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DEVELOPMENT OF A DYNAMIC METHOD FOR DETERMINING THE TERRESTRIAL GRAVITY FIELD FROM SATELLITE DATA

By

MOHAMMAD ASADULLAH KHAN

SEPTEMBER 1968

FINAL REPORT

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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HAWAII INSTITUTE OF GEOPHYSICS
UNIVERSITY OF HAWAII

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ABSTRACT

A method is developed to use satellite data in the problem of obtaining the terrestrial gravity field with specific reference to its localised features. The method makes use of the Hamiltonian function associated with the satellite motion, which is time-invariant for the degenerate case when all perturbations are ignored but which, under special assumptions, has an integrable time-derivative when the effect of the perturbing forces is considered. Since the potential function of the earth appears additively in the Hamiltonian function, it is possible to set up observation equations which can be solved for the spherical harmonic coefficients of the geopotential which appear linearly in these equations. Since the method is primarily meant to determine the short-wavelength features of the gravity field, the long-wavelength components may be considered known from the previous satellite solutions if this be found desirable or essential, and only the coefficients of the higher degree terms may be determined. Such high degree coefficients will describe the gravity field for the area which is immediately below the orbital segments analyzed, since the mass distribution in this area will play a dominant role in perturbing the satellite trajectory over the area. The method of setting up the observation equations seems to minimize the effect of the uncertainties in our knowledge of the values of the perturbing forces and, consequently, may permit the use of satellites at lower altitudes in the determining of the earth's gravity field.

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INTRODUCTION

An accurate knowledge of the gravity field of the earth is required in most geodetic and geophysical investigations. The problems encountered in determining the gravity field of the earth by surface gravity measurements, particularly over the oceanic areas, are well known (see for example, Khan, 1967). Airborne gravimetry seems to be the answer to some of these problems, but it has not developed to the stage where it can be considered as a readily available technique. Now that the capability to launch artificial earth satellites in orbits of arbitrary size, shape, and inclination has been attained, it is generally accepted that the artificial satellites with a variety of orbital elements are the best means to achieve an accurate knowledge of the earth's gravity field. With this as one of its major scientific objectives, a special geodetic satellites program was initiated by the National Aeronautics and Space Administration in cooperation with several other agencies. A number of the geodetic satellites (in the ANNA, GEOS, and PAGEOS series) under this program are already in orbit and are furnishing orbital data for specific geodetic and geophysical uses. In addition, orbital data from several other satellites, initially launched for other purposes, have also been available for these studies.

The techniques employed to determine the gravity field of the earth from the orbital motion of artificial satellites are the traditional techniques of celestial mechanics refined to apply to the close earth-satellites in the presence of a perturbing nonspherical earth, the moon, the sun, and the perturbing effects of the atmosphere (Brouwer, 1959, 1946; Garfinkel, 1958, 1959, 1960; Groves, 1960; Hagihara, 1961; Izsak, 1960, 1961, 1962; Kozai, 1959a, 1959b, 1960, 1961a,b, 1962; Mersman, 1961; Merson, 1961; Musen, 1959, 1960a,b,c; Parkinson, 1960; Poritsky, 1962; Sterne, 1958; Vinti, 1959, and many others). These orbital theories determine the theoretical orbit of the satellite. Any perturbations relative to this computed orbit will be due to factors which have not been considered in the theory used to obtain the computed orbit. Let us assume that, by considering the dynamical effect of atmospheric drag and solar radiation pressure and the perturbations caused by the attraction of the moon and the sun, we can separate the perturbations caused by the terrestrial disturbing function. Then from the analytical expressions which describe the perturbations caused by the various terms in the spherical harmonic expansion of the geopotential, it is possible to determine the coefficients of the terms. In practice, however, there are several difficulties to achieving this. First, there is the difficulty of identifying the part of the perturbation due to the terrestrial gravity disturbing function. Then, of course, there is the problem of decomposing the residual perturbation, caused by the terrestrial gravity, into its harmonic components and identifying each of the components with the appropriate source harmonic of geopotential. The zonal harmonics are relatively easier to determine since they cause secular and long-period changes in the orbital elements. The even-degree zonal harmonics cause secular changes in the longitude of the ascending node Ω and the argument of perigee ω, which are much larger than the long-period effect of odd-degree zonal harmonics and, hence, are best determined from the secular motion of the longitude of the ascending node and the argument of perigee. The effect of the odd-degree zonal harmonics is more pronounced in the inclination i and the eccentricity e. Hence, the odd-degree zonal harmonic coefficients are determined from the amplitude of the long-period variations in e and i. The tesseral harmonics are much more difficult to determine because they cause oscillatory disturbances in the orbit which rapidly change in sign. Hence, to study them, the distribution of observations in space and time needs to be much more dense and uniform, something not entirely feasible as yet with some of the camera-tracking techniques. In addition, the effect of some of the tesseral harmonics is not very distinct in amplitude, and sometimes not even in shape. As pointed out before, to derive the amplitudes and the secular motions from the observations, for the purpose of determining the geopotential coefficients, we must take out other perturbations from all other sources. To avoid interaction terms among perturbations due to the geopotential and to other sources, the use of satellites greatly affected by air drag, lunisolar and radiation pressure perturbations is not desirable, and one is limited to only those satellites which are sufficiently high so that the effect of air drag is not too great on them but are yet not so high that they can not sense the major features of the gravity field of the earth. This, of course, will limit the number of orbits which can be used in the determination of geopotential. The analytical expressions describing the perturbations caused by the various terms in the geopotential depend essentially on the inclination i. Hence, in order to have well conditioned observation equations, it is essential to have data from satellites with a wide variety of inclinations. The latter requirement, of course, will no longer be a serious problem as more and more satellites with a variety of orbital elements become available for these studies.

In view of some of the above difficulties, different tracking systems and computational techniques have been developed. At the Smithsonian Astrophysical Observatory, investigators have been using the Baker-Nunn camera-tracking data and the Differential Orbit Improvement program to obtain the satellite-orbit solutions. Kozai, using these optical data, has obtained several solutions for the zonal harmonic coefficients. Recently he has obtained (Kozai, 1967) a set of zonal harmonic coefficients out to 20th degree which he has again revised (private communication). Gaposhkin (1966), following up on the work of the late Imre Izsak, has obtained a complete set of tesseral harmonic coefficients to 8th degree and order. This solution is already in the process of being improved and extended as a consequence of the availability of more extensive and refined data. At the National Weapons Laboratory, R. J. Anderle and his group have been using the Doppler tracking data. Recently, Anderle has improved and extended his solution obtaining a complete set of tesseral harmonic coefficients out to 15th degree and order. Unfortunately, however, this solution is classified at this time. In addition, there have been various geopotential solutions based on various types of data and computational approaches (Kaula, 1966; Guier and Newton, 1965; Smith, 1963, 1965). Mention should also be made of several attempts to determine the tesseral harmonics of specific degree from their resonant effects on some satellite orbits (Allan, 1965; Anderle, 1965; Murphy and Victor, 1967; Wagner, 1966, 1968; Yionoulis, 1966).

Because of the various types of the input data used and the various types of computational techniques involved, there have been several attempts to make comparative studies of the spherical harmonic coefficients derived from the various satellite-orbit solutions with one another and with those obtained from the surface gravity material, with the aim of judging their reliability. The gravity field defined by surface gravity measurements constitutes a natural standard of comparison for the various satellite solutions. However, the gravimetric coverage is not sufficient to give reliable determinations of the harmonic coefficients either. It is generally felt that the geoidal undulations and the gravity anomalies obtained from the different satellite solutions show better agreement than do the individual harmonic coefficients themselves, probably because the harmonic coefficients obtained from each solution should be considered as a composite set instead, and hence the combined contribution of each of these sets to the geopotential should be compared. Even when this is done, discrepancies of the order of 20 to 30 meters are noted between the various satellite solutions (Khan and Woollard, 1968). The low-degree zonal harmonic coefficients show a favorable comparison while the higher degree zonal harmonic and the tesseral harmonic coefficients in different solutions show rather large discrepancies. As a result of these investigations, and in view of the difficulties inherent in the determination of the tesseral harmonics, as pointed out before, it is now generally agreed that the low-degree zonal harmonic coefficients out to $n = 4$ to 6 have been accurately determined, whereas the values of the higher degree zonal harmonic and all of the tesseral harmonic coefficients should be accepted with caution.

Since the surface gravity measurements will obviously determine the short-wavelength components of the gravity field more accurately, there have been several attempts to integrate the available surface-gravity data with the satellite-determined gravity field, as proposed primarily by Kaula (1966) and by Arnold (1966). While a considerable number of solutions have been obtained by the technique proposed by Kaula, not much has been reported on that suggested by Arnold.

It is reasonable to believe that a part of the perturbation in the orbit of a satellite due to the terrestrial gravity function will be primarily a result of the mass distribution in the area which is nearest the satellite (Khan, 1967). If this component is accurately recorded, it can be used, in theory at least, to yield information on the local mass distribution in the area.

In this report, we outline a method which will accomplish this by making use of the Hamiltonian function associated with the motion of a satellite. The potential function of the earth appears additively in the Hamiltonian function. If we expand it in terms of the spherical harmonics, it is possible to determine the expansion coefficients which appear linearly as multiples of spherical harmonics. The determination of these coefficients from the observations of a small orbital arc is made possible by the fact that the potential function is regarded as made up of two parts: long-wavelength components and short-wavelength components. The long-wavelength components are assumed to be the global characteristic of the earth's gravity field and may be treated as known from the previous satellite-orbit solutions, as pointed out above. The short-wavelength components are assumed to be characteristic of a certain area only and hence should vary from one area to the other. If we have a sufficient number of observations, we can solve for an appropriate number of the unknown coefficients. In this way we will obtain expressions which will describe gravitational potential of the region immediately below the satellite trajectory. Let the region for which the information is primarily weighted be called the 'effective area'.

Then the set of the higher degree harmonic coefficients taken in conjunction with the long-wavelength components will define an equipotential surface which, hopefully, will closely approximate the actual geoidal surface in this 'effective area' but will apparently be merely hypothetical beyond this area. Theoretically, it is possible to cover the surface of the earth by such effective areas. These areas can finally be integrated to obtain a final description of the earth's gravity field.

In this report, I first give the theory of the degenerate case when all perturbations are ignored and gradually modify it by considering the perturbing effects of the attraction of the moon and the sun and the dynamic effects of the atmospheric drag and the solar radiation pressure. The final observation equations are primarily designed to be set up in such a way that the effects of the uncertainties in our knowledge of the perturbing forces are reduced to a minimum, so that the possibility of using the relatively lower altitude satellites may be explored in the problem of determining the finer structure of the earth's gravity field.

THE EXPANSION OF THE POTENTIAL FUNCTION

Consider a body M of finite dimensions and arbitrary mass distribution. Let

$\underline{r} = x_1\underline{i} + x_2\underline{j} + x_3\underline{k}$ = position vector of a material particle at P with mass m.

$\underline{\rho} = x'_1\underline{i} + x'_2\underline{j} + x'_3\underline{k}$ = position vector of the mass element dM at point Q of the body M.

$\underline{\Delta} = \underline{r} - \underline{\rho}$

Then U, the potential of attraction of body M on the mass at P is given by

$$U = G \int_V \frac{D(\underline{\rho})}{\Delta} d^3 (\underline{\rho})$$

where $D(\underline{\rho})$ is the density function and the symbol $\int_V$ indicates that integration is to be carried over the entire volume of the body.

The expression of $\Delta = |\underline{r} - \underline{\rho}|$ in terms of Legendre's polynomials which are function of the angle $\underline{r} \cdot \underline{\rho}/r\rho$ is well-known and is given in several texts on the subject (see for example, MacMillan, 1958; Kellog, 1953). Consequently, U can be expressed as (Khan, 1967)

$$U = \frac{GM}{r} + \frac{G\underline{p} \cdot r}{r^3} + \frac{1}{2} G \sum_{i,j}^{3} Q_{ij} \frac{x_i x_j}{r^5} + \ldots \tag{1}$$

where

$$\underline{p} = \int \underline{\rho}\, D(\underline{\rho})\, d^3 \underline{\rho} = M\underline{r}_c$$

$$Q_{ij} = \int (3\, x'_i\, x'_j - \rho^2\, \delta_{ij})\, D(\underline{\rho})\, d^3\, \underline{\rho} \tag{2}$$

$\underline{r}_c$ = position vector of the center of mass 0 relative to the origin of the coordinate system

and δ_{ij} is the Kronecher δ function, defined as

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

Note that the first term in equation (1) indicates the potential of the body if all its mass M were concentrated at its center.

The second term vanishes if the center of mass 0 of body M is taken as the origin of the coordinate system because in that case $\underline{r}_c = 0$.

Formulas relating the principal moments of inertia and the products of inertia with the elements Q_{ij} are already available in literature (see for example Brouwer and Clemence, 1961; Khan, 1967).

A more familiar representation of U in terms of spherical harmonics is (see for example, Byerly, 1959; Brouwer and Clemence, 1961; Mueller, 1964; Heiskenan and Moritz, 1967)

$$U = \frac{GM}{r} \Big[\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \cdot P_{nm} (\sin \phi) \Big] \tag{3}$$

where P_{nm} $(\sin \phi)$ are called the Associated Legendre's functions.

If the center of mass of the body M is chosen as the origin of the coordinate system, the terms for n = 1 vanish and equation (3) reduces to the form

$$U = \frac{GM}{r} [1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} (\frac{a_e}{r})^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \quad (4)$$

$$P_{nm} (\sin \phi)]$$

A comparison of equation (3) with equation (1) gives some useful relations between the spherical harmonic coefficients C_{nm}, S_{nm} and the physical constants of the earth. These relations are already available in literature (see for example, Brouwer & Clemence, 1961; Mueller, 1964; Khan, 1967; Heiskenan and Moritz, 1967).

A knowledge of the set of quantities in equation (1) or the spherical harmonic coefficients C_{nm} and S_{nm} in equation (3) will determine the potential function U, the gradient of which will give the gravity vector. If the low-degree harmonic coefficients are already known with adequate accuracy, one may seek to compute only the higher frequency components. These components define the relatively more localized structure of the gravity field, as previously pointed out. In this report, we will propose a method which will use the satellite position and velocity vectors to determine the higher frequency terms is the series given in equation (1) or (3), as these terms are believed to be characteristic of the area which the satellite is transiting at a given instant of time.

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THEORY OF THE METHOD

The Ideal Case Ignoring All Perturbations

Let $\underline{V}$ denote the absolute velocity vector of the satellite in an earth-centered, space-fixed system of coordinates (we call it the inertial coordinate system), $\underline{v}$, its relative velocity vector in an earth-centered, earth-fixed coordinate system (we call it the rotating coordinate system), $\underline{\omega}$, the angular velocity of the earth, and $\underline{r}$ the radius vector of the satellite from the origin of the coordinate system. Note that the origin of the system is assumed to lie on the axis of rotation of the earth. Then we can express $\underline{V}$ as (Constant, 1954, p. 72):

$$\underline{V} = \underline{v} + \underline{\omega} \times \underline{r} \tag{5}$$

The Langrangian L of the system is

$$L = \frac{1}{2} m \, \underline{V} \cdot \underline{V} - m \, U \tag{6}$$

where

m = mass of the satellite

U = potential energy of the earth at the location of the satellite.

Substitution of (5) in (6) gives

$$L = \frac{1}{2} m \, (\underline{v} + \underline{\omega} \times \underline{r}) \cdot (\underline{v} + \underline{\omega} \times \underline{r}) - m \, U \tag{7}$$

The canonical momentum p_i is given by

$$p_i = \frac{\partial L}{\partial V_i} = m \, (\underline{v} + \underline{\omega} \times \underline{r})_i$$

In vectorial form the above equation is

$$\underline{p} = m\ (\underline{v} + \underline{\omega} \times \underline{r}) \tag{8}$$

Note that $\underline{r}_i$ have been chosen as the generalized coordinates. The Hamiltonian H of the system is

$$H = \underline{p} \cdot \underline{V} - L \tag{9}$$

Substitute (7) and (8) in (9) and get

$$\begin{aligned} H &= m\ (\underline{v} + \underline{\omega} \times \underline{r}) \cdot (\underline{v} + \underline{\omega} \times \underline{r}) \\ &\quad - \frac{1}{2} m\ (\underline{v} + \underline{\omega} \times \underline{r}) \cdot (\underline{v} + \underline{\omega} \times \underline{r}) + m\ U \\ &= \frac{1}{2} m\ (\underline{v} + \underline{\omega} \times \underline{r}) \cdot (\underline{v} + \underline{\omega} \times \underline{r}) + m\ U \\ &= \frac{1}{2} m\ [\underline{v}^2 + 2\underline{v} \cdot (\underline{\omega} \times \underline{r}) + (\dot{\underline{\omega}} \times \underline{r})^2] + m\ U \end{aligned} \tag{10}$$

Now assume that $\underline{i}$, $\underline{j}$, $\underline{k}$ be the unit vectors along x_1, x_2 and x_3 axis, respectively. The system of coordinates x_i is rigidly attached to the earth. Then

$$\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$$

$$\underline{r} = r_1\underline{i} + r_2\underline{j} + r_3\underline{k}$$

and

$$\underline{\omega} = \omega\underline{k}$$

Consequently,

$$v^2 = v_1^{\ 2} + v_2^{\ 2} + v_3^{\ 2}$$

$$\underline{v} \cdot (\underline{\omega} \times \underline{r}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ 0 & 0 & \omega \\ r_1 & r_2 & r_3 \end{vmatrix} = -\omega \begin{vmatrix} v_1 & v_2 \\ r_1 & r_2 \end{vmatrix} \tag{11}$$

$$(\underline{\omega} \times \underline{r})^2 = \omega^2 \ [\underline{k} \times (r_1\underline{i} + r_2\underline{j} + r_3\underline{k})]^2$$

$$= \omega^2 \ (r_1\underline{j} - r_2\underline{i})^2 = \omega^2 \ (r_1^{\ 2} + r_2^{\ 2})$$

The substitution of equation (11) in (10) will give the function H in a more practical form.

Substituting the expansion series for the potential function U, one gets

$$\frac{H}{m} = \frac{1}{2} [v^2 + 2\underline{v} \cdot (\underline{\omega} \times \underline{r}) + (\underline{\omega} \times \underline{r})^2]$$

$$+ \frac{GM}{r} [\sum_{n=0}^{p} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)$$

$$P_{nm}(\sin \phi) + \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (c_{nm}\cos m\lambda \tag{12}$$

$$+ s_{nm}\sin m\lambda) P_{nm}(\sin \phi)]$$

If the perturbations from all other sources excepting terrestrial gravity were ignored, H will be a constant of the motion. Then if a set of observations $\underline{r}_i$ and $\underline{v}_i$ were available, the coefficients C_{nm} and S_{nm} could be calculated. Minimally, $i - 1 = (n + 1)^2$; in which case $(n + 1)^2$ equations will have to be solved for $(n + 1)^2$ unknowns. But in that case the observation equations will have to be set up by successive subtraction. If one requires to set up each first difference observation equation from two independent measurements, the minimal condition will be $i/2 = (n + 1)^2$. However, the desirable case will be when $i - 1 >> (n + 1)^2$ or $i/2 >> (n + 1)^2$ as the case may be, so that one can use the standard least-squares procedure to solve for the unknowns (see Khan, 1967).

Note that in equation (12) we have denoted the higher degree spherical harmonic coefficients ($n > p$ where the value of p is to be selected depending upon the accuracy of the low-degree coefficients, number of data points, etc.) by small c_{nm}'s and s_{nm}'s. This is done to distinguish them from the familiar global harmonic coefficients. These coefficients are expected to be characteristic of the area which lies below the orbital segment or set of such segments being studied. Hence, we expect them to differ from area to area. Each of these sets of local harmonic coefficients will refer to a hypothetical equipotential surface which will approximate the actual geoidal surface in the area lying below the set of orbital segments from which this particular set of local coefficients was initially obtained. The part of the area for which the solution is satisfactory will be called the 'effective area'. Once such effective areas have covered the surface of the globe, it will be no problem to integrate the individual sets of local coefficients into the more familiar sets of global harmonic coefficients.

It is to be noted that, at least initially, this method will be used primarily for determining the relatively higher frequency harmonics, the low-degree harmonics being already known with adequate accuracy from various satellite orbital analyses (Kozai, 1968; Anderle 1966; Guier and Newton, 1965, Kaula, 1966; Gaposhkin, 1966). Hence, when the actual observation equations are set up, the first few terms in the expression for the potential function U will be treated as known and consequently, the condition for the minimal number of observations, noted above, will be modified accordingly.

Let $\underline{r}_i$ and $\underline{v}_i$ $(i = 1,2,\ldots i)$ be the measured values of the position vector and the relative velocity of the satellite along a short segment of its orbit. By substituting these values in equation (12) and by successive subtractions, the observation equations are:

$$\left.\begin{aligned}
&\frac{1}{2}\Bigg[(v_{i+1}^2 - v_i^{\,2}) + 2\underline{v}_{i+1} \cdot (\underline{\omega} \times \underline{r}_{i+1}) - 2\underline{v}_i \\
&\cdot (\underline{\omega} \times \underline{r}_i) + (\underline{\omega} \times \underline{r}_{i+1})^2 - (\underline{\omega} \times \underline{r}_i)^2\Bigg] \\
&+ GM \sum_{n=0}^{p} \sum_{m=0}^{n} a_e^n \Bigg[C_{nm} \left(\frac{\cos m\lambda_{i+1} P_{nm}(\sin \phi_{i+1})}{r_{i+1}^{n+1}} \right. \\
&\left. - \frac{\cos m\lambda_i P_{nm}(\sin \phi_i)}{r_i^{n+1}} \right) + S_{nm} \left(\frac{\sin m\lambda_{i+1} P_{nm}(\sin \phi_{i+1})}{r_{i+1}^{n+1}} \right. \\
&\left. \left. - \frac{\sin m\lambda_i P_{nm}(\sin \phi_i)}{r_i^{n+1}} \right) \right] + GM \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} a_e^n \Bigg[c_{nm} \\
&\left(\frac{\cos m\lambda_{i+1} P_{nm}(\sin \phi_{i+1})}{r_{i+1}^{n+1}} \right) - \left(\frac{\cos m\lambda_i P_{nm}(\sin \phi_i)}{r_i^{n+1}} \right) \\
&+ s_{nm} \left(\frac{\sin m\lambda_{i+1} P_{nm}(\sin \phi_{i+1})}{r_{i+1}^{n+1}} - \frac{\sin m\lambda_i P_{nm}(\sin \phi_i)}{r_i^{n+1}} \right) \Bigg] \\
&= 0 \qquad\qquad\qquad i = 1,2,\ldots i
\end{aligned}\right\} \quad (13)$$

or

$$\begin{aligned}
&\sum_{n=0}^{p} \sum_{m=0}^{n} (C_{nm}\, g_{nm_{(i)}} + S_{nm}\, h_{nm_{(i)}}) \\
&+ \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} (c_{nm}\, g_{nm_{(i)}} + s_{nm}\, h_{nm_{(i)}}) + f_i = 0
\end{aligned} \qquad (14)$$

where

$$g_{nm(i)} = f\,[GM,\ a_e,\ r,\ \cos\lambda,\ P_{nm}(\sin\phi)]$$

$$= GM\, a_e^n \left(\frac{\cos m\lambda_{i+1} P_{nm}(\sin\phi_{i+1})}{r_{i+1}^{n+1}} - \frac{\cos m\lambda_i P_{nm}(\sin\phi_i)}{r_i^{n+1}} \right) \qquad i = 1,2,\ldots i. \tag{15a}$$

$$h_{nm(i)} = f\,[GM,\ a_e,\ r,\ \sin\lambda,\ P_{nm}(\sin\phi)]$$

$$= GM\, a_e^n \left(\frac{\sin m\lambda_{i+1} P_{nm}(\sin\phi_{i+1})}{r_{i+1}^{n+1}} - \frac{\sin m\lambda_i P_{nm}(\sin\phi_i)}{r_i^{n+1}} \right) \qquad i = 1,2,\ldots i. \tag{15b}$$

and

$$f_i = \frac{1}{2}\Big[(v_{i+1}^2 - v_i^2) + 2\underline{v}_{i+1} \cdot (\underline{\omega} \times \underline{r}_{i+1}) - 2\underline{v}_i \cdot (\underline{\omega} \times \underline{r}_i) + (\underline{\omega} \times \underline{r}_{i+1})^2 - (\underline{\omega} \times \underline{r}_i)^2\Big] \tag{15c}$$

$$i = 1,2,\ldots i$$

Inclusion of Perturbations

If the effects of the main perturbations such as the luni-solar, air drag, radiation pressure etc., are considered, the development of the theory becomes slightly more complicated. We will consider the effects of each of these perturbations one by one to show how the theory has to be modified (Khan, 1967) in order to account for the effect of each of the perturbing forces.

Lunar Effect: Consider the case of a satellite whose motion, though still primarily controlled by the earth's gravitational field, is being perturbed by the lunar attraction. Let

m_1 = mass of the satellite

m_2 = mass of the moon

M = mass of the earth

V_1 = velocity vector of the satellite in the inertial coordinate system

$\underline{v}_1$ = velocity vector of the satellite in the rotating coordinate system

V_2 = velocity vector of the moon in the inertial coordinate system

$\underline{v}_2$ = velocity vector of the moon in the rotating coordinate system

$\underline{r}_1$ = the radius vector of the satellite from the origin of the coordinate system

$\underline{r}_2$ = the radius vector of the moon from the origin of the coordinate system

$\underline{\omega}$ = angular velocity of the earth

L = the total Lagrangian of the system

H = the total Hamiltonian function of the system

Then as before, the Lagrangian L of the system is

$$L = \frac{1}{2} m_1 (\underline{v}_1 + \underline{\omega} \times \underline{r}_1)^2 - m_1 U_1 + \frac{1}{2} m_2 (\underline{v}_2 + \underline{\omega} \times \underline{r}_2)^2 - m_2 U_2 - \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|} \tag{16}$$

The canonical momenta $\underline{p}_1$, $\underline{p}_2$ are

$$\begin{aligned} \underline{p}_1 &= \frac{\partial L}{\partial \underline{V}_1} = m_1 (\underline{v}_1 + \underline{\omega} \times \underline{r}_1) = m_1 V_1 \\ \underline{p}_2 &= \frac{\partial L}{\partial \underline{V}_2} = m_2 (\underline{v}_2 + \underline{\omega} \times \underline{r}_2) = m_2 V_2 \end{aligned} \tag{17}$$

which gives

$$V_1 = \frac{\underline{p}_1}{m_1}$$

$$V_2 = \frac{\underline{p}_2}{m_2}$$

$$\underline{v}_1 = \frac{\underline{p}_1}{m_1} - \underline{\omega} \times \underline{r}_1 \qquad (18)$$

$$\underline{v}_2 = \frac{\underline{p}_2}{m_2} - \underline{\omega} \times \underline{r}_2$$

The Hamiltonian function H of the system is

$$H = \underline{p}_1 \cdot \underline{V}_1 + \underline{p}_2 \cdot \underline{V}_2 - L$$

$$= \underline{p}_1 \cdot \frac{\underline{p}_1}{m_1} + \underline{p}_2 \cdot \frac{\underline{p}_2}{m_2} - \frac{1}{2} \cdot \frac{p_1^2}{m_1}$$

$$- \frac{1}{2} \cdot \frac{p_2^2}{m_2} + m_1 U_1 + m_2 U_2 + \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|} \qquad (19)$$

or

$$H = \frac{1}{2} \frac{p_1^2}{m_1} + \frac{1}{2} \frac{p_2^2}{m_2} + m_1 U_1 + m_2 U_2 + \frac{G m_1 m_2}{|\underline{r}_1 - \underline{r}_2|}$$

Partial differentiation of H with respect to $\underline{p}_1$, $\underline{p}_2$, $\underline{r}_1$ and $\underline{r}_2$ yields

$$\frac{\partial H}{\partial \underline{p}_1} = \frac{\underline{p}_1}{m_1}$$

$$\frac{\partial H}{\partial \underline{p}_2} = \frac{\underline{p}_2}{m_2}$$

$$\frac{\partial H}{\partial \underline{r}_1} = -\underline{F}_1 - \underline{F}_{12}$$

$$\frac{\partial H}{\partial \underline{r}_2} = -\underline{F}_2 + \underline{F}_{12}$$

where

$$\underline{F}_1 = - \frac{G\ m_1\ M}{r_1^3}\ \underline{r}_1$$

$$\underline{F}_2 = - \frac{G\ m_2\ M}{r_2^3}\ \underline{r}_2 \tag{21}$$

and

$$\underline{F}_{12} = + \frac{G\ m_1\ m_2}{\left|\underline{r}_1 - \underline{r}_2\right|^3}\ (\underline{r}_1 - \underline{r}_2)$$

Now we define H_1 as

$$\begin{aligned} H_1 &= \underline{p}_1 \cdot (\underline{v}_1 + \underline{\omega} \times \underline{r}_1) - \frac{1}{2} m_1\ (\underline{v}_1 + \underline{\omega} \times \underline{r}_1)^2 + m_1\ U_1 \\ &= \frac{1}{2} \frac{p_1^2}{m_1} + m_1\ U_1 \end{aligned} \tag{22}$$

and similarly H_2 as

$$\begin{aligned} H_2 &= \underline{p}_2 \cdot (\underline{v}_2 + \underline{\omega} \times \underline{r}_2) - \frac{1}{2} m_2\ (\underline{v}_2 + \underline{\omega} \times \underline{r}_2)^2 + m_2\ U_2 \\ &= \frac{1}{2} \frac{p_2^2}{m_2} + m_2\ U_2 \end{aligned} \tag{23}$$

From equation (22), it is seen that

$$H_1 = H_1 \left(\underline{p}_1, \underline{r}_1\right)$$

Hence,

$$\frac{dH_1}{dt} = \frac{\partial H_1}{\partial \underline{p}_1} \cdot \frac{d\underline{p}_1}{dt} + \frac{\partial H_1}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt} \tag{24}$$

Now the Hamiltonian equations of motions are

$$\frac{dp_s}{dt} = - \frac{\partial H}{\partial r_s} \qquad \text{and} \qquad \frac{dr_s}{dt} = \frac{\partial H}{\partial p_s}$$

With the help of these equations, equation (24) reduces to the form

$$\frac{dH_1}{dt} = - \frac{\partial H_1}{\partial \underline{p}_1} \frac{\partial H}{\partial \underline{r}_1} + \frac{\partial H_1}{\partial \underline{r}_1} \frac{\partial H}{\partial \underline{p}_1} = (H_1, H) \tag{25}$$

where (H_1, H) is the Poisson's bracket expression and is defined as

$$(H_1, H) = \frac{\partial H_1}{\partial \underline{r}_1} \cdot \frac{\partial H}{\partial \underline{p}_1} - \frac{\partial H_1}{\partial \underline{p}_1} \cdot \frac{\partial H}{\partial \underline{r}_1}$$

Equation (22) gives

$$\frac{\partial H_1}{\partial \underline{p}_1} = \frac{\underline{p}_1}{m_1} = \frac{\partial H}{\partial \underline{p}_1} \tag{26}$$

so that the Poisson's bracket expression can be written as

$$(H_1, H) = \frac{dH_1}{dt} = \frac{\partial H}{\partial \underline{p}_1} \left(\frac{\partial H_1}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{r}_1}\right) = \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} (H_1 - H)$$

$$= \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} (-H_2 - H_{12}) \tag{27}$$

$$= - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_2}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1}$$

where

$$H_{12} = \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|}$$

But $\frac{\partial H_2}{\partial \underline{r}_1} = 0$ which gives

$$\frac{dH_1}{dt} = - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1} \tag{28}$$

Substitution of equation (20) in equation (28) gives

$$\frac{dH_1}{dt} = - \frac{\underline{p}_1}{m_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1}$$

Now

$$\frac{\partial H_{12}}{\partial \underline{r}_1} = \frac{\partial}{\partial \underline{r}_1} \left(\frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|} \right) = - \underline{F}_{12}$$

and hence we finally get

$$\frac{dH_1}{dt} = \frac{\underline{p}_1}{m_1} \cdot \underline{F}_{12}$$

$$= \underline{F}_{12} \cdot \underline{V}_1 \tag{29}$$

Air Drag Effect: If we now take into consideration the effect of air drag, the Lagrange's equations of motion become

$$\frac{d}{dt} \cdot \left(\frac{\partial L}{\partial \underline{V}_s} \right) - \frac{\partial L}{\partial \underline{r}_s} = \underline{F}_{Ds} \qquad s = 1, 2 \tag{30}$$

where $\underline{F}_D$ is the generalized force. In this particular case $\underline{F}_D$ has the dimensions of force because $\underline{r}_1$ is a linear coordinate. In general, the generalized force need not have dimensions of force. For example, if $\underline{r}_1$ is an angular coordinate, the generalized force will have dimensions of torque.

Since the canonical momenta are given by

$$\frac{\partial L}{\partial \underline{V}_s} = \underline{p}_s \qquad s = 1, 2$$

The Lagrange's equations can be written as

$$\frac{d\underline{p}_s}{dt} - \frac{\partial L}{\partial \underline{r}_s} = \underline{F}_{Ds} \qquad s = 1, 2 \tag{31}$$

However, H was defined as

$$H = \sum_s \underline{p}_s \cdot \underline{V}_s - L$$

wherefrom

$$\frac{dH}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 + \underline{p}_1 \cdot \frac{d\underline{V}_1}{dt} + \frac{d\underline{p}_2}{dt} \cdot \underline{V}_2 + \underline{p}_2 \cdot \frac{d\underline{V}_2}{dt} - \frac{\partial L}{\partial t}$$

$$- \frac{\partial L}{\partial \underline{V}_1} \cdot \frac{d\underline{V}_1}{dt} - \frac{\partial L}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt} - \frac{\partial L}{\partial \underline{v}_2} \cdot \frac{d\underline{V}_2}{dt} - \frac{\partial L}{\partial \underline{r}_2} \cdot \frac{d\underline{r}_2}{dt}$$

$$= \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 + \frac{d\underline{p}_2}{dt} \cdot \underline{V}_2 - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt} - \frac{\partial L}{\partial \underline{r}_2} \cdot \frac{d\underline{r}_2}{dt} \tag{32}$$

But from equation (31) we get

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} - \underline{F}_{D1}$$

$$\frac{\partial L}{\partial \underline{r}_2} = \frac{d\underline{p}_2}{dt} - \underline{F}_{D2}$$

These substitutions in equation (32) give

$$\begin{aligned} \frac{dH}{dt} &= \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 + \frac{d\underline{p}_2}{dt} \cdot \underline{V}_2 - \frac{\partial L}{\partial t} - \underline{V}_1 \cdot \left(\frac{d\underline{p}_1}{dt} - \underline{F}_{D1}\right) \\ &- \underline{V}_2 \left(\frac{d\underline{p}_2}{dt} - \underline{F}_{D2}\right) = - \frac{\partial L}{\partial t} + \underline{V}_1 \cdot \underline{F}_{D1} + \underline{V}_2 \cdot \underline{F}_{D2} \end{aligned} \tag{33}$$

If L is independent of time, $\frac{dH}{dt}$ obviously becomes

$$\frac{dH}{dt} = \underline{V}_1 \cdot \underline{F}_{D1} + \underline{V}_2 \cdot \underline{F}_{D2} \tag{34}$$

Following the same procedure for H_1, we get

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 + \frac{d\underline{V}_1}{dt} \cdot \underline{p}_1 - \frac{dL_1}{dt} \tag{35}$$

where L_1 has been defined as

$$L_1 = \frac{1}{2} m_1 (\underline{V}_1 \cdot \underline{V}_1) - m_1 U_1$$

That is,

$$L_1 = L_1 (\underline{V}_1, \underline{r}_1)$$

Consequently,

$$\frac{dL_1}{dt} = \frac{\partial L_1}{\partial \underline{V}_1} \cdot \frac{d\underline{V}_1}{dt} + \frac{\partial L_1}{\partial \underline{r}_1} \cdot \frac{d\underline{r}_1}{dt}$$

$$= \underline{p}_1 \cdot \frac{d\underline{V}_1}{dt} + \frac{\partial L_1}{\partial \underline{r}_1} \cdot \underline{V}_1 \tag{36}$$

Substituting the above relation in equation (35), we finally get

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 + \frac{d\underline{V}_1}{dt} \cdot \underline{p}_1 - \underline{p}_1 \cdot \frac{d\underline{V}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} \cdot \underline{V}_1$$

$$= \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 - \frac{\partial L_1}{\partial \underline{r}_1} \cdot \underline{V}_1 \tag{37}$$

Now we define a quantity L_2 such that

$$L_2 = \frac{1}{2} m_2 (\underline{v}_2 + \underline{\omega} \times \underline{r}_2)^2 - m_2 U_2$$

and another quantity L_{12} as

$$L_{12} = - \frac{G\, m_1\, m_2}{\underline{r}_1 - \underline{r}_2} = - H_{12}$$

Then

$$L = L_1 + L_2 + L_{12}$$

which gives

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial L_2}{\partial \underline{r}_1} + \frac{\partial L_{12}}{\partial \underline{r}_1}$$

or

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} - \frac{\partial H_{12}}{\partial \underline{r}_1}$$

Substituting the above in Lagrange's equation of motion we obtain

$$\frac{d\underline{p}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial H_{12}}{\partial \underline{r}_1} = \underline{F}_{D1}$$

or

$$\frac{\partial L_1}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1}$$

wherefrom, equation (37) becomes

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 - \left(\frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1} \right) \cdot \underline{V}_1$$

$$= - \left(\frac{\partial H_{12}}{\partial \underline{r}_1} - \underline{F}_{D1} \right) \cdot \underline{V}_1$$

$$= \underline{F}_{12} \cdot \underline{V}_1 + \underline{F}_{D1} \cdot \underline{V}_1 \qquad (38)$$

In view of equation (29) it is obvious that equation (38) takes into account the effects of both lunar attraction and air drag. Further development of equation (38) is postponed until the effect of all the major disturbing forces is taken into consideration.

The Effects of Solar Attraction and Radiation Pressure: Equation (38) is derived on the assumption that the only disturbing forces (other than the non-spherical part of the gravitational force of the earth) acting on the satellite are air drag and the moon's pull. However, further development of the theory shows the interesting feature that this equation retains its basic form when the effects of solar attraction and radiation pressure are taken into account.

Denote the apparent velocity vector of the sun relative to the earth in the inertial system by $\underline{V}_3$, the corresponding velocity vector in the rotating coordinate system by v_3, and the mass of the sun by m_3. Also let L now denote the total Lagrangian of the new system in which the satellite is moving in the gravitational field of the earth under the perturbing influence of both the sun and the moon. Let H denote the Hamiltonian function of the new system. Then proceeding in the same way as before, we have

$$L = \frac{1}{2} \sum_{i=1}^{3} m_i \left(\underline{v}_i + \underline{\omega} \times \underline{r}_i\right)^2 - \sum_{i=1}^{3} m_i U_i$$

$$- \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|} - \frac{G\, m_1\, m_3}{|\underline{r}_1 - \underline{r}_3|} - \frac{G\, m_2\, m_3}{|\underline{r}_2 - \underline{r}_3|} \qquad (39)$$

and the Hamiltonian function H is

$$H = \sum_{i=1}^{3} \underline{p}_i \cdot \underline{V}_i - L$$

$$= \frac{1}{2} \sum_{i=1}^{3} \frac{p_i^2}{m_i} + \sum_{i=1}^{3} m_i\, U_i + H_{12} + H_{13} + H_{23} \qquad (40)$$

where

$$H_{12} = \frac{G\, m_1\, m_2}{|\underline{r}_1 - \underline{r}_2|}$$

$$H_{13} = \frac{G\, m_1\, m_3}{|\underline{r}_1 - \underline{r}_3|} \qquad (41)$$

and

$$H_{23} = \frac{G\, m_2\, m_3}{|\underline{r}_2 - \underline{r}_3|}$$

This gives as before

$$\frac{\partial H}{\partial \underline{p}_i} = \frac{\underline{p}_i}{m_i} \qquad i = 1, 2, 3. \qquad (42)$$

$$\frac{\partial H}{\partial \underline{r}_1} = - \underline{F}_1 - \underline{F}_{12} - \underline{F}_{13}$$

$$\frac{\partial H}{\partial \underline{r}_2} = - \underline{F}_2 + \underline{F}_{12} - \underline{F}_{23}$$

where

$$\underline{F}_{12} = - \frac{\partial H_{12}}{\partial \underline{r}_1} = \frac{\partial H_{12}}{\partial \underline{r}_2}$$

$$\underline{F}_{13} = - \frac{\partial H_{13}}{\partial \underline{r}_1} = \frac{\partial H_{13}}{\partial \underline{r}_3} \qquad (43)$$

$$\underline{F}_{23} = - \frac{\partial H_{23}}{\partial \underline{r}_1} = \frac{\partial H_{23}}{\partial \underline{r}_3}$$

Define H_i as

$$H_i = \frac{1}{2} \frac{p_i^2}{m_i} + m_i U_i$$

Differentiate this expression to obtain

$$\frac{dH_1}{dt} = \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial}{\partial \underline{r}_1} [- H_2 - H_3 - H_{12} - H_{13} - H_{23}]$$

$$= - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{12}}{\partial \underline{r}_1} - \frac{\partial H}{\partial \underline{p}_1} \cdot \frac{\partial H_{13}}{\partial \underline{r}_1} \qquad (44)$$

$$= \underline{V}_1 \cdot [\underline{F}_{12} + \underline{F}_{13}]$$

Note that in the above expressions $\underline{F}_{12}$ and $\underline{F}_{13}$ are the forces acting on the satellite arising from the disturbing effects of the moon and the sun respectively.

If the radiation pressure is considered as a steady repulsive force, its effect can be taken into account in the same way as above. However, it can also be considered without making this assumption, as follows.

Let $\underline{F}_D$ and $\underline{F}_R$ be the generalized forces arising from the disturbing effect of the air drag and the radiation pressure, then the Lagrange's equations of motion expressed in terms of canonical momenta become

$$\frac{d\underline{p}_i}{dt} - \frac{\partial L}{\partial \underline{r}_i} = \underline{F}_{Di} + \underline{F}_{Ri} \tag{45}$$

Note that in this particular case $\underline{F}_D$ and $\underline{F}_R$ have the dimensions of force, but they do not have to be essentially potential-derived.

If we denote $H_{12} = - L_{12}$ and $H_{13} = - L_{13}$ and define L_1 as in the previous section, the expression for $\partial L/\partial \underline{r}_1$ becomes

$$\frac{\partial L}{\partial \underline{r}_1} = \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial L_{12}}{\partial \underline{r}_1} + \frac{\partial L_{13}}{\partial \underline{r}_1}$$

$$= \frac{\partial L_1}{\partial \underline{r}_1} - \frac{\partial H_{12}}{\partial \underline{r}_1} - \frac{\partial H_{13}}{\partial \underline{r}_1}$$

which on substitution in equations of motion gives

$$\frac{d\underline{p}_1}{dt} - \frac{\partial L_1}{\partial \underline{r}_1} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} = \underline{F}_{D1} + \underline{F}_{R1}$$

wherefrom

$$\frac{\partial L_1}{\partial \underline{r}_1} = \frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1} \tag{46}$$

If this value of $\frac{\partial L_1}{\partial \underline{r}_1}$ is substituted in equation (37), we obtain

$$\frac{dH_1}{dt} = \frac{d\underline{p}_1}{dt} \cdot \underline{V}_1 - \left(\frac{d\underline{p}_1}{dt} + \frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1}\right) \cdot \underline{V}_1$$

$$= - \underline{V}_1 \cdot \left(\frac{\partial H_{12}}{\partial \underline{r}_1} + \frac{\partial H_{13}}{\partial \underline{r}_1} - \underline{F}_{D1} - \underline{F}_{R1}\right) \tag{47}$$

$$= \underline{F}_{12} \cdot \underline{V}_1 + \underline{F}_{13} \cdot \underline{V}_1 + \underline{F}_{D1} \cdot \underline{V}_1 + \underline{F}_{R1} \cdot \underline{V}_1$$

As may be noted, equation (47) takes into consideration the forces arising from the four most important factors perturbing the satellite motion; i.e., lunar and solar attraction, air drag and radiation pressure.

If equation (47) is integrated between $t = t_i$ and $t = t_{i+1}$, we get

$$H_1 (t_{i+1}) - H_1 (t_i) = \int_{t_i}^{t_{i+1}} [\underline{F}_{12} + \underline{F}_{13} + \underline{F}_{D1} + \underline{F}_{R1}] \cdot \underline{V}_1 \, dt \tag{48a}$$

Similarl , integration between $t = t_{i+1}$ and $t = t_{i+2}$ gives

$$H_1 (t_{i+2}) - H_1 (t_{i+1}) = \int_{i+1}^{i+2} [\underline{F}_{12} + \underline{F}_{13} + \underline{F}_{D1} + \underline{F}_{R1}] \cdot \underline{V}_1 \, dt \tag{48b}$$

Now the attraction of the sun and the moon is not likely to vary to any great degree in a small interval of time. The air drag and radiation pressure are more susceptible to variation. If, however, the satellite is not entering or leaving the atmosphere or the earth's shadow zone, air drag and radiation pressure may also be assumed to be approximately constant in a small interval of time Δt. Hence, if $t_{i+2} - t_{i+1} = t_{i+1} - t_i = \Delta t$ where Δt is so small that the disturbing forces $\underline{F}_{12}$, $\underline{F}_{13}$, $\underline{F}_{D1}$, and $\underline{F}_{R1}$ can be regarded independent of time during this interval, equations (48a) and (48b) will become

$$H_1 (t_{i+1}) - H_1 (t_i) = \underline{F}_i^{i+1} \cdot [\underline{r}_1 (t_{i+1}) - \underline{r}_1 (t_i)] \tag{49a}$$

$$H_1 (t_{i+2}) - H_1 (t_{i+1}) = \underline{F}_{i+1}^{i+2} \cdot [\underline{r}_1 (t_{i+2}) - \underline{r} (t_{i+1})] \tag{49b}$$

where

$$\underline{F} = \underline{F}_{12} + \underline{F}_{13} + \underline{F}_{D1} + \underline{F}_{R1}$$

The second order difference equation can be obtained by subtracting equation (49a) from (49b), i.e.,

$$\begin{aligned} &H_1 (t_{i+2}) - 2H_1 (t_{i+1}) + H_1 (t_i) \\ &= \underline{F}_{i+1}^{i+2} \cdot [\underline{r}_1 (t_{i+2}) - \underline{r}_1 (t_{i+1}) - \underline{F}_i^{i+1} \cdot [\underline{r}_1 (t_{i+1}) \\ &\quad - \underline{r}_1 (t_i)] \end{aligned} \tag{50}$$

The higher order difference equations may be obtained in a similar manner. For example, the third order difference equation will be

$$H_1\,(t_{i+3}) - 3H_1\,(t_{i+2}) + 3H_1\,(t_{i+1}) - H_1\,(t_i)$$

$$= \underline{F}_{i+2}^{i+3} \cdot [\underline{r}_1\,(t_{i+3}) - \underline{r}_1\,(t_{i+2})]$$

$$- 2\underline{F}_{i+1}^{i+2} \cdot [\underline{r}_1\,(t_{i+2}) - \underline{r}_1\,(t_{i+1})] \tag{51}$$

$$+ \underline{F}_i^{i+1} \cdot [\underline{r}_1\,(t_{i+1}) - \underline{r}_1\,(t_i)]$$

We will now examine equation (50) in more detail. It may be seen that any inferences drawn from this examination will be equally applicable to higher order difference equations.

Let $\underline{F}_{i+1}^{i+2} = \underline{F}_i^{i+1}$; equation (50) then becomes

$$H_1\,(t_{i+2}) - 2H_1\,(t_{i+1}) + H_1\,(t_i)$$

$$= \underline{F} \cdot [\underline{r}_1\,(t_{i+2}) - 2\underline{r}_1\,(t_{i+1}) + \underline{r}_1\,(t_i)) \tag{52a}$$

$$= \underline{F} \cdot \delta^2\,\underline{r}_1\,(t_{i+2},\,t_i)$$

To elaborate what the above assumption actually entails, let $\underline{F}$ be the mean value of the extra terrestrial disturbing force in the interval t_i to t_{i+2}, equation (50) can then be written as

$$H_1\,(t_{i+2}) - 2H_1\,(t_{i+1}) + H_1\,(t_i) \tag{52b}$$

$$= [\underline{F} + \delta\underline{F}\,(t_{i+2},\,t_{i+1})] \cdot \delta^1\,\underline{r}_1\,(t_{i+2},\,t_{i+1})$$

$$- [\underline{F} + \delta\underline{F}\,(t_{i+1},\,t_i)] \cdot \delta^1\,\underline{r}_1\,(t_{i+1},\,t_i)$$

$$= \underline{F} \cdot \delta^2 \, \underline{r}_1 \, (t_{i+2}, t_i) + \delta\underline{F} \, (t_{i+2}, t_{i+1})$$

$$\cdot \, \delta^1 \, \underline{r}_1 \, (t_{i+2}, t_{i+1}) - \delta\underline{F} \, (t_{i+1}, t_i)$$

$$\cdot \, \delta^1 \, \underline{r}_1 \, (t_{i+1}, t_i)$$

If we now assume that

$$\delta\underline{F} \, (t_{i+2}, t_{i+1}) \cdot \delta^1 \, \underline{r}_1 \, (t_{i+2}, t_{i+1}) - \delta\underline{F} \, (t_{i+1}, t_i) \cdot \delta^1 \, \underline{r}_1 \, (t_{i+1}, t_i) \approx 0 \qquad (52c)$$

equation (52b) becomes the same as (52a). Thus, there is an implicit assumption in the derivation of equation (52a) that the quantity on the left-hand side of equation (52c) is negligibly small.

The idea in invoking the higher order differences is to make the term $\delta^j \, \underline{r}_1$ approach zero, so that equations (50) and (51) can be finally written in the form

$$H_1 \, (t_{i+2}) - 2H_1 \, (t_{i+1}) + H_1 \, (t_i) \approx 0 \qquad (53)$$

and

$$H_1 \, (t_{i+3}) - 3H_1 \, (t_{i+2}) + 3H_1 \, (t_{i+1}) - H_1 \, (t_i) \approx 0 \qquad (54)$$

However, as indicated by equations (52a) and (52b), it seems doubtful at this stage at least, that products like $\underline{F} \cdot \delta^j \underline{r}_1$ can really be made to approach zero, since, as higher order differences are taken, the left-hand side of these equations will also become increasingly small as the Hamiltonian function itself is a function of $\underline{r}_1$ and

$\underline{V}_1$. Hence, in practice, one may not find it feasible to deal with very high order difference equations.

The advantage of the procedure is already clear, however. If the differences $\delta^j \underline{r}_i$ can be made sufficiently small, their products with any uncertainties in $\underline{F}$ will become still smaller. Write $\underline{F}$ as $\underline{F} \pm \delta\underline{F}$ where $\delta\underline{F}$ is the uncertainty in the value of $\underline{F}$. Equation (52a) or (52b) or similar difference equations of higher order will then assume the general form

$$H_1 \ (t_{i+j}, \ \ldots \ t_i) = (\underline{F} \pm \delta\underline{F}) \cdot \delta^j \ \underline{r}_1 = \underline{F} \cdot \delta^j \ \underline{r}_1 \tag{55}$$

provided the products of the type $\delta\underline{F} \cdot \delta^j \ \underline{r}_1$ can be ignored. Here $\underline{F}$ is the mean value of the extra terrestrial disturbing function in the interval $t = t_i$ to $t = t_{i+j}$, and the value of the index j determines the order of the difference equation being handled.

Observation Equations for the Extended Theory

The observation equations in their explicit form can be obtained by merely substituting the value of H from equation (10). For the second order case, equation (52a) or (52b) is the basic form of the observation equations. To write them explicitly, let $\underline{r}_1$ and $\underline{v}_1$ be the observed values of the radius vector and the relative velocity of the satellite. Then substituting these values in the expression for the Hamiltonian function, we obtain the observation equations in the following form:

$$GM \sum_{n=0}^{p} \sum_{m=0}^{n} a_e^n \ [C_{nm} (\frac{\cos \ m\lambda_{i+2}}{r_{i+2}^{n+1}} \ P_{nm(i+2)} \tag{56}$$

$$- \frac{2 \cos m\lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm(i+1)} + \frac{\cos m\lambda_i}{r_i^{n+1}} P_{nm(i)})$$

$$+ S_{nm}(\frac{\sin m\lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm(i+2)} - \frac{2 \sin m\lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm(i+1)}$$

$$+ \frac{\sin m\lambda_i}{r_i^{n+1}} P_{nm(i)})] + GM \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} a_e^n [C_{nm}(\frac{\cos m\lambda_{i+2}}{r_{i+2}^{n+1}}$$

$$P_{nm(i+2)} - \frac{2 \cos m\lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm(i+1)} + \frac{\cos m\lambda_i}{r_i^{n+1}} P_{nm(i)})$$

$$+ S_{nm}(\frac{\sin m\lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm(i+2)} - \frac{2 \sin m\lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm(i+1)}$$

$$+ \frac{\sin m\lambda_i}{r_i^{n+1}} P_{nm(i)})] + (\frac{1}{2} v_{i+2}^2 - v_{i+1}^2 + \frac{1}{2} v_i^2)$$

$$+ \underline{v}_{i+2} \cdot (\underline{\omega} \times \underline{r}_{i+2}) - 2\underline{v}_{i+1} \cdot (\underline{\omega} \times \underline{r}_{i+1})$$

$$+ \underline{v}_i \cdot (\underline{\omega} \times \underline{r}_i) + (\underline{\omega} \times \underline{r}_{i+2})^2 - 2 (\underline{\omega} \times \underline{r}_{i+1})^2$$

$$+ (\underline{\omega} \times \underline{r}_i)^2 = \frac{\underline{F}}{m} \cdot \delta^2 \underline{r} (t_{i+2}, t_i) \qquad i = 1, 2, \ldots i$$

or

$$\left.\begin{aligned}
&\sum_{n=0}^{p} \sum_{m=0}^{n} [C_{nm}\, g_{nm}(t_{i+2}, t_i) + S_{nm}\, h_{nm}\, (t_{i+2}, t_i)] \\
&+ \sum_{n=p+1}^{\infty} \sum_{m=0}^{n} [c_{nm}\, g_{nm}(t_{i+2}, t_i) + s_{nm}\, h_{nm}(t_{i+2}, t_i)] \\
&+ f(t_{i+2}, t_i) = \frac{F}{m} \cdot \delta^2\, \underline{r}(t_{i+2}, t_i) \qquad i = 1, 2, \ldots i
\end{aligned}\right\} \quad (57)$$

where

$$\left.\begin{aligned}
&g_{nm}(t_{i+2}, t_i) = GM\, a_e^n \, [\, \frac{\cos\, m\lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm(i+2)} \\
&- \frac{2\, \cos\, m\lambda_{i+1}}{r_i^{n+1}} P_{nm(i+1)} + \frac{\cos\, m\lambda_i}{r_i^{n+1}} P_{nm(i)} \,]
\end{aligned}\right\} \quad (58)$$

$$\left.\begin{aligned}
&h_{nm}(t_{i+2}, t_i) = GM\, a_e^n \, [\, \frac{\sin\, m\lambda_{i+2}}{r_{i+2}^{n+1}} P_{nm(i+2)} \\
&- \frac{2\, \sin\, m\lambda_{i+1}}{r_{i+1}^{n+1}} P_{nm(i+1)} + \frac{\sin\, m\lambda_i}{r_i^{n+1}} P_{nm(i)} \,]
\end{aligned}\right\} \quad (59)$$

and

$$\left.\begin{aligned}
&f(t_{i+2}, t_i) = [(\, \frac{1}{2}\, v_{i+2}^2 - v_{i+1}^2 + \frac{1}{2}\, v_i^2) \\
&+ \underline{v}_{i+2} \cdot (\underline{\omega} \times \underline{r}_{i+2}) - 2\underline{v}_{i+1} \cdot (\underline{\omega} \times \underline{r}_{i+1}) \\
&+ \underline{v}_i \cdot (\underline{\omega} \times \underline{r}_i) + (\underline{\omega} \times \underline{r}_{i+2})^2 - 2(\underline{\omega} \times \underline{r}_{i+1})^2 \\
&+ (\underline{\omega} \times \underline{r}_i)^2
\end{aligned}\right\} \quad (60)$$

Note that the functions g_{nm}, h_{nm}, and f defined above are different from those used in equation (14). Also note that, in the above observation equations, $\underline{v}_2$ denotes the relative velocity of the satellite at time t_2. This is not to be confused with the relative velocity of the moon for which the same symbol has been used in the theoretical development.

Further simplification of equation (60) is possible with the help of formulas given in equation (11).

APPLICABILITY OF THE THEORY

Scope and Limitations

With our present knowledge of the atmosphere and radiation pressure at satellite altitudes, the corrections which are applied to account for these factors are at best only estimates, and it is desirable that any proposed method of separating the terrestrial disturbing function from the extra-terrestrial disturbing function should aim at minimizing the effect of any uncertainties in our knowledge of the latter. The method of setting up the observation equations presented here seems to meet this requirement to some degree. However, it should be noted that the equations of condition are derived on the assumption that, over small intervals of time, the extra-terrestrial disturbing function is constant. In quantitative terms it would mean that the product of any variation in the function $\underline{F}$ with the second order differences of the position vector of the satellite is negligible. The assumption has its obvious limitations. It may not hold for cases when the satellite is entering or leaving the denser atmosphere in the perigee region or when it is entering or leaving the earth's 'shadow zone'. In other words, whenever the position of the satellite is such that the effects of air resistance or radiation pressure are likely to change substantially over short periods of time, of the order of a few observation intervals, the above assumption may not hold. However, it is hoped that it will be possible to filter the data for such 'critical transit positions'.

The method is primarily designed to determine the localized features of the gravity field. The short-wavelength perturbations in the satellite trajectory, apart from those arising from the extra-terrestrial disturbing function and the long-wavelength components of the terrestrial disturbing function, are produced primarily by mass anomalies in the area which is being traversed by the satellite at a certain instant, provided due consideration is given, of course, to the possible sources of error. If several passes of one satellite or a number of satellites are available for a certain area, it would be possible to retrieve the higher degree gravity coefficients from these residual perturbations. Obviously, from the above discussion, these harmonic coefficients will be characteristic of the area under consideration. They will determine an equipotential surface superimposed on the 'normal equipotential surface' defined by the long-wavelength components of the geopotential. The resulting equipotential surface thus computed will, hopefully, closely approximate the geoidal surface in the area under consideration--though it may be merely hypothetical outside that area. Hence, the solution will be reliable only for the former. By moving from area to area in this way, however, it should be possible to cover the surface of the globe. The areas for which the gravimetric data are available, will provide test cases for the study and will help to investigate the general character, if any, of any possible relationships which may exist between the satellite-determined gravity and the gravimetric data in these areas. Once this is settled, there will be no difficulty in integrating these data with one another and with the surface gravity measurements to obtain an accurate version of the earth's gravity field.

Another advantage of the method seems to be that it could possibly be applied to low-altitude satellites, since the observation equations can be set up such a way that the effect of any uncertainties in the extra-terrestrial disturbing function can be reduced. If this is found feasible, it will offer the possibility of exploiting the relatively higher 'sensing potential' of a low-altitude satellite for the short-wavelength features.

Satellites at different altitudes will 'sense' the gravity field of the earth to different degrees of detail. Obviously, the effect of the short-wavelength components will diminish more rapidly with altitude. Hence, the localized harmonic coefficients will be a slowly varying function of altitude and will reflect the degree of detail of the gravity field for specific height ranges. This offers the possibility of studying the 'lapse rate' of gravity with altitude or in more familiar terms, the problems of the upward and downward continuation of the gravity field.

The method will determine the regional gravity field for different areas. If the gravimetric data are available in a certain area, this regional field can be used to separate the higher frequencies associated with mass anomalies and structural features of interest in prospecting geophysics. Theoretically speaking, the method could also be used to study the time-variant part of the gravity field, the radial asymmetry in the mass distribution of the core--such as the one caused by the convection currents or, for that matter, the differential rotation between the core and the mantle. However, these problems are so interdependent that a study of any one of these will require an accurate knowledge of some of the hitherto unknown or poorly determined parameters, and, in addition, will put very stringent requirements on the accuracy of the satellite data.

It is assumed that the instantaneous orbital elements of a satellite (in terms of its position and velocity vectors) are determinable with a fairly high degree of accuracy. However, this accuracy requirement is not beyond our present observational capability, as discussed later. It is also assumed that the changes in the elements are such that one will get a well conditioned system of observation equations. This will hopefully be possible for areas for which data from several passes of various satellites are available.

The Required Observational Accuracy

An approximate error analysis shows that in order to detect changes in the geopotential of the order represented by the higher degree zonal harmonic coefficients and the tesseral harmonic coefficients, the position of the satellite should be known to better than 0.1 km and its velocity to an accuracy of 10 cm/sec, or better. The positional accuracy is well within our present observational capability. The accuracy requirement on the velocity seems to be somewhat stringent. However, one should be able to use the precisely reduced camera observations, the Doppler data, or the more recent laser tracking data to make at least a tentative application of this theory.

RECOMMENDATIONS FOR FURTHER WORK

The method proposed here may be tested for a few areas for which gravimetric data are available. This involves considerable computerization work as well as collection of satellite and gravity data and standardization of the latter. Gravimetric data for a number of areas are available at the Hawaii Institute of Geophysics. It is recommended that the required satellite data be obtained and that test comparisons be made for a few selected areas.

A method proposed by Arnold (1966) aims at utilizing the available gravimetric data in computing the gravity field of the earth from satellite data. It is suggested that the results obtained by using the method proposed in this report be compared with those obtained by using the procedure proposed by Arnold.

The possibility of studying the various geophysical problems will have to be examined after the test phase of the project is over.

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