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ADJUSTMENT OF SATELLITE TRIANGULATIONS

By

R. A. HIRVONEN

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Prepared for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
UNDER GRANT NO. NGR 12-001-045

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UNIVERSITY OF HAWAII

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ABSTRACT

The elementary problem considered was: Given earth-fixed space directions from two non-intervisible ground stations A and B to two elevated points 1 and 2, find the direction AB. Since more than two elevated points were to be used in the practice, a preliminary adjustment for each side AB was carried out. As the result of this adjustment, the two correlated quantities S_{AB} *and* t_{AB}*, which were obtained, can be used as fictive observations, in the main adjustment of a network of several stations.*

If there are N stations, the main adjustment contains 3N parameters. Of these, four cannot be determined by triangulation. It hardly seems worthwhile, however, to include in the main adjustment such observations as satellite distances, ground triangulations, levelings, and astronomical coordinates (with gravimetric deflections of the vertical). It is suggested, therefore, that four other conditions be used, in order to keep the gravity center of all stations and the mean scale of the approximate coordinates fixed. In this way, the overall accuracy obtainable by the adjustment of the network can be estimated.

1. The Fundamental Problem

We start from the well-known elementary problem of intersection on a plane (see Fig. 1):

Given the coordinates x, y for the corner points A1, A2, B1, and B2 of a quadrilateral,

Wanted the coordinates for the intersection point Q of the diagonals A1 - B1 and A2 - B2.

The solution can be obtained by the following sequence of formulas:

$$\left\{\begin{aligned} K_1 &= x_{A1}\, y_{B1} - x_{B1}\, y_{A1} \\ L_1 &= y_{A1} - y_{B1} \\ M_1 &= x_{B1} - x_{A1} \end{aligned}\right. \qquad \left\{\begin{aligned} K_2 &= x_{A2}\, y_{B2} - x_{B2}\, y_{A2} \\ L_2 &= y_{A2} - y_{B2} \\ M_2 &= x_{B2} - x_{A2} \end{aligned}\right. \tag{1}$$

$$\left\{\begin{aligned} N &= L_1\, M_2 - L_2\, M_1 \\ x_Q &= (M_1\, K_2 - M_2\, K_1)/N \\ y_Q &= (K_1\, L_2 - K_2\, L_1)/N \end{aligned}\right. \tag{2}$$

Suppose now that A and B are two observation stations between which there is a forest or a hill which makes a direct sighting from A to B impossible. Furthermore, suppose that, on the top of the obstacle, there are two fixed target points, 1 and 2, which can be identified and sighted both from A and from B.

Let a and b denote the horizontal and vertical bearings, respectively, observed at the ground stations toward the elevated targets. Assume that the distance AB is so short that the curvature of the earth can be neglected and that the horizontal bearings, observed by a compass, are

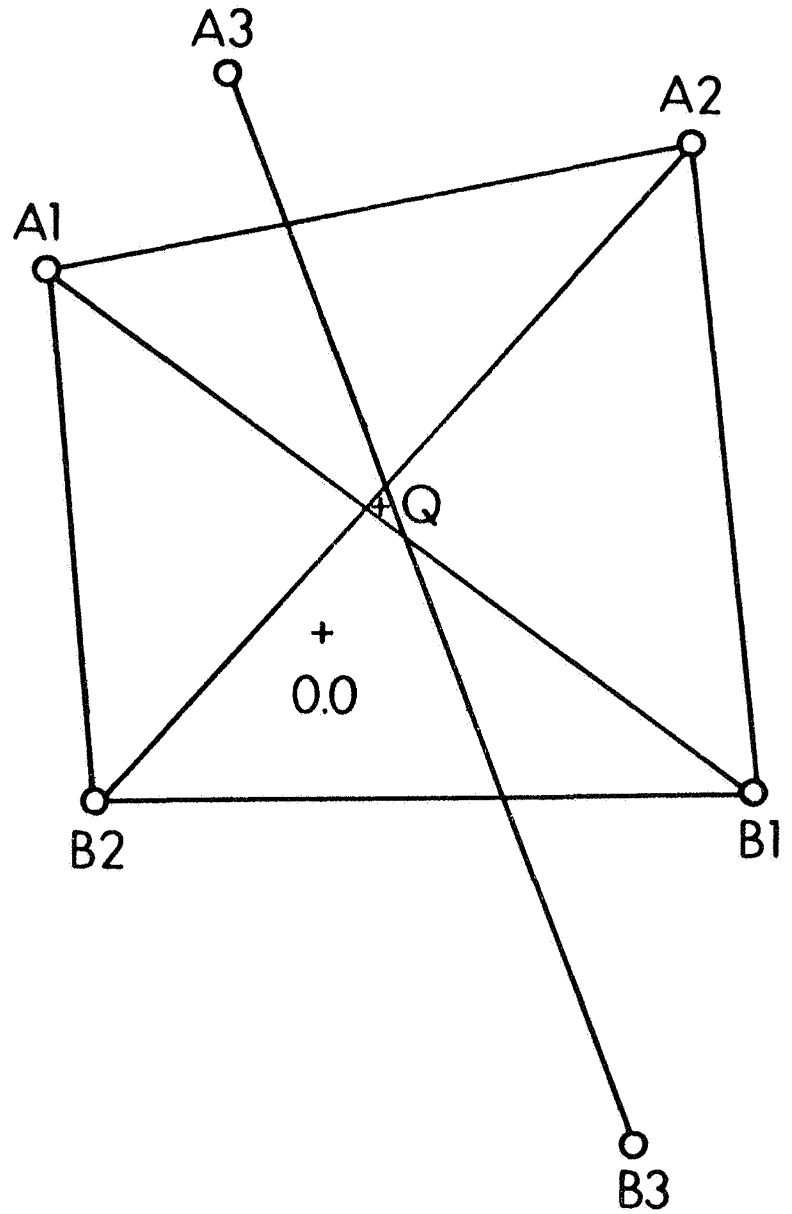

Fig. 1. Intersection of diagonals of a quadrilateral on the celestial sphere gives the earth-fixed space direction between two stations A and B at which two satellite positions 1 and 2 have been observed.

referred to the same vertical zero plane. It is advisable to select a zero plane which is close to direction AB.

Assume that, in the infinity behind station B, there is a vertical screen perpendicular to this zero plane. Then the bearings from A to 1 and 2 give two points A1 and A2 on the screen. The bearings from B to 1 and 2 are taken in reverse, replacing a by 180° + a and b by -b. Then two more points, B1 and B2, are obtained on the screen. The geometric solution of the fundamental problem now gives point Q, or the bearing from A to B.

For the numerical computations, we must consider the screen as a gnomonic projection of the celestial sphere to the tangent plane at zero point:

$$x = \tan a$$
$$y = \frac{\tan b}{\cos a} \tag{3}$$

Substituting these coordinates into (1), formulas (2) give x_Q, y_Q and the reverse solution of (3) gives a_Q and b_Q.

2. The Equatorial Coordinates

In the satellite traingulation, the horizontal coordinates a and b are replaced by equatorial coordinates α and δ. However, the zero plane of α is revolving with respect to the ground stations. Therefore, we introduce the Greenwich hour angles

$$t = \theta - \alpha \tag{4}$$

where θ is the Greenwich sidereal time of the observation.

Because the satellites are moving targets, the observations at both ground stations must be made simultaneously. This can be accomplished by photographing flashing satellites against the background of stars--or chopping the trail of the satellite by synchronized shutters at both stations.

Without going into the technical details of the measurements and their reductions, we suppose here that the positions t_{Ai} and δ_{Ai} of each event i at each station A, corrected for the refraction and other disturbing effects, are already available. In the same way the mean errors m_t and m_δ should be readily estimated.

Writing t instead of a and δ instead of b, we may use formulas (3) and (1) - (2) for the calculation of t_Q and δ_Q, which indicate the earth-fixed space direction from A to B. However, the following sequence of formulas is more convenient for the practical computations:

For i = 1 and i = 2, compute

$$\begin{cases} \kappa_i = \tan\delta_{Bi}\sin t_{Ai} - \tan\delta_{Ai}\sin t_{Bi} \\ \lambda_i = \tan\delta_{Ai}\cos t_{Bi} - \tan\delta_{Bi}\cos t_{Ai} \\ \mu_i = \sin(t_{Bi} - t_{Ai}) \end{cases} \tag{5}$$

Our numerical example has been taken from Veröffentlichung des Geod. Inst. Potsdam, Nr. 29, (1965) referring observations of Echo I in Potsdam and Bucharest. The reduction to the center of the balloon has been computed again for each observation, instead of the mean of each series, and the results differ slightly from those published by Arnold and Schoeps (1965). Similarly, the weights of observations for our adjustment do not agree with those used in Potsdam.

Observations:	A1	B1	A2	B2
t	-27°16'14".70	+ 5°35'53".50	-58°40'00".28	-39°05'57".90
δ	+13°27'18".76	+34°31'11".45	+24°51'40".66	+57°16'22".22
tan δ	0.23925217	0.68779110	0.46336373	1.55603704

sin t	− .45819587	+ .09755152	− .85415716	− .63066790
cos t	+ .88885125	+ .99523048	+ .52001495	+ .77605283

i	1	2	1x2	
κ	−0.33848245	−1.03687155	+0.11898805	$= \nu$
λ	−0.37323293	−0.44956779	−0.44936755	
μ	+0.54271928	+0.33491434	−0.23482380	
cot t_Q	−0.26479004			
t_Q	−75°10'08".45			
sin t_Q	−0.96668510			
tan δ_Q			−0.50515590	
δ_Q			−26°48'03".21	

For the adjustment (see pages 14 and 15), these approximate values have been rounded off to

$$t_Q = -75°10'10" \qquad \text{and} \qquad \delta_Q = -26°48'00"$$

In Arnold and Schoeps (1965), there are 18 satellite positions (numbered 1 - 18) which have been observed simultaneously at Potsdam and at Bucharest against the background of stars. From these double observations, numbers 1 and 12 have been used for the approximate computation before the adjustment of all 18 events.

3. The Preliminary Adjustments

In the practice, more than two events are observed for the determination of the direction of each side AB. These observations usually are independent of those of other sides, e.g., AC or BC. We omit here the slight correlation between sides AB, AC, and BC which occurs if some events are observed simultaneously at three stations A, B, C.

For each side, we perform a preliminary adjustment. Before this adjustment, we need the first approximate values t_Q, δ_Q which can be computed from two selected events as explained above or from approximate

For the estimation of the weights p_i, we suppose that the mean error of one measured-event position is independent of the direction:

$$m_{Ai} = m(\delta_{Ai}) = m(t_{Ai}) \cdot \cos \delta_{Ai}$$

but may be variable from one photograph to another. The variance of l_i is then

$$m_i^2 = \frac{m_{Ai}^2 \sin^2\sigma_{Ai} + m_{Bi}^2 \sin^2\sigma_{Bi}}{\sin^2\sigma_{AB}} \tag{11}$$

where σ means the arc from Q to Ai or Bi or from Ai to Bi, respectively. The computation formulas read

$$\left\{\begin{aligned} &\cos \sigma_{Ai} = \sin \delta_Q \sin \delta_{Ai} + \cos \delta_Q \cos \delta_{Ai} \cos (t_{Ai} - t_Q) \\ &\sin^2\sigma_{AB} = \cos^2\delta_{Ai} \cos^2\delta_{Bi} (\kappa_i^2 + \lambda_i^2 + \mu_i^2) \end{aligned}\right. \tag{12}$$

The weights p_i are, of course, inversely proportional to m_i^2.

Figure 2 gives a graphical table of weights p in the case where $m_{Ai} = m_{Bi}$.

The normal equations of the adjustment read

$$\left\{\begin{aligned} &[paa]\ \xi + [pab]\ \eta + [pal] = 0 \\ &[pab]\ \xi + [pbb]\ \eta + [pbl] = 0 \end{aligned}\right. \tag{13}$$

After the solution of these equations, we have new approximate values

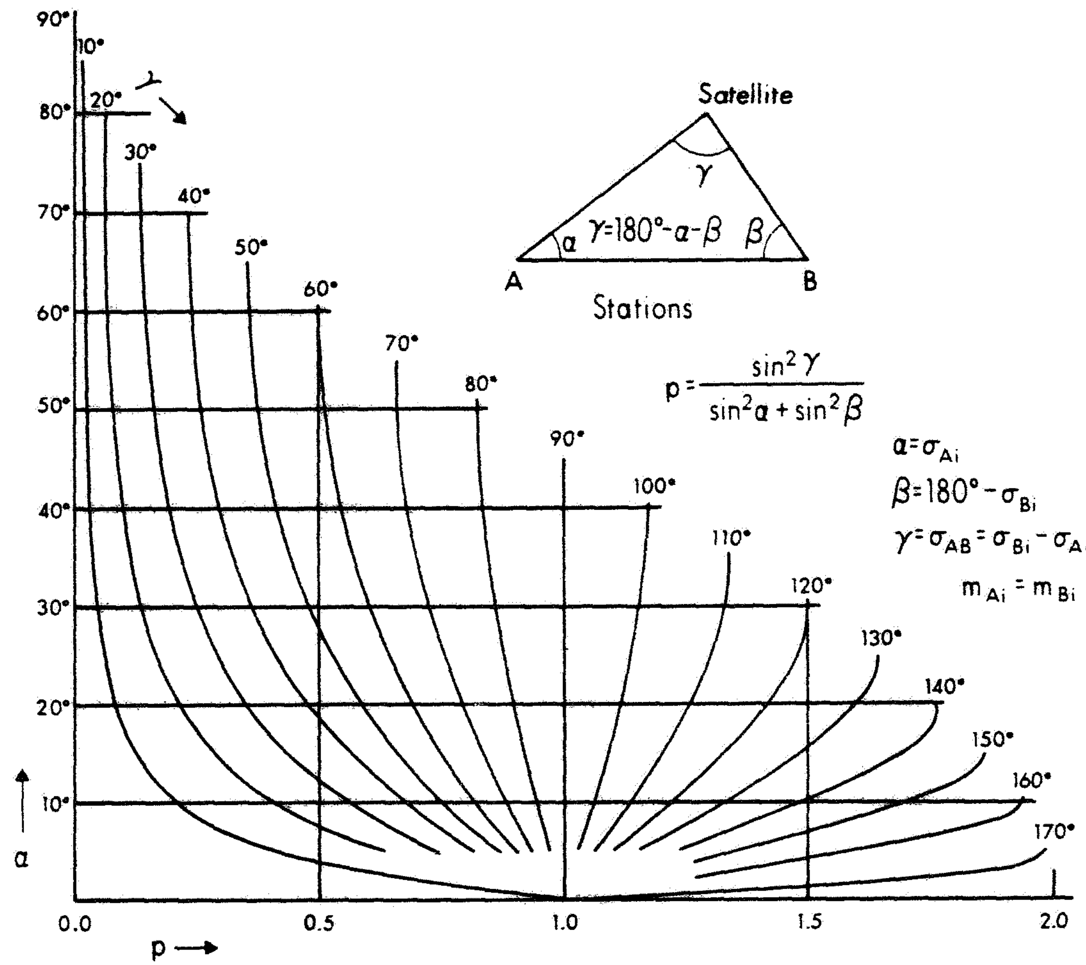

Fig. 2. Weight p of a satellite event as function of angles α = BAS and β = ABS or γ = ASB.

coordinates of stations A and B, as will be explained in the next section. However, the formulas (5) can be applied to all observed events $i = 1, 2, 3 \ldots n$. Then formulas

$$\tan t_i = \frac{\lambda_i}{\kappa_i} , \qquad \tan \delta_i = \frac{\mu_i}{\kappa_i} \cos t_i \tag{7}$$

give the coordinates t_i, δ_i of that point on the celestial sphere which is at the pole of great circle Ai - Bi. The distance of point t_Q, δ_Q from this great circle can be computed by formula

$$l_i = \rho [\sin \delta_Q \sin \delta_i + \cos \delta_Q \cos \delta_i \cos (t_i - t_Q)] \tag{8}$$

where $\rho = 206265$ seconds of arc.

Differentiation of (8) shows that if δ_Q and t_Q are corrected by $d\delta_Q = \xi$ and $dt_Q = \eta/\cos \delta_Q$, respectively, the new distances will be

$$v_i = a_i \xi + b_i \eta + l_i \tag{9}$$

where $a_i^2 + b_i^2 = 1$, approximately. In fact, we have

$$\begin{cases} a_i = \cos \delta_Q \sin \delta_i - \sin \delta_Q \cos \delta_i \cos (t_i - t_Q) \\ b_i = \cos \delta_i \sin (t_i - t_Q) \end{cases} \tag{10}$$

The least-squares solution is based on the condition

$$[pvv] = \sum_i p_i v_i v_i = \text{minimum}$$

$$\begin{cases} \delta_{AB} = \delta_Q + \xi \\ t_{AB} = t_Q + \eta/\cos \delta_Q \end{cases} \tag{14}$$

which are entered as fictitious observations into the main adjustment of a network. In addition, we need the weight matrix

$$P_{AB} = \frac{1}{m^2} \begin{Bmatrix} [paa] & [pab] \\ [pab] & [pbb] \end{Bmatrix} \tag{15}$$

where m = 1 in ideal cases. Alternatively,

$$m^2 = \frac{[pvv]}{n - 2} \tag{16}$$

can be determined separately for each side AB. Here n is the number of events i used in the preliminary adjustment.

The formulas (5) and (7) to (12) give the true geometric quantities if we want to illustrate the events graphically by the diagonals mentioned in the first section (see Fig. 3). Especially

$$\begin{cases} a = \cos \beta \\ b = -\sin \beta \end{cases}$$

where β is the angle between the diagonal and the meridian of point Q.

For the programming of the numerical computations, however, the following sequence of formulas is more convenient:

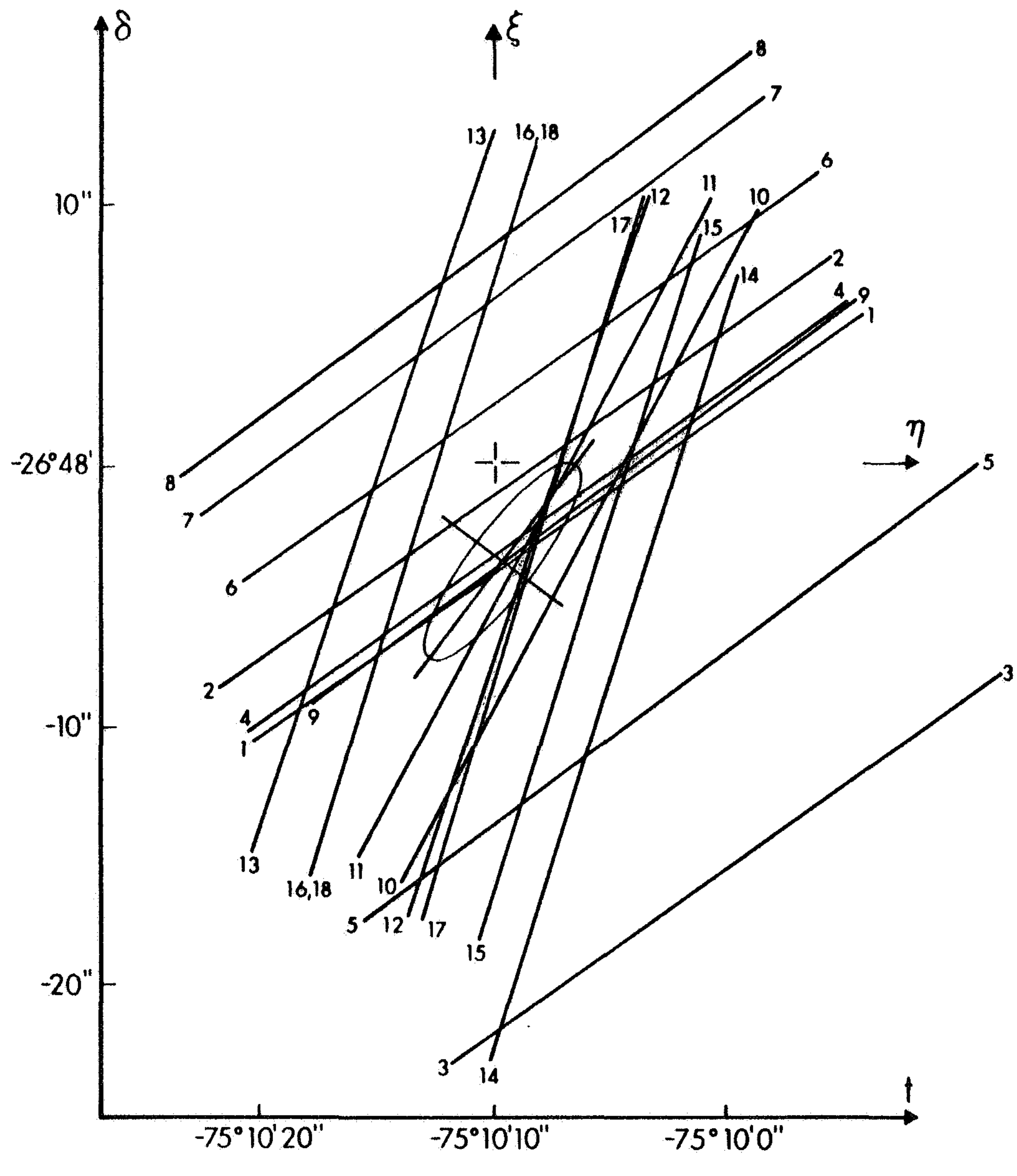

Fig. 3. Error ellipse of the direction Potsdam–Bucharest, as computed from 18 satellite events.

$(Q1) = \sin \delta_Q \qquad (Q2) = \cos \delta_Q$

$(A1) = \sin \delta_{Ai} \qquad (B1) = \sin \delta_{Bi}$

$(A2) = \cos \delta_{Ai} \cos (t_{Ai} - t_Q) \qquad (B2) = \cos \delta_{Bi} \cos (t_{Bi} - t_Q)$

$(A3) = \cos \delta_{Ai} \sin (t_{Ai} - t_Q) \qquad (B3) = \cos \delta_{Bi} \sin (t_{Bi} - t_Q)$

$(A4) = (Q1)\ (A1) + (Q2)\ (A2) \qquad (B4) = (Q1)\ (B1) + (Q2)\ (B2)$

$K = (A3)\ (B1) - (A1)\ (B3)$

$M = (A2)\ (B3) - (A3)\ (B2)$

$P = \sqrt{m_{Ai}^2 \ \{1 = (A4)^2\} + m_{Bi}^2 \ \{1 - (B4)^2\}}$

$L = 206265 \ \{(Q1)M + (Q2)K\}/P$

$A = \{(Q2)M - (Q1)K\}/P$

$B = \{(A1)\ (B2) - (A2)\ (B1)\}/P$

Normal equations: $[A^2]\ \xi + [AB]\ \eta + [AL] = 0$

$[AB]\ \xi + [B^2]\ \eta + [BL] = 0$

$[pvv] = [L^2] + \xi[AL] + \eta[BL]$

The true distances of point Q from arcs Ai - Bi are

$$l_i = \frac{L}{\sqrt{A^2 + B^2}}$$

Hence:

$$v_i = \frac{A\xi + B\eta + L}{\sqrt{A^2 + B^2}}$$

Example: $m_{Ai} = m_{Bi} = 1$

t_Q	-75°10'10".00	o_Q	-26°48'00".00
t_{Ai}	-27°16'14".70	o_{Ai}	+13°27'18".76
t_{Bi}	+ 5°35'53".50	o_{Bi}	+34°31'11".45
$t_{Ai} - t_Q$	+47°53'55".30	$t_{Bi} - t_Q$	+80°46'03".50
(Q1)	-0.45087754	(Q2)	+0.89258582
(A1)	+0.23268520	(B1)	+0.56669168
(A2)	+0.65204128	(B2)	+0.13219026
(A3)	+0.72159530	(B3)	+0.81325659
(A4)	+0.47709027	(B4)	+0.13751740
K	+0.21968928	A	+0.3679
M	+0.43488900	B	-0.2558
P	1.3242	L	+1".53

Adjustment

Operation equations

i	P	A	B	L	v
1	1.324	+0.3679	-0.2558	+1".53	-0".20
2	1.324	3668	- 2582	+0.41	+2.31
3	1.323	3656	- 2606	+7.96	-14.52
4	1.321	3626	- 2666	+1.25	+0.39
5	1.320	3613	- 2690	+4.95	-7.84
6	1.206	+0.3459	-0.2506	-0.78	+5.03
7	1.204	3436	- 2544	-2.14	+8.19
8	1.203	3427	- 2556	-2.88	+9.89
9	1.203	3419	- 2569	+1.37	-0.04
10	1.139	+0.1959	-0.3694	+1.80	-2.29
11	1.138	1930	- 3704	+1.03	-0.48
12	1.287	+0.1431	-0.4260	+1.05	-0.84
13	1.286	1406	- 4259	-1.81	+5.50

14	1.283 v	1356	- 4255	+3.02	-5.34
15	1.282	1332	- 4253	+2.19	-3.51
16	1.281	1295	- 4249	-0.91	+3.43
17	1.280	1283	- 4247	+1.12	-1.13
18	1.279	1271	- 4246	-0.88	+3.37

Normal equations

$1.3391\xi \; -1.3700\eta \; +5.406 = 0$

$-1.3700\xi \; +2.1419\eta \; -5.771 = 0$

$[L^2] = 132.80$

Solution

$\xi = -3''.704 = d\delta_Q$

$\eta = 0.325 \qquad dt_Q = +0''.364$

$[pvv] = 110.90$

Weight coefficients

2.1608	+1.3821
+1.3821	1.3509

$m^2 = 6.931 \quad m = \pm\, 2''.63$

$\delta_{AB} = -26°48'03''.70 \pm 3''.87$

$t_{AB} = -75°10'09''.64 \pm 3''.43$

Error ellipse

Semiaxis	4''.71	direction 37°
	1.48	127

4. Adjustment of a Network

For each station of the network, we need to calculate good approximate coordinates designated as ϕ, λ, H. Using any reference ellipsoid with semiaxes a, b and denoting

$$e^2 = \frac{a^2 - b^2}{a^2}$$

we compute the Cartesian coordinates

$$\begin{cases} x = (N + H) \cos \phi \cos \lambda \\ y = (N + H) \cos \phi \sin \lambda \\ z = (N - e^2 N + H) \sin \phi \end{cases} \tag{17}$$

where

$$\begin{cases} N = \frac{a}{W} \\ W = \sqrt{1 - e^2 \sin^2 \phi} \end{cases}$$

The fictitious observations δ_{AB}, t_{AB} between two stations A and B give residuals

$$\begin{cases} k_{AB} = \arctan \dfrac{z_B - z_A}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}} - \delta_{AB} \\ h_{AB} = \left(\arctan \dfrac{y_B - y_A}{x_B - x_A} - t_{AB}\right) \cos \delta_{AB} \end{cases} \tag{18}$$

Differentiation of (18) shows that if x, y, z are corrected by dx, dy, dz, respectively, the new residuals will be

$$v_k = A_1(dx_B - dx_A) + B_1(dy_B - dy_A) + C_1(dz_B - dz_A) + k_{AB} \tag{19}$$

$$v_h = A_2(dx_B - dx_A) + B_2(dy_B - dy_A) + h_{AB} \tag{20}$$

where

$$
\left\{
\begin{aligned}
A_1 &= -\frac{\rho}{r}\sin\delta_{AB}\cos t_{AB} \\
B_1 &= -\frac{\rho}{r}\sin\delta_{AB}\sin t_{AB} \\
C_1 &= +\frac{\rho}{r}\cos\delta_{AB} \\
A_2 &= -\frac{\rho}{r}\sin t_{AB} \\
B_2 &= +\frac{\rho}{r}\cos t_{AB} \\
r &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}
\end{aligned}
\right. \tag{21}
$$

The weight matrix of (19) and (20) is given by (15). Therefore, we form equations

$$
\left\{
\begin{aligned}
&\frac{(19)[paa] + (20)[pab]}{m\sqrt{[paa]}} = v_1 \\
&\frac{(20)}{m}\sqrt{[pbb] - \frac{[pab]^2}{[paa]}} = v_2
\end{aligned}
\right. \tag{22}
$$

which can be considered as uncorrelated observation equations with weights of 1. Then the normal equations can be formed and solved, in order to obtain corrections dx, dy, dz.

In this adjustment, only the earth-fixed space directions have been used as observed quantities. Therefore, the network cannot be fixed with respect to any absolute zero point nor to the scale. If the corrections of all coordinates are considered to be unknown, the matrix of normal equations becomes singular. In fact, one point remains undetermined and, in addition, one coordinate of some other point.

In the practice, the scale can be determined from the existing ground triangulations between some stations. Perhaps, also, the observed distances of satellites from the ground stations can be used, with appropriate weights. Similarly, the absolute zero point can be determined from existing astronomical observations of latitude and longitude, with appropriate gravimetric corrections for the deflection of the vertical. It is probable that the weights for all these additional observations are smaller than those for the space directions. Therefore, it is hardly feasible to include them in the main adjustment of the satellite network.

In the following adjustments, the missing four coordinates are determined as follows: The observation equations (22) are taken in matrix form

$$Ax + 1 = v \tag{23}$$

but the four coordinates x_1 cannot be determined. Therefore, we write

$$A_1 x_1 + A_2 x_2 + 1 = v \tag{24}$$

and take four more equations

$$B_1 x_1 + B_2^T x_2 = 0 \tag{25}$$

from conditions that the arithmetical mean of all (or, at least, the best) approximate coordinates and of the side lengths will not be changed:

$$[dx] = [dy] = [dz] = [xdx + ydy + zdz] = 0 \tag{26}$$

Because B_1 is a 4 x 4 matrix, we can compute

$$x_1 = -B_1^{-1} B_2^T x_2$$

and (24) becomes

$$(A_2 - A_1 B_1^{-1} B_2) x_2 + 1 = v$$

This system of equations is no longer singular. Therefore, denoting:

$$\begin{cases} G^T = B_1^{-1} B_2^T \\ A = A_2 - A_1 G^T \end{cases}$$

$$\begin{cases} N = A^T A \\ u = A^T 1 \end{cases}$$

we can compute

$$\begin{cases} x_2 = -N^{-1} u \\ x_1 = -G^T x_2 \end{cases}$$

and the weight coefficients:

$$\begin{cases} Q_2 = N^{-1} \\ Q_1 = G^T N^{-1} G \end{cases}$$

Finally, we may point out that the Cartesian coordinates (17) are needed for the computation of residuals (18) only. The observation equations (19) and (20) can be written using the original parameters ϕ, λ, and H in which purpose it is advisable to introduce temporary differentials

$$\begin{aligned} dB &= (M + H)\ d\phi \\ dL &= (N + H) \cos \phi\ d\lambda \end{aligned} \tag{27}$$

where $M = \dfrac{1 - e^2}{W^2} N = \dfrac{1 - e^2}{W^3}\ a$

For the computation of the coefficients of these new observation equations, the following matrix formula can be used:

$$\begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} \cos\phi \; \cos\lambda & - \sin\phi \; \cos\lambda & - \sin\lambda \\ \cos\phi \; \sin\lambda & - \sin\phi \; \sin\lambda & \cos\lambda \\ \sin\phi & \cos\phi & 0 \end{Bmatrix} \begin{Bmatrix} dH \\ dB \\ dL \end{Bmatrix} \tag{28}$$

Summary

There are as yet no extensive observed nets which would provide data that could be used in numerical examples or in testing the methods suggested above. However, I am currently studying a schematic net which is a very good approximation of the global satellite network planned for international cooperation. In brief, I computed a net of 20 spherical triangles with 12 corner points and 30 sides, using the well-known properties of a regular icosaeder. The middle points of each side were then taken as additional corner points. In this way, a network with 80 triangles, 42 stations, and 120 sides was obtained. I hope that the experiences obtainable from the computation of such a regular system will give useful ideas to others for the planning of future computations of the networks actually observed.

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