
KV=4
GO TO 2500
298 CALL TILC(Y(2),B(2),1)
IF(T-Z(1).GT.2.*D) GO TO 301
C CONTROL 12
300 V(1)=XJ*A
TAU(1)=0
V(2)=0
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
V(4)=0
TAU(4)=Z(1)
V(5)=XJ*A
TAU(5)=T
KV=5
KCON=12
GO TO 2500
C CONTROL 13
301 V(1)=XJ*A
TAU(1)=0
V(2)=0
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
V(4)=0
TAU(4)=T-(2.*D)
V(5)=XJ*A
TAU(5)=T
KCON=13
KV=5
GO TO 2500
C TWO ZERO CROSSINGS
311 IF(Y(1).LT.0.) GO TO 314
XJ=-1
GO TO 315
314 XJ=1
315 IF(T.GT.D) GO TO 317
C CONTROL 14
316 V(1)=XJ*A
TAU(1)=Z(1)
V(2)=XJ*A
TAU(2)=Z(2)
V(3)=XJ*A
TAU(3)=T
KV=5
KCON=14
GO TO 2500
317 CALL TALC(Y(1),Y(2),B(2))

```

Subroutine EXPR (cont'd)

```

IF(TA-GE-D) GO TO 327
IF(T-D-GE-2*TA) GO TO 324
IF(Z(2)-Z(1)-LT-(T-D)/2) GO TO 322
GO TO 316
C CALCULATION OF T1 AND T2- T2=T1+(T-D)/2
322 CALL TILC(Y(2),B(2),T)
C CONTROL 18
V(1)=XJ*A
TAU(1)=T1
V(2)=XJ*A
TAU(2)=T2
V(3)=XJ*A
TAU(3)=T
KV=3
KCON=18
GO TO 2500
324 IF(T-GE-3*D) GO TO 326
GO TO 114
326 GO TO 115
327 IF(TA-LE-3*D) GO TO 329
CALL PSI(D)
ALP1=U
CALL PSI(3*D)
ALP2=U
IF(ABS(Y(2)-ALP2)-LT-ABS(Y(2)-ALP1)) GO TO 337
329 CALL TILC(Y(2),B(2),0)
IF(T-2*GE-2*D) GO TO 335
IF(Z(1)-GT-T1) GO TO 3291
GO TO 336
3291 IF(Z(1)-GT-D) GO TO 330
IF(Z(2)-Z(1)-LT-(T-D)/2) GO TO 322
GO TO 316
330 CALL PSI(D)
V1=U
CALL TALC(Y1,Y2,B(2))
IF(TA-D-GE-(T-D)/2) GO TO 331
GO TO 322
331 IF(Z(2)-LT-(T+Z(1))/2) GO TO 333
C CONTROL 16
332 V(1)=XJ*A
TAU(1)=D
V(2)=0
TAU(2)=Z(1)
V(3)=XJ*A
TAU(3)=Z(2)
V(4)=XJ*A
TAU(4)=T
KV=4
KCON=16
GO TO 2500
C CALCULATION OF T1 AND T2- T2=T1+T1/2
333 CALL TILC(Y(2),B(2),T)
C CONTROL 17
334 V(1)=XJ*A
TAU(1)=D
V(2)=0
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
V(4)=XJ*A
TAU(4)=T
KV=4
KCON=17
GO TO 2500
335 IF(T-2*GE-2*D) GO TO 342
IF(Z(1)-GT-T1) GO TO 338
GO TO 343
338 IF(Z(2)-LT-(T+Z(1))/2) GO TO 333
GO TO 332
342 IF(T-Z(2)-LT-2*D) GO TO 343
GO TO 301
C CONTROL 19
343 V(1)=XJ*A
TAU(1)=D
V(2)=0
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
V(4)=0
TAU(4)=Z(2)
V(5)=XJ*A
TAU(5)=T
KV=5
KCON=19
GO TO 2500
C NO ZERO CROSSINGS, ABS(Y(2)) LESS THAN ABS(Y(1))
401 IF(Y(1)-LT-0.1) GO TO 404
XJ=-1
GO TO 405
404 XJ=1
405 IF(T-GE-D) GO TO 407
GO TO 112
407 CALL TALC(Y(1),Y(2),B(2))
IF(TA-GE-D) GO TO 415
IF(T-D-GE-2*TA) GO TO 412
GO TO 322
412 IF(T-GE-3*D) GO TO 115
GO TO 114
415 IF(TA-LE-3*D) GO TO 417
CALL PSI(D)
ALP1=U
CALL PSI(3*D)
ALP2=U
IF(ABS(Y(2)-ALP2)-LT-ABS(Y(2)-ALP1)) GO TO 422
417 CALL TILC(Y(2),B(2),D)
IF(T-2*GE-2*D) GO TO 297
IF(B(2)-GT-D) GO TO 418

```

Subroutine EXTB (cont'd)

GO TO 322	V(2)=XJ*A
418 CALL PSI(D)	TAU(2)=T
V1=U	KV=2
CALL TALC(Y1,Y(2),B(2))	KCON=22
IF(T-D.GT.(T-D)/2.) GO TO 333	GO TO 2500
GO TO 322	607 IF(Z(1).GT.D) GO TO 628
422 CALL TLCL(Y(2),B(2),1)	IF(T-D.GT.2.*Z(1)) GO TO 610
IF(T-T2.GE.2.*D) GO TO 301	GO TO 404
GO TO 333	610 CALL TLCL(Y(3),Y(2),B(2))
ONE EXTREMUM, ABS(Y(2)-Y(1)) GREATER THAN ABS(Y(3)-Y(2))	IF(TB.GT.D) GO TO 622
500 IF(L-1)501,401,701	IF(T-D.GT.2.*TB) GO TO 614
501 IF(ABS(Y(2))<LT.ABS(Y(1))) GO TO 801	GO TO 509
NO ZERO CROSSINGS, ABS(Y(2)) GREATER THAN OR EQUAL TO ABS(Y(1))	614 IF(T.LE.3.*D) GO TO 618
IF(Y(1).GT.0.) GO TO 505	CALL PSI(4D)
XJ=-1	ALP1=U
GO TO 506	CALL PSI(3.*D)
505 XJ=1	ALP2=U
506 IF(T.GT.D) GO TO 508	IF(ABS(Y(2)-ALP2)<LT.ABS(Y(2)-ALP1)) GO TO 268
GO TO 106	618 GO TO 266
CALCULATION OF TB	622 IF(T.GT.3.*D) GO TO 624
508 CALL TLCL(Y(3),Y(2),B(2))	GO TO 509
IF(TB.GT.D) GO TO 520	624 IF(T-TB.GT.2.*D) GO TO 268
IF(T-D.GT.2.*TB) GO TO 512	GO TO 523
CONTROL 20	628 CALL TLCL(Y(3),Y(2),B(2))
509 V(1)=XJ*A	IF(T-Z(1).GT.2.*D) GO TO 630
TAU(1)=(T-D)/2.	CONTROL 21
V(2)=XJ*A	629 V(1)=XJ*A
TAU(2)=T	TAU(1)=D
KV=2	V(2)=0
KCON=20	TAU(2)=Z(1)
GO TO 2500	V(3)=XJ*A
512 IF(T.LE.3.*D) GO TO 514	TAU(3)=T
CALL PSI(D)	KV=3
ALP1=U	KCON=23
CALL PSI(3.*D)	GO TO 2500
ALP2=U	630 IF(T-TB.GT.2.*D) GO TO 268
IF(ABS(Y(2)-ALP2)<LT.ABS(Y(2)-ALP1)) GO TO 518	GO TO 523
516 GO TO 266	TWO ZERO CROSSINGS
518 GO TO 268	701 IF(Y(1).LT.0.) GO TO 704
520 IF(T.GT.3.*D) GO TO 522	XJ=-1
GO TO 509	GO TO 703
522 IF(T-TB.GT.2.*D) GO TO 268	704 XJ=1
CONTROL 21	705 IF(T.GT.D) GO TO 707
523 V(1)=XJ*A	GO TO 514
TAU(1)=D	707 CALL TLCL(Y(3),Y(2),B(2))
V(2)=0	IF(TB.GT.D) GO TO 733
TAU(2)=T-(2.*D)	IF(T-D.GE.2.*TB) GO TO 722
V(3)=XJ*A	IF(Z(1).GT.D) GO TO 714
TAU(3)=T	IF(Z(2)-Z(1)).LT.(T-D)/2.) GO TO 322
KV=3	GO TO 316
KCON=21	714 CALL PSI(D)
GO TO 2500	V1=U
ONE ZERO CROSSING	CALL TALC(Y1,Y(2),B(2))
601 IF(Y(1).LT.0.) GO TO 604	IF(TA-D.GT.(T-D)/2.) GO TO 718
XJ=-1	GO TO 322
GO TO 605	718 IF(Z(2).LT.(Y(2)+Z(1))/2.) GO TO 333
604 XJ=1	GO TO 332
605 IF(T.GT.D) GO TO 607	722 IF(T.LE.3.*D) GO TO 726
CONTROL 22	CALL PSI(D)
606 V(1)=XJ*A	ALP1=U
TAU(1)=Z(1)	CALL PSI(3.*D)

Subroutine EXCR (cont'd)

```

ALP2=U
IF(ABS(Y(2)-ALP2).LT.ABS(Y(2)-ALP1))-GO TO 744
726 CALL TILC(Y(2),B(2),0)
IF(T-T2.GE.2.*D) GO TO 741
IF(Z(1).GT.T1) GO TO 727
GO TO 336
727 IF(Z(1).GT.D) GO TO 733
IF(Z(2)-Z(1).LT.(T-D)/2.) GO TO 322
GO TO 316
733 CALL PSI(D)
Y1=U
CALL TALC(Y1,Y(2),B(2))
IF(TA-D.GT.(T-D)/2.) GO TO 737
GO TO 322
737 IF(Z(2).LT.(T+Z(1))/2.) GO TO 333
GO TO 332
741 IF(T-Z(2).GT.2.*D) GO TO 297
GO TO 336
744 CALL TILC(Y(2),B(2),1)
IF(T-T2.GE.2.*D) GO TO 750
IF(Z(1).GT.T1) GO TO 745
GO TO 343
745 IF(Z(2).LT.(T+Z(1))/2.) GO TO 333
GO TO 332
750 IF(T-Z(2).GT.2.*D) GO TO 301
GO TO 343
753 IF(T-TB.GT.2.*D) GO TO 758
IF(Z(2).LT.(T+Z(1))/2.) GO TO 333
GO TO 332
758 CALL TILC(Y(2),B(2),1)
IF(T-T2.GT.2.*D) GO TO 763
IF(Z(1).GT.T1) GO TO 759
GO TO 343
759 IF(Z(2).LT.(T+Z(1))/2.) GO TO 333
GO TO 332
763 IF(T-Z(2).GT.2.*D) GO TO 301
GO TO 343
C NO ZERO CROSSINGS* ABS(Y(2)) LESS THAN ABS(Y(1))
801 IF(Y(1).LT.0.) GO TO 804
XJ=-1.
GO TO 805
804 XJ=1.
805 IF(T2.GT.D) GO TO 807
GO TO 112
807 CALL TILC(Y(3),Y(2),B(2))
IF(TB.GT.D) GO TO 840
IF(T-D.GE.2.*TB) GO TO 819
IF(B(2).GT.D) GO TO 813
GO TO 322
813 CALL PSI(D)
Y1=U
CALL TALC(Y1,Y(2),B(2))
IF(TA-D.GT.(T-D)/2.) GO TO 333
GO TO 322
819 IF(T-TE.2.*D) GO TO 823
CALL PSI(D)
ALP1=U
CALL PSI(3.*D)
ALP2=U
IF(ABS(Y(2)-ALP2).LT.ABS(Y(2)-ALP1)) GO TO 835
823 CALL TILC(Y(2),B(2),0)
IF(T-T2.GE.2.*D) GO TO 297
IF(B(2).GT.D) GO TO 828
GO TO 322
828 CALL PSI(D)
Y1=U
CALL TALC(Y1,Y(2),B(2))
IF(TA-D.GT.(T-D)/2.) GO TO 333
GO TO 322
835 CALL TILC(Y(2),B(2),1)
IF(T-T2.GE.2.*D) GO TO 301
GO TO 333
840 IF(T-TB.GT.2.*D) GO TO 843
GO TO 333
843 CALL TILC(Y(2),B(2),1)
IF(T-T2.GT.2.*D) GO TO 301
GO TO 333
2500 RETURN
END

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Subroutine EXTRP

```

SUBROUTINE EXTRP(M)
TWO OR MORE EXTREMA
COMMON /BLOK1/ T
COMMON /BLOK3/ U
COMMON /BLOK4/ TA
COMMON /BLOK5/ TB
COMMON /BLOK6/ T1,T2
COMMON /BLOK7/ Y(100),Z(101),B(100),L
COMMON /BLOK8/ M
COMMON /BLOK10/ V(210),TAU(210),KV,KCON
COMMON /BLOK11/ A,D
COMMON /BLOK12/ XJ
COMMON /BLOK13/ IERR
1 FORMAT(14H1ERROR MESSAGE IS)
GO TO (1000,1500),M
TWO EXTREMA (M=4)
TEST
1000 CALL PSI(D)
ALP1=U
CALL PSI(3.*D)
ALP2=U
IF(ABS(Y(3))-Y(2)).LE.ABS(Y(1)-Y(2)) GO TO 1035
CALL TARC(Y(1),Y(2),B(2))
IF(TA-GT.D) GO TO 1030
IF(ABS(Y(4)-Y(3)).LE.ABS(ALP1-Y(3))) GO TO 1100
CALL TARC(ALP1,Y(3),B(3))
IF(TA-LT.3.*D) GO TO 2100
GO TO 1200
1030 IF(TA-LE.3.*D) GO TO 1050
1035 IF(ABS(Y(2)-ALP1).GT.ABS(Y(2)-ALP2)) GO TO 1075
1050 CALL TARC(Y(2),B(2),0)
CALL PSI(T2)
Y1=U
IF(ABS(Y(4)-Y(3)).LE.ABS(Y1-Y(3))) GO TO 1300
CALL TARC(Y1,Y(3),B(3))
IF(TA-T2.LT.2.*D) GO TO 2110
GO TO 1400
1075 CALL TARC(Y(2),B(2),1)
CALL PSI(T2)
Y1=U
IF(ABS(Y(4)-Y(3)).LE.ABS(Y1-Y(3))) GO TO 1090
CALL TARC(Y1,Y(3),B(3))
IF(TA-T2.LT.2.*D) GO TO 2120
IF(Y(2).GT.Y(1)) GO TO 1079
XJ=-1
GO TO 1080
1079 XJ=1
1080 V(1)=XJ*A
TAU(1)=D
IEND=2
GO TO 1801
1090 IF(Y(2).GT.Y(1)) GO TO 1094
XJ=-1
GO TO 1095
1094 XJ=1
1095 V(1)=XJ*A
TAU(1)=D
IEND=2
GO TO 1704
1100 XJ=-1
IF(Y(2).LT.Y(3)) XJ=1
IF(L-1) 1105,1110,1115
1105 IF(ABS(Y(2)).LT.ABS(Y(1))) GO TO 1120
GO TO 1180
1110 IF(Z(L).GT.D) GO TO 1140
GO TO 1120
1115 IF(Z(L).GT.B(3)) GO TO 1160
C NO ZERO CROSSINGS
1120 CALL TARC(Y(4),Y(3),B(3))
IF(T-TB.LT.2.*D) GO TO 1125
1121 CALL TARC(Y(3),B(3),1)
CALL EXTR(6)
GO TO 2500
1125 IF(T-LT.3.*D) GO TO 1130
XJ=XJ
CALL EXTR(3)
GO TO 2500
1130 IF(T-D.LT.2.*TA) GO TO 1135
IF(L-1) 1131,1132,1131
1131 XJ=XJ
CALL EXTR(2)
GO TO 2500
1132 IF(2.*Z(L).LT.T-D) GO TO 1131
1133 XJ=XJ
CALL EXTR(4)
GO TO 2500
1135 IF(L-1) 1138,1133,1136
1136 IF(Z(L)-Z(L-1).LT.(T-D)/2.) GO TO 1138
1137 XJ=XJ
CALL EXTR(11)
GO TO 2500
1138 XJ=XJ
CALL EXTR(12)
GO TO 2500
C ONE ZERO CROSSINGS
1140 CALL TARC(Y(4),Y(3),B(3))
IF(T-TB.LT.2.*D) GO TO 1145
GO TO 1121
1145 IF(T-Z(L).GT.2.*D) GO TO 1147
XJ=XJ
CALL EXTR(5)
GO TO 2500
1147 XJ=XJ
CALL EXTR(3)
GO TO 2500
C TWO ZERO CROSSINGS
1160 CALL TARC(Y(4),Y(3),B(3))
IF(T-TB.GT.2.*D) GO TO 1170
1161 IF(2.*Z(L).LT.T+Z(L-1)) GO TO 1165
CALL EXTR(14)
GO TO 2500
1165 CALL TARC(Y(3),B(3),3)
CALL EXTR(13)
GO TO 2500
1170 CALL TARC(Y(3),B(3),1)
IF(T-T2.GT.2.*D) GO TO 1175
IF(Z(L).LT.T2) GO TO 1161
1171 Z(1)=Z(2)
CALL EXTR(9)
GO TO 2500
1175 IF(T-Z(L).LT.2.*D) GO TO 1171
CALL EXTR(10)

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Subroutine EXMP (cont'd)

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GO TO 2500
NO ZERO CROSSINGS
1180 CALL TBLC(Y(4),Y(3),B(3))
IF(T-TB.GT.2.*D) GO TO 1190
GO TO 1165
1190 CALL TILC(Y(3),B(3),1)
IF(T-T2.LE.2.*D) GO TO 1165
CALL EXTR(10)
GO TO 2500
1200 XJ=-1.
IF(Y(2).LT.Y(3)) XJ=1.
IF(L-1) 1210,1230,1250
1210 IF(ABS(Y(2)),L=ABS(Y(1))) GO TO 1121
GO TO 1190
1230 IF(Z(1).GT.B(3)) GO TO 1231
GO TO 1121
1231 CALL TILC(Y(3),B(3),1)
IF(T-Z(1).GT.2.*D) GO TO 1232
CALL EXTR(9)
GO TO 2500
1232 CALL EXTR(10)
GO TO 2500
1250 IF(Z(L-1).GT.D) GO TO 1170
IF(Z(L).GT.B(3)) GO TO 1270
GO TO 1121
1270 Z(1)=Z(L)
GO TO 1230
1300 XJ=-1.
IF(Y(2).LT.Y(3)) XJ=1.
IF(L-1) 1305,1306,1320
1305 IF(ABS(Y(2)),L=ABS(Y(1))) GO TO 1320
GO TO 1380
1306 IF(Z(L).GT.T2) GO TO 1340
GO TO 1380
1310 IF(Z(L-1).GT.T2) GO TO 1360
IF(Z(L).GT.T2) GO TO 1340
C
NO ZERO CROSSINGS
1320 CALL TBLC(Y(4),Y(3),B(3))
IF(T-TB.LT.2.*D) GO TO 1325
C
CONTROL 26
1321 V(1)=XJ*A
TAU(1)=T1
V(2)=-XJ*A
TAU(2)=T2
CALL TILC(Y(3),B(3),1)
V(3)=0.
TAU(3)=T1
V(4)=XJ*A
TAU(4)=T2
V(5)=0.
TAU(5)=T
KV=5
KCON=26
GO TO 2500
1325 IF(T-T2.LT.2.*D) GO TO 1330
XJ=-XJ
CALL EXTR(8)
GO TO 2500
1330 IF(L.EQ.0) GO TO 1138
IF(Z(2)-Z(1).LT.(T-D)/2.) GO TO 1138
GO TO 1137

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C
ONE ZERO CROSSING
1340 CALL TBLC(Y(4),Y(3),B(3))
IF(T-TB.LT.2.*D) GO TO 1345
GO TO 1321
1345 IF(T-Z(L).GT.2.*D) GO TO 1347
Z(1)=Z(L)
XJ=-XJ
CALL EXTR(7)
GO TO 2500
1347 XJ=-XJ
CALL EXTR(8)
GO TO 2500
C
TWO ZERO CROSSINGS
1360 CALL TBLC(Y(4),Y(3),B(3))
IF(T-TB.GT.2.*D) GO TO 1370
1361 IF(2.*Z(L).LT.T-Z(L-1)) GO TO 1361
C
CONTROL 25
1362 V(1)=XJ*A
TAU(1)=T1
V(2)=-XJ*A
TAU(2)=T2
V(3)=0.
TAU(3)=Z(L-1)
V(4)=XJ*A
TAU(4)=Z(L)
V(5)=-XJ*A
TAU(5)=T
KV=5
KCON=25
GO TO 2500
C
CONTROL 26
1365 V(1)=XJ*A
TAU(1)=T1
V(2)=-XJ*A
TAU(2)=T2
CALL TILC(Y(3),B(3),1)
V(3)=0.
TAU(3)=T1
V(4)=XJ*A
TAU(4)=T2
V(5)=-XJ*A
TAU(5)=T
KV=5
KCON=26
GO TO 2500
1370 CALL TILC(Y(3),B(3),1)
IF(T-T2.GT.2.*D) GO TO 1375
IF(Z(L).GT.T2) GO TO 1372
CALL TILC(Y(2),B(2),0)
GO TO 1361
C
CONTROL 27
1372 CALL TILC(Y(2),B(2),0)
V(1)=XJ*A
TAU(1)=T1
V(2)=-XJ*A
TAU(2)=T2
CALL TILC(Y(3),B(3),1)
V(3)=0.
TAU(3)=T1
V(4)=XJ*A
TAU(4)=T2

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Subroutine EXTRP (cont'd)

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V(5)=0.
TAU(5)=Z(L)
V(6)=XJ*A
TAU(6)=T
KV=6
KCON=27
GO TO 2500
1375 IF(T-Z(L).LT.2.*D) GO TO 1372
1376 Z(L)=T-(2.*D)
GO TO 1372
'NO ZERO CROSSINGS
1380 CALL TLCL(Y(4),Y(3),B(3))
IF(T-TB.GT.2.*D) GO TO 1390
GO TO 1365
1390 CALL TLCL(Y(3),B(3),1)
IF(T-T2.GT.2.*D) GO TO 1395
CALL TLCL(Y(2),B(2),0)
GO TO 1365
1395 L=1
GO TO 1376
1400 XJ=-1.
IF(Y(2).LT.Y(3)) XJ=1.
IF(L-1) 1410,1430,1450
1410 IF(ABS(Y(2)).LT.ABS(Y(1))) GO TO 1321
GO TO 1390
1430 IF(Z(L).LT.B(2)) GO TO 1390
1435 IF(T-Z(L).LT.2.*D) GO TO 1372
GO TO 1376
1450 IF(Z(L-1).GT.T2) GO TO 1370
IF(Z(L).GT.B(3)) GO TO 1470
GO TO 1321
1470 Z(L)=Z(L)
L=1
GO TO 1435
GENERAL CASE, MORE THAN TWO EXTREMA (N=5 OR MORE)
TEST
1500 N2=N-2
IF(N.EQ.5) GO TO 1525
CALL TLCL(Y(3),B(3),1)
DO 1520 IT2=4,N2
AT2=T2
CALL TLCL(Y(IT2),B(IT2),1)
IF(T1.LT.AT2) GO TO 2130
1520 CONTINUE
1525 CALL TLCL(Y(3),B(3),1)
AT3=T1
IF(ABS(Y(3)-Y(2)).LE.ABS(Y(1)-Y(2))) GO TO 1535
CALL TALC(Y(1),Y(2),B(2))
IF(TA.GT.D) GO TO 1530
IF(AT3.LT.D) GO TO 2140
GO TO 1600
1530 IF(TA.LE.3.*D) GO TO 1540
1535 CALL PSI(D)
ALP1=U
CALL PSI(3.*D)
ALP2=U
IF(ABS(Y(2)-ALP1).GT.ABS(Y(2)-ALP2)) GO TO 1550
1540 CALL TLCL(Y(2),B(2),0)
IF(AT3.LT.T2) GO TO 2150
GO TO 1600
1550 CALL TLCL(Y(2),B(2),1)

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IF(AT3.LT.T2) GO TO 2160
COMPUTATION OF THE EXTREMAL CONTROL (BEGINNING)
1600 IF(Y(2).GT.Y(1)) GO TO 1604
XJ=-1.
GO TO 1605
1604 XJ=1.
1605 IF(ABS(Y(3)-Y(2)).LE.ABS(Y(1)-Y(2))) GO TO 1625
CALL TALC(Y(1),Y(2),B(2))
IF(TA.GT.D) GO TO 1620
V(1)=XJ*A
TAU(1)=D
ISTART=2
GO TO 1650
1620 IF(TA.LE.3.*D) GO TO 1630
1625 CALL PSI(D)
ALP1=U
CALL PSI(3.*D)
ALP2=U
IF(ABS(Y(2)-ALP1).GT.ABS(Y(2)-ALP2)) GO TO 1640
1630 CALL TLCL(Y(2),B(2),0)
V(1)=XJ*A
TAU(1)=T1
V(2)=XJ*A
TAU(2)=T2
ISTART=3
GO TO 1650
1640 CALL TLCL(Y(2),B(2),1)
V(1)=XJ*A
TAU(1)=D
V(2)=0.
TAU(2)=T1
V(3)=XJ*A
TAU(3)=T2
ISTART=4
C. CALCULATION OF THE EXTREMAL CONTROL (MIDDLE)
1650 N4=N-4
IEXT=3
IZERO=ISTART
IMAG=ISTART+1
ICOM=1
1675 IF(ICOM.EQ.N4) GO TO 1690
CALL TLCL(Y(IEXT),B(IEXT),1)
V(IZERO)=0.
TAU(IZERO)=T1
V(IMAG)=((-1.)**ICOM)*XJ*A
TAU(IMAG)=T2
IZERO=IZERO+2
IMAG=IMAG+2
IEXT=IEXT+1
ICOM=ICOM+1
GO TO 1675
1690 IEND=IZERO
C. CALCULATION OF THE EXTREMAL CONTROL (END)
1700 CALL TLCL(Y(IEXT),B(IEXT),1)
CALL PSI(T2)
Y1=U
IF(ABS(Y(N)-Y(N-1)).GT.ABS(Y1-Y(N-1))) GO TO 1800
1704 XJ=-1.
IF(Y(N-2).LT.Y(N-1)) XJ=1.
IF(L-1) 1705,1710,1710
1705 IF(ABS(Y(N-2)).LT.ABS(Y(N-3))) GO TO 1720

```

Subroutine ZCWP (cont'd)

```

GO TO 1780
1710 IF(Z(L).GT.B(N-1)) GO TO 1760
IF(Z(L).GT.T2) GO TO 1740
IF(XJ*Y(N-1).GT.0.) GO TO 1720
GO TO 1780
C NO ZERO CROSSINGS
1720 CALL TBLC(Y(N),Y(N-1),B(N-1))
IF(T-TB.LE.2.*D) GO TO 1725
C CONTROL 28
1721 V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
CALL TILC(Y(N-1),B(N-1),1)
V(IEND+2)=0.
TAU(IEND+2)=T1
V(IEND+3)=XJ*A
TAU(IEND+3)=T2
V(IEND+4)=0.
TAU(IEND+4)=T
KV=IEND+4
KCON=28
GO TO 2500
1725 IF(T-T2.LT.2.*D) GO TO 1730
C CONTROL 29
1726 V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
V(IEND+2)=0.
TAU(IEND+2)=T-(2.*D)
V(IEND+3)=XJ*A
TAU(IEND+3)=T
KV=IEND+3
KCON=29
GO TO 2500
1730 IF(L-1) 1735,1735,1731
1731 IF(Z(L).LT.B(N-2)) C: TO 1735
IF(2.*Z(L).LT.Z(L-1)+T) GO TO 1735
C CONTROL 30
1732 V(IEND)=0.
TAU(IEND)=Z(L-1)
V(IEND+1)=-XJ*A
TAU(IEND+1)=Z(L)
V(IEND+2)=XJ*A
TAU(IEND+2)=T
KV=IEND+2
KCON=30
GO TO 2500
C CONTROL 31
1735 CALL TILC(Y(N-2),B(N-2),3)
1736 V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
V(IEND+2)=XJ*A
TAU(IEND+2)=T
KV=IEND+2
KCON=31
GO TO 2500
C ONE ZERO CROSSING

```

```

1740 CALL TBLC(Y(N),Y(N-1),B(N-1))
IF(T-TB.GT.2.*D) GO TO 1721
IF(T-Z(L).GT.2.*D) GO TO 1726
C CONTROL 32
1741 V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
V(IEND+2)=0.
TAU(IEND+2)=Z(L)
V(IEND+3)=XJ*A
TAU(IEND+3)=T
KV=IEND+3
KCON=32
GO TO 2500
C TWO ZERO CROSSINGS
1760 CALL TBLC(Y(N),Y(N-1),B(N-1))
IF(T-TB.LE.2.*D) GO TO 1765
1761 CALL TILC(Y(N-1),B(N-1),1)
IF(T-T2.LT.2.*D) GO TO 1765
IF(T-Z(L).GT.2.*D) GO TO 1763
C CONTROL 33
1762 CALL TILC(Y(N-2),B(N-2),1)
V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
CALL TILC(Y(N-1),B(N-1),1)
V(IEND+2)=0.
TAU(IEND+2)=T1
V(IEND+3)=XJ*A
TAU(IEND+3)=T2
V(IEND+4)=0.
TAU(IEND+4)=Z(L)
V(IEND+5)=-XJ*A
TAU(IEND+5)=T
KV=IEND+5
KCON=33
GO TO 2500
C CONTROL 34
1763 CALL TILC(Y(N-2),B(N-2),1)
V(IEND)=0.
TAU(IEND)=T1
V(IEND+1)=-XJ*A
TAU(IEND+1)=T2
CALL TILC(Y(N-1),B(N-1),1)
V(IEND+2)=0.
TAU(IEND+2)=T1
V(IEND+3)=XJ*A
TAU(IEND+3)=T2
V(IEND+4)=0.
TAU(IEND+4)=T-(2.*D)
V(IEND+5)=-XJ*A
TAU(IEND+5)=T
KV=IEND+5
KCON=34
GO TO 2500
1765 IF(Z(L).GT.T2) GO TO 1762
1766 IF(2.*Z(L).LT.Z(L-1)+T) GO TO 1768
1767 CALL TILC(Y(N-2),B(N-2),1)
V(IEND)=0.

```

Subroutine EXTP (cont'd)

```

TAU(IEND)=1
V(IEND)=1-JWA
TAU(IEND+1)=T2
IEND=IEND+2
XJ=-1+XJ
GO TO 1732
768 CALL TILC(Y(N-2),B(N-2),1)
V(IEND)=0
TAU(IEND)=T1
V(IEND+1)=XJ+2
TAU(IEND+1)=T2
IEND=IEND+2
XJ=-1+XJ
CALL TILC(Y(N-1),B(N-1),3)
GO TO 1736
NO ZERO CROSSINGS
780 CALL TILC(Y(N),Y(N-1),B(N-1))
IF(T-B*LE.2.*D) GO TO 1768
785 CALL TILC(Y(N-1),B(N-1),1)
IF(T-T2*GT.2.*D) GO TO 1763
GO TO 1768
800 CALL TILC(Y1,Y(N-1),B(N-1))
IF(TA-T2*LT.2.*D) GO TO 2170
CALCULATION OF THE EXTREMAL CONTROL (END)
801 XJ=-1
IF(Y(N-2)*LT.Y(N-1)) XJ=1
IF(L-1) 1805,1810,1815
805 IF(ABS(Y(N-2))-LT.ABS(Y(N-3))) GO TO 1721
GO TO 1785
810 IF(Z(L)*GT.B(N-1)) GO TO 1761
IF(XJ*Y(N-1)*GT.0.) GO TO 1721
GO TO 1785
815 IF(Z(L-1)*GT.T2) GO TO 1761
GO TO 1810
1100 IERR=2
WRITE(6,1)IERR
WRITE(6,2101)
1101 FORMAT(18HOTA(B(1)) IS LESS THAN OR EQUAL TO 1/K, BUT TAU1/KJ IS L
LESS THAN 3/K
GO TO 2500
1110 IERR=3
WRITE(6,1)IERR
WRITE(6,2111)
1111 FORMAT(18HOT2(B(2))=2*T1+1/K BUT TAU(T2)-T2 IS LESS THAN 2/K
GO TO 2500
1120 IERR=4
WRITE(6,1)IERR
WRITE(6,2121)
1121 FORMAT(18HOT2(B(2))=T1+2*1/K BUT TAU(T2)-T2 IS LESS THAN 2/K
GO TO 2500
1130 IERR=5
WRITE(6,1)IERR
WRITE(6,2131)
1131 FORMAT(71HOMORE THAN 3 EXTREMA, BUT T1(B(K)) IS LESS THAN T2(B(K)-
1)) K=4+5*...N-2
GO TO 2500
1140 IERR=6
WRITE(6,1)IERR
WRITE(6,2141)
1141 FORMAT(98HOMORE THAN 2 EXTREMA AND TA(B(1)) IS LESS THAN OR EQUAL
1 TO 1/K, BUT T1(B(3)) IS LESS THAN TA(B(1))

```

APPENDIX D

NUMERICAL RESULTS FOR STEP (1) AND STEP (2)

In this appendix, some of the numerical results that were obtained for the illustrative example of Chapter IV are presented. Table D.1 gives the values of the angles ϕ_i , $i = 1, 2, 3$, that were used to calculate the initial adjoint vectors. Tables D.2 through D.7 gives the computed values of the elements of the matrices $\bar{C}_k$ and $\tilde{C}_k$, $k=1, 2, \dots, 118$.

Table D.1. Numerical Values of the Angles

ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3	ϕ_1	ϕ_2	ϕ_3
Degrees			Degrees			Degrees			Degrees			Degrees		
120.	50.	10.	150.	30.	10.	40.	10.	0.	100.	50.	30.	90.	90.	60.
90.	90.	80.	90.	50.	10.	90.	40.	10.	20.	70.	10.	0.	-70.	-30.
0.	-40.	-60.	90.	90.	-60.	90.	90.	-40.	90.	70.	10.	90.	50.	20.
90.	60.	30.	90.	50.	40.	60.	30.	10.	40.	10.	70.	70.	-40.	70.
60.	-50.	-10.	90.	90.	0.	90.	70.	50.	90.	70.	30.	90.	60.	40.
50.	40.	60.	90.	30.	60.	70.	10.	0.	20.	70.	60.	20.	70.	-40.
70.	40.	30.	70.	50.	-30.	50.	70.	-10.	90.	90.	50.	60.	60.	10.
70.	40.	20.	90.	-40.	-10.	90.	-70.	-10.	90.	40.	40.	90.	-70.	-40.
90.	30.	70.	90.	60.	70.	90.	-60.	-70.	90.	-20.	70.	90.	-30.	60.
50.	-30.	70.	90.	-50.	50.	70.	60.	-10.	70.	70.	0.	70.	50.	10.
60.	70.	20.	70.	50.	20.	60.	60.	50.	70.	50.	30.	70.	50.	40.
100.	-70.	0.	70.	70.	10.	100.	-60.	-10.	90.	-10.	10.	90.	0.	-20.
90.	10.	-20.	90.	10.	-30.	90.	30.	-40.	90.	40.	0.	90.	50.	-30.
90.	60.	-50.	90.	-60.	30.	90.	70.	-50.	90.	70.	40.	90.	-50.	20.
90.	-20.	-10.	90.	-30.	-20.	90.	50.	-50.	90.	60.	70.	90.	10.	-50.
90.	70.	-60.	90.	0.	-30.	90.	-50.	30.	90.	60.	40.	90.	-60.	50.
50.	-70.	50.	90.	-70.	70.	100.	-70.	-10.	100.	-60.	30.	70.	70.	30.
00.	-40.	-40.	10.	70.	70.	70.	60.	50.	80.	40.	50.	00.	10.	-50.
30.	70.	60.	70.	50.	50.	70.	60.	40.	90.	90.	70.	90.	20.	0.
90.	30.	0.	90.	50.	-10.	90.	60.	0.	90.	-60.	10.	90.	70.	-10.
90.	-70.	10.	90.	-30.	0.	90.	-40.	10.	90.	40.	0.	90.	-50.	0.
90.	-20.	0.	100.	60.	-30.	80.	30.	60.	30.	-60.	70.	40.	70.	70.
00.	10.	-60.	60.	-60.	70.	80.	0.	60.	70.	0.	70.	80.	20.	60.
10.	70.	-70.	80.	60.	60.	70.	-70.	10.	70.	50.	0.	70.	-70.	30.
60.	-20.	40.	80.	-40.	50.	60.	-70.	10.	60.	-70.	60.	60.	-70.	40.
80.	-20.	50.	80.	-20.	60.	100.	70.	10.	100.	70.	20.	60.	-70.	-20.
90.	90.	-20.	90.	90.	-50.	100.	-40.	-70.	80.	-20.	10.	70.	-70.	0.
80.	-20.	20.	80.	-10.	10.	80.	-40.	-10.	70.	-60.	10.	70.	-70.	-30.
70.	-40.	-40.	70.	-70.	-20.	70.	0.	-60.	100.	70.	40.	80.	-40.	70.
60.	-70.	40.	80.	-70.	0.	80.	-70.	-10.	80.	-60.	40.	80.	30.	10.
60.	10.	-10.	80.	-20.	-10.	80.	30.	0.	80.	10.	30.	70.	30.	-70.
50.	60.	-70.	100.	-70.	40.	70.	70.	-20.	70.	60.	50.	60.	70.	-50.
60.	40.	10.	80.	-60.	-30.	100.	70.	60.	80.	60.	50.	80.	50.	-50.
80.	50.	-10.	80.	60.	0.	80.	50.	10.	80.	60.	10.	80.	50.	30.
80.	10.	-70.	70.	-60.	-70.	80.	70.	-20.	80.	60.	30.	00.	-70.	-30.
00.	-60.	-50.	80.	60.	40.	70.	70.	50.	80.	70.	50.	80.	-70.	50.
00.	50.	-70.	100.	60.	-50.	110.	-60.	-70.	100.	40.	60.	80.	50.	60.
80.	-30.	70.	100.	70.	-20.	80.	10.	70.	80.	50.	70.	80.	-60.	40.
80.	-50.	60.	80.	-70.	-30.	80.	-70.	-40.	80.	60.	60.	70.	70.	-70.
80.	-40.	-70.	80.	-20.	-70.	80.	-60.	0.	80.	50.	10.	80.	-30.	-30.
70.	-70.	-40.	100.	70.	50.	70.	-70.	-50.	80.	50.	60.	80.	-60.	-50.
80.	-50.	-40.	80.	50.	-40.	80.	10.	-60.	80.	40.	50.	90.	90.	-8.
70.	70.	-50.	80.	70.	-40.	80.	70.	-50.	80.	70.	0.	80.	60.	-20.
60.	70.	10.	80.	-60.	70.	100.	70.	-70.	100.	-60.	70.	80.	70.	60.
00.	-70.	-70.	80.	70.	20.	100.	-70.	-40.	80.	70.	40.	80.	70.	-70.

Table D.2. Computed Values of the Elements $\tilde{c}_{11} - \tilde{c}_{15}$

T_k Seconds	Element				
	$\tilde{c}_{11}$	$\tilde{c}_{12}$	$\tilde{c}_{13}$	$\tilde{c}_{14}$	$\tilde{c}_{15}$
1.30	0.25036E 06	0.15712E 06	0.41343E 01	0.56945E 04	0.36726E 03
1.35	0.14680E 06	0.97120E 05	0.26856E 05	0.38705E 04	0.25495E 03
1.40	0.12134E 06	0.83826E 05	0.24270E 05	0.37047E 04	0.27165E 03
1.45	0.99307E 05	0.70860E 05	0.21199E 05	0.33727E 04	0.26632E 03
1.50	0.73476E 05	0.54020E 05	0.16607E 05	0.27216E 04	0.22258E 03
1.55	0.51035E 05	0.38758E 05	0.12266E 05	0.20743E 04	0.17491E 03
1.60	0.38174E 05	0.30121E 05	0.99061E 04	0.17640E 04	0.15965E 03
1.65	0.30499E 05	0.24789E 05	0.83907E 04	0.15465E 04	0.14621E 03
1.70	0.21490E 05	0.18106E 05	0.63371E 04	0.12163E 04	0.11870E 03
1.75	0.16278E 05	0.14229E 05	0.51639E 04	0.10280E 04	0.10276E 03
1.80	0.13906E 05	0.12424E 05	0.46417E 04	0.94461E 03	0.95610E 02
1.85	0.12002E 05	0.11034E 05	0.42060E 04	0.86446E 03	0.87612E 02
1.90	0.10009E 05	0.94071E 04	0.36580E 04	0.75999E 03	0.76966E 02
1.95	0.84086E 04	0.80826E 04	0.32086E 04	0.67398E 03	0.68371E 02
2.00	0.71050E 04	0.69993E 04	0.28430E 04	0.60601E 03	0.62178E 02
2.05	0.57267E 04	0.57788E 04	0.24005E 04	0.52053E 03	0.54312E 02
2.10	0.45933E 04	0.47390E 04	0.20082E 04	0.44140E 03	0.46631E 02
2.20	0.30825E 04	0.33008E 04	0.14437E 04	0.32247E 03	0.34360E 02
2.30	0.22756E 04	0.25345E 04	0.11510E 04	0.26798E 03	0.30157E 02
2.40	0.16850E 04	0.19626E 04	0.93280E 03	0.23522E 03	0.29229E 02
2.50	0.13014E 04	0.15833E 04	0.78671E 03	0.21317E 03	0.28489E 02
2.60	0.91369E 03	0.11580E 04	0.59803E 03	0.17250E 03	0.24352E 02
2.70	0.66333E 03	0.88052E 03	0.47576E 03	0.14343E 03	0.20677E 02
2.80	0.45783E 03	0.62919E 03	0.35026E 03	0.10667E 03	0.15273E 02
2.90	0.28557E 03	0.40454E 03	0.23022E 03	0.70481E 02	0.99811E 01
3.00	0.22252E 03	0.32905E 03	0.19540E 03	0.63122E 02	0.91905E 01
3.10	0.19861E 03	0.30469E 03	0.18810E 03	0.63115E 02	0.93462E 01
3.20	0.15965E 03	0.24903E 03	0.15583E 03	0.51694E 02	0.75309E 01
3.30	0.10775E 03	0.17061E 03	0.10738E 03	0.35369E 02	0.50646E 01
3.40	0.86528E 02	0.14710E 03	0.10048E 03	0.38321E 02	0.59722E 01
3.50	0.77660E 02	0.13928E 03	0.10100E 03	0.41791E 02	0.67715E 01
3.60	0.63911E 02	0.11693E 03	0.86575E 02	0.36127E 02	0.58933E 01
3.70	0.51842E 02	0.96281E 02	0.72224E 02	0.30192E 02	0.49620E 01
3.80	0.41788E 02	0.78603E 02	0.59579E 02	0.24834E 02	0.40784E 01
3.90	0.33962E 02	0.64690E 02	0.49571E 02	0.20770E 02	0.33945E 01
4.00	0.27251E 02	0.52521E 02	0.40654E 02	0.17142E 02	0.27975E 01
4.10	0.22305E 02	0.43784E 02	0.34465E 02	0.14942E 02	0.24646E 01
4.20	0.18919E 02	0.38224E 02	0.31188E 02	0.14106E 02	0.23447E 01
4.30	0.15660E 02	0.32490E 02	0.27448E 02	0.12694E 02	0.21405E 01
4.40	0.12268E 02	0.25921E 02	0.22429E 02	0.10344E 02	0.17391E 01
4.50	0.96849E 01	0.21189E 02	0.18901E 02	0.93570E 01	0.15890E 01
4.60	0.86623E 01	0.19776E 02	0.18073E 02	0.10036E 02	0.17929E 01
4.70	0.78483E 01	0.18464E 02	0.17128E 02	0.10283E 02	0.19203E 01
4.80	0.64484E 01	0.15487E 02	0.14583E 02	0.91362E 01	0.17621E 01
4.90	0.51162E 01	0.12541E 02	0.11983E 02	0.77646E 01	0.15293E 01
5.00	0.41456E 01	0.10363E 02	0.10046E 02	0.67161E 01	0.13460E 01
5.10	0.35037E 01	0.88863E 01	0.87218E 01	0.59655E 01	0.12061E 01
5.20	0.30456E 01	0.78073E 01	0.77353E 01	0.53715E 01	0.10815E 01
5.30	0.26685E 01	0.69112E 01	0.69039E 01	0.48653E 01	0.97607E 00
5.40	0.23246E 01	0.60952E 01	0.61336E 01	0.44086E 01	0.88475E 00
5.50	0.20068E 01	0.52901E 01	0.53696E 01	0.38961E 01	0.78166E 00
5.60	0.16900E 01	0.44616E 01	0.45711E 01	0.33307E 01	0.66604E 00
5.70	0.13947E 01	0.36856E 01	0.38173E 01	0.27963E 01	0.55752E 00
5.80	0.11381E 01	0.29971E 01	0.31515E 01	0.23029E 01	0.45882E 00
5.90	0.93356E 00	0.24544E 01	0.26165E 01	0.19208E 01	0.38025E 00
6.10	0.64057E 00	0.17062E 01	0.18450E 01	0.14310E 01	0.28550E 00
6.30	0.46356E 00	0.12752E 01	0.13860E 01	0.11671E 01	0.23212E 00
6.50	0.38271E 00	0.11146E 01	0.11680E 01	0.11312E 01	0.22918E 00
6.70	0.30375E 00	0.90881E 00	0.92692E 00	0.98710E 00	0.20227E 00

Table D.2. (cont'd.)

T_k Seconds	Element				
	$\tilde{c}_{11}$	$\tilde{c}_{12}$	$\tilde{c}_{13}$	$\tilde{c}_{14}$	$\tilde{c}_{15}$
6.9	0.22035E+00	0.65892E 00	0.66537E 00	0.75051E 00	0.15854E 00
7.1	0.15600E 00	0.46026E 00	0.46256E 00	0.54698E 00	0.11859E 00
7.3	0.11917E 00	0.35427E 00	0.33733E 00	0.44097E 00	0.97390E-01
7.5	0.97880E-01	0.30196E 00	0.25454E 00	0.39173E 00	0.88470E-01
7.7	0.82350E-01	0.26541E 00	0.19667E 00	0.34885E 00	0.79930E-01
7.9	0.70870E-01	0.23745E 00	0.15481E 00	0.30996E 00	0.71550E-01
8.1	0.64460E-01	0.22549E 00	0.12250E 00	0.29335E 00	0.68290E-01
8.3	0.61660E-01	0.22272E 00	0.98310E-01	0.28651E 00	0.67680E-01
8.5	0.61270E-01	0.23009E 00	0.81660E-01	0.28920E 00	0.69370E-01
8.7	0.65300E-01	0.25616E 00	0.67380E-01	0.31413E 00	0.76150E-01
8.9	0.68780E-01	0.27418E 00	0.53350E-01	0.32411E 00	0.79870E-01
9.1	0.65240E-01	0.26167E 00	0.46850E-01	0.29430E 00	0.73650E-01
9.3	0.57710E-01	0.23375E 00	0.32400E-01	0.24748E 00	0.62160E-01
9.5	0.50320E-01	0.20559E 00	0.28000E-01	0.20233E 00	0.50360E-01
9.7	0.44780E-01	0.18434E 00	0.26980E-01	0.16457E 00	0.40230E-01
9.9	0.41170E-01	0.17259E 00	0.29450E-01	0.13810E 00	0.32810E-01
10.1	0.38210E-01	0.16566E 00	0.35230E-01	0.12006E 00	0.27540E-01
10.3	0.35270E-01	0.15993E 00	0.43190E-01	0.10823E 00	0.24240E-01
10.5	0.32950E-01	0.15677E 00	0.51200E-01	0.10318E 00	0.23220E-01
10.7	0.32250E-01	0.15776E 00	0.59290E-01	0.10479E 00	0.24060E-01
10.9	0.31600E-01	0.15956E 00	0.66890E-01	0.10994E 00	0.25710E-01
11.1	0.31330E-01	0.16297E 00	0.75570E-01	0.11628E 00	0.26770E-01
11.3	0.31020E-01	0.16471E 00	0.82060E-01	0.12263E 00	0.28210E-01
11.5	0.30460E-01	0.16560E 00	0.85440E-01	0.12941E 00	0.30160E-01
11.7	0.29400E-01	0.16331E 00	0.94720E-01	0.13383E 00	0.31440E-01
11.9	0.28010E-01	0.15801E 00	0.83070E-01	0.13567E 00	0.32060E-01
12.1	0.26520E-01	0.15228E 00	0.81260E-01	0.13675E 00	0.32420E-01
12.3	0.25100E-01	0.14787E 00	0.86000E-01	0.13898E 00	0.32840E-01
12.5	0.24150E-01	0.14563E 00	0.79780E-01	0.14324E 00	0.33650E-01
12.7	0.23110E-01	0.14130E 00	0.76690E-01	0.14480E 00	0.34060E-01
12.9	0.22130E-01	0.13606E 00	0.72130E-01	0.14449E 00	0.34120E-01
13.1	0.20580E-01	0.12879E 00	0.65340E-01	0.14057E 00	0.33530E-01
13.3	0.19410E-01	0.12236E 00	0.58850E-01	0.13716E 00	0.32900E-01
13.5	0.18370E-01	0.11670E 00	0.52550E-01	0.13380E 00	0.32190E-01
13.7	0.17470E-01	0.11149E 00	0.46310E-01	0.12981E 00	0.31270E-01
13.9	0.16570E-01	0.10608E 00	0.39560E-01	0.12465E 00	0.30150E-01
14.1	0.15520E-01	0.10001E 00	0.32150E-01	0.11828E 00	0.28860E-01
14.3	0.14390E-01	0.94130E-01	0.24850E-01	0.11194E 00	0.27550E-01
14.5	0.13410E-01	0.89670E-01	0.19770E-01	0.10583E 00	0.26220E-01
14.7	0.12680E-01	0.86390E-01	0.16880E-01	0.97850E-01	0.24860E-01
14.9	0.12070E-01	0.83370E-01	0.14520E-01	0.94030E-01	0.23370E-01
15.1	0.11450E-01	0.80210E-01	0.11930E-01	0.88290E-01	0.21970E-01
15.3	0.10860E-01	0.76920E-01	0.10460E-01	0.81710E-01	0.20370E-01
15.5	0.10280E-01	0.73660E-01	0.94500E-02	0.75110E-01	0.18840E-01
15.7	0.97500E-02	0.70750E-01	0.85400E-02	0.69490E-01	0.17530E-01
15.9	0.92900E-02	0.68210E-01	0.79700E-02	0.64380E-01	0.16310E-01
16.1	0.88700E-02	0.66120E-01	0.77200E-02	0.60450E-01	0.15360E-01
16.3	0.85000E-02	0.64390E-01	0.80200E-02	0.57130E-01	0.14460E-01
16.5	0.82200E-02	0.63070E-01	0.90200E-02	0.54620E-01	0.13570E-01
16.7	0.79300E-02	0.61850E-01	0.10010E-01	0.52770E-01	0.12860E-01
16.9	0.77500E-02	0.61070E-01	0.11870E-01	0.51820E-01	0.12590E-01
17.1	0.75000E-02	0.60110E-01	0.13400E-01	0.50880E-01	0.12330E-01
17.3	0.72900E-02	0.59180E-01	0.14850E-01	0.50240E-01	0.12210E-01
17.5	0.70100E-02	0.57950E-01	0.16050E-01	0.49550E-01	0.12120E-01
17.7	0.67500E-02	0.56740E-01	0.17170E-01	0.49130E-01	0.12090E-01
17.9	0.66000E-02	0.55850E-01	0.18790E-01	0.49770E-01	0.12140E-01
18.1	0.64600E-02	0.55080E-01	0.19470E-01	0.50720E-01	0.12240E-01
18.3	0.63000E-02	0.54280E-01	0.19330E-01	0.51770E-01	0.12390E-01
18.5	0.60800E-02	0.53140E-01	0.19160E-01	0.52210E-01	0.12510E-01

Table D.3. Computed Values of the Elements $\tilde{c}_{22} - \tilde{c}_{33}$

τ_k	Element				
Seconds	$\tilde{c}_{22}$	$\tilde{c}_{23}$	$\tilde{c}_{24}$	$\tilde{c}_{25}$	$\tilde{c}_{33}$
1.30	0.10134E 06	0.27597E 05	0.39670E 04	0.27121E 03	0.78513E 04
1.35	0.86442E 05	0.19157E 05	0.29083E 04	0.20660E 03	0.58164E 04
1.40	0.59664E 05	0.17924E 05	0.28618E 04	0.22220E 03	0.56353E 04
1.45	0.51962E 05	0.16075E 05	0.26619E 04	0.21996E 03	0.51835E 04
1.50	0.40811E 05	0.12970E 05	0.22087E 04	0.18817E 03	0.42948E 04
1.55	0.30287E 05	0.99224E 04	0.17434E 04	0.15280E 03	0.33918E 04
1.60	0.24425E 05	0.82970E 04	0.15268E 04	0.14217E 03	0.29321E 04
1.65	0.20658E 05	0.72006E 04	0.13651E 04	0.13187E 03	0.26022E 04
1.70	0.15647E 05	0.56405E 04	0.11104E 04	0.11019E 03	0.21083E 04
1.75	0.12752E 05	0.47644E 04	0.97278E 03	0.98931E 02	0.18453E 04
1.80	0.11439E 05	0.43921E 04	0.91786E 03	0.94780E 02	0.17454E 04
1.85	0.10364E 05	0.40556E 04	0.85800E 03	0.89054E 02	0.16415E 04
1.90	0.90291E 04	0.36039E 04	0.77257E 03	0.80439E 02	0.14882E 04
1.95	0.79302E 04	0.32292E 04	0.70065E 03	0.73222E 02	0.13599E 04
2.00	0.70299E 04	0.29264E 04	0.64457E 03	0.68174E 02	0.12587E 04
2.05	0.59467E 04	0.25325E 04	0.56835E 03	0.61223E 02	0.11151E 04
2.10	0.49880E 04	0.21684E 04	0.49408E 03	0.53961E 02	0.97556E 03
2.20	0.36084E 04	0.16208E 04	0.37643E 03	0.41622E 02	0.75474E 03
2.30	0.28800E 04	0.13415E 04	0.32473E 03	0.37852E 02	0.64670E 03
2.40	0.23323E 04	0.11359E 04	0.29796E 03	0.38225E 02	0.57119E 03
2.50	0.19534E 04	0.99743E 03	0.28046E 03	0.38503E 02	0.52114E 03
2.60	0.14972E 04	0.79050E 03	0.23666E 03	0.34222E 02	0.42893E 03
2.70	0.11922E 04	0.65826E 03	0.20624E 03	0.30476E 02	0.37326E 03
2.80	0.88317E 03	0.50326E 03	0.16018E 03	0.23608E 02	0.29540E 03
2.90	0.58791E 03	0.34429E 03	0.11131E 03	0.16341E 02	0.20934E 03
3.00	0.49860E 03	0.30405E 03	0.10336E 03	0.15532E 02	0.19204E 03
3.10	0.47689E 03	0.30069E 03	0.10530E 03	0.15991E 02	0.19514E 03
3.20	0.39586E 03	0.25264E 03	0.87315E 02	0.13034E 02	0.16590E 03
3.30	0.27623E 03	0.17783E 03	0.61548E 02	0.90038E 01	0.11842E 03
3.40	0.25573E 03	0.17811E 03	0.71593E 02	0.11474E 02	0.12753E 03
3.50	0.25426E 03	0.18678E 03	0.81010E 02	0.13441E 02	0.13997E 03
3.60	0.21771E 03	0.16311E 03	0.71443E 02	0.11932E 02	0.12470E 03
3.70	0.18209E 03	0.13820E 03	0.60742E 02	0.10219E 02	0.10720E 03
3.80	0.15368E 03	0.11557E 03	0.50708E 02	0.85275E 01	0.90768E 02
3.90	0.12569E 03	0.97445E 02	0.42974E 02	0.71998E 01	0.77497E 02
4.00	0.10335E 03	0.80939E 02	0.35898E 02	0.60036E 01	0.65166E 02
4.10	0.87766E 02	0.69837E 02	0.31743E 02	0.53551E 01	0.57209E 02
4.20	0.78779E 02	0.64808E 02	0.30656E 02	0.52042E 01	0.54880E 02
4.30	0.68723E 02	0.58378E 02	0.28254E 02	0.48637E 01	0.51091E 02
4.40	0.55905E 02	0.48553E 02	0.23538E 02	0.40488E 01	0.43608E 02
4.50	0.47486E 02	0.42332E 02	0.22333E 02	0.39087E 01	0.39105E 02
4.60	0.46260E 02	0.42035E 02	0.24909E 02	0.45780E 01	0.39514E 02
4.70	0.44463E 02	0.40891E 02	0.26081E 02	0.49845E 01	0.38901E 02
4.80	0.38112E 02	0.35484E 02	0.23622E 02	0.46486E 01	0.34318E 02
4.90	0.31559E 02	0.29723E 02	0.20506E 02	0.41126E 01	0.29259E 02
5.00	0.26636E 02	0.25364E 02	0.18087E 02	0.36859E 01	0.25410E 02
5.10	0.23215E 02	0.22314E 02	0.16391E 02	0.33499E 01	0.22703E 02
5.20	0.20656E 02	0.20000E 02	0.14937E 02	0.30355E 01	0.20511E 02
5.30	0.18516E 02	0.18041E 02	0.13579E 02	0.27681E 01	0.18612E 02
5.40	0.16567E 02	0.16224E 02	0.12449E 02	0.25390E 01	0.17110E 02
5.50	0.14497E 02	0.14288E 02	0.11066E 02	0.22564E 01	0.15273E 02
5.60	0.12305E 02	0.12192E 02	0.95028E 01	0.19321E 01	0.13248E 02
5.70	0.10234E 02	0.10195E 02	0.80154E 01	0.16305E 01	0.11306E 02
5.80	0.83791E 01	0.84058E 01	0.66296E 01	0.13448E 01	0.95680E 01
5.90	0.69246E 01	0.69793E 01	0.55575E 01	0.11226E 01	0.81606E 01
6.10	0.50114E 01	0.49811E 01	0.42748E 01	0.87600E 00	0.61227E 01
6.30	0.39898E 01	0.38667E 01	0.36701E 01	0.75032E 00	0.49405E 01
6.50	0.37373E 01	0.34374E 01	0.37542E 01	0.77940E 00	0.43515E 01
6.70	0.32147E 01	0.28002E 01	0.33887E 01	0.71302E 00	0.36054E 01

Table D.3. (cont'd.)

$\frac{m}{\mu}$	Element					
Seconds	$\tilde{c}_{22}$	$\tilde{c}_{23}$	$\tilde{c}_{24}$	$\tilde{c}_{25}$	$\tilde{c}_{26}$	$\tilde{c}_{27}$
6.7	0.24647E 01	0.20229E 01	0.26497E 01	0.57867E 00	0.27824E 01	
7.1	0.13476E 01	0.14161E 01	0.19911E 01	0.45027E 00	0.21417E 01	
7.3	0.15414E 01	0.10726E 01	0.18774E 01	0.38514E 00	0.17221E 01	
7.5	0.14264E 01	0.87191E 00	0.15490E 01	0.36640E 00	0.14240E 01	
7.7	0.13598E 01	0.74600E 00	0.15047E 01	0.35146E 00	0.12183E 01	
7.9	0.13059E 01	0.65821E 00	0.14347E 01	0.33495E 00	0.10663E 01	
8.1	0.12992E 01	0.59118E 00	0.14335E 01	0.33745E 00	0.93772E 00	
8.3	0.13346E 01	0.53824E 00	0.14511E 01	0.34599E 00	0.84345E 00	
8.5	0.13350E 01	0.49007E 00	0.15004E 01	0.36192E 00	0.76463E 00	
8.7	0.14162E 01	0.42881E 00	0.16154E 01	0.39419E 00	0.71034E 00	
8.9	0.14431E 01	0.36258E 00	0.16377E 01	0.40604E 00	0.67039E 00	
9.1	0.13807E 01	0.31450E 00	0.15259E 01	0.38271E 00	0.65020E 00	
9.3	0.12639E 01	0.28484E 00	0.13678E 01	0.34326E 00	0.64039E 00	
9.5	0.11894E 01	0.27154E 00	0.12146E 01	0.30120E 00	0.63776E 00	
9.7	0.11197E 01	0.27412E 00	0.10946E 01	0.26323E 00	0.64242E 00	
9.9	0.10912E 01	0.28323E 00	0.99462E 00	0.24962E 00	0.64226E 00	
10.1	0.10891E 01	0.30741E 00	0.95402E 00	0.22710E 00	0.64515E 00	
10.3	0.10997E 01	0.34366E 00	0.93965E 00	0.22355E 00	0.65121E 00	
10.5	0.11183E 01	0.37424E 00	0.95141E 00	0.22676E 00	0.65215E 00	
10.7	0.11331E 01	0.39679E 00	0.96913E 00	0.23191E 00	0.65041E 00	
10.9	0.11509E 01	0.42100E 00	0.99361E 00	0.23768E 00	0.65278E 00	
11.1	0.11750E 01	0.45159E 00	0.10244E 01	0.24450E 00	0.66271E 00	
11.3	0.11910E 01	0.47381E 00	0.10589E 01	0.25417E 00	0.67298E 00	
11.5	0.12072E 01	0.48737E 00	0.10924E 01	0.26307E 00	0.67798E 00	
11.7	0.12024E 01	0.47997E 00	0.11094E 01	0.26773E 00	0.67264E 00	
11.9	0.11813E 01	0.46623E 00	0.11180E 01	0.27001E 00	0.66681E 00	
12.1	0.11680E 01	0.45805E 00	0.11385E 01	0.27269E 00	0.66232E 00	
12.3	0.11669E 01	0.46122E 00	0.11527E 01	0.27711E 00	0.66516E 00	
12.5	0.11701E 01	0.46850E 00	0.11785E 01	0.28221E 00	0.67096E 00	
12.7	0.11494E 01	0.45437E 00	0.11782E 01	0.28191E 00	0.66070E 00	
12.9	0.11128E 01	0.42579E 00	0.11575E 01	0.27753E 00	0.63854E 00	
13.1	0.10719E 01	0.39659E 00	0.11262E 01	0.27129E 00	0.61265E 00	
13.3	0.10291E 01	0.36155E 00	0.10981E 01	0.26560E 00	0.57975E 00	
13.5	0.99136E 00	0.32851E 00	0.10673E 01	0.25926E 00	0.54593E 00	
13.7	0.95557E 00	0.29581E 00	0.10336E 01	0.25088E 00	0.51133E 00	
13.9	0.91899E 00	0.26049E 00	0.99413E 00	0.24148E 00	0.47541E 00	
14.1	0.88177E 00	0.22047E 00	0.95364E 00	0.23238E 00	0.43719E 00	
14.3	0.85357E 00	0.18384E 00	0.92573E 00	0.22675E 00	0.40266E 00	
14.5	0.84550E 00	0.16466E 00	0.90210E 00	0.22155E 00	0.38054E 00	
14.7	0.82152E 00	0.15794E 00	0.87763E 00	0.21446E 00	0.36057E 00	
14.9	0.80728E 00	0.15537E 00	0.84737E 00	0.20801E 00	0.36203E 00	
15.1	0.79285E 00	0.14822E 00	0.82511E 00	0.20285E 00	0.35669E 00	
15.3	0.77310E 00	0.14517E 00	0.77044E 00	0.19456E 00	0.35564E 00	
15.5	0.75218E 00	0.14223E 00	0.75371E 00	0.18570E 00	0.35518E 00	
15.7	0.73576E 00	0.13743E 00	0.72553E 00	0.17904E 00	0.35577E 00	
15.9	0.72202E 00	0.13262E 00	0.70062E 00	0.17321E 00	0.35583E 00	
16.1	0.71399E 00	0.12802E 00	0.68655E 00	0.17023E 00	0.35303E 00	
16.3	0.70763E 00	0.12390E 00	0.67560E 00	0.16773E 00	0.35341E 00	
16.5	0.70214E 00	0.12743E 00	0.66812E 00	0.16554E 00	0.34816E 00	
16.7	0.69882E 00	0.12700E 00	0.66762E 00	0.16467E 00	0.34262E 00	
16.9	0.69412E 00	0.13204E 00	0.66485E 00	0.16396E 00	0.33869E 00	
17.1	0.68941E 00	0.13550E 00	0.66021E 00	0.16255E 00	0.33401E 00	
17.3	0.68430E 00	0.14056E 00	0.65721E 00	0.16170E 00	0.33035E 00	
17.5	0.67757E 00	0.14591E 00	0.65498E 00	0.16012E 00	0.32773E 00	
17.7	0.67509E 00	0.15172E 00	0.64874E 00	0.15904E 00	0.32595E 00	
17.9	0.66898E 00	0.16151E 00	0.64326E 00	0.15981E 00	0.32330E 00	
18.1	0.66324E 00	0.16339E 00	0.63879E 00	0.16128E 00	0.32079E 00	
18.3	0.65773E 00	0.16040E 00	0.63652E 00	0.16302E 00	0.32497E 00	
18.5	0.65311E 00	0.15977E 00	0.63377E 00	0.16394E 00	0.32997E 00	

Table D.4. Computed Values of the Elements $\tilde{c}_{34} - \tilde{c}_{55}$

T_k Seconds	Element				
	$\tilde{c}_{24}$	$\tilde{c}_{25}$	$\tilde{c}_{34}$	$\tilde{c}_{45}$	$\tilde{c}_{55}$
1.30	0.11932E 04	0.87543E 02	0.19812E 03	0.16247E 02	0.17970E 01
1.35	0.94168E 03	0.72678E 02	0.16849E 03	0.14680E 02	0.17560E 01
1.40	0.95179E 03	0.78888E 02	0.17571E 03	0.16108E 02	0.19390E 01
1.45	0.90287E 03	0.78711E 02	0.17047E 03	0.16199E 02	0.19490E 01
1.50	0.76773E 03	0.68653E 02	0.14823E 03	0.14337E 02	0.17190E 01
1.55	0.62543E 03	0.57370E 02	0.12446E 03	0.12294E 02	0.14900E 01
1.60	0.56276E 03	0.54266E 02	0.11548E 03	0.11831E 02	0.14530E 01
1.65	0.51193E 03	0.50853E 02	0.10697E 03	0.11194E 02	0.13860E 01
1.70	0.42935E 03	0.43587E 02	0.92463E 02	0.98200E 01	0.12260E 01
1.75	0.38995E 03	0.40611E 02	0.87591E 02	0.95970E 01	0.12790E 01
1.80	0.37821E 03	0.40128E 02	0.87907E 02	0.99061E 01	0.14160E 01
1.85	0.36116E 03	0.38685E 02	0.86096E 02	0.98930E 01	0.14925E 01
1.90	0.33278E 03	0.35904E 02	0.81372E 02	0.95020E 01	0.14982E 01
1.95	0.30818E 03	0.33440E 02	0.76846E 02	0.90754E 01	0.14736E 01
2.00	0.28966E 03	0.31824E 02	0.73676E 02	0.88361E 01	0.14740E 01
2.05	0.26187E 03	0.29336E 02	0.68358E 02	0.83870E 01	0.14473E 01
2.10	0.23300E 03	0.26491E 02	0.62223E 02	0.77732E 01	0.13812E 01
2.20	0.18434E 03	0.21285E 02	0.50877E 02	0.65033E 01	0.12126E 01
2.30	0.16439E 03	0.19947E 02	0.47255E 02	0.63231E 01	0.12374E 01
2.40	0.15702E 03	0.20855E 02	0.48739E 02	0.70616E 01	0.14980E 01
2.50	0.15281E 03	0.21575E 02	0.50175E 02	0.76281E 01	0.17007E 01
2.60	0.13360E 03	0.19797E 02	0.46473E 02	0.73519E 01	0.17429E 01
2.70	0.12162E 03	0.18393E 02	0.44633E 02	0.72234E 01	0.18149E 01
2.80	0.98184E 02	0.14846E 02	0.37510E 02	0.61342E 01	0.15925E 01
2.90	0.71266E 02	0.10781E 02	0.28641E 02	0.47416E 01	0.12476E 01
3.00	0.68329E 02	0.10525E 02	0.28471E 02	0.47598E 01	0.12427E 01
3.10	0.70850E 02	0.10963E 02	0.29693E 02	0.49443E 01	0.12740E 01
3.20	0.59327E 02	0.90131E 01	0.24565E 02	0.40271E 01	0.10267E 01
3.30	0.42623E 02	0.64175E 01	0.18289E 02	0.30119E 01	0.84150E 00
3.40	0.52817E 02	0.85818E 01	0.26013E 02	0.45627E 01	0.14523E 01
3.50	0.61834E 02	0.10345E 02	0.32113E 02	0.57527E 01	0.18897E 01
3.60	0.55509E 02	0.93380E 01	0.29316E 02	0.52923E 01	0.17482E 01
3.70	0.47832E 02	0.80984E 01	0.25543E 02	0.46488E 01	0.15291E 01
3.80	0.40404E 02	0.68354E 01	0.21693E 02	0.39587E 01	0.12908E 01
3.90	0.34624E 02	0.58290E 01	0.18683E 02	0.33996E 01	0.11060E 01
4.00	0.29226E 02	0.49076E 01	0.15787E 02	0.28607E 01	0.94203E 00
4.10	0.26222E 02	0.44331E 01	0.14328E 02	0.25913E 01	0.87761E 00
4.20	0.26013E 02	0.44156E 01	0.14612E 02	0.26358E 01	0.92387E 00
4.30	0.24634E 02	0.42316E 01	0.14197E 02	0.25923E 01	0.91344E 00
4.40	0.20967E 02	0.35923E 01	0.12341E 02	0.22689E 01	0.83358E 00
4.50	0.20188E 02	0.35018E 01	0.13258E 02	0.25027E 01	0.10696E 01
4.60	0.22624E 02	0.41042E 01	0.16333E 02	0.31970E 01	0.14528E 01
4.70	0.23830E 02	0.44896E 01	0.18045E 02	0.36205E 01	0.16962E 01
4.80	0.21794E 02	0.42225E 01	0.17121E 02	0.35088E 01	0.18427E 01
4.90	0.19097E 02	0.37662E 01	0.15560E 02	0.32345E 01	0.20170E 01
5.00	0.16989E 02	0.33992E 01	0.14322E 02	0.30148E 01	0.22041E 01
5.10	0.15435E 02	0.31100E 01	0.13333E 02	0.28268E 01	0.21836E 01
5.20	0.14163E 02	0.28401E 01	0.12420E 02	0.26212E 01	0.19130E 01
5.30	0.13060E 02	0.26083E 01	0.11604E 02	0.24423E 01	0.15322E 01
5.40	0.12046E 02	0.24050E 01	0.10895E 02	0.22937E 01	0.12560E 01
5.50	0.10797E 02	0.21533E 01	0.98784E 01	0.20786E 01	0.10507E 01
5.60	0.93432E 01	0.18553E 01	0.86614E 01	0.18200E 01	0.92379E 00
5.70	0.79398E 01	0.15716E 01	0.75065E 01	0.15807E 01	0.85079E 00
5.80	0.66371E 01	0.13067E 01	0.63802E 01	0.13423E 01	0.79044E 00
5.90	0.56130E 01	0.10935E 01	0.55149E 01	0.11581E 01	0.73643E 00
6.10	0.42729E 01	0.83207E 00	0.46031E 01	0.98143E 00	0.67098E 00
6.30	0.36081E 01	0.69580E 00	0.42740E 01	0.90853E 00	0.58711E 00
6.50	0.35345E 01	0.69238E 00	0.46326E 01	0.99209E 00	0.60647E 00
6.70	0.30811E 01	0.60788E 00	0.43974E 01	0.94962E 00	0.60553E 00

Table D.4. (cont'd.)

T_k	Element				
Seconds	c_{44}	c_{45}	c_{44}	c_{45}	c_{55}
6.9	0.23501E 01	0.47092E 00	0.36484E 01	0.81255E 00	0.59111E 00
7.1	0.17390E 01	0.35094E 00	0.29371E 01	0.67229E 00	0.56329E 00
7.3	0.13958E 01	0.28245E 00	0.26207E 01	0.60721E 00	0.55682E 00
7.5	0.11922E 01	0.24291E 00	0.25578E 01	0.59773E 00	0.58269E 00
7.7	0.10413E 01	0.21242E 00	0.24843E 01	0.58302E 00	0.58089E 00
7.9	0.91395E 00	0.18582E 00	0.23803E 01	0.55999E 00	0.56657E 00
8.1	0.80705E 00	0.16390E 00	0.23849E 01	0.56433E 00	0.59669E 00
8.3	0.71131E 00	0.14466E 00	0.24050E 01	0.57474E 00	0.64501E 00
8.5	0.62280E 00	0.12629E 00	0.24496E 01	0.59119E 00	0.71759E 00
8.7	0.52477E 00	0.10536E 00	0.25639E 01	0.62537E 00	0.83678E 00
8.9	0.42589E 00	0.83710E-01	0.25330E 01	0.62666E 00	0.92242E 00
9.1	0.35439E 00	0.66670E-01	0.23241E 01	0.58074E 00	0.94797E 00
9.3	0.31044E 00	0.55210E-01	0.20687E 01	0.51758E 00	0.92510E 00
9.5	0.28833E 00	0.47990E-01	0.18321E 01	0.45474E 00	0.87492E 00
9.7	0.28435E 00	0.44650E-01	0.16265E 01	0.39778E 00	0.80512E 00
9.9	0.28790E 00	0.45350E-01	0.14806E 01	0.35949E 00	0.73775E 00
10.1	0.30834E 00	0.50990E-01	0.13938E 01	0.33785E 00	0.66525E 00
10.3	0.34235E 00	0.59560E-01	0.13647E 01	0.33077E 00	0.61062E 00
10.5	0.37025E 00	0.66600E-01	0.13780E 01	0.33404E 00	0.59207E 00
10.7	0.38757E 00	0.71220E-01	0.14014E 01	0.33986E 00	0.60976E 00
10.9	0.40321E 00	0.75190E-01	0.14261E 01	0.34585E 00	0.65821E 00
11.1	0.42040E 00	0.78800E-01	0.14560E 01	0.35364E 00	0.72462E 00
11.3	0.43446E 00	0.83040E-01	0.14938E 01	0.36353E 00	0.78445E 00
11.5	0.44325E 00	0.86180E-01	0.15232E 01	0.37076E 00	0.82714E 00
11.7	0.43540E 00	0.85050E-01	0.15395E 01	0.37527E 00	0.84429E 00
11.9	0.42639E 00	0.83210E-01	0.15547E 01	0.37932E 00	0.85719E 00
12.1	0.42726E 00	0.83710E-01	0.15697E 01	0.38273E 00	0.84918E 00
12.3	0.44341E 00	0.87400E-01	0.15932E 01	0.38720E 00	0.81380E 00
12.5	0.46802E 00	0.92960E-01	0.16228E 01	0.39251E 00	0.73947E 00
12.7	0.47205E 00	0.94150E-01	0.16291E 01	0.39333E 00	0.68125E 00
12.9	0.46055E 00	0.91770E-01	0.16150E 01	0.39018E 00	0.65343E 00
13.1	0.44248E 00	0.87840E-01	0.15866E 01	0.38419E 00	0.65578E 00
13.3	0.41992E 00	0.82650E-01	0.15676E 01	0.38074E 00	0.67752E 00
13.5	0.39622E 00	0.77480E-01	0.15477E 01	0.37629E 00	0.70555E 00
13.7	0.36741E 00	0.71370E-01	0.15158E 01	0.36816E 00	0.72337E 00
13.9	0.33019E 00	0.63460E-01	0.14726E 01	0.35793E 00	0.73186E 00
14.1	0.28432E 00	0.53390E-01	0.14289E 01	0.34832E 00	0.73714E 00
14.3	0.23983E 00	0.43440E-01	0.14031E 01	0.34347E 00	0.73932E 00
14.5	0.21087E 00	0.36860E-01	0.13718E 01	0.33655E 00	0.71181E 00
14.7	0.19337E 00	0.32720E-01	0.13240E 01	0.32474E 00	0.65975E 00
14.9	0.17971E 00	0.29530E-01	0.12762E 01	0.31322E 00	0.62276E 00
15.1	0.16300E 00	0.25750E-01	0.12351E 01	0.30372E 00	0.61637E 00
15.3	0.15217E 00	0.23230E-01	0.11734E 01	0.28902E 00	0.61548E 00
15.5	0.14357E 00	0.21140E-01	0.11083E 01	0.27356E 00	0.61797E 00
15.7	0.13621E 00	0.19350E-01	0.10573E 01	0.26144E 00	0.61052E 00
15.9	0.13108E 00	0.17860E-01	0.10120E 01	0.25051E 00	0.61918E 00
16.1	0.12655E 00	0.16830E-01	0.98520E 00	0.24412E 00	0.62521E 00
16.3	0.12797E 00	0.17200E-01	0.96529E 00	0.23900E 00	0.60926E 00
16.5	0.13571E 00	0.19370E-01	0.95571E 00	0.23596E 00	0.59291E 00
16.7	0.13861E 00	0.20380E-01	0.95574E 00	0.23548E 00	0.58789E 00
16.9	0.14527E 00	0.22360E-01	0.95577E 00	0.23518E 00	0.59072E 00
17.1	0.14827E 00	0.23290E-01	0.95408E 00	0.23480E 00	0.61965E 00
17.3	0.15124E 00	0.23880E-01	0.95361E 00	0.23448E 00	0.66561E 00
17.5	0.15285E 00	0.24300E-01	0.94686E 00	0.23251E 00	0.71908E 00
17.7	0.15442E 00	0.24930E-01	0.94238E 00	0.23128E 00	0.78688E 00
17.9	0.16156E 00	0.26850E-01	0.95142E 00	0.23352E 00	0.86995E 00
18.1	0.16256E 00	0.27120E-01	0.96213E 00	0.23631E 00	0.98003E 00
18.3	0.15986E 00	0.26600E-01	0.97774E 00	0.23919E 00	0.10633E 01
18.5	0.16018E 00	0.26830E-01	0.97687E 00	0.24028E 00	0.11052E 01

Table D.5. Computed Values of the Elements $\bar{c}_{11} - \bar{c}_{15}$

$\frac{R}{K}$	Element				
Seconds	$\bar{c}_{11}$	$\bar{c}_{12}$	$\bar{c}_{13}$	$\bar{c}_{14}$	$\bar{c}_{15}$
1.10	0.25729E 06	0.16314E 06	0.43253E 05	0.59044E 04	0.35429E 03
1.35	0.15287E 06	0.10241E 06	0.28575E 05	0.40710E 04	0.24901E 03
1.40	0.11882E 06	0.81844E 05	0.23436E 05	0.34399E 04	0.22449E 03
1.45	0.86945E 05	0.61705E 05	0.18176E 05	0.27650E 04	0.19295E 03
1.50	0.62627E 05	0.45758E 05	0.13843E 05	0.21776E 04	0.16048E 03
1.55	0.43168E 05	0.32716E 05	0.10245E 05	0.16828E 04	0.13184E 03
1.60	0.33071E 05	0.26048E 05	0.84837E 04	0.14684E 04	0.12430E 03
1.65	0.26959E 05	0.21738E 05	0.72343E 04	0.12832E 04	0.11252E 03
1.70	0.18548E 05	0.15419E 05	0.52625E 04	0.95602E 03	0.85128E 02
1.75	0.13340E 05	0.11426E 05	0.39961E 04	0.73790E 03	0.65804E 02
1.80	0.10074E 05	0.88685E 04	0.31733E 04	0.59447E 03	0.53179E 02
1.85	0.79798E 04	0.72085E 04	0.26397E 04	0.50392E 03	0.45846E 02
1.90	0.67429E 04	0.62530E 04	0.23496E 04	0.45648E 03	0.42008E 02
1.95	0.59376E 04	0.56533E 04	0.21832E 04	0.43139E 03	0.40101E 02
2.00	0.52661E 04	0.51508E 04	0.20465E 04	0.41226E 03	0.38893E 02
2.05	0.45520E 04	0.45614E 04	0.18572E 04	0.38131E 03	0.36662E 02
2.10	0.38630E 04	0.39561E 04	0.16450E 04	0.34360E 03	0.33652E 02
2.20	0.27200E 04	0.29014E 04	0.12529E 04	0.26959E 03	0.27168E 02
2.30	0.19428E 04	0.21747E 04	0.98589E 03	0.22525E 03	0.24215E 02
2.40	0.14112E 04	0.16511E 04	0.78212E 03	0.18955E 03	0.21715E 02
2.50	0.98629E 03	0.11992E 04	0.59891E 03	0.14935E 03	0.17864E 02
2.60	0.62111E 03	0.78413E 03	0.39762E 03	0.10320E 03	0.12474E 02
2.70	0.42711E 03	0.56070E 03	0.29483E 03	0.78268E 02	0.95466E 01
2.80	0.30866E 03	0.42212E 03	0.23084E 03	0.64190E 02	0.81321E 01
2.90	0.22863E 03	0.32539E 03	0.18478E 03	0.53969E 02	0.71068E 01
3.00	0.18000E 03	0.26630E 03	0.15703E 03	0.48268E 02	0.66166E 01
3.10	0.15031E 03	0.22980E 03	0.13994E 03	0.44970E 02	0.63674E 01
3.20	0.12102E 03	0.19069E 03	0.11939E 03	0.39667E 02	0.57076E 01
3.30	0.96348E 02	0.15719E 03	0.10167E 03	0.35309E 02	0.51937E 01
3.40	0.79584E 02	0.13468E 03	0.90490E 02	0.33057E 02	0.50160E 01
3.50	0.67226E 02	0.11782E 03	0.82201E 02	0.31513E 02	0.49307E 01
3.60	0.54106E 02	0.96753E 02	0.68998E 02	0.26328E 02	0.40058E 01
3.70	0.45287E 02	0.83038E 02	0.60627E 02	0.23779E 02	0.37041E 01
3.80	0.37344E 02	0.70057E 02	0.52146E 02	0.20993E 02	0.33390E 01
3.90	0.30602E 02	0.58964E 02	0.44850E 02	0.18803E 02	0.30982E 01
4.00	0.24730E 02	0.48673E 02	0.37656E 02	0.16088E 02	0.26660E 01
4.10	0.20514E 02	0.41328E 02	0.32614E 02	0.14356E 02	0.23991E 01
4.20	0.17023E 02	0.35133E 02	0.28380E 02	0.12868E 02	0.21597E 01
4.30	0.13946E 02	0.29465E 02	0.24355E 02	0.11334E 02	0.19261E 01
4.40	0.11210E 02	0.24202E 02	0.20387E 02	0.97310E 01	0.16687E 01
4.50	0.90413E 01	0.20054E 02	0.17261E 02	0.86452E 01	0.15107E 01
4.60	0.75644E 01	0.17265E 02	0.15181E 02	0.80505E 01	0.14421E 01
4.70	0.63151E 01	0.14796E 02	0.13285E 02	0.73822E 01	0.13461E 01
4.80	0.52120E 01	0.12508E 02	0.11435E 02	0.66177E 01	0.12251E 01
4.90	0.43279E 01	0.10612E 02	0.98535E 01	0.58986E 01	0.11069E 01
5.00	0.37013E 01	0.92292E 01	0.86844E 01	0.53333E 01	0.10124E 01
5.10	0.31187E 01	0.79004E 01	0.75376E 01	0.47286E 01	0.90574E 00
5.20	0.25913E 01	0.66831E 01	0.64619E 01	0.41590E 01	0.80458E 00
5.30	0.21798E 01	0.57364E 01	0.56089E 01	0.37385E 01	0.73224E 00
5.40	0.18510E 01	0.49737E 01	0.49042E 01	0.33957E 01	0.67442E 00
5.50	0.15901E 01	0.43467E 01	0.43162E 01	0.30849E 01	0.62012E 00
5.60	0.13650E 01	0.37931E 01	0.37895E 01	0.27960E 01	0.56588E 00
5.70	0.11739E 01	0.46339E 01	0.46737E 01	0.31668E 01	0.87543E 00
5.80	0.10155E 01	0.29228E 01	0.29613E 01	0.23695E 01	0.48628E 00
5.90	0.88806E 00	0.26036E 01	0.26498E 01	0.22110E 01	0.45420E 00
6.10	0.66703E 00	0.20223E 01	0.20615E 01	0.18525E 01	0.38695E 00
6.30	0.51290E 00	0.16037E 01	0.16265E 01	0.15693E 01	0.33196E 00
6.50	0.41598E 00	0.13428E 01	0.13330E 01	0.14087E 01	0.30317E 00
6.70	0.32677E 00	0.10814E 01	0.10457E 01	0.12031E 01	0.25647E 00

Table D.5. (cont'd.)

τ_k	Element				
Seconds	$\bar{c}_{11}$	$\bar{c}_{12}$	$\bar{c}_{13}$	$\bar{c}_{14}$	$\bar{c}_{15}$
6.9	0.24483E 00	0.82607E 00	0.77806E 00	0.96270E 00	0.20955E 00
7.1	0.18252E 00	0.62620E 00	0.57048E 00	0.75790E 00	0.16743E 00
7.3	0.13922E 00	0.48309E 00	0.42012E 00	0.60426E 00	0.13339E 00
7.5	0.11004E 00	0.39100E 00	0.31237E 00	0.50248E 00	0.11130E 00
7.7	0.91150E-01	0.32950E 00	0.23905E 00	0.42933E 00	0.95290E-01
7.9	0.79010E-01	0.29165E 00	0.18675E 00	0.37888E 00	0.84180E-01
8.1	0.69570E-01	0.26306E 00	0.14525E 00	0.33775E 00	0.75540E-01
8.3	0.62150E-01	0.24112E 00	0.11404E 00	0.30329E 00	0.67900E-01
8.5	0.55940E-01	0.22367E 00	0.88520E-01	0.27320E 00	0.61250E-01
8.7	0.50670E-01	0.20716E 00	0.70400E-01	0.24336E 00	0.54730E-01
8.9	0.46430E-01	0.19407E 00	0.57980E-01	0.21784E 00	0.49130E-01
9.1	0.42870E-01	0.18295E 00	0.49630E-01	0.19500E 00	0.44020E-01
9.3	0.40040E-01	0.17447E 00	0.44850E-01	0.17595E 00	0.39630E-01
9.5	0.37990E-01	0.16958E 00	0.43360E-01	0.16196E 00	0.36260E-01
9.7	0.35950E-01	0.16383E 00	0.43560E-01	0.14803E 00	0.32750E-01
9.9	0.34220E-01	0.15910E 00	0.44440E-01	0.13683E 00	0.29830E-01
10.1	0.32830E-01	0.15576E 00	0.46260E-01	0.12864E 00	0.27620E-01
10.3	0.31680E-01	0.15310E 00	0.49040E-01	0.12289E 00	0.26350E-01
10.5	0.30730E-01	0.15133E 00	0.52400E-01	0.11970E 00	0.25140E-01
10.7	0.29810E-01	0.14920E 00	0.55870E-01	0.11740E 00	0.24450E-01
10.9	0.28930E-01	0.14692E 00	0.59090E-01	0.11611E 00	0.23970E-01
11.1	0.28090E-01	0.14468E 00	0.61890E-01	0.11587E 00	0.23750E-01
11.3	0.27220E-01	0.14214E 00	0.63950E-01	0.11623E 00	0.23910E-01
11.5	0.26360E-01	0.13990E 00	0.65420E-01	0.11715E 00	0.24210E-01
11.7	0.25480E-01	0.13734E 00	0.66620E-01	0.11815E 00	0.24690E-01
11.9	0.24490E-01	0.13430E 00	0.68010E-01	0.11901E 00	0.25140E-01
12.1	0.23480E-01	0.13090E 00	0.69000E-01	0.11955E 00	0.25570E-01
12.3	0.22480E-01	0.12765E 00	0.69650E-01	0.11990E 00	0.25950E-01
12.5	0.21520E-01	0.12440E 00	0.61980E-01	0.12025E 00	0.26310E-01
12.7	0.20510E-01	0.12054E 00	0.59560E-01	0.11986E 00	0.26440E-01
12.9	0.19420E-01	0.11585E 00	0.56060E-01	0.11803E 00	0.26170E-01
13.1	0.18190E-01	0.11000E 00	0.51750E-01	0.11405E 00	0.25400E-01
13.3	0.17150E-01	0.10518E 00	0.47680E-01	0.11136E 00	0.24830E-01
13.5	0.16200E-01	0.10119E 00	0.44050E-01	0.10875E 00	0.24280E-01
13.7	0.15260E-01	0.96830E-01	0.39950E-01	0.10530E 00	0.23400E-01
13.9	0.14420E-01	0.92900E-01	0.35910E-01	0.10183E 00	0.22510E-01
14.1	0.13590E-01	0.88850E-01	0.31710E-01	0.97600E-01	0.21580E-01
14.3	0.12830E-01	0.85090E-01	0.27580E-01	0.93340E-01	0.20820E-01
14.5	0.12180E-01	0.82020E-01	0.24020E-01	0.89440E-01	0.20100E-01
14.7	0.11570E-01	0.79010E-01	0.21020E-01	0.84750E-01	0.19200E-01
14.9	0.11020E-01	0.76630E-01	0.18880E-01	0.81220E-01	0.18550E-01
15.1	0.10550E-01	0.74530E-01	0.17030E-01	0.77610E-01	0.17900E-01
15.3	0.10120E-01	0.72600E-01	0.15820E-01	0.73920E-01	0.17170E-01
15.5	0.97300E-02	0.70880E-01	0.15010E-01	0.70460E-01	0.16450E-01
15.7	0.93700E-02	0.69230E-01	0.14490E-01	0.67100E-01	0.15700E-01
15.9	0.90600E-02	0.67840E-01	0.14290E-01	0.64340E-01	0.15060E-01
16.1	0.87500E-02	0.66420E-01	0.14410E-01	0.61660E-01	0.14360E-01
16.3	0.84700E-02	0.65170E-01	0.14540E-01	0.59410E-01	0.13780E-01
16.5	0.82300E-02	0.64010E-01	0.14920E-01	0.57630E-01	0.13340E-01
16.7	0.79900E-02	0.63040E-01	0.15490E-01	0.56250E-01	0.12840E-01
16.9	0.77700E-02	0.61980E-01	0.15870E-01	0.54920E-01	0.12520E-01
17.1	0.75600E-02	0.61080E-01	0.16740E-01	0.54200E-01	0.12270E-01
17.3	0.73600E-02	0.60060E-01	0.17150E-01	0.53300E-01	0.12040E-01
17.5	0.71700E-02	0.59230E-01	0.17540E-01	0.52870E-01	0.11920E-01
17.7	0.69800E-02	0.58220E-01	0.17920E-01	0.52370E-01	0.11840E-01
17.9	0.67900E-02	0.57230E-01	0.17910E-01	0.51910E-01	0.11730E-01
18.1	0.65900E-02	0.56110E-01	0.17760E-01	0.51430E-01	0.11560E-01
18.3	0.64000E-02	0.55000E-01	0.17610E-01	0.51000E-01	0.11550E-01
18.5	0.62000E-02	0.53840E-01	0.17470E-01	0.50520E-01	0.11520E-01

Table D.6. Computed Values of the Elements $\bar{c}_{22} - \bar{c}_{33}$

T_k Seconds	Element				
	$\bar{c}_{22}$	$\bar{c}_{23}$	$\bar{c}_{24}$	$\bar{c}_{25}$	$\bar{c}_{33}$
1.30	0.10600E 06	0.28973E 05	0.41054E 04	0.25939E 03	0.82340E 04
1.35	0.70630E 05	0.20437E 05	0.30468E 04	0.19964E 03	0.61883E 04
1.40	0.57984E 05	0.17199E 05	0.26365E 04	0.18276E 03	0.53331E 04
1.45	0.45071E 05	0.13761E 05	0.21846E 04	0.16097E 03	0.43949E 04
1.50	0.34434E 05	0.10804E 05	0.17704E 04	0.13684E 03	0.35473E 04
1.55	0.25566E 05	0.83097E 04	0.14185E 04	0.11570E 03	0.28278E 04
1.60	0.21108E 05	0.71114E 04	0.12702E 04	0.11054E 03	0.24979E 04
1.65	0.17989E 05	0.61749E 04	0.11255E 04	0.10083E 03	0.22030E 04
1.70	0.13181E 05	0.46509E 04	0.86821E 03	0.78857E 02	0.17108E 04
1.75	0.10077E 05	0.36513E 04	0.69386E 03	0.63222E 02	0.13832E 04
1.80	0.80422E 04	0.29837E 04	0.57558E 03	0.52654E 02	0.11591E 04
1.85	0.67041E 04	0.25444E 04	0.50011E 03	0.46511E 02	0.10113E 04
1.90	0.59605E 04	0.23179E 04	0.46431E 03	0.43818E 02	0.94295E 03
1.95	0.55216E 04	0.22024E 04	0.44953E 03	0.43026E 02	0.91711E 03
2.00	0.51591E 04	0.21132E 04	0.44062E 03	0.42958E 02	0.90195E 03
2.05	0.46766E 04	0.19615E 04	0.41761E 03	0.41593E 02	0.85656E 03
2.10	0.41438E 04	0.17743E 04	0.38483E 03	0.39099E 02	0.79384E 03
2.20	0.31653E 04	0.14077E 04	0.31544E 03	0.33071E 02	0.65207E 03
2.30	0.24895E 04	0.11620E 04	0.27708E 03	0.30985E 02	0.56460E 03
2.40	0.19752E 04	0.96267E 03	0.24347E 03	0.28930E 02	0.48779E 03
2.50	0.14924E 04	0.75478E 03	0.20019E 03	0.24831E 02	0.39729E 03
2.60	0.10172E 04	0.53372E 03	0.14609E 03	0.18441E 02	0.29335E 03
2.70	0.75806E 03	0.41355E 03	0.11665E 03	0.14961E 02	0.23721E 03
2.80	0.59502E 03	0.33772E 03	0.99962E 02	0.13298E 02	0.20162E 03
2.90	0.47730E 03	0.28107E 03	0.87262E 02	0.12016E 02	0.17398E 03
3.00	0.40527E 03	0.24707E 03	0.80377E 02	0.11452E 02	0.15779E 03
3.10	0.36044E 03	0.22604E 03	0.76497E 02	0.11197E 02	0.14782E 03
3.20	0.30786E 03	0.19806E 03	0.69145E 02	0.10262E 02	0.13262E 03
3.30	0.26247E 03	0.17412E 03	0.63423E 02	0.96066E 01	0.12000E 03
3.40	0.23290E 03	0.15998E 03	0.61252E 02	0.95503E 01	0.11375E 03
3.50	0.21067E 03	0.14978E 03	0.60194E 02	0.96627E 01	0.10981E 03
3.60	0.17641E 03	0.12804E 03	0.51303E 02	0.80061E 01	0.95887E 02
3.70	0.15515E 03	0.11510E 03	0.47385E 02	0.75571E 01	0.87996E 02
3.80	0.13392E 03	0.10117E 03	0.42730E 02	0.64482E 01	0.78773E 02
3.90	0.11581E 03	0.89304E 02	0.39282E 02	0.66204E 01	0.70951E 02
4.00	0.97682E 02	0.76557E 02	0.34318E 02	0.58149E 01	0.61898E 02
4.10	0.84844E 02	0.67779E 02	0.31287E 02	0.53403E 01	0.55844E 02
4.20	0.73940E 02	0.60324E 02	0.28705E 02	0.49440E 01	0.50806E 02
4.30	0.63510E 02	0.52930E 02	0.25891E 02	0.44973E 01	0.45592E 02
4.40	0.53374E 02	0.45268E 02	0.22772E 02	0.39938E 01	0.39780E 02
4.50	0.45508E 02	0.39357E 02	0.20847E 02	0.37264E 01	0.35347E 02
4.60	0.40347E 02	0.35546E 02	0.19953E 02	0.36512E 01	0.32563E 02
4.70	0.35520E 02	0.31856E 02	0.18754E 02	0.34907E 01	0.29776E 02
4.80	0.30801E 02	0.28037E 02	0.17224E 02	0.32533E 01	0.26698E 02
4.90	0.26739E 02	0.24648E 02	0.15645E 02	0.30033E 01	0.23372E 02
5.00	0.23681E 02	0.22068E 02	0.14422E 02	0.27884E 01	0.21689E 02
5.10	0.20639E 02	0.19447E 02	0.13003E 02	0.25364E 01	0.19428E 02
5.20	0.17830E 02	0.16964E 02	0.11672E 02	0.22999E 01	0.17233E 02
5.30	0.15660E 02	0.15009E 02	0.10715E 02	0.21374E 01	0.15474E 02
5.40	0.13902E 02	0.13384E 02	0.99393E 01	0.20090E 01	0.13970E 02
5.50	0.12400E 02	0.11979E 02	0.91927E 01	0.18787E 01	0.12647E 02
5.60	0.11041E 02	0.10687E 02	0.84799E 01	0.17447E 01	0.11415E 02
5.70	0.12504E 02	0.12719E 02	0.86005E 01	0.27984E 01	0.13479E 02
5.80	0.88821E 01	0.86245E 01	0.74367E 01	0.15494E 01	0.94709E 01
5.90	0.80853E 01	0.78422E 01	0.70529E 01	0.14717E 01	0.87130E 01
6.10	0.65717E 01	0.63048E 01	0.61217E 01	0.12982E 01	0.71667E 01
6.30	0.54394E 01	0.51278E 01	0.53584E 01	0.11496E 01	0.59517E 01
6.50	0.47479E 01	0.43353E 01	0.49564E 01	0.10702E 01	0.50695E 01
6.70	0.39886E 01	0.34914E 01	0.43579E 01	0.94158E 00	0.41505E 01

Table D.6. (cont'd.)

T_k	Element				
Seconds	$\bar{c}_{22}$	$\bar{c}_{23}$	$\bar{c}_{24}$	$\bar{c}_{25}$	$\bar{c}_{33}$
6.9	0.31931E 01	0.26628E 01	0.35949E 01	0.79303E 00	0.32811E 01
7.1	0.25490E 01	0.20052E 01	0.29211E 01	0.65416E 00	0.25911E 01
7.3	0.20683E 01	0.15190E 01	0.23951E 01	0.53563E 00	0.20698E 01
7.5	0.17721E 01	0.11811E 01	0.20681E 01	0.46288E 00	0.16755E 01
7.7	0.15651E 01	0.94711E 00	0.18240E 01	0.40761E 00	0.13908E 01
7.9	0.14420E 01	0.78328E 00	0.16644E 01	0.37105E 00	0.11765E 01
8.1	0.13523E 01	0.65322E 00	0.15387E 01	0.34421E 00	0.10084E 01
8.3	0.12858E 01	0.55654E 00	0.14371E 01	0.32121E 00	0.88768E 00
8.5	0.12364E 01	0.47570E 00	0.13544E 01	0.30259E 00	0.79682E 00
8.7	0.11819E 01	0.41729E 00	0.12614E 01	0.28185E 00	0.73956E 00
8.9	0.11391E 01	0.37808E 00	0.11830E 01	0.26458E 00	0.70731E 00
9.1	0.11019E 01	0.35350E 00	0.11112E 01	0.24856E 00	0.69293E 00
9.3	0.10751E 01	0.34245E 00	0.10528E 01	0.23513E 00	0.69093E 00
9.5	0.10649E 01	0.34261E 00	0.10157E 01	0.22607E 00	0.69508E 00
9.7	0.10486E 01	0.34812E 00	0.97496E 00	0.21507E 00	0.70269E 00
9.9	0.10350E 01	0.35166E 00	0.94329E 00	0.20592E 00	0.70677E 00
10.1	0.10268E 01	0.35636E 00	0.92287E 00	0.19961E 00	0.70683E 00
10.3	0.10193E 01	0.36274E 00	0.90880E 00	0.19519E 00	0.70352E 00
10.5	0.10149E 01	0.36957E 00	0.90316E 00	0.19331E 00	0.69616E 00
10.7	0.10074E 01	0.37599E 00	0.89846E 00	0.19154E 00	0.68506E 00
10.9	0.99754E 00	0.38006E 00	0.89380E 00	0.18956E 00	0.67146E 00
11.1	0.98748E 00	0.38194E 00	0.89085E 00	0.18802E 00	0.65780E 00
11.3	0.97592E 00	0.38079E 00	0.88877E 00	0.18752E 00	0.64453E 00
11.5	0.96778E 00	0.37986E 00	0.88805E 00	0.18880E 00	0.63477E 00
11.7	0.95852E 00	0.38156E 00	0.88891E 00	0.19041E 00	0.62903E 00
11.9	0.94899E 00	0.37239E 00	0.88941E 00	0.19185E 00	0.62067E 00
12.1	0.93764E 00	0.36387E 00	0.88826E 00	0.19335E 00	0.61517E 00
12.3	0.92862E 00	0.35845E 00	0.88802E 00	0.19496E 00	0.61227E 00
12.5	0.91914E 00	0.35257E 00	0.88915E 00	0.19692E 00	0.60943E 00
12.7	0.90534E 00	0.34369E 00	0.88675E 00	0.19742E 00	0.60453E 00
12.9	0.88510E 00	0.32834E 00	0.87489E 00	0.19542E 00	0.59341E 00
13.1	0.85608E 00	0.30783E 00	0.84925E 00	0.19006E 00	0.58246E 00
13.3	0.83337E 00	0.29053E 00	0.83445E 00	0.18634E 00	0.56585E 00
13.5	0.81872E 00	0.27904E 00	0.82376E 00	0.18425E 00	0.54966E 00
13.7	0.79925E 00	0.26214E 00	0.80744E 00	0.17936E 00	0.52776E 00
13.9	0.78178E 00	0.24655E 00	0.79132E 00	0.17521E 00	0.50467E 00
14.1	0.76222E 00	0.22807E 00	0.77056E 00	0.17022E 00	0.47899E 00
14.3	0.74358E 00	0.20862E 00	0.75003E 00	0.16604E 00	0.45305E 00
14.5	0.72406E 00	0.19192E 00	0.73306E 00	0.16343E 00	0.43046E 00
14.7	0.71307E 00	0.17757E 00	0.70645E 00	0.15838E 00	0.41242E 00
14.9	0.70583E 00	0.17204E 00	0.69844E 00	0.15791E 00	0.40076E 00
15.1	0.69802E 00	0.16474E 00	0.68555E 00	0.15645E 00	0.39173E 00
15.3	0.69033E 00	0.16122E 00	0.67036E 00	0.15406E 00	0.38640E 00
15.5	0.68297E 00	0.15922E 00	0.65548E 00	0.15152E 00	0.38420E 00
15.7	0.67613E 00	0.15850E 00	0.64092E 00	0.14849E 00	0.38421E 00
15.9	0.67062E 00	0.15899E 00	0.63042E 00	0.14602E 00	0.38431E 00
16.1	0.66464E 00	0.16109E 00	0.61983E 00	0.14284E 00	0.38798E 00
16.3	0.65948E 00	0.16157E 00	0.61190E 00	0.14065E 00	0.38701E 00
16.5	0.65346E 00	0.16314E 00	0.60557E 00	0.13878E 00	0.39665E 00
16.7	0.65000E 00	0.16557E 00	0.60317E 00	0.13679E 00	0.38480E 00
16.9	0.64527E 00	0.16554E 00	0.59840E 00	0.13552E 00	0.37897E 00
17.1	0.64180E 00	0.16971E 00	0.59836E 00	0.13504E 00	0.37535E 00
17.3	0.63516E 00	0.16890E 00	0.59393E 00	0.13368E 00	0.36936E 00
17.5	0.63216E 00	0.16821E 00	0.59277E 00	0.13335E 00	0.36219E 00
17.7	0.62621E 00	0.16619E 00	0.58897E 00	0.13281E 00	0.35591E 00
17.9	0.62903E 00	0.16185E 00	0.58490E 00	0.13189E 00	0.34976E 00
18.1	0.61399E 00	0.15827E 00	0.57939E 00	0.13097E 00	0.34580E 00
18.3	0.60747E 00	0.15462E 00	0.57432E 00	0.12967E 00	0.34309E 00
18.5	0.59976E 00	0.14983E 00	0.56786E 00	0.12898E 00	0.34058E 00

Table D.7. Computed Values of the Elements $\bar{c}_{34} - \bar{c}_{55}$

T_k	Element				
Seconds	$\bar{c}_{34}$	$\bar{c}_{35}$	$\bar{c}_{44}$	$\bar{c}_{45}$	$\bar{c}_{55}$
1.30	0.12264E 04	0.82683E 02	0.19839E 03	0.14946E 02	0.15240E 01
1.35	0.97715E 03	0.69246E 02	0.16945E 03	0.13579E 02	0.14930E 01
1.40	0.86409E 03	0.64261E 02	0.15324E 03	0.12757E 02	0.14190E 01
1.45	0.73658E 03	0.57844E 02	0.13467E 03	0.11721E 02	0.13260E 01
1.50	0.61229E 03	0.50067E 02	0.11468E 03	0.10274E 02	0.11620E 01
1.55	0.50692E 03	0.43397E 02	0.98050E 02	0.90980E 01	0.10430E 01
1.60	0.46489E 03	0.41919E 02	0.92297E 02	0.88780E 01	0.10290E 01
1.65	0.41650E 03	0.38414E 02	0.83456E 02	0.81410E 01	0.94100E 00
1.70	0.33132E 03	0.30925E 02	0.68007E 02	0.66950E 01	0.77500E 00
1.75	0.27329E 03	0.25639E 02	0.57537E 02	0.57190E 01	0.67600E 00
1.80	0.23276E 03	0.21945E 02	0.50021E 02	0.50136E 01	0.60509E 00
1.85	0.20697E 03	0.19836E 02	0.45497E 02	0.46490E 01	0.58130E 00
1.90	0.19706E 03	0.19235E 02	0.44648E 02	0.46439E 01	0.52686E 00
1.95	0.19574E 03	0.19460E 02	0.45756E 02	0.49521E 01	0.70481E 00
2.00	0.19708E 03	0.20030E 02	0.47631E 02	0.53179E 01	0.80354E 00
2.05	0.19148E 03	0.19920E 02	0.47695E 02	0.54783E 01	0.86466E 00
2.10	0.18038E 03	0.19158E 02	0.46100E 02	0.54217E 01	0.88494E 00
2.20	0.15409E 03	0.16919E 02	0.41193E 02	0.50323E 01	0.86385E 00
2.30	0.14205E 03	0.16607E 02	0.40670E 02	0.52798E 01	0.97591E 00
2.40	0.12996E 03	0.16075E 02	0.39418E 02	0.53791E 01	0.10512E 01
2.50	0.11110E 03	0.14322E 02	0.35550E 02	0.50448E 01	0.10382E 01
2.60	0.85264E 02	0.11236E 02	0.28881E 02	0.42246E 01	0.91381E 00
2.70	0.71421E 02	0.95969E 01	0.25493E 02	0.38299E 01	0.86363E 00
2.80	0.63637E 02	0.88399E 01	0.23890E 02	0.36947E 01	0.86031E 00
2.90	0.57383E 02	0.82079E 01	0.22461E 02	0.35524E 01	0.84502E 00
3.00	0.54189E 02	0.79700E 01	0.21965E 02	0.35445E 01	0.86257E 00
3.10	0.52468E 02	0.78856E 01	0.21848E 02	0.35788E 01	0.89045E 00
3.20	0.48367E 02	0.73514E 01	0.20663E 02	0.34132E 01	0.86304E 00
3.30	0.45465E 02	0.70332E 01	0.20163E 02	0.33805E 01	0.88326E 00
3.40	0.45066E 02	0.71493E 01	0.20958E 02	0.35939E 01	0.99720E 00
3.50	0.45429E 02	0.73940E 01	0.22171E 02	0.38929E 01	0.11517E 01
3.60	0.39515E 02	0.62504E 01	0.19470E 02	0.33346E 01	0.10059E 01
3.70	0.37141E 02	0.59879E 01	0.18783E 02	0.32702E 01	0.10150E 01
3.80	0.34015E 02	0.55797E 01	0.17606E 02	0.31108E 01	0.98524E 00
3.90	0.31793E 02	0.53939E 01	0.17075E 02	0.31191E 01	0.10202E 01
4.00	0.28203E 02	0.48042E 01	0.15451E 02	0.28344E 01	0.94358E 00
4.10	0.26108E 02	0.44733E 01	0.14696E 02	0.27066E 01	0.92439E 00
4.20	0.24381E 02	0.42092E 01	0.14165E 02	0.26306E 01	0.93312E 00
4.30	0.22394E 02	0.38931E 01	0.13433E 02	0.25169E 01	0.92961E 00
4.40	0.20005E 02	0.35060E 01	0.12433E 02	0.23526E 01	0.90891E 00
4.50	0.18590E 02	0.33132E 01	0.12206E 02	0.23468E 01	0.96082E 00
4.60	0.17998E 02	0.32749E 01	0.12446E 02	0.24327E 01	0.10468E 01
4.70	0.17114E 02	0.31589E 01	0.12360E 02	0.24462E 01	0.11020E 01
4.80	0.15872E 02	0.29653E 01	0.11962E 02	0.23944E 01	0.11266E 01
4.90	0.14585E 02	0.27540E 01	0.11410E 02	0.23072E 01	0.11258E 01
5.00	0.13509E 02	0.25737E 01	0.10833E 02	0.22069E 01	0.11003E 01
5.10	0.12287E 02	0.23568E 01	0.10100E 02	0.20718E 01	0.10579E 01
5.20	0.11106E 02	0.21467E 01	0.94257E 01	0.19505E 01	0.10243E 01
5.30	0.10240E 02	0.19991E 01	0.90080E 01	0.18821E 01	0.10142E 01
5.40	0.95139E 01	0.18785E 01	0.86834E 01	0.18321E 01	0.10134E 01
5.50	0.88142E 01	0.17574E 01	0.82945E 01	0.17639E 01	0.99036E 00
5.60	0.81365E 01	0.16302E 01	0.79008E 01	0.16885E 01	0.97090E 00
5.70	0.90353E 01	0.25084E 01	0.69308E 01	0.25732E 01	0.16689E 01
5.80	0.71446E 01	0.14445E 01	0.74538E 01	0.16053E 01	0.98632E 00
5.90	0.67717E 01	0.13663E 01	0.73077E 01	0.15738E 01	0.98605E 00
6.10	0.58330E 01	0.11905E 01	0.67338E 01	0.14680E 01	0.98174E 00
6.30	0.50501E 01	0.10389E 01	0.62179E 01	0.13669E 01	0.96526E 00
6.50	0.45643E 01	0.94232E 00	0.60478E 01	0.13329E 01	0.98154E 00
6.70	0.39086E 01	0.80161E 00	0.55910E 01	0.12307E 01	0.97071E 00

Table D.7. (cont'd.)

T_k					
Seconds	$\bar{c}_{34}$	$\bar{c}_{35}$	$\bar{c}_{44}$	$\bar{c}_{45}$	$\bar{c}_{55}$
6.9	0.31315E 01	0.65054E 00	0.48528E 01	0.10869E 01	0.95601E 00
7.1	0.24728E 01	0.51496E 00	0.41444E 01	0.93974E 00	0.94211E 00
7.3	0.19607E 01	0.40258E 00	0.35731E 01	0.80939E 00	0.92523E 00
7.5	0.15895E 01	0.32243E 00	0.32234E 01	0.73133E 00	0.94435E 00
7.7	0.13118E 01	0.26270E 00	0.29443E 01	0.66844E 00	0.95228E 00
7.9	0.10930E 01	0.21588E 00	0.27385E 01	0.62214E 00	0.95448E 00
8.1	0.90313E 00	0.17578E 00	0.25631E 01	0.58546E 00	0.96368E 00
8.3	0.74694E 00	0.14184E 00	0.24042E 01	0.55039E 00	0.96710E 00
8.5	0.60922E 00	0.11136E 00	0.22608E 01	0.51912E 00	0.97624E 00
8.7	0.50253E 00	0.87720E-01	0.20938E 01	0.48264E 00	0.97236E 00
8.9	0.42553E 00	0.70880E-01	0.19408E 01	0.44920E 00	0.96212E 00
9.1	0.37270E 00	0.59380E-01	0.17947E 01	0.41658E 00	0.94817E 00
9.3	0.34297E 00	0.52910E-01	0.16680E 01	0.38758E 00	0.93107E 00
9.5	0.33290E 00	0.50660E-01	0.15729E 01	0.36512E 00	0.91219E 00
9.7	0.33530E 00	0.50820E-01	0.14831E 01	0.34242E 00	0.89238E 00
9.9	0.34145E 00	0.51650E-01	0.14159E 01	0.32452E 00	0.87984E 00
10.1	0.35271E 00	0.53820E-01	0.13734E 01	0.31257E 00	0.87415E 00
10.3	0.36680E 00	0.56950E-01	0.13483E 01	0.30499E 00	0.86711E 00
10.5	0.38032E 00	0.60320E-01	0.13411E 01	0.30243E 00	0.86056E 00
10.7	0.39069E 00	0.63190E-01	0.13416E 01	0.30195E 00	0.86008E 00
10.9	0.39489E 00	0.64390E-01	0.13439E 01	0.30191E 00	0.86347E 00
11.1	0.39311E 00	0.63960E-01	0.13466E 01	0.30206E 00	0.87223E 00
11.3	0.38654E 00	0.62590E-01	0.13487E 01	0.30270E 00	0.87864E 00
11.5	0.37827E 00	0.61120E-01	0.13485E 01	0.30388E 00	0.88612E 00
11.7	0.37343E 00	0.60580E-01	0.13458E 01	0.30455E 00	0.88761E 00
11.9	0.35946E 00	0.58050E-01	0.13406E 01	0.30451E 00	0.89436E 00
12.1	0.34898E 00	0.56640E-01	0.13321E 01	0.30398E 00	0.89403E 00
12.3	0.34459E 00	0.56590E-01	0.13218E 01	0.30284E 00	0.89390E 00
12.5	0.34370E 00	0.57270E-01	0.13152E 01	0.30258E 00	0.89616E 00
12.7	0.34308E 00	0.58230E-01	0.13071E 01	0.30134E 00	0.89504E 00
12.9	0.33671E 00	0.57640E-01	0.12889E 01	0.29729E 00	0.89137E 00
13.1	0.32694E 00	0.56370E-01	0.12642E 01	0.29216E 00	0.92352E 00
13.3	0.32080E 00	0.54800E-01	0.12495E 01	0.28793E 00	0.94052E 00
13.5	0.31897E 00	0.54520E-01	0.12381E 01	0.28499E 00	0.95588E 00
13.7	0.30933E 00	0.52240E-01	0.12229E 01	0.27990E 00	0.96485E 00
13.9	0.29687E 00	0.49840E-01	0.12039E 01	0.27493E 00	0.96794E 00
14.1	0.27859E 00	0.46080E-01	0.11828E 01	0.26970E 00	0.97275E 00
14.3	0.25570E 00	0.41430E-01	0.11615E 01	0.26544E 00	0.97885E 00
14.5	0.23304E 00	0.36890E-01	0.11422E 01	0.26263E 00	0.98850E 00
14.7	0.20863E 00	0.31740E-01	0.11020E 01	0.25471E 00	0.96837E 00
14.9	0.19653E 00	0.29820E-01	0.10888E 01	0.25323E 00	0.96349E 00
15.1	0.18086E 00	0.26860E-01	0.10642E 01	0.24928E 00	0.96142E 00
15.3	0.16983E 00	0.24840E-01	0.10327E 01	0.24313E 00	0.95652E 00
15.5	0.16226E 00	0.23340E-01	0.10002E 01	0.23658E 00	0.95710E 00
15.7	0.15786E 00	0.22660E-01	0.96751E 00	0.22930E 00	0.95124E 00
15.9	0.15781E 00	0.22570E-01	0.94156E 00	0.22299E 00	0.93858E 00
16.1	0.16077E 00	0.22440E-01	0.91774E 00	0.21628E 00	0.94181E 00
16.3	0.16292E 00	0.23360E-01	0.90109E 00	0.21181E 00	0.94305E 00
16.5	0.16819E 00	0.24390E-01	0.89054E 00	0.20859E 00	0.94997E 00
16.7	0.17416E 00	0.25860E-01	0.88646E 00	0.20620E 00	0.94860E 00
16.9	0.17644E 00	0.26730E-01	0.88008E 00	0.20431E 00	0.94191E 00
17.1	0.18333E 00	0.28910E-01	0.88414E 00	0.20498E 00	0.93973E 00
17.3	0.18206E 00	0.28740E-01	0.88100E 00	0.20396E 00	0.95076E 00
17.5	0.18031E 00	0.28560E-01	0.88283E 00	0.20453E 00	0.96345E 00
17.7	0.17600E 00	0.27760E-01	0.87993E 00	0.20416E 00	0.96984E 00
17.9	0.16750E 00	0.25660E-01	0.87362E 00	0.20258E 00	0.97201E 00
18.1	0.16046E 00	0.23970E-01	0.86556E 00	0.20102E 00	0.97989E 00
18.3	0.15370E 00	0.22230E-01	0.85677E 00	0.19889E 00	0.99070E 00
18.5	0.14672E 00	0.20720E-01	0.84536E 00	0.19698E 00	0.10051E 01