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POWER SERIES SOLUTIONS OF THE THIELE-BURRAU REGULARIZED PLANAR RESTRICTED THREE-BODY PROBLEM

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RESTRICTED THREE-BODY PROBLEM

Regularizing transformations which remove the singularities at the attracting masses from the equations of motion of the planar restricted problem of three-bodies are well known. They have been developed by Thiele, Burrau, Levi-Civita, Birkhoff, Lemaitre, etc.,[3] to study the solution of the equations of motion at a collision and to facilitate numerical integration of collision orbits.

The planar restricted problem of three bodies constrains two primary masses to move in circular orbits about one another while a third body of infinitesimal mass moves in their plane of motion and exerts no force on the primaries. Regularization calls for the elimination of one or both of the two singularities r_1^{-1} and r_2^{-1}, where r_1 and r_2 are the distances between the two primaries and the third body. By simultaneous, or global regularization, we mean the process which removes both singularities with one transformation, whereas by local regularization we mean the removal of only one singularity by a single transformation.

Global regularizations generally result in equations of motion which are considerably more complicated than either the original equations or the equations obtained when a local transformation such as that of Levi-Civita is used. To effect a solution by power series, the need for higher-order derivatives exists. Szebehely [4] has proposed a general method based on differentiating the equation of motion which is straightforward but somewhat cumbersome. It is particularly inadequate when an optimal power series integration technique, such as that of Van Flandern[5] is desired. With this method, the integration step size at each step is optimized with respect to computing time so that a large number of high-order derivatives may be needed.

The global transformation proposed by Thiele (1895) and Burrau (1906) has advantages over the other global regularizations since the equations of motion in this system contain only derivatives and transcendental functions of the dependent variables. Since the trigonometric and hyperbolic functions in the equations themselves satisfy simple differential equations, the Thiele-Burrau regularization is particularly amenable to the calculation of the series coefficients by recurrent power series techniques. [1]

THIELE-BURRAU REGULARIZATION

The equations of motion in a coordinate system with the origin at the mid-point between the primaries and rotating with the mean motion are[4]

$$\ddot{z} + 2i\dot{z} = \nabla_z \Phi \tag{1}$$

where

$$z = x + iy$$

$$\dot{z} = dz/dt$$

$$\nabla_z \Phi = \frac{\partial \Phi}{\partial x} + i \frac{\partial \Phi}{\partial y}$$

$$\Phi = \frac{1}{2}\left[(1-\mu)r_1^2 + \mu r_2^2\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

and where μ is the ratio of the mass of the smaller primary to the sum of the two primary masses. Here r_1 and r_2 represent the distances between the primaries and the infinitesimal third body, and assuming that the more massive primary is to the right of the origin we obtain

$$r_1 = |z - \tfrac{1}{2}| \tag{2}$$

$$r_2 = |z + \tfrac{1}{2}| \; .$$

The first integral of the equations of motion is

$$J = 2\Phi - |\dot{z}|^2 \tag{3}$$

where J is the Jacobi constant of motion.

To regularize the equations of motion, transformations

$$z = f(w) \tag{4}$$

and

$$dt = \left| \frac{df(w)}{dw} \right|^2 d\tau \tag{5}$$

are introduced into Equation (1), where $w = \mu + iv$ and where the function $f(w)$ is required to be such that the singularities of the equations of motion are removed. The result of this procedure is[4]

$$w'' + 2i \left| \frac{df}{dw} \right|^2 w' = \nabla_w \Phi^* \tag{6}$$

where

$$w' = dw/d\tau ,$$

$$\Phi^* = \left| \frac{df}{dw} \right|^2 \left[\Phi - \frac{J}{2} \right]. \tag{7}$$

The global regularizing transformation of Thiele-Burrau is defined by Equation (4) with

$$z = f(w) = \frac{1}{2} \cos w \tag{8}$$

We then find immediately from Equations (2), (6), and (7)

$$r_1 = \frac{1}{2}(\cosh v - \cos u) \tag{9}$$

$$r_2 = \frac{1}{2}(\cosh v + \cos u) \tag{10}$$

$$\left| \frac{df}{dw} \right|^2 = r_1 r_2 = \frac{1}{4}(\cosh^2 v - \cos^2 u) \tag{11}$$

$$\Phi^* = r_1 r_2 \left(\Phi - \frac{1}{2} J\right). \tag{12}$$

The real and imaginary parts of Equation (6) yield

$$u'' - 2 r_1 r_2 v' = \frac{\partial \Phi^*}{\partial u} \tag{13}$$

$$v'' + 2 r_1 r_2 u' = \frac{\partial \Phi^*}{\partial v} \tag{14}$$

Substituting Equations (9), (10) and (1) into Equation (12), we have

$$\begin{aligned} \Phi^* &= r_1 r_2 \left(\Phi - \frac{1}{2} J\right) \\ &= \frac{1}{2} r_1 r_2 \left[(1-\mu) r_1^2 + \mu r_2^2 - J\right] + (1-\mu) r_2 + \mu r_1 \\ &= \frac{1}{16}(\cosh 2v - \cos 2u) \left[\frac{1}{8}(\cosh 2v + \cos 2u) - \left(\frac{1}{2} - \mu\right) \cos u \cosh v \right. \\ &\qquad \left. + \frac{1}{4} - J\right] + \frac{1}{2} \cosh v + \left(\frac{1}{2} - \mu\right) \cos u \end{aligned} \tag{15}$$

and it follows that

$$\begin{aligned} u'' - 2 r_1 r_2 v' &= \frac{1}{16} \sin u \left[8(2\mu - 1) - 4 J \cos u \right. \\ &\qquad \left. + (2\mu - 1) \cosh v\,(3 \cos^2 u - \cosh^2 v) + 2 \cos^3 u\right] \end{aligned} \tag{16}$$

$$\begin{aligned} v'' + 2 r_1 r_2 u' &= \frac{1}{16} \sinh v \left[8 - 4 J \cosh v \right. \\ &\qquad \left. + (2\mu - 1) \cos u\,(3 \cosh^2 v - \cos^2) + 2 \cosh^3 v\right] \end{aligned} \tag{17}$$

which are the globally regularized equations of motion for the infinitesimal third body under the transformation of Thiele-Burrau. The dimensionless coordinates x, y and time t may be recovered by the formulas

$$x = \frac{1}{2} \cos u \cosh v \tag{18}$$

$$y = -\frac{1}{2} \sin u \sinh v \tag{19}$$

$$t = \frac{1}{4} \int_{\tau_0}^{\tau} (\cosh^2 v - \cos^2 u)\, d\tau, \tag{20}$$

while the dimensionless velocities may be obtained by applying Equation (5) to Equations (18) and (19). The result is

$$\dot{x} = \frac{(v' \cos u \sinh v - u' \sin u \cosh v)}{2 r_1 r_2} \tag{21}$$

$$\dot{y} = -\frac{(v' \sin u \cosh v + u' \cos u \sinh v)}{2 r_1 r_2} \tag{22}$$

RECURRENT POWER SERIES SOLUTION

The solution of Equations (16) and (17) is assumed to be represented by the power series

$$u = \sum_{i=0}^{\infty} u_i \tau^i \tag{23}$$

$$v = \sum_{i=0}^{\infty} v_i \tau^i \tag{24}$$

where the coefficients are to be determined recursively.

Define

$$a = \sin u = \sum_{i=0}^{\infty} a_i \tau^i \tag{25}$$

$$b = \cos u = \sum_{i=0}^{\infty} b_i \tau^i \tag{26}$$

$$c = \sinh v = \sum_{i=0}^{\infty} c_i \tau^i \tag{27}$$

$$d = \cosh v = \sum_{i=0}^{\infty} d_i \tau^i \tag{28}$$

$$p = b^2 = \sum_{i=0}^{\infty} p_i \tau^i \tag{29}$$

$$q = d^2 = \sum_{i=0}^{\infty} q_i \tau^i \tag{30}$$

$$f = -\frac{1}{2}J + \frac{(2\mu - 1)}{8} d\,(3p - q) + \frac{1}{4} bp = \sum_{i=0}^{\infty} f_i \tau^i \tag{31}$$

$$g = -\frac{1}{2}J + \frac{(2\mu - 1)}{8} b\,(3q - p) + \frac{1}{4} dq = \sum_{i=0}^{\infty} g_i \tau^i . \tag{32}$$

Then Equations (16) and (17) may be written as

$$2u'' + (p - q)\,v' = (2\mu - 1)\,a + af \tag{33}$$

$$2v'' - (p - q)\,u' = c + cg . \tag{34}$$

Further, from Equations (25)-(28) we have the relations

$$a' = bu' \tag{35}$$

$$b' = -au' \tag{36}$$

$$c' = dv' \tag{37}$$

$$d' = cv' \tag{38}$$

Substituting the series expansions into the auxiliary relations (35)-(38), (31), (32) and into the equations of motion, (33) and (34) and equating coefficients of equal powers of τ we arrive at the recurrence relations

$$i\, a_i = \sum_{j=1}^{i} j\, u_j\, b_{i-j} \tag{39}$$

$$i\, b_i = -\sum_{j=1}^{i} j\, u_j\, a_{i-j} \tag{40}$$

$$i\, c_i = \sum_{j=1}^{i} j\, v_j\, d_{i-j} \tag{41}$$

$$i\, d_i = \sum_{j=1}^{i} j\, v_j\, c_{i-j} \tag{42}$$

for $i \geq 1$, and

$$p_i = \sum_{j=0}^{i} b_j\, b_{i-j} \tag{43}$$

$$q_i = \sum_{j=0}^{i} d_j \, d_{i-j} \tag{44}$$

$$f_i = -\frac{1}{2} J \, b_i + \frac{1}{8} \sum_{j=0}^{i} \left[\left(3(2\mu - 1) \, d_j + 2b_j \right) p_{i-j} - (2\mu - 1) \, d_j \, q_{i-j} \right] \tag{45}$$

$$g_i = -\frac{1}{2} J \, d_i + \frac{1}{8} \sum_{j=0}^{i} \left[\left(3(2\mu - 1) \, b_j + 2d_j \right) q_{i-j} - (2\mu - 1) \, b_j \, p_{i-j} \right] \tag{46}$$

$$2(i+1)(i+2) \, u_{i+2} = (2\mu - 1) \, a_i + \sum_{j=0}^{i} \left[a_j \, f_{i-j} - (j+1) \, v_{j+1} (p_{i-j} - q_{i-j}) \right] \tag{47}$$

$$2(i+1)(i+2) \, v_{i+2} = c_i + \sum_{j=0}^{i} \left[c_j \, g_{i-j} + (j+1) \, u_{j+1} (p_{i-j} - q_{i-j}) \right] \tag{48}$$

for $i \geq 0$. It is to be noted that for optimal computing advantage may be taken of the symmetry in Equations (43) and (44) to nearly halve the arithmetic operations involved in these recursions.

Initial conditions of the problem yield u_0, v_0, u_1 and v_1. Then

$$a_0 = \sin u_0$$

$$b_0 = \cos u_0$$

$$c_0 = \sinh v_0$$

$$d_0 = \cosh v_0 \, .$$

With these values at hand, proceeding through the above algorithm will determine all higher-order power series coefficients in terms of the preceding coefficients, so that the solution may be extended optimally by analytic continuation using a variable step size and a variable number of terms in the power series expansion for each integration step.

This power series method for the solution of the Thiele-Burrau Regularized Planar Restricted Three-Body Problem was programmed in double precision on an IBM 360-91 computer using the optimal integration procedures outlined in Reference (1) and compared with the non-regularized recursive power series method of the same paper. As an indication of applicability, the periodic orbit of Figure 1[2] in the Earth-Moon system ($\mu = 1/82.45$) with period

$$P = 6.19216933313196$$

and initial conditions

$$x_0 = -.7121285627653111$$

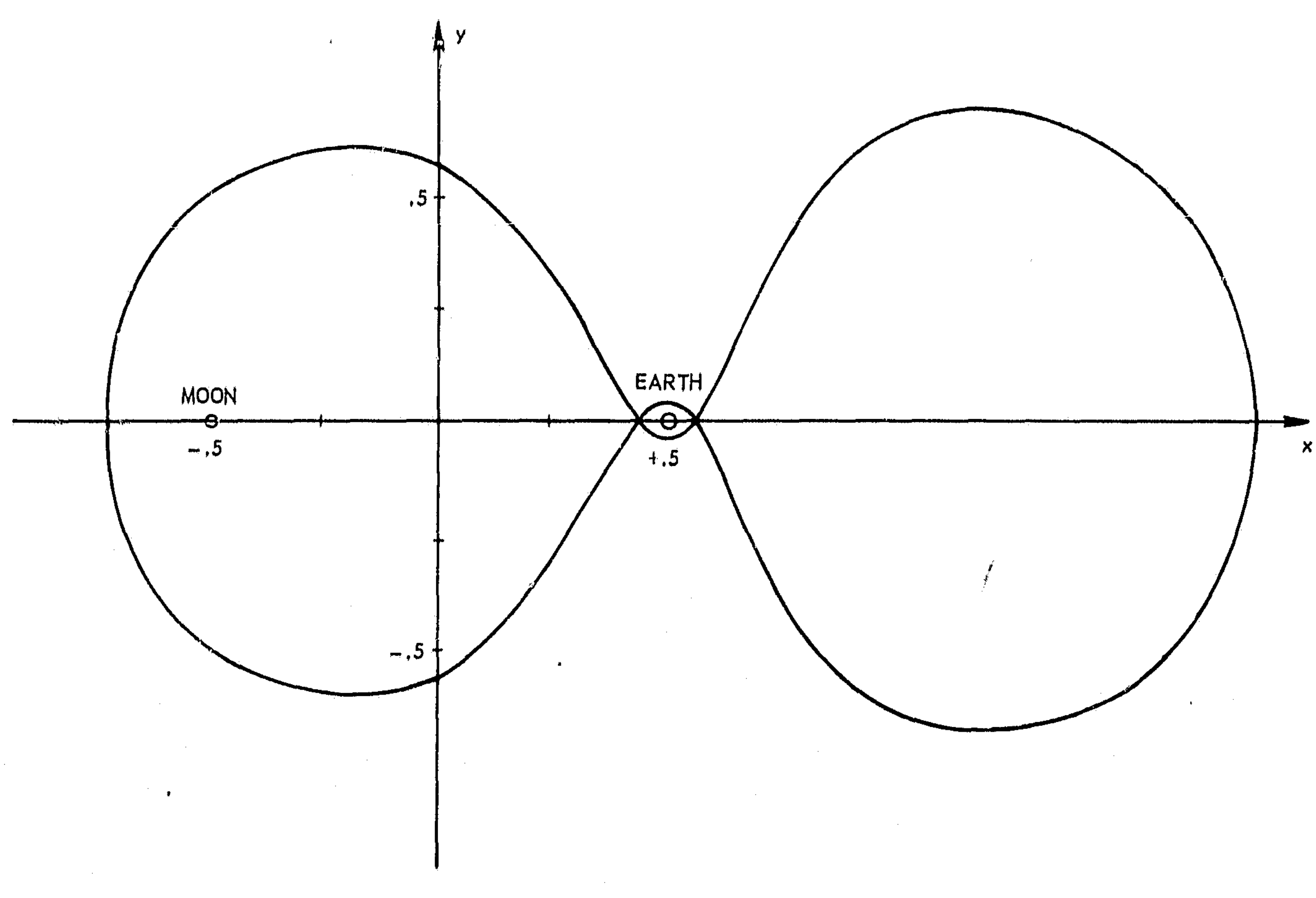

Figure 1

$$y_0 = 0.$$

$$\dot{x}_0 = 0.$$

$$\dot{y}_0 = 1.04935750983033$$

reveals a 33% decrease in computing time as well as two more digits of accuracy in the Jacobi constant after one complete orbital integration.

Thus, although the full power of this integration technique will be realized with collisions or near collisions involving both primaries, the above example demonstrates remarkable results in less extreme problems.

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