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COMPUTATION OF OPTIMAL SINGULAR CONTROLS

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ABSTRACT

A class of singular control problems is made non-singular by the addition of an integral quadratic functional of the control to the cost functional; a parameter $\epsilon > 0$ multiplies this added functional. The resulting non-singular problem is solved for a monotone decreasing sequence of ϵ's; $\epsilon_1 > \epsilon_2 \ldots > \epsilon_k > 0$. As $k \longrightarrow \infty$, and $\epsilon_k \longrightarrow 0$, the solution of the modified problem tends to the solution of the original singular problem. A variant of the method which does not require that $\epsilon \longrightarrow 0$ is also presented. Four illustrative numerical examples are described.

1. Introduction

In recent years singular control problems have received attention [1]-[8].† However, researchers have concerned themselves mainly with necessary conditions of optimality, and the computation of singular extremals appears to have been ignored except for the experiments of Bass [4], Kelley [9], Johansen [10] and Pagurek [11]. Johansen has pointed out that the convergence of the gradient method on singular control problems is slow indeed. Jacobson and Lele solved singular problems arising in [12] by using the conjugate gradient method [13]; the convergence rate was acceptable, but in that class of problems no control constraints are present, so that the conjugate gradient method can be applied without modification. When control constraints are present, the conjugate gradient method should be modified‡; one possible modification has been suggested [11].

In [15] a second-order algorithm is described for solving optimal, non-singular, control problems with control variable inequality constraints. Being second-order, the algorithm exhibits quadratic convergence when the priming trajectory is sufficiently close to the optimal. In this paper, a class of singular control problems is made non-singular by the addition of an integral quadratic functional of the control to the cost functional; a parameter $\epsilon > 0$ multiplies this added functional. The algorithm of [15]‡‡ is then used to successively solve

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† Many additional references are given in [3], [5] and [6].

‡ Alternatively, the control constraints can be transformed away, using Valentine's device [14].

‡‡ The control inequality constraints could be transformed away, using Valentine's device; this would allow one to use an algorithm for unconstrained problems which is also described in [15].

this modified problem for a monotone decreasing sequence of ϵ's; $\epsilon_1 > \epsilon_2 \ldots > \epsilon_k > 0$. As $k \longrightarrow \infty$, and $\epsilon_k \longrightarrow 0$, the solution of the modified problem tends to the solution of the original singular problem. A proof of convergence of the method is given in Section 5. The method is similar to the penalty function techniques of solving state variable constrained problems [16], [17] and [18]. However, here the 'penalty' is successively reduced rather than increased. As ϵ tends to zero, so the modified problem tends to become singular, which may result in computational sensitivity. A variant† of the modified problem is described which allows the use of a small value of ϵ, which is not made to approach zero whilst still ensuring convergence to the required singular control. A proof of convergence of this variant is given in Section 5. The techniques described in this paper are equally applicable to purely bang-bang control problems, and provide alternatives to the methods described in [21] and [22]. Four small control problems are solved to illustrate the usefulness of the methods. It is hoped that in a future paper, the computation of a seven state variable, three control, nearly singular, model of a binary distillation column will be described.

2. Preliminaries

Consider a dynamical system described by the differential equation:

$$\dot{x} = f_1(x,t) + f_u(x,t)u \quad ; \quad x(t_o) = x_o \tag{1}$$

Here, x is an n-dimensional state vector and u is a scalar control. f_1 and f_u are n-dimensional vector functions. The control u is constrained in the following way:

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† Similar notions have been used by Ho [19] and Powell [20] in penalty function methods.

$$|u(t)| \leq 1 \quad ; \quad t \in [t_o, t_f] \tag{2}$$

The performance index, or cost functional, is:

$$\hat{V}[u(\cdot)] = \int_{t_o}^{t_f} L(x,t)dt + F(x(t_f)) \tag{3}$$

where the final time t_f is given explicitly. The functions f_1, f_u, L and F are assumed to be three times continuously differentiable in each argument.

The object of the control problem is to choose the control function $u(\cdot)$ to satisfy (2) and minimize $\hat{V}[u(\cdot)]$. It is well known that the optimal control function for this class of problems consists, in general, of bang-bang and singular sub-arcs. Whilst purely bang-bang optimal controls can be calculated, using, for example, the methods described in [21] and [22], determination of optimal controls consisting of both bang-bang and singular sub-arcs is not straightforward.

3. The ϵ-Algorithm

In place of (3), consider the modified cost functional

$$V[u(\cdot), \epsilon_k] = \int_{t_o}^{t_f} [L(x,t) + \frac{\epsilon_k}{2} u^2]dt + F(x(t_f)) \tag{4}$$

where

$$\epsilon_k > 0 \tag{5}$$

Define the following as the "ϵ-problem": minimize (4) subject to (1), (2). The ϵ-problem is non-singular, and can be solved using, say, the second-order algorithm described in [15].

Description of the ϵ-Algorithm:

1) Choose a starting value $\epsilon_1 > 0^{\dagger}$, and a nominal control function $\bar{u}_1(\cdot)$.

$\dagger$ The starting value ϵ_1 may be chosen heuristically.

2) Solve the resulting ϵ_k-problem ($k = 1$, initially) using the algorithm of [15]; this yields a minimizing control function $u_k(\cdot)$.

3) Choose $\epsilon_{k+1} < \epsilon_k$ †, and set $\bar{u}_{k+1}(\cdot) = u_k(\cdot)$, $k = k + 1$ and go to 2).

4) Computation is terminated when:

either, a) ϵ_k is so small that numerical instability occurs

or, b) $\epsilon_k < \sigma$, σ a small, pre-determined positive quantity.

In Section 5 it is proved that as $k \longrightarrow \infty$, so $u_k(\cdot) \longrightarrow u^o(\cdot)$ -- the optimal control function of the original problem.

In the above algorithm, as $k \longrightarrow \infty$, so the modified problem tends to become singular; this can lead to numerical difficulties ‡ when using the algorithm of [15] to solve the modified problem. A variant of the 'ϵ-Algorithm' which overcomes these difficulties is described in the next section.

4. The ϵ-$\alpha(\cdot)$-Algorithm

The algorithm of Section 3 is used until ϵ is reduced to a small value. ‡‡ Let this value of ϵ be denoted ϵ_N.

Consider the cost functional:

$$W[u(\cdot), \epsilon_N, \alpha_s(\cdot)] = \int_{t_o}^{t_f} [L(x,t) + \frac{\epsilon_N}{2}(u - \alpha_s)^2]dt + F(x(t_f)) \quad . \tag{6}$$

Define the following as the "ϵ-$\alpha(\cdot)$-problem": minimize (6) subject to (1), (2), where $\alpha(\cdot)$ is some piecewise continuous function defined on $t_o \leqslant t \leqslant t_f$.

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† The amount by which ϵ is reduced at each step 3) depends on the particular problem; setting $\epsilon_{k+1} = \epsilon_k/10$ was found to be adequate in the problems tried (Section 6).

‡ Similar difficulties occur when using penalty functions [16]-[19].

‡‡ Two possible criteria for deciding the smallness of ϵ are given in Section 6.

Description of the ϵ-$\alpha(\cdot)$-Algorithm:

1) Set $\alpha_1(\cdot) = u_N(\cdot)$ and $\bar{u}_1(\cdot) = u_N(\cdot)$.

2) Solve the resulting ϵ_N-$\alpha_s(\cdot)$-problem (s = 1 initially) using the algorithm of [15]. This yields a minimizing control function $u_s(\cdot)$.

3) Set $\bar{u}_{s+1} = u_s(\cdot)$, $\alpha_{s+1}(\cdot) = u_s(\cdot)$, and s = s + 1 and go to 2).

4) Computation is terminated when:

$$\int_{t_o}^{t_f} (u_s - \alpha_s)^2 dt < \sigma \quad ; \quad \sigma \text{ a small positive quantity.}$$

In Section 5 it is proved that as $s \longrightarrow \infty$, so $u_s(\cdot) \longrightarrow u^o(\cdot)$, provided that ϵ_N is sufficiently small.

5. Proofs of Convergence

a) ϵ-Algorithm

Minimize:
$u(\cdot)$

$$V[u(\cdot), \epsilon_k] = \int_{t_o}^{t_f} [L(x,t) + \frac{\epsilon_k}{2} u^2] dt + F(x(t_f)) \tag{7}$$

where:

$$\dot{x} = f_1(x,t) + f_u(x,t)u \tag{8}$$

and:

$$|u(t)| \leqslant 1 \quad ; \quad t \in [t_o, t_f] \tag{9}$$

for:

$$\epsilon_k > \epsilon_{k+1} > 0 \quad \text{and} \quad \underset{k \longrightarrow \infty}{\text{Limit}}\, \epsilon_k = 0 \tag{10}$$

Assumptions i)-ii)

Let R be the set of piecewise continuous control functions in the interval $[t_o, t_f]$ which satisfy inequality (9).

i) $\hat{V}[u(\cdot)]$ is a continuous functional of $u(\cdot)$ $\forall u(\cdot) \in R$.

ii) $\inf_R \hat{V}[u(\cdot)] = \min_R \hat{V}[u(\cdot)] = v_o > -\infty$.

Lemma 1: $\inf_R V[u(\cdot), \epsilon_k] > \inf_R \hat{V}[u(\cdot)] = v_o$ †

Proof: $\inf_R V[u(\cdot), \epsilon_k] \geq \inf_R \hat{V}[u(\cdot)] + \inf_R \frac{\epsilon_k}{2} \int_{t_o}^{t_f} u^2(t)dt$

$$> v_o \quad †$$

Assumption iii) $\inf_R V[u(\cdot), \epsilon_k] \equiv V[u_k(\cdot), \epsilon_k]$ is obtained in R $\forall k$.
i.e., $u_k(\cdot) \in R \ \forall k$.

Lemma 2: For $k > \ell$ $(\epsilon_k < \epsilon_\ell)$:

$$V[u_k(\cdot), \epsilon_k] < V[u_\ell(\cdot), \epsilon_\ell] \quad .$$

Proof: $V[u_k(\cdot), \epsilon_k] = \min_R V[u(\cdot), \epsilon_k] \leq V[u_\ell(\cdot), \epsilon_k]$

$$< V[u_\ell(\cdot), \epsilon_\ell] \quad , \quad \text{as } \epsilon_\ell > \epsilon_k$$

Theorem 1: For a positive sequence $\{\epsilon_k\} : \epsilon_k > \epsilon_{k+1} > 0$ and $\lim_{k\to\infty} \epsilon_k = 0$ and under conditions i)-iii);

$$\lim_{k\to\infty} V[u_k(\cdot), \epsilon_k] = v_o$$

Corollary:

1) $$\lim_{k\to\infty} \epsilon_k \int_{t_o}^{t_f} u_k^2(t)dt = 0 \quad .$$

2) $$\lim_{k\to\infty} \hat{V}[u_k(\cdot)] = v_o$$

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† If the minimizing $u(\cdot)$ for the ϵ-problem is identically zero then these strict inequalities become equalities; however an identically zero minimizing $u(\cdot)$ for the ϵ-problem would also be the optimal solution of the original singular control problem, so that there is no loss of generality in considering $u(\cdot) \not\equiv 0$. In the following proofs we assume that $u(\cdot) \not\equiv 0$.

Proof: Given any $\eta > 0$; by assumption i) and Lemma 1, $\exists u^*(\cdot) \in R$ such that:

$$\hat{V}[u^*(\cdot)] < v_o + \frac{\eta}{2}$$

Choose ϵ_ℓ such that:

$$\frac{\epsilon_\ell}{2} \int_{t_o}^{t_f} [u^*(t)]^2 dt < \frac{\eta}{2}$$

Then for $k > \ell$ ($\epsilon_k < \epsilon_\ell$);

$$V[u_k(\cdot), \epsilon_k] < V[u_\ell(\cdot), \epsilon_\ell] \qquad \text{by Lemma 2}$$

$$\leq V[u^*(\cdot), \epsilon_\ell]$$

$$< v_o + \frac{\eta}{2} + \frac{\eta}{2} = v_o + \eta$$

Also:

$$V[u_k(\cdot), \epsilon_k] > \hat{V}[u_k(\cdot)] \geq v_o > v_o - \eta$$

Therefore for any $\eta > 0$:

$$|V[u_k(\cdot), \epsilon_k] - v_o| < \eta \qquad \forall k > \ell$$

b) ϵ-$\alpha(\cdot)$-Algorithm

Minimize: $u(\cdot)$

$$W[u(\cdot), \epsilon_N, \alpha_s(\cdot)] = \int_{t_o}^{t_f} [L(x,t) + \frac{\epsilon_N}{2}(u - \alpha_s)^2] dt + F(x(t_f)) \tag{11}$$

where:

$$\left.\begin{aligned} \bar{u}(\cdot) &= u_N(\cdot) \\ \alpha_1 &= u_N(\cdot) \\ \alpha_{s+1} &= u_s(\cdot) \quad , \quad s \geq 1 \end{aligned}\right\} \tag{12}$$

and:

$$s = 1, 2, \ldots \tag{13}$$

Lemma 3:

$$v_o \leqslant W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] < W[u_{s-1}(\cdot), \epsilon_N, \alpha_{s-1}(\cdot)]$$

for $\alpha_s(\cdot) \neq \alpha_{s-1}(\cdot)$.

Proof:

$$W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] \equiv \min_R[\hat{V}[u(\cdot)] + \frac{\epsilon_N}{2}\int_{t_o}^{t_f}(u - \alpha_s)^2 dt]$$

$$\geqslant \min_R \hat{V}[u(\cdot)] + \min_R[\frac{\epsilon_N}{2}\int_{t_o}^{t_f}(u - \alpha_s)^2 dt] \geqslant v_o$$

Also:

$$W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] \equiv \min_R W[u(\cdot), \epsilon_N, \alpha_s(\cdot)]$$

$$\leqslant W[u_{s-1}(\cdot), \epsilon_N, \alpha_s]$$

$$< W[u_{s-1}(\cdot), \epsilon_N, \alpha_{s-1}]$$

It must be noted that to guarantee that the right hand inequality is strict, ϵ_N must be sufficiently small so that all the $u_s(\cdot)$'s fall in some neighbourhood of the optimal control $u^o(\cdot)$.† Theorem 1 guarantees this for ϵ_N sufficiently small. This leads to:

Assumption iv) ϵ_N is sufficiently small so that all the $u_s(\cdot)$'s lie in some neighbourhood of $u^o(\cdot)$.

Theorem 2: Under assumptions i)-iv),

$$\lim_{s \to \infty} W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] = v_o$$

---

† Otherwise the algorithm may converge to a stationary, non-optimal, solution of the singular problem.

Corollary:

1) $\underset{s \to \infty}{\text{Limit}}\ \epsilon_N \int_{t_o}^{t_f} (u_s - \alpha_s)^2 dt = 0$

2) $\underset{s \to \infty}{\text{Limit}}\ \hat{V}[u_s(\cdot)] = v_o$

Proof: For any $\eta > 0$, by assumption i) and Lemma 3, $\exists u_{\ell-1}(\cdot)$ such that:

$$\hat{V}[u_{\ell-1}(\cdot)] < v_o + \eta \quad .$$

Then, for $s > \ell$,

$$W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] < W[u_\ell(\cdot), \epsilon_N, \alpha_\ell(\cdot)] \quad \text{for } \alpha_s(\cdot) \neq \alpha_\ell(\cdot)$$

$$\leq W[u_{\ell-1}(\cdot), \epsilon_N, \alpha_\ell(\cdot)]$$

$$= \hat{V}[u_{\ell-1}(\cdot)] + \frac{\epsilon_N}{2} \int_{t_o}^{t_f} (u_{\ell-1} - \alpha_\ell)^2 dt$$

$$= v_o + \eta \quad , \quad \text{since } \alpha_\ell = u_{\ell-1}$$

Also:

$$W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] > \hat{V}[u_s(\cdot)] \geq v_o > v_o - \eta \quad .$$

Therefore for any $\eta > 0$,

$$|W[u_s(\cdot), \epsilon_N, \alpha_s(\cdot)] - v_o| < \eta \quad , \quad \forall s > \ell$$

6. Computed Examples

Four numerical examples were solved using the ϵ- and ϵ-$\alpha(\cdot)$-algorithms. Several important numerical-analysis details had to be resolved. These were:

i) the choice of the sequence $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$,

ii) the choice of ϵ_N,

iii) the choice of an adequate 'convergence criterion.'

The first three control problems are known to have singular arcs in their optimal solutions, whilst the optimal control function of the fourth problem is known to be bang-bang.

The non-singular ϵ- and ϵ-$\alpha(\cdot)$-problems were solved using the Differential Dynamic Programming algorithms described in [15]. The differential equations arising in the algorithms were solved using a simple Euler integration scheme. One hundred integration steps were employed in the solution of the first three problems, whilst the fourth problem was solved using four hundred integration steps. Details of the numerical solution of the four problems are given in the following pages. Graphs which display the results of using the ϵ-algorithm and the ϵ-$\alpha(\cdot)$-algorithm are presented. Each graph depicts $u(\cdot)$ computed by the two algorithms and, where possible, $H_u(\cdot)$ is displayed (H is the Hamiltonian of the original $\hat{V}[u(\cdot)]$ problem). In the cases where $H_u(\cdot)$ is not displayed, it is indistinguishable from zero on the singular control segments.

Problem 1:

Minimize:
$u(\cdot)$

$$\hat{V}[u(\cdot)] = \int_0^2 x^2 \, dt \qquad (14)$$

where:

$$\dot{x} = u \quad ; \quad x(0) = 1 \qquad (15)$$

and:

$$|u| \leqslant 1 \qquad (16)$$

The ϵ-$\alpha(\cdot)$-problem is: $\underset{u(\cdot)}{\text{Minimize}}$

$$W[u(\cdot), \epsilon, \alpha(\cdot)] = \int_0^2 x^2 dt + \frac{\epsilon}{2} \int_0^2 (u - \alpha)^2 dt \tag{17}$$

subject to (15) and (16).

Using the second-order control inequality constraint algorithm of [15] to solve the ϵ-$\alpha(\cdot)$-problem (for ϵ and $\alpha(\cdot)$ assumed known), the following equations must be solved iteratively:

$$-\dot{a} = W_x(u - \bar{u}) + \frac{\epsilon}{2}[(u - \alpha)^2 - (\bar{u} - \alpha)^2] \quad ; \quad a(t = 2) = 0 \tag{18}$$

$$-\dot{W}_x = 2\bar{x} + W_{xx}(u - \bar{u}) \quad ; \quad W_x(t = 2) = 0 \tag{19}$$

$$-\dot{W}_{xx} = \begin{cases} 2 - \frac{1}{\epsilon} W_{xx}^2 & \text{if} \quad |u| < 1 \\ 2 & \text{if} \quad |u| = 1 \end{cases} \quad ; \quad W_{xx}(t = 2) = 0 \ . \tag{20}$$

The new control function is obtained (see [15]) from $\bar{u}(\cdot)$ and

$$u(t) = \text{sat}[\alpha - W_x/\epsilon] \tag{21}$$

If $|a(t = 0)|$ is zero (or, in practice, 'sufficiently small') then $\bar{u}(\cdot)$ is the solution of the ϵ-$\alpha(\cdot)$-problem [15]. If $|a(t = 0)|$ is not zero then the above equations are used to produce a new improved control function $u(\cdot)$ (see [15] for details).

The initial $\bar{u}(\cdot)$ and $\alpha(\cdot)$ were chosen to be identically zero; thus, initially, the ϵ-problem was solved. The initial value of ϵ was 0.5. Whenever $|a(t = 0)|$ became less than 10^{-4}, ϵ was replaced by its value divided by 5; that is, $\epsilon_1 = 0.5$, $\epsilon_k = \epsilon_{k-1}/5$.

The ϵ-algorithm was used until it was found that for both ϵ_{N-1} and ϵ_N, the same $\bar{u}(\cdot)$ caused $|a(t = 0)|$ to be less than 10^{-4}; this $\bar{u}(\cdot)$ was then considered to be the solution of the ϵ-problem. $\alpha(\cdot)$ was then set equal to $\bar{u}(\cdot)$ and the ϵ-$\alpha(\cdot)$-algorithm was invoked. After only one iteration, the optimal control function of the original $\hat{V}$ problem was obtained (to four decimal places). The result of this iteration as well as the output of the ϵ-algorithm are shown in Fig. 1. The value of $\hat{V}$ that was produced by the ϵ-algorithm was 0.3434. After the one iteration of the ϵ-$\alpha(\cdot)$-algorithm this was reduced to 0.3234.

Problem 2: Minimize: $\underset{u(\cdot)}{}$

$$\hat{V}[u(\cdot)] = \int_0^5 (x_1^2 + x_2^2)dt \tag{22}$$

where:

$$\begin{aligned} \dot{x}_1 &= x_2 \quad ; \quad x_1(0) = 0 \\ \dot{x}_2 &= u \quad ; \quad x_2(0) = 1 \end{aligned} \tag{23}$$

and:

$$|u| \leq 1 \quad . \tag{24}$$

The equations to be solved for the ϵ-$\alpha(\cdot)$-algorithm are:

$$-\dot{a} = W_{x_2}(u - \bar{u}) + \frac{\epsilon}{2}[(u - \alpha)^2 - (\bar{u} - \alpha)^2] \quad ; \quad a(t = 5) = 0 \tag{25}$$

$$-\dot{W}_{x_1} = 2\bar{x}_1 + W_{x_1 x_2}(u - \bar{u}) \quad ; \quad W_{x_1}(t = 5) = 0 \tag{26}$$

$$-\dot{W}_{x_2} = 2\bar{x}_2 + W_{x_1} + W_{x_2 x_2}(u - \bar{u}) \quad ; \quad W_{x_2}(t = 5) = 0 \tag{27}$$

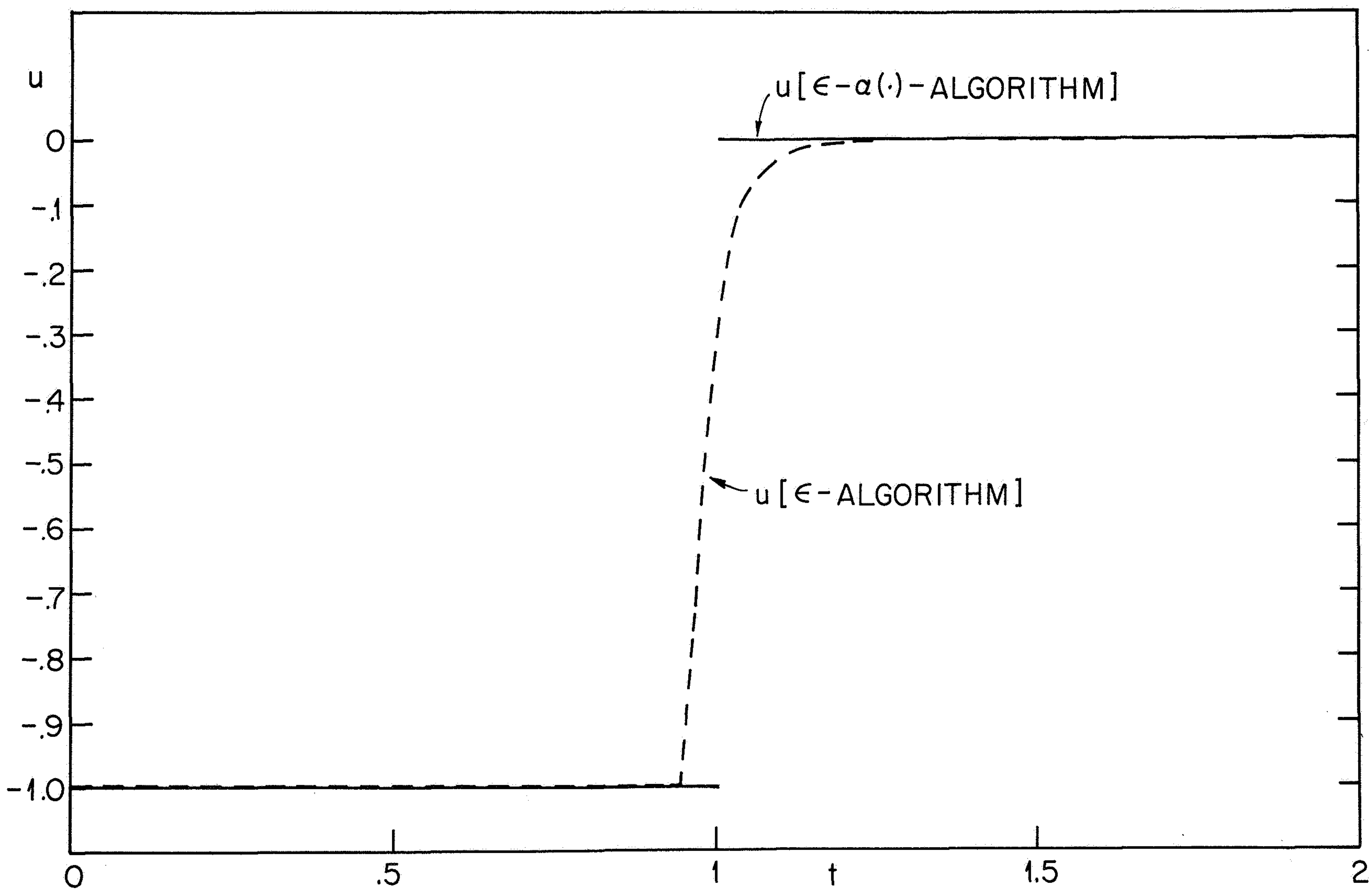

FIG. 1 u vs. t

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
-\dot{W}_{x_1x_1} = 2 - W^2_{x_1x_2}/\epsilon \\
-\dot{W}_{x_1x_2} = W_{x_1x_1} - W_{x_1x_2}W_{x_2x_2}/\epsilon \\
-\dot{W}_{x_2x_2} = 2 + 2W_{x_1x_2} - W^2_{x_2x_2}/\epsilon
\end{array}\right\} \text{if } |u| < 1 \\
\\
\left.\begin{array}{l}
-\dot{W}_{x_1x_1} = 2 \\
-\dot{W}_{x_1x_2} = W_{x_1x_1} \\
-\dot{W}_{x_2x_2} = 2 + 2W_{x_1x_2}
\end{array}\right\} \text{if } |u| = 1
\end{array}\right\}
\begin{array}{l}
W_{x_1x_1}(t=5) = 0 \\
W_{x_1x_2}(t=5) = 0 \\
W_{x_2x_2}(t=5) = 0
\end{array}
\qquad (28)
$$

The new control function is obtained (see [15]) from $\overline{u}(\cdot)$ and

$$u(t) = \text{sat}[\alpha - W_{x_2}/\epsilon] \qquad . \qquad (29)$$

Implementation of the algorithms was similar to that used for Problem 1; however, there were minor differences. Here, $\epsilon_1 = 5.$, and $\epsilon_k = \epsilon_{k-1}/10$. Moreover, N was found by reducing ϵ until the differential equations for the components of W_{xx} become ill behaved; the last value of ϵ for which these equations were well behaved was selected to be ϵ_N, its value being 0.005.

$\hat{V}$ after using the ϵ-algorithm was 0.828517, and this was reduced to 0.828514 by the ϵ-$\alpha(\cdot)$-algorithm. The ϵ-algorithm required 16 iterations including 4 reductions of ϵ and one increase (when $\epsilon = .0005$ was found to be too small). The ϵ-$\alpha(\cdot)$-algorithm required 3 iterations and two changes of $\alpha(\cdot)$.† Not much is gained by using the ϵ-$\alpha(\cdot)$-algorithm except for a slight sharpening of the switching in the control function (see Fig. 2).

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† An "iteration" is one iteration of the second-order algorithm of [15].

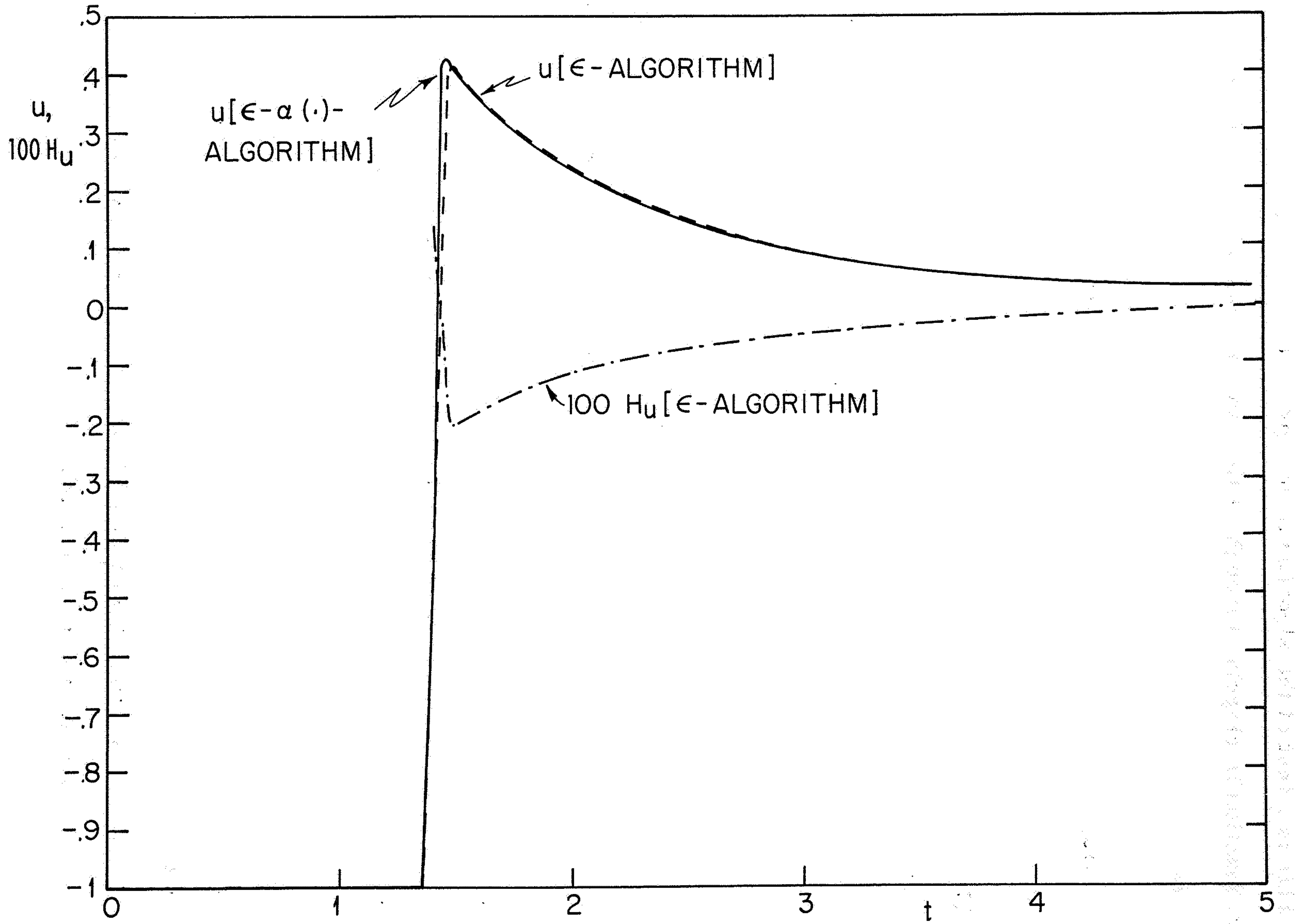

FIG. 2 u AND 100 H_u vs. t

Problem 3:

This is a test of the algorithms on a problem whose optimal control function is singular throughout the time interval over which the system is run. Minimize:

$$\underset{u(\cdot)}{\text{Minimize}}\quad \hat{V}[u(\cdot)] = \int_0^5 \{[x_2 - (\tfrac{3}{4}t + 1)]^2 + [x_1 - (\tfrac{3}{8}t^2 + t)]^2\} dt \tag{30}$$

subject to (23) and (24).

Clearly the solution is $u(t) = \frac{3}{4}$; $t \in [0, 5]$ and the optimum $\hat{V} = 0$. The relevant equations are:

$$-\dot{a} = W_{x_2}(u - \bar{u}) + \frac{\epsilon}{2}[(u - \alpha)^2 - (\bar{u} - \alpha)^2] \quad ; \quad a(t = 5) = 0 \tag{31}$$

$$-\dot{W}_{x_1} = 2(\bar{x}_1 - \tfrac{3}{8}t^2 - t) + W_{x_1x_2}(u - \bar{u}) \quad ; \quad W_{x_1}(t = 5) = 0 \tag{32}$$

$$-\dot{W}_{x_2} = 2(\bar{x}_2 - \tfrac{3}{4}t - 1) + W_{x_2x_2}(u - \bar{u}) + W_{x_1} \; ; \; W_{x_2}(t = 5) = 0 \tag{33}$$

where the components of W_{xx} satisfy (28), and u(t) is given by (29).

The criterion of optimality was taken to be $|a(t = 0)| \leq 2 \times 10^{-5}$, and ϵ_1 was set as 0.5. In 21 iterations the ϵ-algorithm yielded $\hat{V} = 0.0025$, which was reduced to 0.00152 by the ϵ-$\alpha(\cdot)$-algorithm. The latter algorithm required two iterations and one change of $\alpha(\cdot)$. After another change of $\alpha(\cdot)$ and another two iterations, $\hat{V} = .00147$. The results of this last iteration and of the ϵ-algorithm are shown in Fig. 3. Evidently the ϵ-$\alpha(\cdot)$-algorithm seems to contribute more to sharpening the control function than to reducing the cost. The deviation of the control function from $\frac{3}{4}$ at $t \cong 0$ and $t \approx 5$ is attributed to the crude Euler integration routine used.

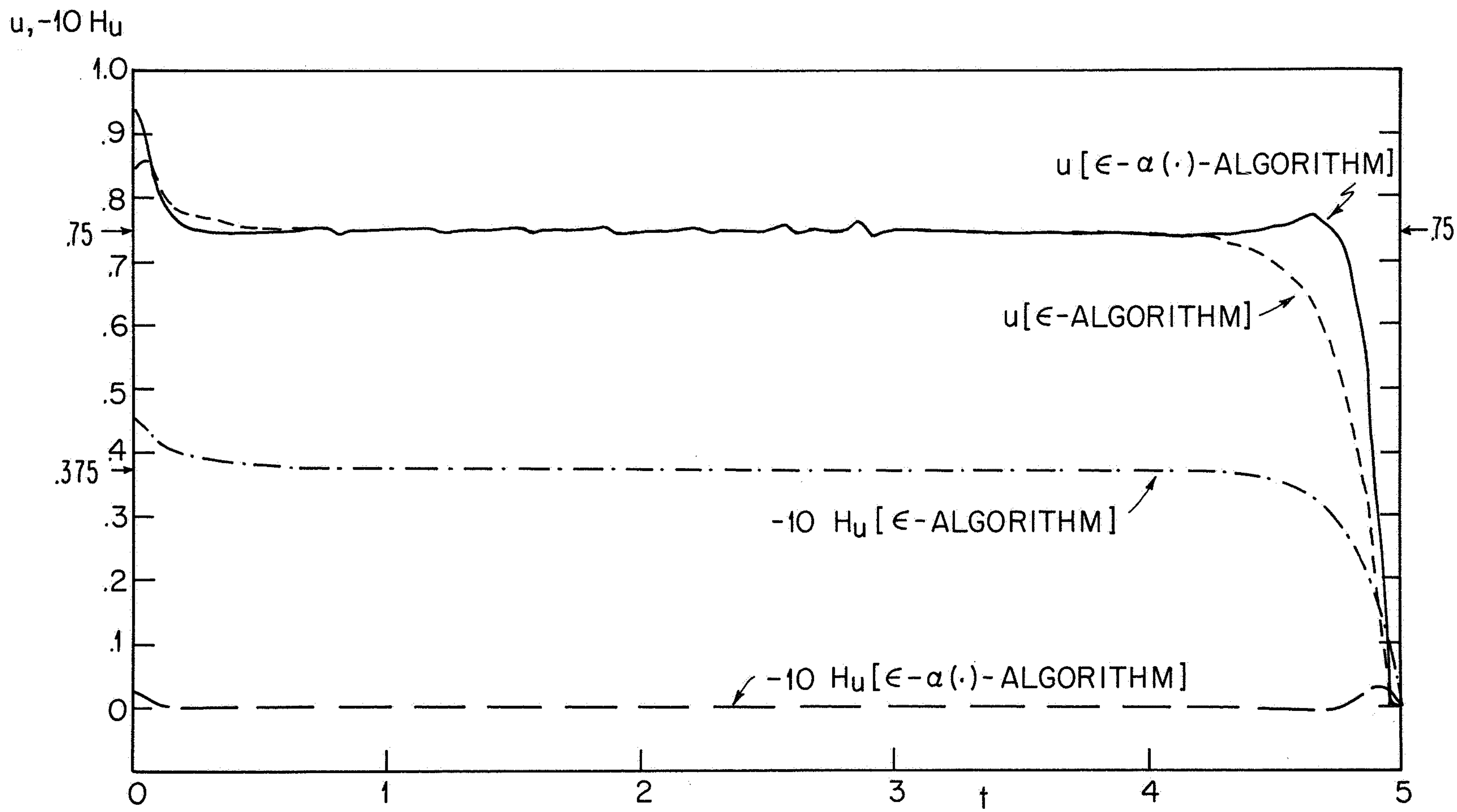

FIG. 3 u AND $-10\ H_u$ vs. t

Problem 4: Minimize: $\underset{u(\cdot)}{}$

$$\hat{V}[u(\cdot)] = \int_0^5 x_1^2 dt \tag{34}$$

subject to (23) and (24).

The relevant equations are (31), (32) and (28) and

$$-\dot{W}_{x_2} = W_{x_1} + W_{x_2x_2}(u - \bar{u}) \quad ; \quad W_{x_2}(t = 5) = 0 \tag{35}$$

Here, ϵ_1 was set as 2., and $\epsilon_k = \epsilon_{k-1}/10$. From previous experience (Problems 1 to 3) a value of .02 was chosen for ϵ_N. The criterion of optimality was chosen as $|a(t = 0)| < 10^{-4}$.

It is known that the optimal solution to (34) is bang-bang (for an infinite upper limit of integration, the optimal control function exhibits an infinite number of switchings and the state x of the system tends to zero. Fuller [23] gives expressions for the optimal cost function for the infinite time case, and because x tends rapidly to zero, these expressions are useful for predicting the optimal value of $\hat{V}$.)

A guess of the form of the optimal control function was made, and $\alpha_o(\cdot)$ was set equal to this guess, i.e.:

$$\begin{aligned} \alpha_o(t) &= -1 \quad , \quad 0 \leq t < 1.7 \\ \alpha_o(t) &= 0 \quad , \quad 1.7 \leq t \leq 5 \end{aligned} \tag{36}$$

Seven iterations and two reductions of ϵ were required by the ϵ-algorithm†; the results are shown in Fig. 4. The minimum value of $\hat{V}$ is 0.2777. After 5 more iterations and 3 changes of $\alpha(\cdot)$ in the ϵ-$\alpha(\cdot)$-algorithm, $\hat{V}$ was reduced to 0.2771. The resulting control

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† Note that in this variant of the ϵ-algorithm, $\alpha(\cdot)$ is non-zero and is given by (36).

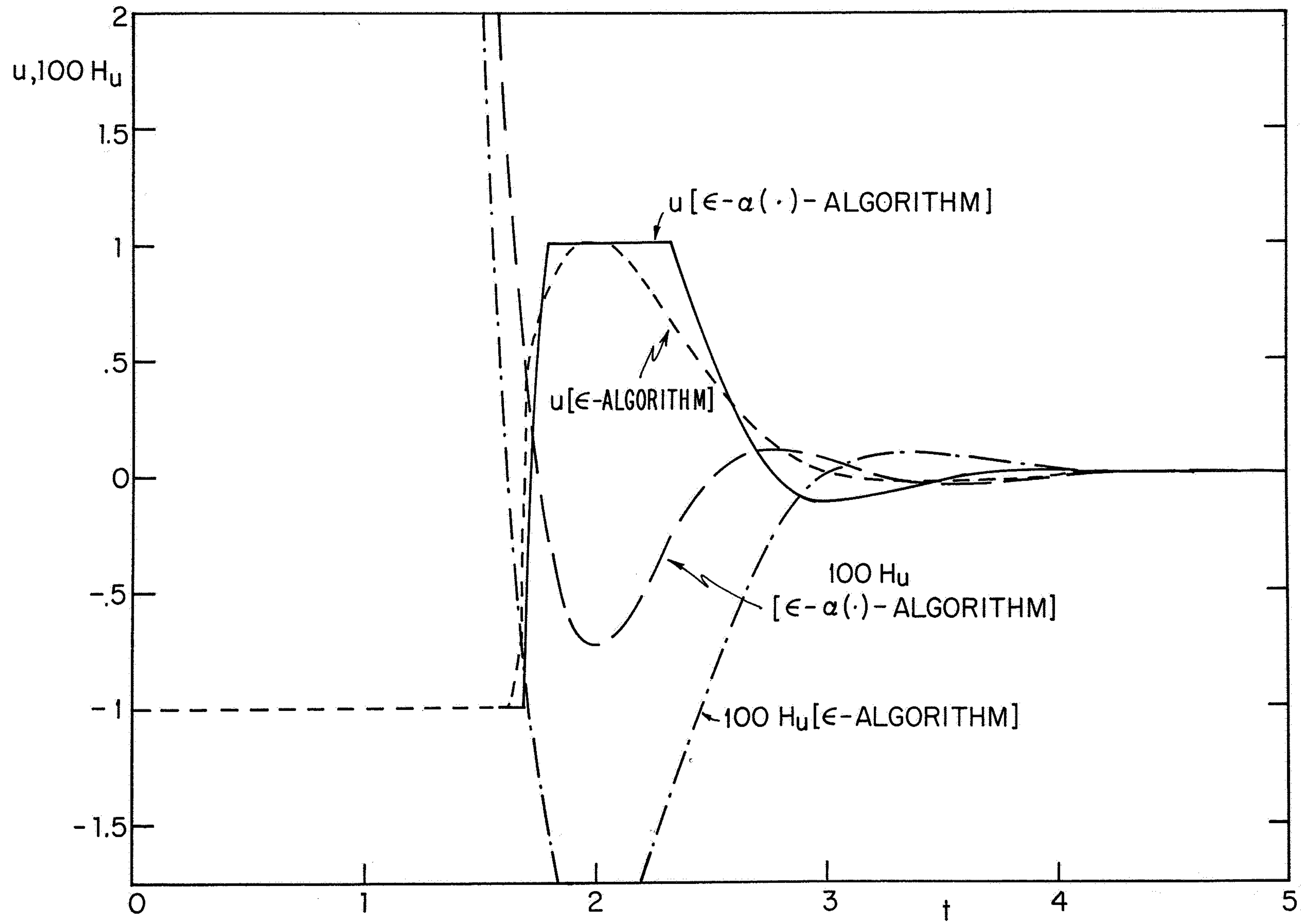

FIG. 4 u AND 100 H_u vs. t

function and H_u as a function of time are also shown in Fig. 4. Note that the ϵ-$\alpha(\cdot)$-algorithm sharpens considerably the switchings of the control function, but does not affect markedly the cost $\hat{V}$.

The final control function shown in Fig. 4 is clearly not bang-bang; however in the non-bang-bang region, $H_u(t)$ is extremely small, indicating that the Pontryagin Principle is very nearly satisfied by this non-bang-bang control function. This suggests that the optimal bang-bang control should not produce a $\hat{V}$ which differs much from the $\hat{V}$ obtained using our approximately optimal control (i.e. $\hat{V}[u(\cdot)]$ is very 'flat' as a functional of $u(\cdot)$ in the neighbourhood of the optimum). The above conjecture is confirmed by the fact that, for the given initial conditions, the analytic expressions, given in [23], yield the optimal cost for the infinite time problem as 0.278 which is not very different from that obtained by the ϵ-$\alpha(\cdot)$-algorithm.

7. Conclusion

The addition of a quadratic functional of the control $u(\cdot)$ to the performance index (cost functional) of a certain class of singular control problems, results in a non-singular control inequality constrained problem which can be solved by an existing second-order successive approximation method. A parameter $\epsilon > 0$, multiplying the additional quadratic functional, is allowed to tend to zero, whilst the optimality of the non-singular problem is maintained. In the limit as $\epsilon \longrightarrow 0$, the solution of the original singular control problem is approached. A variant of the ϵ-problem is described which allows the use of a small value of ϵ which is not required to approach zero, whilst still ensuring convergence to the required singular control function. Four illustrative numerical examples are presented to demonstrate the usefulness of the proposed algorithms.

It is believed that, except for the gradient and a modified conjugate gradient algorithm, the proposed algorithms are the only ones currently available for computing optimal singular controls.

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13. ABSTRACT

A class of singular control problems is made non-singular by the addition of an integral quadratic functional of the control to the cost functional; a parameter $\epsilon > 0$ multiplies this added functional. The resulting non-singular problem is solved for a monotone decreasing sequence of ϵ's; $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_k > 0$. As $k \to \infty$, and $\epsilon_k \to 0$, the solution of the modified problem tends to the solution of the original singular problem. A variant of the method which does not require that $\epsilon \to 0$ is also presented. Four illustrative numerical examples are described.

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