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SOLUTIONS FOR THE DIFFRACTION OF A PLANE SHOCK WAVE BY GENERAL TWO-DIMENSIONAL DISTURBANCES

Y. S. Pan

OCTOBER 1968

This work was supported by the
National Aeronautics and Space Administration
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FOREWORD

This report was prepared under the National Aeronautics and Space Administration Grant No. NGL-33-016-119, which Grant supports the research involving the problem of sonic boom. The purpose of this research is to study the diffraction of a plane shock wave by general two-dimensional weak disturbances.

ABSTRACT

Solutions for the diffraction of a plane shock wave by general two-dimensional weak disturbances are obtained. The technique employed is an extension of the method developed by Ting and Ludloff for the solution of aerodynamics of blasts. Disturbances due to a solid body are prescribed by distributions of sources, doublets and vortices in a two-dimensional case and by a distribution of point sources in an axisymmetric case. The disturbance pressure behind an advancing shock is expressed by integrals of distributions. The shape of diffracted shock and other disturbance quantities are expressed in terms of disturbance pressure behind the shock and disturbance velocity components ahead of the shock. Application to shock diffraction of thin structure in still air is shown. Some other applications are indicated.

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NOMENCLATURE

$\bar{a}_o$	defined in Eq. (5.11)
A	numerical number, Eq.(2.4)
A_1, A_2	quantities, determined by Eqs.(5.29)
B_1, B_2	quantities, determined by Eqs.(5.30)
c	speed of sound
C_1, C_2	quantities, determined by Eqs.(5.45)
$D_{x,t}$	differential operator, Eq.(4.12)
f	source distribution (two-dimensional)
F	distributions, defined under Eq.(5.8)
g	source distribution (axisymmetric)
G	defined in Eq.(4.13)
h	shape of thin structure Section VII
H	defined in Section V
M	Mach number
p	disturbance pressure
P	undisturbed pressure
$\vec{q}$	velocity relative to curved shock
r	defined under Eq.(5.38)
R	undisturbed density
s	curved shock front
t	time
u	disturbance velocity (x-components)
U	undisturbed velocity
v	disturbance velocity (y-components)
x,y	moving coordinates
x', y'	fixed coordinates

γ	ratio of specific heats
Γ	arbitrary functions
δ	increment
θ	shock angle
$\bar{\lambda}$	defined in Section V
$\bar{\lambda}_o$	defined in Eq.(5.12)
$\bar{\lambda}_1, \bar{\lambda}_2$	two roots of Eq.(5.28)
μ	doublet distribution
ν	vortex distribution
ξ, η, τ	integration variables
ρ	disturbance density
Φ	velocity potential
ψ	stream function
$\Omega_o, \Omega_1, \Omega_2$	defined under Eqs.(4.10)
ζ	defined in Eq. (5)

<u>Subscripts</u>

a	axisymmetric
d	doublet
e	even solution
i=0,1,2	indices
n	normal component
o	odd solution
s	source or conditions of shock
τ	tangential component
x,y,t	partial differentiation with respect to x,y,t
0	conditions ahead of shock

Superscripts

(function)$'$	derivatives of a function with respect to its argument
$\overline{\text{(coordinates)}}$	Lorentz transformation, Eqs.(5.1)
$\overline{\text{(variables)}}$	flow variables, Eqs.(3.1),(3.2) and (4.5)

I. INTRODUCTION

In this report, the solutions for the diffraction of a plane shock wave by general two-dimensional weak disturbances are presented. The technique employed here is an extension of the method developed for the aerosynamics of blasts[1,2]. The present solutions can be applied to many practically interesting shock diffraction problems, such as blast effects on aircrafts and on wings at angles of attack, moving subsonically or supersonically, diffraction of shock due to turbulances in atmosphere, diffraction of shock due to non-smooth walls in shock tubes and diffraction of sonic-booms due to non-planar surfaces on the ground.

Disturbances can be described by the distributions of sources (or sinks), doublets, and vortices in a two-dimensional diffraction problem, and by distribution of point sources (or sinks) in an axisymmetric diffraction problem. The disturbances caused by solid bodies are considered in the present report. For this kind of disturbances, the velocities of distributions with respect to the shock wave remain unchanged when the shock passes over, while the strengths of distributions which represent the disturbances of the solid body are changing across the shock. The diffractions of shock due to atmospheric turbulance will be presented in a separate report.

The disturbance pressure p behind the shock wave is governed by a simple wave equation in three variables (x,y,t), where the coordinates are fixed with the undisturbed flow behind the shock. The shock condition across the slightly disturbed shock front yield a boundary condition $D_{x,t}p = G(y,t)$ at $x = Ut$. $D_{x,t}$ is a second order linear hyperbolic differential operator of x,t with constant coefficients and G(y,t) is a given function related to the prescribed disturbances ahead of the shock. By means of a Lorentz transformation of variables $\bar{x},\bar{y},\bar{t}$, the wave equation is preserved and the boundary condition at the shock $\bar{x} = 0$ reduces to $\bar{D}_{\bar{x},\bar{t}}p = \bar{G}(\bar{y},\bar{t})$, where the operator $\bar{D}_{\bar{x},\bar{t}}$ is of the same type as $D_{x,t}$.

The prescribed general two-dimensional disturbances can be split into even, odd and axisymmetric functions of y; and accordingly, the disturbance pressure p behind the shock can be divided in the same manner to even, odd and axisymmetric solutions. They will be determined separately.

For the even solution of p, $p_{\bar{y}}(\bar{x} < 0,\ \bar{y} = 0,\ \bar{t})$ has either to vanish before the shock hits the leading edge of the prescribed source distribution or to be determined from the prescribed source distribution when the shock passes over it. The disturbance pressure in $\bar{y} > 0$ as a solution of the wave equation can be expressed as an integral of the known distribution $p_{\bar{y}}\ (\bar{x} < 0,\ \bar{y} = 0^{+},\ \bar{t})$ and an integral of an unknown fictitious distribution $p_{\bar{y}}(\bar{x} > 0,\ \bar{y} = 0^{+},\bar{t})$ which is determined by the boundary condition at the shock. When the even disturbance is represented as an integral of the source (or sink) distribution $f(x,t)$ on the plane $y = 0$, the boundary condition at the shock reduces to a differential equation $\bar{D}_{\bar{x},\bar{t}}\ p_{\bar{y}}\ (\bar{x} > 0,\ 0^{+},\ \bar{t}) = \bar{G}(\bar{y},\bar{t})$. $\bar{G}(\bar{y},\bar{t})$ is a known function related to $f(x,t)$. Since the differential operator $\bar{D}_{\bar{x},\bar{t}}$ can be written as $\left(\frac{\partial}{\partial\bar{x}} - \bar{\lambda}_1 \frac{\partial}{\partial\bar{t}}\right)\left(\frac{\partial}{\partial\bar{x}} - \bar{\lambda}_2 \frac{\partial}{\partial\bar{t}}\right)$ with $\bar{\lambda}_1,\bar{\lambda}_2$ real, distinct and positive, solutions for $p_y(\bar{x} > 0,\ 0^{+},\ \bar{t})$ and hence for $p(\bar{x} \le 0,\bar{y},\ \bar{t})$ are obtained. For odd solution, the same technique is used with the source distribution replaced by a doublet or a vortex distribution.

When the prescribed disturbance is axisymmetric with the axis normal to the shock front, the disturbance pressure behind the shock is also axisymmetric and is governed by an axisymmetric wave equation. The boundary condition at the shock $\bar{x} = 0$ remains the same as the two-dimensional case. The disturbance pressure can again be represented as integral of a known distribution for the region $\bar{x} < 0$ which is related to the prescribed disturbance, and an integral of an unknown fictitious distribution for the region $\bar{x} > 0$ which is again determined by the source distribution ahead of the shock, through the shock boundary condition.

In Section II, the prescribed disturbances are presented and are expressed in a moving coordinate system fixed with respect to the undisturbed flow behind the shock. Governing equations for disturbance pressure and other disturbance quantities are derived in Section III. Boundary and initial conditions for the governing equation for disturbance pressures are obtained in Section IV. In Section V, the Lorentz transformation is first introduced. Solutions for disturbance pressure are expressed by Possio integrals for two-dimensional cases and by retarded potentials for the axisymmetric case. Analytic expressions of fictitious distributions are obtained. Final solutions of disturbance pressure and other quantities are summarized in Section VI. The present solution is applied to the shock diffraction by a thin structure in Section VII. The result for this simple example reduces to the solution for the aerodynamics of blasts of Refs. 1 and 2.

The author would like to thank Dr. L. Ting for suggesting the problem and for his help in the initial stage of this work.

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II. PRESCRIBED DISTURBANCES

We choose a coordinate system (x', y') fixed with the undisturbed flow ahead of the advancing shock wave. The x'-axis coincides with the shock propagating direction and the y'-axis, being perpendicular to the x'-axis, is the second coordinate in the general two-dimensional problem (Fig. 1). We assume that the disturbances are generally weak in the sense that the disturbance velocities are much smaller than the speed of sound ahead of the shock. The disturbances are irrotational and stationary with respect to x', y'. Therefore the disturbance velocity potential and stream function exist and they satisfy the Laplace equation[3].

We further assume that the disturbances can be expressed by a combination of distributions of sources (or sinks), doublets, and vortices along x'-axis for a two-dimensional shock diffraction problem, and by a distribution of point sources (or sinks) along x'-axis in an axisymmetric shock diffraction problem. Since the problem is linear, we can treat diffraction due to each distribution separately. In the following, we shall present the velocity potentials or stream functions for each distribution.

A. Source Distribution in Two-Dimensional Disturbances

Suppose the source distribution along x'-axis is specified by $f_o(x')$. It is well known that the velocity potential at any point $P(x', y')$ due to an element of the source distribution $f_o(x_1')\delta x_1'$ at $(x_1', 0)$ is

$$\delta\Phi_s(x', y') = \frac{1}{4\pi} f_o(x_1') \; \delta x_1' \, \ell n \, [(x' - x_1')^2 + y'^2] \tag{2.1}$$

The velocity potential $\Phi_s(x',y')$ due to the entire source distribution can be obtained by integrating over x_1',

$$\Phi_s(x',y') = \frac{1}{4\pi}\int_{-\infty}^{\infty} f_o(x_1')\, dx_1' \, \ln\,[(x' - x_1')^2 + y'^2] \tag{2.2}$$

Here the integral is assumed to exist.

The above expression can be rewritten in the following form,

$$\Phi_s(x',y') = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{c(t'-t_1')\geq E} \frac{c\, f_o(x_1')dt_1'dx_1'}{[c^2(t'-t_1')^2 - (x'-x_1')^2 - y'^2]^{\frac{1}{2}}} + \Phi_{os}, \tag{2.3}$$

with $E = [(x' - x_1')^2 + y'^2]^{\frac{1}{2}}$

and $$\Phi_{os} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_o(x_1')dx_1' \ln\left\{A + [A^2 - (x' - x_1')^2 - y'^2]^{\frac{1}{2}}\right\}_{A\to\infty} \tag{2.4}$$

Here t' and c are respectively the time and the speed of sound behind the shock. Φ_{os} which is due to the lower limit of the integration of t_1' is a number of very large magnitude. Note that any order of its differentiation with respect to its argument are zero. Moreover, we shall see it later that we are interested only in the disturbance velocity which is related to the derivative of the velocity potential. Therefore, we may disregard the existence of this quantity. Now we have written the two-dimensional velocity potential in an "unsteady-like" expression. The reason for this step will be seen clearly later.

B. Doublet Distribution in Two-Dimensional Disturbances

If the doublet distribution along x'-axis is $\mu_o(x')$. The velocity potential $\Phi_d(x',y')$ due to this distribution is

$$\Phi_d(x',y') = \frac{1}{4\pi}\frac{\partial}{\partial y'}\int_{-\infty}^{\infty} \mu_o(x_1')dx_1' \,\ell n\, [(x' - x_1')^2 + y'^2] \quad (2.5)$$

Again, we can rewrite the above expression in an "unsteady-like" form.

$$\Phi_d(x',y') = -\frac{1}{2\pi}\frac{\partial}{\partial y'}\int_{-\infty}^{\infty}\int_{-\infty}^{c(t'-t_1')\geq E} \frac{c\mu_o(x_1')dt_1'dx_1'}{[c^2(t'-t_1')^2 - (x'-x_1')^2 - y'^2]^{\frac{1}{2}}} + \Phi_{od} \quad (2.6)$$

with

$$\Phi_{od} = \frac{1}{2\pi}\frac{\partial}{\partial y'}\int_{-\infty}^{\infty} \mu_o(x_1')dx_1' \,\ell n\, \left\{A + [A^2 - (x' - x_1')^2 - y'^2]^{\frac{1}{2}}\right\}_{A \to \infty} = 0$$

C. Vortex Distribution in Two-Dimensional Disturbances

If the vortex distribution along x'-axis is $\nu_o(x')$. The stream function at $P(x',y')$ due to an element of vortex distribution $\nu_o(x_1')\delta x_1'$ at $(x_1',0)$ is

$$\delta\psi(x',y') = -\frac{1}{4\pi}\nu_o(x_1')dx_1' \,\ell n\, [(x' - x_1')^2 + y'^2] \quad (2.7)$$

The stream function $\psi(x',y')$ due to the entire vortex distribution can be obtained by integrating over x'.

$$\psi(x',y') = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \nu_o(x_1')dx_1' \,\ell n\, [(x' - x_1')^2 + y'^2] \quad (2.8)$$

The above expression can be rewritten in the following form.

$$\psi(x',y') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{c(t'-t_1')\geq E} \frac{c\,\nu_o(x_1')dt_1'dx_1'}{[c^2(t'-t_1')^2 - (x' - x_1')^2 - y'^2]^{\frac{1}{2}}} + \psi_o \quad (2.9)$$

with

$$\psi_o = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \nu_o(x_1')dx_1' \,\ell n\, \left\{A + [A^2 - (x' - x_1')^2 - y'^2]^{\frac{1}{2}}\right\}_{A \to \infty} \quad (2.10)$$

Here ψ_o has the same properties of Φ_{os} as discussed above in Eq.(2.3)

D. Source Distribution in Axisymmetric Disturbances

If the point source distribution along x'-axis is $g_o(x')$, the velocity potential at any point $P(x',y')$ is

$$\Phi_a(x',y') = -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{g_o(x_1')\,dx_1'}{[(x'-x_1')^2 + y'^2]^{\frac{1}{2}}} \tag{2.11}$$

Knowing the velocity potential or stream function, we can determine disturbance velocity components, pressure and density. The velocity components are

$$u_o(x',y') = \frac{\partial\Phi}{\partial x'} = \frac{\partial\psi}{\partial y} \tag{2.12a}$$

$$v_o(x',y') = \frac{\partial\Phi}{\partial y'} = -\frac{\partial\psi}{\partial x'} \tag{2.12b}$$

Disturbance pressure can be found from the linearized Bernoulli's equation[3]. Ahead of the shock, since the disturbances are stationary and the main flow is at rest in x',y', therefore both disturbance pressure and density are zero.

$$p_o = c_o^2\rho_o = 0 \tag{2.13}$$

The prescribed disturbances shall be used to find a boundary conditions in order to solve the disturbed flow field behind the advancing shock. Therefore it is useful to write the velocity potentials and stream function in a moving coordinate system (x,y,t) fixed with the undisturbed flow behind the shock. The relations between the moving and the stationary coordinates are

$$x' = x + (U_o - U)t$$
$$y' = y \qquad (2.14)$$
$$t' = t$$

Here U_o is the shock velocity, and $(U_o - U)$ is the velocity of the undisturbed flow behind the shock with respect to x', y'.

Hence the expressions of velocity potentials and stream function can be written in x,y,t coordinates. By changing the order of integrations, they are

$$\Phi_s(x,y,t) = -\frac{1}{2\pi}\int_{-\infty}^{t-(y/c)} dt_1 \int_{x-[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}}^{x+[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}} dx_1 \frac{c\ f_o[x_1+(U_o-U)t]}{[c^2(t-t_1)^2-(x-x_1)^2-y^2]^{\frac{1}{2}}} \qquad (2.15)$$

$$\Phi_d(x,y,t) = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int_{-\infty}^{t-(y/c)} dt_1 \int_{x-[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}}^{x+[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}} dx_1 \frac{c\ \mu_o[x_1+(U_o-U)t]}{[c^2(t-t_1)^2-(x-x_1)^2-y^2]^{\frac{1}{2}}} \qquad (2.16)$$

$$\psi(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{t-(y/c)} dt_1 \int_{x-[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}}^{x+[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}} dx_1 \frac{c\ \nu_o[x_1+(U_o-U)t]}{[c^2(t-t_1)^2-(x-x_1)^2-y^2]^{\frac{1}{2}}} \qquad (2.17)$$

and

$$\Phi_a(x,y,t) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} dx_1 \frac{g_o[x_1+(U_o-U)t]}{[(x-x_1)^2+y^2]} \qquad (2.18)$$

Here, the integration constants Φ_{os}, Φ_{od} and ψ_o have been dropped. Now it is clear that from Eqs.(2.15) to (2.18), they represent the velocity potential or the stream function at a point (x,y,t) due to moving source, doublet or vortex distribution. The expression of disturbance velocity components are preserved.

$$u_o(x,y,t) = \frac{\partial\Phi}{\partial x} = \frac{\partial\psi}{\partial y} \tag{2.19a}$$

$$v_o(x,y,t) = \frac{\partial\Phi}{\partial y} = - \frac{\partial\psi}{\partial x} \tag{2.19b}$$

When the shock wave passes over the distributions which represent the disturbances due to a solid body, the strengths of these distributions behind the shock change. These distributions are related to the shape of the body and the undistributed flow velocity relative to the body. The velocity potential and the stream potential have the similar expressions of Eqs.(2.15) to (2.18) provided that f_o, μ_o, ν_o and g_o are replaced by f, μ, ν and g respectively, and lower limit of time integral is replaced by a constant.

III. GOVERNING EQUATIONS

Referring to the coordinates x,y,t fixed with the undisturbed flow field behind the shock, we can write the differential equations determining the general two-dimensional unsteady rotational flow behind the shock in the following:

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\bar{\rho}\bar{u}) + y^{-j}\frac{\partial}{\partial y}(\bar{\rho}\bar{v}y^{j}) = 0 \tag{3.1a}$$

$$\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial x} \tag{3.1b}$$

$$\frac{\partial\bar{v}}{\partial t} + \bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} = -\frac{1}{\bar{\rho}}\frac{\partial\bar{p}}{\partial y} \tag{3.1c}$$

$$\left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}\right]\left(\frac{\bar{p}}{\bar{\rho}^{\gamma}}\right) = 0 \tag{3.1d}$$

where $j = 0$ for the two-dimensional problem and $j = 1$ for the axisymmetric problem.

Since the disturbances are weak in comparison with the undisturbed quantities, we may write:

$$\bar{p} = P + p(x,y,t) \tag{3.2a}$$

$$\bar{\rho} = R + \rho(x,y,t) \tag{3.2b}$$

$$\bar{u} = u(x,y,t) \tag{3.2c}$$

$$\bar{v} = v(x,y,t) \tag{3.2d}$$

Substituting the expressions (3.2) into Eqs.(3.1), we have the linearized differential equations for disturbances

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(Ru) + y^{j}\frac{\partial}{\partial y}(Rvy^{j}) = 0 \tag{3.3a}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{R}\frac{\partial p}{\partial x} \tag{3.3b}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{R}\frac{\partial p}{\partial y} \tag{3.3c}$$

$$\frac{\partial p}{\partial t} = c^{2}\frac{\partial\rho}{\partial t} \tag{3.3d}$$

where c is the speed of sound in the undisturbed flow behind the shock

$$c^{2} = \frac{\gamma P}{R} \tag{3.4}$$

Eliminating from the above equations (3.3) three out of four variables, we have the wave equation

$$\Box\, p = \left[\frac{\partial^{2}}{\partial x^{2}} + y^{-j}\frac{\partial}{\partial y}\left(y^{j}\frac{\partial}{\partial y}\right) - \frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}\right] p = 0 \tag{3.5}$$

ρ,u,v are satisfied by the following equations

$$\frac{\partial}{\partial t}\Box\,\rho = \frac{\partial}{\partial t}\Box\, u = \frac{\partial}{\partial t}\Box\, v = 0 \tag{3.6}$$

It is now necessary to find two initial conditions and one boundary condtion for p to satisfy uniquely the wave equation (3.5).

IV. BOUNDARY AND INITIAL CONDITIONS

A. Boundary Condition at Shock Front

Relative to the undisturbed flow behind the shock, the disturbances in the front of the shock are moving with constant velocity $-(U_o-U)$, while the undisturbed shock front moves with the velocity $+U$. The disturbed shock front can be expressed by the equation (Fig. 2)

$$x = Ut + s(y,t) \tag{4.1}$$

Since the disturbances are weak, s is a higher order quantity in comparison with the first order quantity Ut. The shock angle is given by

$$\theta = -\frac{\partial x}{\partial y} = -s_y\ (y,t) \tag{4.2}$$

a higher order quantity.

The shock velocity which is directed normal to the shock front at any point may be written in an x-component U_s and a y-component v_s.

$$U_s = \frac{dx}{dt} = U + s_t\ (y,t) \tag{4.3}$$

$$s_s = U_s \tan\theta \cong U\theta \tag{4.4}$$

The oblique shock conditions on the shock front $x = Ut + s(y,t)$ are

$$\bar{\rho}\bar{q}_n = \rho_o \bar{q}_{no} \tag{4.5a}$$

$$\bar{p} + \bar{\rho}\bar{q}_n^2 = \bar{p}_o + \bar{\rho}_o\ \bar{q}_{no}^2 \tag{4.5b}$$

$$\frac{1}{2}\ \bar{q}_n^2 + \frac{\gamma}{\gamma-1}\left(\frac{\bar{p}}{\bar{\rho}}\right) = \frac{1}{2}\ \bar{q}_{no}^2 + \frac{\gamma}{\gamma-1}\ \left(\frac{\bar{p}_o}{\bar{\rho}_o}\right) \tag{4.5c}$$

$$\bar{q}_\tau = \bar{q}_{\tau_o} \tag{4.5d}$$

where $\bar{q}_n$ = the normal component of the velocity relative to the shock

behind the shock.

$$= (\bar{v} - v_s) \sin\theta + (\bar{u} - U_s) \cos\theta \tag{4.6a}$$

$$= -U - s_t + u$$

q_τ = the tangential component of the velocity relative to the

shock behind the shock

$$= (\bar{v} - v_s) \cos\theta - (\bar{u} - U_s) \sin\theta \tag{4.6b}$$

$$= v$$

$\bar{q}_{no}$ = the normal component of the velocity relative to the

shock ahead of the shock

$$= -v_x \sin\theta - [(U_o - U) + U_s] \cos\theta \tag{4.6c}$$

$$= -U_o + u_o - s_t$$

$\bar{q}_{\tau o}$ = the tangential component of the velocity relative to the

shock ahead of the shock

$$= -v_s \cos\theta + [(U_o - U) + U_s] \cos\theta \tag{4.6d}$$

$$= - (U_o - U)\, s_y + v_o$$

Substituting Eqs.(4.6) and Eqs.(3.2) into Eqs.(4.5) and neglecting higher order quantities, we have

$$-RU + R[-s_t + u] - \rho U = -R_o U_o + R_o(u_o - s_t) \tag{4.7a}$$

$$P + p + RU^2 + \rho U^2 + 2RU\,(s_t - u)$$

$$= P_0 + R_o U_o^2 - 2R_o U_o (u_o - s_t) \tag{4.7b}$$

$$U^2 + 2U(s_t - u) + \frac{2\gamma}{\gamma-1}\left[\frac{P}{R} + \frac{P}{R}\left(\frac{p}{P} - \frac{\rho}{R}\right)\right]$$

$$= U_o^2 - 2U_o\,(u_o - s_t) + \frac{2\gamma}{\gamma-1}\frac{P_o}{R_o} \tag{4.7c}$$

$$v = -\,(U_o - U)s_y + s_o \tag{4.7d}$$

The undisturbed quantities must satisfy the usual Rankine-Hugoniot normal shock relations; namely

$$RU = R_o U_o \tag{4.8a}$$

$$P + RU^2 = P_o + R_o U_o^2 \tag{4.8b}$$

$$\tfrac{1}{2}U^2 + \frac{\gamma}{\gamma-1}\frac{P}{R} = \tfrac{1}{2}U_o^2 + \frac{\gamma}{\gamma-1}\frac{P_o}{R_o} \tag{4.8c}$$

Hence the shock relations (4.7) reduce to

$$R[-s_t + u] - \rho U = R_o(u_o - s_t) \tag{4.9a}$$

$$p + \rho U^2 + 2RU(s_t - u) = p_o - R_o U_o (u_o - s_t) \tag{4.9b}$$

$$U(s_t - u) + \frac{\gamma}{\gamma-1}\left(\frac{p}{P} - \frac{\rho}{R}\right) = -\,U_o(u_o - s_t) \tag{4.9c}$$

$$v = -\,(U_o - U)\,s_y + v_o \tag{4.9d}$$

From Eqs.(4.9), we can solve for ρ, u, s_t and s_y in terms of p, u_o and v_o,

$$c^2 \rho = (1 + \Omega_o)\, p \tag{4.10a}$$

$$u = \frac{\Omega_1}{Rc}\, p + u_o \tag{4.10b}$$

$$-(U_o - U) s_t = \frac{\Omega_2}{R}\, p + \frac{2c}{\gamma+1} \frac{M^2-1}{M}\, u_o \tag{4.10c}$$

$$-(U_o - U) s_y = v - v_o \tag{4.10d}$$

where $\Omega_o = -\dfrac{(\gamma-1)\ (M^2-1)^2}{M^2[(\gamma-1)M^2+2]}$

$$\Omega_1 = \frac{(3\gamma-1)M^2+(3-\gamma)}{2M[(\gamma-1)M^2+2]} \qquad \text{and} \qquad \Omega_2 = \frac{M^2-1}{2M^2}$$

By using differential equations (3.3) and (3.5) one can eliminate u,v,ρ and s from Eqs.(4.10). In this way, a boundary condition for p alone, to be applied at the shock x = Ut, can be formulated,

$$D_{x,t}\, p(x=Ut,y,t) = G(x=Ut,y,t) \tag{4.11}$$

where $D_{x,t}$ is a linear differential operator defined as

$$D_{x,t} = (\Omega_1 + M + \Omega_2 M) \frac{\partial^2}{\partial t^2} + (1+M^2+2M\,\Omega_1)c \frac{\partial^2}{\partial x \partial t}$$

$$+ M(1+\Omega_1 M-\Omega_2)\ c^2 \frac{\partial^2}{\partial x^2} \tag{4.12}$$

G is an expression of prescribed disturbances

$$G(x=Ut,y,t) = -Rc\left[\frac{\partial^2 u_o}{\partial t^2} + Mc \frac{\partial^2 u_o}{\partial x \partial t} + \frac{2(1-M^2)}{\gamma+1}\, c^2 \frac{\partial^2 u_o}{\partial t^2}\right] \tag{4.13}$$

B. Boundary Condition on x-axis

Since prescribed disturbances can be categorized as even (line source distribution), odd (line doublet or vortex distribution) and axisymmetric (point source distribution) disturbances, the disturbances behind the shock can be considered as even, odd and axisymmetric respectively. In the two-dimensional problem, we can solve the wave equation in the region $-\infty < x < Ut$ and $0 \le y < \infty$. Furthermore, the prescribed disturbances which are stationary with respect to x', y' may be taken over by the advancing shock. Therefore, a boundary condition on the x-axis behind the shock should be specified.

As stated previously, the disturbances considered are due to the motion of a body. The strengths of disturbances are changing while velocities are unchanged across the shock. The velocity potential at any point $(x \le Ut, y, t)$ behind the shock due to the source distribution $f(x,t)$ is

$$\Phi_s(x \le Ut, y, t) = -\frac{1}{2\pi}\int_0^{t-(y/c)} dt_1 \int_{x-[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}}^{x+[c^2(t-t_1)^2-y^2]^{\frac{1}{2}}} dx_1 \quad \frac{cf[x_1+(U_o-U)t]}{[c^2(t-t_1)^2-(x-x_1)^2-y^2]^{\frac{1}{2}}} \tag{4.14}$$

where $t = 0$ in the instant that the shock hits the leading edge of the source distribution, and

$$f[x + (U_o-U)t] = 0 \qquad \text{for } x > Ut$$

i.e. the velocity potential at $(x \le Ut, y, t)$ is affected only by the distribution behind the shock. As $y \to 0$, Φ_s can be found as[4]

$$\Phi_s(x \le Ut, y \to 0, t) = \tfrac{1}{2}(ct-y)\ f[x + (U_o - U)t] \tag{4.15}$$

Then the disturbance velocity, by Eq.(2.19a)

$$v(x \le Ut,\ y \to 0,t) = \tfrac{1}{2}f[x + (U_o - U)t] \tag{4.16}$$

By applying Eq.(3.3c), we have the boundary condition for p on x-axis

$$p_y(x \le Ut,0,t) = -\frac{Rc}{2}\,\frac{U_o-U}{c}\,f'[x + (U_o - U)t] \tag{4.17}$$

where the prime of a function denotes the differentiation with respect to its argument.

Similarly, the boundary condition on the axis behind the shock for a doublet distribution can be obtained

$$p(x \le Ut,0,t) = -\frac{Rc}{2}\,\frac{U_o-U}{c}\,\mu'[x + (U_o-U)t] \tag{4.18}$$

For a vortex distribution,

$$p(x \le Ut,0,t) = \frac{Rc}{2}\,\frac{U_o-U}{c}\,\nu[x + (U_o-U)t] \tag{4.19}$$

and for a source distribution in axisymmetric case

$$yp_y(x \le Ut,0,t) = -\frac{Rc}{2\pi}\,\frac{U_o-U}{c}\,g'[x + (U_o-U)t] \tag{4.20}$$

C. Boundary Condition at Infinity

$$p\ (x \to -\infty,\ y \to \infty,t) = 0 \tag{4.21}$$

D. Initial Conditions

$$p\ (x \le Ut,y,t \to -\infty) = 0 \tag{4.22}$$

$$p_t(x \le Ut,y,t \to -\infty) = 0 \tag{4.23}$$

V. DERIVATION OF ANALYTIC SOLUTION FOR DISTURBANCE PRESSURE

A. The Lorentz Transformation

The boundary condition at shock, Eq.(4.11), is prescribed at the plane $x = Ut$. We may introduce a new coordinate system $(\bar{x},\bar{y},\bar{t})$ related to the old variable (x,y,t) by the Lorentz transformation

$$\bar{x} = (x - Ut)\ (1 - M^2)^{-\frac{1}{2}} \quad , \quad \bar{y} = y, \qquad \bar{t} = (ct - Mx)\ (1 - M^2)^{-\frac{1}{2}} \tag{5.1}$$

The plane $\bar{x} = 0$ corresponds to the plane $x = Ut$, and the wave equation remains unchanged,

$$\left[\frac{\partial^2}{\partial \bar{t}^2} - \frac{\partial^2}{\partial \bar{x}^2} - \bar{y}^{-j} \frac{\partial}{\partial \bar{y}} \left(\bar{y}^j \frac{\partial}{\partial \bar{y}}\right)\right] p = 0 \tag{5.2}$$

The boundary condition at infinity, Eq.(4.21) becomes

$$p \to 0 \qquad\qquad \text{as } (\bar{x}^2 + \bar{y}^2)^{\frac{1}{2}} \to \infty \tag{5.3}$$

The boundary condition at shock $(\bar{x} = 0)$ becomes (Appendix A)

$$\bar{D}_{\bar{x},\bar{t}}\ p(\bar{x} = 0,\bar{y},\bar{t}) = \bar{G}(\bar{x} = 0,\bar{y},\bar{t}) \tag{5.4}$$

Here $\bar{D}_{\bar{x},\bar{t}}$ is a linear differential operator of hyperbolic type

$$\bar{D}_{\bar{x},\bar{t}} = \frac{1}{M_o^2} \frac{\partial^2}{\partial \bar{t}^2} + 2M \frac{\partial^2}{\partial \bar{x} \partial \bar{t}} + \frac{\partial^2}{\partial \bar{x}^2} \tag{5.5}$$

with $M_o^2 = \dfrac{2\gamma M^2-(\gamma-1)}{(\gamma-1)M^2+2}$ the Mach number of undisturbed flow ahead of the shock. $\bar{G}(\bar{x} = 0,\bar{y},\bar{t})$ can be related to prescribed disturbances. For an even disturbance-source distribution.

$$\bar{G}(\bar{x}=0,\bar{y},\bar{t}) = \frac{Rc}{\pi}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{f_o'(\zeta)\;\; \bar{x} = 0}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.6}$$

$$\text{where } \zeta = (1-M^2)^{-\frac{1}{2}}\,[M\tau + \xi + M\,\frac{U_o-U}{U}\,\bar{t} + M^2\,\frac{U_o-U}{U}\,\bar{x}] \tag{5.7}$$

For an odd disturbance - doublet or vortex distribution

$$\bar{G}(\bar{x}=0,\bar{y},\bar{t}) = -\frac{Rc}{\pi}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{[(\bar{t}-\tau)^2-\bar{y}^2]} d\xi \; \frac{F_o''(\zeta)\;\; \bar{x} = 0}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.8}$$

where $F_o(\zeta) = -\mu_o'(\zeta)$ for doublet distribution, and $F_o(\zeta) = \nu_o(\zeta)$ for vortex distribution. For an axisymmetric disturance - point source distribution

$$\bar{G}(\bar{x}=0,\bar{y},\bar{t}) = \frac{Rc}{2\pi}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M(1-M^2)^{\frac{1}{2}}}\right]\int_{-\infty}^{\infty} d\xi\; \frac{g_o'''(\zeta)\bar{x}=0}{[\xi^2+\bar{y}^2]^{\frac{1}{2}}} \tag{5.9}$$

The boundary conditions, Eqs.(4.17) to (4.20) are also transformed. For an even disturbance from Eq.(4.17),

$$p_{\bar{y}}(\bar{x}\le 0,0,\bar{t}) = -\frac{Rc}{2}\,\frac{U_o-U}{c}\, f'[\bar{a}_o(\bar{\lambda}_o\bar{x}+\bar{t})], \tag{5.10}$$

where

$$\bar{a}_o = U_o/c\;(1-M^2)^{\frac{1}{2}} \tag{5.11}$$

$$\bar{\lambda}_o = \left(1-M^2 + \frac{MU_o}{c}\right)\frac{c}{U_o} \tag{5.12}$$

For an odd disturbance, from Eq.(4.18) or (4.19)

$$p(\bar{x} \le 0,0,\bar{t}) = \frac{Rc}{2}\;\frac{U_o-U}{c}\; F[\bar{a}_o(\bar{\lambda}_o\bar{x}+\bar{t})] \tag{5.13}$$

For an axisymmetric disturbance from Eq.(4.20)

$$\bar{y}p_{\bar{y}}\ (\bar{x} \le 0,0,\bar{t}) = -\frac{Rc}{2\pi}\frac{U_o - U}{c}\, g'[\bar{a}_o(\bar{\lambda}_o \bar{x}+\bar{t})] \tag{5.14}$$

The two initial conditions Eqs.(4.22) and (4.23) are now

$$p = p_{\bar{t}} = 0 \qquad \text{for } \bar{t} \to -\infty \tag{5.15}$$

B. <u>The Possio Integral - Two-dimensional Problem</u>

In general, the solution of such two-dimensional boundary initial value problems as one defined by Eqs.(5.2),(5.3),(5.4) and (5.10) [or (5.13)] can be solved in terms of "temporary sources (or doublet)" spread over a certain area in the $\bar{x}$-$\bar{t}$ plane characterizing the motion of the disturbances. Such solutions can be written as Possio integrals[1]. For the even solution

$$p(\bar{x},\bar{y},\bar{t}) = -\frac{1}{\pi}\iint \frac{p_{\bar{y}}(\xi,0,\tau)\,d\tau\, d\xi}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.16}$$

and for the odd solution,

$$p(\bar{x},\bar{y},\bar{t}) = -\frac{1}{\pi}\frac{\partial}{\partial \bar{y}}\iint \frac{p(\xi,0,\tau)\,d\tau\, d\xi}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.17}$$

Here in numerator $p_{\bar{y}}$ and p of the integrals represent respectively the "temporary source strength" and the "temporary doublet strength"; the denominator represents the "pseudo distance" between "source (or doublet)" $\xi,0,\tau$ and point $\bar{x},\bar{y},\bar{t}$. The integration area in the ξ,τ plane is confined by the hyperbola

$$\bar{t} - \tau = [(\bar{x}-\xi)^2 + \bar{y}^2]^{\frac{1}{2}}$$

and the straight line $\tau = -\infty$ (see Fig. 3).

Equation (5.16) can be written in the following form

$$p(\bar{x} \leq 0, \bar{y} > 0, \bar{t}) = -\frac{1}{\pi}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{x-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{0} \frac{p_{\bar{y}}(\xi \leq 0,0,\tau > 0)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{x}+(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi$$

$$\cdot \frac{p_{\bar{y}}(\xi > 0,0,\tau)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.18}$$

Equation (5.17) becomes

$$p(\bar{x} \leq 0, y > 0, \bar{t}) = -\frac{1}{\pi}\frac{\partial}{\partial \bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{0} d\xi \frac{p(\xi \leq 0,0,\tau > 0)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$-\frac{1}{\pi}\frac{\partial}{\partial \bar{y}}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \frac{p(\xi > 0,0,\tau)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$\tag{5.19}$$

It is obvious that the method is applicable only if $p_{\bar{y}}$ or p is prescribed on the entire plane $\bar{y} = 0$. However, $p_{\bar{y}}$ or p is given for the left half of the plane $\bar{y} = 0$ ($\bar{x} \leq 0$), while it is unknown in the right half of the plane ($\bar{x} > 0$). The next step is to find an equation for fictidious distribution $p_{\bar{y}}$ ($\bar{x} > 0,0,\bar{t}$) or $p(\bar{x} > 0,0,\bar{t})$ which will replace shock boundary condition Eq.(5.4) prescribed on the plane $\bar{x} = 0$.

C. Retarded Potential - Axisymmetric Problem

By the Kirchhoff's theorem, a solution of the wave equation can be written in terms of retarded potential[2,5]

$$p(\bar{x} \le 0,\bar{r},\bar{t}) = -\frac{1}{4\pi}\int\int \frac{p_{\bar{r}}\left[\xi,0,\bar{t}-[(\bar{x}-\xi)^2+\bar{r}^2]^{\frac{1}{2}}\right]\bar{r}\,d\theta\, d\xi}{[(\bar{x}-\xi)^2+\bar{\gamma}^2]^{\frac{1}{2}}}$$

For an axisymmetric problem

$$p(\bar{x} \le 0,\bar{y},\bar{t}) = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\bar{y}p_{\bar{y}}\left[\xi,0,\bar{t}-[(\bar{x}-\xi)^2+\bar{y}^2]^{\frac{1}{2}}\right]d\xi}{[(\bar{x}-\xi)^2+\bar{y}^2]^{\frac{1}{2}}} \tag{5.20}$$

Again this method is applicable only if $p_{\bar{y}}$ is prescribed on the entire $\bar{x}$-axis. $p_{\bar{y}}$ is given for the left half of the axis ($\bar{x} \le 0$), while it is unknown on the right half of the axis. The next step is to find an equation for the fictitious distribution $p_{\bar{y}}(\bar{x} > 0,0,\bar{t})$ which will replace shock condition prescribed on the plane $\bar{x} = 0$.

D. Evaluation of Fictitious Distributions

For even disturbances, by substituting Eq.(5.10) into Eq.(5.18) we have

$$p(\bar{x} \le 0,\ y > 0,\bar{t}) = -\frac{1}{\pi}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi\ \frac{p_{\bar{y}}(\xi > 0,0,\tau)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$+\frac{Rc}{2\pi}\frac{U_o-U}{c}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{0} d\xi\ \frac{f'[\bar{a}_o(\bar{\lambda}_o\xi+\tau)]}{[(\bar{t}-t)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.21}$$

To find an equation for $p_{\bar{y}}(\xi > 0,0,t)$, let us apply shock boundary condition, Eq.(5.4)

$$- \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \, \frac{\bar{D}_{\xi,\tau} p_{\bar{y}}(\xi > 0,0,\tau)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}]^{\frac{1}{2}}} + \frac{Rc}{2} \frac{U_o-U}{c} \bar{a}_o^2 \quad H(-\bar{\lambda}_o)$$

$$\cdot \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \, \frac{f'''[\bar{a}_o(\tau-\bar{\lambda}_o\xi)]}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$- \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \, \frac{[p_{\bar{y}\bar{\xi}}(0^+,0,\tau)+2Mp_{\bar{y}\bar{t}}(0^+,0,\tau) + \frac{Rc}{2}\frac{U_o-U}{c}\bar{a}_o(\bar{\lambda}_o+2M)f''(\bar{a}_o\bar{t})]}{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$= Rc \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right] \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \, \frac{[f_o'''(\zeta_{\xi=\xi})_{\bar{x}=0} + f_o'''(\zeta_{\xi=-\xi})_{\bar{x}=0}]}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.22}$$

where $H(\bar{\lambda}_o) = \frac{1}{M_o^2} - 2M\,\bar{\lambda}_o + \lambda_o^2$. This shock boundary condition can be satisfied, provided that

$$\bar{D}_{\bar{x},\bar{t}} p_{\bar{y}}(\bar{x} > 0,0,\bar{t}) = \frac{Rc}{2}\frac{U_o-U}{c}\bar{a}_o H(-\lambda_o) f'''[\bar{a}_o(\bar{t}-\bar{\lambda}_o\bar{x})]$$

$$- Rc \, \frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \left\{ f'''[\bar{a}_o(\bar{t}+\bar{\lambda}\,\bar{x})] + f'''[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})] \right\} \tag{5.23}$$

with $\bar{a}_o = U_o/c(1-M^2)^{\frac{1}{2}}$ and $\bar{\lambda} = c/U_o$; and

$$p_{\bar{y}\bar{x}}(0^+,0,\bar{t}) + 2Mp_{\bar{y}\bar{t}}(0^+,0,\bar{t}) = -\frac{Rc}{2}\frac{U_o-U}{c}\bar{a}_o(\bar{\lambda}_o + 2M)f''(\bar{a}_o\bar{t}) \tag{5.24}$$

Now the problem is reduced to obtain a function $p_{\bar{y}}(\bar{x} > 0,0,\bar{t})$ which satisfies differential equation (5.23) and two boundary conditions at $\bar{x} = 0^+, \bar{y} = 0, \bar{t}$. These two boundary conditions can be obtained by a kind of "mean value theorem" at $\bar{x} = 0$[1] and Eq.(5.24). They are, (see Appendix B)

$$p_{\bar{y}}(0^+,0,\bar{t}) = Rc\Big[\Big(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{1}{2}\frac{U_o-U}{c}\Big) f'(\bar{a}_o\bar{t})$$

$$- \Big(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{4M}{\gamma+1}\Big) f_o'(\bar{a}_o\bar{t})\Big] \tag{5.25}$$

and

$$p_{\bar{y}\bar{x}}(0^+,0,\bar{t}) = - Rc\Big\{\Big[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{1}{2}\frac{U_o-U}{c}(\bar{\lambda}_o + 4M)\Big]\bar{a}_o f''(\bar{a}_o\bar{t})$$

$$- \Big[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{8M^2}{\gamma+1}\Big]\bar{a}_o f''(\bar{a}_o\bar{t})\Big\} \tag{5.26}$$

It is shown in Appendix C that the solution for Eq.(5.23) has the form

$$p_{\bar{y}}(\bar{x} > 0,\bar{y}=0,t) = Rc\Big\{\sum_{i=0,1,2} A_i f'[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})] + \sum_{i=1,2} B_i f_o'[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})]\Big\}$$

$$- Rc\frac{1}{\bar{a}_o^2}\Big[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\Big]\Big\{\frac{1}{H(-\bar{\lambda})} f_o'[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})] + \frac{1}{H(\bar{\lambda})} f_o'[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\Big\} \tag{5.27}$$

Here $\bar{a}_o = U_o/c(1-M^2)^{\frac{1}{2}}$, $\bar{\lambda} = c/U_o$, $\bar{\lambda}_o = \Big(1-M^2+\frac{MU_o}{c}\Big)\frac{c}{U_o}$, $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are two real, distinct and positive roots of the quadratic equation

$$H(\bar{\lambda}) = \bar{\lambda}^2 + 2M\bar{\lambda} + \frac{1}{M_o^2} = 0 \tag{5.28}$$

$A_o = \frac{U_o-U}{2c}\frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}$, A_1 and A_2 are the solutions of the two simultaneous linear equations

$$A_1 + A_2 = \frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{U_o - U}{2c}\left[1 - \frac{H(-\lambda_o)}{H(\bar{\lambda}_o)}\right] \tag{5.29a}$$

and

$$\bar{\lambda}_1 A_1 + \bar{\lambda}_2 A_2 = \frac{4M^3}{M^2-1}\,\frac{U_o}{c} + \frac{U_o - U}{2c}\left[\bar{\lambda}_o\left(1 - \frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}\right) + 4M\right] \tag{5.29b}$$

and B_1 and B_2 are the solutions of the other two simultaneous linear eqautions

$$B_1 + B_2 = -\frac{2M^2}{M^2-1}\,\frac{U_o}{c} - \frac{4M}{\gamma+1} + \frac{1}{\bar{a}_o^2}\left[\frac{4(\gamma M^2-1)}{(\gamma+1)^2 M}\right]\left[\frac{1}{H(-\bar{\lambda})} + \frac{1}{H(\bar{\lambda})}\right] \tag{5.30a}$$

and

$$\bar{\lambda}_1 B_1 + \bar{\lambda}_2 B_2 = -\frac{4M^3}{M^2-1}\,\frac{U_o}{c} - \frac{8M^2}{\gamma+1} - \frac{\bar{\lambda}}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\left[\frac{1}{H(-\bar{\lambda})} - \frac{1}{H(\bar{\lambda})}\right] \tag{5.30b}$$

For odd disturbances, by substituting Eq.(5.13) into Eq.(5.19), we have

$$p(\bar{x} \le 0, \bar{y} > 0, t) = -\frac{1}{\pi}\frac{\partial}{\partial \bar{y}}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \frac{p(\xi > 0,0,\tau)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$-\frac{Rc}{2\pi}\,\frac{U_o-U}{c}\,\frac{\partial}{\partial \bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{0} d\xi \frac{F[\bar{a}_o(\bar{\lambda}_o\xi+\tau)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \tag{5.31}$$

To find an equation for $p(\xi > 0,0,\tau)$, let us apply shock boundary condition Eq. (5.4)

$$\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{\bar{D}_{\xi,\tau}p(\xi>0,0,\bar{\tau})}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}} + \frac{Rc}{2}\,\frac{U_o-U}{c}\,\bar{a}_o^2\, H(-\bar{\lambda}_o)\,\frac{\partial}{\partial\bar{y}}$$

$$\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{F''[\bar{a}_o(\bar{\tau}-\bar{\lambda}_o\xi)]}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$+\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \; \frac{p_\xi(0^+,0,\tau)+2Mp_\tau(0^+,0,\tau)-\frac{Rc}{2}\,\frac{U_o-U}{c}\,\bar{a}_o(\bar{\lambda}_o+2M)F'(\bar{a}_o\bar{t})}{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$= Rc\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_0^{[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{[F_o''(\zeta_{\xi=\xi})_{\bar{x}=0} + F_o''(\zeta_{\xi=-\xi})_{\bar{x}=0}]}{[(\bar{t}-\tau)^2-\xi^2-\bar{y}^2]^{\frac{1}{2}}}$$

(5.32)

Again this boundary condition can be satisfied provided that

$$\bar{D}_{\bar{x},\bar{t}}\; p(\bar{x}>0,0,\bar{t}) = -\frac{Rc}{2}\,\frac{U_o-U}{c}\,\bar{a}_o H(-\bar{\lambda}_o)F''[\bar{a}_o(\bar{t}-\bar{\lambda}_o\bar{x})]$$

$$+ Rc\,\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\left\{F_o''[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})] + F_o''[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\right\} \qquad (5.33)$$

and

$$p_{\bar{x}}(0^+,0,\bar{t}) + 2Mp_{\bar{t}}(0^+,0,\bar{t}) = \frac{Rc}{2}\,\frac{U_o-U}{c}\,\bar{a}_o(\bar{\lambda}_o+2M)F'(\bar{a}_o\bar{t}) \qquad (5.34)$$

Now the problem is reduced to obtain a function $p(\bar{x}>0,0,t)$ which satisfies differential equation (5.33) and two boundary conditions at $\bar{x}=0^+$, $\bar{y}=0,\bar{t}$. These two boundary conditions are (See Appendix B)

$$p(0^+,0,\bar{t}) = -Rc\left[\left(\frac{2M^2}{M^2-1}\frac{U_o}{c}+\frac{U_o-U}{2c}\right)F(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\frac{U_o}{c}+\frac{4M}{\gamma+1}\right)F_o(\bar{a}_o\bar{t})\right] \tag{5.35}$$

and

$$p_{\bar{x}}(0^+,0,\bar{t}) = Rc\left\{\left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{U_o-U}{2c}(\bar{\lambda}_o+4M)\right]\bar{a}_oF'(\bar{a}_o\bar{t})\right.$$

$$\left. - \left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{8M^2}{\gamma+1}\right]\bar{a}_oF'(\bar{a}_o\bar{t})\right\} \tag{5.36}$$

It is shown in Appendix C that the solution for Eq.(5.33) has the form

$$p(\bar{x}>0,\bar{y}=0,\bar{t}) = -Rc\left\{\sum_{i=0,1,2}A_iF[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})] + \sum_{i=1,2}B_iF_o[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})]\right\}$$

$$+ Rc\frac{1}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2M}\right]\left\{\frac{1}{H(-\bar{\lambda})}F_o[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})] + \frac{1}{H(\bar{\lambda})}F_o[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\right\} \tag{5.37}$$

where A_i, B_i and λ_i are defined in the case of even disturbances.

For axisymmetric disturbances, by substituting Eq.(5.14) into Eq.(5.20), we have

$$p(\bar{x}\le 0,\bar{y},\bar{t}) = -\tfrac{1}{2}\int_0^\infty d\xi\,\frac{\bar{y}p_{\bar{y}}(\xi>0,0,\bar{t}-\bar{r})}{\bar{r}} + \frac{Rc}{4\pi}\frac{U_o-U}{c}\int_{-\infty}^0 d\xi\,\frac{g'[\bar{a}_o(\lambda_o\xi+\bar{t}-\bar{r})]}{\bar{r}} \tag{5.38}$$

where $\bar{r} = [(\bar{x}-\xi)^2+\bar{y}^2]^{\frac{1}{2}}$. Now let us apply shock boundary condition Eq.(5.4) to find $\bar{y}p_{\bar{y}}$ $(\xi>0,0,\bar{t})$.

$$- \int_0^\infty d\xi \; \frac{\bar{D}_{\xi,t}[\bar{y}p_{\bar{y}}(\xi > 0,0,\bar{t}-\bar{r})]}{\bar{r}} + \frac{Rc}{2\pi}\frac{U_o-U}{c} \; \bar{a}_o H(-\bar{\lambda}_o) \int_0^\infty d\xi \; \frac{g''[\bar{a}_o(\bar{t}-\bar{r}-\bar{\lambda}_o\xi)]}{\bar{r}}$$

$$- \frac{1}{\bar{y}} \left\{ 2M\bar{y}p_{\bar{y}\bar{t}}(0^+,0,\bar{t}-\bar{y}) + \bar{y}p_{\bar{y}\bar{x}}(0^+,0,\bar{t}-\bar{y}) + \frac{Rc}{2\pi}\frac{U_o-U}{c} \; \bar{a}_o(2M+\bar{\lambda}_o)g''[\bar{a}_o(\bar{t}-\bar{y})] \right\}$$

$$= \frac{Rc}{\pi}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right] \int_0^\infty d\xi \; \frac{[g_o'''(\zeta_{\xi=\xi})_{\bar{x}=0} + g_o'''(\zeta_{\xi=-\xi})_{\bar{x}=0}]}{\bar{r}} \tag{5.39}$$

Again, this boundary condition can be satisfied provided that

$$\bar{D}_{\bar{x},\bar{t}}\left[\bar{y}p_{\bar{y}}(\bar{x} > 0,0,\bar{t})\right] = \frac{Rc}{2\pi}\frac{U_o-U}{c} \; \bar{a}_o H(-\bar{\lambda}_o)g'''[\bar{a}_o(\bar{t}-\bar{\lambda}_o\bar{x})]$$
$$- \frac{Rc}{\pi}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(r+1)^2 M}\right]\left\{g_o'''[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})] + g_o'''[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\right\} \tag{5.40}$$

and

$$\bar{y}p_{\bar{y}}(0^+,0,\bar{t}) + 2M\bar{y}\,p_{\bar{y}\bar{t}}(0^+,0,\bar{t}) = -\frac{Rc}{2\pi}\frac{U_o-U}{c} \; \bar{a}_o(\bar{\lambda}_o+2M)g'''(\bar{a}_o\bar{t}) \tag{5.41}$$

It is shown in Appendix B, that two boundary conditions for Eq.(5.40) are

$$\bar{y}p_{\bar{y}}(0^+,0,\bar{t}) = \frac{Rc}{\pi}\left[\left(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{1}{2}\frac{U_oU}{c}\right)g'(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{4M}{\gamma+1}\right)g_o'(\bar{a}_2\bar{t})\right) \tag{5.42}$$

and

$$\bar{y}p_{\bar{y}\bar{x}}(0^+,0,\bar{t}) = -\frac{Rc}{\pi}\left\{\left[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{1}{2}\frac{U_o-U}{c}(\bar{\lambda}_o+4M)\right]\bar{a}_o g''(\bar{a}_o t)\right.$$
$$\left. - \left[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{8M^2}{\gamma+1}\right]\bar{a}_o g''(\bar{a}_o\bar{t})\right\} \tag{5.43}$$

As shown in Appendix C, the solution of Eq.(5.40) has the form

$$\bar{q}p_{\bar{q}}\ (x > 0, \bar{y} = o, \bar{t}) = \frac{Rc}{\pi}\left\{\sum_{i=0,1,2} A_i g'[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})] + \sum_{i=1,2} C_i g_o'[\bar{a}_o(\bar{t}-\bar{\lambda}_i\bar{x})]\right\}$$

$$-\frac{Rc}{\pi}\frac{1}{\bar{a}_o^{\,2}}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right]\left\{\frac{1}{H(-\bar{\lambda})}\,g_o'[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})] + \frac{1}{H(\bar{\lambda})}g_o'[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\right\}$$

(5.44)

where A_o, λ_i are defined in the case of even disturbances. C_1 and C_2 are the stations of the following two simultaneous linear equations

$$C_1 + C_2 = -\frac{2M^2}{M^2-1}\frac{U_o}{c} - \frac{4M}{\gamma+1} + \frac{1}{\bar{a}_o^{\,2}}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right]\left[\frac{1}{H(-\bar{\lambda})} + \frac{1}{H(\bar{\lambda})}\right]$$

(5.45a)

and

$$\bar{\lambda}_1 C_1 + \bar{\lambda}_2 C_2 = -\frac{4M^3}{M^2-1}\frac{U_o}{c} - \frac{8M^2}{\gamma+1} - \frac{\bar{\lambda}}{\bar{a}_o^{\,2}}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right]\left[\frac{1}{H(-\bar{\lambda})} - \frac{1}{H(\bar{\lambda})}\right]$$

(5.45b)

VI. FINAL RESULTS

The results obtained in Section V can be summarized as follows:

For even disturbances

$$p(\bar{x},\bar{y},\bar{t}) = \frac{Rc}{\pi}\frac{U_o - U}{2c}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}]^{\frac{1}{2}}}^{0} d\xi \; \frac{f'[\bar{a}_o(\tau+\bar{\lambda}_o\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}]^{\frac{1}{2}}}$$

$$-\frac{Rc}{\pi}\sum_{i=0,1,2} A_i \int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{f'[\bar{a}_o(\tau-\bar{\lambda}_i\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$-\frac{Rc}{\pi}\sum_{i=1,2} B_i \int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{x+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{f_o'[\bar{a}_o(\tau-\bar{\lambda}_i\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$+\frac{Rc}{\pi}\frac{1}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\left\{\frac{1}{H(-\lambda)}\int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{f_o'[\bar{a}_o(\tau+\bar{\lambda}\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}\right.$$

$$\left.+\frac{1}{H(\bar{\lambda})}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{x+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{f_o'[\bar{a}_o(\tau-\bar{\lambda}\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]}\right\} \tag{6.1}$$

For odd disturbances

$$
\begin{aligned}
p(\bar{x},\bar{y},\bar{t}) = & -\frac{Rc}{\pi}\frac{U_o-U}{2c}\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{0} d\xi \; \frac{F[\bar{a}_o(\tau+\bar{\lambda}_o\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \\
& + \frac{Rc}{\pi}\sum_{i=0,1,2} A_i \frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{F[\bar{a}_o(\tau-\bar{\lambda}_i\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \\
& + \frac{Rc}{\pi}\sum_{i=1,2} B_i \frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-[\bar{x}^2+\bar{y}^2]^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{F_o[\bar{a}_o(\tau-\bar{\lambda}_i\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} \\
& - \frac{Rc}{\pi}\frac{1}{\bar{a}_o^{\,2}}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]\left\{\frac{1}{H(-\bar{\lambda})}\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{F_o[\bar{a}_o(\tau+\bar{\lambda}\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}\right. \\
& \left. + \frac{1}{H(\bar{\lambda})}\frac{\partial}{\partial\bar{y}}\int_{-\infty}^{\bar{t}-(\bar{x}^2+\bar{y}^2)^{\frac{1}{2}}} d\tau \int_{0}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \; \frac{F_o[\bar{a}_o(\tau-\bar{\lambda}\xi)]}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}}\right\}
\end{aligned}
\tag{6.2}
$$

For axisymmetric disturbances

$$p(\bar{x},\bar{y},\bar{t}) = \frac{Rc}{2\pi}\,\frac{U_o-U}{2c}\int_{-\infty}^{0} d\xi\;\frac{g'[\bar{a}_o(\bar{t}-\bar{r}+\bar{\lambda}_o\xi)]}{\bar{r}}$$

$$-\frac{Rc}{2\pi}\sum_{i=0,1,2} A_i\int_0^{\infty} d\xi\;\frac{g'[\bar{a}_o(\bar{t}-\bar{r}-\bar{\lambda}_i\xi)]}{\bar{r}}$$

$$-\frac{Rc}{2\pi}\sum_{i=1,2} C_i\int_0^{\infty} d\xi\;\frac{g_o'[\bar{a}_o(\bar{t}-\bar{r}-\bar{\lambda}_i\xi)]}{\bar{r}}$$

$$+\frac{Rc}{2\pi}\,\frac{1}{\bar{a}_o^{\,2}}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right]\left\{\frac{1}{H(-\bar{\lambda})}\int_0^{\infty} d\xi\;\frac{g_o'[\bar{a}_o(\bar{t}-\bar{r}+\bar{\lambda}\xi)]}{\bar{r}}\right.$$

$$\left.+\frac{1}{H(\bar{\lambda})}\int_0^{\infty} d\xi\;\frac{g_o'[\bar{a}_o(\bar{t}-\bar{r}-\bar{\lambda}\xi)]}{\bar{r}}\right\} \qquad (6.3)$$

By using the transformation (5.1), the disturbance pressure $p(x,y,t)$ can be obtained from $p(\bar{x},\bar{y},\bar{t})$. From differential equations (3.3) and shock relations, Eqs.(4.10), the following expressions are obtained.

The disturbance density

$$\rho(x,y,t) = \frac{1}{c^2}\,p(x,y,t) + \frac{\Omega_o}{c^2}\,p\left(x,y,t=\frac{x}{U}\right) \qquad (6.4)$$

The x-component disturbance velocity

$$u(x,y,t) = -\frac{1}{R}\int_{x/U}^{t} p_y(x,y,\tau)\,d\tau + \frac{\Omega_1}{Rc}\,p\left(x,y,t=\frac{x}{U}\right) + u_o(x,y,t) \qquad (6.5)$$

where $u_o(x,y,t)$ depending on the prescribed disturbances is defined by Eq.(2.19a).

The form of shock front

$$x = Ut - \frac{\Omega_2}{R(U_o-U)}\int_{-\infty}^{t} p(x=U\tau,y,\tau)\,d\tau - \frac{2c}{U_o-U}\,\frac{M^2-1}{(\gamma+1)M}\int_{-\infty}^{t} u_o(x=U\tau,y,\tau)\,d\tau$$

$$(6.6)$$

The y-component disturbance velocity

$$v(x,y,t) = -\frac{1}{R}\int_{x/U}^{t} p_y(x,y,t)\,d\tau + \frac{\Omega_2}{R}\int_{-\infty}^{x/U} p_y(x=U\tau,y,\tau)\,d\tau$$

$$+\frac{2c}{\gamma+1}\,\frac{M^2-1}{M}\int_{-\infty}^{x/U} u_{oy}(x=U\tau,\tau)\,d\tau + v_o(x,y,t)$$

$$(6.7)$$

where $v_o(x,y,t)$ and $u_o(x,y,t)$ are related to the prescribed disturbances by Eqs.(2.19).

VII. CONCLUDING REMARKS

Solutions for the diffraction of a plane shock by general two-dimensional weak disturbances are obtained analytically by a method developed for the aerodynamics of blasts[1,2]. The disturbances which are caused by moving bodies can be described by a combination of distribution of sources (or sinks), doublets and vortices in a two-dimensional case, and by a distribution of point sources (or sinks) in an axisymmetric case. The solution for the disturbance pressure behind an advancing shock in each case is expressed as integrals of known distributions*. The shape of diffracted shock and other disturbance quantities are expressed in terms of disturbance pressure and disturbance velocity components ahead of the shock.

The present solutions can be applied to many practically interesting shock diffraction problems. The simplest example is the diffraction of a shock by a thin structure** in still air. Suppose that the shape of the thin structure is given as $y = h(x')$ or $h(x + (U_o - U)t)$, we can determine the corresponding source distributions based on this shape. Ahead of the shock the air is still, there is no disturbance caused by the pressure of this structure. The source distribution $f_o(x + (U_o - U)t)$ for $x > Ut$ which represents the "zero" disturbance is zero. Behind the shock, the undisturbed flow velocity is (U_o-U) relative to the structure. The disturbance due to the structure can be measured by the y-component disturbance velocity

*Dr. Ting pointed out that the solution of the problem can be interpreted as images of moving strengths.

**For example, a wedge with small wedge angle.

$$v(x \leq Ut,\ y \to 0,\ t) = (U_o - U)\ h'[x + (U_o - U)t] \tag{7.1}$$

This disturbance velocity component should be equal to the value due to a distribution of source on the axis given by

$$v(x \leq Ut,\ y \to 0, t) = \tfrac{1}{2} f[x + (U_o - U)t] \tag{4.16}$$

Equating Eqs.(7.1) and (4.16), we have the corresponding source distribution behind the shock

$$f[x + (U_o - U)t] = 2(U_o - U)\ h'[x + (U_o - U)t] \tag{7.2}$$

If we substitute Eq.(7.2) and $f_o = 0$ into our general solution in Sections V and VI, we will have exactly the same result as given in Ref. 1.

Using the similar procedure, we can determine a point source distribution that represents a slender axisymmetric body. The results of diffraction of a shock by this slender axisymmetric body is consistent with those in Ref. 2. For the case of moving bodies, the distributions corresponding to a body ahead of a shock is no longer zero. Their strengths can be determined from the relative flow velocity and the body shape. Results for shock diffraction of moving bodies shall be presented later in a separate report.

VIII. REFERENCES

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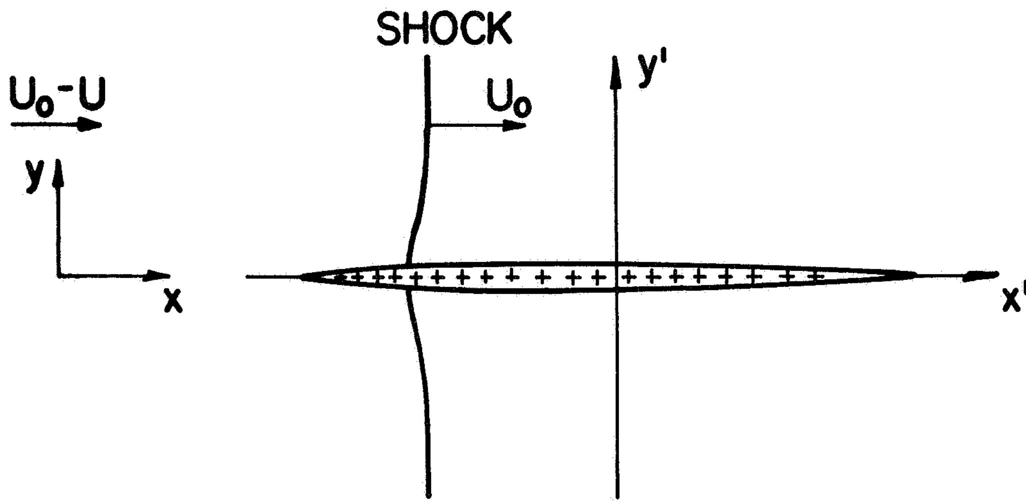
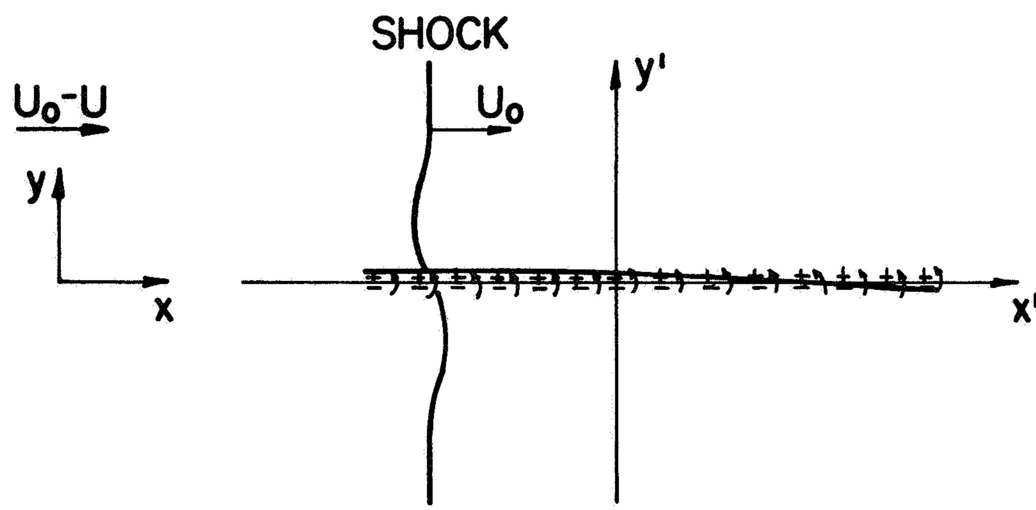


(a)



(b)

Fig. 1 COORDINATES AND PRESCRIBED DISTURBANCES

(a) Two-dimensional or axisymmetric source distribution

(b) Two-dimensional doublet or vortex distribution

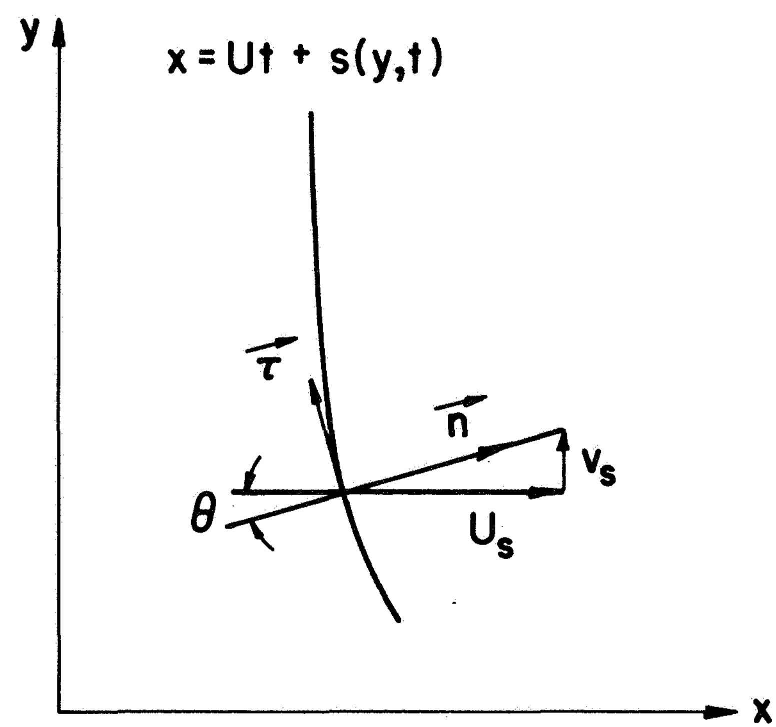


Fig. 2 CURVED SHOCK FRONT

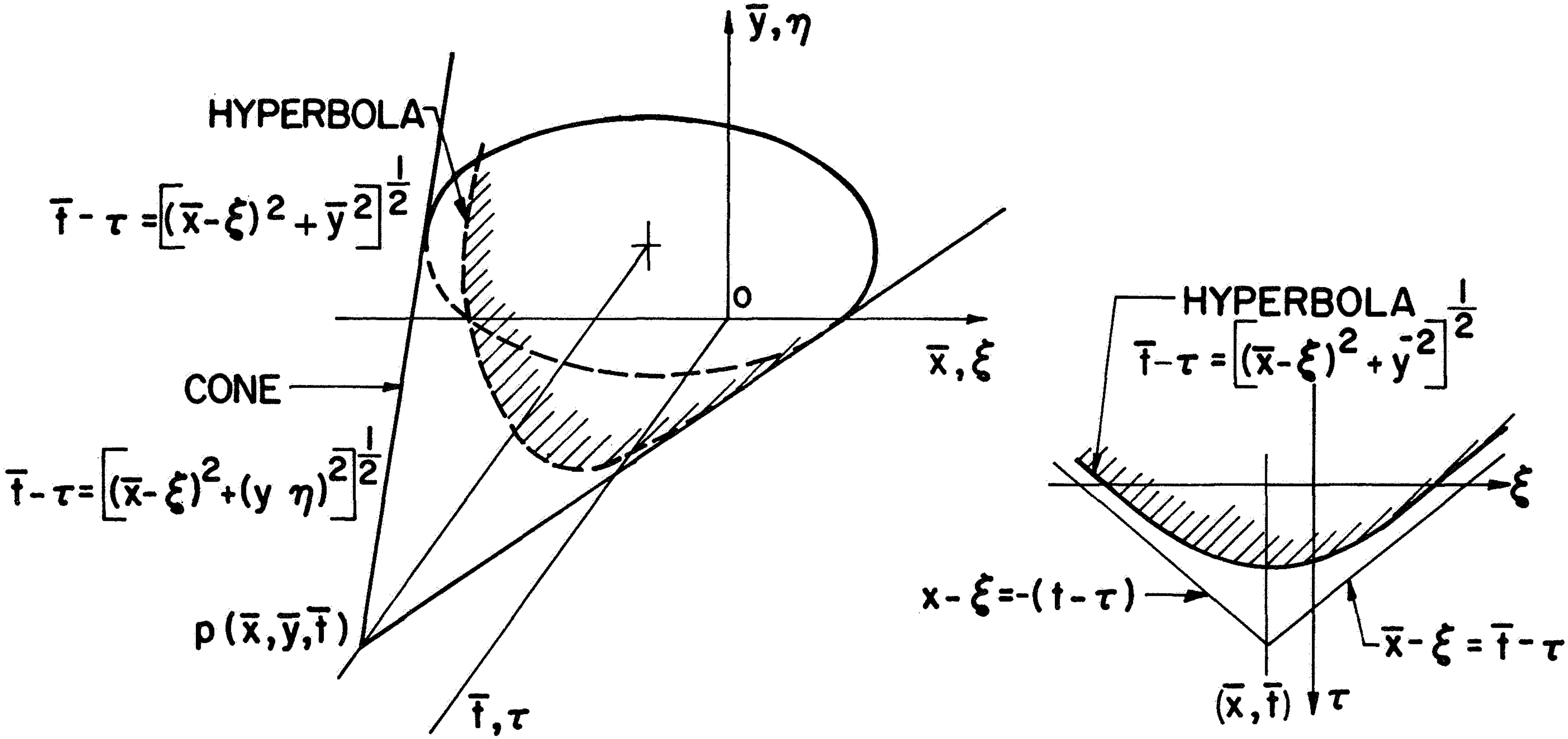


Fig. 3 CONE OF INFLUENCE AND DOMAIN OF INTEGRATION

APPENDIX A

SHOCK BOUNDARY CONDITION

The shock boundary condition in the x,y,t coordinate system has been found to be (Section IV)

$$D_{x,t}\,p(x = Ut,y,t) = G(x = Ut,y,t) \tag{4.11}$$

with

$$D_{x,t} = (\Omega_1 + M + \Omega_2 M)\frac{\partial^2}{\partial t^2} + (1 + M^2 + 2M\Omega_1)c\,\frac{\partial^2}{\partial x\partial t} + M(1 + \Omega_1 M - \Omega_2)\,c^2\,\frac{\partial^2}{\partial x^2} \tag{4.12}$$

and

$$G(x = Ut,y,t) = -Rc\left[\frac{\partial^2 u_o}{\partial t^2} + Mc\,\frac{\partial^2 u_o}{\partial x\partial t} + \frac{2(1-M^2)}{\gamma+1}\,c^2\,\frac{\partial^2 u_o}{\partial x^2}\right] \tag{4.13}$$

By using Lorentz transformation, Eq.(4.11) becomes

$$\bar{D}_{\bar{x},\bar{t}}\;p(\bar{x} = 0,\bar{y},\bar{t}) = \bar{G}(\bar{x} = 0,\bar{y},\bar{t}) \tag{A.1}$$

where

$$\bar{D}_{\bar{x},\bar{t}} = \frac{1}{M_o^2}\,\frac{\partial^2}{\partial\bar{t}^2} + 2M\,\frac{\partial^2}{\partial\bar{t}\partial\bar{x}} + \frac{\partial^2}{\partial\bar{x}^2} \tag{A.2}$$

and

$$\bar{G}(\bar{x} = 0,\bar{y},\bar{t}) = -\,Rc\,\frac{2M}{1-M^2}\left[\frac{2}{\gamma+1}\;\frac{\partial^2 u_o}{\partial\bar{x}^2} - \frac{\gamma+5}{\gamma+1}\,M\,\frac{\partial^2 u_o}{\partial\bar{x}\partial\bar{t}} + \left(1 + \frac{2M^2}{\gamma+1}\right)\frac{\partial^2 u_o}{\partial\bar{t}^2}\right] \tag{A.3}$$

$\bar{G}(\bar{x} = 0, \bar{y}, \bar{t})$ can be written in terms of distributions

Transforming from Eq.(2.15), two-dimensional source distribution is

$$\Phi_s(\bar{x},\bar{y},\bar{t}) = -\frac{1}{2\pi}\int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}}^{\bar{x}+[(\bar{t}-\tau)^2-\bar{y}^2]^{\frac{1}{2}}} d\xi \quad \frac{f_o(\zeta)}{[(\bar{t}-\tau)^2-(\bar{x}-\xi)^2-\bar{y}^2]^{\frac{1}{2}}} + \Phi_{os} \tag{A.4}$$

with

$$\zeta = (1-M^2)^{-\frac{1}{2}}\left[M\tau + \xi + M\frac{U_o-U}{U}\bar{t} + M^2\frac{U_o-U}{U}\bar{x}\right] \tag{A.5}$$

To find disturbance velocities, let us change variables in Eq.(A.4)

$$\bar{t} - \tau = \bar{\tau}, \quad \bar{x} - \xi = \bar{\xi}$$

$$\Phi_s(\bar{x},\bar{y},\bar{t}) = -\frac{1}{2\pi}\int_{\bar{y}}^{\infty} d\bar{\tau}\int_{-(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}}^{(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}} d\bar{\xi} \quad \frac{f_o(\bar{\zeta})}{[\bar{\tau}^2-\bar{\xi}^2-\bar{y}^2]^{\frac{1}{2}}} + \Phi_{os} \tag{A.6}$$

with $\quad \bar{\zeta} = (1-M^2)^{-\frac{1}{2}}\left[M\frac{U_o}{U}\bar{t} + \left(1+M^2\frac{U_o-U}{U}\right)\bar{x} - M\bar{\tau} - \bar{\xi}\right]$ (A.7)

$$u_o(\bar{x},\bar{y},\bar{t}) = \frac{\partial\Phi}{\partial x} = \frac{\partial\Phi}{\partial\bar{x}}\frac{\partial\bar{x}}{\partial x} + \frac{\partial\Phi}{\partial\bar{t}}\frac{\partial\bar{t}}{\partial x}$$

$$= -\frac{1}{2\pi}\int_{\bar{y}}^{\infty} d\bar{\tau}\int_{-(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}}^{(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}} d\bar{\xi}\,\frac{f_o'(\bar{\zeta})}{[\bar{\tau}^2-\bar{\xi}^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$\frac{\partial^2 u_o}{\partial\bar{x}^2} = -\frac{1}{2\pi}\frac{1}{(1-M^2)}\left[1 + M^2\frac{U_o-U}{U}\right]^2\int_{\bar{y}}^{\infty} d\bar{\tau}\int_{-(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}}^{(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}} d\bar{\xi}\,\frac{f_o'''(\bar{\zeta})}{[\bar{\tau}^2-\bar{\xi}^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$\frac{\partial^2 u_o}{\partial\bar{x}\partial\bar{t}} = -\frac{1}{2\pi}\frac{M}{(1-M^2)}\frac{U_o}{U}\left[1 + M^2\frac{U_o-U}{U}\right]\int_{\bar{y}}^{\infty} d\bar{t}\int_{-(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}}^{(\bar{\tau}^2-\bar{y}^2)^{\frac{1}{2}}} d\bar{\xi}\,\frac{f_o'''(\bar{\zeta})}{[\bar{\tau}^2-\bar{\xi}^2-\bar{y}^2]^{\frac{1}{2}}}$$

$$\frac{\partial^2 u_o}{\partial \bar{t}^2} = - \frac{M^2}{(1-M^2)} (U_o/U)^2 \left[1 + M^2 \frac{U_o - U}{U}\right]^2 \int_{\bar{y}}^{\infty} d\bar{\tau} \int_{-(\bar{\tau}^2 - \bar{y})^{\frac{1}{2}}}^{(\bar{\tau}^2 - \bar{y}^2)^{\frac{1}{2}}} d\xi \frac{f_o'''(\bar{\zeta})}{[\bar{\tau}^2 - \bar{\xi}^2 - \bar{y}^2]^{\frac{1}{2}}}$$

substituting into Eq.(A.3), we have

$$\bar{G}(\bar{x},\bar{y},\bar{t}) = \frac{Rc}{\pi} \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \right] \int_{\bar{y}}^{\infty} d\bar{\tau} \int_{-(\bar{\tau}^2 - \bar{y}^2)^{\frac{1}{2}}}^{(\bar{\tau}^2 - \bar{y}^2)^{\frac{1}{2}}} d\bar{\xi} \frac{f_o'''(\bar{\zeta})}{[\bar{\tau}^2 - \bar{\xi}^2 - \bar{y}^2]^{\frac{1}{2}}}$$

or

$$\bar{G}(\bar{x},\bar{y},\bar{t}) = \frac{Rc}{\pi} \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \right] \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{\bar{x}-[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}}^{\bar{x}+[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}} d\xi \frac{f_o'''(\zeta)}{[(\bar{t}-\tau)^2 - (\bar{x}-\xi)^2 - \bar{y}^2]^{\frac{1}{2}}}$$

(A.8)

At shock $\bar{x} = 0$

$$G(\bar{x}=0,\bar{y},\bar{t}) = \frac{Rc}{\pi} \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \right] \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}}^{[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}} d\xi \frac{f_o'''(\zeta)_{\bar{x}=0}}{[(\bar{t}-\tau)^2 - \xi^2 - \bar{y}^2]^{\frac{1}{2}}}$$

(A.9)

Following the similar procedure, we can find $\bar{G}(x = 0,\bar{y},\bar{t})$ for other distributions. For two-dimensional doublet distribution,

$$G(\bar{x}=0,\bar{y},\bar{t}) = \frac{Rc}{\pi} \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \right] \frac{\partial}{\partial \bar{y}} \int_{-\infty}^{\bar{t}-\bar{y}} d\tau \int_{-[(\bar{t}-\tau)^2 - \bar{y}]^{\frac{1}{2}}}^{[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}} d\xi \frac{\mu_o'''(\zeta)_{x=0}}{[(\bar{t}-t)^2 - \xi^2 - \bar{y}^2]^{\frac{1}{2}}}$$

(A.10)

For two-dimensional vortex distribution,

$$G(\bar{x}=0,\bar{y},\bar{t}) = - \frac{Rc}{\pi} \left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \right] \frac{\partial}{\partial \bar{y}} \int_{-\infty}^{\bar{t}-y} d\tau \int_{-[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}}^{[(\bar{t}-\tau)^2 - \bar{y}^2]^{\frac{1}{2}}} d\xi \frac{\nu_o'''(\zeta)_{\bar{x}=0}}{[(\bar{t}-\tau)^2 - \xi^2 - \bar{y}^2]^{\frac{1}{2}}}$$

(A.11)

For axisymmetric source distribution,

$$G(\bar{x}=0,\bar{y},\bar{t}) = \frac{Rc}{2\pi}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right]\int_{-\infty}^{\infty} d\xi \, \frac{g_o'''(\zeta)_{\bar{x}=0}}{[\xi^2+\bar{y}^2]^{\frac{1}{2}}} \qquad \text{(A.12)}$$

APPENDIX B

BOUNDARY CONDITIONS FOR EQS.(5.23) (5.33) AND (5.40)

Following the argument of Ref. 1, a missing boundary condition can be found from a kind of "mean value theorem" at $\bar{x} = 0^+$, $y = 0$, $\bar{t}$. For the case of even disturbances, it gives the relation

$$p_{\bar{y}}(0^+,0,\bar{t}) + p_{\bar{y}}(0^-, 0,\bar{t}) = 2p_{\bar{y}}(0,0^+,\bar{t}) \tag{B.1}$$

$p_{\bar{y}}(0^-,0,\bar{t})$ is obtained from the boundary on $\bar{x}$-axis, Eq.(5.10)

$$p_{\bar{y}}(0^-,0,\bar{t}) = - Rc \frac{U_o-U}{\pi} f'(\bar{a}_o\bar{t}) \tag{B.2}$$

$p_{\bar{y}}(0,0^+,\bar{t})$ can be derived from shock relations Eqs.(4.10c) and (4.10d)

$$p_y = \frac{R}{\Omega_2}\left[(v_t - v_{o_t}) - \frac{2c}{\gamma+1}\frac{M^2-1}{M} u_{o_y}\right] \tag{B.3}$$

Disturbance velocity components can be derived from the prescribed distributions. Hence

$$p_y(x = Ut,\ 0^+,\ t) = \frac{R}{\Omega_2}\left\{\frac{U_o}{2}\left[f'(U_o t) - f_o'(U_o t)\right] - \frac{2c}{\gamma+1}\frac{M^2-1}{M} f_o'(U_o t)\right\}$$

or

$$p_{\bar{y}}(0,0^+,\bar{t}) = Rc\left[\frac{M^2}{M^2-1}\frac{U_o}{c} f'(\bar{a}_o\bar{t}) - \left(\frac{M^2}{M^2-1}\frac{U_o}{c} + \frac{2M}{\gamma+1}\right)f_o'(\bar{a}_o\bar{t})\right] \tag{B.4}$$

Substituting Eqs.(B.2) and (B.4) into Eq.(B.1), we have one of the two boundary conditions for Eq.(5.23).

$$p_{\bar{y}}(0^+,0,\bar{t}) = Rc\left[\left(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{U_o-U}{2c}\right)f'(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\frac{U_o}{c} + \frac{4M}{\gamma+1}\right)f_o'(\bar{a}_o\bar{t})\right] \tag{B.5}$$

Differentiating Eq.(B.5) with respect to $\bar{t}$ and substituting into Eq.(5.24), we have the second boundary condition

$$p_{\bar{y}\bar{x}}(0^+,0,\bar{t}) = -Rc\left\{\left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{U_o-U}{2c}(\bar{\lambda}_o+4M)\right]\bar{a}_o f''(\bar{a}_o\bar{t}) - \left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{8M^2}{\gamma+1}\right]\bar{a}_o f_o''(\bar{a}_o\bar{t})\right\} \tag{B.6}$$

For the case of odd disturbances, the "mean value theorem" at $\bar{x}=0$, $\bar{y}=0^+$, $\bar{t}$ gives the relation

$$p(0^+,0,\bar{t}) + p(0^-,0,\bar{t}) = 2p(0,0^+,\bar{t}) \tag{B.7}$$

$p(0^-,0,\bar{t})$ is obtained from Eq.(5.13),

$$p(0^-,0,\bar{t}) = Rc\,\frac{U_o-U}{2c}F(\bar{a}_o\bar{t}) \tag{B.8}$$

$p(0,0^+,\bar{t})$ can be derived from shock relations Eqs.(4.10)

$$p(0,0^+,\bar{t}) = -Rc\left[\frac{M^2}{M^2-1}\frac{U_o}{c}F(\bar{a}_o\bar{t}) - \left(\frac{M^2}{M^2-1}\frac{U_o}{c}+\frac{2M}{\gamma+1}\right)F_o(\bar{a}_o\bar{t})\right] \tag{B.9}$$

Then from Eqs.(B.7) and (5.34), we have two boundary conditions for Eq.(5.33),

$$p(0^+,0,\bar{t}) = -Rc\left[\left(\frac{2M^2}{M^2-1}\frac{U_o}{c}+\frac{U_o-U}{2c}\right)F(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\frac{U_o}{c}+\frac{4M}{\gamma+1}\right)F_o(\bar{a}_o\bar{t})\right] \tag{B.10}$$

and

$$p_{\bar{y}}(0^+,0,\bar{t}) = Rc\left\{\left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{U_o-U}{2c}(\bar{\lambda}_o+4M)\right]\bar{a}_o F'(\bar{a}_o\bar{t}) - \left[\frac{4M^3}{M^2-1}\frac{U_o}{c}+\frac{8M^2}{\gamma+1}\right]\bar{a}_o F'(\bar{a}_o\bar{t})\right\} \tag{B.11}$$

For the case of axisymmetric disturbances, the "mean value theorem" gives

$$\bar{y}p_{\bar{y}}\,(0^+,0,\bar{t}) + \bar{y}p_{\bar{y}}(0^-,0,\bar{t}) = 2\bar{y}p_{\bar{y}}(0,0^+,\bar{t}) \tag{B.12}$$

$\bar{y}p_{\bar{y}}(0^+,0,\bar{t})$ is obtained from Eq.(5.14)

$$\bar{y}p_{\bar{y}}\,(0^+,0,\bar{t}) = -\frac{Rc}{\pi}\,\frac{U_o-U}{2c}\,g'(\bar{a}_o\bar{t}) \tag{B.13}$$

$\bar{y}p_{\bar{y}}\,(0,0^+,\bar{t})$ can be obtained from shock relations Eq.(4.10)

$$\bar{y}p_{\bar{y}}(0,0^+,\bar{t}) = \frac{Rc}{\pi}\left[\left(\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{U_o-U}{2c}\right) g'(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{4M}{\gamma+1}\right) g_o'\,(\bar{a}_o\bar{t})\right] \tag{B.14}$$

Then from Eqs.(B.12) and (5.41), we have two boundary conditions for Eq.(5.40)

$$\bar{y}p_{\bar{y}}(0^+,0,\bar{t}) = \frac{Rc}{\pi}\left[\left(\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{U_o-U}{2c}\right) g'(\bar{a}_o\bar{t}) - \left(\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{4M}{\gamma+1}\right) g_o'(\bar{a}_o\bar{t})\right]$$

and

$$\bar{y}p_{\bar{y}\bar{x}}\,(0^+,0,\bar{t}) = -\frac{Rc}{\pi}\left\{\left[\frac{4M^3}{M^2-1}\,\frac{U_o}{c} + \frac{U_o-U}{2c}\,(\bar{\lambda}_o + 4M)\right]\bar{a}_o g''(\bar{a}_o\bar{t}) - \left[\frac{4M^3}{M^2-1}\,\frac{U_o}{U} + \frac{8M^2}{\gamma+1}\right]\bar{a}_o g''(\bar{a}_o\bar{t})\right\} \tag{B.16}$$

APPENDIX C

SOLUTIONS OF EQS.(5.23),(5.33) AND (5.40) WITH THEIR BOUNDARY CONDITIONS

A. Even Disturbances

The solution of differential equation (5.23) canbe written in the following form

$$p_{\bar{y}}(\bar{x} > 0,0,\bar{t}) = \Gamma_{e1}(\bar{t}-\bar{\lambda}_1\bar{x}) + \Gamma_{e2}(\bar{t}-\bar{\lambda}_2\bar{x})$$

$$+ \text{Rc}\,\frac{U_o-U}{2c}\cdot\frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}\, f'[\bar{a}_o(\bar{t}-\bar{\lambda}_o\bar{x})]$$

$$- \text{Rc}\,\frac{1}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\right]$$

$$\left\{\frac{1}{H(-\bar{\lambda})}\, f_o'[\bar{a}_o(\bar{t}+\bar{\lambda}\bar{x})]\right. \tag{C.1}$$

$$\left.+\frac{1}{H(\bar{\lambda})}\, f_o'[\bar{a}_o(\bar{t}-\bar{\lambda}\bar{x})]\right\}$$

where $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are two real, distinct and possitive roots of the quadratic equat ion,

$$H(\bar{\lambda}) = \bar{\lambda}^2 - 2M\,\bar{\lambda} + \frac{1}{M_o^2} = 0 \tag{C.2}$$

Γ_{e1} and Γ_{e2} are two arbitrary functions to be determined by two boundary conditions Eqs.(5.25) and (5.26)

$$\Gamma_{e1}(\bar{t}) + \Gamma_{e2}(\bar{t}) = \text{Rc}\left\{\left[\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{U_o-U}{2c}\left(1 - \frac{H(-\lambda_o)}{H(\lambda_o)}\right)\right] f'(\bar{\lambda}_o\bar{t})\right.$$

$$\left.- \left[\frac{2M^2}{M^2-1}\,\frac{U_o}{c} + \frac{4M}{\gamma+1} - \frac{1}{\bar{a}_o^2}\,\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\left(\frac{1}{H(-\lambda)} + \frac{1}{H(\lambda)}\right)\right] f_o'(\bar{a}_o\bar{t})\right\} \tag{C.3}$$

$$\bar{\lambda}_1\Gamma'_{e1}(\bar{t}) + \bar{\lambda}_2\Gamma'_{e2}(\bar{t}) = Rc\left\{\left[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{U_o-U}{2c}\left(\bar{\lambda}_o[1 - \frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}] + 4M\right)\right]\bar{a}_o f''(\bar{a}_o\bar{t})\right.$$

$$\left. - \left[\frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{8M^2}{\gamma+1} + \frac{\bar{\lambda}}{\bar{a}_o^2}\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\left(\frac{1}{H(-\bar{\lambda})} - \frac{1}{H(\bar{\lambda})}\right)\right]\bar{a}_o f''_o(\bar{a}_o\bar{t})\right\} \tag{C.4}$$

This leads to

$$\Gamma_{e1}(\bar{t}) = Rc\ [A_1 f'(\bar{a}_o\bar{t}) + B_1 f'_o(\bar{a}_o\bar{t})] \tag{C.5}$$

$$\Gamma_{e2}(\bar{t}) = Rc\ [A_2 f'(\bar{a}_o\bar{t}) + B_2 f'_o(\bar{a}_o\bar{t})] \tag{C.6}$$

Substituting Eqs.(C.5) and (C.6) into Eqs.(C.3) and (C.4) and equating coefficients of $f'(\bar{a}_o\bar{t})$ and $f'_o(\bar{a}_o\bar{t})$, we have

$$A_1 + A_2 = \frac{2M^2}{M^2-1}\frac{U_o}{U} + \frac{U_o-U}{2c}\left(1 - \frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}\right) \tag{C.7}$$

$$\bar{\lambda}_1 A_1 + \bar{\lambda}_2 A_2 = \frac{4M^3}{M^2-1}\frac{U_o}{c} + \frac{U_o-U}{2c}\left[\bar{\lambda}_o\left(1 - \frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)}\right) + 4M\right] \tag{C.8}$$

and

$$B_1 + B_2 = -\frac{2M^2}{M^2-1}\frac{U_o}{c} - \frac{4M}{\gamma+1} + \frac{1}{\bar{a}_o^2}\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\left[\frac{1}{H(-\bar{\lambda})} + \frac{1}{H(\bar{\lambda})}\right] \tag{C.9}$$

$$\bar{\lambda}_1 B_1 + \bar{\lambda}_2 B_2 = -\frac{4M^3}{M^2-1}\frac{U_o}{c} - \frac{8M^2}{\gamma+1} - \frac{\bar{\lambda}}{\bar{a}_o}\frac{4(\gamma M^2+1)}{(\gamma+1)^2 M}\left[\frac{1}{H(-\bar{\lambda})} - \frac{1}{H(\bar{\lambda})}\right] \tag{C.10}$$

since $\begin{vmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{vmatrix} \neq 0$

Therefore A_1, A_2, B_1 and B_2 can be determined uniquely from Eqs.(C.7) to (C.10).

B. Odd Disturbances

The solution of differential equation (5.33) can be written as

$$p(\bar{x} > 0,0,\bar{t}) = \Gamma_{01}(\bar{t} - \bar{\lambda},\bar{x}) + \Gamma_{02}(\bar{t} - \lambda_2\bar{x})$$

$$- Rc \frac{U_o-U}{2c} \frac{H(-\bar{\lambda}_o)}{H(\bar{\lambda}_o)} F[\bar{a}_o(\bar{t} - \bar{\lambda}_o\bar{x})]$$

$$+ Rc \frac{1}{\bar{a}_o^2} \frac{4(\gamma M^2+1)}{(\gamma+1)^2 M} \left\{ \frac{1}{H(-\bar{\lambda})} F_o[\bar{a}_o(\bar{t} + \bar{\lambda}\bar{x})] \right. \tag{C.11}$$

$$\left. + \frac{1}{H(\bar{\lambda})} F_o[\bar{a}_o(\bar{t} - \bar{\lambda}\bar{x})] \right\}$$

Two arbitrary functions Γ_{01} and Γ_{02} can be written in the following form

$$\Gamma_{01}(\bar{t}) = - Rc\, [A_1F(\bar{a}_o\bar{t}) + B_1F_o(\bar{a}_o\bar{t})] \tag{C.12}$$

$$\Gamma_{02}(\bar{t}) = - Rc\, [A_2F(\bar{a}_o\bar{t}) + B_2F_o(\bar{a}_o\bar{t})] \tag{C.13}$$

A_1, A_2, B_1 and B_2 are determined uniquely from Eqs.(C.7) to (C.10).

C. Axisymmetric Disturbances

The solution of differential equation (5.40) can be written as

$$\bar{y}p_{\bar{y}}(\bar{x} > 0,0,\bar{t}) = \Gamma_{a1}(\bar{t} - \bar{\lambda}_1\bar{x}) + \Gamma_{a2}(\bar{t} - \bar{\lambda}_2\bar{x})$$

$$+ \frac{Rc}{\pi} \frac{U_o-U}{2c} \frac{H(-\lambda_o)}{H(\bar{\lambda}_o)} g'[\bar{a}_o(\bar{t} - \bar{\lambda}_o\bar{x})]$$

$$- \frac{Rc}{\pi} \frac{1}{\bar{a}_o^2} \frac{4(\gamma M^2+1)}{(1-M^2)^{1/2}(\gamma+1)^2 M} \left\{ \frac{1}{H(-\bar{\lambda})} g_o'[\bar{a}_o(\bar{t} + \bar{\lambda}\bar{x})] \right. \tag{C.14}$$

$$\left. + \frac{1}{H(\bar{\lambda})} g_o'[\bar{a}_o(\bar{t} - \bar{\lambda}\bar{x})] \right\}$$

Two arbitrary functions Γ_{a1} and Γ_{a2} can be written in the following form

$$\Gamma_{a1}(\bar{t}) = \frac{Rc}{\pi}\left[A_1 g'(\bar{a}_o\bar{t}) + C_1 g_o'(\bar{a}_o\bar{t})\right] \quad (C.15)$$

$$\Gamma_{a2}(\bar{t}) = \frac{Rc}{\pi}\left[A_2 g'(\bar{a}_o\bar{t}) + C_2 g_o'(\bar{a}_o\bar{t})\right] \quad (C.16)$$

A_1 and A_2 are determined from Eqs. (C.7) and (C.8). C_1 and C_2 are determined from

$$C_1 + C_2 = -\frac{2M^2}{M^2-1}\frac{U_o}{c} - \frac{4M}{\gamma+1} + \frac{1}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right] \quad (C.17)$$

$$\left[\frac{1}{H(-\bar{\lambda})} + \frac{1}{H(\bar{\lambda})}\right]$$

and

$$\bar{\lambda}_1 C_1 + \bar{\lambda}_2 C_2 = -\frac{4M^3}{M^2-1}\frac{U_o}{c} - \frac{8M^2}{\gamma+1} - \frac{\bar{\lambda}}{\bar{a}_o^2}\left[\frac{4(\gamma M^2+1)}{(1-M^2)^{\frac{1}{2}}(\gamma+1)^2 M}\right] \quad (C.18)$$

$$\left[\frac{1}{H(-\bar{\lambda})} - \frac{1}{H(\bar{\lambda})}\right]$$

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13. ABSTRACT

Solutions for the diffraction of a plane shock wave by general two-dimensional weak disturbances are obtained. The technique employed is an extension of the method developed by Ting and Ludloff for the solution of aerodynamics of blasts. Disturbances due to a solid body are prescribed by distributions of sources, doublets and vortices in a two-dimensional case and by a distribution of point sources in an axisymmetric case. The disturbance pressure behind an advancing shock is expressed by integrals of distributions. The shape of diffracted shock and other disturbance quantities are expressed in terms of disturbance pressure behind the shock and disturbance velocity components ahead of the shock. Application to shock diffraction of thin structure in still air is shown. Some other applications are indicated.

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It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

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