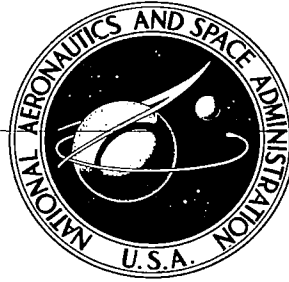


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SERIES SOLUTION FOR
HYPERSONIC FLOW ABOUT
BLUNT BODIES

by Y. S. Chou

Prepared by

LOCKHEED MISSILES & SPACE COMPANY

Palo Alto, Calif.

for Ames Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • FEBRUARY 1969



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By Y. S. Chou

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CONTENTS

Section	Page
NOMENCLATURE	v
PART I. ADIABATIC INVISCID CASES	
1 INTRODUCTION	1
2 GOVERNING EQUATIONS	4
3 SERIES SOLUTIONS	9
3.1 Formulation	9
3.2 Solution for Velocity Field	12
3.3 Solution for Pressure Distribution	17
3.4 Iterative Solutions	19
4 EXAMPLES	22
4.1 Inverse Problem	22
4.2 Direct Problem	22
5 CONCLUSIONS	33
6 REFERENCES	34
Appendix	
I COEFFICIENTS IN THE POWER SERIES REPRESENTATION	36
II A LIST OF COEFFICIENTS A'S AND B'S	40
PART II. VISCOUS RADIATING-ABSORBING CASES	
1 INTRODUCTION	44
2 ANALYSIS	46
2.1 Governing Equations	46
2.2 Gas Properties	50

Section		Page
	2.3 Series Formulation	57
	2.4 Method of Solution	64
3	CONCLUSIONS	67
4	REFERENCES	68
Appendix		
III	A LIST OF COEFFICIENTS FOR PART II	69
IV	ERROR ESTIMATION FOR THE QUASI-ONE-DIMENSIONAL APPROXIMATION	80

NOMENCLATURE

$A_1, A_2 \dots$	constant coefficients given in Appendix II
$A_{i,1}, A_{i,2} \dots$	coefficients in the series expansion for integrated intensity
B	blackbody radiation
B_b	body bluntness $(b/a)^2$ (see Fig. 1)
B_ν	Planck function
$B_1, B_2 \dots$	constant coefficients given in Appendix II
C	speed of light
$E_{i,0} \dots E_{i,j}$	universal function for integrated intensity
$f_1, f_2 \dots$	coefficients in the series expansion for tangential velocity
$g_1, g_2 \dots$	coefficients in the series expansion for total enthalpy
H	total enthalpy
h	specific enthalpy, Planck constant
I_ν	spectral radiation intensity
I_o	integrated intensity
k	body curvature, Boltzman constant
M	Mach number
m	$m = 1$ for inverse problem, $m = 0$ for direct problem
N	mass density viscosity ratio $\rho_s \mu_s / \rho \mu$
N_N	number density of nitrogen atoms
p	pressure

$p_{s,t}$	pressure at 1 atm
Pr	Prandtl number
$Q_{i,1}, Q_{i,2}$	coefficients in the series expansion for radiative heat flux
$(q_x)_\nu, (q_y)_\nu$	spectral radiative heat flux in the direction of x and y , respectively
$\dot{q}$	normalized total heat flux
r	distance measured from axis of symmetry
R_b	body radius of curvature at stagnation point
R_s	shock radius of curvature at stagnation point
Re	Reynolds number $\rho_\infty U_\infty R_s / \mu_{s(G)}$
T	temperature
u	velocity component parallel to the body
v	velocity component perpendicular to the body
$v_1, v_2 \dots$	coefficients in the series expansion of normal velocity
x	distance parallel to the body measured from stagnation line
$X_{i,0} \dots X_{i,j}$	universal function for velocity profile
y	distance normal to the body
$Y_0 \dots Y_j$	universal function for velocity profile
$Z_0 \dots Z_j$	universal function for radiative heat flux
α_ν	absorption coefficient
β	degree of dissociation
γ	specific heat ratio C_p/C_v
Δ	shock layer thickness
ρ	density
ψ	stream function

ξ	normalized distance parallel to the body
η	normalized stream function [Eq. (6), Part I]
ξ_N	quantum mechanical correction factor for absorption coefficient
μ	coefficient of viscosity
φ	degree of ionization
θ_d	dissociation energy
θ_i	ionization energy
ϵ	stagnation point density ratio across shock
δ	$3-2\epsilon$
ω	parametric variable defined by Eq. (31a) Part I
τ	normalized skin friction

Subscripts

s	quantities at shock
b	quantities at body
o	quantities at stagnation line
i	iterative solution

PART I. ADIABATIC INVISCID CASES

Section 1

INTRODUCTION

A considerable amount of work has been done on the subject of a blunt-nosed body moving in a uniform equilibrium gas at supersonic speeds [Chester (1956), Freeman (1956), Belotserkovskii (1961), and others]. Despite all efforts made, a definitive solution for the blunt body flow is yet to be found. A simple and accurate solution is still in great demand in view of the complicated numerical scheme one usually encounters. It is therefore the purpose of this paper to present such a solution. It seems almost certain that such a solution would be the result of some approximation; in the present case, the approximation is based on the flow being hypersonic, which implies the density ratio (ρ_∞/ρ_s) across the shock wave is small. We further restrict our consideration to axisymmetric bodies for which the solution can be written in analytic form.

A series approach is employed. It differs from the existing series formulations [Swigart (1963), Van Dyke (1965), Conti (1966), Kao (1967)] both in form and in concept. In Swigart and Van Dyke's formulations, the series was expanded in parabolic coordinates, while Conti and Kao have written their series in physical polar coordinates. Swigart treated the density and stream functions as dependent variables. Later, Van Dyke and Conti chose pressure instead of density and the expression of the series was improved by Van Dyke to increase the convergence of the series. It is well known that because the governing equations are elliptic in nature, truncation of the series is necessary; this method is therefore called the "method of series truncation." The process of truncation in the method is, however, somewhat arbitrary; it is more out of necessity than logical consequence. Kao has suggested a more rational way to truncate the series by implicitly taking advantage of the smallness of the density ratio. In his scheme, the truncated quantities are of a lower order of magnitude. Kao is able to present his solution in analytic form; however, the complicated algebra involved in his solution seems rather discouraging.

The accuracy given by his solution also is not completely satisfactory. A comparison will be made in the text. In the present paper, the standard body (direct problem) or shock (inverse problem) oriented coordinate system will be used. A von Mises transformation is made and the normalized stream function η and normalized distance along body (or shock) ξ will be the independent variables. The series then is written in powers of ξ with coefficients as functions of η . This form of series is compatible with the Rankine-Hugoniot shock conditions and is in the same spirit as the Blasius series in the classical boundary layer theory.

In the hypersonic limit, the y-momentum equation can be simplified by neglecting the convective terms in that equation, with the resulting set of equations usually referred to as the "thin layer equation." An immediate consequence of this simplification is, mathematically, to change the original elliptical governing equation into one of parabolic form. Presumably then, an initial value problem can be formulated. However, in an unpublished report by Chou (1967), it was shown that it is not possible to formulate completely such an initial value problem. The backward upstream influence still exists, and is a fundamental feature for the blunt body flow problem. It is then understood that taking the upstream influence into account will be essential for any approximate method proposed for the problem, regardless of whether the equations have been reduced to parabolic form or not. The basic elliptic nature of blunt body flow can also become clear by an order of magnitude consideration of the y-momentum equation. Such an examination reveals that in the stagnation region the convective terms in the y-momentum equation are of the same order of magnitude as the rest of the terms in that equation. Thus, the parabolic type thin layer equation is not strictly applicable in that region. Based on these considerations, we therefore performed the following steps in finding a simple approximate yet accurate solution for the blunt body flow problem. First, we solve the thin layer equations. Then an approximate iterative solution based on the thin layer solution is obtained. The thin layer solution is obtained without requiring truncation of the series. The upstream influence is taken into account in the determination of the stagnation point shock curvature R_s . For the iterative solution, truncation of the series is again not

necessary; however, it has actually been made implicitly in the course of the approximation. The quantities being implicitly truncated are of order ϵ^2 and the truncation is a logical consequence of demanding that the solution be consistent.

Throughout the analysis, perfect gas thermodynamics are assumed. The three term thin layer solution and the two term iteration solution are presented in closed form. Examples are given with emphasis on the direct problem. An inverse problem of flow behind a paraboloidal shock is also given. Extensive comparison with other solutions as well as available experimental data show that the present solution yields accurate results up to the sonic line for various body shapes, except for the shape that will produce a hyperboloidal shock. For this kind of body shape, the series solution is found to converge slowly. The iterative solution is found to improve the thin layer solution as well as to increase the radius of convergence of the series. Finally, the simplicity of the solution and the small effort needed in the calculation will be emphasized.

Section 2

GOVERNING EQUATIONS

For the axisymmetric case, the inviscid equations written in body (or shock) oriented coordinates (Fig. 1) are

$$\frac{\partial \rho u r}{\partial x} + \frac{\partial \rho v r \tilde{k}}{\partial y} = 0 \quad (1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \tilde{k} \frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} - \rho k u v \quad (2)$$

$$\frac{\tilde{k}}{\rho} \frac{\partial p}{\partial y} = k u^2 - u \frac{\partial v}{\partial x} - k v \frac{\partial v}{\partial y} \quad (3)$$

$$H = h + \frac{1}{2} (u^2 + v^2) = H_s = \text{constant} \quad (4)$$

with

$$\tilde{k} = 1 + k y \quad (5)$$

where ρ is the density, u, v the velocity component along x and y , respectively, p is the pressure, k is the body (or shock) curvature. The above equations are the continuity, x -momentum, y -momentum, and energy equations, respectively. We have replaced the energy equation by the simple statement that the total enthalpy H is constant.

We now proceed to normalize the variables. The distances x, y and the distance from the axis, r , are normalized by the stagnation point shock radius R_s , the velocities u, v by the free stream velocity u_∞ , the density ρ by the free stream

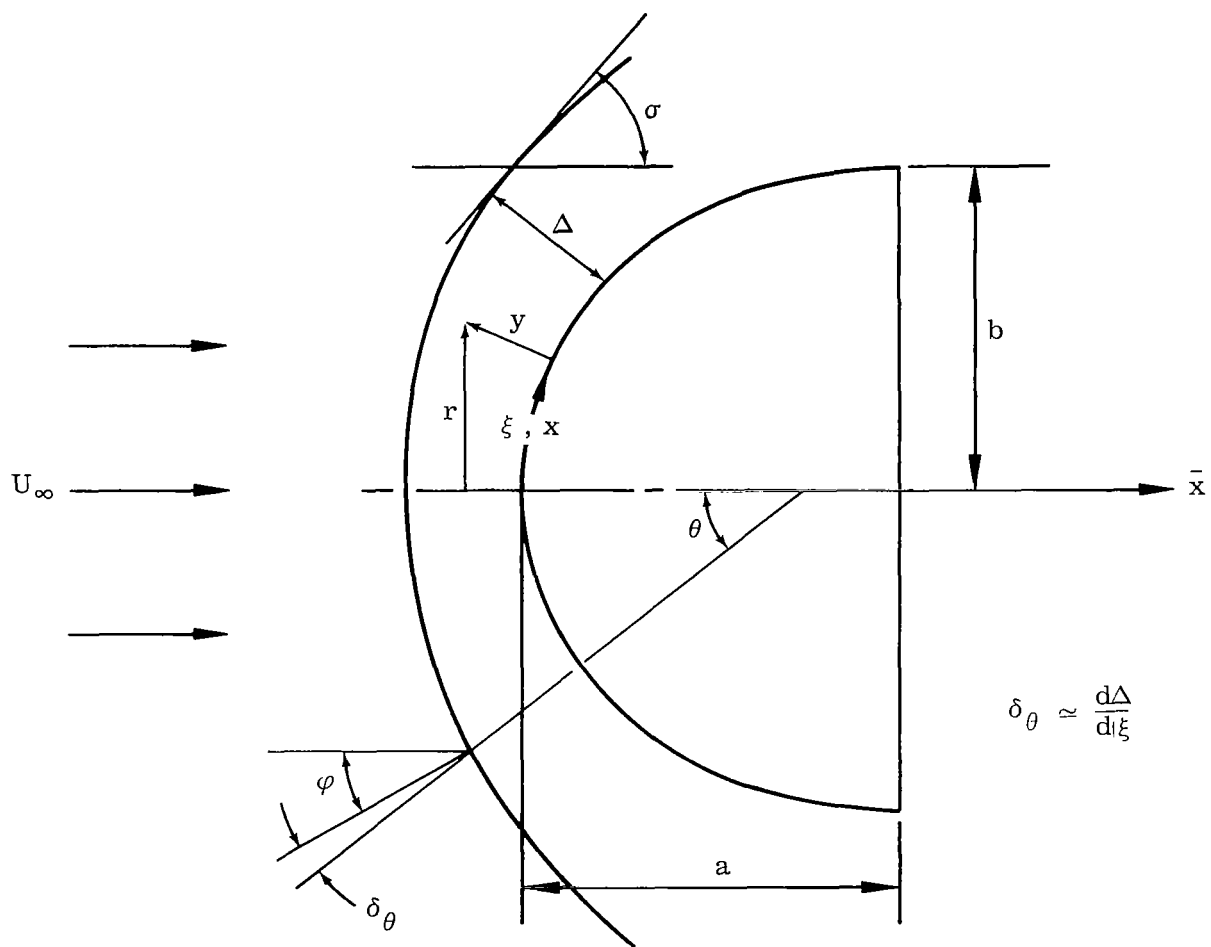


Fig. 1 Sketch of geometry

density ρ_∞ , the pressure p by twice the free stream kinetic pressure $\rho_\infty u_\infty^2$, the body (or shock) curvature k by $1/R_s$, the total enthalpy H as well as the static enthalpy h by H_s and, finally, the stream function ψ by $\rho_\infty u_\infty R_s^2$. From here on, all the equations are considered in nondimensional form.

The following transformation is made:

$$\xi(x) = x, \quad \eta(x, y) = \frac{\psi}{r_s^2}, \quad d\psi = \rho u r dy - \rho v r k dx \quad (6)$$

where subscript s denotes quantities at the shock, and subscript b denotes quantities at the body. Since $\psi_s = (1/2)r_s^2$ and $\psi_b = 0$, the range of η is from 0 to 1/2.

In view of Eq. (6), we obtain the following operational transformation:

$$\frac{\partial}{\partial y} = \frac{\rho u r}{r_s^2} \frac{\partial}{\partial \eta} \quad (7)$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \xi} \right)_\eta \frac{d\xi}{dx} - \left(\frac{\rho v k r}{r_s^2} + \frac{2\eta}{r_s} \frac{dr_s}{d\xi} \frac{d\xi}{dx} \right) \left(\frac{\partial}{\partial \eta} \right)_\xi \quad (8)$$

After using Eqs. (7) and (8) together with the following hypersonic approximation, namely

$$\frac{U_\infty^2}{H_s} \simeq 2 \quad (9)$$

the momentum and energy equations become

$$\rho u \left(r_s \frac{\partial u}{\partial \xi} - 2\eta \frac{dr_s}{d\xi} \frac{\partial u}{\partial \eta} \right) = - \left[r_s \frac{\partial p}{\partial \xi} - \left(\rho v \frac{k r}{r_s^2} + 2\eta \frac{dr_s}{d\xi} \right) \frac{\partial p}{\partial \eta} \right] - \rho u v k r_s \quad (10)$$

$$\frac{\partial p}{\partial \eta} = \frac{r_s}{r k} \left[r_s k u - \left(r_s \frac{\partial v}{\partial \xi} - 2\eta \frac{dr_s}{d\xi} \frac{\partial v}{\partial \eta} \right) \right] \quad (11)$$

$$h = 1 - (u^2 + v^2) \quad (12)$$

These equations, together with the following perfect gas thermodynamics, form a complete set of equations.

$$\frac{\rho}{\rho_s} = \frac{p h}{p_s h} \quad (13)$$

The streamline can be obtained from Eq. (6) as

$$y(\xi) = r_s^2 \int_0^\eta \frac{d\eta}{\rho u r} \quad (14)$$

The shock wave location or body shape (inverse problem) can be found as

$$\Delta(\xi) = r_s^2 \int_0^{1/2} \frac{d\eta}{\rho u r} \quad (15)$$

Also, the velocity component in the y direction is given by

$$v = - \frac{1}{\rho r k} \left(\frac{\partial \psi}{\partial x} \right)_y$$

but

$$\frac{\partial \psi}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{x}} \left(\eta \, r_s^2 \right) = 2 r_s \, \eta \, \frac{d\mathbf{r}_s}{d\xi} + r_s^2 \left(\frac{\partial \eta}{\partial \xi} \right)_y$$

thus

$$\mathbf{v} = - \frac{r_s}{\rho r \tilde{k}} \left[2 \frac{d\mathbf{r}_s}{d\xi} \eta + r_s \left(\frac{\partial \eta}{\partial \xi} \right)_y \right] \quad (16)$$

Section 3 SERIES SOLUTIONS

3.1 FORMULATION

Let us now first examine the Rankine-Hugoniot oblique shock relations. Clearly, all the variables u_s , p_s , ρ_s are functions of either $\sin \varphi$ or $\cos \varphi$, where φ is the shock angle. A Taylor expansion of $\sin \phi$ and $\cos \phi$ in powers of distance along the shock is possible. Thus, all the variables behind the shock can be represented by a power series in terms of distance along the shock measured from the stagnation point. Based on this consideration, we assume the following series representation of variables behind the shock.

$$\begin{aligned}
 u_s &= u_o (\xi + u_1 \xi^3 + u_2 \xi^5 + \dots) \\
 r_s &= u_o (\xi + r_1 \xi^3 + r_2 \xi^5 + \dots) \\
 k_s &= 1 + k_1 \xi^2 + k_2 \xi^4 + \dots \\
 \rho_s &= \frac{1}{\epsilon} (1 - \rho_1 \xi^2 - \rho_2 \xi^4 \dots) \\
 p_s &= (1 - \epsilon) (1 - p_1 \xi^2 - p_2 \xi^4 \dots)
 \end{aligned} \tag{17}$$

with $\epsilon = [(\gamma - 1)M_\infty^2 + 2]/(\gamma + 1)M_\infty^2$, $\gamma = C_p/C_v$, and M_∞ is the free stream Mach number. The quantity u_o equals $(d\xi_s/d\xi)_o$ for the direct problem and equals unity for the inverse problem. The variable ξ_s is the distance along the shock and ξ the distance along the body. The subscript o denotes quantities at the stagnation point.

In view of Eq. (17), it is logical to assume that inside the layer the velocity can also be represented by a power series in ξ . We write

$$u = u_o \left[f_1(\eta)\xi + u_1 f_2(\eta)\xi^3 + u_2 f_3(\eta)\xi^5 + \dots \right] \quad (18)$$

The boundary condition is simply: at $\eta = 1/2$, $f_1 = 1$, $f_2 = 1$, and $f_3 = 1$. The coefficients u_1 , r_1 and k_1 , etc., depend on the specific body shape (or the given shock shape). The quantities ρ_1 , ρ_2 depend on the free stream Mach number and γ . These coefficients are obtained in Appendix I for a general conical body (or shock). The relations between ξ and the body angle θ (or φ) for this class of body (or shock) are also presented in Appendix I.

By knowing the expression for ρ (or p) we can also obtain series for y and Δ from Eqs. (14) and (15). We write

$$y = \epsilon \left(y_o + y_1 \xi^2 + y_2 \xi^4 + \dots \right) \quad (19a)$$

$$\Delta = \epsilon \left(\Delta_o + \Delta_1 \xi^2 + \Delta_2 \xi^4 + \dots \right) \quad (19b)$$

where y_o and y_1 will be determined later. The quantities Δ_o and Δ_1 are y_o and y_1 evaluated at $\eta = 1/2$. We also note that y_o , y_1 are normalized distances measured from the body along the line $\xi = \text{constant}$.

Obviously $\epsilon \Delta_o$ is the normalized standoff distance. From the expression for Δ [Eq. (19b)] and the geometry of Fig. 1, we can derive an approximate formula for R_s/R_b as follows. By definition

$$\frac{d\varphi}{d\xi_s} = 1$$

But

$$\varphi \simeq \theta_b - \epsilon \delta \simeq \theta_b - 2\xi \epsilon \Delta_1$$

hence

$$\frac{d}{d\xi} \left(\theta_b - 2\xi \epsilon \Delta_1 \right) \frac{d\xi}{d\xi_s} = \left(\frac{R_s}{R_b} - 2\epsilon \Delta_1 \right) \frac{d\xi}{d\xi_s} = 1 \quad (20a)$$

This then yields

$$\frac{R_s}{R_b} = \left(\frac{d\xi}{d\xi_s} \right)_o + 2\epsilon \Delta_1 \quad (20b)$$

At this point, it is interesting to note that the implicit form of upstream influence defined by Chou (1967) clearly appears here. Namely, the stagnation point shock curvature depends on an unknown quantity $(d\xi_s/d\xi)_o$ which can be either arbitrarily given or determined by some condition downstream (singularities, for example). We also note that for inverse problems since R_s is known and R_b is not needed for the solution and can be found after the body shape is determined, then the implicit form of an upstream influence thus does not exist for inverse problems.

In this study, we will approximate $(d\xi_s/d\xi)_o$ by assuming the shock is ccncentric with the body at the stagnation point. Therefore,

$$\left(\frac{d\xi}{d\xi_s} \right)_o = \frac{R_b}{R_b + \epsilon \Delta_o R_s} \quad (21)$$

Substituting Eq. (21) into Eq. (20b) yields

$$\frac{R_s}{R_b} = \frac{1 + 2\epsilon \Delta_1}{1 - \epsilon \Delta_o} \quad (22)$$

3.2 SOLUTION FOR VELOCITY FIELD

We will now proceed to seek the solution only up to the order ϵ . Therefore, one can simplify the governing equations, (10) and (11), as

$$\rho u \left(r_s \frac{\partial u}{\partial \xi} - 2\eta \frac{dr_s}{d\xi} \frac{\partial u}{\partial \eta} \right) = - \left(r_s \frac{\partial p}{\partial \xi} - 2\eta \frac{dr_s}{d\xi} \frac{\partial p}{\partial \eta} \right) \quad (23)$$

$$\frac{\partial p}{\partial \eta} = r_s k u \quad (24)$$

We note here that $1/\rho \sim 0(\epsilon)$, hence $v \sim 0(\epsilon)$. Also, $r_s/r \sim 1 + 0(\epsilon)$, $\tilde{k} = 1 + 0(\epsilon)$.

Equations (23) and (24) are the so-called "thin layer" equations. These equations are not strictly valid in the stagnation regions where the velocity u itself is of the order $(\epsilon)^{1/2}$; in turn, the term neglected in Eq. (24) has an order of ϵ effect on u in this region. Therefore, these equations are not adequate for the description of flow around a body with very large bluntness. We will solve these equations first; then, based on these solutions, we will approximate the effects of the convection term in the y-momentum Eq. (11) in the next section with the hope that the solutions can be improved as well as be extended to body shapes with large bluntness.

In order to make the solution as simple as possible while not sacrificing much accuracy, Maslen's approximation (1964) will now be employed by replacing ku in Eq. (24) by $k_s u_s$. This approximation has been shown to yield satisfactory results for paraboloidal and spherical shocks. Equation (24) then can be integrated to give the relation

$$p = p_s + \left(\eta - \frac{1}{2} \right) (k_s u_s r_s) \quad (25)$$

By using Eqs. (25), (13), (12), and (17), one can find a series representation for ρ as

$$\frac{1}{\rho} = \epsilon \left(1 + \bar{\rho}_1 \xi^2 + \bar{\rho}_2 \xi^4 + \dots \right) \quad (26a)$$

with

$$\bar{\rho}_1 = \rho_1 + u_o^2 \left[1 - f_1^2 + \frac{\left(\frac{1}{2} - \eta \right)}{1 - \epsilon} \right] \quad (26b)$$

$$\begin{aligned} \bar{\rho}_2 = & \rho_1^2 + \rho_2 + u_o^2 \left\{ 2u_1(1 - f_1 f_2) + u_o^2(1 - f_1^2) - \frac{\left(\eta - \frac{1}{2} \right)}{(1 - \epsilon)} \left[k_1 + r_1 + u_1 + p_1 \right. \right. \\ & \left. \left. + (1 - f_1^2)u_o^2 \right] + \rho_1 \left[1 - f_1^2 - \frac{\left(\eta - \frac{1}{2} \right)}{1 - \epsilon} \right] + u_o^2 \frac{\left(\eta - \frac{1}{2} \right)^2}{(1 - \epsilon)^2} \right\} \end{aligned} \quad (26c)$$

Consequently, y_o , y_1 , and y_2 are found to be (after setting $r \approx r_s$)

$$y_o = \int_o^\eta \frac{d\eta}{f_1} \quad (27a)$$

$$y_1 = (\rho_1 + r_1)y_o + \int_o^\eta \left\{ u_o^2 \left[1 + \frac{\left(\frac{1}{2} - \eta \right)}{1 - \epsilon} - f_1^2 \right] - \frac{u_1 f_2}{f_1} \right\} \frac{d\eta}{f_1} \quad (27b)$$

$$y_2 = (r_2 - r_1^2)y_o + r_1 y_1 - \int_o^\eta \left[\frac{u_2 f_3}{f_1} + \frac{u_1 f_2 \bar{\rho}_1}{f_1} - \bar{\rho}_2 - \left(\frac{u_1 f_2}{f_1} \right)^2 \right] \frac{d\eta}{f_1} \quad (27c)$$

Now after inserting Eqs. (17), (18), (25), and (26), and the relation

$$\frac{\partial p_s}{\partial \xi} = -2(1 - \epsilon)u_s \frac{\partial u_s}{\partial \xi}$$

from the Rankine-Hugoniot conditions into Eq. (23), collecting terms of same order of ξ , one finally obtains the governing equations for f_1 , f_2 , and f_3 as

$$2\eta f_1 \frac{df_1}{d\eta} - f_1^2 = -\epsilon \delta \quad (28)$$

$$2\eta f_1 \frac{df_2}{d\eta} - 3f_1 f_2 - \frac{\epsilon \delta f_2}{f_1} = -\epsilon A_1 - \frac{4r_1 f_1 \eta}{u_1} \frac{df_1}{d\eta} + \epsilon A_2 \eta + \frac{\epsilon \delta f_1^2 u_o^2}{u_1} \quad (29)$$

$$\begin{aligned} 2\eta f_1 \frac{df_3}{d\eta} - 5f_1 f_3 - \frac{\epsilon \delta f_3}{f_1} &= \frac{\epsilon}{u_2} \left\{ \frac{u_1^2 f_2}{f_1} \left(A_1 - A_2 \eta - \frac{\delta f_2}{f_1} \right) - B_3 + 2\eta B_4 + u_1 u_o^2 \delta f_1 f_2 \right. \\ &\quad - \left[\frac{\delta u_o^4}{1 - \epsilon} + 2(k_1 + u_1 - r_1) u_o^2 \right] \eta f_1^2 + u_o^2 \delta \left[u_o^2 (1 + \rho_1 \right. \\ &\quad \left. \left. + \frac{1}{2(1 - \epsilon)}) - r_1 \right] f_1^2 - \frac{u_o^2}{1 - \epsilon} \left[2(k_1 + u_1 - r_1) - \frac{u_o^2 \delta}{1 - \epsilon} \right] \eta^2 \right\} \\ &\quad - 4\eta f_1 \frac{df_2}{d\eta} \frac{r_1 u_1}{u_2} \quad (30) \end{aligned}$$

with $\delta = 3 - 2\epsilon$, the coefficients A and B are all constants but are rather complicated algebraic functions of u_1 , r_1 , etc. They are listed in Appendix II.

The boundary conditions are simply at $\eta = 1/2$, $f_1 = f_2 = 1$.

It is now convenient to transform the variable further from η to ω by the relation

$$\frac{d\eta}{d\omega} = f_1 \quad (31a)$$

The range of ω will be set from one to ω_o , and ω_o is determined by the relation

$$\eta(\omega_o) = 0 \quad (31b)$$

Now in terms of ω , the solution to Eq. (28) can be written as

$$f_1 = \omega(1 - \epsilon\delta) + \epsilon\delta \quad (32a)$$

$$\eta = \frac{1}{2} [\omega^2 - \epsilon\delta(\omega - 1)^2]$$

Hence

$$\omega_o = \frac{\sqrt{\epsilon\delta}}{1 + \sqrt{\epsilon\delta}} \quad (32c)$$

The standoff distance $\epsilon\Delta_o$ from Eq. (27a) is therefore

$$\epsilon\Delta_o = \frac{\epsilon}{1 + \sqrt{\epsilon\delta}} \quad (33)$$

Using the solution given by Eq. (32), one can find the solution to f_2 as

$$f_2 = \frac{1}{f_1} \left(\frac{\epsilon A_1}{4} + A_3 \eta - \frac{\epsilon \delta u_o^2}{4u_1} f_1^2 + A_4 \eta^2 \right) \quad (34)$$

Similarly, the solution to f_3 is found to be

$$\begin{aligned} f_3 = \frac{1}{u_2 f_1} \left[B_6 + \frac{B_7}{f_1^2} + \left(B_8 + \frac{B_9}{f_1^2} \right) \eta + \left(B_{10} + \frac{B_{11}}{f_1^2} \right) \eta^2 + \left(B_{12} + \frac{B_{13}}{f_1^2} \right) \eta^3 \right. \\ \left. + B_{14} \eta^3 \ln \frac{2(1 - \epsilon\delta)f_1^2}{\eta} + B_{15} f_1^2 + \frac{\epsilon B_5}{2} \eta f_1^2 \right] \end{aligned} \quad (35)$$

Again, the coefficients A and B are constants given in Appendix II. Once f_2 and f_3 are known, y_1 and y_2 can be found from Eqs. (27b) and (27c). The integration involved in Eqs. (27b) and (27c) can be performed and y_1 and y_2 written in terms of f_1 , η , and ω . For the sake of simplicity, however, we will leave the integral representations for y_1 and y_2 .

The normal component of velocity v can be written [from Eq. (16)] as

$$v = -\frac{1}{\rho} \left[2\eta \frac{dr_s}{d\xi} + r_s \left(\frac{\partial \eta}{\partial \omega} \right) \left(\frac{\partial \omega}{\partial \xi} \right)_y \right]$$

Now since $y = \epsilon(y_o + y_1 \xi^2 + y_2 \xi^4 + \dots)$, we have, to the order of ξ^3

$$\left(\frac{\partial \omega}{\partial \xi} \right)_y = -2\xi \left[y_1 + \left(2y_2 - y_1 \frac{dy_1}{d\omega} \right) \xi^2 \right]$$

With use of the series expressions, Eqs. (17) and (18), and $\partial \eta / \partial \omega = f_1$, v can be written to the order of ξ^4 as

$$v = -\epsilon u_o \left(2\eta + v_1 \xi^2 + v_2 \xi^4 \right) \quad (36a)$$

with

$$v_1 = 2\eta \bar{\rho}_1 + 6\eta r_1 - 2(y_1 - m\Delta_1) f_1 \quad (36b)$$

$$v_2 = 2\eta(5r_2 + \bar{\rho}_2 + 3r_1 \bar{\rho}_1) - 4f_1(y_2 - m\Delta_2) - 2u_1 f_2(y_1 - m\Delta_1) \quad (36c)$$

Note here that y_1 and y_2 are measured from the body; hence for the inverse problem y_1 should be replaced by $(y_1 - \Delta_1)$, similarly for y_2 and y_o . Thus, in Eqs. (36b) and (36c) $m = 1$ for the inverse problem and $m = 0$ for the direct problem.

3.3 SOLUTION FOR PRESSURE DISTRIBUTION

For the determination of the pressure distribution, $r \simeq r_s$ and $\tilde{k} \simeq 1$ cannot be assumed. Expressions for r and $\tilde{k}$ are necessary. From geometry, we have $r = r_b + y \sin \theta$ for direct problems and $r = r_s + (y - k\Delta) \sin \varphi$ for inverse problems. Also, by definition we have $k = 1 + k(y - m\Delta)$. Now after using the series expression for r_b , y , k , Δ , and expanding $\sin \varphi$ in powers of ξ , one can obtain the following expression:

$$\frac{r_s}{rk} = \frac{(1 + \bar{r}_1 \xi^2 + \dots)}{[1 - \epsilon(\Delta_o - y_o)][1 + \epsilon(y_o - m\Delta_o)a_o]} \quad (37)$$

where $\bar{r}_1$ is a function of ω , y_1 , and Δ_1 . Since it is associated with ξ^2 and is of order ϵ , we will not write it down and it will be neglected in the calculation for the sake of simplicity. Equation (12) then can be integrated by using the solutions for velocities and Eq. (37) to obtain the following expression for the pressure distribution to the order ϵ .

$$p = p_s + \frac{\epsilon}{2} (1 - \eta^2) - u_o^2 [F_1 \xi^2 + G_1 \xi^4 + G_2 \xi^6 + \dots] \quad (38a)$$

$$F_1 = \int_{\omega}^1 f_1^2 \left\{ 1 + \epsilon [\Delta_o (1 + ma_o) - y_o (1 + a_o)] \right\} d\omega - 4\epsilon \int_{\omega}^1 v_1 f_1 d\omega + 2\eta\epsilon(v_1 + 3r_1\eta) - \epsilon(4.5 r_1 + \rho_1 - 2\Delta_1(1 - m)) \quad (38b)$$

$$G_1 = \int_{\omega}^1 \left\{ [u_1 f_2 + f_1(r_1 + k_1)] [1 + \epsilon\Delta_o(1 + ma_o) - y_o\epsilon(1 + a_o)] + 8\epsilon r_1 v_1 \right\} f_1 d\omega + 6\epsilon\eta r_1 v_1 - 3\epsilon r_1 v_1(1) \quad (38c)$$

$$G_2 = \frac{a_o(1-f_1^3)}{3(1-\epsilon\delta)}(k_2 + r_2 + k_1 r_1) - 2\epsilon [5r_2 v_1(1) + 3r_1 v_2(1)] + \int_{\omega}^1 \left\{ a_o [u_1(r_1 + k_1)f_2 + u_2 f_3] \right. \\ \left. + 12\epsilon r_2 v_1 + 10\epsilon r_1 v_2 \right\} f_1 d\omega \quad (38d)$$

The expression given by Eq. (38d) is not complete since the contribution due to normal velocity v to this term has been neglected. The pressure so obtained, however, is consistent with the three term velocity solution.

For the direct problem, p_s is approximated by

$$p_s = (1 - \epsilon)(1 - u_s^2) \quad (38e)$$

The parameter a_o is given by $a_o = m + (1 - m) R_s/R_b$, and $m = 1$ for the inverse problem, $m = 0$ for the direct problem. This parameter is introduced due to the fact that the curvature k is normalized by R_s .

From the pressure given by Eq. (38), one can obtain the corrected density from the perfect gas relation (13). To the order ϵ , the density given by (26b) and (26c) becomes

$$\bar{\rho}_1 = \rho_1 + u_o^2 \left(1 - f_1^2 + \frac{F_1}{1 - \epsilon} \right) \quad (39a)$$

$$\bar{\rho}_2 = \rho_1^2 + \rho_2 + u_o^2 \left\{ 2u_1(1 - f_1 f_2) + u_o^2(1 - f_1^2) + F_1/(1 - \epsilon) \left[\frac{u_o^2 F_1}{1 - \epsilon} + \frac{G_1}{F_1} + u_o^2(2 - f_1^2) \right] \right. \\ \left. + \rho_1 \left(1 - f_1^2 + \frac{F_1}{1 - \epsilon} \right) \right\} \quad (39b)$$

The physical coordinate y together with the shock (or body) location can be then determined by evaluating the integral given by Eq. (14). In the evaluation of this integral, in order to be consistent with the solution for velocity, we set $r \simeq r_s$.

For the sake of identification, we will name the solution obtained in this and previous sections as the thin layer solution since it is based on the thin layer equation.

3.4 ITERATIVE SOLUTIONS

We have mentioned earlier that the thin layer equations are not strictly valid in the stagnation region where u itself is of order $\epsilon^{1/2}$. Thus, the neglected second order terms in y -momentum equation have actually first order effects on u in this region. Therefore, the thin layer solution is not expected to be valid for a body shape with large bluntness, for which the velocity u is order of $\epsilon^{1/2}$ in most parts of the shock layer. In order to seek improvement of the thin layer solution, we propose the following approximation. First, we rearrange Eq. (10) to the following form:

$$\rho u \left(r_s \frac{\partial u}{\partial \xi} - 2\eta \frac{dr_s}{d\xi} \frac{\partial u}{\partial \eta} \right) = -r_s \frac{\partial p}{\partial \xi} + \frac{r_s}{rk} \left(\frac{\partial \eta}{\partial \xi} \right)_y \left(2y \frac{dr_s}{d\xi} \frac{\partial v}{\partial \eta} - r_s \frac{\partial v}{\partial \xi} \right) + \frac{2k r_s}{rk} \eta \frac{dr_s}{d\xi} \quad (40)$$

Then the right hand side of Eq. (40) is to be approximated by its value at the body. The density ρ , and $(\partial p / \partial \xi)_b$ are given by the thin layer solution, Eqs. (38) and (39).

This approximation is based on the observation made by Cheng (1966) that the right hand side of Eq. (40) will be of lower order except in a very thin layer close to the body where the streamlines have originated from the region $r_s \sim o(\epsilon^{1/2})$ and hence u in that region will be of order $\epsilon^{1/2}$.

Based on this approximation and using the series expressions given by Eq. (17), one can find the equations for f_{1i} and f_{2i} similar to those thin layer equations (here the subscript i denotes the iterative solution).

The solutions for f_{1i} and f_{2i} to the order of ϵ are given as follows:

$$\eta_i = \frac{1}{2} \left[\omega^2 - \epsilon \delta_i (\omega - 1)^2 \right] \quad (41)$$

$$f_{1i} = (1 - \epsilon \delta_i) \omega + \epsilon \delta_i \quad (42)$$

$$f_{2i} = \frac{1}{f_{1i}} \left\{ A_5 + A_6 \eta_i - \frac{\epsilon \delta_i u_o^2 f_{1i}}{4(1 - \epsilon) u_1} + \frac{\epsilon \delta_i u_o^2 \eta_i^2 (1 - \epsilon \delta_i)}{4 u_1 (1 - \epsilon) (\epsilon \delta_i)^{1/2}} \ln \left[\frac{f_{1i} (\epsilon \delta_i)^{1/2} - \epsilon \delta_i}{f_{1i} (\epsilon \delta_i)^{1/2} + \epsilon \delta_i} \right] \right. \\ \left. - \frac{\epsilon^2 \delta_i u_o^2 (3r_1 + u_o) \eta_i^2 \ln \eta_i}{u_1 (1 - \epsilon)} + \frac{u_o^2 \epsilon \delta_i F_1(\omega)}{4(1 - \epsilon) u_1} - \frac{\epsilon \delta_i f_{1i}^2 u_o^2}{4 u_1} + A_7 \eta_i^2 \right\} \quad (43)$$

with coefficients $A_5 - A_7$ again listed in Appendix II, and

$$\delta_i = 2 [(1 - \epsilon) + F_1(\omega_o)] \quad (44)$$

In this iterative solution $\omega_{oi} = [(\epsilon \delta_i)^{1/2} / (1 + (\epsilon \delta_i)^{1/2})]$ and consequently the standoff distance is

$$\epsilon \Delta_{oi} = \frac{\epsilon}{1 + (\epsilon \delta_i)^{1/2}} \quad (45)$$

It is now interesting to see that for very small ϵ , δ_i can be approximated by $\delta_i \simeq \delta_i(\epsilon \rightarrow 0) = 8/3$. The expression Eq. (45) then is identical to the constant density solution given in the monograph by Hayes and Probstein (1966).

Similarly, y_{1i} and v_i can be obtained by using the density ρ is given by Eq. (39) as

$$y_{1i} = (\rho_1 + r_1 + u_o^2)(\omega - \omega_{oi}) + \frac{u_o^2}{1 - \epsilon} \int_{\omega_{oi}}^{\omega} F_1(\omega) d\omega - \frac{u_o^2}{3(1 - \epsilon \delta_i)} [f_{1i}^3 - (\epsilon \delta_i)^{3/2}] \\ - u_1 \int_{\omega_{oi}}^{\omega} \frac{f_{2i}}{f_{1i}} d\omega \quad (46)$$

$$v_i = -\epsilon \left(2\eta_i + v_{1i} \xi^2 \right) u_o \quad (47a)$$

$$v_{1i} = 2\eta_i \left[\left(1 - \frac{F_1}{1-\epsilon} - f_{1i}^2 \right) u_o^2 + \rho_1 \right] + 6r_1 \eta_i - 2(y_{1i} - m\Delta_{1i}) f_{1i} \quad (47b)$$

An expression identical to Eq. (38) can be obtained for pressure p_i by simply changing f_1 to f_{1i} , f_2 to f_{2i} , η to η_i , v_1 to v_{1i} , δ to δ_i , $v_2 = 0$, and $G_2 = 0$. Similarly, Eq. (39) less the term of order ξ is also valid for ρ_i (to the order of ϵ) by changing f_1 to f_{1i} and F_1 to F_i . Again, the physical coordinate y_i is obtained by evaluating Eq. (10) under the approximation $r \simeq r_s$. We note again that the pressure so obtained is not strictly complete since in the expression for G_{1i} the contribution due to v_2 has been disregarded; nevertheless, it is consistent with the two-term velocity solution. We also note here that only two terms of the iteration series will require four terms of the thin layer series. The examples given below, showing that the two-term iterative solutions yield satisfactory results, indicate that the proposed iterative procedure also improves the convergence of the series.

Section 4

EXAMPLES

4.1 INVERSE PROBLEM

First we calculate the flow behind a paraboloidal shock wave at $M_\infty = \infty$ and $\gamma = 1.4$. This case has been regarded as a "test case" by Van Dyke (1965). The solutions for surface values are plotted on Fig. 2. Comparisons are made with Van Dyke, Lomax, and Inouye (1964) as well as with Kao (1967). We see all the solutions agree well in the body shape. The three term thin layer solution gives a pressure distribution which is in good agreement with Kao's solution as well as with Van Dyke's solution up to the sonic point. The two term iterative pressure distribution, however, diverges before it reaches the sonic point. The three term thin layer solution for Mach number shows a few percent deviation from Lomax and Inouye's solution and is superior to Kao's solution. On the other hand, the two-term iterative solution yields improvement on Mach number, which agrees well with the numerical solution in its region of validity (before the pressure becomes divergent). From this example, we learn that the present series solution converges rather slowly in the present case. Three terms of the series for velocities and four terms for pressure seem necessary for the solution to be accurate up to the body sonic point. We also learn that the proposed iteration yields improvement on the thin layer solution, especially in the velocity field. In comparison with the two-term thin layer solution (not shown), the iteration also increases the radius of convergence of the series.

4.2 DIRECT PROBLEM

4.2.1 Sphere

We have calculated three cases for spheres, at $M_\infty = \infty$, 10, and 5. In all the cases γ is taken to be 1.4. The results are plotted on Figs. 3 to 5, respectively. In the

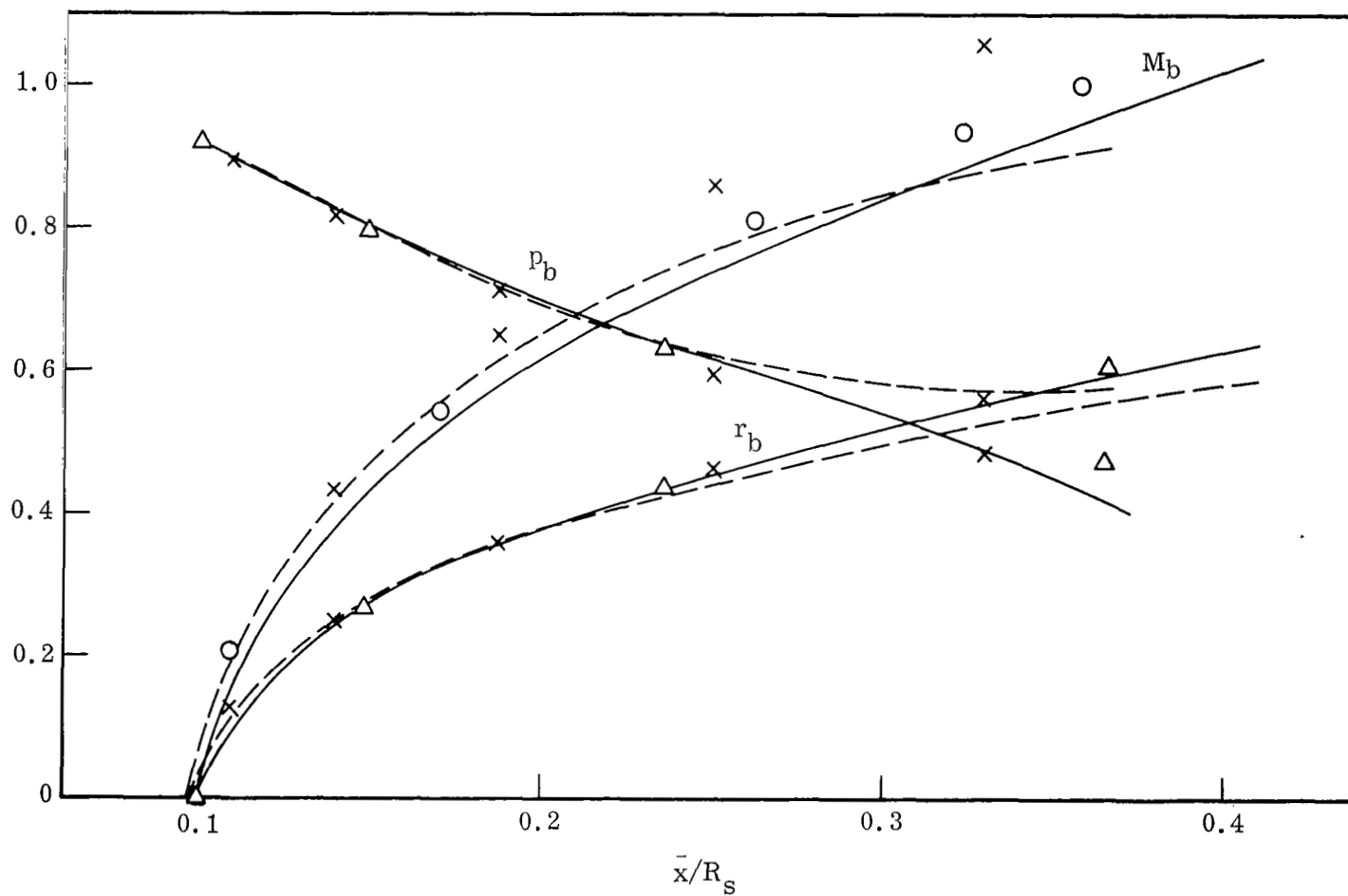


Fig. 2 Body surface values for paraboloidal shock wave at $M_\infty = \infty$, $\gamma = 1.4$
 — Three term thin layer solution. - - - Two term iterative solution.
 Δ Van Dyke. $\times$ Kao. $\circ$ Lomax and Inouye.

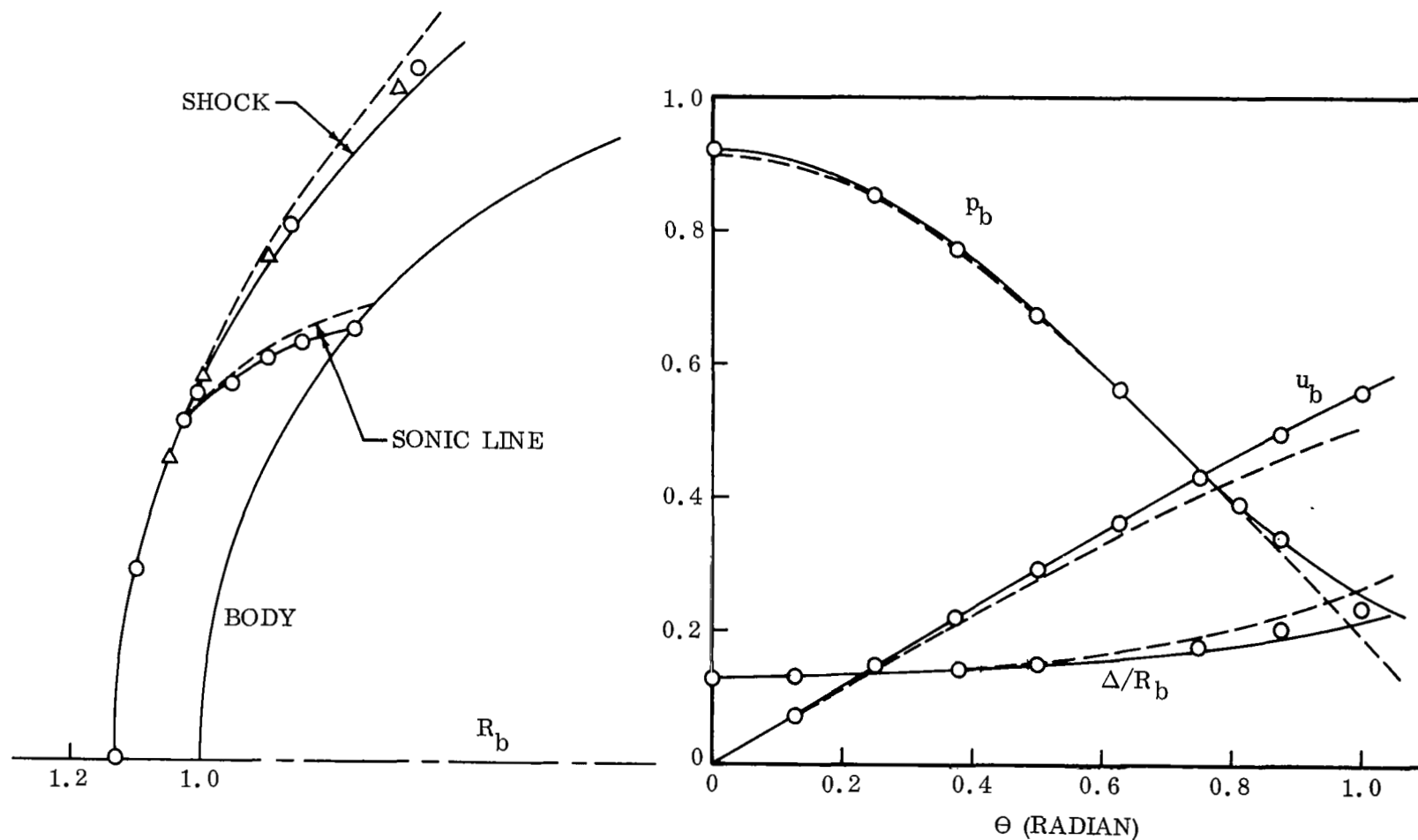


Fig. 3a Sonic line, shock shape and body surface values for sphere at $M_\infty = \infty$, $\gamma = 1.4$. — Two term iterative solution. ---- Three term thin layer solution. $\circ$ Belotserkovskii. Δ Assumed shock shape.

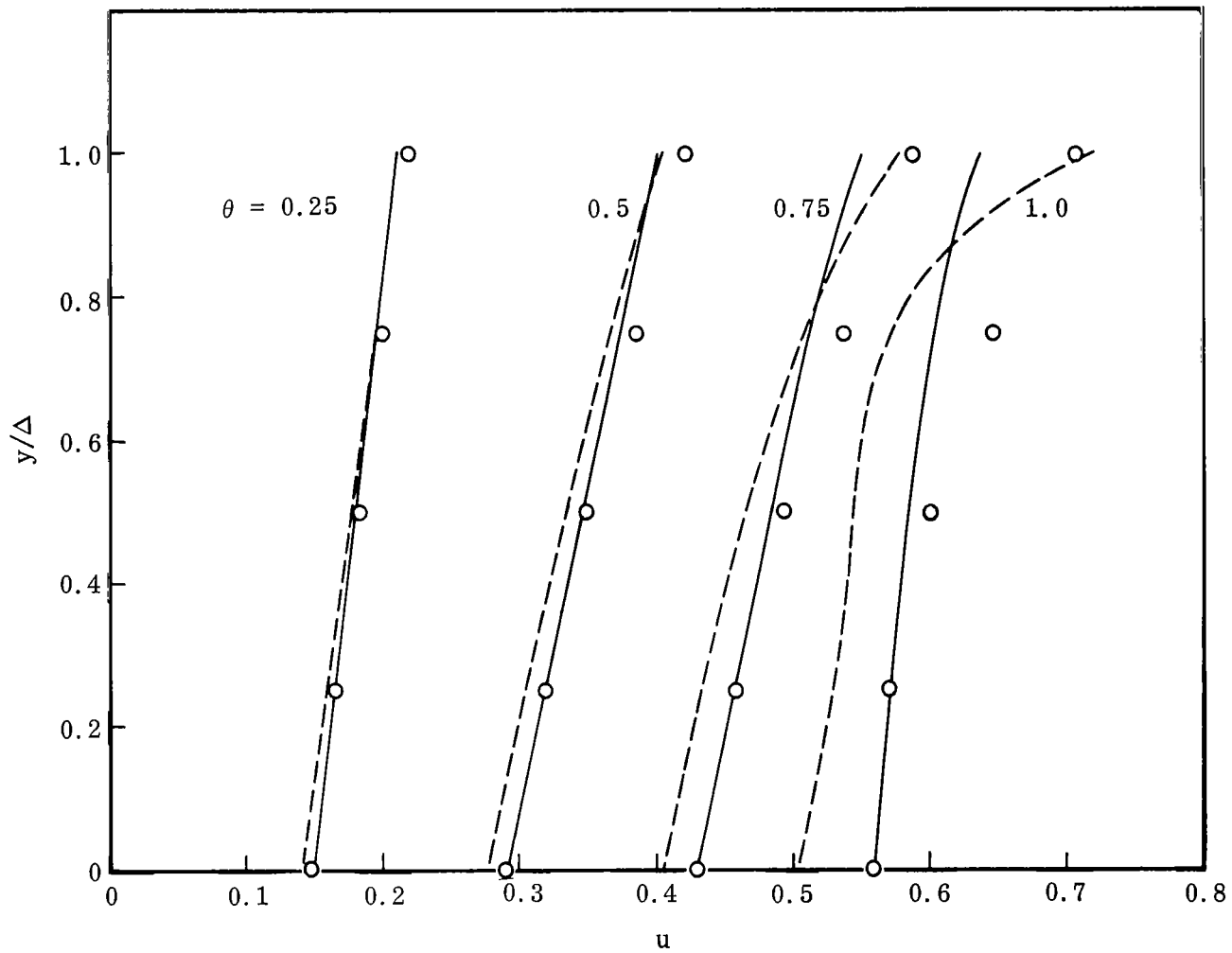


Fig. 3b Tangential Velocity Profile for Sphere at $M_\infty = \infty$, $\gamma = 1.4$, — Two-Term Iterative Solution, - - - Three-Term Thin-Layer Solution, $\circ$ Belotserkouskii

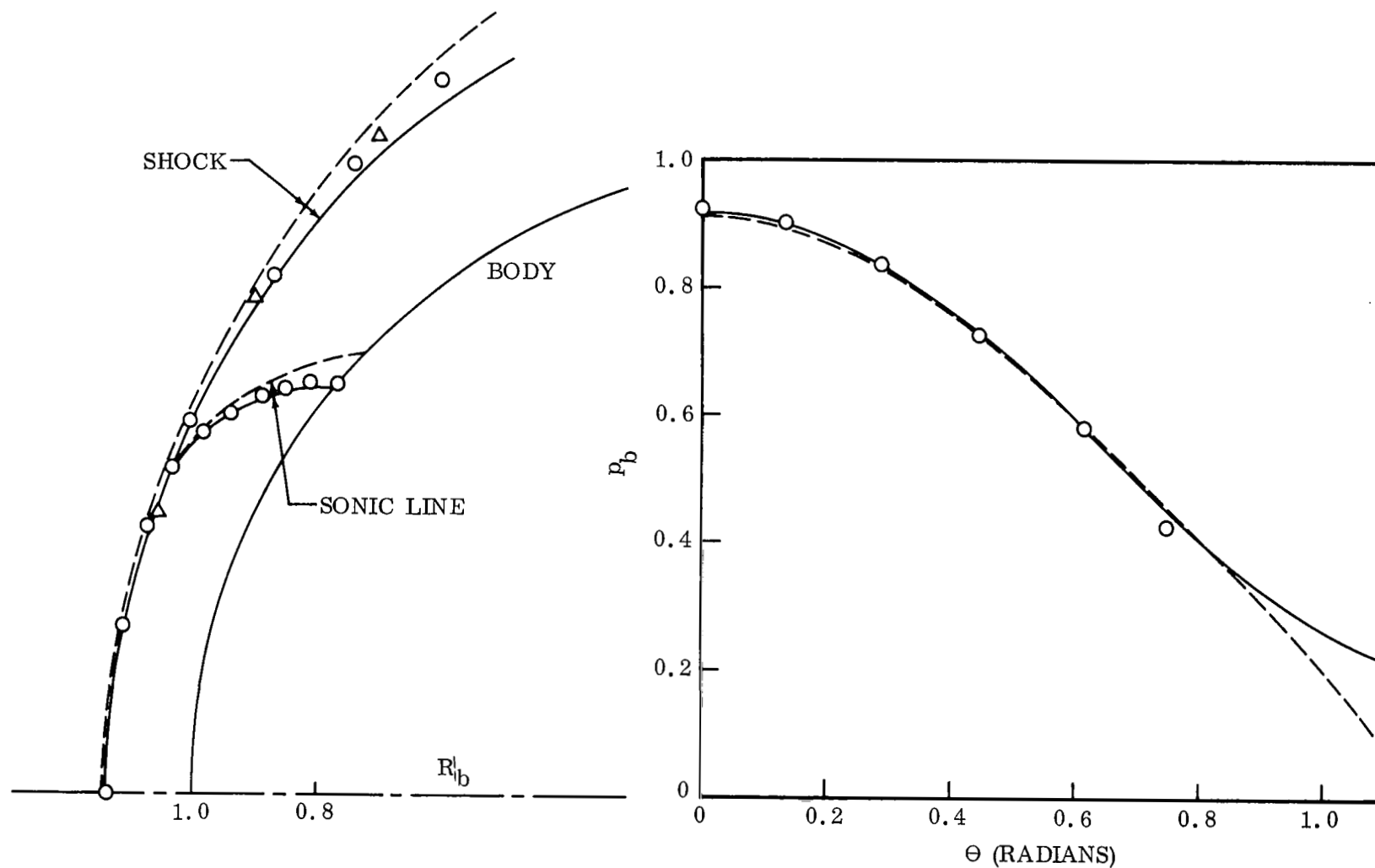


Fig. 4 Sonic line, shock shape and body surface pressure for sphere at $M_\infty = 10$, $\gamma = 1.4$. — Two term iterative solution. ---- Three term thin layer solution. $\circ$ Van Dyke. Δ Assumed shock shape.

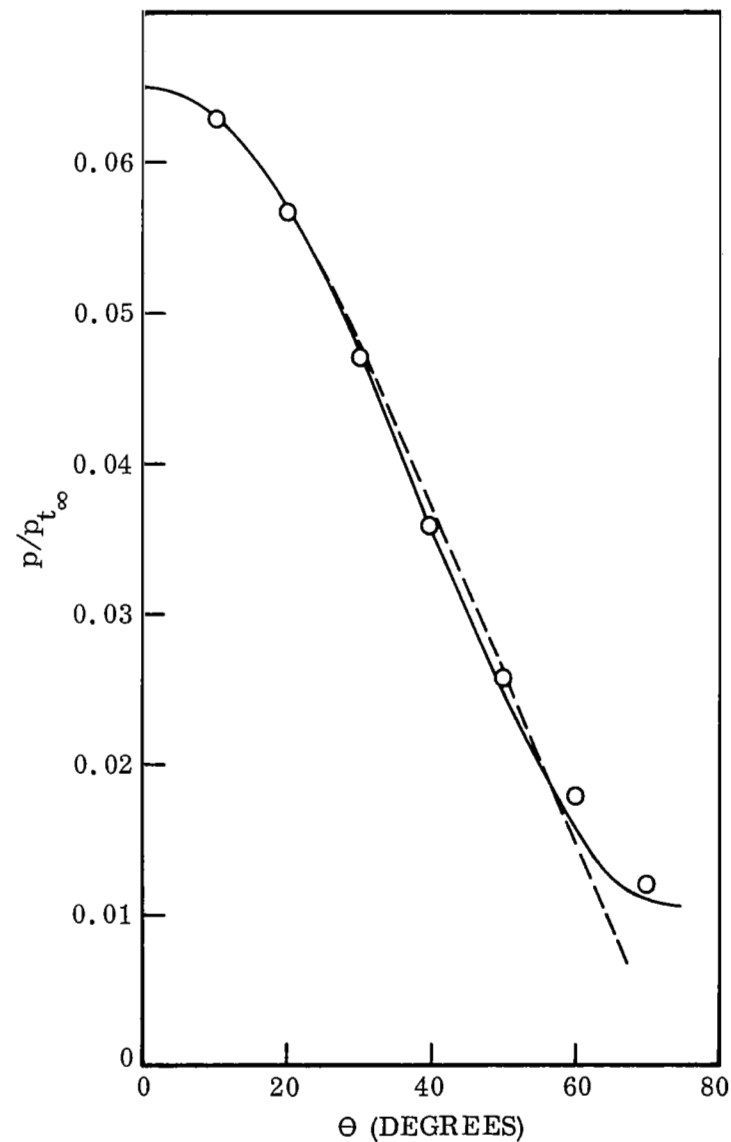
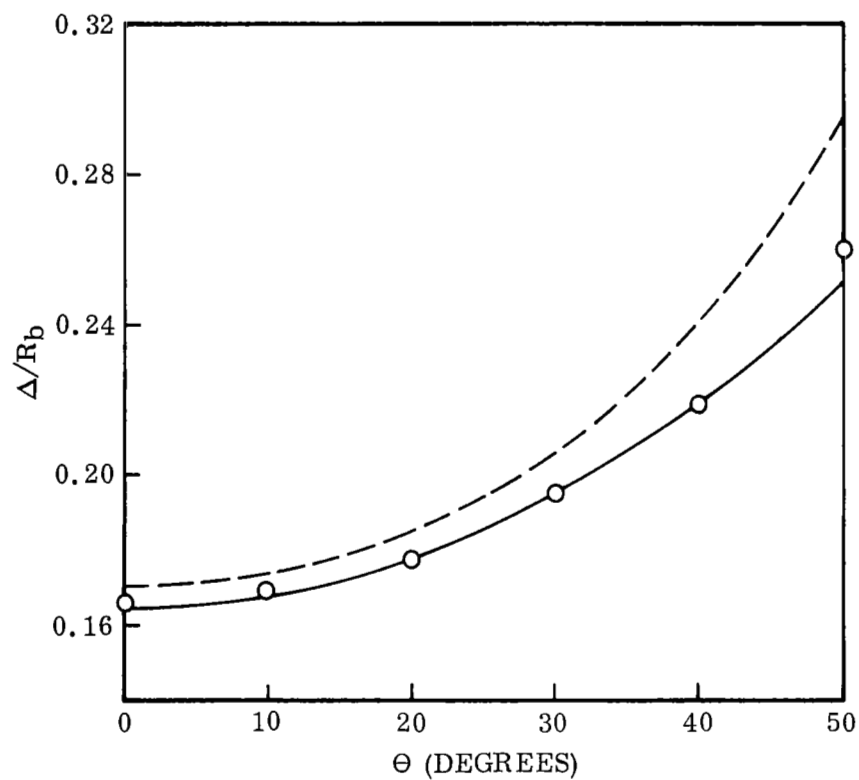


Fig. 5 Shock location and body surface pressure for sphere at $M_\infty = 5$, $\gamma = 1.4$.
 — Two term iterative solution. - - - Three term thin layer solution.
 ○ Experimental data (Xerikos and Anderson).

case of $M_\infty = \infty$, comparison is made with the numerical solution by Belotserkovskii (1961). We see that the agreement between these two solutions is excellent. The iterative solution shows clearly an improvement over the thin layer solution on the velocity along the body and the shock location. The calculation (not shown) based on the two-term thin layer solution as well as on the concentric shock assumption (replace B_s by B_b) also yields satisfactory results up to the sonic line. For the case of $M_\infty = 10$ (Fig. 4), the comparison is made with the solution by Van Dyke and Gordon (1959); agreement is again found to be excellent. In Fig. 5, the results for shock shape and body pressure distribution are presented for the case of $M_\infty = 5$. This solution is compared with the experimental data obtained by Xerikos and Anderson (1965). The pressure in this plot is normalized by p_{t_∞} which is the total pressure at infinity instead of $\rho_\infty U_\infty^2$. We have correlated the pressure between p_{t_∞} and $\rho_\infty U_\infty^2$ by equating the calculated pressure to the measured one. The agreement is good up to 60 degrees or so. Despite the good agreement in pressure and shock shape in this low Mach number case, an accurate prediction of the velocity profile is not expected.

These calculations show, surprisingly, that the thin layer equation seems to work best for spheres. Also, unlike the case of the paraboloid, the series solution converges rapidly and only two terms of the series are needed for the solution to be good up to the sonic line.

4.2.2 Ellipsoid

$R_b = 0.25$, $M_\infty = \infty$, $\gamma = 1.4$. This body shape is shown to produce a nearly paraboloidal shock wave. The results are shown in Fig. 6 and are compared with Belotserkovskii's solution. Again the iterative solution shows improvement on velocity along the body surface over the thin layer solution. Similar to the inverse problem discussed in the last section, the iterative pressure distribution along the body diverges; the sonic line also deviates somewhat from the numerical solution. Hence, for body shape with bluntness parameter less than 0.25, or for those shapes that will produce a hyperboloidal shock, the present solution (both the thin layer solution and iterative

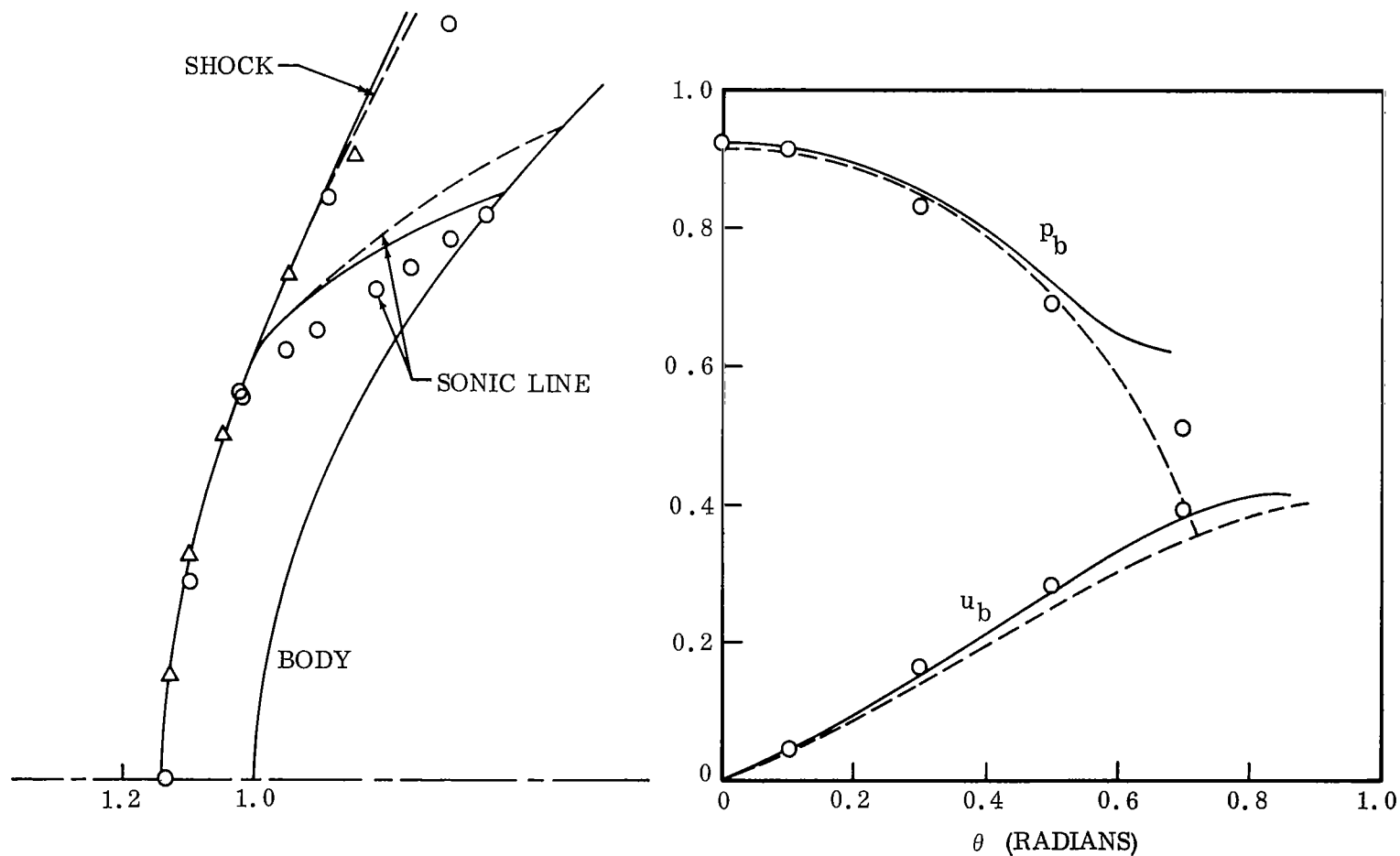


Fig. 6 Sonic line, shock shape and body surface values for ellipsoid with $B_b = 0.25$ at $M_\infty = \infty$, $\gamma = 1.4$. — Two term iterative solution. ---- Three term thin layer solution. $\circ$ Belotserkovskii. Δ Assumed shock shape.

solution) will not be adequate to predict accurate results up to the sonic line.

$R_b = 2.25$, $M_\infty = \infty$, $\gamma = 1.4$. The results for this case are shown in Fig. 7.

Comparisons are made again with the numerical solution by Belotserkovskii. The agreement is found to be excellent; the iterative solution does not offer such improvement in this case.

$R_b = 4$, $M_\infty = 3$, $\gamma = 1.4$. Strictly speaking, the solutions obtained are not applicable to the present case due to the fact that the free stream Mach number is low. It is interesting to see, however, the results. This particular case is chosen because experimental data are available. The results are shown on Fig. 8. The body pressure plotted is normalized by the stagnation point pressure. Comparison is made with experiments given by Perry and Pasiuk (1966). The agreement is surprisingly good for both the thin layer solution and the iterative solution. (The stagnation point pressure deviates somewhat among different solutions. An instance of this is where the experimental value of $p_{b_o} = 0.954$, the thin layer gives $p_{b_o} = 0.865$, and the iterative solutions gives $p_{b_o} = 0.886$. The discrepancy between present solutions and the experiment at the stagnation point is of the order of ϵ^2 , as expected. In the present case, $\epsilon = 0.259$.) The good agreement seems to suggest that the three term thin layer solution is also applicable to the present moderate bluntness case. On the same plot, we also show the sonic line and shock position. There are some discrepancies between the thin layer solution and the iterative solution, as shown. No other existing solution is available to make comparisons in this respect.

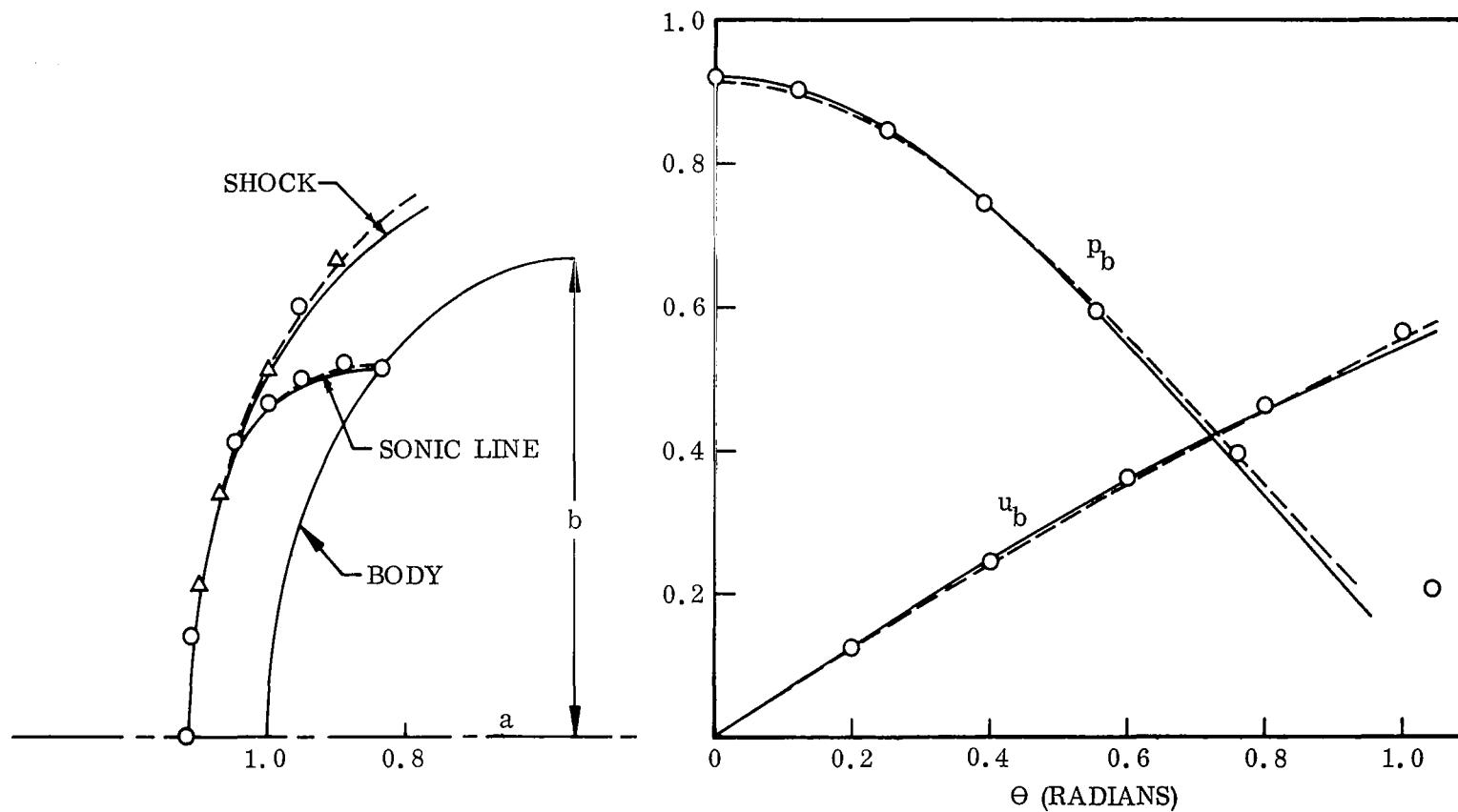


Fig. 7 Sonic line, shock shape and body surface values for ellipsoid with $B_b = 2.25$, at $M_\infty = \infty$, $\gamma = 1.4$. — Two term iterative solution. ---- Three term thin layer solution. $\circ$ Belotserkovskii. $\triangle$ Assumed shock shape.

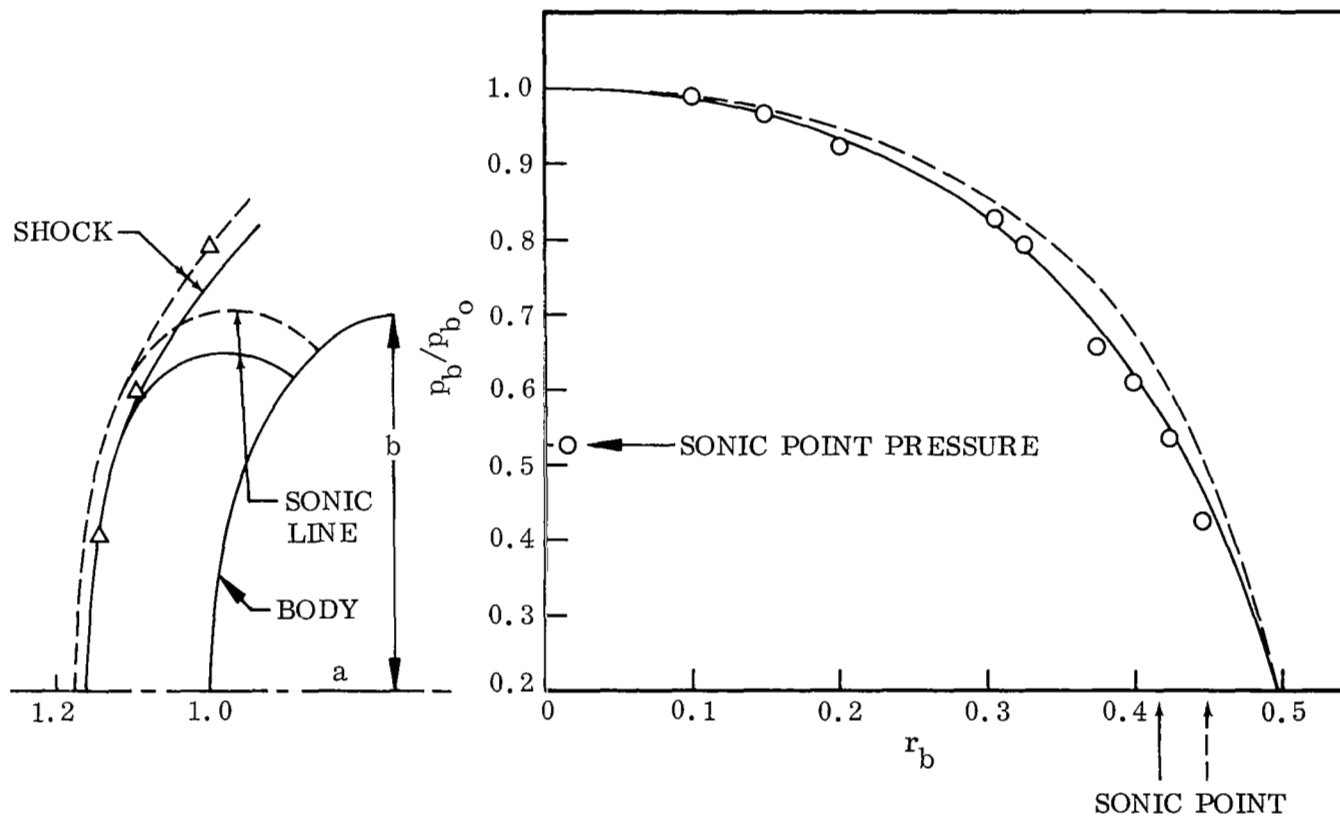


Fig. 8 Sonic line, shock shape and body pressure distribution for ellipsoid with $B_b = 4$ at $M_\infty = 3$, $\gamma = 1.4$. — Two term iterative solution. - - - Three term thin layer solution. $\circ$ Experiment (NOL, Perry and Pasiuk). $\triangle$ Assumed shock shape.

Section 5

CONCLUSIONS

The Blasius type series formulation for hypersonic perfect gas flow over an axisymmetric blunt body has been made. A three-term series solution, based on the thin layer equations and a two-term iterative solution based on the three-term thin layer solution have been obtained. From the calculated examples, we reach the following conclusions:

- (1) The present series solution seems most effective for a sphere; the convergence of the series is fast. The first two terms of the thin layer series are sufficient to describe the flow field accurately up to the sonic line and no iterative solution will be necessary. The concentric shock will be a good first approximation for a sphere.
- (2) For paraboloids, and for those body shapes will produce hyperboloidal shock waves, the present series solution is found to converge rather slowly, especially for the pressure field. The first three terms of the series seem necessary for accurate results up to the sonic line. The proposed iterative solution shows improvement over the thin layer solution, yet the three-term iterative solution requires a four-term thin layer solution. To compute the fourth term of the thin layer series is rather tedious.
- (3) The thin layer equation is found to be applicable to ellipsoids with moderate bluntness such as the ones discussed in the examples. The iterative solution gives a more accurate velocity field, especially in the neighborhood of the body surface. For a body with very large bluntness (in the limit of a flat-nosed body, for example), the thin layer equation is not expected to yield good solutions, and the effectiveness of the proposed approximate iteration is also doubtful.
- (4) The solution is not applicable at low free stream Mach numbers due to the hypersonic assumption made in the analysis as well as due to the fact that we have neglected the terms of order ϵ^2 .

Section 6

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APPENDIX I

COEFFICIENTS IN THE POWER SERIES REPRESENTATION

In this appendix we will determine the coefficients u_1 , r_1 , k_1 , ρ_1 , etc. in the series represented by Eq. (17) for a conical shape body (or shock), and the relation between distance along the body ξ (or shock) and the body angle θ (or shock angle φ).

The equation for a conical shape body (or shock) can, in general, be written as

$$\frac{r^2}{2\bar{x}} + \frac{1}{2}b_o\bar{x} = a_o \quad (A1)$$

With $a_o = 1$, $b_o = B_s$ for the shock-oriented coordinates (inverse problem) $b_o = B_b$, $a_o = R_b/R_s$, for the body-oriented coordinates (direct problem), and with $B_b = (b/a)^2$ (we here refer all the symbols to Fig. 1). We recall all the lengths r , and $\bar{x}$ are normalized by R_s . The parameter B_b (or B_s) is called the body (or shock) bluntness parameter. $B_b = 0$ represents paraboloids, $B_b = 1$ represents a sphere, $B_b > 1$ represents ellipsoids.

We first determine coefficients u_1 , r_1 , etc. From the oblique shock relation and geometry, we have the following relations:

$$u_s = \sin \varphi(\xi) \quad , \quad p_s = (1 - \epsilon) \cos^2 \varphi(\xi_s) \quad , \quad \rho_s = \frac{(\gamma + 1)M_\infty^2 \cos^2 \varphi}{(\gamma - 1)M_\infty^2 \cos^2 \varphi + 2}$$

$$k_s = -\frac{d\sigma}{d\xi_s} \quad , \quad r_s = \int_0 \sin \sigma d\xi_s \quad (A2)$$

In the determination of u_1 , r_1 , and k_1 , we will approximate that the shock shape for a conical body is also conical. This has been shown by Lomax and Inouye (1964) to be an excellent approximation for free stream Mach numbers greater than 5 where γ ranged from 1.1 to 1.6667. We then formally expanded u_s , k_s , r_s in the following form:

$$u_s = (u_s)_o + \left(\frac{du_s}{d\xi_s} \right)_o u_o \xi + \frac{1}{2} \left(\frac{d^2 u_s}{d\xi_s^2} \right)_o u_o^2 \xi^2 + \dots \quad (A3)$$

A similar expression for k_s and r_s can be written. Where $u_o = (d\xi_s/d\xi)_o$ for the direct problem and $u_o = 1$ for the inverse problem. From Eq. (A2) we see that the derivatives of u_s are related to the derivatives of φ . They can be found from Eq. (A1), after taking $a_o = 1$ and $b_o = B_s$, as

$$\text{Cot } \varphi = \frac{dr_s}{d\bar{x}} = \frac{1 - B_s \bar{x}}{\sqrt{2\bar{x} - B_s \bar{x}^2}} \quad (A4)$$

Also from the relation $d\xi_s = \sqrt{(d\bar{x})^2 + (dr_s)^2}$ we obtain

$$\frac{d\xi_s}{d\bar{x}} = \sqrt{1 - B_s + \frac{1}{2\bar{x} - \bar{x}^2 B_s}} \quad (A5)$$

It is then not difficult to find

$$\left(\frac{d\varphi}{d\xi_s} \right)_o = 1, \quad \left(\frac{d^2 \varphi}{d\xi_s^2} \right)_o = 0, \quad \left(\frac{d^3 \varphi}{d\xi_s^3} \right)_o = -3(1 - B_s), \quad \left(\frac{d^4 \varphi}{d\xi_s^4} \right)_o = 0, \text{ etc.} \quad (A6)$$

Consequently, Eqs. (A2) yield the relations

$$\begin{aligned}
\left(\frac{du_s}{d\xi_s}\right)_o &= 1, \quad \left(\frac{d^2u_s}{d\xi_s^2}\right)_o = 0, \quad \left(\frac{d^3u_o}{d\xi_s^3}\right)_o = -[1 + 3(1 - B_s)] \left(\frac{dk_s}{d\xi_s}\right)_o = 0, \\
\left(\frac{d^2k_s}{d\xi_s^2}\right)_o &= -3(1 - B_s) \left(\frac{dr_s}{d\xi_s}\right)_o = 1, \quad \left(\frac{d^2r_s}{d\xi_s^2}\right)_o = 0, \quad \left(\frac{d^3r_s}{d\xi_s^3}\right)_o = -1
\end{aligned} \tag{A7}$$

By comparing Eq. (17) with Eq. (A3), we finally obtain

$$\begin{aligned}
u_1 &= \frac{u_o^2}{6} [1 + 3(1 - B_s)] \\
r_1 &= -\frac{u_o^2}{6} \\
k_1 &= -\frac{3u_o^2}{2} (1 - B_s) \\
p_1 &= u_o^2 \\
\rho_1 &= \frac{2u_o^2}{(\gamma - 1)M_\infty^2 + 2} \\
u_2 &= \frac{u_o^4}{5!} [1 + 3(1 - B_s)(29 - 15 B_s)] \\
r_2 &= \frac{u_o^4}{5!} [1 + 12(1 - B_s)] \\
k_2 &= \frac{(1 - B_s)(19 - 15 B_s)}{8} \\
p_2 &= 2 u_1 u_o^2 \\
\rho_2 &= \frac{2u_o^2}{(\gamma - 1)M_\infty^2 + 2} 2u_1 + \frac{(\gamma - 1)M_\infty^2 u_o^2}{(\gamma - 1)M_\infty^2 + 2}
\end{aligned} \tag{A8}$$

Higher order coefficients can be obtained in a similar way if so desired.

The bluntness parameter B_s is related to the shock parameter A_s defined by Lomax and Inouye (1964) ($B_s = 8A_s$, A_s defined here, should not be confused with the constant given in Appendix II). The relation between B_b and B_s can be found for various cases in their report. B_s can also be found by first guessing a value then adjusting it to fit the calculated shock shape. The values of B_s used in this paper are all taken directly or extrapolated from the curve given in Lomax and Inouye's report.

The relation between ξ and θ (or φ) is given by Eqs. (A4) and (A5). They can be rewritten for the body oriented coordinates as

$$\cot \theta = \frac{1 - B_b \bar{x}'}{\sqrt{2\bar{x}' - B_b \bar{x}'^2}}$$

$$\xi = a_o \int_0^{\bar{x}'} \sqrt{1 - B_b + \frac{1}{2\bar{x}' - \bar{x}'^2 B_b}} d\bar{x}' \quad (A9)$$

Here $\bar{x}'$ is acting as a parametric variable, $\bar{x}' = \bar{x}/a_o$.

For a paraboloid ($B_b = 0$) and a sphere ($B_b = 1$), Eq. (A9) can be integrated. Consequently, the relations for $\xi(\theta)$ are

paraboloids: $\xi = a_o/2 [\sqrt{2\bar{x}(1 + 2\bar{x}')} + \ln [\sqrt{2\bar{x}} + \sqrt{1 + 2\bar{x}'}]]$

$$\cot \theta = \frac{1}{\sqrt{2\bar{x}'}} \quad (A10)$$

$$\text{sphere:} \quad \xi = a_o \theta \quad (A11)$$

For ellipsoids, however, Eq. (A9) has to be integrated numerically. For inverse problems, (A9) to (A11) are valid by taking $a_o = 1$, changing θ onto φ , and B_b into B_s .

APPENDIX II
A LIST OF COEFFICIENTS A's AND B's

$$A_1 = \frac{\delta \rho_1}{u_1} + 8(1-\epsilon) + 2\left(1 + \frac{k_1}{u_1} + \frac{r_1}{u_1}\right) + \frac{u_o^2 \delta}{u_1} \left(1 + \frac{1}{2(1-\epsilon)}\right)$$

$$A_2 = u_o \left(1 + \frac{k_1}{u_1} - \frac{r_1}{u_1} + \frac{u_o^2 \delta}{2u_1(1-\epsilon)}\right)$$

$$A_3 = \frac{2r_1}{u_1} (1-\epsilon\delta) - \epsilon \left[1 + \frac{k_1}{u_1} - \frac{r_1}{u_1} + \frac{\delta u_o^2}{2u_1} \left(\frac{2}{1-\epsilon} + 1 - \epsilon\delta\right)\right]$$

$$A_4 = 4 \left[1 - \frac{\epsilon}{4} A_1 - \frac{A_3}{2} + \frac{\epsilon \delta u_o^2}{4u_1}\right]$$

$$A_5 = \frac{\epsilon}{4} \left\{ \frac{\delta_i r_1}{u_1} + \frac{8\epsilon^2(\Delta_{1m})^2}{u_1} \delta_i + 4 \left[2(1-\epsilon) + \frac{G_1(w_o)}{u_1} \right] + \frac{\delta_i u_o^2}{u_1} + \frac{\delta_i \rho_1}{u_1} \right\}$$

$$A_6 = \frac{2r_1}{u_1} (1-\epsilon\delta_i) - \frac{\epsilon \delta_i}{2u_1} (1-\epsilon\delta_i) u_o^2$$

$$A_7 = 4 \left\{ 1 - A_5 - A_6/2 + \frac{\epsilon \delta_i u_o^2}{8(1-\epsilon)u_1} - \frac{\epsilon \delta_i u_o^2 (1-\epsilon\delta_i)}{16u_1(1-\epsilon)(\epsilon\delta_i)^{1/2}} \right.$$

$$\left. \ln \left[\frac{(\epsilon\delta_i)^{1/2} - \epsilon\delta_i}{(\epsilon\delta_i)^{1/2} + \epsilon\delta_i} \right] - \frac{\epsilon^2 \delta_i u_o^2 (3r_1 + u_o) \ln 2}{u_1(1-\epsilon)} + \frac{\epsilon \delta_i u_o^2}{4u_1} \right\}$$

$$\begin{aligned}
B_1 = & \left[r_1 + \rho_1 + u_o^2 \left(1 + \frac{1}{2(1-\epsilon)} \right) \right] \left[8u_1(1-\epsilon) + 2u_o(u_1 + k_1 + r_1) \right] \\
& + 6(1-\epsilon)(2u_2 + u_1^2) + 3 \left[r_2 + u_1 + k_2 + k_1 r_1 + u_1(k_1 + r_1) \right] \\
& + \delta \left\{ r_1 \left[\rho_1 + u_o^2 \left(1 + \frac{1}{2(1-\epsilon)} \right) \right] + \rho_2 + 2u_1 u_o^2 + \frac{u_o^2}{2(1-\epsilon)} \left[k_1 + r_1 + u_1 + 2u_o \right] \right. \\
& \left. + u_o^4 \rho_1 \left(1 + \frac{1}{2(1-\epsilon)} \right) + u_o^4 \left(1 + \frac{1}{4(1-\epsilon)^2} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
B_2 = & k_2 + u_2 - r_2 + k_1 u_1 + r_1(k_1 + u_1 + r_1) \\
& + (k_1 + u_1 - r_1) \left[\rho_1 + u_o^2 \left(1 + \frac{1}{2(1-\epsilon)} \right) \right] \\
& - \frac{\delta u_o^2}{2(1-\epsilon)} \left[k_1 + r_1 + u_1 + u_o^2 \left(2 + \rho_1 + \frac{1}{1-\epsilon} \right) \right]
\end{aligned}$$

$$B_3 = B_1 - u_1 r_1 A_1$$

$$B_4 = B_2 - \frac{A_2 u_1 r_1}{2} + \frac{2r_1^2(1-\epsilon\delta)}{\epsilon} - \frac{4r_2(1-\epsilon\delta)}{\epsilon}$$

$$B_5 = \frac{\epsilon u_o^2}{2} \left[\frac{\delta u_o^2}{1-\epsilon} + 2(k_1 + u_1 - r_1) \right]$$

$$B_6 = \frac{\epsilon}{6} \left[B_1 - u_1 r_1 A_1 - \frac{u_1 u_o^2 A_1}{4} + \delta u_1^2 \left(\frac{\epsilon \delta u_o^2}{4u_1} \right)^2 \right]$$

$$B_7 = \frac{u_1^2 \epsilon A_1}{32} \left[\frac{(\epsilon \delta)^2 u_o^2}{u_1} - \epsilon A_1 \right]$$

$$B_8 = \frac{-A_1 A_3 u_1^2}{16 \delta} + u_1 \left[2r_1(1-\epsilon\delta) - \frac{\epsilon A_2 u_1}{2} \right] \left[\frac{\epsilon \delta u_o^2}{8 u_1} - \frac{A_1}{\epsilon \delta} \right] - \frac{5\epsilon \delta u_1 u_o^2 A_3}{16}$$

$$+ \frac{u_1 u_o^2 (1-\epsilon\delta) A_1}{8 \delta} + r_1 u_1 A_3 + \frac{\epsilon \delta u_o^2 (1-\epsilon\delta)}{3} \left[\frac{\epsilon \delta}{8} - \frac{2(1+\rho_1)(1-\epsilon)+1}{4(1-\epsilon)} - \frac{r_1}{u_o^2} \right]$$

$$B_9 = - \frac{\epsilon A_1 A_3 u_1^2}{8}$$

$$B_{10} = \frac{\epsilon A_1 A_3 u_1^2 (1-\epsilon\delta)}{4} - \frac{\epsilon u_1^2 A_1 A_4}{4\epsilon\delta} + \frac{\epsilon u_1^2 A_2}{2} \left(\frac{A_3}{\epsilon\delta} - \frac{2(1-\epsilon\delta)}{(\epsilon\delta)^2} \right)$$

$$+ \frac{u_1^2}{2} \left[\frac{2A_3^2}{\epsilon\delta} - \frac{u_o^2 (1-\epsilon\delta) A_1}{\epsilon\delta^2 u_1} + \frac{(1-\epsilon\delta) A_3 u_o^2}{2u_1} \right] + u_1 A_4 \left(4r_1 - \frac{\epsilon \delta u_o^2}{2} \right)$$

$$+ \frac{2r_1 u_1 (1-\epsilon\delta)}{\epsilon\delta} \left(\frac{A_1 (1-\epsilon\delta)}{2\delta} - A_3 \right) + \epsilon B_5 (1-\epsilon\delta)$$

$$B_{11} = - \frac{u_1^2}{2} \left(A_3^2 + \frac{A_1 A_4}{4} \right)$$

$$B_{13} = \frac{u_1^2}{2} \left(\frac{\epsilon \delta A_4^2}{1-\epsilon\delta} - A_3 A_4 \right)$$

$$B_{14} = \frac{\epsilon u_1^2 (1-\epsilon\delta) A_1}{2(\epsilon\delta)^2} \left(A_4 - \frac{A_3 (1-\epsilon\delta)}{\epsilon\delta} \right) - \frac{\epsilon u_1^2 A_2}{2\epsilon\delta} \left[\frac{2A_3 (1-\epsilon\delta)}{\epsilon\delta} - A_4 \right.$$

$$\left. - \frac{\epsilon A_1 (1-\epsilon\delta)^2}{(\epsilon\delta)^2} \right] - \frac{u_1^2 \epsilon \delta}{2} \left[\frac{2\epsilon A_1 u_o^2 (1-\epsilon\delta)^3}{u_1 (\epsilon\delta)^4} + \frac{4A_3^2 (1-\epsilon\delta)}{(\epsilon\delta)^2} \right.$$

$$\left. + \frac{A_3 u_o^2 (1-\epsilon\delta)^2}{u_1 (\epsilon\delta)^2} + \frac{A_4 u_o^2}{4u_1} \right] + \frac{2u_1 r_1 (1-\epsilon\delta)}{\epsilon\delta} \left[\frac{2A_3 (1-\epsilon\delta)}{\epsilon\delta} \right.$$

$$\left. - A_4 - \frac{\epsilon A_1 (1-\epsilon\delta)^2}{(\epsilon\delta)^2} \right]$$

$$B_{15} = \frac{1}{6} \epsilon \delta u_o^2 \left[r_1 + \frac{\epsilon \delta u_o^2}{4} - u_o^2 \left(1 + \rho_1 + \frac{1}{2(1-\epsilon)} \right) \right]$$

$$B_{12} = 8 \left[u_2 - B_6 - B_7 - \frac{1}{2} (B_8 + B_9) - \frac{1}{4} (B_{10} + B_{11}) - \frac{B_{14}}{8} \ln 4 (1-\epsilon \delta) \right. \\ \left. - B_{15} - \frac{\epsilon B_5}{4} - \frac{B_{13}}{8} \right]$$

PART II. VISCOUS RADIATING-ABSORBING CASES

Section 1 INTRODUCTION

In Part I, the series formulation has been applied to the inviscid case and an analytic solution obtained for three terms of the series. The results show good agreement between the solution and existing numerical ones. In Part I, the thin layer equations were solved first and then an approximate iterative procedure was followed to improve the thin layer solution as well as to extend the validity of the solutions to the class of bodies with large bluntness. In Part II, the same series formulation will be applied to the case of viscous, radiating-absorbing layers. The density ratio ϵ in this class of shock layer is usually small. The smallness of ϵ suggests further simplification by introducing a "thin layer" concept. This concept implies that in the evaluation of the velocity field, the outer boundary of the shock layer can be considered to be adjacent to (or "at") the body, which is similar to the first order boundary-layer theory in the sense that the flow solution at the outer edge of the boundary layer is obtained by evaluating the inviscid solution at the body.

The formulation made here is essentially the same as that given in Part I. However, only two terms of the series have been formulated, and only the thin layer equations are treated. The series however is written in such a form that the body shape dependence is "grouped out" and universal functions have been found.

The radiative transfer equation is solved by the well known differential approximation (Vincenti and Kruger 1965), and simplification is made assuming a thin layer which is consistent with those made in the fluid mechanics. A three-band model is used for the continuum absorption coefficient. In each band analytic expressions are available (Hoshizaki and Wilson, 1966) that give explicitly the frequency dependence of the absorption coefficients. Thus proper integration can be performed in each of the frequency ranges and mean quantities such as radiative heat flux can be found in terms

of the local thermodynamics. No line or molecular band radiation is included in this study. The body wall is assumed to be highly cooled in comparison with the gas behind the shock. Perfect gas thermodynamics are not assumed. The equation of state is obtained by curve fitting the data based on numerical evaluation for air of Neel and Lewis, 1964.

It seems needless to say that the algebraic manipulations in the present series formulation are complicated and tedious, especially for the treatment of the radiative transfer and absorption coefficients. Any repetition of the manipulation is indeed not advisable, and we are therefore hesitant to suggest that the present series formulation represents an effective method in solving the radiating-absorbing thin shock layer. The justification of the present study, however; is that in the present formulation, universal functions can be found. The solution of such functions depends only on some independent parameters (as will become apparent) and is independent of the specific shape of the body. We can therefore evaluate the functions for a range of those parameters and perhaps find some simple formulas for quantities of interest such as heat flux to the wall, skin friction, etc. , to be applied to arbitrary body shapes. Such formulas will be useful should the approximation and simplification made in this study prove to be valid (by comparison with more exact solutions). Judging from inviscid solutions, the validity of such formulas will also be extended beyond the stagnation region. To find and examine the accuracy of such formulas is the basic purpose of the present study.

Section 2 ANALYSIS

2.1 GOVERNING EQUATIONS

The fluid conservation equations (mass, momentum, energy) for axisymmetric bodies, written in a body oriented coordinate system, are:

$$\frac{\partial \rho u r}{\partial x} + \frac{\partial \rho v r \tilde{k}}{\partial y} = 0 \quad (1)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \tilde{k} \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) - \rho k u v \quad (2)$$

$$\tilde{k} \frac{\partial p}{\partial x} = k u^2 \rho \quad (3)$$

$$\rho u \frac{\partial H}{\partial x} + \tilde{k} \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} \frac{\partial H}{\partial y} \right) + \frac{\partial}{\partial y} \left[\frac{\mu \left(1 - \frac{1}{Pr} \right)}{2} \frac{\partial u^2}{\partial y} \right] + \int_0^\infty \alpha_\nu \left[\int_0^{4\pi} I_\nu d\Omega - 4\pi B_\nu \right] d\nu \quad (4)$$

The y-momentum equation has been simplified by retaining only the terms of order unity.

The radiation field is coupled with the fluid mechanics through the integral term in the energy Eq. (4) which represents the radiative energy loss or gain per unit volume.

In seeking solutions of the radiative transfer equation, we will use the differential approximation. In the present case of a thin hot gas with a cold wall and cool gas in front of the shock, the differential approximation is a good, valid approximation. Furthermore, it yields the simplest set of governing equations for radiative transfer

among all the approximations. In the framework of the differential approximation, the radiative transfer equation can be written as

$$\nabla \cdot \mathbf{q}_\nu = -\alpha_\nu [(I_o)_\nu - 4\pi B_\nu] \quad (5a)$$

with

$$\frac{\partial (I_o)_\nu}{\partial y} = -3\alpha_\nu (q_y)_\nu \quad (6)$$

$$\frac{\partial (I_o)_\nu}{\partial x} = -3\alpha_\nu (q_x)_\nu \quad (7)$$

In the case of the thin shock layer, $\partial/\partial x \ll \partial/\partial y$, and we make the approximation that*

$$\nabla \cdot \mathbf{q}_\nu \simeq \frac{\partial (q_y)_\nu}{\partial y}$$

Thus, Eq. (5a) becomes

$$\frac{\partial (q_y)_\nu}{\partial y} = -\alpha_\nu [(I_o)_\nu - 4\pi B_\nu] \quad (5b)$$

One immediate consequence of this thin layer simplification is that Eqs. (5b) and (6) are sufficient for the determination of q_y and I_o ; Eq. (7) is only needed for calculating q_x .

We will now consider only the absorption coefficient α_ν for continuum radiation and neglect the line radiation and molecular band radiation. We will also take the three band model for α_ν ; namely, we approximate α_ν by three analytic expressions in three different frequency ranges of the spectrum. The specific form of α_ν and the

*The error introduced by this approximation is shown to be of the order ϵ^2 (Appendix IV).

range of frequency interval will be presented in the next section. Here we will represent the frequency interval generally by $(\Delta h\nu)_i$, $i = 1, 2, 3$. We then define

$$(\alpha q_y)_i = \int_{(\Delta h\nu)_i} \alpha_\nu (q_y)_\nu d\nu, \quad (\alpha I_o)_i = \int_{(\Delta h\nu)_i} \alpha_\nu (I_o)_\nu d\nu, \quad B_i = \int_{(\Delta h\nu)_i} \alpha_\nu B_\nu d\nu \quad (8)$$

Clearly

$$\begin{aligned} \alpha q_y &= \int_0^\infty \alpha_\nu (q_y)_\nu d\nu = \sum_i (\alpha q_y)_i \\ \alpha I_o &= \int_0^\infty \alpha_\nu B_\nu d\nu = \sum_i (\alpha I_o)_i \\ B &= \int_0^\infty \alpha_\nu B_\nu d\nu = \sum_i B_i \end{aligned} \quad (9)$$

With substitution of Eqs. (9), Eq. (4) and the radiative transfer Eqs. (5b) and (6) (after integration with respect to frequency) then become

$$\rho u \frac{\partial H}{\partial x} + \tilde{k} \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} \frac{\partial H}{\partial y} \right) + \frac{\partial}{\partial y} \left[\mu \left(1 - \frac{1}{Pr} \right) \frac{\partial}{\partial y} \left(\frac{u^2}{2} \right) \right] + \sum_{i=1}^3 [(\alpha I_o)_i - 4\pi B_i] \quad (10)$$

$$\frac{\partial (q_y)_i}{\partial y} = - [(\alpha I_o)_i - 4\pi B_i] \quad (11)$$

$$\frac{\partial (I_o)_i}{\partial y} = - 3(\alpha q_y)_i \quad (12)$$

The variables are now normalized in the same manner as in Part I. In addition, the temperature is normalized by the temperature immediately behind the normal shock $T_{S(0)}$, the viscosity by its value immediately behind the normal shock $\mu_{S(0)}$ and

q , I_o , B are all normalized by the quantity $[kT_{s(o)}]^4/h^3c^2$. From here on, the equations are all written in nondimensional form.

Introducing the following transformation

$$\xi(x) = \epsilon \int_0^{\rho_s \mu_s} dx, \quad \eta(x, y) = \frac{\psi}{r_s^2}, \quad d\psi = \rho u r dy - \rho v r \tilde{k} dx \quad (13)$$

the conservation equations can then be written in ξ, η coordinates as

$$\rho u \left(r_s^2 \frac{\partial u}{\partial \xi} - \eta \frac{dr_s^2}{d\xi} \frac{\partial u}{\partial \eta} \right) = - \left(r_s^2 \frac{\partial p}{\partial \xi} - \eta \frac{dr_s^2}{d\xi} \frac{\partial p}{\partial \eta} \right) + \frac{N}{\epsilon Re} \rho u \frac{\partial}{\partial \eta} \left(u \frac{\partial u}{\partial \eta} \right) \quad (14)$$

$$\frac{\partial p}{\partial \eta} = k u r_s \quad (15)$$

$$\epsilon \rho_s \mu_s u \left[r_s^2 \frac{\partial H}{\partial \xi} - \eta \frac{dr_s^2}{d\xi} \frac{\partial H}{\partial \eta} - \frac{N}{\epsilon Re Pr} \frac{\partial}{\partial \eta} \left(u \frac{\partial H}{\partial \eta} \right) - \frac{N}{\epsilon Re} \left(1 - \frac{1}{Pr} \right) \frac{\partial}{\partial \eta} \left(u^2 \frac{\partial u}{\partial \eta} \right) \right] = r_s^2 \sum_i \Gamma_i [(\alpha I_o)_i - 4\pi B_i] \quad (16)$$

$$u \frac{\partial (q_\eta)_i}{\partial \eta} = -\lambda_i r_s [(\alpha I_o)_i - 4\pi B_i] \quad (17)$$

$$u \frac{\partial (I_o)_i}{\partial \eta} = -3\lambda_i r_s (\alpha q_\eta)_i \quad (18)$$

with $i = 1, 2, 3$ and $Re = \rho_\infty U_\infty R_s / \mu_{s(o)}$.

In Eqs. (14) and (16) the following approximation and assumption were made; namely, $r \simeq r_s$ and $\rho\mu/\rho_s\mu_s = N = \text{constant } \lambda_1$.

Expressions for the radiation parameters λ_i and Γ_i will be given in the next section after α_p is specified.

Equations (14) and (18) are to be solved by satisfying the proper boundary conditions and a state equation (given in the next section). The boundary conditions for the flow field are given as follows:

at the shock,

$$\eta = 1/2, \quad u = u_s, \quad p = p_s, \quad \rho = \rho_s, \quad H = H_s \cdot u_s, \quad p_s, \quad \rho_s, \\ \text{and } H_s \text{ are given by oblique shock relations}$$

at the body,

$$\eta = 0, \quad u = 0, \quad H = 0; \text{ here the condition of a cold wall is assumed.}$$

For the radiation field, the differential approximation gives us the condition:

at the shock,

$$\text{for cold gas ahead of the shock, we have } I_o - 2q_y = 0, \text{ which} \\ \text{then implies } (I_o)_i - 2(q_y)_i = 0 \text{ for all } i.$$

at the body,

$$\text{the cold wall assumption } I_o + 2q_y = 0. \text{ Thus, } (I_o)_i + 2(q_y)_i = 0, \\ \text{for all } i.$$

2.2 GAS PROPERTIES

Curve fitting of the equilibrium properties of air calculated in Ref. 3 suggests the following approximate equation of state:

$$p = 10^{-2.5} \rho \left(\frac{h}{R} \right)^{0.84} \quad (19a)$$

with p in atm., ρ in amagt, and h/R in degree Kelvin. This then implies that in the normalized form, the equation of state is

$$\frac{p}{p_s} = \frac{\rho}{\rho_s} \left(\frac{h}{h_s} \right)^{0.84} \quad (19b)$$

Another equation of state for air is also available, namely

$$\rho = 1.65 \times 10^{-5} p (T/10^4)^{-1.72} \frac{\text{gm}}{\text{cm}^3} \quad (20)$$

where p is in atms and T is in $^{\circ}\text{K}$. Equations (19) and (20) thus completely specify the equilibrium thermodynamic properties in air.

In the hypersonic limit, i.e., $h_{\infty} \ll 1/2 U_{\infty}^2$ ($H_{\infty} = H_s \simeq 1/2 U_{\infty}^2$), the thermodynamic properties immediately behind the shock at the stagnation point are completely determined by ρ_{∞} and U_{∞} . This is shown as follows: From Eq. (20), after normalizing T by $T_{s(o)}$, p by $\rho_{\infty} U_{\infty}^2$, ρ by ρ_{∞} , one can write

$$T = C_1 \left(\frac{p}{\epsilon \rho} \right)^{\frac{1}{1.72}} \quad (21a)$$

with

$$C_1 = \frac{10^4}{T_{s(o)}} \left(\frac{1.65 \epsilon U_{\infty}^2}{10^{11}} \right)^{\frac{1}{1.72}} \quad (21b)$$

We note all quantities in Eq. (21b) should be in c.g.s. $^{\circ}\text{K}$ units. Now since at the stagnation point $T = 1$, $p = 1 - \epsilon$ and $\rho = 1/\epsilon$, Eq. (21a) immediately yields a relation

$$C_1 = (1 - \epsilon)^{-\frac{1}{1.72}} \simeq 1 \text{ (for } \epsilon \ll 1) \quad (22)$$

Applying Eq. (19a) at the stagnation point behind the shock, we have the relation for hypersonic flow $\left[\text{i.e., } h_{s(o)} \approx \frac{1}{2} U_{\infty}^2 \right]$

$$\epsilon(1 - \epsilon) = \frac{10^{-2.5}}{U_{\infty}^2} \left(\frac{U_{\infty}^2}{2R} \right)^{0.84} \quad (23)$$

in Eq. (23). R is the universal gas constant and the unit of U_∞^2/R should always be in $^\circ\text{K}$.

Equations (21b) and (23) thus give us two relations to determine ϵ and $T_{s(o)}$ for a given ρ_∞ and U_∞ .

From these considerations, we conclude that in the hypersonic limit, the independent parameters in our problem are the flight condition (ρ_∞, U_∞) , Reynolds number, Prandtl number, viscosity-density product ratio, and characteristic length R_s . R_s becomes an independent parameter since it appears in the radiative transfer parameters explicitly.

The analytic expression for the absorption coefficient in our three band model is taken from Hoshizaki and Wilson. In fact, in that work the expressions are given for more than three bands, for accuracy. However, we will group them into only three. In addition, the contributions from ions are neglected and we treat oxygen as if it were nitrogen. In other words, in the calculation of absorption coefficients, we model air by a pure nitrogen gas. In this model, we write

$$\alpha_\nu = (1 - e^{-h\nu/kT})(K_\nu)_N \quad (24a)$$

with

$$\left. \begin{aligned} (K_\nu)_N &= 4.5 \tilde{a} N_N kT e^{-(14.3 - h\nu)/kT} \frac{\xi_N}{(h\nu)^3} & 0 \leq h\nu \leq 4.22 \\ (K_\nu)_N &= 4.5 \tilde{a} N_N kT \frac{\xi_N}{(h\nu)^3} e^{-10.08/kT} & 4.22 \leq h\nu \leq 10.8 \end{aligned} \right\} \quad (24b)$$

$$(K_\nu)_N = N_N \cdot \varphi_{N,1} \quad 10.8 < h\nu \leq 12.0 \quad (24c)$$

$$(K_\nu)_N = N_N (\varphi_{N,1} + \varphi_{N,2}) \quad 12.0 < h\nu \leq \infty \quad (24d)$$

and

$$\varphi_{N,1} = 5.16 \times 10^{-17} e^{-3.5/kT} / (4 + 10 e^{-2.38/kT} + 6 e^{-3.57/kT}) \quad (24e)$$

$$\varphi_{N,2} = 6.4 \times 10^{-17} e^{-2.3/kT} / (4 + 10 e^{-2.38/kT} + 6 e^{-3.57/kT})$$

$$\tilde{a} = 7.25 \times 10^{-16} \text{ cm}^2 - \text{eV}^2$$

where N_N is the number density of nitrogen atoms and ξ_N is the quantum mechanical correction factor. We approximate ξ_N by

$$\xi_N = 0.24 + 0.0426 (h\nu - 4.22)^2 \quad (25)$$

The ranges of the three-band model are chosen as $(\Delta h\nu)_1$, $0 \leq h\nu \leq 10.8$, $(\Delta h\nu)_2$, $10.8 \leq h\nu \leq 12$, $(\Delta h\nu)_3$, $12 \leq h\nu \leq \infty$.

In the first band $(\Delta h\nu)_1$, α_ν is small, and the optical depth (in this range of frequency and based on the shock layer thickness) is much less than unity. Hence it is reasonable to approximate α_ν in this range by its partial Planck mean, defined as

$$\alpha'_1 = \frac{\int_{(\Delta h\nu)_1} \alpha_\nu B_\nu d\nu}{\int_{(\Delta h\nu)_1} B_\nu d\nu} \quad (26)$$

In the second and third bands, $h\nu/kT$ is large, consequently $e^{-h\nu/kT} \ll 1$. And we can take α'_2 , α'_3 as independent of frequency, i.e.,

$$\alpha'_2 = N_N \varphi_{N,1} \quad (26a)$$

$$\alpha'_3 = N_N (\varphi_{N,1} + \varphi_{N,2}) \quad (26b)$$

By using Eqs. (25), (26a) and (26b), one finds

$$\begin{aligned}(\alpha' q'_y)_i &= \alpha'_i (q'_y)_i \\ (\alpha' I'_o)_i &= \alpha'_i (I'_o)_i\end{aligned}\tag{27}$$

and after performing the integration, α'_1 , B'_1 , B'_2 , B'_3 are found to be (kT is of order 1)

$$\alpha'_1 = \frac{9 \tilde{a} N_N e^{-14.3/kT} \left\{ 2.1 + kT [0.24 + 0.0853 (kT)^2] \right\}}{(kT)^3 \left\{ \frac{2\pi^4}{15} - 2 e^{-10.8/kT} \left[\left(\frac{10.8}{kT} \right)^3 + 3 \left(\frac{10.8}{kT} \right)^2 + 6 \left(\frac{10.8}{kT} \right) + 6 \right] \right\}}\tag{28}$$

$$B'_1 = \frac{9 \tilde{a} N_N kT}{h^3 c^2} e^{-14.3/kT} \left\{ 2.1 + kT [0.24 + 0.0853 (kT)^2] \right\}\tag{29}$$

$$\begin{aligned}B'_2 &= \frac{2N_N \varphi_{N,1}}{h^3 c^2} (kT)^4 \left\{ e^{-10.8/kT} \left[\left(\frac{10.8}{kT} \right)^3 + 3 \left(\frac{10.8}{kT} \right)^2 + 6 \left(\frac{10.8}{kT} \right) + 6 \right] \right. \\ &\quad \left. - e^{-12/kT} \left[\left(\frac{12}{kT} \right)^2 + 3 \left(\frac{12}{kT} \right) + 6 \left(\frac{12}{kT} \right) + 6 \right] \right\}\end{aligned}\tag{30}$$

$$B'_3 = \frac{2N_N (\varphi_{N,1} + \varphi_{N,2})}{h^3 c^3} (kT)^4 e^{-12/kT} \left[\left(\frac{12}{kT} \right)^3 + 3 \left(\frac{12}{kT} \right)^2 + 6 \left(\frac{12}{kT} \right) + 6 \right]\tag{31}$$

The superscript (') denotes dimensional quantities.

The atom number density N_N is related to the total number density (and eventually density) by taking the following model. Namely, we assume that there will be no ionization before the gas is totally dissociated. In the dissociation phase, we relate the atom number density to the total number density by using the Lighthill's ideal

dissociation model. For the ionization phase, we assume the gas is in Saha equilibrium. Hence, for the dissociation phase, we have

$$N_N = \beta N_t \quad (32a)$$

$$\frac{\beta^2}{1 - \beta^2} = \frac{\rho_d R_{A_2} T e^{-\theta_d/T}}{p}, \quad R_{A_2} = \frac{k}{2m} \quad (32b)$$

and for the ionization phase

$$N_N = (1 - \varphi) N_t \quad (33a)$$

$$\frac{\varphi^2}{1 - \varphi^2} = C \frac{T^{5/2}}{p} e^{-\theta_i/T} \quad (33b)$$

$$C = 2 \left(\frac{2\pi m e}{h^2} \right)^{2/3} k^{5/2} \quad (33c)$$

By using the relations (26) to (33), one can find the expressions (in nondimensional form) for α 's, B 's, λ 's and Γ 's involved in the energy Eq. (16) and the radiative transfer Eqs. (17) and (18) are as follows:

$$\alpha_1 = \left(\frac{\beta}{1 - \varphi} \right) \frac{T^{-3} e^{-C_4/T} [2.1 + C_5 + C_6 T^3]}{\frac{2\pi^4}{15} - 2 e^{-a/T} \left[\left(\frac{a}{T} \right)^3 + 3 \left(\frac{a}{T} \right)^2 + \frac{6a}{T} + 6 \right]} \quad (34)$$

$$\alpha_2 = \left(\frac{\beta}{1 - \varphi} \right) e^{-C_7/T} / \left(4 + 10 e^{-C_8/T} + 6 e^{-C_7/T} \right) \quad (35)$$

$$\alpha_3 = \left(1 - \frac{\beta}{\varphi}\right) e^{-C_7/T} \left(1 + 1.24 e^{C_9/T}\right) / \left(4 + 10 e^{-C_8/T} + 6 e^{-C_7/T}\right) \quad (36)$$

$$B_1 = \left(1 - \frac{\beta}{\varphi}\right) T e^{-C_4/T} (2.1 + C_5 + C_6 T^3) \quad (37)$$

$$B_2 = 2T \left\{ e^{-a/T} (a^3 + 3a^2 T + 6aT^2 + 6T^3) - e^{-b/T} [(b^3 + 3b^2 T + 6bT^2 + 6T^3)] \right\} \quad (38)$$

$$B_3 = 2T e^{-b/T} (b^3 + 3b^2 T + 6bT^2 + 6) \quad (39)$$

$$\Gamma_1 = \frac{9 \tilde{a} R_s kT_{s(o)}}{m h^2 c^2 H_s U_\infty}, \quad \Gamma_2 = \frac{R_s [kT_{s(o)}]^4 5.16 \times 10^{-17}}{m h^3 c^2 H_s U_\infty} \quad (40)$$

$$\lambda_1 = \frac{9 \tilde{a} R_s \rho_\infty}{m [kT_{s(o)}]^3}, \quad \lambda_2 = \lambda_3 = \frac{5.16 \times 10^{-17} \rho_\infty R_s}{m} \quad (41)$$

$$\Gamma_2 / \Gamma_1 = \lambda_2 / \lambda_1 \quad (42)$$

with

$$\begin{aligned} a &= \frac{10.8}{kT_{s(o)}}, \quad b = \frac{12}{kT_{s(o)}}, \quad C_4 = \frac{14.3}{kT_{s(o)}}, \quad C_5 = 0.24 kT_{s(o)} \\ C_6 &= 0.058 [kT_{s(o)}]^2, \quad C_7 = \frac{2.38}{kT_{s(o)}}, \quad C_8 = \frac{2.38}{kT_{s(o)}}, \quad C_9 = \frac{1.2}{kT_{s(o)}} \end{aligned} \quad (43)$$

The factor $\left(1 - \frac{\beta}{\varphi}\right)$ in front of Eqs. (34) to (37) denotes that during the dissociation phase, the value β should be used and during the ionization phase $(1 - \varphi)$ should be used.

2.3 SERIES FORMULATION

Similar to Part I, one can write the following series expression immediately behind the shock:

$$\begin{aligned}
 u_s &= \xi + u_1 \xi^3 + \dots \\
 r_s &= \xi + r_1 \xi^3 + \dots \\
 p_s &= 1 - \xi^2 + \dots \\
 \rho_s &= \frac{1}{\epsilon} (1 - \rho_1 \xi^2 + \dots) \\
 k_s &= 1 + k_1 \xi^2 + \dots \\
 \mu_s &= 1 - \mu_1 \xi^2 + \dots
 \end{aligned} \tag{44}$$

For the direct problem, these expressions are valid under the approximation of the thin layer concept, i.e., we have approximated the factor u_o which should appear as a factor in the expressions of u_s and r_s by taking $u_o = (d\xi_s/d\xi)_o = 1$. This approximation is believed to be reasonable for small values of ϵ . We also note that in the expression for p_s , we have neglected the term ϵ , by setting $1 - \epsilon \simeq 1$.

Based on Eq. (44), we assume the dependent variables can be written as

$$\begin{aligned}
 u &= f_1(\eta) \xi + u_1 f_2(\eta) \xi^3 + \dots \\
 H &= g_1(\eta) + g_2(\eta) \xi^2 + \dots \\
 (I_o)_i &= A_{i,1}(\eta) + A_{i,2}(\eta) \xi^2 + \dots \\
 (q_\eta)_i &= Q_{i,1}(\eta) + Q_{i,2}(\eta) \xi^2 + \dots
 \end{aligned} \tag{45}$$

Also, from the discussion in the previous section, we can write

$$T = T_1(1 + T_2\xi^2 + T_3\xi^4 + \dots) \quad (46)$$

$$\beta = \beta_1(1 + \beta_2\xi^2 + \dots) \quad (47)$$

$$\varphi = \varphi_1(1 + \varphi_2\xi^2 + \dots) \quad (48)$$

After using the following typical approximation for the exponential function

$$e^{-C_4/T} = e^{-C_4/T_1(1-T_2\xi^2)} = e^{-C_4/T_1} \left(1 + \frac{C_4}{T_1} T_2 \xi^2 + \dots \right)$$

we obtain

$$B_i = B_{i,0}(B_{i,1} + B_{i,2}\xi^2 + \dots) \quad (49)$$

$$\alpha_i = \alpha_{i,0}(\alpha_{i,1} + \alpha_{i,2}\xi^2 + \dots) \quad (50)$$

$i = 1, 2, 3$.

The coefficients T_i , β_i , $B_{i,j}$, $\alpha_{i,j}$ are known functions of f 's and g 's. They are listed in the Appendix III. We note that T_1 , β_1 , φ_1 , $\beta_{i,0}$, $\beta_{i,1}$, $\alpha_{i,0}$ and $\alpha_{i,1}$ are all independent of body shape (i.e., independent of u_1 , r_1 , k_1 , etc.). For T_2 , β_2 and φ_2 , one can group out the body shape dependence as follows:

$$\begin{aligned} T_2 &= T_{2,0} + 0.49 \frac{g_2}{g_1} + \frac{\rho_1}{1.72} \\ B_2 &= \beta_{2,0} + \beta_{2,1} g_2 + \beta_{2,2} \rho_1 \\ \varphi_2 &= \varphi_{2,0} + \varphi_{2,1} g_2 + \varphi_{2,2} \rho_1 \end{aligned} \quad (51)$$

All the coefficients $T_{2,0}$, $\beta_{2,0}$, etc. are functions of f_1 , g_1 , only. They are also listed in Appendix III. Substituting Eqs. (44) to (51) into the governing Eqs. (14) to (18), and collecting the same power of ξ , we obtain the equations for f_1 , g_1 , $A_{i,1}$, $Q_{i,1}$, f_2 , g_2 , $A_{i,2}$, and $Q_{i,2}$ [we note the pressure is obtained by integrating the Eq. (15)].

$$f_1 \left(f_1 - 2\eta \frac{df_1}{d\eta} \right) = 3\epsilon g_1^{0.84} + \frac{N}{\epsilon Re} f_1 \frac{d}{d\eta} \left(f_1 \frac{df_1}{d\eta} \right) \quad (52)$$

$$2\eta f_1 \frac{dg_1}{d\eta} + \frac{N}{\epsilon Re Pr} f_1 \frac{d}{d\eta} \left(f_1 \frac{dg_1}{d\eta} \right) + F_1(g_1) = 0 \quad (53)$$

$$f_1 \frac{dQ_{1,1}}{d\eta} = -\lambda_1 \alpha_{1,1} (\alpha_{1,0} A_{1,1} - 4\pi B_{1,0}) \quad (54)$$

$$f_1 \frac{dQ_{i,1}}{d\eta} = -\lambda_i \alpha_{i,0} \alpha_{i,1} (A_{i,1} - 4\pi B_{i,1}) \quad i = 2, 3 \quad (55)$$

$$f_1 \frac{dA_{i,1}}{d\eta} = -3\lambda_i \alpha_{i,0} \alpha_{i,1} Q_{i,1} \quad i = 1, 2, 3 \quad (56)$$

$$\begin{aligned} & \frac{N}{\epsilon Re} f_1 \frac{d}{d\eta} \left(f_2 \frac{df_1}{d\eta} + f_1 \frac{df_2}{d\eta} \right) - f_1 \left(3f_2 - 2\eta \frac{df_2}{d\eta} \right) - \frac{3\epsilon f_2 g_1^{0.84}}{f_1} \\ & + 2\epsilon g_1^{0.84} (5 - \eta) + 2.52 W_1 \epsilon g_1^{-0.16} g_2 + \sum_{j=1}^4 W_j S_j = 0 \end{aligned} \quad (57)$$

$$\begin{aligned} & f_1 \left\{ \frac{N}{\epsilon Re Pr} \left[\frac{d}{d\eta} \left(f_1 \frac{dg_2}{d\eta} \right) + u_1 \frac{d}{d\eta} \left(f_2 \frac{dg_1}{d\eta} \right) \right] + 2\eta \frac{dg_2}{d\eta} \right\} + F_4 g_2 - \frac{u_1 f_2 F_1}{f_1} + \Gamma_1 \alpha_{1,1} \alpha_{1,0} A_{1,2} \\ & + \Gamma_2 \alpha_{2,0} (\alpha_{2,1} A_{2,2} + \alpha_{3,1} A_{3,2}) + F_2 + \sum_{j=3}^5 \Omega_j V_j = 0 \end{aligned} \quad (58)$$

$$f_1 \frac{dQ_{i,2}}{d\eta} + u_1 f_2 \frac{dQ_{i,1}}{d\eta} = -\lambda_i (k_{i,1} + k_{i,2} \rho_1 + 2k_{i,3} r_1 + k_{i,4} g_2 + k_{i,5} A_{i,2}) \quad (59)$$

$$f_1 \frac{dA_{i,2}}{d\eta} + u_1 f_2 \frac{dA_{i,1}}{d\eta} = -3\lambda_i (N_{i,1} + N_{i,2} \rho_1 + 2N_{i,3} r_1 + N_{i,4} g_2 + N_{i,5} Q_{i,2})$$

$$i = 1, 2, 3 \quad (60)$$

The coefficients W_j , Ω_j are functions of body shape (u_1 , r_1 , k_1 , etc.) only, and S_j , V_j , F 's, K 's, N 's are functions of f_1 , g_1 only.

In order to find universal functions, we split f_2 , g_2 , $A_{i,2}$ and $Q_{i,2}$ into the following:

$$f_2 = Y_0 + \sum_{j=1}^5 W_j Y_j$$

$$g_2 = Z_0 + \sum_{j=1}^5 \Omega_j Z_j$$

$$Q_{i,2} = X_{i,0} + \sum_{j=1}^5 \Omega_j X_{i,j}$$

$$A_{i,2} = E_{i,0} + \sum_{j=1}^5 \Omega_j E_{i,j}$$

$$(61)$$

Then after grouping the terms with the same coefficients, we obtain the following governing equations for universal functions:

$$D_{f_2} Y_0 + 2.52 \epsilon g^{-0.16} Z_1 + 2 \epsilon (5 - \eta) g_1^{0.84} = 0 \quad (62)$$

$$D_{f_2} Y_1 + 2.52 g_1^{-0.16} Z_0 + S_1 = 0 \quad (63)$$

$$D_{f_2} Y_j + 2.52 g_1^{-0.16} Z_j + S_j = 0 \quad j \neq 0, 1 \quad (64)$$

$$\begin{aligned} D_{g_2} Z_0 + \frac{N}{\epsilon \text{RePr}} f_1 \frac{d}{d\eta} \left(Y_1 \frac{dg_1}{d\eta} \right) - \frac{Y_1 F_1}{f_1} + F_2 + \Gamma_1 \alpha_{1,1} E_{1,0} \\ + \Gamma_2 \alpha_{2,0} (\alpha_{2,1} E_{2,0} + \alpha_{3,1} E_{3,0}) = 0 \end{aligned} \quad (65)$$

$$\begin{aligned} D_{g_2} Z_1 + \frac{N}{\epsilon \text{RePr}} f_1 \frac{d}{d\eta} \left(Y_0 \frac{dg_1}{d\eta} \right) - \frac{Y_0 F_1}{f_1} + \Gamma_1 \alpha_{1,1} E_{1,1} \\ + \Gamma_2 \alpha_{2,0} (\alpha_{2,1} E_{2,1} + \alpha_{3,1} E_{3,1}) = 0 \end{aligned} \quad (66)$$

$$\begin{aligned} D_{g_2} Z_j + \frac{N}{\epsilon \text{RePr}} f_1 \frac{d}{d\eta} \left(Y_j \frac{dg_1}{d\eta} \right) - \frac{Y_j F_1}{f_1} + \Gamma_1 \alpha_{1,1} E_{i,j} \\ + \Gamma_2 \alpha_{2,0} (\alpha_{2,1} E_{2,j} + \alpha_{3,1} E_{3,j}) + V_j = 0 \quad j \neq 0, 1 \end{aligned} \quad (67)$$

$$f_1 \frac{dX_{i,0}}{d\eta} + Y_1 \frac{dQ_{i,1}}{d\eta} = -\lambda_i (K_{i,1} + K_{i,4} Z_0 + K_{i,5} E_{i,0}) \quad (68)$$

$$f_1 \frac{dX_{i,1}}{d\eta} + Y_0 \frac{dQ_{i,1}}{d\eta} = -\lambda_i (K_{i,4} Z_1 + K_{i,5} E_{i,1}) \quad (69)$$

$$f_1 \frac{dX_{i,j}}{d\eta} + Y_j \frac{dQ_{i,1}}{d\eta} = -\lambda_i (M_{i,j} + K_{i,4} Z_j + K_{i,5} E_{i,j}) \quad j \neq 0, 1 \quad (70)$$

$$f_1 \frac{dE_{i,0}}{d\eta} + Y_1 \frac{dA_{i,1}}{d\eta} = -3\lambda_i(N_{i,1} + N_{i,4} Z_0 + N_{i,5} X_{i,0}) \quad (71)$$

$$f_1 \frac{dE_{i,1}}{d\eta} + Y_0 \frac{dA_{i,1}}{d\eta} = -3\lambda_i(N_{i,4} Z_1 + N_{i,5} X_{i,1}) \quad (72)$$

$$f_1 \frac{dE_{i,j}}{d\eta} + Y_j \frac{dA_{i,1}}{d\eta} = -3\lambda_i(0_{i,j} + N_{i,4} Z_j + N_{i,5} X_{i,j}) \quad j \neq 0, 1 \quad (73)$$

With D_{f_2} , and D_{g_2} being differential operators defined as

$$D_{f_2} = \frac{N}{\epsilon Re} f_1^2 \frac{d^2}{d\eta^2} + \left(2 \frac{N}{\epsilon Re} f_1 \frac{df_1}{d\eta} + 2\eta f_1 \right) \frac{d}{d\eta} + \left(\frac{N}{\epsilon Re} f_1 \frac{d^2 f_1}{d\eta^2} - 3f_1 - \frac{3\epsilon g_1^{0.84}}{f_1} \right) \quad (74)$$

$$D_{g_2} = \frac{N}{\epsilon Re Pr} f_1^2 \frac{d}{d\eta} + f_1 \left(\frac{N}{\epsilon Re Pr} \frac{df_1}{d\eta} + 2\eta \right) \frac{d}{d\eta} + F_4 \quad (75)$$

All the coefficients, i.e., W_j , Ω_j , S_j , V_j , F 's, $K_{i,j}$, $N_{i,j}$, $0_{i,j}$ can be found in Appendix III. The system of Eqs. (52) to (75) are to be solved subject to the following boundary conditions.

at wall,

$$\eta = 0, f_1 = 0, g_1 = 0, Y_j = 0, Z_j = 0 \text{ for all } j. \\ A_{i,1} + 2Q_{i,1} = 0, E_{i,j} + 2X_{i,j} = 0 \text{ for all } i, j.$$

at shock,

(76)

$$\eta = 1/2, f_1 = 1, g_1 = 1, Y_0 = 1, Z_0 = 0, Y_j = 0, \\ Z_j = 0 \quad j \neq 0 \\ A_{i,1} - 2Q_{i,1} = 0, E_{i,j} - 2X_{i,j} = 0 \text{ for all } i, j.$$

The streamline can be found by the transformations (13), namely

$$y = \int_0^{\eta} \frac{r_s^2 d\eta}{\rho u r} \approx r_s \int_0^{\eta} \frac{d\eta}{\rho u} \quad (77)$$

and the shock location $\Delta(\xi)$ can be obtained by evaluating Eq. (77) at $\eta = 1/2$.

It has been shown in Part I that in order to find the ratio R_s/R_b , the second derivative of $\Delta(\xi)$ at the stagnation point with respect to ξ is needed. For this purpose, y is therefore to be expanded in a power series of ξ . A series for density ρ can be obtained from the stage Eq. (19b) and the pressure from Eq. (15). Then similar to that of Part I, y can be written as

$$y = \epsilon(y_0 + y_1 \xi^2 + \dots) \quad (78a)$$

with

$$y_0 = \int_0^{\eta} \frac{g_1^{0.84}}{f_1} d\eta \quad (78b)$$

$$y_1 = y_{1,0} + \sum_{j=1}^5 \Omega_j y_{i,j} \quad (78c)$$

and

$$\begin{aligned}
y_{1,0} &= y_0 + \int_0^{\eta} \frac{g_1^{0.84}}{f_1} \left[0.84 \left(1 + \frac{Z_0 - f_1^2}{g_1} \right) - \eta + 1/2 - \frac{Y_1}{f_1} \right] d\eta \\
y_{1,1} &= y_0 + \int_0^{\eta} \frac{g_1^{0.84}}{f_1} \left(\frac{0.84 Z_1}{g_1} - \frac{Y_0}{f_1} \right) d\eta \\
y_{1,2} &= y_0 + \int_0^{\eta} \frac{g_1^{0.84}}{f_1} \left(\frac{0.84 Z_2}{g_1} - \frac{Y_2}{f_1} \right) d\eta \\
y_{1,3} &= \frac{y_0}{2} + \int_0^{\eta} \frac{g_1^{0.84}}{f_1} \left(\frac{0.84 Z_3}{g_1} - \frac{Y_3}{f_1} \right) d\eta \\
y_{1,j} &= \int_0^{\eta} \frac{g_1^{0.84}}{f_1} \left(\frac{0.84 Z_j}{g_1} - \frac{Y_j}{f_1} \right) d\eta \quad j = 4, 5
\end{aligned} \tag{78d}$$

Similarly one can also write the shock layer thickness as

$$\Delta = \epsilon \left[\Delta_0 + \left(\Delta_{1,0} + \sum_{j=1}^5 \Omega_j \Delta_{1,j} \right) \xi^2 + \dots \right] \tag{79}$$

Δ_0 , $\Delta_{1,0}$, $\Delta_{1,j}$ is to be evaluated from Eqs. (78b) to (78d) at $\eta = 1/2$, respectively. Once Δ_1 is found, the shock radius of curvature at stagnation line R_s can be determined by using the equation presented in Part I.

2.4 METHOD OF SOLUTION

From Eqs. (52) to (56), we observe that it is convenient to transform the variable from η to ω , by the transformation

$$\frac{d\eta}{d\omega} = f_1 \tag{80}$$

Equation (80) is solved subject to the condition at $\eta = 1/2$, $\omega = 1$, and the range of variable ω is thus found from 1 to ω_0 , where ω_0 corresponds to the point $\eta = 0$ and is determined by Eq. (80). It can then be shown that Eq. (52) can be solved in terms of integrals in which the integrand depends on η and g_1 . Also, g_1 can be solved in terms of integrals in which the integral depends on η and F_1 . An iteration is therefore set up to evaluate those integrals.

For f_1 , it is clear from Eq. (52) that $f_1 = \omega$ is a solution for $\epsilon = 0$; in this case η is found to be $(1/2)\omega^2$. These solutions then provide us the necessary information for iteration. For g_1 , we start the iteration by first setting $F_1 = 0$. Physically it means the iteration is started from the solution of the nonradiating case. Once f_1 and g_1 are found by solving Eqs. (52) and (53), simultaneously, Eqs. (54) to (56) (which also can be written in integral form) can be solved and F_1 can be determined. This computation cycle is then continued until the desired accuracy is obtained. A similar procedure is employed for the solution of Y 's, Z 's, $X_{i,j}$ and $A_{i,j}$. The difference in the latter case, is that the function $\eta(\omega)$ and the range of ω have already been determined, but the initial profile of Y 's and Z 's must be guessed to start the iteration.

Once the solutions for f_1 , y_j , g_1 , Z_j , etc. have been found, the desired information can be drawn from these solutions. Since the skin friction and the heat transfer to the wall are of particular interest, we will therefore derive their expressions as follows:

The symbol τ will be denoted as the nondimensional skin friction (normalized by $\rho_\infty U_\infty^2$); then by definition, one can show that

$$\tau \text{ Re} = \left(\mu \frac{\partial u}{\partial y} \right)_0 \quad (81a)$$

After using Eqs. (77) and (80), one can show that

$$\tau \text{ Re} = \frac{N \rho_s \mu_s}{r_s} \left(\frac{u}{f_1} \frac{\partial u}{\partial \omega} \right)_0 \quad (81b)$$

written in series

$$\begin{aligned} \tau \text{ Re} = \frac{N\xi}{\epsilon} & \left\{ f_1'(\omega_0) - \left[(\rho_1 + \mu_1 + r_1) f_1'(\omega_0) - 2U_1 Y_0'(\omega_0) - 2 Y_1'(\omega_0) \right. \right. \\ & \left. \left. - 2 \sum_{j=2}^5 \Omega_j Y_j'(\omega_0) \right] \xi^2 + \dots \right\} \end{aligned} \quad (81c)$$

Here superscript (') indicates differentiation with respect to ω and $f_1'(\omega_0)$ is $df_1/d\omega$ evaluating at $\omega = \omega_0$, etc.

The total heat transfer to the wall $\dot{q}$ (normalized by $1/2 \rho_\infty U_\infty^3$), in the hypersonic limit, can be written as

$$\dot{q} = \frac{N}{\text{PrRe}} \frac{\rho_s \mu_s}{r_s} \left(\frac{u}{f_1} \frac{\partial H}{\partial \omega} \right)_0 + \Gamma \dot{q}_r(\omega_0) \quad (82a)$$

with

$$\Gamma = \frac{2 [kT_{s(0)}]^4}{h^3 c^2 \rho_\infty U_\infty^3}$$

The first term of Eq. (82a) represents the convective heat transfer to the wall while the second term represents the radiative heating. The parameter Γ appears because the radiative heating $\dot{q}_r$ is normalized by $(kT_{s(0)})^4/h^3 c^2$.

Written in series, we found

$$\begin{aligned} \dot{q} = \frac{N}{\text{PrRe}\epsilon} g_1'(\omega_0) + \Gamma \sum_{i=1}^3 Q_{i,1} + & \left\{ \frac{N}{\text{PrRe}\epsilon} \left[g_1'(\omega_0) \left(\frac{u_1 f_2'(\omega_0)}{f_1'(\omega_0)} - \rho_1 - \mu_1 - r_1 \right) + Z_0'(\omega_0) \right. \right. \\ & \left. \left. + \sum_{j=1}^5 \Omega_j Z_j'(\omega_0) \right] + \Gamma \left(\sum_{i=1}^3 X_{i,0} + \sum_{i=1}^3 \sum_{j=1}^5 \Omega_j X_{i,j} \right) \right\} \xi^2 + \dots \end{aligned} \quad (82b)$$

Section 3

CONCLUSIONS

A series formulation for the viscous, radiating-absorbing hypersonic shock layer under the three-band model of continuum radiation has been completed. As a result of this formulation, equations for universal functions which determine the velocity and enthalpy profiles across the layer are obtained. Solutions to these highly coupled equations have to be obtained numerically. Expressions for the skin friction as well as the convective and radiative heat transfer to the wall are also presented. These latter quantities are functions of the following parameters: Reynolds number, Re ; Prandtl number, Pr ; density ratio, ϵ ; body nose radius, R_b ; viscosity-density ratio, N ; and the free-stream conditions. The specific form of their functional dependence will not be known until the equations for the universal functions are solved. The universal functions also depend on the above mentioned parameters. A numerical computation program is needed for their solution. Despite the complicated appearance of the equations and the coefficients listed in the Appendix, the computation effort for each specific case is expected to be reasonable.

Section 4
REFERENCES

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2. H. Hoshizaki and K. H. Wilson, "Convective and Radiative Heat Transfer During Superorbital Entry," Lockheed Rept. 4-43-65-5, June 1966
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APPENDIX III

A LIST OF COEFFICIENTS FOR PART II

$$T_1 = g_1^{0.58}, \beta_1 = \left[\frac{c_2 T_1 e^{-\theta d/T_1}}{1 + c_2 T_1 e^{-\theta d/T_1}} \right]^{1/2}, \varphi_1 = \left[\frac{c_3 T_1^{5/2} e^{-\theta i/T_1}}{1 + c_3 (T_1)^{5/2} e^{-\theta i/T_1}} \right]^{1/2}$$

$$c_2 = \frac{2.46 \times 10^7 B_u (1.16 k T_{S(0)})^{0.86}}{\sqrt{\epsilon} (k T_{S(0)})^3}, c_3 = 1.92 \times 10^{-3} (k T_{S(0)}) c_2$$

$$B_u = \frac{2.78 \times 10^{-6} T_{S(0)}^4}{\rho_\infty U_\infty^3}, \theta_d = \frac{9.7}{k T_{S(0)}}, \theta_i = \frac{14.53}{k T_{S(0)}}, k T_{S(0)} \text{ in ev.}$$

$$B_{1,0} = \begin{pmatrix} \beta_1 T_1 e^{-C_4/T_1} \\ T_1 e^{-C_4/T_1} \end{pmatrix}, \tilde{b} = 2.1 + c_5 T_1 + c_6 T_1^3$$

$$B_{1,1} = \begin{pmatrix} \tilde{b} \\ (1 - \varphi_1) \tilde{b} \end{pmatrix}$$

$$B_{1,2} = \begin{pmatrix} \beta_2 \tilde{b} + T_2 \left[\left(1 + \frac{C_4}{T_1} \right) \tilde{b} + c_5 T_1 + 3 c_6 T_1^3 \right] \\ (1 - \varphi_1) T_2 \left[\left(1 + \frac{C_4}{T_1} \right) \tilde{b} + c_5 T_1 + 3 c_6 T_1^3 \right] - \varphi_1 \varphi_2 \tilde{b} \end{pmatrix}$$

$$\alpha_{1,0} = \begin{pmatrix} \beta_1 \\ 1 \end{pmatrix} e^{-\frac{c_4}{T_1}} \left[\frac{2\pi^4 T_1^3}{15} - 2e^{-\frac{a}{T_1}} (a^3 + 3a^2 T_1 + 6a T_1^2 + 6T_1^3) \right]^{-1}$$

$$\alpha_{1,1} = \begin{pmatrix} 1 \\ 1-\varphi_1 \end{pmatrix} \tilde{b}$$

$$\begin{aligned} \alpha_{1,2} = & \begin{pmatrix} 1 \\ 1-\varphi_1 \end{pmatrix} T_2 \left\{ \frac{c_4}{T_1} \tilde{b} + c_5 T_1 + 3c_6 T_1^3 - \left[\frac{\tilde{b} 2\pi^4 T_1^3}{5} - 2\tilde{b} e^{-\frac{a}{T_1}} \left(\frac{a^4}{T_1} + 3a^3 \right. \right. \right. \\ & \left. \left. \left. + 9a^2 T_1 + 18a T_1^2 + 18 T_1^3 \right) \right] / \left[\frac{2\pi^4 T_1^3}{15} - 2e^{-\frac{a}{T_1}} (a^3 + 3a^2 T_1 + 6a T_1^2 + 6 T_1^3) \right] \right\} \\ & + \begin{pmatrix} \beta_2 \\ -\varphi_1 \varphi_2 \end{pmatrix} \tilde{b} \end{aligned}$$

$$T_{2,0} = 0.49 \left(1 - \frac{f_1^2}{g_1} \right) - \frac{1}{1.72}$$

$$\beta_{2,0} = \frac{\frac{3}{2} - \eta + (1 + \frac{\theta_d}{T_1}) T_{2,0}}{2(1 + c_2 T_1 e^{-\frac{\theta_d}{T_1}})}$$

$$\beta_{2,1} = \frac{0.49 \left(1 + \frac{\theta_d}{T_1} \right)}{2g_1 \left(1 + c_2 T_1 e^{-\frac{\theta_d}{T_1}} \right)}$$

$$\beta_{2,2} = 1.184 \ g_1 \ \beta_{2,1}$$

$$\varphi_{2,0} = \frac{\frac{3}{2} - \eta + 2.5 \ T_{2,0} \left(1 + \frac{\theta_i}{T_1^{2.5}}\right)}{2 \left[1 + T_1^{2.5} \ c_3 \ e^{-\theta_i/T_1^{2.5}}\right]}$$

$$\varphi_{2,1} = \frac{1.225 \left(1 + \frac{\theta_i}{T_1^{2.5}}\right)}{2g_1 \left[1 + T_1^{2.5} \ c_3 \ e^{-\theta_i/T_1^{2.5}}\right]}$$

$$\varphi_{2,2} = 1.184 \ \varphi_{2,1} \ g_1$$

$$B_{2,0} = 1$$

$$B_{2,1} = 2T_1 \left[e^{-\frac{a}{T_1}} (a^3 + 3 \ a T_1 + 6a \ T_1^2 + 6 \ T_1^3) - e^{-\frac{b}{T_1}} (b^3 + 3b^2 T_1 + 6 \ b T_1^2 + 6 \ T_1^3) \right]$$

$$B_{2,2} = 2T_2 \left\{ \frac{B_{2,1}}{2} + e^{-\frac{a}{T_1}} \left[a^4 + 3 \ a^3 T_1 + 9a^2 T_1^2 + 20 \ a \ T_1^3 + 18 \ T_1^4 \right] \right. \\ \left. - e^{-\frac{b}{T_1}} \left[b^4 + 3b^3 T_1 + 9 \ b^2 T_1^2 + 20 \ b \ T_1^3 + 18 \ T_1^4 \right] \right\}$$

$$\alpha_{2,0} = \binom{\beta_1}{1} \frac{e^{-C_7/T_1}}{4 + 10 \ e^{-C_8/T_1} + 6 \ e^{-C_7/T_1}}$$

$$\alpha_{2,1} = \binom{1}{1-\varphi_1}$$

$$\alpha_{2,2} = \begin{pmatrix} \beta_2 \\ -\varphi_1 \varphi_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 - \varphi_1 \end{pmatrix} \frac{T_2}{T_1} \left[C_7 - \frac{10 C_8 e^{-C_8/T_1} + 6 C_7 e^{-C_7/T_1}}{4 + 10 e^{-C_8/T_1} + 6 e^{-C_7/T_1}} \right]$$

$$B_{3,0} = 1$$

$$B_{3,1} = 2 T_1 e^{-b/T_1} (b^3 + 3 b T_1 + 6 T_1^2 + 6 T_1^3)$$

$$B_{3,2} = 2 T_2 \left[\frac{B_{3,1}}{2} + e^{-b/T_1} (b^4 + 3 b^3 T_1 + 9 b^2 T_1^2 + 20 b T_1^3 + 18 T_1^4) \right]$$

$$\alpha_{3,0} = \alpha_{2,0}$$

$$\alpha_{3,1} = \left(1 + 1.24 e^{C_9/T_1} \right) \alpha_{2,1}$$

$$\alpha_{3,2} = \left(1 + 1.24 e^{C_9/T_1} \right) \alpha_{2,2} - \frac{1.24 C_9 T_2}{T_1} \alpha_{2,1} e^{C_9/T_1}$$

$$W_1 = \frac{1}{u_1}, \quad W_2 = \frac{\rho_1}{u_1}, \quad W_3 = \frac{2r_1}{u_1}, \quad W_4 = \frac{K_1}{u_1}, \quad W_5 = \frac{2r_1 + \mu_1 + \rho_1}{u_1}$$

$$\Omega_1 = u_1, \quad \Omega_2 = \rho_1, \quad \Omega_3 = 2r_1, \quad \Omega_4 = K_1, \quad \Omega_5 = 2r_1 + \mu_1 + \rho_1$$

$$s_1 = 3\epsilon_{g_1}^{0.84} \left[0.84 \left(1 - \frac{f_1^2}{g_1} \right) + \frac{1}{2} - \eta \right]$$

$$s_2 = 3\epsilon_{g_1}^{0.84}$$

$$s_3 = \epsilon_{g_1}^{0.84} (4 + \eta) - f_1^2 + 4\eta \frac{df}{d\omega}$$

$$s_4 = 2(1 - \eta) \epsilon_{g_1}^{0.84}$$

$$s_5 = 0$$

$$v_1 = 0$$

$$v_2 = F_5$$

$$v_3 = 4\eta \frac{dg_1}{d\eta} = \frac{4\eta}{f_1} \frac{dg_1}{d\omega}$$

$$v_4 = 0$$

$$v_5 = F_1$$

$$M_{i,2} = K_{i,2} , \quad M_{i,3} = K_{i,3} , \quad M_{i,4} = 0 , \quad M_{i,5} = 0$$

$$O_{i,2} = N_{i,2} , \quad O_{i,3} = N_{i,3} , \quad O_{i,4} = 0 , \quad O_{i,5} = 0$$

$$F_1 = \Gamma_1 \alpha_{1,1} (\alpha_{1,0} A_{1,1}^{-4\pi B_{1,0}}) + \Gamma_2 \alpha_{2,0} \alpha_{2,1} (A_{2,1}^{-4\pi B_{2,1}}) + \Gamma_2 \alpha_{2,0} \alpha_{3,1} (A_{3,1}^{-4\pi B_{3,1}})$$

$$\begin{aligned} F_2 = & B(f_1^2 f_1')' f_1 + \Gamma_1 \alpha_{1,0} A_{1,1} \left\{ \frac{E_1 T_{2,0} + E_2 \beta_{2,0}}{E_1 (1-\varphi_1) T_{2,0} - E_2 \varphi_1 \varphi_{2,0}} \right\}^{-4\pi B_{1,0}} \Gamma_1 \left\{ \frac{E_3 T_{2,0} + E_4 \beta_{2,0}}{E_3 (1-\varphi_1) T_{2,0} - E_4 \varphi_1 \varphi_{2,0}} \right\} \\ & - \Gamma_2 \alpha_{2,0} \left\{ 4\pi (\alpha_{2,1} E_5 + \alpha_{3,1} E_3) T_{2,0} + (A_{3,1}^{-4\pi B_{3,1}}) \frac{1.24}{T_1} c_9 e^{\frac{c_9}{T_1}} \alpha_{2,0} T_{2,0} \right. \\ & \left. - (A_{2,1}^{-4\pi B_{2,1}} + (A_{3,1}^{-4\pi B_{3,1}}) (1 + 1.24 e^{\frac{c_9}{T_1}})) \left\{ \frac{\beta_{2,0} + E_6 T_{2,0}}{(1-\varphi_1) E_6 T_{2,0} - \varphi_1 \varphi_{2,0}} \right\} \right\} \end{aligned}$$

$$\begin{aligned}
F_4 = & -2f_1 + \Gamma_1 \alpha_{1,0} A_{1,1} \left\{ \frac{E_2 \beta_{2,2} + \frac{0.84 E_1}{1.72 g_1}}{\frac{E_1 (1-\varphi_1) 0.84}{1.72 g_1} - E_2 \varphi_1 \varphi_{2,2}} \right\} - 4\pi B_{1,0} \Gamma_1 \left\{ \frac{E_4 \beta_{2,2} + \frac{0.84 E_3}{1.72 g_1}}{\frac{0.84 E_3 (1-\varphi_1)}{1.72 g_1} - E_4 \varphi_1 \varphi_{2,2}} \right\} \\
& - \Gamma_2 \alpha_{2,0} \left[\frac{1.68 \pi}{1.72 g_1} (\alpha_{2,1} E_5 + \alpha_{3,1} E_7) - \left[A_{2,1} - 4\pi B_{2,1} + (A_{3,1} - 4\pi B_{3,1}) (1 + 1.24 e^{\frac{C_9}{T_1}}) \right] \right. \\
& \left. \left\{ \frac{\beta_{2,2} + \frac{0.84 E_6}{1.72 g_1}}{\frac{0.84 E_6 (1-\varphi_1)}{1.72 g_1} - \varphi_1 \varphi_{2,2}} \right\} + \frac{1.04}{1.72 g_1} e^{\frac{C_9}{T_1}} \frac{C_9}{T_1} \alpha_{2,1} (A_{3,1} - 4\pi B_{3,1}) \right\} \\
F_5 = & \Gamma_1 \alpha_{1,0} A_{1,1} \left\{ \frac{E_2 \beta_{2,3} + \frac{E_1}{1.72}}{\frac{E_1 (1-\varphi_1)}{1.72} - E_2 \varphi_1 \varphi_{2,3}} \right\} - 4\pi B_{1,0} \Gamma_1 \left\{ \frac{\frac{E_3}{1.72} + E_4 \beta_{2,3}}{\frac{E_3 (1-\varphi_1)}{1.72} - \varphi_1 E_4 \varphi_{2,3}} \right\} \\
& - \Gamma_2 \alpha_{2,0} \left[\frac{4\pi}{1.72} (E_5 \alpha_{2,1} + E_7 \alpha_{3,1}) + \frac{1.24}{1.72} \frac{C_9}{T_1} \alpha_{2,1} e^{\frac{C_9}{T_1}} (A_{3,1} - 4\pi B_{3,1}) \right. \\
& \left. - \left[(A_{2,1} - 4\pi B_{2,1}) + (A_{3,1} - 4\pi B_{3,1}) \right] \left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \frac{\beta_{2,3} + \frac{E_6}{1.72}}{\frac{E_6 (1-\varphi_1)}{1.72} - \varphi_1 \varphi_{2,3}} \right\} \right]
\end{aligned}$$

with

$$\begin{aligned}
E_1 = & \frac{C_4}{T_1} \tilde{b} + (C_5 T_1 + 3 C_6 T_1^3) \\
& - \frac{\tilde{b} \left\{ \frac{2\pi^4 T_1^3}{5} - 2e^{-\frac{a}{T_1}} [(a^3 + 3a^2 T_1 + 6a T_1^2 + 6T_1^3) \frac{a}{T_1} + 3(a^2 T_1 + 4a T_1^2 + 6T_1^3)] \right\}}{\frac{2\pi^4 T_1^3}{15} - 2e^{-\frac{a}{T_1}} (a^3 + 3a^2 T_1 + 6a T_1^2 + 6T_1^3)}
\end{aligned}$$

$$E_2 = \tilde{b}$$

$$E_3 = \left(1 + \frac{c_4}{T_1}\right)\tilde{b} + c_5 T_1 + 3c_6 T_1^3$$

$$E_4 = E_2$$

$$E_5 = B_{2,1} + 2e^{-\frac{a}{T_1}} [3(a^2 T_1 + 4a T_1^2 + 6 T_1^3) T_1 + a(a^3 + 3a^2 T_1 + 6a T_1^2 + 6 T_1^3)]$$

$$- 2e^{-\frac{b}{T_1}} [3(b^2 T_1 + 4b T_1^2 + 6 T_1^3) T_1 + b(b^3 + 3b^2 T_1 + 6b T_1^2 + 6 T_1^3)]$$

$$E_6 = \frac{c_7}{T_1} - \frac{10e^{-\frac{c_8}{T_1}} \frac{c_8}{T_1} + 6e^{-\frac{c_7}{T_1}} \frac{c_7}{T_1}}{4 + 10e^{-\frac{c_8}{T_1}} + 6e^{-\frac{c_7}{T_1}}}$$

$$E_7 = B_{3,1} + 2e^{-\frac{b}{T_1}} [3(b^2 T_1 + 4b T_1^2 + b T_1^3) T_1 + b(b^3 + 3b^2 T_1 + 6b T_1^2 + 6 T_1^3)]$$

$$K_{1,1} = \alpha_{1,0} A_{1,1} \alpha_{1,1} \left\{ \frac{E_1 T_{2,0} + E_2 \beta_{2,0}}{E_1 (1 - \varphi_1) T_{2,0} - E_2 \varphi_1 \varphi_{2,0}} \right\} - 4\pi B_{1,0} \left\{ \frac{E_3 T_{2,0} + E_4 \beta_{2,0}}{E_3 (1 - \varphi_1) T_{2,0} - E_4 \varphi_1 \varphi_{2,0}} \right\}$$

$$K_{1,2} = \alpha_{1,0} A_{1,1} \alpha_{1,1} \left\{ \frac{E_1 \beta_{2,2} + \frac{E_1}{1.72}}{-E_2 \varphi_1 \varphi_{2,2} + \frac{(1 - \varphi_1) E_1}{1.72}} \right\} - 4\pi B_{1,0} \left\{ \frac{E_4 \beta_{2,2} + \frac{E_3}{1.72}}{-E_4 \varphi_1 \varphi_{2,2} + \frac{(1 - \varphi_1) E_3}{1.72}} \right\}$$

$$K_{1,3} = \frac{1}{2} (\alpha_{1,0} A_{1,1} \alpha_{1,1} - 4\pi B_{1,0} B_{1,1})$$

$$K_{1,4} = \alpha_{1,0} A_{1,1} \alpha_{1,1} \left\{ \begin{array}{l} \frac{E_2 \beta_{2,1} + \frac{0.84}{1.72} \frac{E_1}{g_1}}{\frac{E_1 (1-\varphi_1) 0.84}{1.72 g_1} - E_2 \varphi_1 \varphi_{2,1}} \end{array} \right\} - 4\pi B_{1,0} \left\{ \begin{array}{l} \frac{E_4 \beta_{2,1} + \frac{0.84}{1.72} \frac{E_3}{g_1}}{\frac{E_3 (1-\varphi_1) 0.84}{1.72 g_1} - E_4 \varphi_1 \varphi_{2,1}} \end{array} \right\}$$

$$K_{1,5} = \alpha_{1,0} \alpha_{1,1}$$

$$K_{2,1} = \alpha_{2,0} (A_{2,1} - 4\pi B_{2,1}) \left\{ \begin{array}{l} \beta_{2,0} + \frac{E_6}{1.72} T_{2,0} \\ (1-\varphi_1) T_{2,0} E_6 - \varphi_1 \varphi_{2,0} \end{array} \right\} - 4\pi \alpha_{2,1} E_5 T_{2,0} \alpha_{2,0}$$

$$K_{2,2} = \alpha_{2,0} (A_{2,1} - 4\pi B_{2,1}) \left\{ \begin{array}{l} \beta_{2,2} + \frac{E_6}{1.72} \\ -\varphi_1 \varphi_{2,2} + \frac{E_6 (1-\varphi_1)}{1.72} \end{array} \right\} - 4\pi \alpha_{2,0} \alpha_{2,1} \frac{E_5}{1.72}$$

$$K_{2,3} = \frac{1}{2} \alpha_{2,0} \alpha_{2,1} (A_{2,1} - 4\pi B_{2,1})$$

$$K_{2,4} = \alpha_{2,0} (A_{2,1} - 4\pi B_{2,1}) \left\{ \begin{array}{l} \frac{E_6 0.84}{\beta_{2,1} + \frac{0.84}{1.72} \frac{E_1}{g_1}} \\ \frac{E_5 0.84 (1-\varphi_1)}{1.72 g_1} - \varphi_1 \varphi_{2,1} \end{array} \right\} - 4\pi \alpha_{2,1} \alpha_{2,0} \frac{0.84}{1.72} \frac{E_5}{g_1}$$

$$K_{2,5} = \alpha_{2,0} \alpha_{2,1}$$

$$K_{3,1} = \alpha_{2,0} (A_{3,1} - 4\pi B_{3,1}) \left[\left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \begin{array}{l} \beta_{2,0} E_6 T_{2,0} \\ (1-\varphi_1) T_{2,0} E_6 - \varphi_1 \varphi_{2,0} \end{array} \right\} - 1.24 \alpha_{2,1} T_{2,0} \frac{C_9}{T_1} e^{\frac{C_9}{T_1}} \right]$$

$$- 4\pi \alpha_{2,0} E_7 T_{2,0} \alpha_{3,1}$$

$$K_{3,2} = \alpha_{2,0} (A_{3,1}^{-4\pi B_{3,1}}) \left[\left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \begin{array}{l} \beta_{2,2} + \frac{E_6}{1.72} \\ -\varphi_1 \varphi_{2,2} + \frac{E_6(1-\varphi_1)}{1.72} \end{array} \right\} + 1.24 \frac{C_9 \alpha_{2,1}}{1.72 T_1} e^{\frac{C_9}{T_1}} \right] \\ - 4\pi \alpha_{2,0} \alpha_{3,1} \frac{E_7}{1.72}$$

$$K_{3,3} = \frac{1}{2} \alpha_{2,0} \alpha_{3,1} (A_{3,1}^{-4\pi B_{3,1}})$$

$$K_{3,4} = \alpha_{2,0} (A_{3,1}^{-4\pi B_{3,1}}) \left[\left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \begin{array}{l} \beta_{2,1} + \frac{0.84 E_6}{1.72 g_1} \\ \frac{0.84 E_6(1-\varphi_1)}{1.72 g_1} - \varphi_1 \varphi_{2,1} \end{array} \right\} - \frac{1.24 C_9 0.84 \alpha_{2,1} e^{\frac{C_9}{T_1}}}{1.72 g_1 T_1} \right] \\ - 4\pi \alpha_{2,0} \alpha_{3,1} \frac{0.84 E_7}{1.72 g_1}$$

$$K_{3,5} = \alpha_{2,0} \alpha_{3,1}$$

$$N_{1,1} = \alpha_{1,0} Q_{1,1} \left\{ \begin{array}{l} E_1 T_{2,0} + E_2 \beta_{2,0} \\ E_1 (1-\varphi_1) T_{2,0} - E_2 \varphi_1 \varphi_{2,0} \end{array} \right\}$$

$$N_{1,2} = \alpha_{1,0} Q_{1,1} \left\{ \begin{array}{l} E_2 \beta_{2,2} + \frac{E_1}{1.72} \\ -E_2 \varphi_1 \varphi_{2,2} + \frac{(1-\varphi_1) E_1}{1.72} \end{array} \right\}$$

$$N_{1,3} = \frac{1}{2} \alpha_{1,0} Q_{1,1} \alpha_{1,1}$$

$$N_{1,4} = \alpha_{1,0} Q_{1,1} \left\{ \begin{array}{l} E_2 \beta_{2,1} + \frac{0.84}{1.72 g_1} E_1 \\ \frac{0.84 E_1 (1-\varphi_1)}{1.72 g_1} - E_2 \varphi_1 \varphi_{2,1} \end{array} \right\}$$

$$N_{1,5} = \alpha_{1,0} \alpha_{1,1} = K_{1,5}$$

$$N_{2,1} = \alpha_{2,0} Q_{2,1} \left\{ \begin{array}{l} \beta_{2,0} + E_6 T_{2,0} \\ (1-\varphi_1) T_{2,0} E_6 - \varphi_1 \varphi_{2,0} \end{array} \right\}$$

$$N_{2,2} = \alpha_{2,0} Q_{2,1} \left\{ \begin{array}{l} \beta_{2,2} + \frac{E_6}{1.72} \\ -\varphi_1 \varphi_{2,2} + \frac{E_6(1-\varphi_1)}{1.72} \end{array} \right\}$$

$$N_{2,3} = \frac{1}{2} \alpha_{2,0} \alpha_{2,1} Q_{2,1}$$

$$N_{2,4} = \alpha_{2,0} Q_{2,1} \left\{ \begin{array}{l} \beta_{2,1} + \frac{0.84 E_6}{1.72 g_1} \\ \frac{0.84 E_6(1-\varphi_1)}{1.72 g_1} - \varphi_1 \varphi_{2,1} \end{array} \right\}$$

$$N_{2,5} = \alpha_{2,0} \alpha_{2,1} = K_{2,5}$$

$$N_{3,1} = \alpha_{2,0} Q_{3,1} \left[\left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \begin{array}{l} \beta_{2,0} + E_6 T_{2,0} \\ (1-\varphi_1) T_{2,0} E_6 - \varphi_1 \varphi_{2,0} \end{array} \right\} - \frac{1.24 C_9 e^{\frac{C_9}{T_1}} \alpha_{2,1} T_{2,0}}{T_1} \right]$$

$$N_{3,2} = \alpha_{2,0} Q_{3,1} \left[\left(1 + 1.24 e^{\frac{C_9}{T_1}} \right) \left\{ \begin{array}{l} \beta_{2,2} + \frac{E_6}{1.72} \\ -\varphi_1 \varphi_{2,2} + \frac{E_6(1-\varphi_1)}{1.72} \end{array} \right\} + \frac{1.24 C_9 e^{\frac{C_9}{T_1}} \alpha_{2,1}}{1.72 T_1} \right]$$

$$N_{3,3} = \frac{1}{2} Q_{3,1} \alpha_{3,1} \alpha_{2,0}$$

$$N_{3,4} = \alpha_{2,0} Q_{3,1} \left[\left(1 + 1.24 e^{\frac{c_9}{T_1}} \right) \left\{ \begin{aligned} &\beta_{2,1} + \frac{0.84 E_6}{1.72 g_1} \\ &\frac{0.84 E_6 (1 - \varphi_1)}{1.72 g_1} - \varphi_1 \varphi_{2,1} \end{aligned} \right\} - \frac{0.84 \times 1.24 c_9 \alpha_{2,1} e^{\frac{c_9}{T_1}}}{1.72 T_1 g_1} \right]$$

$$N_{3,5} = \alpha_{2,0} \alpha_{3,1} = K_{3,5}$$

Appendix IV

ERROR ESTIMATION FOR THE QUASI-ONE-DIMENSIONAL APPROXIMATION

In this appendix we will show that within the framework of differential approximation, the quasi-one-dimensional approximation that leads to Eq. (5b) introduces an error of the order ϵ^2 , which is small.

Equation (5a) can be written in body oriented coordinates as follows:

$$\frac{1}{r} \left[\frac{\partial r(q_x)_\nu}{\partial x} + \frac{\partial r\tilde{k}(q_y)_\nu}{\partial y} \right] = -\alpha_\nu [(I_o)_\nu - 4\pi B_\nu] \quad (A11)$$

For a thin shock layer, one may approximate $\tilde{k} \simeq 1$, $r \approx r_s$. Thus Eq. (A11) becomes

$$\frac{1}{r_s} \frac{\partial r_s(q_x)_\nu}{\partial x} + \frac{\partial (q_y)_\nu}{\partial y} = -\alpha_\nu [(I_o)_\nu - 4\pi B_\nu] \quad (A12)$$

After frequency integration and transforming x and y to ξ and η , we find

$$\frac{\epsilon \rho_s \mu_s}{\rho} \frac{\partial r_s(q_x)_i}{\partial \xi} - \left(v + 2\eta \frac{dr_s}{d\xi} \frac{\epsilon \rho_s \mu_s}{\rho} \right) \frac{\partial (q_x)_i}{\partial \eta} + u \frac{\partial (q_y)_i}{\partial \eta} = -\lambda_i r_s [(\alpha I_o)_i - 4\pi B_i] \quad (A13)$$

Equation (7) is now needed for the determination of q_x . After frequency integration and transformation into ξ and η variables, Eq. (2) becomes

$$\frac{\epsilon \rho_s \mu_s r_s}{\rho} \frac{\partial (I_o)_i}{d\xi} - \left(v + 2\eta \frac{dr_s}{d\xi} \frac{\epsilon \rho_s \mu_s}{\rho} \right) \frac{\partial (I_o)_i}{\partial \eta} = -3\lambda_i r_s (\alpha q_x)_i \quad (A14)$$

Similar to the series given by Eqs. (44) and (45), we also write

$$\begin{aligned} (q_x)_i &= \xi q_{i,1} + \xi^3 q_{i,2} + \dots \\ v &= \epsilon (-2\eta + v_2 \xi^2 + \dots) \end{aligned} \quad (A15)$$

Note that $\rho_s \mu_s / \rho \simeq 1 + 0(\epsilon)$. For simplicity we approximate

$$\frac{\rho_s \mu_s}{\rho} \simeq 1$$

Substituting Eq. (A15) into Eqs. (A14) and (A13), and after using Eqs. (44) and (45), collecting terms of the same order in ξ , we obtain

$$2\epsilon q_{i,1} + f_1 \frac{dQ_{i,1}}{d\eta} = -\lambda_i (\alpha_{i,0} \alpha_{i,1} A_{i,1} - 4\pi B_{i,0} B_{i,1}) \quad (A16)$$

$$2\epsilon A_{i,2} - \epsilon (v_2 + 6\eta r_1) \frac{dA_{i,1}}{d\eta} = -3\lambda_i \alpha_{i,0} \alpha_{i,1} q_{i,1} \quad (A17)$$

$$f_1 \frac{dA_{i,1}}{d\eta} = -3\lambda_i \alpha_{i,0} \alpha_{i,1} Q_{i,1} \quad (56)$$

$$f_1 \frac{dA_{i,2}}{d\eta} = 3u_1 \left(\frac{f_2}{f_1} \right) \lambda_i \alpha_{i,0} \alpha_{i,1} Q_{i,1} - 3\lambda_i \alpha_{i,0} [\alpha_{i,1} Q_{i,2} + (r_1 \alpha_{i,1} + \alpha_{i,2}) Q_{i,1}] \quad (60)$$

From the above equations, it is seen that if we assume $Q_{i,1}$, $Q_{i,2}$ to be of order one, and since $\alpha_{i,1}$, $\alpha_{i,2}$ are of the same order of magnitude, we then see from Eq. (60) that $A_{i,2}$ is of the order $(\lambda_i \alpha_{i,0} \alpha_{i,1})$ (note that $f_1 \frac{d}{d\eta} = \frac{d}{d\omega}$, $d\omega$ is $0(1)$ across the shock layer). In view of Eq. (56), we conclude from Eq. (A17) that $q_{i,1}$ is of the order ϵ^2 . Therefore, our approximation which leads to Eq. (5b) introduces an error of the order ϵ^2 which is indeed small for our radiating thin shock layer problems.

We also observe that our quasi-linear approximation is equivalent to the first order truncation solution of the method of series truncation regardless of the magnitude of ϵ .

Similar demonstrations can be made for the terms involving $q_{i,2}$.