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Analog Sensitivity Design

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CONTENTS

ABSTRACT

The sensitivity of the response of a dynamic system to changes in its parameters is of great importance in the design of systems. Several methods of sensitivity analysis are reviewed: Direct Simulation, Sensitivity Operators, Eigenvalue Sensitivity. These methods are shown to have various merits as tools of analysis, but cannot be used for the systematic design of a low sensitivity system.

In order to develop a systematic design procedure, sensitivity coefficients are introduced. These coefficients are in general the elements of a matrix and are given by the derivative of the system trajectory with respect to the parameters of interest. The sensitivity equations which describe these coefficients are derived and the sensitivity design problem is defined for a constant gain, linear feedback system.

The method of Analog Sensitivity Design (ASD) is developed to solve the sensitivity design problem. This method employs the simultaneous solution of the state and sensitivity equations of the dynamic system on a repetitive operation analog computer and an automated hill climbing iterative solution technique. A structural method is described where ASD can be used when the response of an existing physical device with an unknown mathematical description is to be desensitized to one or more of its parameters. Some of the other unique advantages of ASD are discussed and a simple second order example is solved.

A case study involving the design of a feedback control system for the Saturn V - Apollo is presented. The control system is to be insensitive to changes in the natural frequencies of the bending modes of the vehicle. The equations of motion are derived, a brief discussion of wind disturbances acting on the vehicle is given and a design wind selected. ASD is then applied to a seventh order fixed time model of the booster with a single bending mode included. The resulting desensitized system is compared with an optimal system for nominal and off nominal values of bending mode frequency. The desensitized system maintains adequate control of the vehicle for bending frequency perturbations which are sufficient to drive the optimal system unstable. The same two control systems are then applied to a full time varying eleventh order booster model including three bending modes and a design wind disturbance. Again the desensitized control system is shown to be far more tolerant to changes in bending mode frequencies than is the optimal system.

CHAPTER I

INTRODUCTION

1.1 General

In recent years many powerful new mathematical techniques have been developed for the analysis and design of dynamic processes. In addition to these new techniques, classical techniques have been refined and improved for application to large and sophisticated systems. Before any of these techniques can be applied, however, a mathematical description or model of the process must be obtained. This model usually consists of one or a set of differential equations, in general nonlinear and time varying. Using standard techniques[4] these equations can be reduced to a set of first order differential equations as in (1.1-1).

$$F(x, \dot{x}, m, q, t) = 0 \tag{1.1-1}$$

where x is an n-vector, the dependent variable

$\dot{x}$ is the derivative of x with respect to t

m is an m-vector, the forcing function or control

q is an r-vector, a parameter of the system

t is the independent variable, taken as time

The appropriate design and analysis techniques can then be applied to (1.1-1) with the results assumed to apply to the original physical system. The question to be examined here is the success of this application.

No matter how carefully the mathematical model (1.1-1) has been formulated, there always exist differences between the model and the physical dynamic process that it has been designed to represent. For this reason the solutions to the system of differential equations (1.1-1) cannot be said to represent the true behavior of the physical system. There can be many sources of this difference. First it may be impossible or simply impractical to realize mathematically either the functions or the parameter values of the dynamic process, even assuming they are known exactly. Second, in virtually all practical problems the values of the parameter q is not known exactly; all physical components have non-zero tolerances. Furthermore the equations (1.1-1) are generally solved on either a digital or an analog computer. As is well known, either of these extremely useful machines leads to finite errors in the implementation of the solution of (1.1-1) even in the unlikely event that it did represent the physical dynamic process exactly.

For these reasons it is important to develop a method of control system design that will insure that the response of the physical system will be "close" to the response of the mathematical model in spite of the small but finite differences in the value of parameters between the two. If such a design technique is possible, then full advantage can be taken of the powerful mathematical tools of system analysis and design available today. The properties of the dynamic system could then be successfully predicted and its behavior evaluated with confidence before actual device construction is begun. The method of Analog Sensitivity Design, the subject of this paper, makes this possible.

1.2 Historical Review

The most common method used in classical design to decrease the sensitivity of a control system to changes in parameter values is the addition of negative feedback or in cases where feedback is already present, increasing its magnitude. H. W. Bode,[1] in his book which laid the foundation for modern control theory introduced the concept of parameter sensitivity. With few exceptions, however, the concept of sensitivity remained dormant until comparatively recently.

The use of feedback loops to reduce the sensitivity of plant output to variations in the plant transfer function was suggested by Horowitz[9] as a method of obtaining the advantages of adaptive systems without paying the penalty of their complexity.

The major emphasis in these early studies was on the transform approach to the study of sensitivity.[9,10,13,21] The sensitivity measure was taken as the transfer function relating the percentage change of the system transfer function to the percentage change of the parameter of interest. The pole-zero and root-locus sensitivity problem was also examined by several authors, among them Kuo[13] and Huang.[10] The major difficulties with all of these frequency domain techniques were their inapplicability to time varying or nonlinear systems and the difficulty of drawing meaningful conclusions about the time response sensitivity of the systems.

The study of time response or trajectory sensitivity of dynamic systems was begun in earnest in the field of differential analyzers where accuracy has always been of great importance. Miller and Murry[14] particularly, formulated the time domain sensitivity problem in a meaningful way and derived the original differential equations describing the sensitivity coefficients of a system.

Recently Dorato,[5] Rohrer and Sobral,[18] and Pagurek[15] have used the concept of sensitivity for the study of optimal control of processes with partially unknown parameters. Their primary concern was to make the value of the optimal index of performance insensitive to variations in the system parameters. Again there was little emphasis placed on the sensitivity of the time response or the development of useful design techniques. Holtzman and Horing[8] have used sensitivity analysis to examine the effect of parameter variations on the solution to the fixed terminal point optimal control problem.

Pagurek[15] has shown that under certain conditions the open loop and closed loop sensitivities of a given linear system are identical. This result is strictly true only for infinitesimal parameter variations, however, and does not hold true in general.

Cruz and Perkins[3,16] have recently attacked the problem of plant sensitivity through the use of sensitivity operators. They show that for an open loop and a closed loop plant having identical responses when parameters are at their nominal values, the closed loop plant is less sensitive to particular parameter variations than is the open loop plant if a particular operator, which is a function of the parameter variation, is a contraction operator. This technique appears difficult to apply to a general system and does not seem to offer any promise of leading to a usable procedure.

The problem of synthesizing insensitive systems is a difficult one and not many new results have been obtained in this area. Tomovic[19,20] and Kokotovic[11,12] have indicated several time domain techniques that appear promising. The major emphasis of their work, however, has been in the analysis of the parameter sensitivity of existing systems. Kokotovic's[11] method of sensitivity point analysis makes possible the simultaneous evaluation of all of the sensitivity coefficients of a dynamic process.

Tuel[22] developed a synthesis procedure which is valid for both linear and nonlinear plants. His method results in an open loop controller which limits trajectory dispersion due to uncertainities in plant parameter values. The standard state variables are augmented with sensitivity variables representing the sensitivity coefficients of the open loop system. Standard control signal optimization techniques[12,17] were used to solve the resulting optimal control problem. The major disadvantage of this method is the open loop structure of the controller and the consequent loss of nominal response acceptability.

Dougherty[6] attacked the synthesis problem in a similar fashion except for the use of closed loop sensitivity coefficients. The use of closed loop coefficients required the specification a priori of the structure of the controller. Once this was done, the closed loop sensitivity coefficients and their describing differential equations could be adjoined to the state equations to form an augmented optimal control problem. The index of performance was made a function of both the states and the sensitivity coefficients such that the system response was optimized while simultaneously minimizing a measure of the sensitivity. The parameters over which the optimization took place were the gains in the feedback controller. The problem was solved using the techniques of parameter optimization. The chief difficulty of this technique is the necessity of solving the two point boundary value problem that results from the parameter optimization and the very slow convergence of the gradient technique used.

CHAPTER 2

ANALYSIS OF SYSTEM SENSITIVITY

2.1 General

In this chapter several techniques of sensitivity analysis will be examined and the possibility of their providing a useful method of synthesis discussed. The object is to examine a dynamic system for the effects of inadequate knowledge of parameter values. The nominal value of the parameters can be assumed known, but their actual values are unknown and unmeasureable.

Consider for example the trajectory of a particular dynamic process in state space. The initial and terminal points are fixed at X_o and X_T as shown in Figure 2-1. The solid line indicates the trajectory the process will travel between X_o and X_T when all parameters take on their nominal values. When the parameters are perturbed the trajectories exhibit dispersion about the nominal path. The locus of trajectories caused by "small" perturbations in the plant parameters form some sort of tube in state space. Sensitivity analysis attempts to place bounds on the diameter of the tube once bounds on the parameter variations are given. The smaller the diameter of the tube of trajectories, the less sensitive to parameter variations is the dynamic process.

2.2 Direct Simulation

The sensitivity of dynamic systems can be studied using either digital or analog computers. The most straight forward and commonly used method is by direct simulation. The differential equations of the system (1.1-1) are solved automatically for a range of parameter values

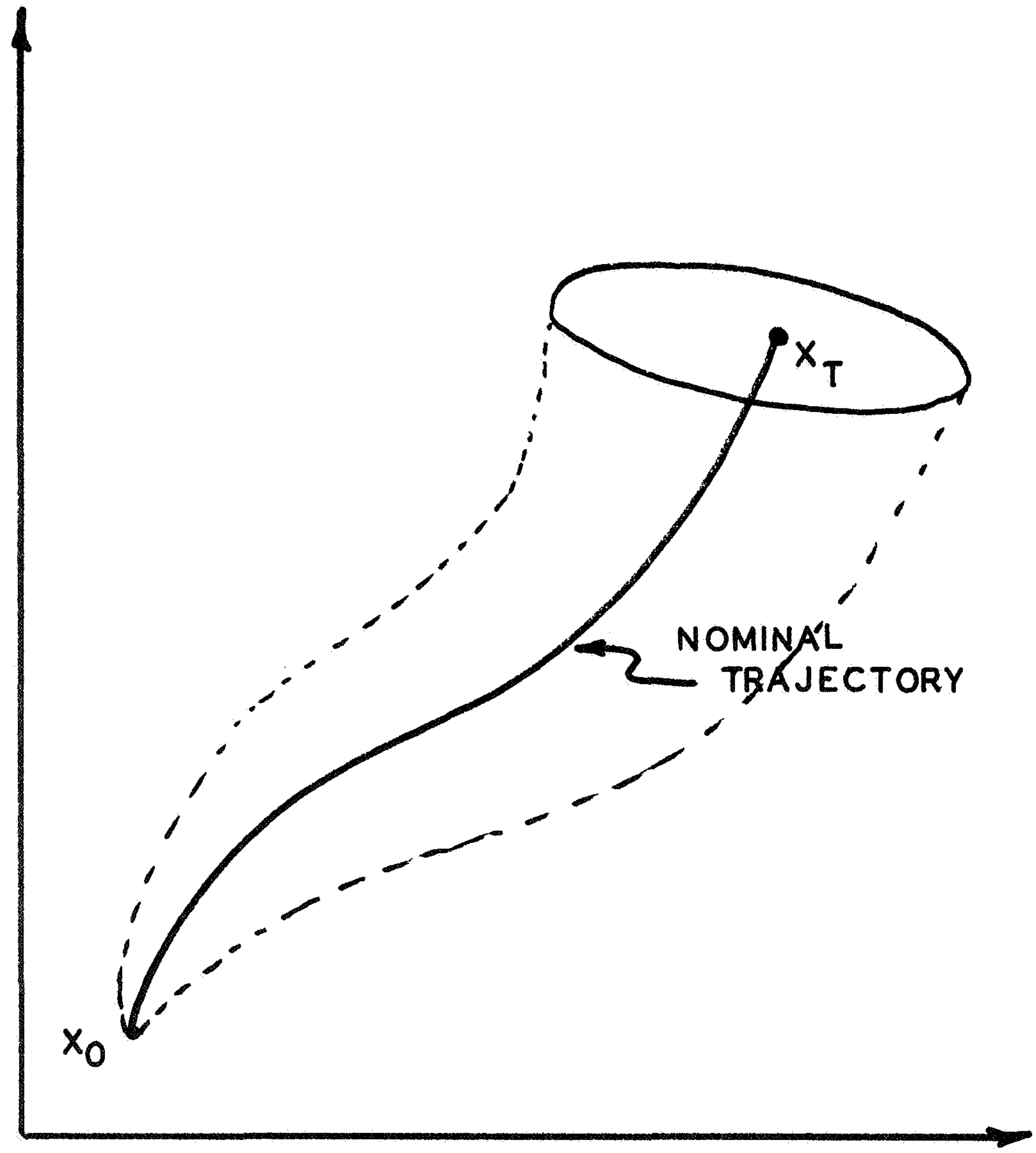


FIG. 2-1 DISPERSION

$q_o + \Delta q_i \quad i = 1, 2, \ldots, N$ where $q = q_o$ is the nominal value of the parameter.

$$
\begin{aligned}
&F(x, x, q_o, t) = 0 \\
&F(x, x, q_o + \Delta q_1, t) = 0 \\
&\qquad\vdots \\
&F(x, x, q_o + \Delta q_N, t) = 0
\end{aligned}
\tag{2.2-1}
$$

These parameter values are selected to cover the entire expected range of perturbations. If the solutions of (2.2-1) are all "close" in some respect, are all acceptable in terms of the purpose of the system and do not exceed component structural limits, then the system described by (2.2-1) is said to be insensitive to variations in the parameter q. If, however, the solutions are not close in the sense of Figure 2-1 or if some of the solutions are unstable, then the system as it stands is obviously unacceptable. The question then arises as to how the system can be made less sensitive. This simple method of sensitivity analysis cannot answer that question. In spite of this drawback, however, the ease with which the equations (2.2-1) can be solved by direct integration makes this technique a good first step in the analysis of the sensitivity of the mathematical model of dynamic processes. Often the system (2.2-1) is found to be sufficiently insensitive to the expected parameter variations and analysis need proceed no further.

2.3 Sensitivity Operators[3,17]

Traditionally the problem of decreasing the sensitivity of the response of a dynamic system to parameter perturbations within the system is solved by adding additional negative feedback around the process. This section examines the effect of doing this and the possibility of developing a design technique utilizing this procedure.

In order to examine the effect of additional feedback on parameter trajectory sensitivity the two systems of Figure 2-2 and Figure 2-3 will be studied.

The controllers C and C_o are so constructed that for nominal parameter values $P = P(q_o, t)$ the response of the two systems to identical inputs is identical. $P(q, t)$ can be regarded as a linear, time varying plant operator dependent on the parameter q.

The output of the open loop plant for nominal parameter values, when forced by the input $r(t)$ is given by

$$x_o(t) = P(q_o, t)\, u_o(t)$$

or

$$x_o(t) = P(q_o, t)\, C_o(t)\, r(t) \tag{2.3-1}$$

Similarly, the output of the closed loop plant under the same conditions of nominal parameter values and forced by the input $r(t)$ is given by

$$x_c(t) = P(q_o, t)\, C(t) \left[r(t) - H(t)\, x_c(t) \right]$$

or, solving for $x_c(t)$

$$x_c(t) = \left[I + P(q_o, t)\, C(t)\, H(t) \right]^{-1} P(q_o, t)\, C(t)\, r(t) \tag{2.3-2}$$

where I is the identity operator and the superscript -1 indicates the inverse operator.

Let $P(q, t) = P(q_o + \Delta q, t)$ be the plant operator for the perturbed parameter values $q = q_o + \Delta q$. Then the output of the plant will also be perturbed from its nominal value. Let this perturbation be $e_o(t)$ and $e_c(t)$ for the open loop and closed loop

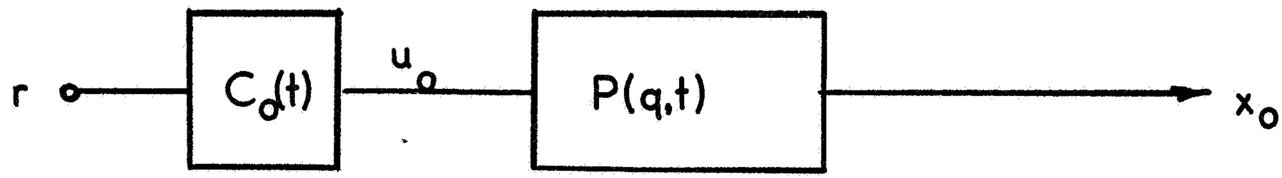


FIG. 2-2 OPEN LOOP

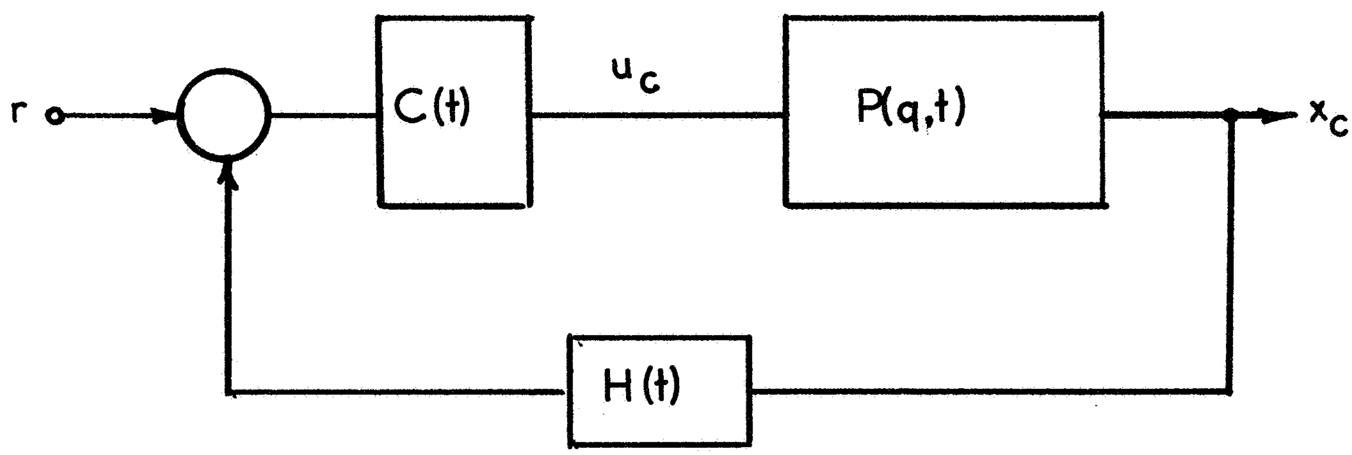


FIG. 2-3 CLOSED LOOP

respectively. Then these perturbations are given by

$$
\begin{aligned}
e_o(q, t) &= x_o(q_o, t) - x_o(q_o + \Delta q, t) \\
e_c(q, t) &= x_c(q_o, t) - x_c(q_o + \Delta q, t)
\end{aligned}
\tag{2.3-3}
$$

The closed loop perturbation can be written as

$$
e_c(q,t) = \left\{ \left[I+P(q_o)\ CH \right]^{-1} P(q_o)\ C - \left[I+P(q_o+\Delta q)\ CH \right]^{-1} P(q_o+\Delta q) \right\} r
\tag{2.3-4}
$$

where the time variable has been suppressed for brevity.

Define the operators

$$
\begin{aligned}
\Delta P &= P(q_o + \Delta q) - P(q_o) \\
T(q) &= \left[I + P(q)\ CH \right]^{-1} P(q)\ C
\end{aligned}
\tag{2.3-5}
$$

then the closed loop trajectory dispersion is given by

$$
e_c = \left[I + P(q_o + \quad q)\ CH \right]^{-1} \left[\Delta PC(HT(q_o) - I \right] r
\tag{2.3-6}
$$

Recall that for nominal parameter values $q = q_o$ the open loop and closed loop systems give identical outputs for identical inputs. Therefore

$$
x_o(q_o) = x_c(q_o)
$$

and

$$
u_o(q_o) = u_c(q_o) = C \left[r(t) - H\ x_c(t) \right]
\tag{2.3-7}
$$

Using the fact that

$$
x_c = \left[I + PCH \right]^{-1} PC\ r(t)
\tag{2.3-8}
$$

gives the relation

$$u_o(q_o) = Cr - CH \left[I + PCH \right]^{-1} PC\ r(t) \tag{2.3-9}$$

which when combined with the definitions (2.3-5) gives the following expression for the open loop plant input function

$$u_o(q_o) = C \left[I - HT(q_o) \right] \ r(t) \tag{2.3-10}$$

From the diagram of the open loop system, Figure 2-2, it is evident that

$$e_o(q_o) = \left[P(q_o) - P(q_o + \Delta q) \right] \ u_o(t) \tag{2.3-11}$$

Using (2.3-11) with (2.3-10) and (2.3-5) gives an expression for the open loop trajectory dispersion

$$e_o(q,\ t) = \Delta PC \left[HT(q) - I \right] \ r(t) \tag{2.3-12}$$

Finally, combining the expressions for closed loop trajectory dispersion (2.3-6) and open loop trajectory dispersion (2.3-12) gives the following relation

$$e_c = \left[I + P(q_o + \Delta q,\ t)\ CH \right]^{-1} e_o \tag{2.3-13}$$

Thus the perturbations of the closed loop system due to a parameter variation are related by a time varying linear operator (2.3-14) to those of the open loop system.

$$L(q_o + \Delta q,\ t) = \left[I + P(q_o + \Delta q,\ t)\ CH \right]^{-1} \tag{2.3-14}$$

What does this indicate about the relative sensitivity of the two systems to parameter variations? If the norm of the output dispersion is taken as a measure of the sensitivity, then from (2.3-13)

$$\|e_c\| \leq \|L(q_o + \Delta q, t)\| \; \|e_o\| \tag{2.3-15}$$

Thus if $\|L(q_o + \Delta q)\| < 1$ then the closed loop trajectory dispersion is less than the open loop trajectory dispersion and the closed loop system is said to be less sensitive than the open loop system for that particular parameter variation Δq. Unfortunately this gives no information about the relative sensitivities for any other parameter variation. To get a complete picture of the parameter sensitivity of the closed loop system (2.3-14) would have to be evaluated for many values of Δq just as was (2.2-1). Thus the method of sensitivity operators appears to have no practical advantage over sensitivity analysis by direct simulation. In general, (2.3-14) is much more difficult to evaluate than is (2.2-1). Additionally, there is again no way to proceed in a systematic fashion if a particular closed loop design proves to be more sensitive than the original open loop design. Thus while the technique may prove to be useful theoretically, there is some doubt as to its practical utility.

2.4 Eigenvalue Sensitivity

A very useful technique for studying the properties of linear, time varying systems involves the plotting of the poles and zeros of the system transfer function. It is well known that the poles or eigenvalues of this transfer function completely determine the stability and shape of the system transient response.[4] Thus it is natural to examine the sensitivity of the eigenvalues to changes in the parameters of the system. In what follows the parameter q will be assumed to be a scalar in order to simplify the notation.

Assume the linear, time invariant dynamic system is described by the homogeneous, constant coefficient differential equation (2.4-1).

$$\begin{aligned} \dot{x}(t) &= A(q)\, x \\ x(0) &= c \end{aligned} \qquad (2.4\text{-}1)$$

where x is the n dimensional state vector

$\dot{x}$ is the time derivative of x

$A(q)$ is the $n \times n$ system matrix dependent on q

q is the scalar parameter of interest with nominal value q_o

The eigenvalues λ_i and the corresponding eigenvector v_i for the nominal value of the parameter q are defined by

$$A(q_o)\, v_i(q_o) = \lambda_i(q_o)\, v_i(q_o) \qquad i = 1, 2, \ldots, n \qquad (2.4\text{-}2)$$

The perturbed system, with $q = q_o + \Delta q$ has eigenvalues and eigenvectors given by

$$A(q_o + \Delta q)\, v_i(q_o + \Delta q) = \lambda_i(q_o + \Delta q)\, v_i(q_o + \Delta q) \qquad (2.4\text{-}3)$$

$$i = 1, 2, \ldots, n$$

Subtracting (2.4-2) from (2.4-3) gives

$$A(q_o + \Delta q)\, v_i(q_o + \Delta q) - A(q_o)\, v_i(q_o) \qquad (2.4\text{-}4)$$

$$- \lambda_i(q_o + \Delta q)\, v_i(q_o + \Delta q) - \lambda_i(q_o)\, v_i(q_o) = 0$$

Equation (2.4-4) can be expanded in a Taylor Series about the nominal parameter value $q = q_o$.

$$\frac{\partial}{\partial q}\Big[A(q_o)\, v_i(q_o)\Big](q-q_o) - \frac{\partial}{\partial q}\Big[\lambda_i(q_o)\, v_i(q_o)\Big](q-q_o) + 0\,(q-q_o)^2 \qquad (2.4\text{-}5)$$

where $O(q-q_o)^2$ indicates an error term of the order $(q-q_o)^2$.

For sufficiently small $|q-q_o|$ the linear terms of (2.4-5) will dominate and the equation becomes

$$\frac{\partial}{\partial q}(Av_i) - \frac{\partial}{\partial q}(\lambda_i v_i) = 0 \qquad i = 1,2,\ldots,n$$

Taking the indicated partial derivatives yields

$$\frac{\partial A}{\partial q}(q_o)\, v_i(q_o) + A(q_o)\frac{\partial v_i}{\partial q}(q_o) = \frac{\partial \lambda_i}{\partial q}(q_o)\, v_i(q_o) + \lambda_i(q_o)\frac{\partial v_i}{\partial q}(q_o)$$

$$i = 1,2,\ldots,n \qquad (2.4\text{-}6)$$

Next let u_i be the i^{th} eigenvector of A^T. Recall that A^T has identical eigenvalues with A but the eigenvectors are different. The u_i are then given by

$$A^T u_i = \lambda_i u_i \qquad i = 1, 2,\ldots,n \qquad (2.4\text{-}7)$$

Transposing (2.4-6) and postmultiplying by u_i gives

$$v_i^T\left(\frac{\partial A}{\partial q}\right)^T u_i + \left(\frac{\partial v_i}{\partial q}\right)^T A^T u_i = v_i^T\left(\frac{\partial \lambda_i}{\partial q}\right)^T u_i + \left(\frac{\partial v_i}{\partial q}\right)^T \lambda_i^T u_i$$

$$(2.4\text{-}8)$$

where all terms are to be evaluated at $q = q_o$. Equation (2.4-7) is next combined with (2.4-8) and the fact that $\lambda_i^T = \lambda_i$ used to obtain

$$\frac{\partial \lambda_i}{\partial q}(q_o) = \frac{v_i^T(q_o)\frac{\partial A}{\partial q}(q_o)\, u_i(q_o)}{v_i^T(q_o)\, u_i(q_o)} \qquad (2.4\text{-}9)$$

Equation (2.4-9) expresses the eigenvalue sensitivity of the system (2.4-1) with all terms evaluated at the nominal value of the parameter.

This is an efficient method of investigating sensitivity since unlike the previous methods only one evaluation is needed with the results holding for all small parameter variations Δq. If a system is found to be highly sensitive with respect to a particular eigenvalue it is not clear how the system should be redesigned to reduce this sensitivity without increasing the sensitivity of the other eigenvalues to unacceptable levels. Thus while (2.4-9) provides a rapid and simple tool for the comparison of the relative eigenvalue sensitivity of competing systems to small parameter variations, it does not provide a systematic procedure for the systhesis of an insensitive system.

CHAPTER 3

SENSITIVITY DESIGN

3.1 General

The previous chapter examined several methods of analyzing the sensitivity of dynamic systems to parameter variations but none of the methods showed any promise of being extendable to a tractable design technique. In this chapter the problem of sensitivity design is investigated. To this end sensitivity coefficients are defined and the differential equations describing their behavior are derived. It is shown that these sensitivity equations are always linear regardless of the form of the equations describing the dynamic process being examined. Finally, Dougherty's method[6] of using parameter optimization for sensitivity design is briefly reviewed and a method suggested for overcoming its chief shortcoming which is the difficulty of solving the resulting two point boundary value problem.

3.2 Sensitivity Coefficients[14,19]

When the sensitivity of a dynamic process to certain parameters is to be investigated, it is necessary to examine the dispersion of the solutions of (3.2-1) for varying values of the parameter q.

$$F(x, x, m, q, t) = 0 \tag{3.2-1}$$

In order to facilitate this examination (3.2-1) is rewritten as a set of couple first order differential equations

$$\begin{aligned} \dot{x} &= f(x, m, q, t) \\ x(0) &= c \end{aligned} \tag{3.2-2}$$

where x is an n-dimensional state vector

m is an m-dimensional control vector

q is an r-dimensional parameter vector

c is an n-dimensional initial condition

It is assumed that a unique solution $x(t)$ with $x(0) = c$ exists once the control $m(t)$ and the parameter q have been specified.

Since it is generally agreed that closed loop or feedback control is desirable in the control of dynamic systems the control law is assumed to be of the form

$$m = m(x, t) \tag{3.2-3}$$

For a given control strategy let the corresponding nominal trajectory for $q = q_o$ be given by

$$x = x_o(m, q_o, t) \tag{3.2-4}$$

A quantitative measure of the dependence of the solution of (3.2-2) on the parameter q can be obtained by expanding the solution of $x(q, t)$ about the nominal value of the parameter $q = q_o$

$$x(q, t) = x(q_o, t) + \frac{\partial x}{\partial q}(q_o, t)(q-q_o) + 0\,(q-q_o)^2 \tag{3.2-5}$$

where the partial derivatives are evaluated at q_o and $0(q-q_o)^2$ indicates that the remainder terms are of second order in $(q-q_o)$. The partial derivative symbol actually indicates a matrix which is defined by

$$\left(\frac{\partial x}{\partial q}\right)_{ij} = \frac{\partial x_i}{\partial q_j} \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, r \end{array} \tag{3.2-6}$$

The elements of this matrix are defined as the sensitivity coefficients of the process and are denoted by

$$Z_i(j) = \frac{\partial x_i}{\partial q_j} \qquad \begin{array}{l} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, r \end{array} \tag{3.2-7}$$

In a sufficiently small neighborhood about the nominal value of the parameter q_o these sensitivity coefficients specify the deviation of the solutions of (3.2-2) from nominal due to the parameter variation $q = (q-q_o)$. Thus the norm of the sensitivity vector $\| z^{(j)} \|$ can be taken as a measure of the trajectory dispersion of the dynamic system due to the perturbations of the parameter q_j.

There is another manner in which the definition of sensitivity coefficients can be approached. In Section 2.2 the sensitivity of (3.2-1) to changes in parameter values was examined by means of direct simulation of the differential equation on an analog or digital computer for various values of the parameter q. Consider the results of two such simulations having the solutions x_o and x where

$$\begin{aligned} x_o &= x(q_o, t) \\ x &= x(q_o + \Delta q, t) \end{aligned} \tag{3.2-8}$$

and the parameter variation has taken place only for a single element q_j of the parameter vector q.

Comparing these two resultant trajectories give the following expression for the relative sensitivity to the parameter q_j

$$\frac{x(q_o + \Delta q_j, t) - x_o(q_o, t)}{\Delta q_j}$$

If this has a limit as $\Delta q_j \to 0$, then

$$\lim_{\Delta q_j \to 0} \frac{x(q_o + \Delta q_j, t) - x_o(q_o, t)}{\Delta q_j} = \frac{\partial x(q_o, t)}{\partial q_j} = z^{(j)} \qquad (3.2\text{-}9)$$

This is defined as the sensitivity coefficient of the state vector x with respect to the parameter q_j and the norm $\| z^{(j)} \|$ gives a measure of the sensitivity of the trajectory $x = x(q, t)$ to small changes in q_j.

3.3 Sensitivity Equations[14,19]

If the sensitivity coefficients are to be used as an aid in system design there must be an efficient method of calculating them. Assuming the state equations (3.2-2) are known, a set of differential equations describing the sensitivity coefficients can be derived. It will be shown later that the sensitivity coefficients can be obtained by simulation even when the state equations are not known.

Taking the derivative of the state equations (3.2-2) with respect to the parameter q gives

$$\frac{\partial}{\partial q}\left(\frac{\partial x}{\partial t}\right) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial m}\frac{\partial m}{\partial q} + \frac{\partial f}{\partial q} \qquad (3.3\text{-}1)$$

Using the closed loop control strategy of (3.2-3) gives

$$\frac{\partial m}{\partial q} = \frac{\partial m}{\partial x}\frac{\partial x}{\partial q} \qquad (3.3\text{-}2)$$

Also, since $\frac{\partial x}{\partial q}$, $\frac{\partial x}{\partial t}$, and $\frac{\partial \dot{x}}{\partial q}$ are all assumed to be continuous functions of q and t, order of differentiation can be interchanged and

$$\frac{\partial}{\partial q}\left(\frac{\partial \dot{x}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial q}\right) \qquad (3.3\text{-}3)$$

Combining (3.3-2) and (3.3-3) with (3.3-1) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial x}{\partial q}\right) = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial m}\frac{\partial m}{\partial x}\right)\frac{\partial x}{\partial q} + \frac{\partial f}{\partial q} \tag{3.3-4}$$

where all partial derivatives are understood to be evaluated at the nominal parameter value $q = q_o$. If the definition of the sensitivity vector z of (3.2-9) is used, equation (3.3-4) can be written more compactly as

$$\dot{z} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial m}\frac{\partial m}{\partial x}\right) z + \frac{\partial f}{\partial q} \tag{3.3-5}$$

Since the parameters are assumed to have no effect on the initial conditions of the system, then

$$z(0) = \lim_{\Delta q \to 0} \frac{x(q_o + \Delta q, 0) - x(q_o, 0)}{\Delta q} = 0 \tag{3.3-6}$$

When the parameter variations are such that the initial conditions of the system can be effected then (3.3-6) no longer holds. For a discussion of this case see Appendix A.

The sensitivity coefficient vector $z(t)$ can thus be obtained by the solution of (3.3-5) with $z(0) = 0$. The sensitivity equation is always linear, regardless of the linearity of nonlinearity of the dynamic system (3.2-2). An examination of the sensitivity equation also reveals that the homogeneous part is identical to the state equation. This interesting observation is the basis of the structural method of solving the sensitivity equations that will be discussed in Section 4.3. It should also be noted that (3.3-5) is valid only if the solution of the state equation (3.2-2) depends analytically on the parameter q. This specifically excludes systems in which q can vary in such a manner as to change the order of the state equation. That is, no variation of q can make the coefficient of the highest derivative of x in the original differential equation of the dynamic process equal to zero. Appendix B discusses this in more detail.

At this point it is well to recapitulate. For the set of differential equations

$$\dot{x} = (f(x, m, q, t) \qquad x(0) = c \tag{3.3-7}$$

a measure of the relative sensitivity of the solution $x(q, t)$ to variations in the parameter q_j is given by the norm of the sensitivity vector $\| z^{(j)} \|$ where

$$z^{(j)} = (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial m} \frac{\partial m}{\partial x}) z^{(j)} + \frac{\partial f}{\partial q_j} \tag{3.3-8}$$

It is clear that a system that is to be insensitive to parameter variations q_j must in some sense make $\| z^{(j)} \|$ as small as possible consistent with the other criteria of system performance. Dougherty attackedthis problem using an optimization technique which is briefly described in the next section.

3.4 Optimal Sensitivity Design

Dougherty[6] sought to develop a procedure to synthesize a control system that was insensitive to parameter variations but did not require on line computation in the form of a sensitivity computer as do other optimal techniques.

The dynamic system state equations (3.3-7) and the corresponding sensitivity equations (3.3-8) are put into the form of a parameter optimization problem in the following manner.

Assume the control law $m(x, t)$ is chosen to be directly proportional to the state vector of the dynamic process

$$m(x, t) = - K^T x(t) \tag{3.4-1}$$

where K^T is an $(m \times n)$ time invariant gain matrix. The object of the optimization is first to obtain satisfactory nominal response of the system and second to limit the sensitivity of the process to changes in parameter values. This may be expressed mathematically as

$$\min_K (J + \hat{J})$$

where

$$\begin{aligned} J &= \int_0^{t_f} f_o(x, -K^T x, q_o, t)\, dt \\ \hat{J} &= \int_0^{t_f} g_o(z, t)\, dt \end{aligned} \tag{3.4-2}$$

where

$$\begin{aligned} \dot{x} &= f(x, m, q, t) \qquad x(0) = c \\ \dot{z} &= \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial m} K^T\right) z + \frac{\partial f}{\partial q}; z(0) = 0 \\ \dot{K} &= 0 \end{aligned} \tag{3.4-3}$$

The usual techniques of parameter optimization are than applied to these equations yielding the two point boundary value problem[6,12]

$$
\begin{aligned}
&\dot{x} = f(x, m, q, t) && x(0) = c \\
&m = - K^T x \\
&-\dot{p} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial m} K^T\right)^T p + \Psi_s^T && p(t_f) = 0 \qquad (3.4\text{-}4) \\
&-\dot{s} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial m} K^T\right)^T s && s(t_f) = 0 \\
&\int_0^{t_f} \left[-\Lambda_{1k} x_\alpha - \Gamma_{1k} z_\alpha \right] dt = 0 && \alpha = 1,2,\ldots,n
\end{aligned}
$$

where

$$\Lambda_{1k} = \frac{\partial f_o}{\partial m_k} + \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial m_R} p_i + \theta_{ik} s_i\right) \qquad k = 1,2,\ldots,m$$

$$\Gamma_{1k} = \sum_{i=1}^{n} s_i \frac{\partial f_i}{\partial m_R} \qquad k = 1,2,\ldots,m$$

$$\Psi_{ij} = \sum_{\alpha=1}^{n} \left[\frac{\partial^2 f}{\partial x_j \partial x_\alpha} - \sum_{k=1}^{m} \frac{\partial^2 f_i}{\partial x_j \partial m_R} K^T_{k\alpha} \right] z_\alpha + \frac{\partial^2 f_i}{\partial x_j \partial q} \qquad i,j=1,2,\ldots,n$$

$$\theta_{ij} = \sum_{\alpha=1}^{n} \left[\frac{\partial^2 f_i}{\partial m_j \partial x_\alpha} - \sum_{k=1}^{m} \frac{\partial^2 f_i}{\partial m_j \partial m_R} K^T_{k\alpha} \right] z_\alpha + \frac{\partial^2 f}{\partial m_j \partial q} \qquad \begin{matrix} i=1,2,\ldots,n \\ j=1,2,\ldots,m \end{matrix}$$

Needless to say, the solution of this two point boundary value problem is a formidable task. Dougherty applied a relaxation method based on steepest descent or gradient technique with some success. The chief difficulty with the gradient method is the very slow convergence in a neighborhood of the optimum. For the simple second order example shown in Chapter 4 the solution time was about 15 minutes on an IBM 360/50 digital computer. For large scale problems the time becomes prohibitive.

In addition to the difficulty of numerical solution of the two point boundary value problem of (3.4-4), there is the problem of choosing the weighting functions $f_o(x, m, q_o, t)$ and $g_o(z, t)$ which specify the desired performance of the dynamic system. If the problem being examined is described by a set of linear differential equations and the performance indices are taken to be quadratic functionals of state, control and sensitivity, this becomes the simplest form of optimization problem. Even then, and when a single varying parameter is involved, for an n^{th} order process it is still necessary to select a state weighting matrix of n^2 elements, a control weighting matrix of m^2 elements and a sensitivity weighting matrix of n^2 elements.

In short, Dougherty's method is useful for relatively simple problems when a lot of digital computation time is available and when the designer's experience with the techniques of optimization allows him to quickly choose a good set of performance index weightings. For larger problems a more efficient method of solving the sensitivity design problem of (3.4-2) and (3.4-3) is needed. One way would be to speed up the numerical solution of the two point boundary value problem (3.4-4) using a technique such as second variations. There are many problems associated with applying second variations to (3.4-4) however and it is not at all clear that there would be a great saving of computation time without a great deal of increased programming complexity. A more efficient way is to solve the sensitivity design problem directly using automated analog computer techniques. This method is discussed in the next chapter.

CHAPTER 4

ANALOG SENSITIVITY DESIGN

4.1 General

Chapter 3 introduced sensitivity coefficients and the differential equations which describe them. It was seen that the norm of the sensitivity vector $\| z \|$ gives a measure of the trajectory sensitivity of a dynamic process to parameter variations. Dougherty used this fact to factor sensitivity limiting into the standard parameter optimization problem. The solution of the two point boundary value problem which results from this approach is difficult enough to make the utility of this method questionable for large scale problems. The method of Analog Sensitivity Design described in this chapter achieves the goal of developing a systematic and efficient design procedure of limiting trajectory dispersion which is applicable to realistic problems. The procedure is such that the designer can directly observe the tradeoff between system nominal response and insensitivity to parameter variations. A method of completely automating the design procedure is given and a simple illustrative example is solved to illustrate the Analog Sensitivity Design method and compare it with other techniques. For simplicity the development will assume a single input system with a single varying parameter although neither of these restrictions is necessary.

4.2 Basic Technique

Analog Sensitivity Design offers a simple, direct method of solving the sensitivity design problem posed in Equations (3.4-2) and (3.4-3).

That is, to determine the set of feedback gains K that give the best compromise, in some sense, between performance of the dynamic system for nominal parameter values and insensitivity to variations in these parameter values. From this point, the linearized or incremental form of the equations of the dynamic system will be used. Also, the performance criteria will be assumed to be expressible in terms of the time integral of quadratic functions of the state and sensitivity vectors. Both of these restrictions are made only to make more definite the details of what follows and are in no way necessary for the application of the Analog Sensitivity Design Technique. The sensitivity design problem can now be stated as follows. Determine the set of feedback gains K^* such that

$$K^* = \left\{ K \;\middle|\; \min_{K^T} (J + \hat{J}) \right\}$$

where

$$J = \int_0^{t_f} (x^T sx + m^T Rm)\, dt$$

$$\hat{J} = \int_0^{t_f} z^T wz\, dt$$

$$\dot{x} = A(q,\ t)\, x + B(q,\ t)\, m \qquad x(0) = c \qquad (4.2\text{-}1)$$

$$m = -\,K^T x$$

$$\dot{z} = (A - BK^T)\, z + \left(\frac{\partial A}{\partial q}(q_o) - \frac{\partial B}{\partial q}(q_o)\, K^T\right) x \qquad z(0) = 0$$

$$\dot{K} = 0$$

This last restriction, that feedback gains be constant is made for several reasons. First, most actual control systems use constant or piecewise constant gains. Second, if time varying gains are considered then it becomes necessary to incorporate online computation in the form of a sensitivity computer with a consequent increase in complexity and cost and a decrease in reliability. Third, constant gains allow the use of greatly simplified analog computer techniques.

Combining the control law $m = - K^T x$ and the state equation of (4.2-1) gives the modified state equation.

$$\dot{x} = (A - BK^T)\, x$$

$$x(0) = c \tag{4.2-2}$$

This is the homogeneous part of the sensitivity equation

$$\dot{z} = (A - BK^T)\, z + \left(\frac{\partial A}{\partial q}(q_o) - \frac{\partial B}{\partial q}(q_o)\, K^T\right) x \tag{4.2-3}$$

$$z(0) = 0$$

The first step in the solution of the sensitivity design problem (4.2-1) is wiring the analog computer to solve (4.2-2) and (4.2-3) simultaneously. This is shown schematically in Figure 4-1. This shows the simultaneous or parallel solution of the two sets of equations.

To perform the solutions simultaneously two complete models of the dynamic process (4.2-2) are required, one for the state equations and one for the sensitivity equations. It is possible, however, to solve these sequentially. This is feasible because the solution of the state equation (4.2-2) is independent of the sensitivity vector described by (4.2-3). To perform the sequential solution, (4.2-2) is first solved by analog simulation and the state vector $x(t)$ recorded.

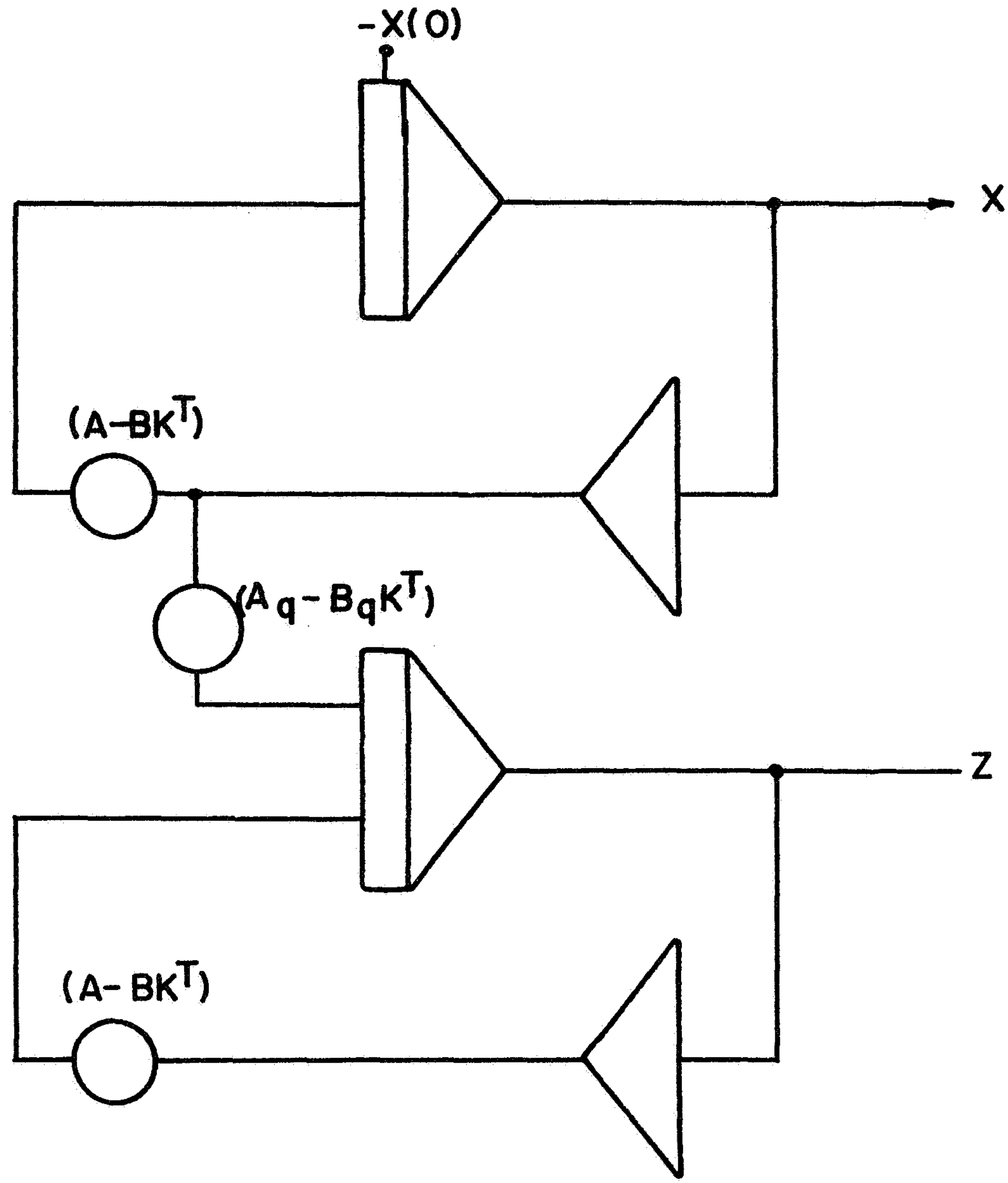


FIG. 4-1 SOLUTION OF STATE AND SENSITIVITY EQUATIONS

This recorded $x(t)$ is passed through the gain w where

$$w = (\frac{\partial A}{\partial q}(q_o) - \frac{\partial B}{\partial q}(q_o)\, K^T) \tag{4.2-4}$$

and then applied to the same simulation network as a forcing function. The output of the simulator is now $z(t)$, the sensitivity vector solution of (4.2-3). This serial technique is not directly applicable to the synthesis procedure of Analog Sensitivity Design. For this the parallel solution method must be used and two separate models are required.

Once the solutions of the state and sensitivity equations are available for some initial value of K, it is a simple matter to compute the values of the performance criteria J and $\hat{J}$

$$J = \int_0^{t_f} (x^T s x + m^T R m)\, dt \tag{4.2-5}$$

$$\hat{J} = \int_0^{t_f} (z^T w z)\, dt \tag{4.2-6}$$

Since the three variables x, m, and z of (4.2-5) and (4.2-6), are all functions of the gain vector K, these criteria can be minimized by adjusting the elements of K. This is the technique of Analog Sensitivity Design. The elements of K are adjusted in the manner described by the flowchart of Figure 4-2.

The state and sensitivity equations (4.2-2) and (4.2-3) are solved for a particular value of $K = K_1$ and the value of the performance index $I = J + \hat{J}$ computed. Each gain element is adjusted in turn by discrete

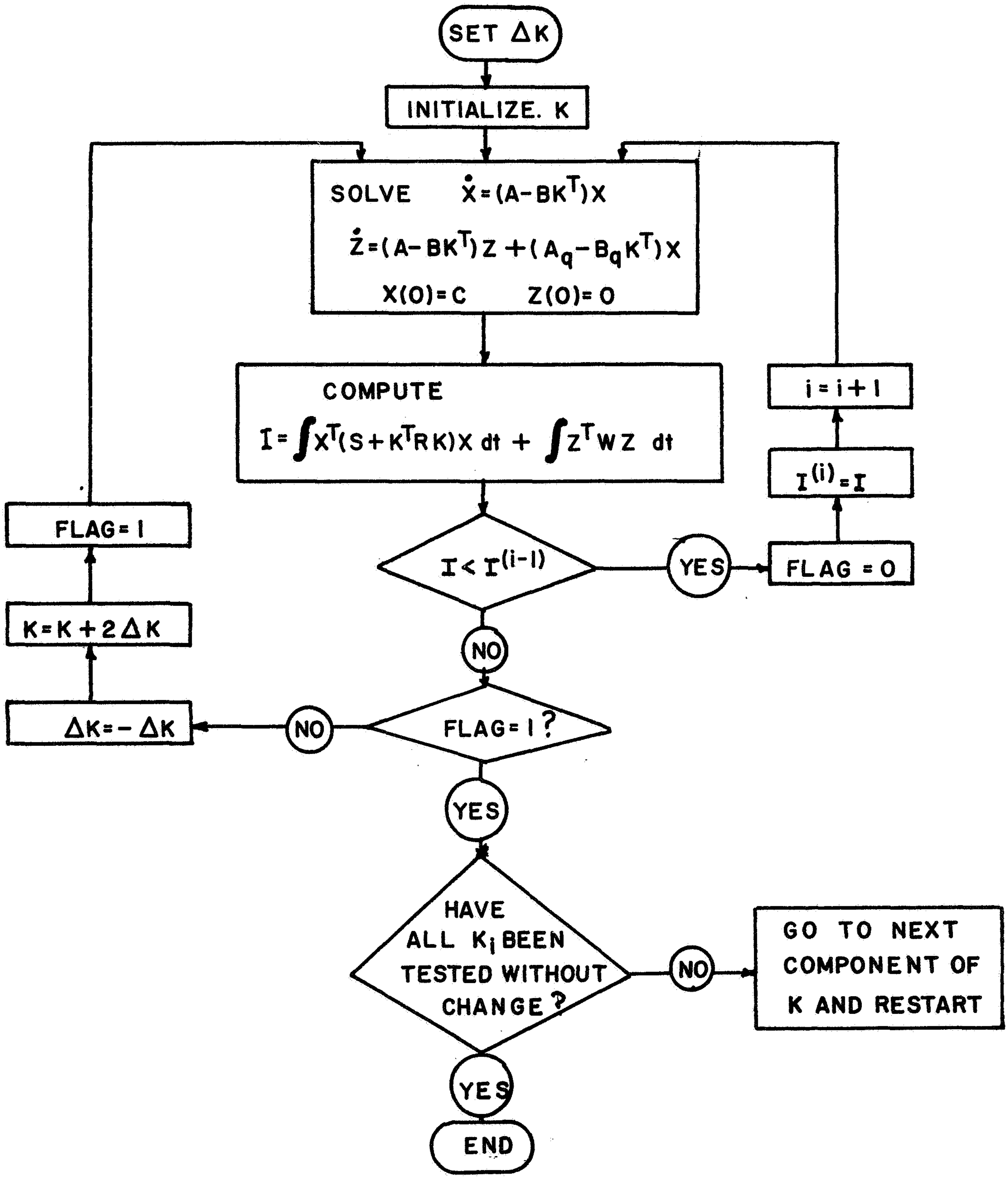


FIG. 4-2 ANALOG SENSITIVITY DESIGN

steps of magnitude ΔK and in either a positive or a negative direction until I can no longer be decreased. Then the next element of K is adjusted in a similar manner. The process continues, readjusting each element of K in turn until no increment of $\pm \Delta K$ in any element will decrease I. For sufficiently small ΔK, the $K = K^*$ which is the solution of the sensitivity design problem (4.2-1) lies within a hypercube of side $2\Delta K$ of this value. If more accuracy is desired, K can then be initialized at a value within this hypercube and the step size ΔK reduced. Continuing this process will yield K^* to within the accuracy of the computing equipment.

This technique is a variation of the hill climbing method. Stability and convergence are guaranteed because the surface formed by the performance index I is convex and its value is decreased at every iteration. Figure 4-3 illustrates a typical sequence of gain adjustments for a two dimensional gain vector.

The operations described for the Analog Sensitivity Design solution of the sensitivity problem (4.2-1) can be carried out manually or if the analog computer has even elementary logic capability they can be completely automated using Figure 4-2 as a guide.

This method of solution would not be feasible on a digital computer because of the many solutions of the state and sensitivity equations required. In this respect the Analog Sensitivity Design Technique is not an efficient one. On a modern repetitive operation analog computer, however, a complete solution of these equations even for a 100 second interval can be carried out at least five times per second <u>independent</u>

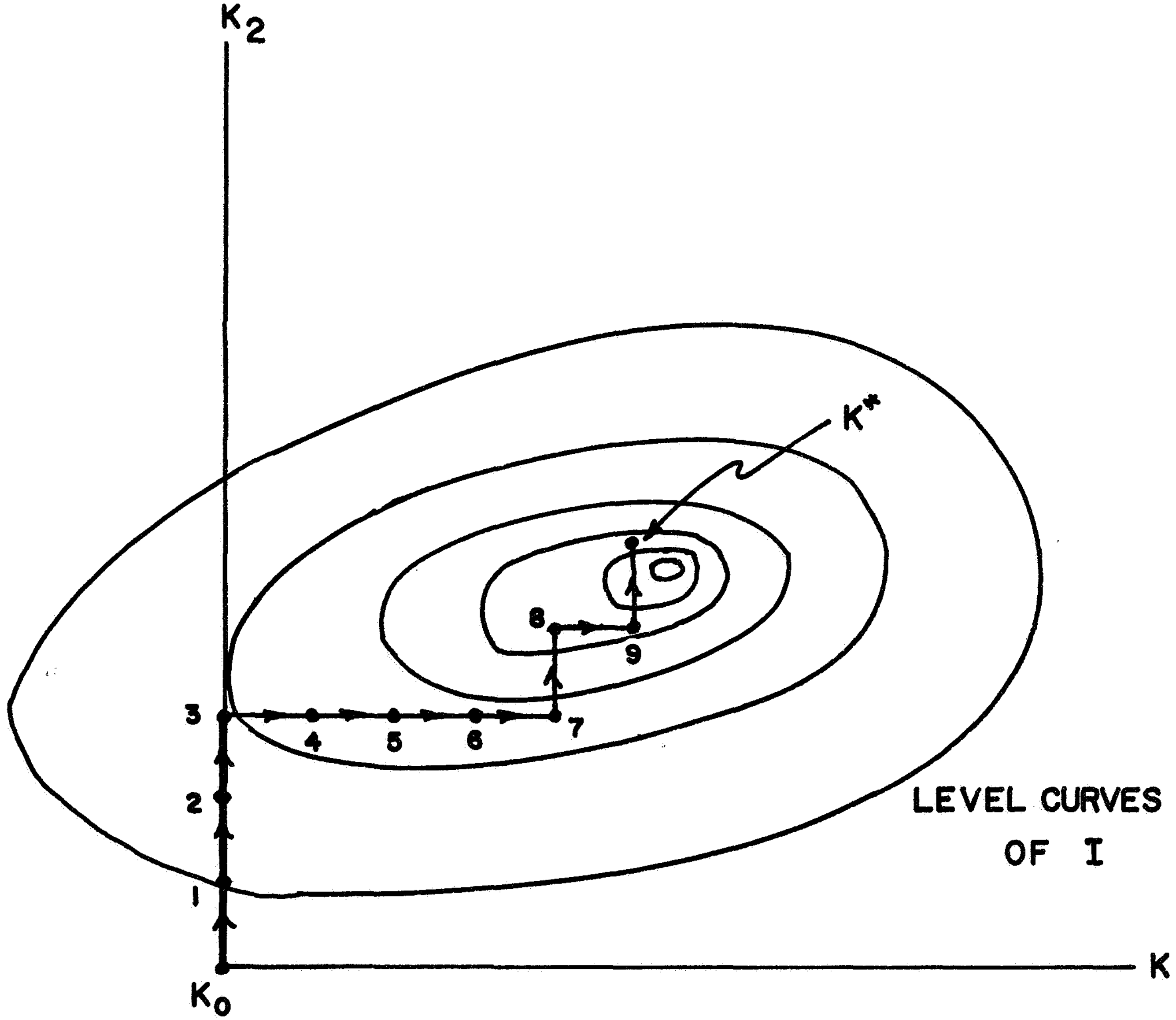


FIG. 4-3 HILL CLIMBING SOLUTION

<u>of the order of the equations</u>. Thus the hill climbing solution of the sensitivity design problem on the analog computer can be performed at about 300 iterations per minute for a dynamic system of virtually any order having a settling time of 100 seconds or less. Of course the larger the dynamic system being investigated, and the higher the order of the describing differential equation, the more analog computation equipment is required to implement the solution.

4.3 The Structural Method

Analog Sensitivity Design as presented above requires a mathematical description of the dynamic process. The starting point for the procedure is the set of differential equations (4.2-2). There are often cases, however, when the engineer is presented with an actual subsystem and is required to minimize the sensitivity of that physical device to a particular parameter by setting the values of certain gains. Using Analog Sensitivity Design this desensitization can be performed directly on the device without having to resort to a mathematical model of questionable accuracy. This section describes a structural method[23] for doing this for linear dynamic systems.

Consider the dynamic system composed of n linear elements as shown in Figure 4-4. The input and output of the system are $x(t)$ and $y(t)$ respectively. The input signal to the i^{th} element is denoted by x_i and the output of the i^{th} element is denoted by y_i. The response of the i^{th} element at time t to an impulse at time τ is denoted by $h_i(t,\tau)$. The dynamic characteristics of the i^{th} element can then be

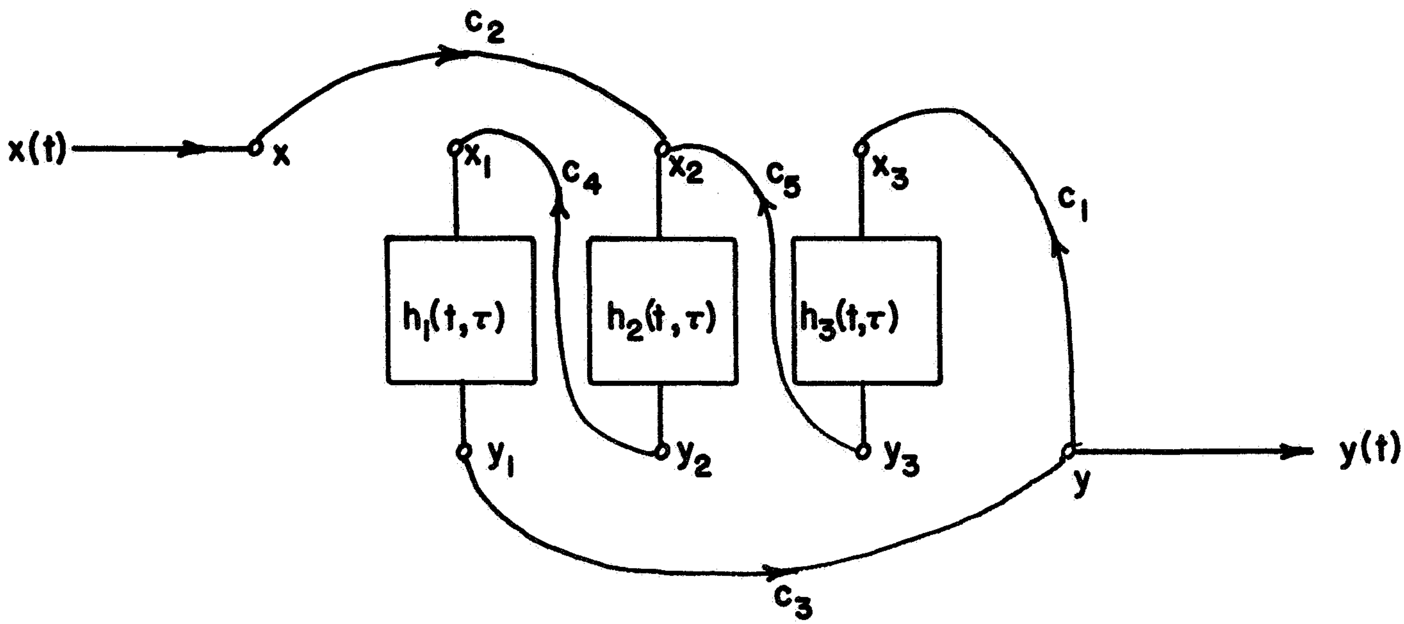


FIG. 4-4 LINEAR DYNAMIC SYSTEM

denoted by the relation

$$y_i(t) = \int_0^t h_i(t,\tau)\, x_i(\tau)\, d\tau \qquad i = 1, 2, \ldots, n \qquad (4.3\text{-}1)$$

Furthermore, the topology of the system or the interconnections between the elements can be expressed by the algebraic matrix equation

$$Au + Bv = 0 \qquad (4.3\text{-}2)$$

where

$$u = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x \end{bmatrix} \qquad v = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y \end{bmatrix} \qquad (4.3\text{-}3)$$

The A and B matrices are constructed in the following manner.

Assume the system (4.3-1) has m separate interconnections between elements with each signal flow path counting as one interconnection. For example in Figure 4-4 there are five interconnections labelled c_1 through c_5, and therefore $m = 5$. Then for each of these interconnections or signal flow paths a row of A and a row of B are defined. If interconnection c_i goes from x_j to y_k then $a_{ij} = -1$ and $b_{ik} = +1$. If c_i goes from y_k to x_j, then $a_{ij} = +1$ and $b_{ik} = -1$. Thus the A and B matrices for the system of Figure 4-3 are

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad (4.3\text{-}4)$$

These matrices completely determine the structure of the physical system.

Now the numbering of the elements of the system is done in such a manner that the first $r \leq n$ elements depend on the parameter q while the remaining $(n-r)$ do not.

The sensitivity vector is defined as

$$z = \frac{\partial v}{\partial q} \tag{4.3-5}$$

where now the outputs of the linear elements take the role of state variables.

Next define a completely different system which is called the sensitivity system. The structure of this sensitivity system can also be completely described by a topological equation

$$A' \phi + B' \theta = 0 \tag{4.3-6}$$

In order to keep the characteristics of the sensitivity system similar to those of the original system, their structures are arbitrarily made identical. That is

$$A' = A \qquad\qquad B' = B$$

The sensitivity structure is thus defined by

$$A \phi + B \theta = 0 \tag{4.3-7}$$

If (4.3-2) is differentiated with respect to the parameter q and the assumption made that no variation of the parameter changes the interconnections between system elements then

$$A \frac{\partial u}{\partial q} + B \frac{\partial v}{\partial q} = 0 \tag{4.3-8}$$

If this equation is compared with (4.3-7) it can be seen that under the restrictions made

$$\phi = \frac{\partial u}{\partial q} \qquad \Theta = \frac{\partial v}{\partial q} \tag{4.3-9}$$

Thus the variables in the sensitivity system are the desired sensitivity functions of the original system. It remains, however, to determine the characteristics of the elements in the sensitivity structure.

Differentiating the defining equation of the linear elements (4.3-1) with respect to the parameter q gives

$$\frac{\partial y_i}{\partial q} = \int_0^t h_i(t,\tau)\, \frac{\partial x_i}{\partial q}(\tau)\, d\tau \qquad i = r+1,\ldots,n \tag{4.3-10}$$

for the elements independent of the parameter and

$$\frac{\partial y_i}{\partial q} = \int_0^t h_i(t,\tau)\, \frac{\partial x_i}{\partial q}(\tau)\, d\tau + \int_0^t \frac{\partial h_i(t,\tau)}{\partial q}\, x_i(\tau)\, d\tau \tag{4.3-11}$$

$$i = 1, 2, \ldots, r$$

for the elements of the dynamic system that depend on the value of the q.

But using (4.3-9) and (4.3-3) these derivatives can be written as

$$\begin{aligned} \frac{\partial y_i}{\partial q} &= \frac{\partial v_i}{\partial q} = \Theta_i \\ \frac{\partial x_i}{\partial q} &= \frac{\partial u_i}{\partial q} = \phi_i \end{aligned} \tag{4.3-12}$$

Using these relations (4.3-10) and (4.3-11) become

$$\Theta_i = \begin{cases} \displaystyle\int_0^t h_i(t,\tau)\, \phi_i(\tau)\, d\tau + \int_0^t \frac{\partial h_i(t,\tau)}{\partial q}\, x_i(\tau)\, d\tau & i=1,2,\ldots,r \\ \\ \displaystyle\int_0^t h_i(t,\tau)\, \phi_i(\tau)\, d\tau & i = r+1,\ldots,n \end{cases} \tag{4.3-13}$$

This says simply that the elements in the sensitivity system are identical with those in the original dynamic system when those elements are independent of the parameter q. When the elements of the dynamic system do depend on the value of the parameter the elements in the sensitivity system are still the same as those in the original dynamic system with one exception. Added to the output of these elements is a signal formed by passing the input to the corresponding element in the dynamic system through a new element having the impulse response.

$$k_i(t,\tau) = \frac{\partial h_i(t,\tau)}{\partial q}(q_o) \tag{4.3-14}$$

When this structural method is applied to the system of Figure 4-4 and it is assumed that only the element h_3 depends on the parameter q the sensitivity structure of Figure 4-5 results. This example indicates the simplicity of the method and the ease with which it can be applied.

The one drawback to this method of obtaining the sensitivity coefficients is immediately obvious. It is necessary to know how the impulse response of the linear elements changes with variations in the parameter q in order to construct the k_i of (4.3-14). Once the system has been broken down to the elemental level, however, this information would usually be available or could be constructed experimentally with acceptable accuracy.

In summary, the sensitivity coefficients z_i of a physical system consisting of the interconnections of linear elements can be determined in the following manner. A sensitivity model is used which is identical to the model of the dynamic process except for the addition of extra signals at the output of each element which is dependent on the parameter q.

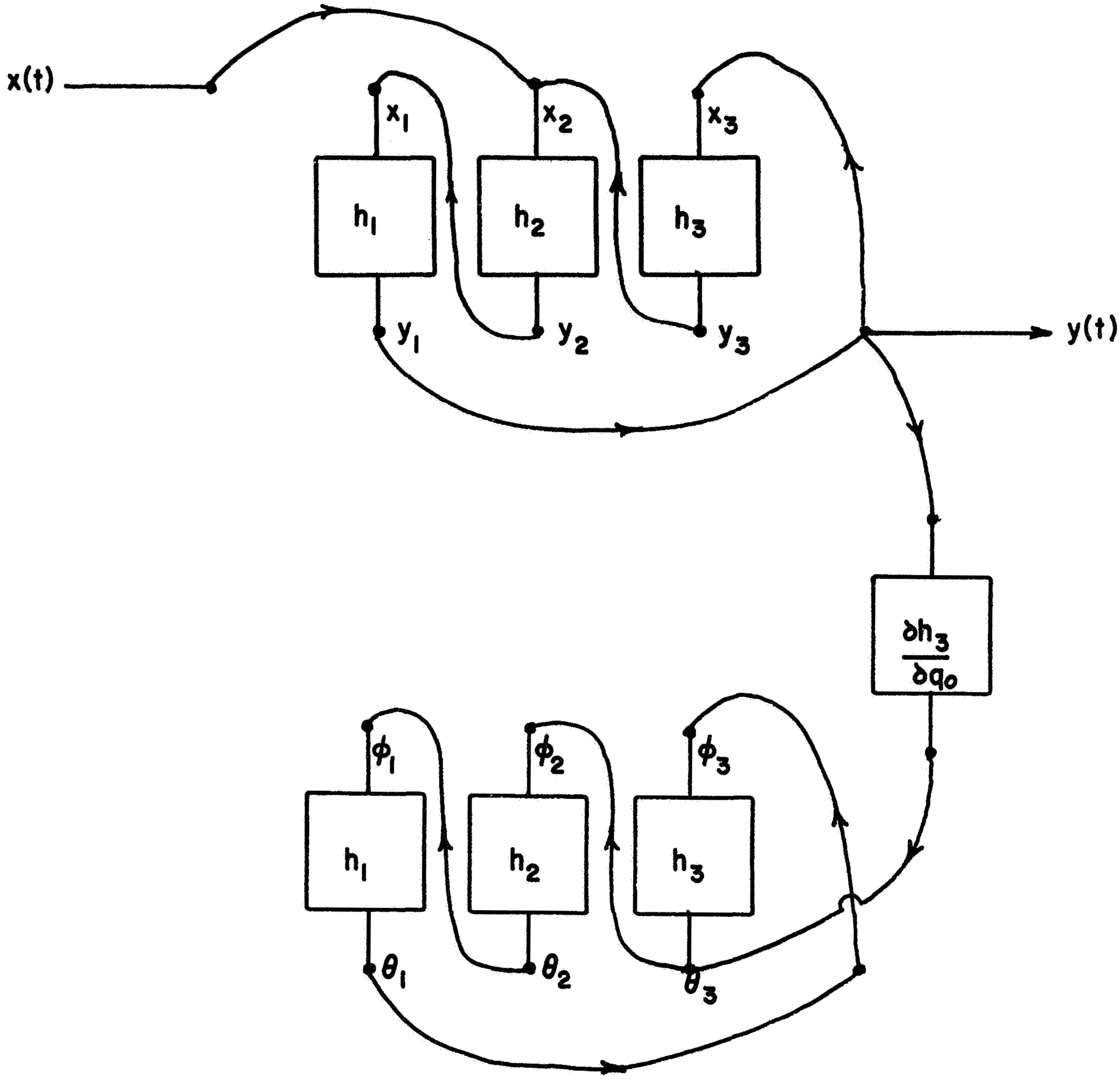


FIG. 4-5 SENSITIVITY SYSTEM

These signals are the inputs of the corresponding elements in the physical system passed through additional elements with the impulse response

$$k_i(t,\tau) = \frac{\partial h_i(t,\tau)}{\partial q_i}$$

Once the sensitivity coefficients have been constructed using the sensitivity system, the Analog Sensitivity Design technique of Section 4.2 can be applied directly with one unfortunate difference. When dealing with the physical system rather than an analog simulation it is in general impossible to speed up the solution. Each iteration of Figure 4-1 must take as long as the system settling time. This is the price that must be paid for the design of a desensitized system without knowledge of the differential equations describing the over-all system.

4.4 Performance Index Sensitivity

After a design has been accomplished it is of interest to determine just how insensitive the system has been made. The main measure of this is of course the relative magnitude of the sensitivity coefficients before and after the desensitization. Some times, however, the relative sensitivity of the performance index is of interest, especially in those cases when the index has a valid physical interpretation. Analog Sensitivity Design extends to this problem quite easily.

Since the J portion of the criterion of the sensitivity design problem (4.2-1) is determined by the state response it is the part whose sensitivity is of interest. If the control law $m = - K^T x$ is substituted into the criterion it becomes

$$J = \int_0^{t_f} x^T(S + K^T RK)\, x\, dt \tag{4.4-1}$$

This can be differentiated with respect to the parameter q to give the performance index sensitivity

$$\frac{\partial J}{\partial q}(q_o) = \int_0^{t_f} \left[\frac{\partial x^T}{\partial q} (S + K^T RK)x + x^T(S + K^T RK) \frac{\partial x}{\partial q} \right] dt \tag{4.4-2}$$

The vector $\frac{\partial x}{\partial q}$ is recognized as the sensitivity vector z. Equation (4.4-2) can then be rewritten as

$$\frac{\partial J}{\partial q}(q_o) = \int_0^{t_f} z^T \left[(S + K^T RK) + (S + K^T RK)^T \right] x\, dt \tag{4.4-3}$$

If, as is most often the case, S and R are diagonal matrices, or at least symmetrical, this becomes

$$\frac{\partial J}{\partial q}(q_o) = 2 \int_0^{t_f} z^T(S + K^T RK)x\, dt \tag{4.4-4}$$

Equation (4.4-4) allows the computation of the performance index sensitivity at the nominal value of the parameter $q = q_o$ once the state and sensitivity vectors have been calculated. Using Analog Sensitivity Design permits calculation of (4.4-4) simultaneously with the state sensitivity coefficients. If desired, this performance index sensitivity integral can be adjoined to J and $\hat{J}$ to form a new index of performance when maximum flatness of the original index J is desired.

This brings to light an important point. When dealing with Nature every improvement has its price and sensitivity design is no exception. The price paid for decreased system sensitivity to parameter changes is degradation in the response of the system when parameters have their nominal values. The system designed solely on the basis of nominal parameters will be "better" in some sense for these nominal values and in a small neighborhood of them. There is a certain critical parameter variation, however, beyond which the system designed on the basis of sensitivity concepts will be "better". If the performance index J is taken as a measure of the worth of a system this trade off can be expressed in the two curves of J vs. q sketched in Figure 4-6. This shows that the performance index of the system designed using sensitivity techniques is flatter than that of the system designed without them, termed the optimal system, but its minimum value is not as low. For this example the optimal system is best for $q \in (q_1, q_2)$ and the sensitivity design system is better for q outside this range.

4.5 Trimming the Design

One of the unique advantages of the Analog Sensitivity Design Technique is the ease with which an initial design arrived at by analytical or iterative means can be "trimmed" to meet actual response sensitivity requirements. As mentioned above, one of the major disadvantages of optimal control or any method which arrives at a system design by minimization of an index of performance is the difficulty of selecting a meaningful performance criterion. As presented here, the automatic iteration scheme of Analog Sensitivity Design also suffers from this problem. However when the design process of Figure 4-2 has converged on a gain vector K^* which minimizes the performance index $I = J + \hat{J}$ it is then a simple matter to trim these gains about K^*

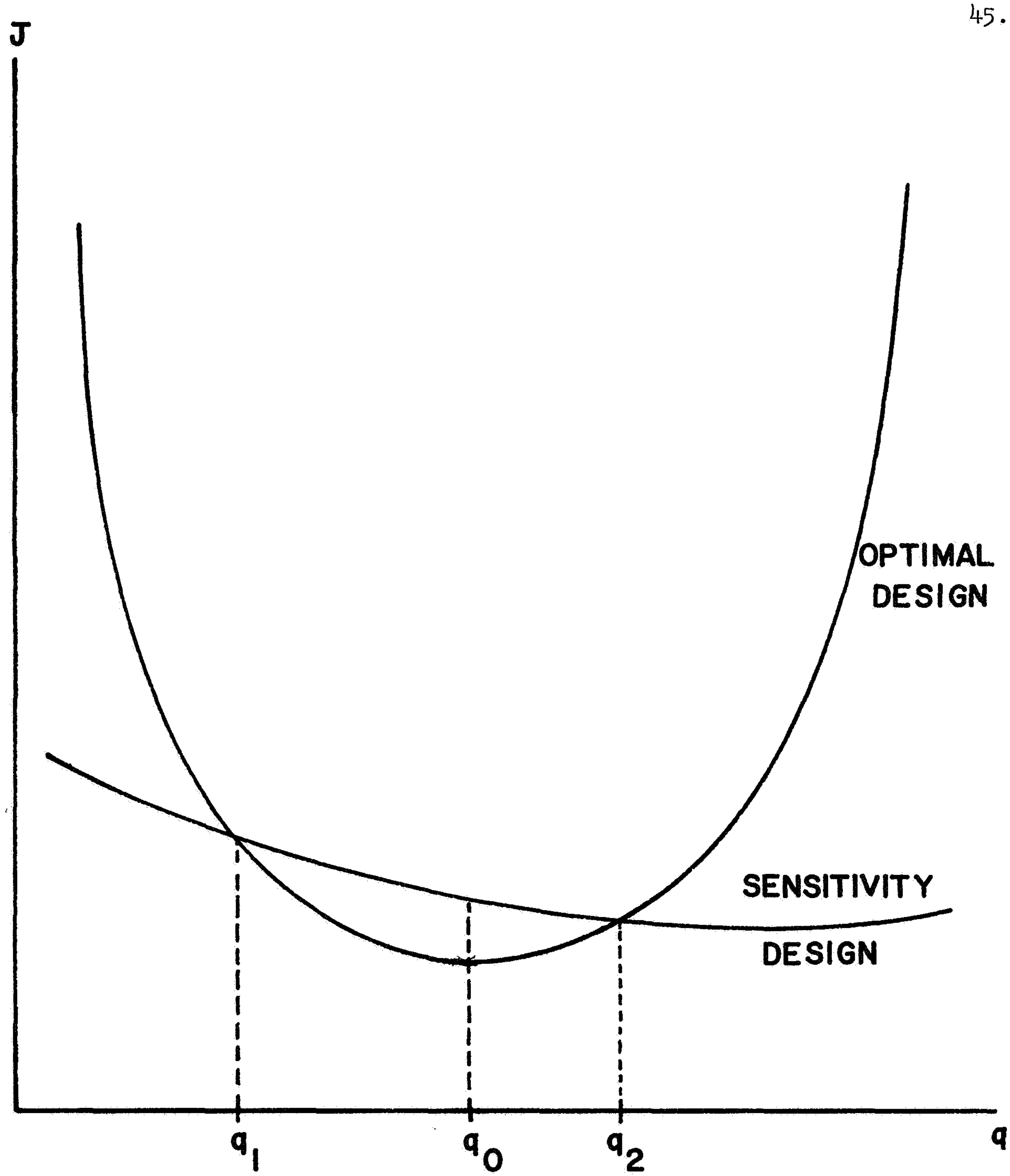


FIG. 4-6 COMPARISON OF PERFORMANCE INDICES

manually while observing the effect directly on an output device such as an oscilloscope. Since both the state and sensitivity equations are being solved simultaneously the designer can examine both the nominal response and the sensitivity coefficients. If (4.4-4) is implemented the initial slope of the performance index J at $q = q_o$ can also be displayed. Using all of this information the designer can employ his judgement and experience with similar systems to arrive at a final set of gains that afford the best compromise between nominal response and insensitivity to parameter variations.

4.6 Second Order Example

In order to illustrate the method of Analog Sensitivity Design and compare the resulting system with other techniques the following example was used:

Consider the second order differential equation given by (4.6-1)

$$\dot{x} = \begin{bmatrix} q-1 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} m \tag{4.6-1}$$

$$x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Assuming the nominal value of the parameter q is equal to one, $q_o = 1$, makes (4.6-1) a harmonic oscillator with nominal damping equal to zero. The control input $m(t)$ is to be formed by feedback from displacement and rate variables $x_2(t)$ and $x_1(t)$ respectively.

$$m = -\begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.6-2}$$

Using equation (4.6-1) in the general form of the sensitivity equation for linear systems with state feedback control (4.2-3) gives the equation for the sensitivity coefficients

$$\dot{z} = \left[\begin{pmatrix} q_o-1 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} K^T \right] z + \frac{\partial}{\partial q} \begin{pmatrix} q_o-1 & -1 \\ 0 & 1 \end{pmatrix} x \qquad (4.6\text{-}3)$$

Using the nominal value of the parameter $q_o = 1$ in (4.6-1) and (4.6-3), the control law (4.6-2), and performing the indicated differentiation yields the final nominal state equation (4.6-4) and sensitivity equation (4.6-5)

$$\dot{x} = \begin{pmatrix} -K_1 & -(K_2+1) \\ 1 & 0 \end{pmatrix} x \qquad x(0) = \begin{matrix} 0 \\ -1 \end{matrix} \qquad (4.6\text{-}4)$$

$$\dot{z} = \begin{pmatrix} -K_1 & -(K_2+1) \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x \qquad z(0) = \begin{matrix} 0 \\ 0 \end{matrix} \qquad (4.6\text{-}5)$$

Note that as required, the homogeneous part of the sensitivity equation is identical to the state equation.

The performance index selected to represent the desired response of the system (4.6-4) is

$$J = \frac{1}{2} \int_0^{10} x^T \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} x + m^2 \quad dt \qquad (4.6\text{-}6)$$

The sensitivity vector is given considerably higher weighting than the state vector in order to make the system insensitive to changes in q, the damping parameter, by requiring the magnitude of the sensitivity vector to be kept small.

$$\hat{J} = \frac{1}{2} \int_0^{10} z^T \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} z \, dt \qquad (4.6\text{-}7)$$

This problem was first solved using digital parameter optimization without consideration of the sensitivity index (4.6-7) or constraining equation (4.6-5). The resulting system response for various parameter values, shown in Figure 4-7 and Figure 4-8, will serve as a basis with which desensitization can be compared. The optimal gains were computed to be

$$K^* = \begin{pmatrix} 2.93 \\ 2.44 \end{pmatrix} \qquad (4.6\text{-}8)$$

Figure 4-7 and Figure 4-8 show the rate and displacement initial condition response respectively for $q = 0.8$, 1.0, and 1.2. It is obvious that there is considerable trajectory dispersion for this range of parameter values. Figure 4-9 shows the response of the sensitivity coefficients for this undesensitized case. These were obtained by the solution of the sensitivity equation (4.6-5) simultaneously with the state equation using the gains (4.6-8).

Next the sensitivity problem, with the sensitivity index included, was solved using Dougherty's digital parameter optimization scheme described in Section 3.4. Recall that this technique solves the two point boundary value problem (3.4-4) resulting from the application of the equations of parameter optimization using a first order gradient method.

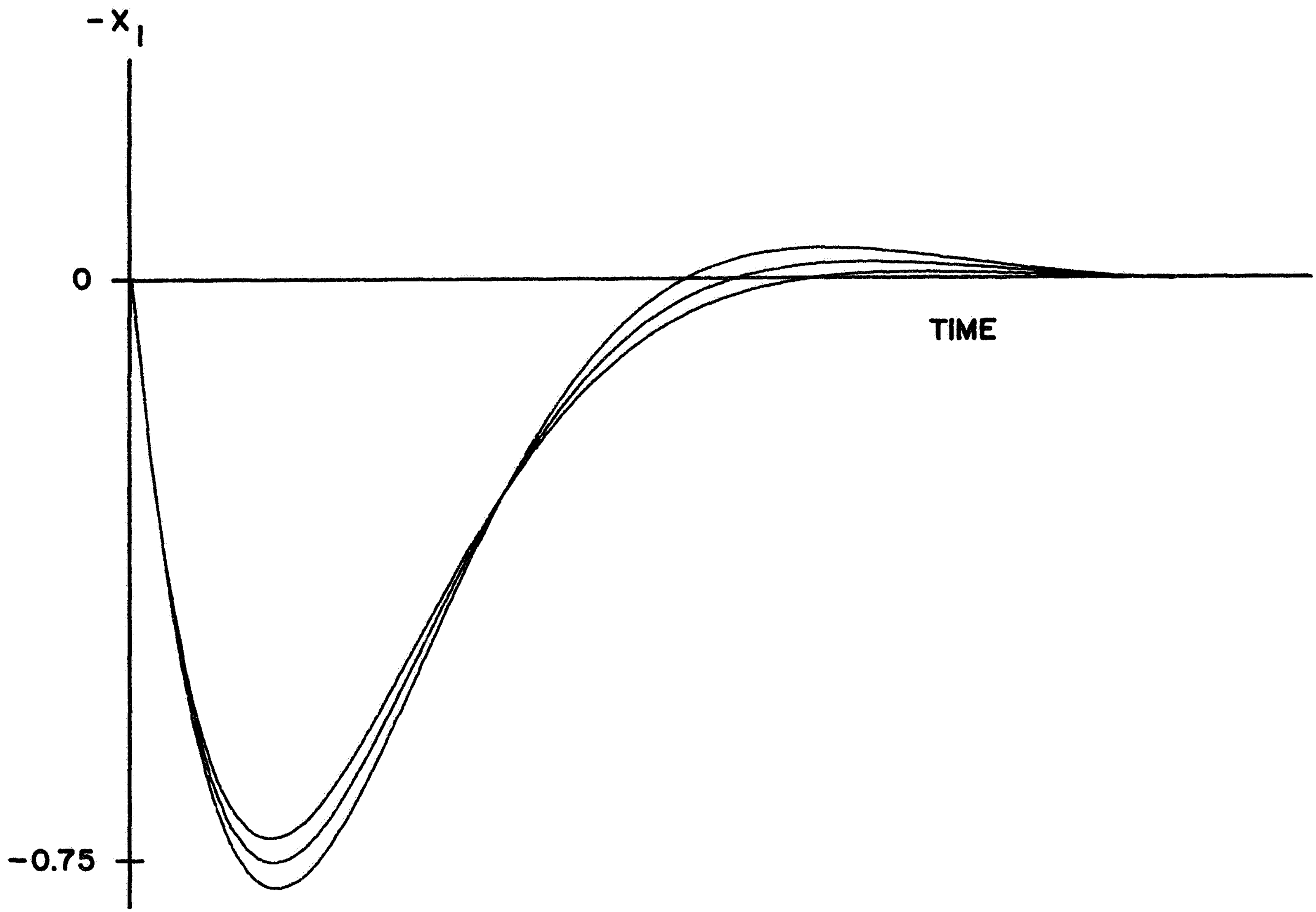


FIG. 4-7 OPTIMAL SYSTEM RATE

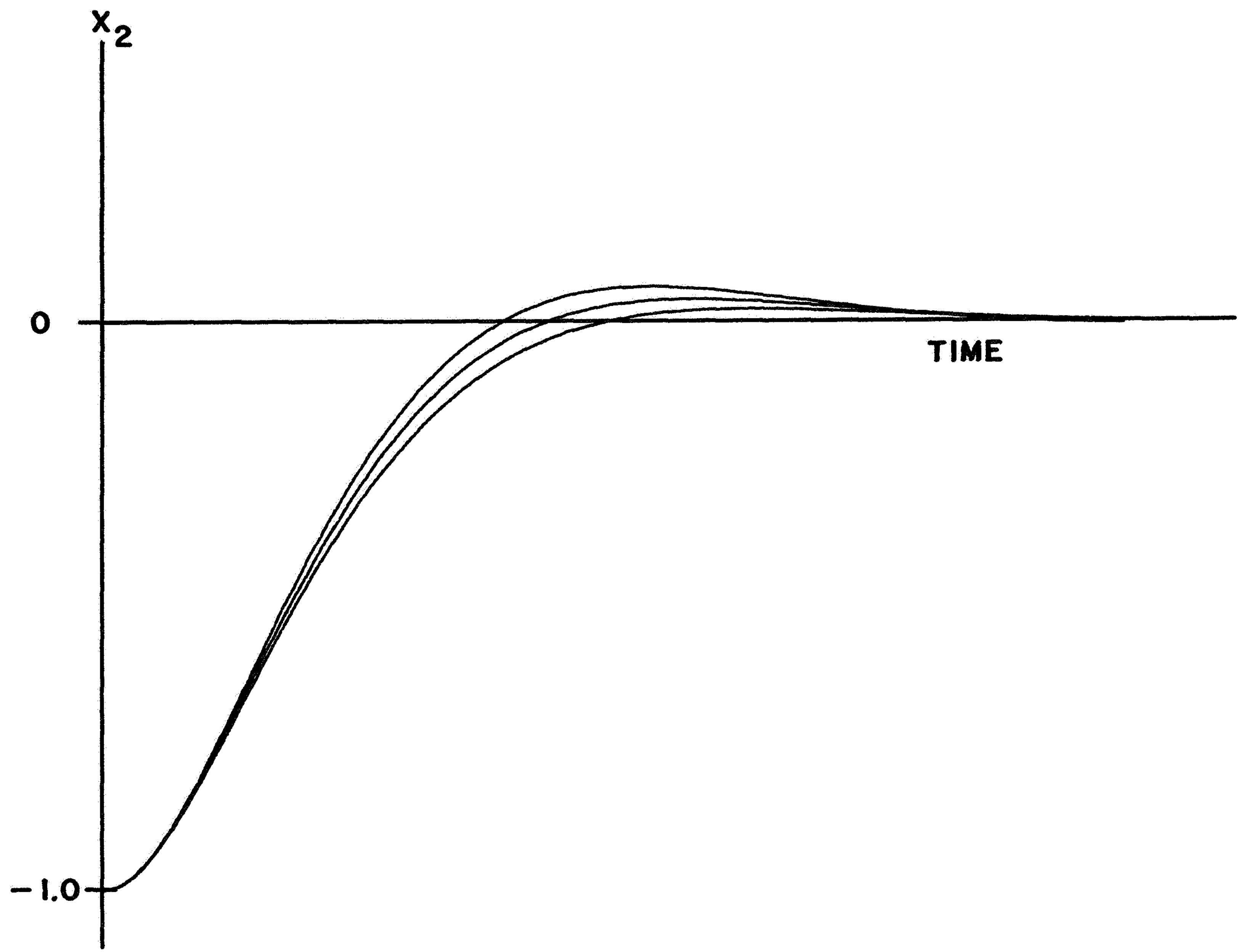


FIG. 4-8 OPTIMAL SYSTEM DISPLACEMENT

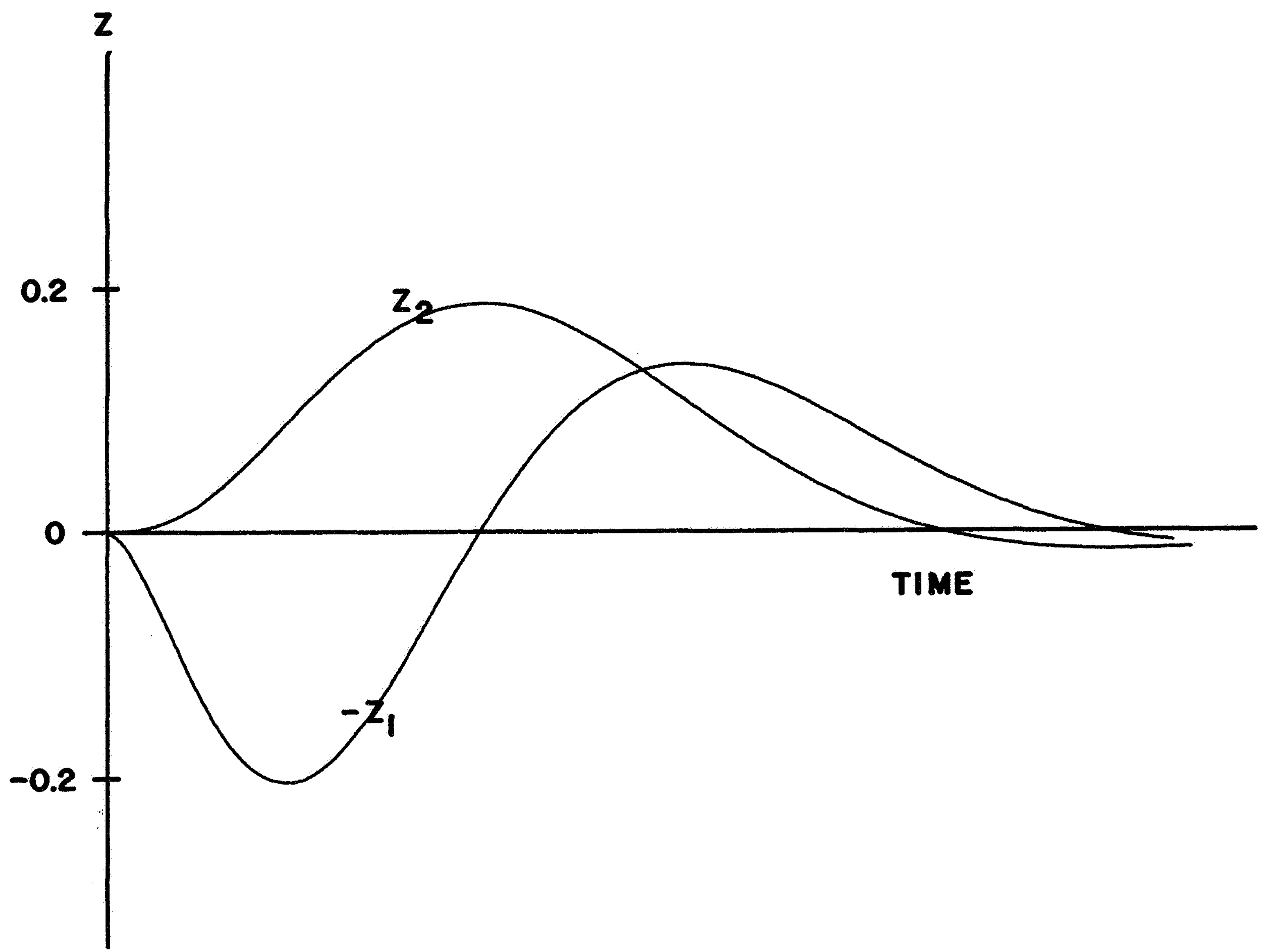


FIG. 4-9 OPTIMAL SYSTEM SENSITIVITY

Starting from the origin in gain space with both K_1 and K_2 equal to zero, convergence to the optimal gains took approximately 15 minutes on an IBM 360/50 digital computer. These optimal gains were

$$K^* = \begin{pmatrix} 4.16 \\ 2.76 \end{pmatrix} \tag{4.6-9}$$

The initial condition response of the system using these gains is shown in Figure 4-10 and Figure 4-11. Now the trajectory dispersion is somewhat less than before so the system has been desensitized to some extent. It can also be seen, however, that the system is somewhat slower responding than before. This is the price that has been paid for the desensitization; nominal response has suffered. Figure 4-12 shows the sensitivity coefficients for this system and their reduction in magnitude over those of Figure 4-8 is immediately apparent.

The identical problem was next solved using the Analog Sensitivity Design Technique. The simulation diagram is shown in Figure 4-13. Because of the lack of logic capability on the two EAI TR 20's available, the Analog Sensitivity Design Technique could not be automated as described in Section 4.2. However, manual solution of the problem was carried out exactly as if the flowchart of Figure 4-2 had been implemented directly for this system. As in the digital case the gains were initialized at zero. The initial step size was taken as $K = 0.5$. Using this step size the algorithm converged in 19 iterations to

$$K = \begin{pmatrix} 4.0 \\ 3.0 \end{pmatrix} \tag{4.6-10}$$

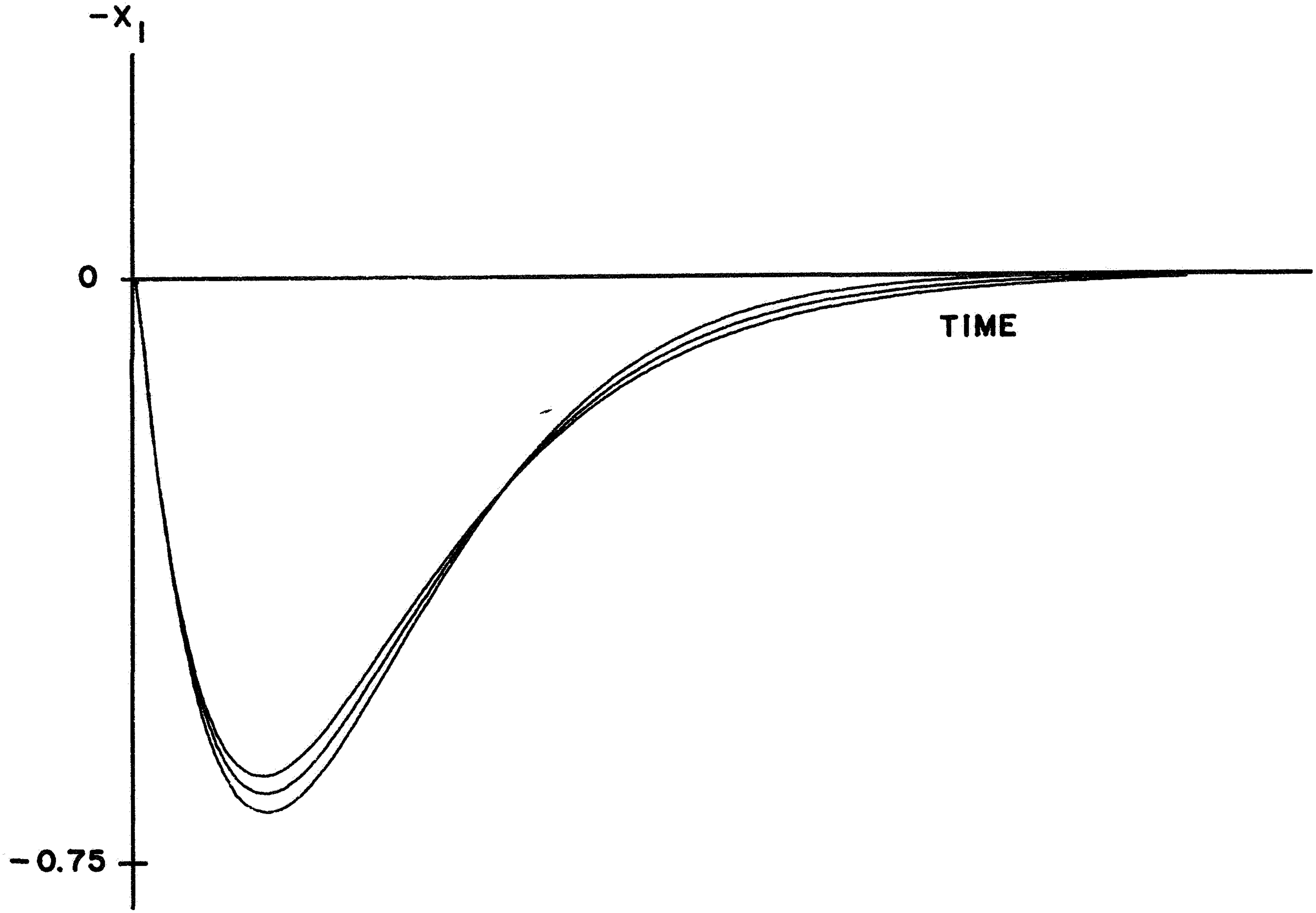


FIG. 4-10 DIGITAL SENSITIVITY DESIGN RATE

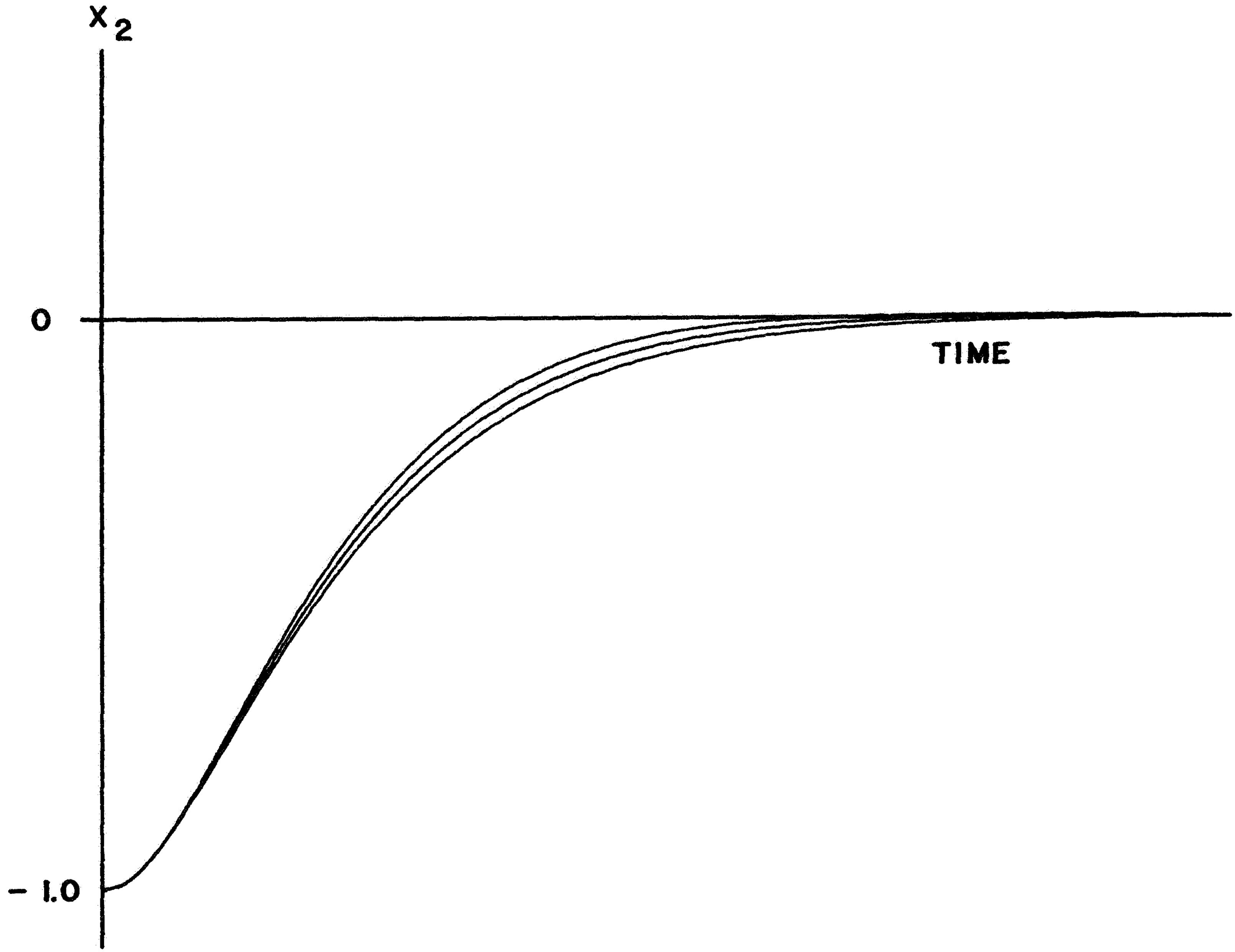


FIG. 4-11 DIGITAL SENSITIVITY DESIGN DISPLACEMENT

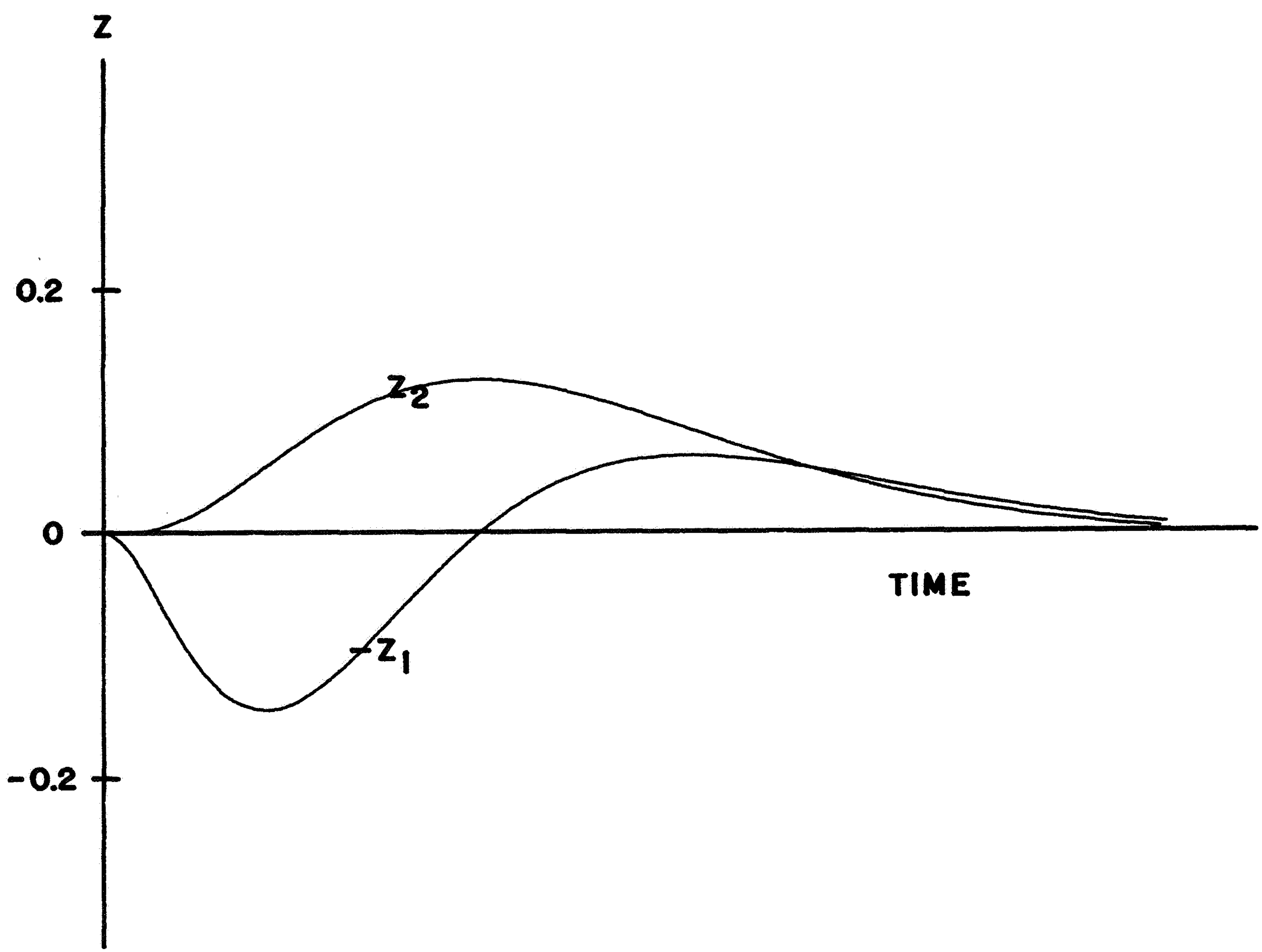


FIG. 4-12 DIGITAL SENSITIVITY DESIGN SENSITIVITY

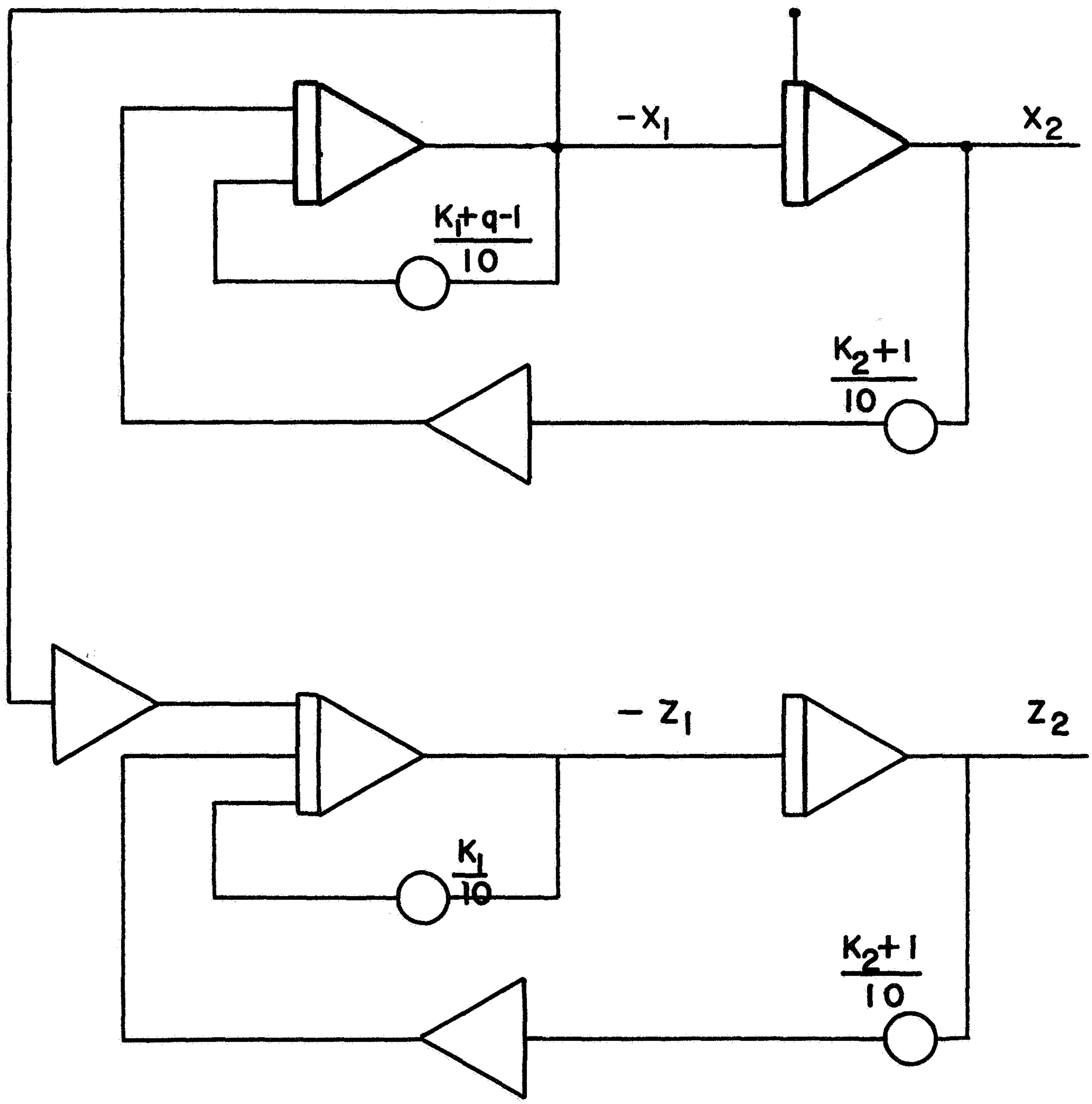


FIG. 4-13 SIMULATION DIAGRAM

The step size was then reduced to $K = 0.1$ and starting from the gains of (4.6-10) convergence to

$$K = \begin{pmatrix} 4.2 \\ 2.8 \end{pmatrix} \tag{4.6-11}$$

took 6 iterations. This convergence to two significant figures took a total of 25 iterations. Since the repetitive operation mode of the EAI TR 20 speeds up solutions by a factor of 500, these 25 iterations if performed automatically would take about 0.5 seconds. This figure does not of course allow for the time necessary to change the gain potentiometer settings but even so, it makes dramatically clear the advantage of Analog Sensitivity Design over other methods. Even with manual adjustment of the gains, the solution took less than two minutes on the analog computer.

The resulting trajectories of the Analog Sensitivity Design are exactly the same as those of the parameter optimization method since the resulting gains were essentially the same. The only advantage up to this point has been increased speed of solution. Now, however, it is possible to trim the values of the feedback gains manually to further improve the design without having to experiment with many different sets of weighting matrices. The following figures show the results of trimming the gains while observing the effect on the nominal response and sensitivity coefficients and selecting those which gave the best compromise between adequate nominal response and minimization of the sensitivity coefficients. These compromise gains are

$$K^* = \begin{pmatrix} 6.0 \\ 3.7 \end{pmatrix} \tag{4.6-12}$$

Figure 4-14 and 4-15 show the initial condition response of the system for various values of the damping parameter q. It is readily seen that these responses are less sensitive to variations in q than were those of the system shown in Figure 4-10 and Figure 4-11 using the gains of (4.6-9). The sensitivity coefficients are shown in Figure 4-16 and as expected they have the smallest magnitude of the three systems.

It is interesting to compare the eigenvalue sensitivities for the two desensitized systems using the method of Section 2.4. Recall that the eigenvalue sensitivity of the system

$$\dot{x} = Ax \tag{4.6-13}$$

is given by

$$\frac{\partial \lambda_i}{\partial q}(q_o) = \frac{v_i^T(q_o)\,\frac{\partial A}{\partial q}(q_o)\,u_i(q_o)}{v_i^T(q_o)\,u_i(q_o)} \tag{4.6-14}$$

where (λ_i, v_i) are the eigenvalue-eigenvector pairs of the matrix A and the u_i are the corresponding eigenvectors of the matrix A^T.

For the present example the gains (4.6-9) the A matrix is

$$A = \begin{pmatrix} -4.16 & -3.76 \\ 1 & 0 \end{pmatrix} \tag{4.6-15}$$

and the eigenvalue sensitivities which are computed in Appendix C are

$$\frac{\partial \lambda_i}{\partial q}(q_o) = 1.9 \qquad \frac{\partial \lambda_2}{\partial q}(q_o) = -\ 0.9 \tag{4.6-16}$$

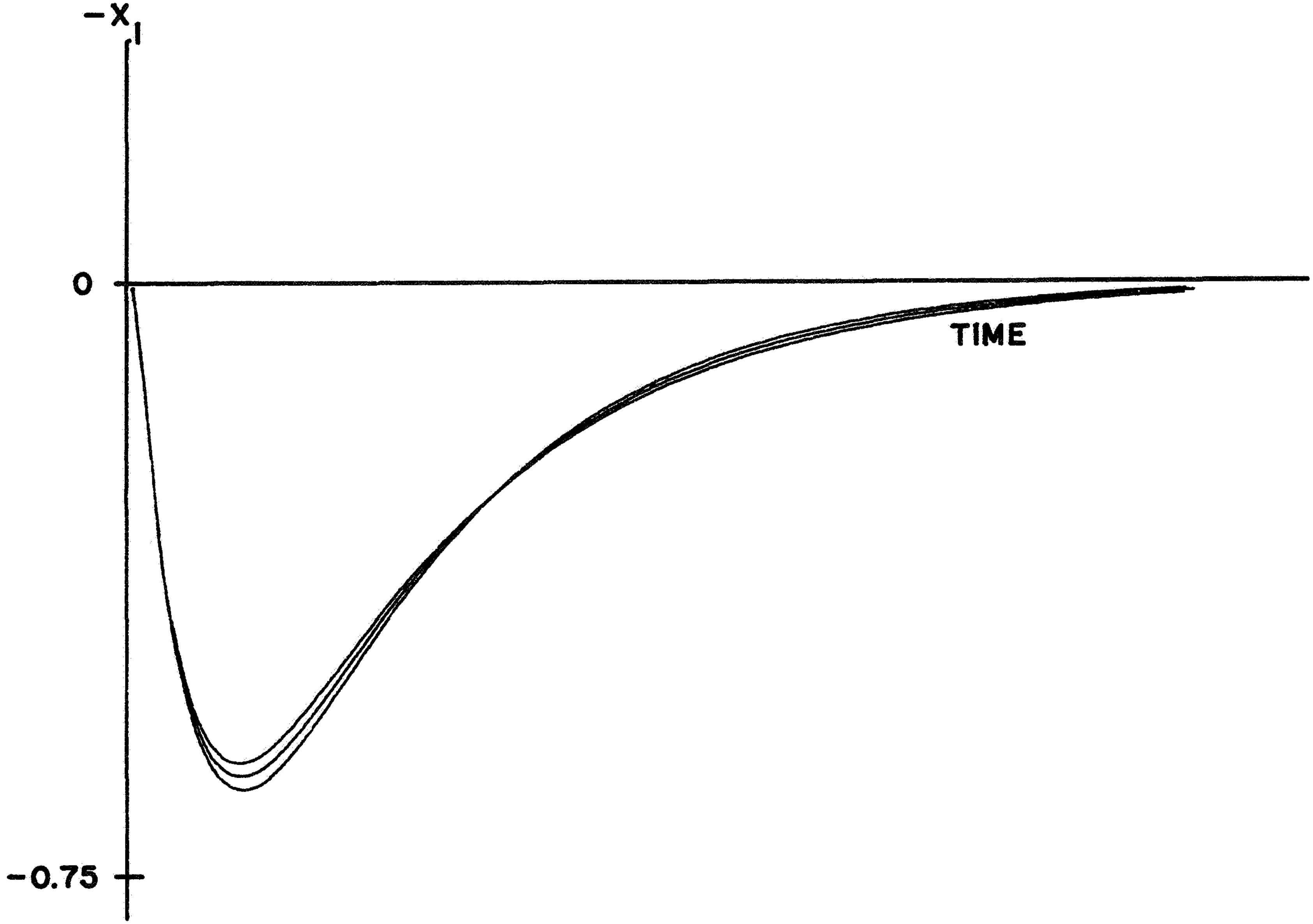


FIG. 4-14 ANALOG SENSITIVITY DESIGN RATE

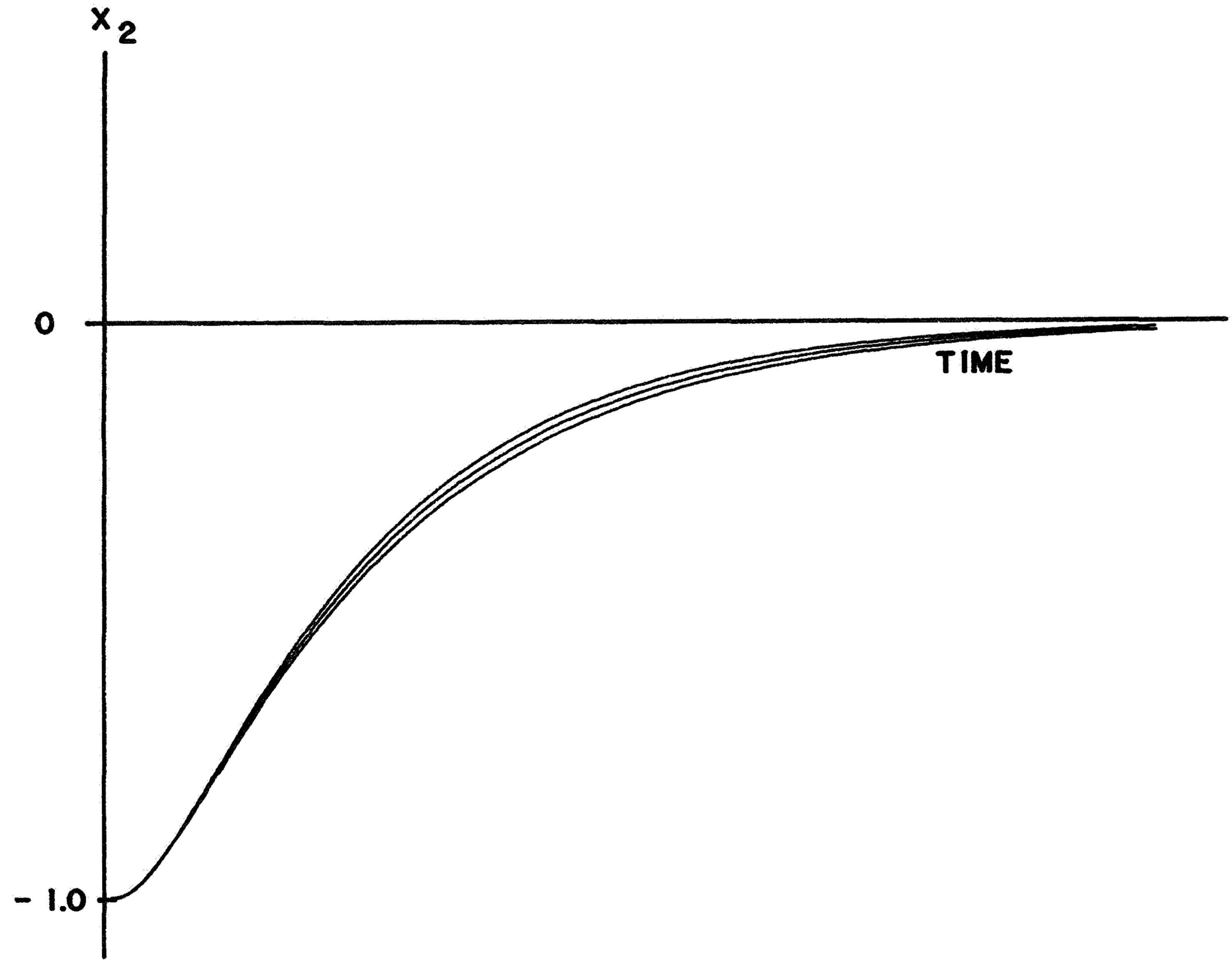


FIG. 4-15 ANALOG SENSITIVITY DESIGN DISPLACEMENT

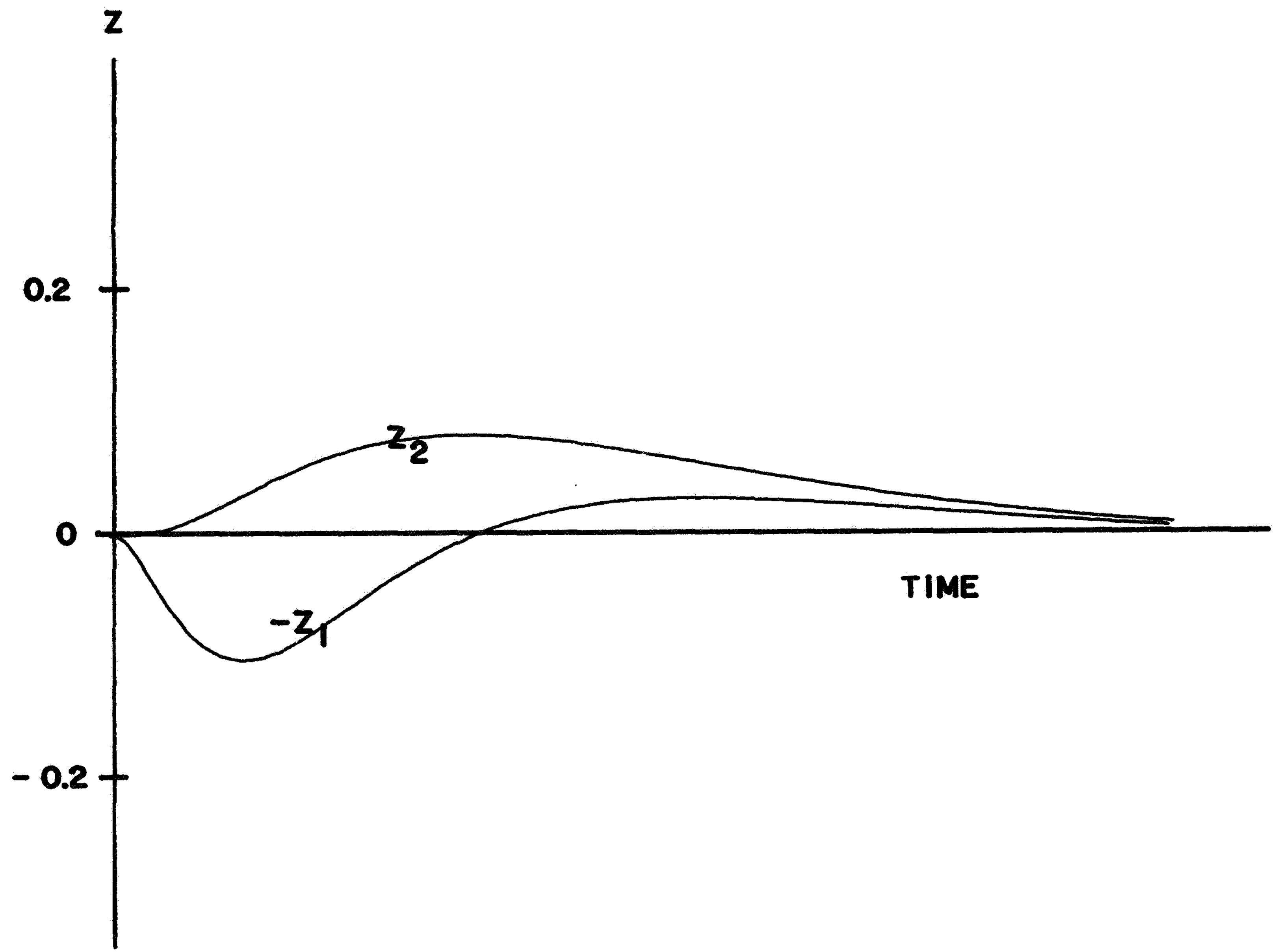


FIG. 4-16 ANALOG SENSITIVITY DESIGN SENSITIVITY

Similarly, for the gains (4.6-12) obtained by using Analog Sensitivity Design the eigenvalue sensitivities are

$$\frac{\partial \lambda_1}{\partial q}(q_o) = 1.2 \qquad\qquad \frac{\partial \lambda_2}{\partial q}(q_o) = -\ 0.2 \qquad\qquad (4.6\text{-}17)$$

Thus as could be expected by comparing the trajectories of the two systems, trimming the gains manually after the automatic iterative design was complete improved the eigenvalue insensitivity of the system (4.6-1) as well as the trajectory insensitivity. These two improvements appear to go hand in hand.

CHAPTER 5

CASE STUDY: FLEXIBLE BOOSTER CONTROL

5.1 General

In order to demonstrate the utility of the Analog Sensitivity Design Technique developed in Chapter 4, a realistic case study was undertaken. This chapter examines the problem of developing a feedback control system for a large flexible booster and desensitizing it to variations in bending mode frequency. Analog Sensitivity Design is used on a frozen time point model of the flexible booster to design the control system. The resulting system is tested by digital simulation using a time varying model excited by a worst case design wind. This design wind is constructed to excite any instabilities that are inherent in the system design.

5.2 The Problem[25]

As launch vehicles become progressively larger and more complex it becomes more and more difficult to determine the exact values of the many parameters which effect the performance characteristics. One of the most difficult sets of data to obtain are those relating to the flexural modes of the vehicle. It is well known from elementary mechanics that as the length of an object is increased and its diameter decreased, bending response to any off axial forces becomes more pronounced. Typically the length to diameter ratio of today's launch vehicles is ten to one or higher. Thus the vehicles are quite flexible and this characteristic must be taken into account in the design of a control system. If it is not taken into account then the structural loading in regions of high dynamic pressure may be such that the elastic limit of the structure is exceeded and the vehicle destroyed.

Until now the bending characteristic of the vehicle have been determined by dynamic testing of the actual booster. This involves mounting the entire vehicle in a huge tower and shaking it. The resulting deflections at different stations along the vehicle are recorded and a complete bending parameter analysis performed. The tremendous size of the Saturn V - Apollo configuration shown in Figure 5-1 makes this operation just marginally possible. For larger vehicles it may not be possible. Even for the Saturn V - Apollo, however, changing mission requirements and changing payloads cause the actual bending characteristics to differ slightly in each vehicle. Thus the bending characteristics, particularly the natural frequency of each mode, may not be known accurately enough for successful control of the vehicle. This is where Analog Sensitivity Design can be used. The problem is to design a control system for the Saturn V - Apollo that gives adequate control when the bending frequency may differ from the nominal value by as much as 20%.

To complicate matters further, the rigid body mode of the vehicle is aerodynamically unstable. This is a result of the center of pressure being forward of the center of gravity. The aerodynamic forces tend to rotate the vehicle away from the nominal trajectory. Figure 5-2 shows plots of the center of pressure and center of gravity during the boost phase of the flight. The vehicle is obviously unstable for all but a short time around 60 seconds where the center of pressure briefly moves

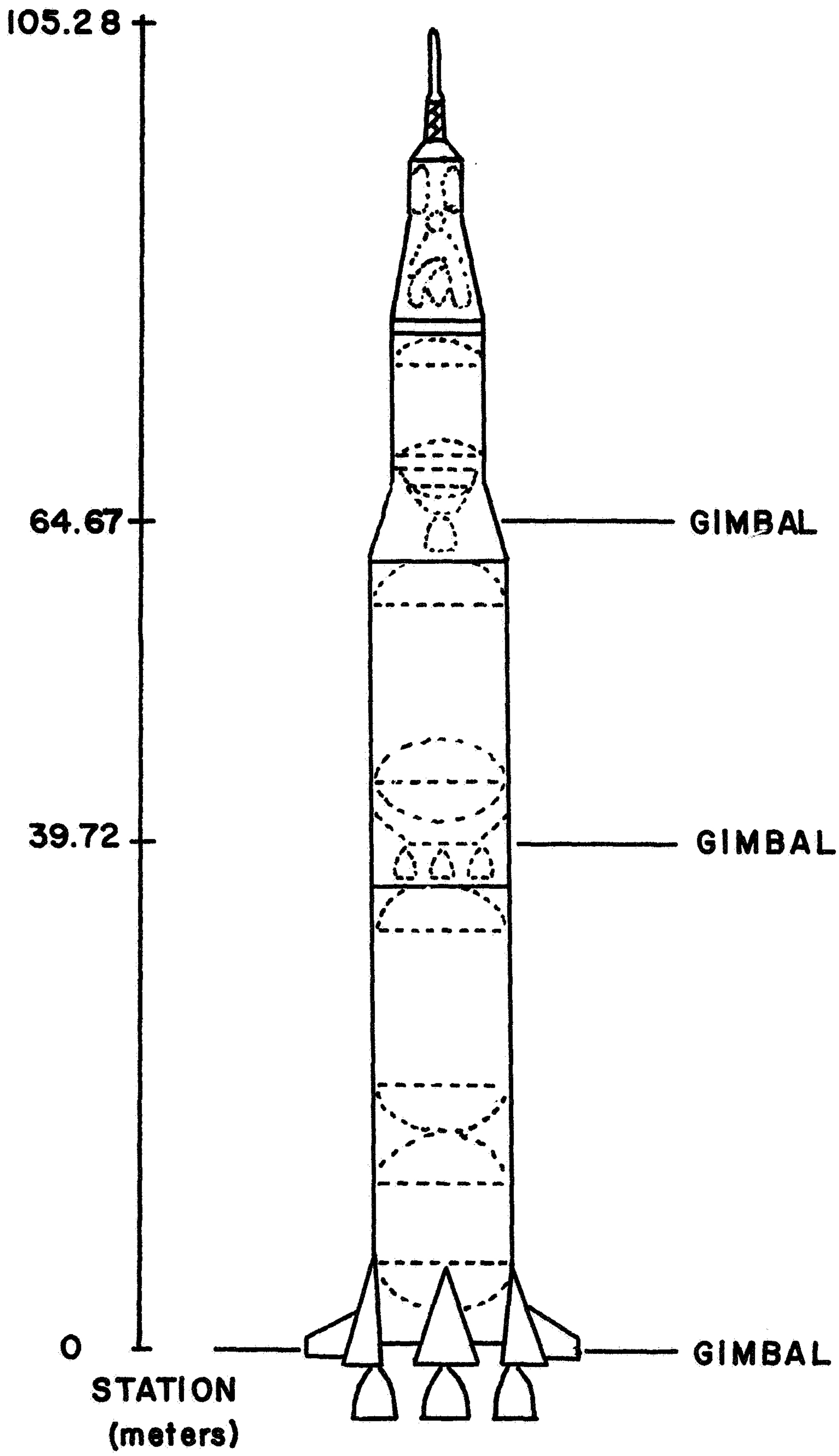


FIG. 5-1 VEHICAL CONFIGURATION

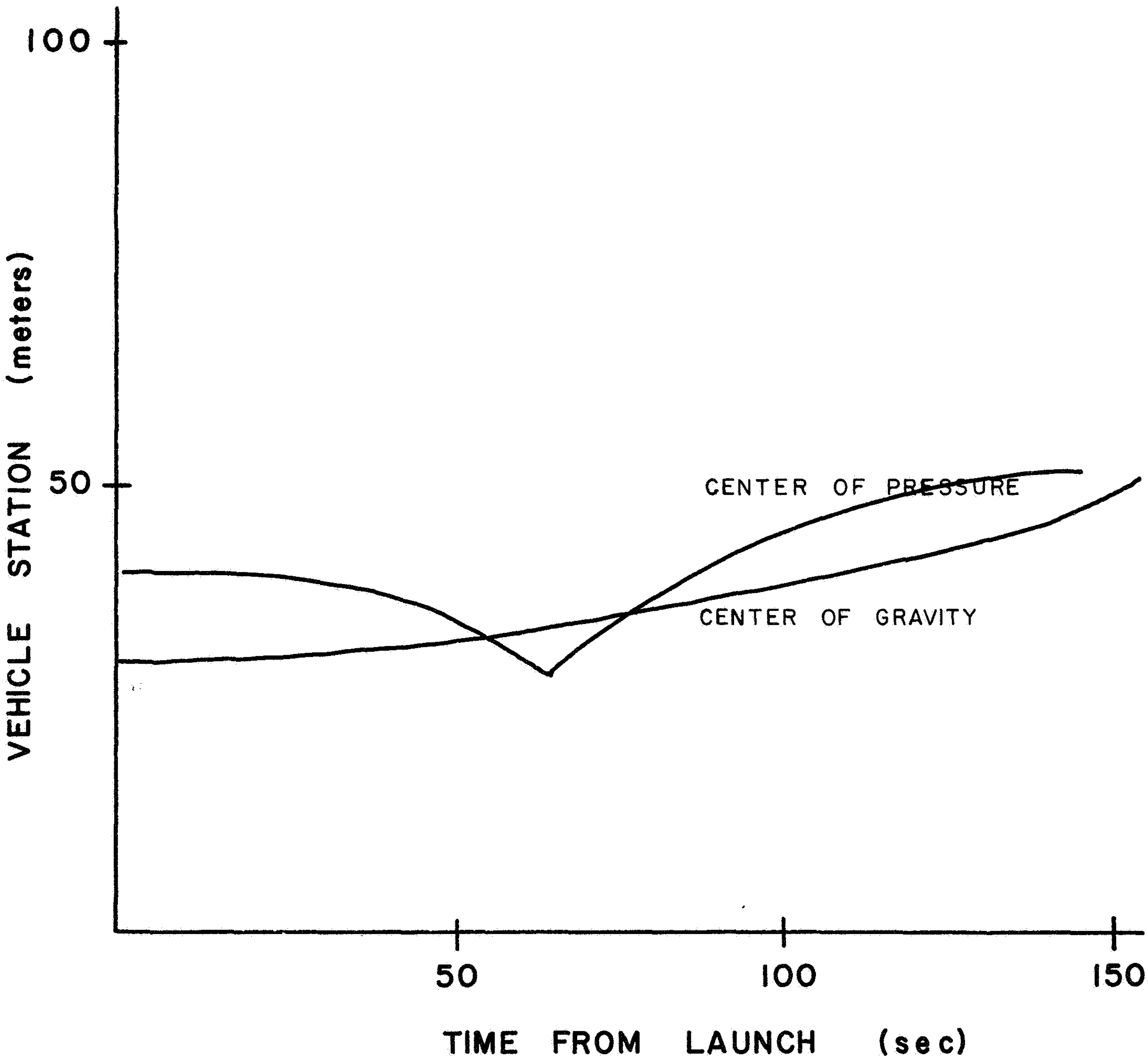
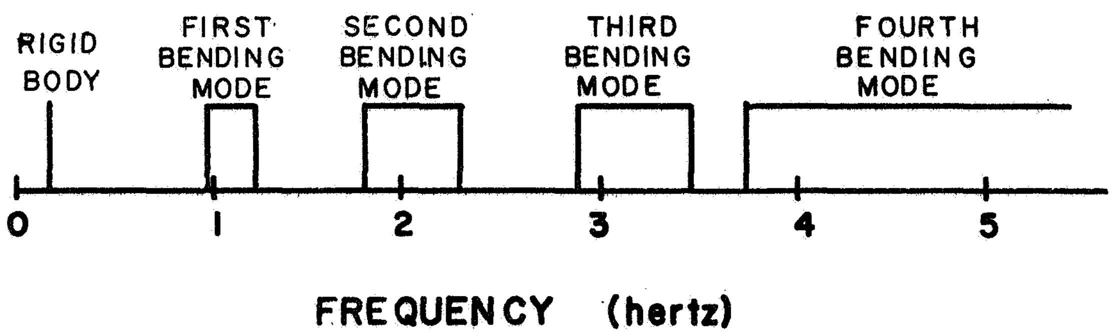


FIG. 5-2 CENTERS OF PRESSURE AND GRAVITY

off of the center of gravity. Thus continuous engine gimbal angle control must be used to keep the vehicle in the nominal orientation.

Figure 5-3 shows the frequency spectrum of the Saturn V - Apollo during the boost phase. This spectrum shows the frequencies of all important modes of the rigid, bending and sloshing bodies. The slosh modes, which will not be considered here, are a result of the fuel moving in the tanks as the vehicle flies. The frequencies are spread out into bands rather than lines at particular frequencies because the dynamic characteristics of the vehicle change with time.

In general there are two types of feedback control systems that can be considered for the Saturn V: drift minimum and load relief. The Drift minimum system takes as its major objective the control and minimization of lateral drift away from the reference trajectory. This involves the use of pitch, pitch rate and lateral velocity feedback. This type of control is used where flexural loading does not play an important part and lateral drift is detrimental to the mission. Unfortunately, under certain conditions a drift minimum control system can cause excessive structural loads on the Saturn V. Thus it is necessary to go to a load relief type of control system. The simplest of these uses only pitch and pitch rate feedback to control the vehicle. This allows the vehicle to drift with the wind avoiding the buildup of large bending moments. Other types of load relief systems are possible, but this one involving only pitch and pitch rate feedback will be considered here because of its simplicity. Also primarily for simplicity a control system using constant feedback gains for the entire flight



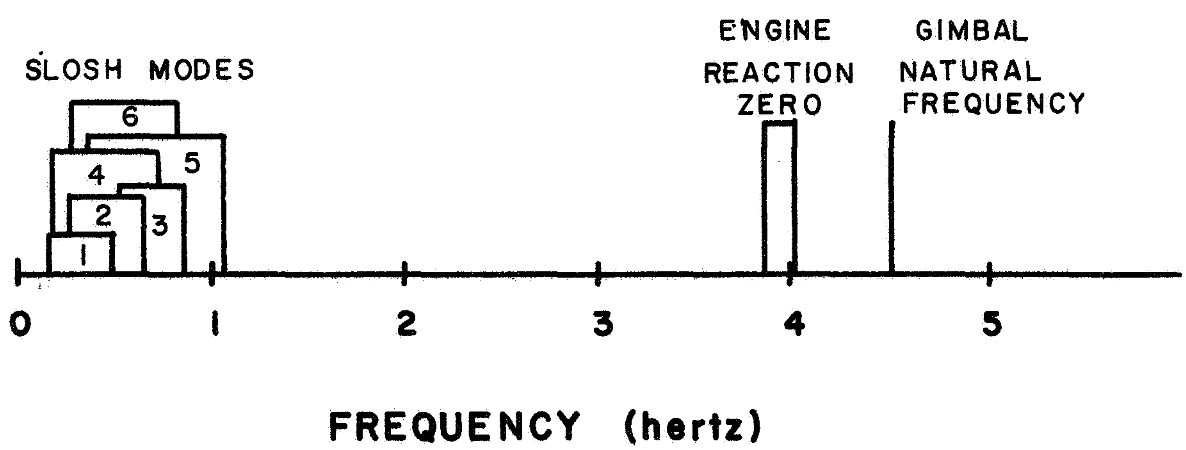


FIG. 5-3 BOOSTER FREQUENCY SPECTRUM

will be sought. Several control schemes using programmed gains have been developed but this will not be attempted here. As will be seen later in this chapter it is possible to find an entirely successful control system desensitized to changes in bending mode frequency using constant feedback gains.

5.3 Equations of Motion[7,25]

The first step in the development of a control system for the Saturn V - Apollo is the derivation of the differential equations describing the behavior of the vehicle. First the rigid body equations are derived assuming a flat earth and considering only the pitch plane of the vehicle.

As usual in this type of problem it is necessary to work with several co-ordinate systems. The first of these has its origin at the launch point with its X and Y axes at the local horizontal and local vertical respectively. This is known as the inertial co-ordinate system. The second set of axes moves with the origin at the vehicle center of gravity. These are the x-y axes with the x axis lying along the center line of the vehicle and the y axis perpendicular to it in the pitch plane. This is the body fixed co-ordinate system. A third co-ordinate system has its X_n axis tangential to the nominal trajectory and its Y_n axis perpendicular to it in the pitch plane. This is called the nominal co-ordinate system. Figure 5-4 shows these co-ordinate systems with a free body diagram of the vehicle.

Summing forces in the X_n direction gives

$$F_{X_n} = (F+R' \cos \beta - D) \cos \phi - N \sin \phi - R' \sin \beta \sin \phi - mg \cos (x_c - x) \qquad (5.3\text{-}1)$$

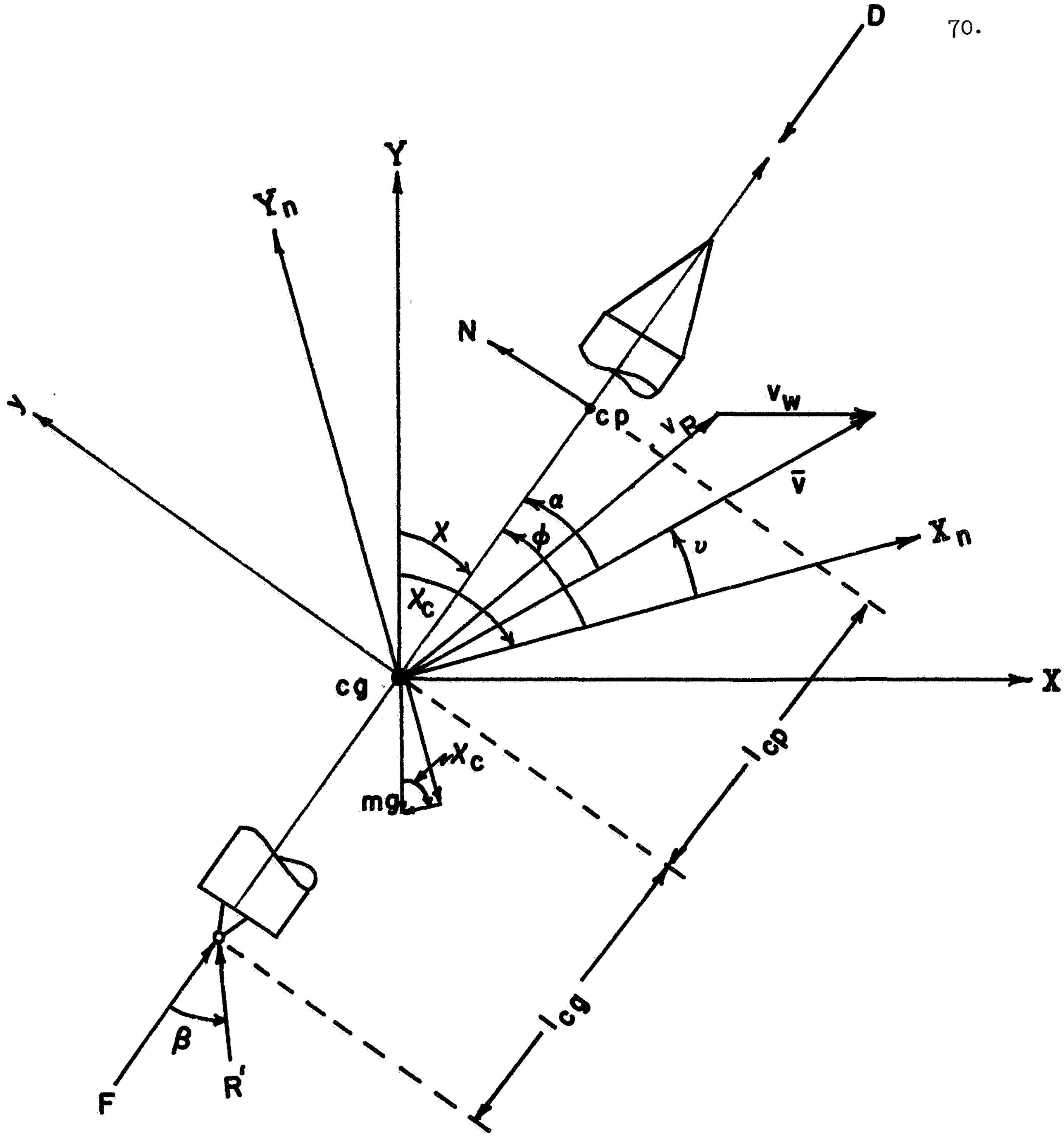


FIG. 5-4 FREE BODY DIAGRAM

Summing forces in the Y_n direction gives

$$F_{Y_n} = (F+R' \cos \beta - D) \sin \phi + N \cos \phi + R' \sin \beta \cos \phi - mg \sin(x_c - x) \qquad (5.3\text{-}2)$$

Summing torques about the center of gravity gives

$$I \ddot{\phi} = - R' l_{cg} \sin \beta - N l_{cp} \qquad (5.3\text{-}3)$$

The angle X_c is the pitch command angle and is determined by the desired mission profile.

The velocity of the vehicle v can be expressed in terms of the nominal co-ordinate system as

$$\bar{v} = v \cos \nu \, \bar{i} + v \sin \nu \, \bar{j} \qquad (5.3\text{-}4)$$

where $v = |\bar{v}|$ and the $\bar{i}$ and $\bar{j}$ vectors are unit vectors in the X_n and Y_n directions respectively.

The acceleration of the vehicle, $\bar{a}$, is then given by the time derivative of (5.3-4)

$$\bar{a} = \frac{dv}{dt} \cos \nu \, \bar{i} - v \sin \nu \frac{d\nu}{dt} \bar{i} + v \cos \nu \frac{d\bar{i}}{dt}$$

$$+ \frac{dv}{dt} \sin \nu \, \bar{j} + v \cos \nu \frac{d\nu}{dt} \bar{j} + v \sin \nu \frac{d\bar{j}}{dt} \qquad (5.3\text{-}5)$$

The derivatives of the unit vectors are shown in Appendix D to be given by

$$\frac{d\bar{i}}{dt} = \bar{\omega} \times \bar{i} = - \dot{X}_c \bar{k} \times \bar{i} = - \dot{X}_c \bar{j}$$

$$\frac{d\bar{j}}{dt} = \bar{\omega} \times \bar{j} = - \dot{X}_c \bar{k} \times \bar{j} = \dot{X}_c \bar{i}$$

Using these relations the acceleration of the vehicle may be expressed as

$$\bar{a} = \left[\dot{v} \cos\nu - v \sin\nu\,\dot{\nu} + v \sin\nu\,\dot{X}_c \right] \bar{i} + \left[\dot{v} \sin\nu + v \cos\nu\,\dot{\nu} - v \cos\nu\,\dot{X}_c \right] \bar{j} \qquad (5.3\text{-}6)$$

Recognizing the following relations

$$\ddot{X}_n = \frac{d}{dt}(\dot{X}_n) = \frac{d}{dt}(v \cos\nu) = \dot{v} \cos\nu - v \sin\nu\,\dot{\nu}$$

$$\ddot{Y}_n = \frac{d}{dt}(\dot{Y}_n) = \frac{d}{dt}(v \sin\nu) = \dot{v} \sin\nu + v \cos\nu\,\dot{\nu} \qquad (5.3\text{-}7)$$

allows the vehicle acceleration (5.3-6) to be written as

$$\bar{a} = (\ddot{X}_n + v \sin\nu\,\dot{X}_c)\,\bar{i} + (\ddot{Y}_n - v \cos\nu\,\dot{X}_c)\,\bar{j} \qquad (5.3\text{-}8)$$

If this is substituted into the force balances (5.3-1) and (5.3-2) the final equations for the motion of the vehicle in terms of the nominal co-ordinate system result from Newton's Law.

$$m(\ddot{X}_n + v \sin\nu\,\dot{X}_c) = (F + R' \cos\beta - D) \cos\phi - N \sin\phi - R' \sin\beta \sin\phi - mg \cos X_c \qquad (5.3\text{-}9)$$

$$m(\ddot{Y}_n - v \cos\nu\,\dot{X}_c) = (F + R' \cos\beta - D) \sin\phi + N \cos\phi + R' \sin\beta \cos\phi - mg \sin X_c \qquad (5.3\text{-}10)$$

These equations can be linearized by making the usual small angle approximations

$$\sin\phi = \phi \qquad \sin\beta = \beta \qquad \sin\beta \sin\phi = 0$$

$$\cos\phi = 1 \qquad \cos\beta = 1$$

Under these assumptions the linearized versions of the vehicle equations of motion are

$$\ddot{X}_n = \frac{F + R' - D}{m} - g \cos X_c \tag{5.3-11}$$

$$\ddot{Y}_n = \frac{F + R' - D}{m} \phi + \frac{N}{m} + \frac{R'}{m} \beta + v \dot{X}_c - g \sin X_c \tag{5.3-12}$$

If the origin of the nominal co-ordinate system is allowed to move with the vehicle in the X_n direction this eliminates the X_n degree of freedom leaving only (5.3-12) and (5.3-3) to describe the motion of the rigid vehicle.

The aerodynamic normal force, N, of (5.3-12) is proportional to the angle of attack α and is thus given by

$$N = N' \alpha$$

If the total thrust of the vehicle is denoted by $T = F + R'$ then (5.3-12) can be written as

$$\ddot{Y}_n = \left(\frac{T-D}{m}\right) \phi + \frac{N'}{m} \alpha + \frac{R'}{m} \beta + \left[v \dot{X}_c - g \sin X_c \right] \tag{5.3-13}$$

Usually the vehicle is allowed to fly a gravity turn trajectory in which case the pitch command angle is given by

$$\dot{X}_c = \frac{g \sin X_c}{v} \tag{5.3-14}$$

and the last two terms of (5.3-13) cancel. Thus the final equation becomes

$$\ddot{Y}_n = \left(\frac{T-D}{m}\right) \phi + \frac{N'}{m} \alpha + \frac{R'}{m} \beta \tag{5.3-15}$$

Making the small angle approximation on the moment equation (5.3-3) gives the pitch angle equation

$$\ddot{\phi} = -\left(\frac{R' \, l_{cg}}{I}\right) \beta - \left(\frac{N' \, l_{cp}}{I}\right) \alpha \tag{5.3-16}$$

Finally there is the equation relating the pitch angle and the angle of attack

$$\alpha - \alpha_w = \phi - \frac{\dot{Y}_n}{v} \tag{5.3-17}$$

These three equations (5.3-15), (5.3-16) and (5.3-17) completely describe the linearized rigid body motion of the Saturn V about its nominal trajectory.

Next the bending effects must be examined. Three bending modes will be considered to be of significance here: the first, second, and third. For simplicity the equations describing these three modes are assumed to be those of a linear oscillator driven by a forcing function proportional to the gimbal angle β. These equations are written in terms of normalized co-ordinates such that the deformation at any station along the vehicle is given by the value of the normal co-ordinate multiplied by the mode shape coefficient for that station. The equations are

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 = \frac{R' \, Y_i(x_\beta)}{m_i} \beta \qquad i=1,2,3 \tag{5.3-18}$$

The normalization is taken with respect to the gimbal plane so that the solution of (5.3-18) gives the actual deflection at the gimbal directly. Notice that the forcing function depends on $Y(x_\beta)$ the mode shape at the gimbal station.

As mentioned above, it was decided to use pitch and pitch rate feedback to control the vehicle. The linear control law is given by

$$\beta = - K_1 \phi - K_2 \dot{\phi} \tag{5.3-19}$$

Unfortunately, because of the flexible nature of the Saturn V it is impossible to measure these quantities directly. Pitch and pitch rate are measured by gyros placed somewhere on the vehicle frame. For the Saturn V these positions are

$$x_D = 79.8 \text{ meters}$$
$$x_R = 67.3 \text{ meters}$$

respectively. Thus the gyros can measure only conditions at these particular points, that is, local pitch and pitch rate. These measurements are corrupted by bending information and are in general impossible to extract from it. The actual control law that must be implemented using available measurements is

$$\beta = - K_1 \phi_D - K_2 \dot{\phi}_R \tag{5.3-20}$$

where ϕ_D is the output signal of the pitch gyro and $\dot{\phi}_R$ is the output signal of the pitch rate gyro. For the vehicle with three bending modes these are, respectively

$$\phi_D = \phi + \sum_{i=1}^{3} Y_i{}'(x_D)\eta_i$$

$$\dot{\phi}_R = \dot{\phi} + \sum_{i=1}^{3} Y_i{}'(x_R)\dot{\eta}_i \qquad i = 1, 2, 3 \tag{5.3-21}$$

The terms $Y_i{}'(x_D)$ and $Y_i{}'(x_R)$ are the mode slopes for the i^{th} mode at the pitch and pitch rate gyro stations.

The equations describing the flexible booster can then be summarized as

$$\ddot{Y}_n = (\frac{T-D}{m})\,\phi + \frac{N'}{m}\,\alpha + \frac{R'}{m}\,\beta \tag{5.3-22}$$

$$\ddot{\phi} = -\,(\frac{R'\,l_{cg}}{I})\,\beta - (\frac{N'\,l_{cp}}{I})\,\alpha \tag{5.3-23}$$

$$\ddot{\eta}_i + 2\zeta_i\,\omega_i\,\dot{\eta}_i + \omega_i^2 \quad = \frac{R'\,Y_i(x_\beta)}{m_i}\,\beta \qquad i=1,2,3 \tag{5.3-24}$$

$$\beta = -\,K_1\,\phi_D - K_2\,\dot{\phi}_R \tag{5.3-25}$$

$$\phi_D = \phi + \sum_{i=1}^{3} Y_i'(x_D)\,\eta_i \tag{5.3-26}$$

$$\dot{\phi}_R = \dot{\phi} + \sum_{i=1}^{3} Y_i'(x_R)\,\dot{\eta}_i \tag{5.3-27}$$

$$\alpha - \alpha_w = \phi - \frac{\dot{Y}_n}{v} \tag{5.3-28}$$

The variable Y_n can be eliminated from these equations to give a more compact set. Solving (5.3-28) for $\dot{Y}_n$ and differentiating with respect to time gives

$$\ddot{Y}_n = v\,\dot{\phi} - \dot{v}(\alpha - \alpha_w) - v(\dot{\alpha} - \dot{\alpha}_w) + \phi\,\dot{v} \tag{5.3-29}$$

Equating this with (5.3-22) and solving for α gives

$$\dot{\alpha} = -\,(\frac{T-D}{mv} - \frac{\dot{v}}{v})\,\phi + \dot{\phi} - (\frac{N'}{mv} + \frac{\dot{v}}{v})\,\alpha - \frac{R'}{mv}\,\beta + (\frac{\dot{v}}{v}\,\alpha_w + \dot{\alpha}_w) \tag{5.3-30}$$

Thus the rigid booster is described by (5.3-30) and (5.3-23).

5.4 State Equations of the Flexible Booster

Equations (5.3-23), (5.3-30) and (5.3-24) through (5.3-27) completely define the system dynamics of the flexible booster model used in this case study. These can be put into state equation form by defining the following state equation

$$\dot{x} = Ax + b\,\beta + u(t) \tag{5.4-1}$$

where x is the state vector, β the scalar gimbal angle, A the vehicle state matrix, b the controller vector and $u(t)$ a disturbance vector. These given by

$$x = \begin{pmatrix} \phi \\ \dot{\phi} \\ \alpha \\ \eta_i \\ \dot{\eta}_i \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{N'\,l_{cp}}{I} & 0 & 0 \\ -\left(\dfrac{T-D}{mv} - \dfrac{\dot{v}}{v}\right) & 1 & -\left(\dfrac{N'}{mv} + \dfrac{\dot{v}}{v}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_i^2 & -2\zeta_i\,\omega_i \end{pmatrix} \tag{5.4-2}$$

$$b = \begin{pmatrix} 0 \\ -\dfrac{R'\,l_{cg}}{I} \\ -\dfrac{R'}{mv} \\ 0 \\ \dfrac{R'\,Y_i(x_\beta)}{m_i} \end{pmatrix} \qquad u(t) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\dot{v}}{v}\,\alpha_w + \dot{\alpha}_w \\ 0 \\ 0 \end{pmatrix}$$

The control law for the state variable model is written as

$$\beta = - K^T \Gamma x \tag{5.4-3}$$

where Γ is a measurement matrix which takes into account the fact that neither pitch nor pitch rate can be observed directly but are corrupted by bending. Thus the measurement matrix Γ is given by

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 & Y_i'(x_D) & 0 \\ 0 & 1 & 0 & 0 & Y_i'(x_R) \end{pmatrix} \tag{5.4-4}$$

All of these matrix equations have included only one bending mode for simplicity. The actual model uses all three modes simultaneously which makes the state equations somewhat more involved but of the same form.

The state equations of (5.4-2) with the control law of (5.4-3) are more difficult to deal with than is necessary. The root of the difficulty lies in the measurement matrix Γ . If the states x_1 and x_2 of (5.4-2) could be measured directly then Γ would be of the form

$$\Gamma = \left(I \ \vdots \ 0 \right)$$

and could be dropped by simply adjoining extra zero gains to the feedback vector

$$K^T = \left(K_1 \quad K_2 \ \vdots \ 0 \quad 0 \quad 0 \right)$$

The state equations would then be of the form

$$\begin{aligned} \dot{x} &= Ax + b\,\beta + u(t) \\ \beta &= - K^T x \\ x(0) &= c \end{aligned} \tag{5.4-5}$$

or more simply

$$\dot{x} = (A - bK^T)\, x + u(t)$$

$$x(0) = c \qquad (5.4\text{-}6)$$

This simplification can be accomplished by choosing as the state variable x_1 and x_2 the actual measured quantities rather than rigid body pitch and pitch rate. Using the output of the pitch and pitch rate gyros as the first and second state variables causes the quantities of (5.4-6) to become

$$x = \begin{pmatrix} \phi + Y_i{}'(x_D)\eta_i \\ \dot{\phi} + Y_i{}'(x_R)\dot{\eta}_i \\ \alpha \\ \eta_i \\ \dot{\eta}_i \end{pmatrix} \qquad (5.4\text{-}7a)$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & Y_i{}'(x_D)-Y_i{}'(x_R) \\ 0 & 0 & -\frac{N' l_{cp}}{I} & -Y_i{}'(x_D)\omega_i{}^2 & -2\zeta_i\,\omega_i\,Y_i{}'(x_R) \\ -(\frac{T-D}{mv} - \frac{\dot{v}}{v}) & 1 & -(\frac{N'}{mv} + \frac{\dot{v}}{v}) & y_i{}'(x_D)(\frac{T-D}{mv} - \frac{\dot{v}}{v}) & -Y_i{}'(x_R) \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega_i{}^2 & -2\zeta_i\,\omega_i \end{pmatrix} \qquad (5.4\text{-}7b)$$

$$
b = \begin{pmatrix} 0 \\ -\dfrac{R' l_{cg}}{I} + Y_i{}'(x_R)\,\dfrac{R'\; Y_i(x_\beta)}{m_i} \\ -\dfrac{R'}{mV} \\ 0 \\ \dfrac{R'\; Y_i(x_\beta)}{m_i} \end{pmatrix} \qquad (5.4\text{-}7c)
$$

$$
K^T = \begin{pmatrix} K_1 & K_2 & 0 & 0 & 0 \end{pmatrix} \qquad (5.4\text{-}7d)
$$

$$
u = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\dot{V}}{V}\,\alpha_w + \dot{\alpha}_w \\ 0 \\ 0 \end{pmatrix} \qquad (5.4\text{-}7e)
$$

Equation (5.4-6) with the definitions of (5.4-7) represents the linear time varying description of the Saturn V - Apollo vehicle. A table of the time varying values of these parameters is included in Appendix F.

The disturbance term, u(t), contains a function of α_w and α_w which are thus seen to play the role of external disturbances acting on the vehicle. The selection of a meaningful wind for use in the evaluation of any control system design is important and is considered briefly in the next section.

5.5 Wind Disturbance

The state equations (5.4-6) and (5.4-7) make it clear that the only external disturbance acting on the booster in flight is wind. The wind alters the apparent angle of attack by an amount α_w. This can be related to the vehicle velocity and wind velocity by examining Figure 5-5 which is a detail of Figure 5-4 for the case when $\alpha = \phi = 0$. That is when the vehicle is on the nominal trajectory. From Figure 5-5

$$\alpha_w = \frac{v_w \cos X_c}{v - v_w \sin X_c} \tag{5.5-1}$$

where v_w is the wind velocity, v is the vehicle velocity and X_c is the pitch command angle measured from launch vertical. Using the nominal values of v and X_c and either measured or assumed values of v_w, a wind angle of attack profile can be constructed for use in the forcing function $u(t)$ or (5.4-7). A synthetic wind speed profile, shown in Figure 5-6, was constructed. This profile has wind magnitudes that exceed those of 95% of the measured winds in the May to November period at Cape Kennedy, Florida.[2] For this reason it is referred to as a 95% wind. In addition, a severe gust was added in the region of expected maximum dynamic pressure (max q). This gust will tend to excite any unstable modes of the vehicle as it passes through this region. Physically, this gust occurs just about the jet stream region which is thus taken into account in the simulation.

The wind induced angle of attack, α_w, that results from this synthetic wind profile is shown in Figure 5-7. This is the external disturbance acting on the vehicle and included in all time varying simulations.

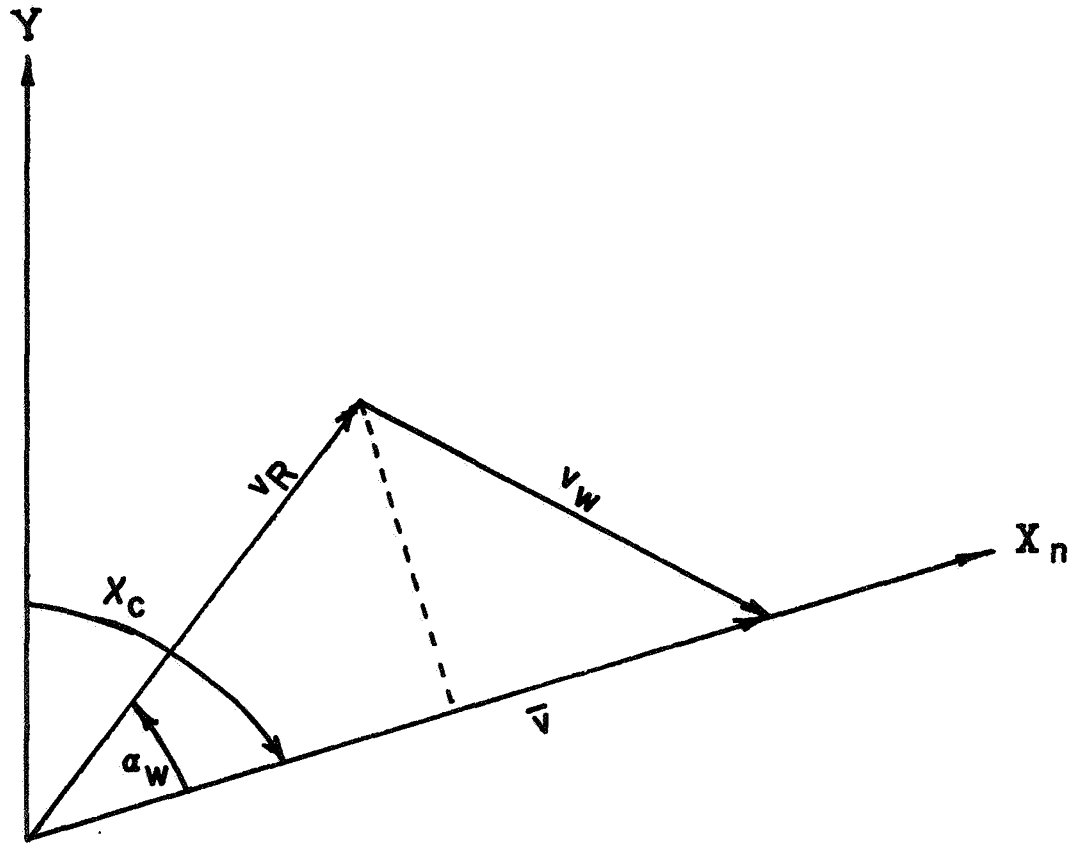


FIG. 5-5 WIND ANGLE OF ATTACK

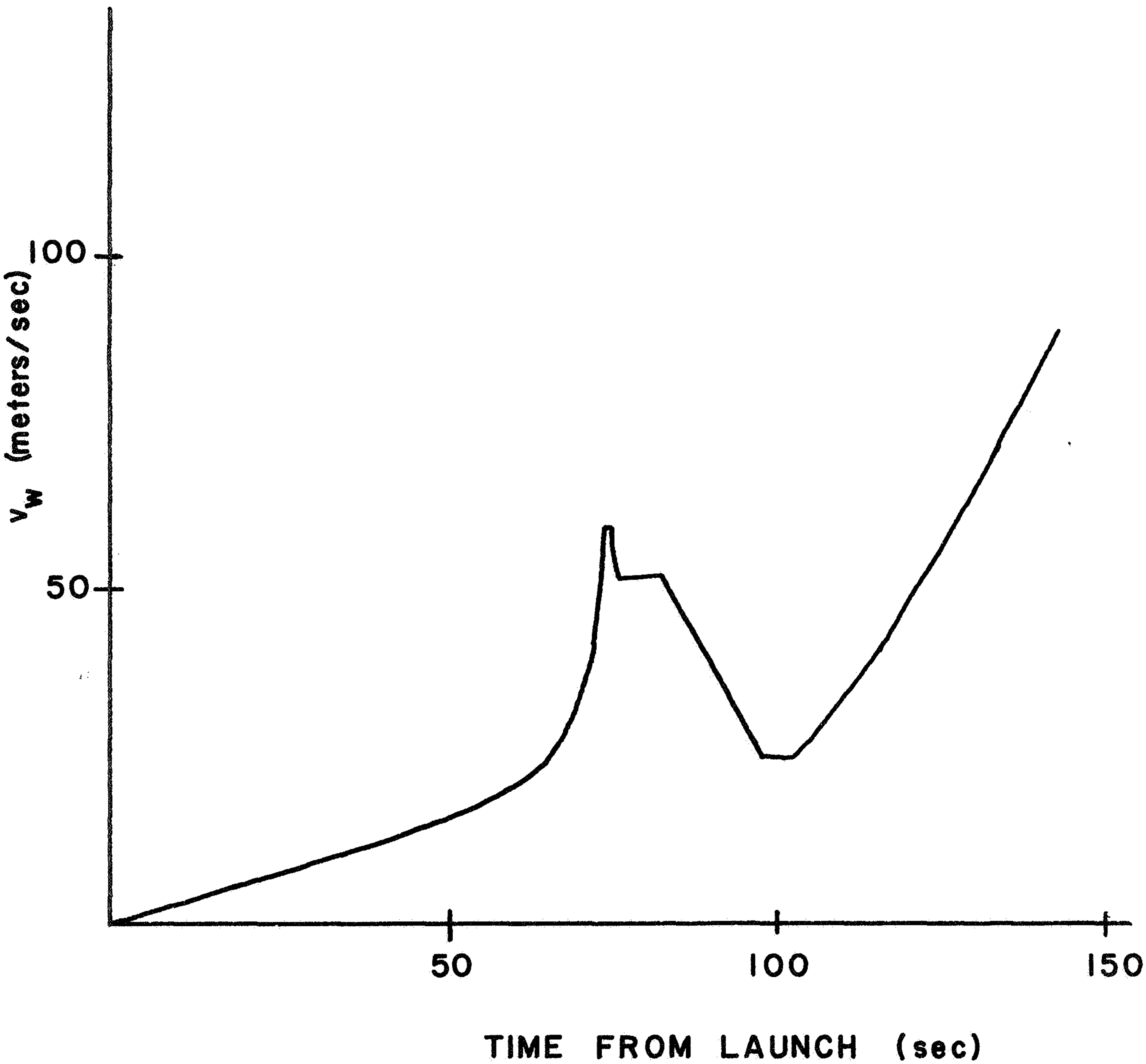


FIG. 5-6 SYNTHETIC WIND

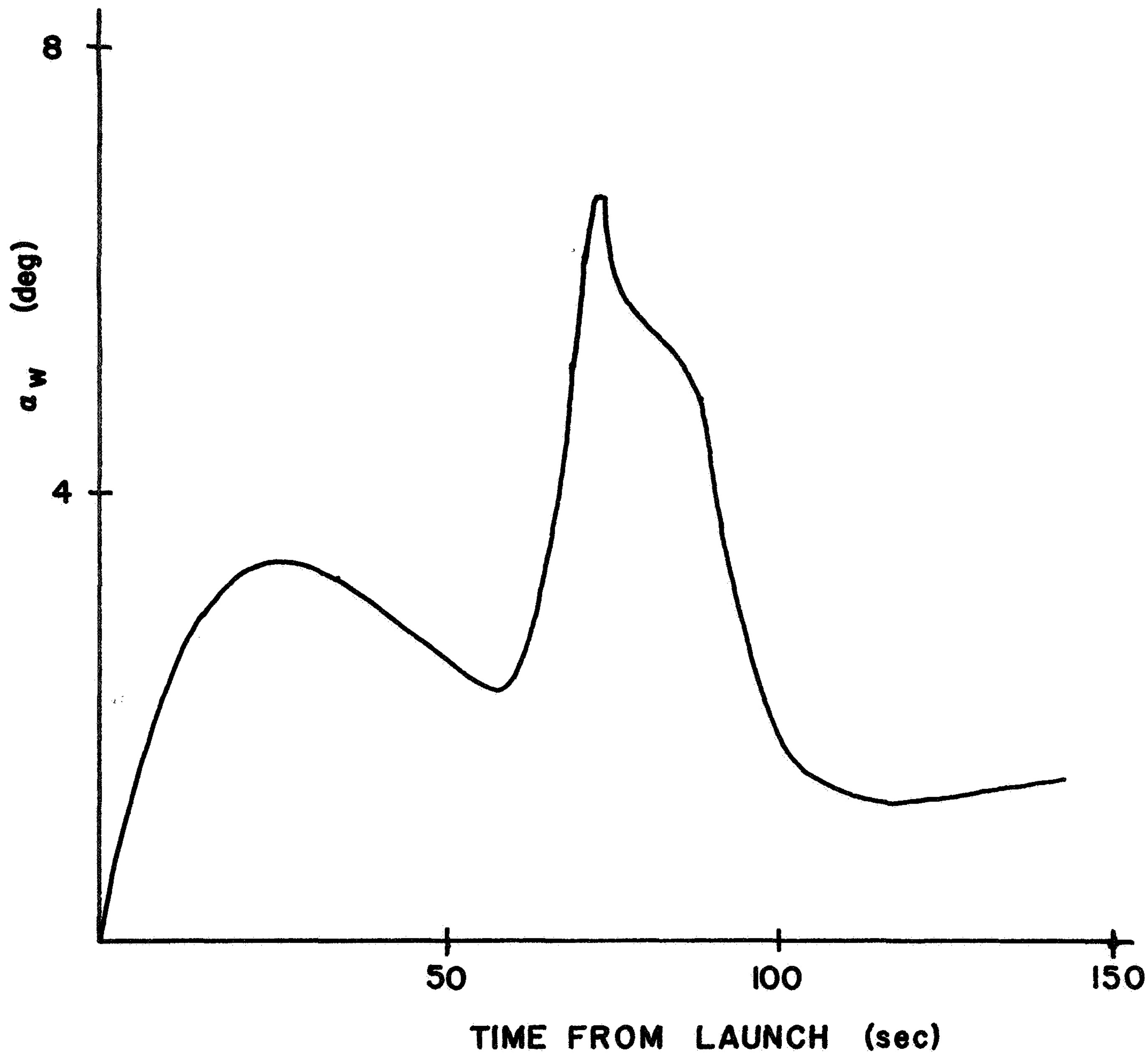


FIG. 5–7 WIND ANGLE OF ATTACK

5.6 The Control Law

An obvious simplification in the design of a control system for the Saturn V - Apollo would result if the control signal β could be made a function of only the rigid body pitch and pitch rate rather than of the measured pitch and pitch rate given by (5.3-26) and (5.3-27). Reference to the frequency spectrum of Figure 5-3 makes this appear possible since the rigid body frequencies and the bending mode frequencies are separated by almost a decade of frequency. It would appear at first that a simple low pass filter placed in the feedback loop with a cutoff frequency of about 0.3 hertz would separate the rigid and bending modes nicely. The difficulty, however, lies in the slosh modes. These extend in frequency from the rigid body modes to the first bending mode. Any low pass filter with a cutoff frequency in this range could add enough phase shift at these slosh mode frequencies to drive the slosh modes unstable. Although consideration of the effects of these slosh is beyond the scope of this work it was decided to allow for their presence by restricting the cutoff frequency of any low pass filter to be above one hertz. With this restriction it was felt that the phase shift caused by the filter at the slosh frequencies would be small enough to avoid slosh stability problems.

With this restriction on the cutoff frequency of the filter the resulting control system becomes sensitive to negative perturbation in bending frequency, particularly of the first mode. The reason for this is clear upon examination of Figure 5-8 which is a Bode plot of the simple second order low pass filter that will be used. The filter has the transfer function

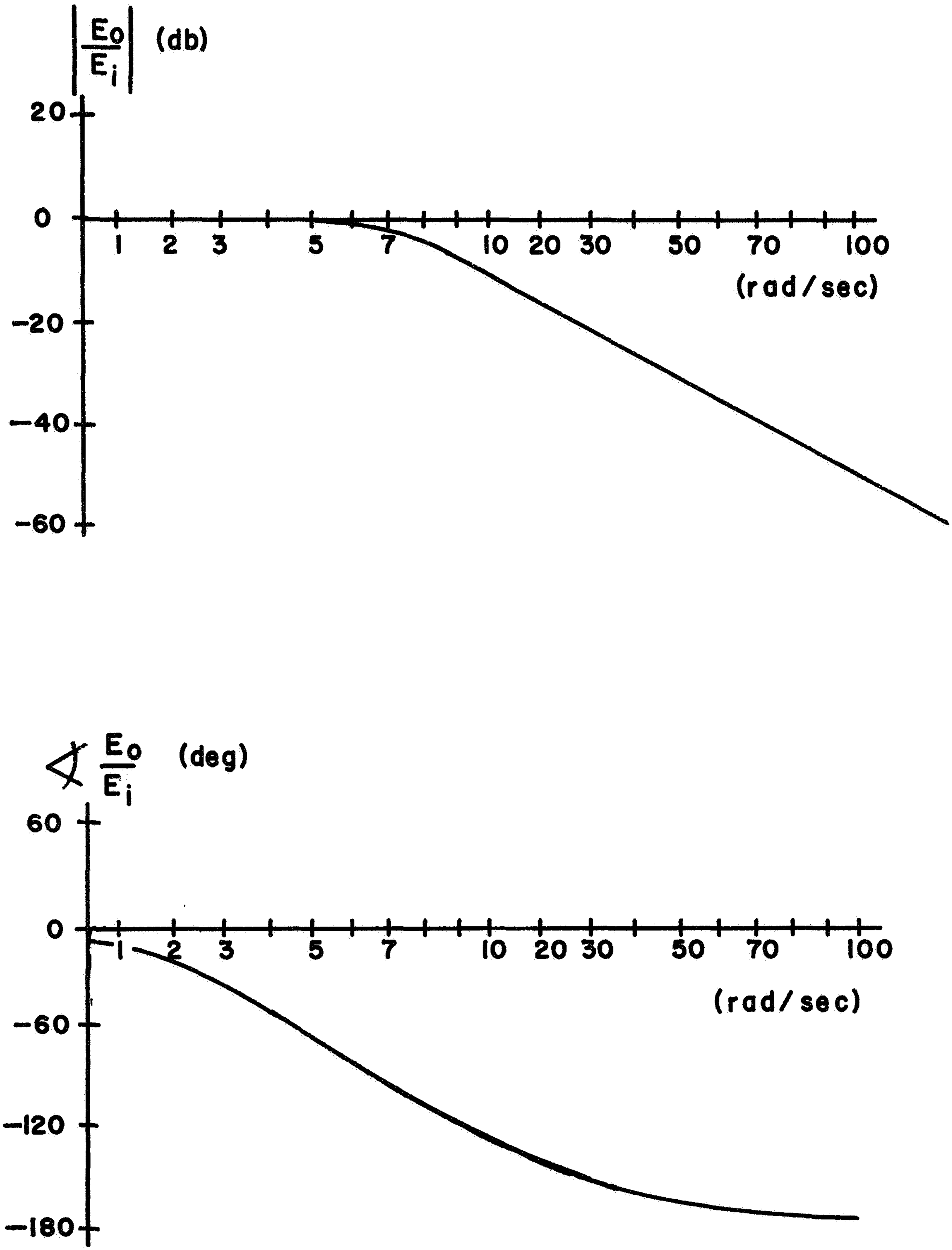


FIG. 5-8 LOW PASS FILTER

$$\frac{E_o(s)}{E_i(s)} = \frac{50}{s^2 + 10s + 50} \tag{5.6-1}$$

The break point frequency of the Bode plot is within the frequency band of the first bending mode. Thus when the first mode frequency is perturbed to a lower value, the effective gain at the bending mode frequencies is increased. This gain increase can be enough to drive one of the closed loop poles into the right half s-plane with resulting instability. It will be shown that Analog Sensitivity Design makes possible a control system that is sufficiently insensitive to changes in bending mode frequency that this instability does not occur.

With this filter (5.6-1) in the feedback control loop the equations of the nominal system with $\alpha_w = 0$ become

$$\ddot{\phi} = -\left(\frac{R' l_{cg}}{I}\right)\beta - \left(\frac{N' l_{cp}}{I}\right)\alpha \tag{5.6-2a}$$

$$\dot{\alpha} = -\left(\frac{T-D}{mv} - \frac{\dot{v}}{v}\right)\phi + \dot{\phi} - \left(\frac{N'}{mv} + \frac{\dot{v}}{v}\right)\alpha - \frac{R'}{mv}\beta \tag{5.6-2b}$$

$$\ddot{\eta}_i = -2\zeta_i\omega_i\dot{\eta}_i - \omega_i^2\eta_i + \frac{R' \, Y_i(x_\beta)}{m_i}\beta \qquad i=1,2,3 \tag{5.6-2c}$$

$$\beta_c = -K_1\phi_D - K_2\dot{\phi}_R \tag{5.6-2d}$$

$$\phi_D = \phi + \sum_{i=1}^{3} Y_i'(x_D)\eta_i \tag{5.6-2e}$$

$$\dot{\phi}_R = \dot{\phi} + \sum_{i=1}^{3} Y_i'(x_R)\dot{\eta}_i \tag{5.6-2f}$$

$$\ddot{\beta} + 10\dot{\beta} + 50\beta = 50\beta_c$$

5.7 Application of Analog Sensitivity Design

Equations (5.6-2) represent the dynamic system to which the techniques of Analog Sensitivity Design are to be applied. Because of the limited analog computation facilities available it was necessary to design the control system using a frozen time point model of the vehicle. This involves the choice of a time representative of the critical portions of the flight at which to evaluate all the time varying coefficients of equations (5.6-2). The control system is then designed for the resulting time invariant model. In order to insure the applicability of the resulting control system it is then tested by digital simulation using a full time varying model of the flexible booster.

For the present problem $t = 80$ seconds after liftoff was taken as the frozen time point. This represents adequately the portion of the flight centering around maximum dynamic pressure without exhibiting the extreme values the vehicle parameters take on exactly at max q. This portion of the flight is considered the most critical. This choice turned out to be justified when the system designed using this frozen point model proved to be able to control the time varying model adequately.

Figure 5-9a and Figure 5-9b show the analog computer simulation diagram for the booster and sensitivity equations. Details of these simulations and the necessary magnitude scaling are given in Appendix E. It should be noted that the sensitivity equations simulation of Figure 5-9b is identical to the vehicle simulation of Figure 5-9a except for the two additional inputs added to the ω_i^2 and $2\zeta_i \omega_i$ feedback

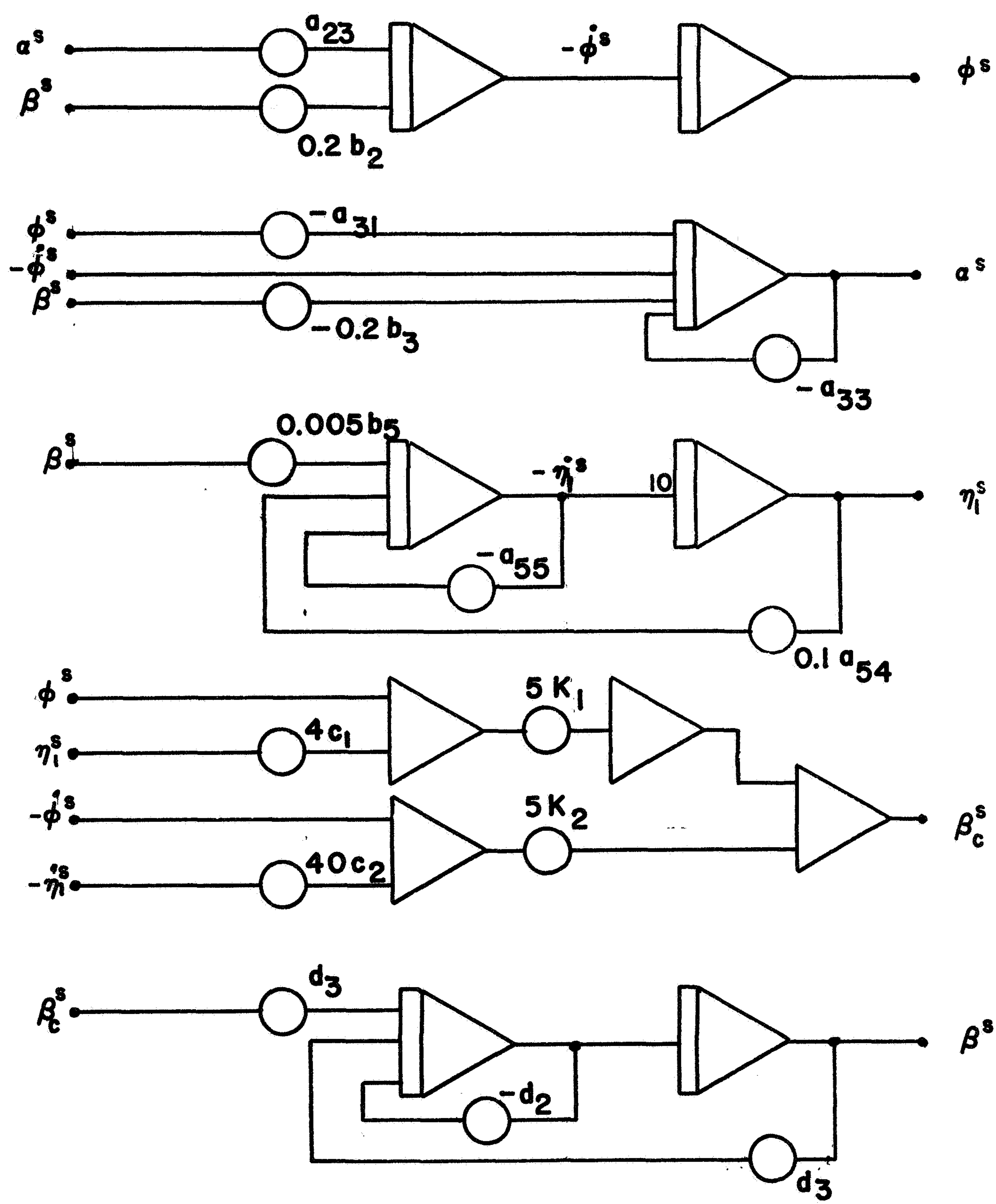


FIG. 5-9a VEHICLE SIMULATION

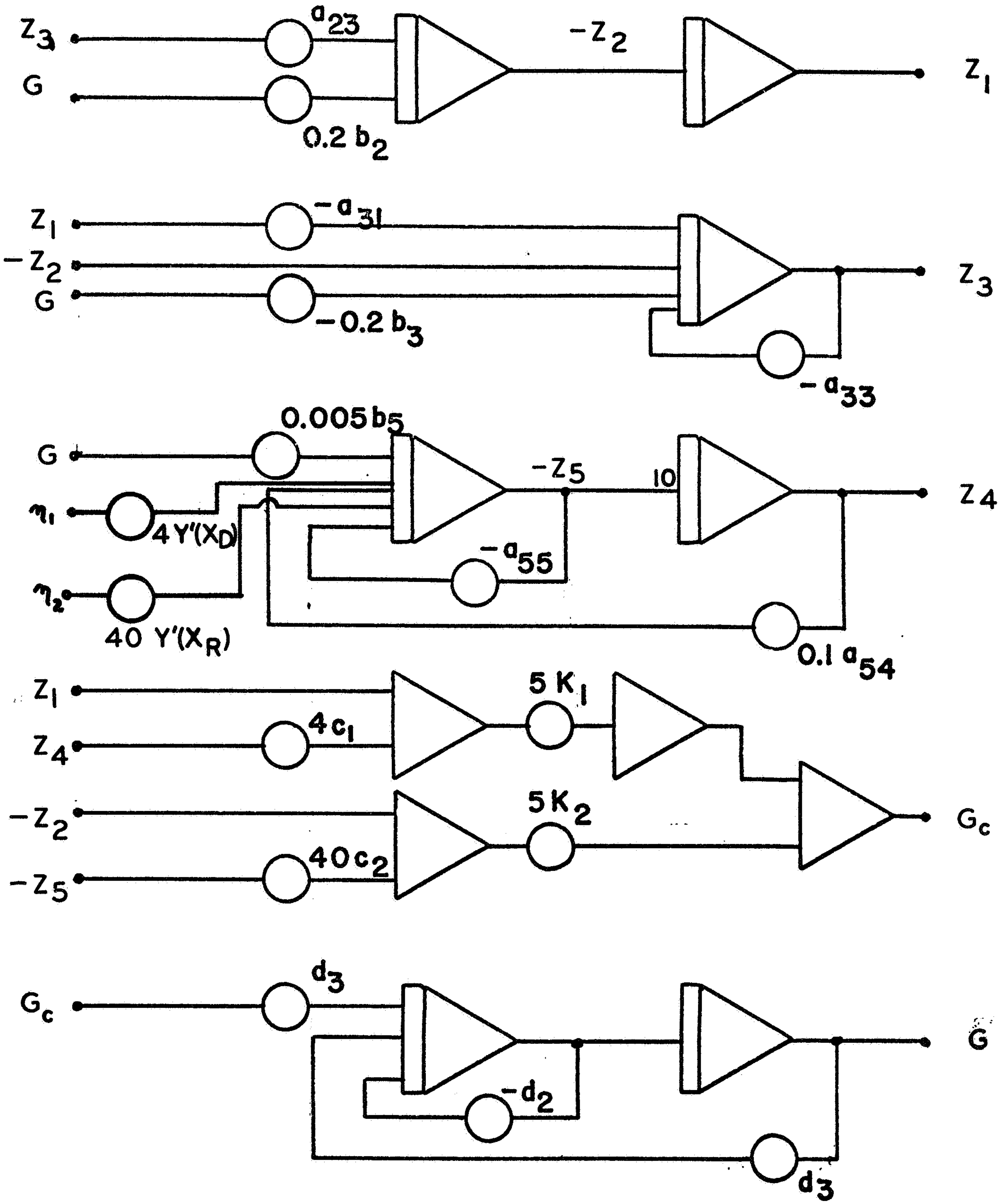


FIG. 5-9b SENSITIVITY SIMULATION

terms. This is exactly what the structural method of Section 4.3 indicates. Thus it is not necessary to develop the simulation equations of Figure 5-9a and Figure 5-9b separately. All that must be done is start with the vehicle simulation diagram of Figure 5-9a and apply the structural rules of Section 4.3.

The first step was to obtain a set of optimal pitch and pitch rate gains. This was to give a yardstick against which the nominal and off nominal performance of the desensitized system could be measured. Since the first bending mode appeared to be the most critical it was the mode implemented for this study. These optimal gains were obtained using a pitch initial condition of 5°. The criterion of optimality was a smooth, well damped rigid body response with stable bending not exceeding 1.0 meters at the gimbal plane. The optimal gains were determined to be

$$K_{opt} = -\begin{pmatrix} 0.8 \\ 0.8 \end{pmatrix} \tag{5.7-1}$$

The simulation results for $t = 80$ seconds with these optimal gains are shown in Figure 5-10 for the nominal values of the first bending mode frequency, $\omega = \omega_o$. The initial value of pitch is seen to damp out quite rapidly and smoothly to zero. The induced angle of attack also damps out to zero smoothly although not quite so rapidly as does pitch. Addition of angle of attack feedback might speed up the decay of the attack angle but this was not attempted, primarily because of the lack of adequate angle of attack sensors on the Saturn V. Normalized bending and gimbal angle are also shown. The bending stays quite small, less than 0.9 m. at the gimbal plane. Gimbal angle shows a rather large

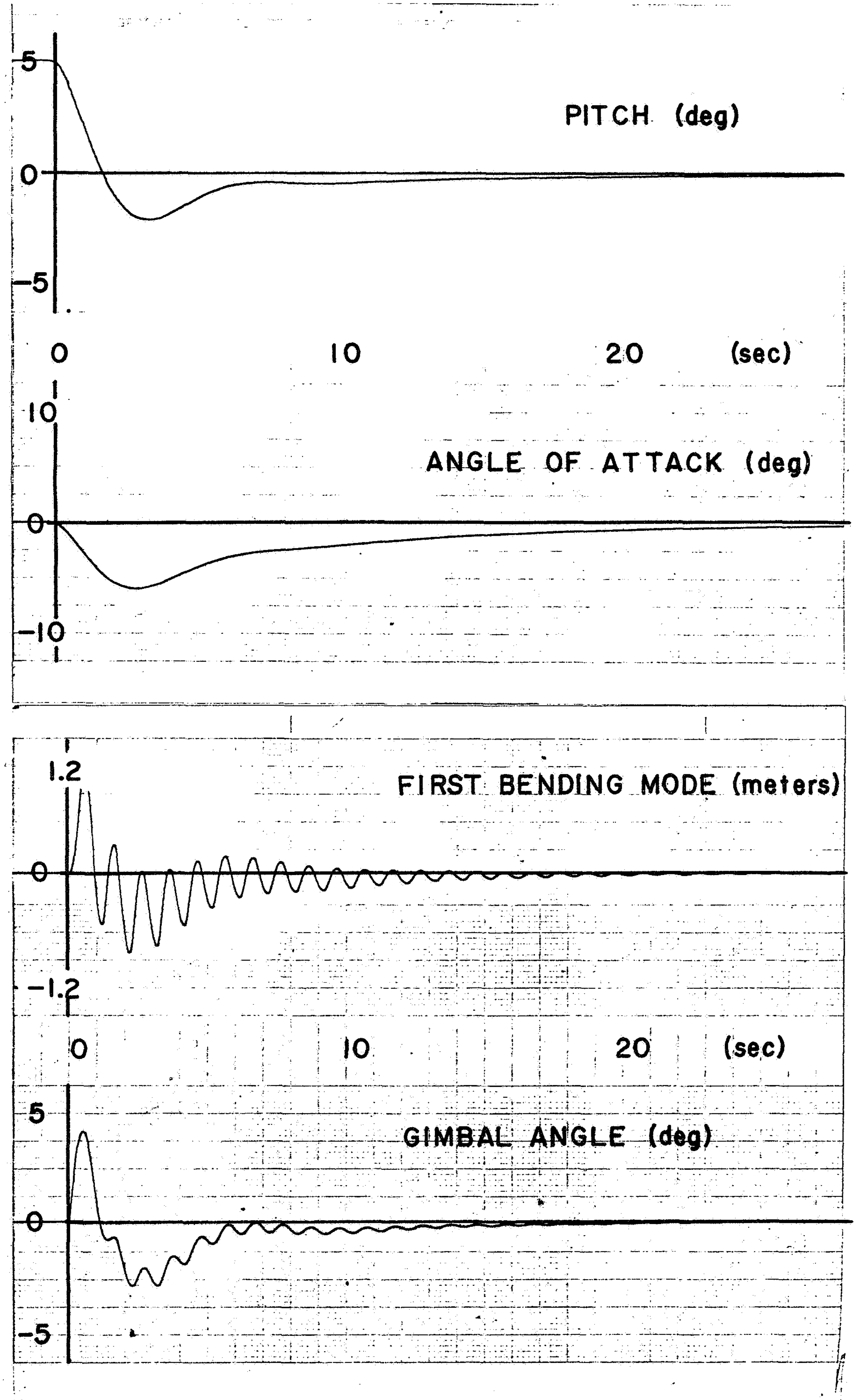


FIG. 5-10 OPTIMAL BOOSTER $\omega = \omega_0$

transient of about four degrees and the bending frequency feedback through the gyros is evident superimposed on it. This transient results from the rather large initial condition of five degrees in pitch, but even so it remains within the design limit of $\pm 5^{\circ}$.

Figure 5-11 shows the response of the same system when the actual first bending mode frequency is ten percent less than the nominal or design frequency, $\omega = 0.9\ \omega_o$. The rigid body response is essentially unchanged. The bending, however, has increased considerably over that of Figure 5-10 and takes longer to decay. The bending is coupled strongly into the gimbal angle and very small oscillations at the bending frequency are just visible superimposed on the rigid body response. The very low damping of the oscillations in the gimbal angle indicate that stability is becoming marginal.

Figure 5-12 shows the response of the optimal system when $\omega = 0.8\ \omega_o$. It is clear that the point of unstability has been reached. The bending reaches a large value almost immediately and this is fed back to the gimbal through the gyro coupling terms of (5.6-2e) and (5.6-2f). This in turn drives the rigid body unstable. The recordings were terminated at the point where the amplifiers of the analog computer reached saturation.

Next it was attempted to desensitize the vehicle control system by trimming the feedback gains about the optimal values while observing the sensitivity coefficients of Figure 5-9b. Two of the most significant sensitivity coefficients labelled z_4 and z_5 are shown in Figure 5-13a for the optimal system where

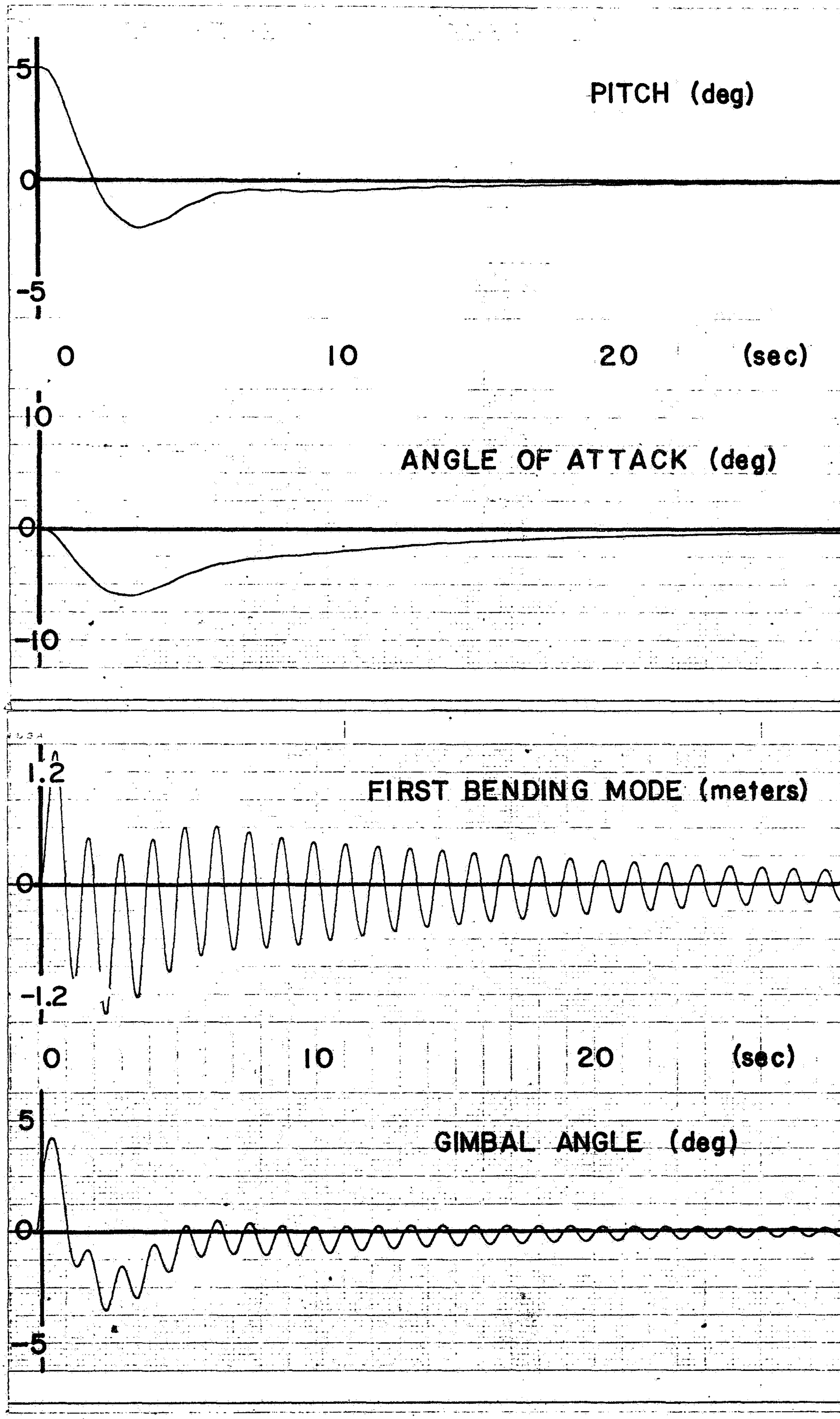


FIG. 5-11 OPTIMAL BOOSTER $\omega = 0.9\omega_o$

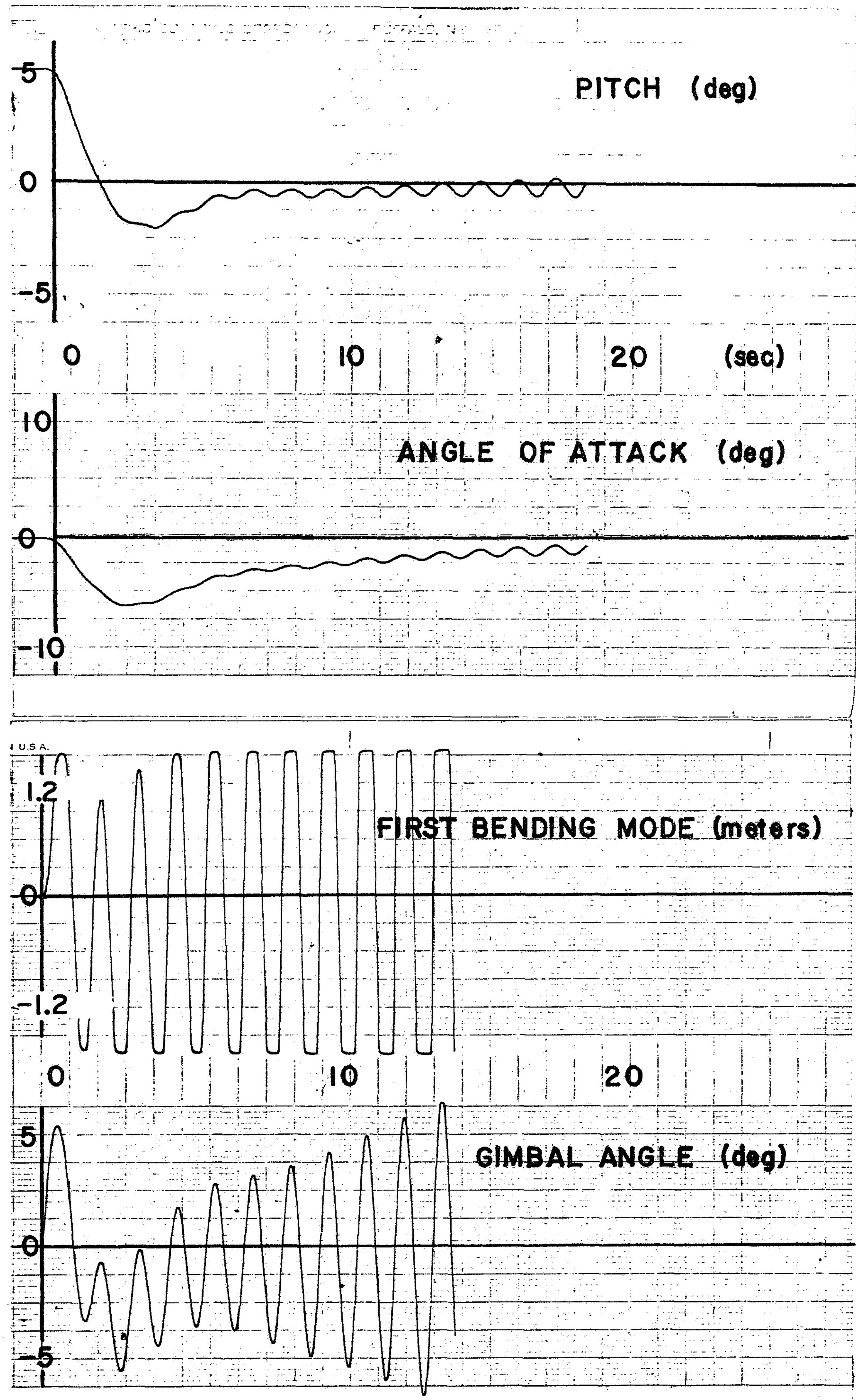


FIG. 5-12 OPTIMAL BOOSTER $\omega = 0.8\omega_0$

$$z_4 = \left.\frac{\partial x_4}{\partial \omega}\right|_{\omega=\omega_o} = \left.\frac{\partial \eta}{\partial \omega}\right|_{\omega=\omega_o}$$

$$z_5 = \left.\frac{\partial x_5}{\partial \omega}\right|_{\omega=\omega_o} = \left.\frac{\partial \dot{\eta}}{\partial \omega}\right|_{\omega=\omega_o} \tag{5.7-2}$$

Recall that ω is the natural frequency of the first bending mode, the parameter to which the system is to be desensitized. (See Appendix D)

By trimming the feedback gains K_1 and K_2 it was possible to reduce these sensitivity coefficients in magnitude by a factor of about two without causing the nominal vehicle response of becoming unacceptable. These desensitized feedback gains are

$$K_{SENS} = - \binom{0.5}{0.4} \tag{5.7-3}$$

and the resulting sensitivity coefficients are shown in Figure 5-13b.

As expected, there is a definite tradeoff between the magnitude of the sensitivity coefficients and the optimality of the nominal response of the control system. By reducing the sensitivity coefficients as indicated in Figure 5-13, the nominal response of the vehicle was degraded somewhat. The nominal response to a pitch initial condition for the desensitized system is shown in Figure 5-14. It is immediately evident that the rigid body response is not so well damped as the optimal system. It is, however, still satisfactory. The bending is slightly less than for the optimal system and like the optimal system is well within design limits. The gimbal angle, while having an envelope which indicates that slightly more energy was used than in the optimal case,

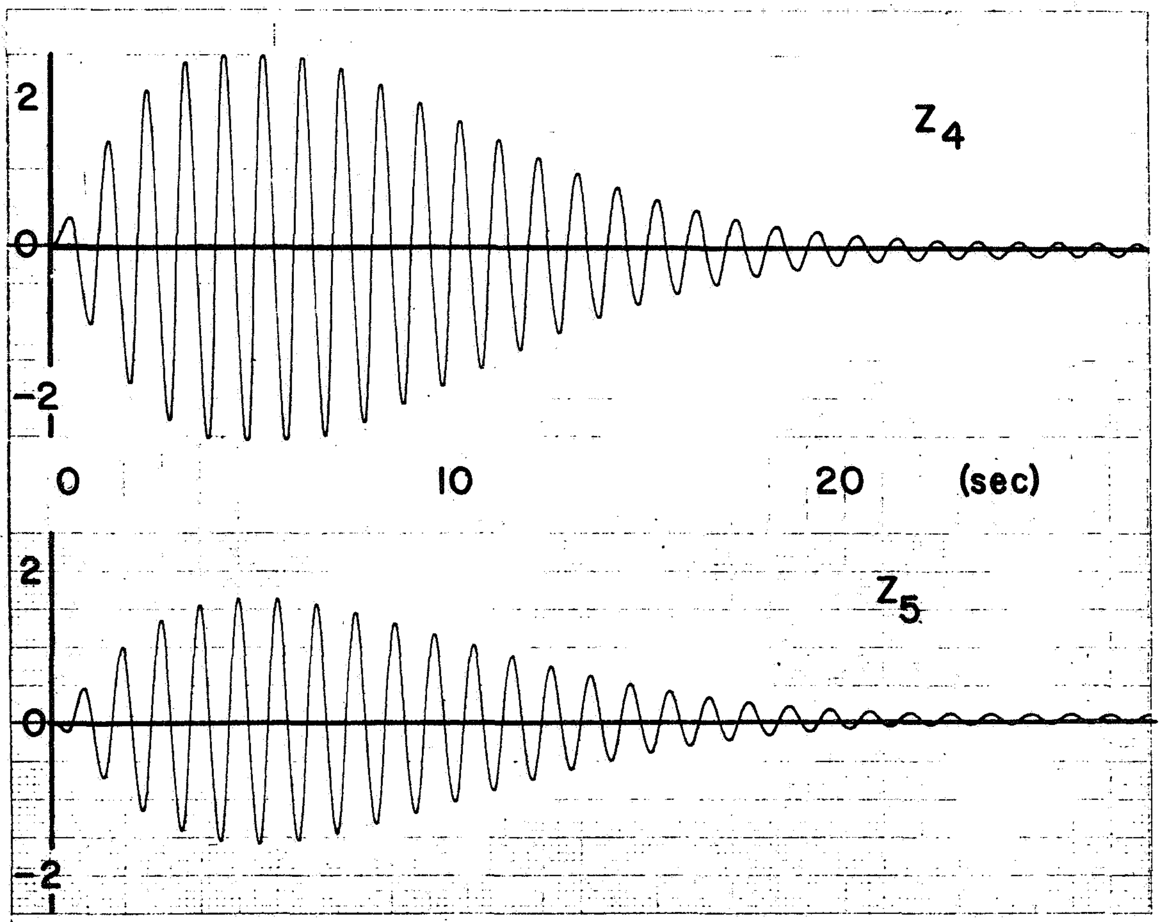


FIG. 5-13a OPTIMAL SENSITIVITY

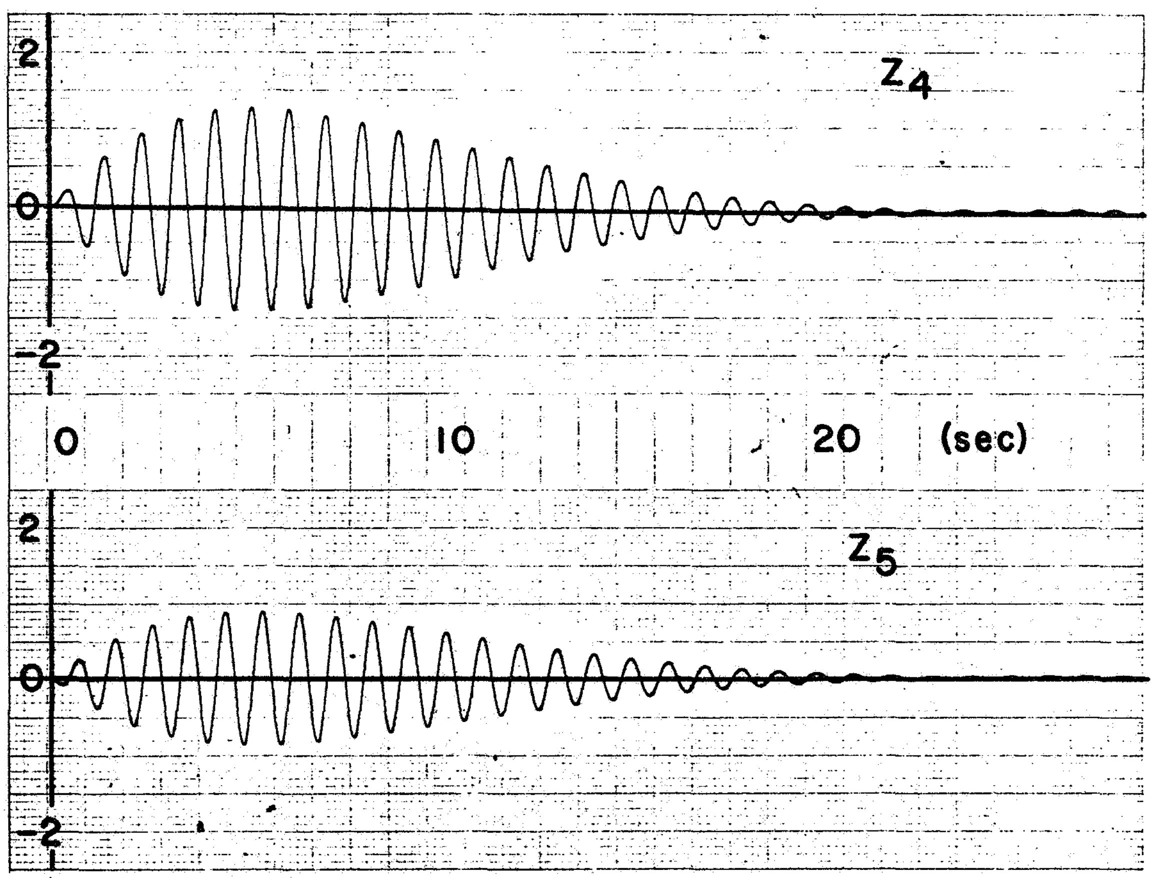


FIG. 5-13b DESENSITIZED SENSITIVITY

is of the same general shape as before and the initial transient is somewhat smaller in magnitude.

While the nominal response of the vehicle is slightly worse for the desensitized gains (Figure 5-14) than for the optimal gains (Figure 5-10) it is still satisfactory. If it were not, it would be a simple matter to trade some of the decrease in the magnitude of the sensitivity coefficients for a nominal response which more nearly approaches the optimal. This decision is entirely in the hands of the designer. By trimming the values of the gains and observing the response of the sensitivity coefficients and the nominal system simultaneously, the designer can directly observe the trade off between the two.

Figure 5-15 shows the response of the desensitized system for $\omega = 0.9\ \omega_o$. The response is virtually the same as nominal except for the bending which is slightly higher.

The desensitized system for $\omega = 0.8\ \omega_o$ is shown in Figure 5-16. Again the bending has grown slightly larger, although still quite small, and damps out more slowly. At this point the bending oscillations have become visible superimposed on the gimbal angle but the rigid body response is still smooth with no trace of the bending apparent. This should be compared with the optimal system response with $\omega = 0.8\ \omega_o$ of Figure 5-12. Clearly Analog Sensitivity Design has extended the permissible range of bending frequency parameter variation.

Figure 5-17 shows the response of the desensitized system for $\omega = 0.7\ \omega_o$, a thirty percent deviation in the value of the parameter. The bending is quite pronounced and the oscillations of the gimbal angle

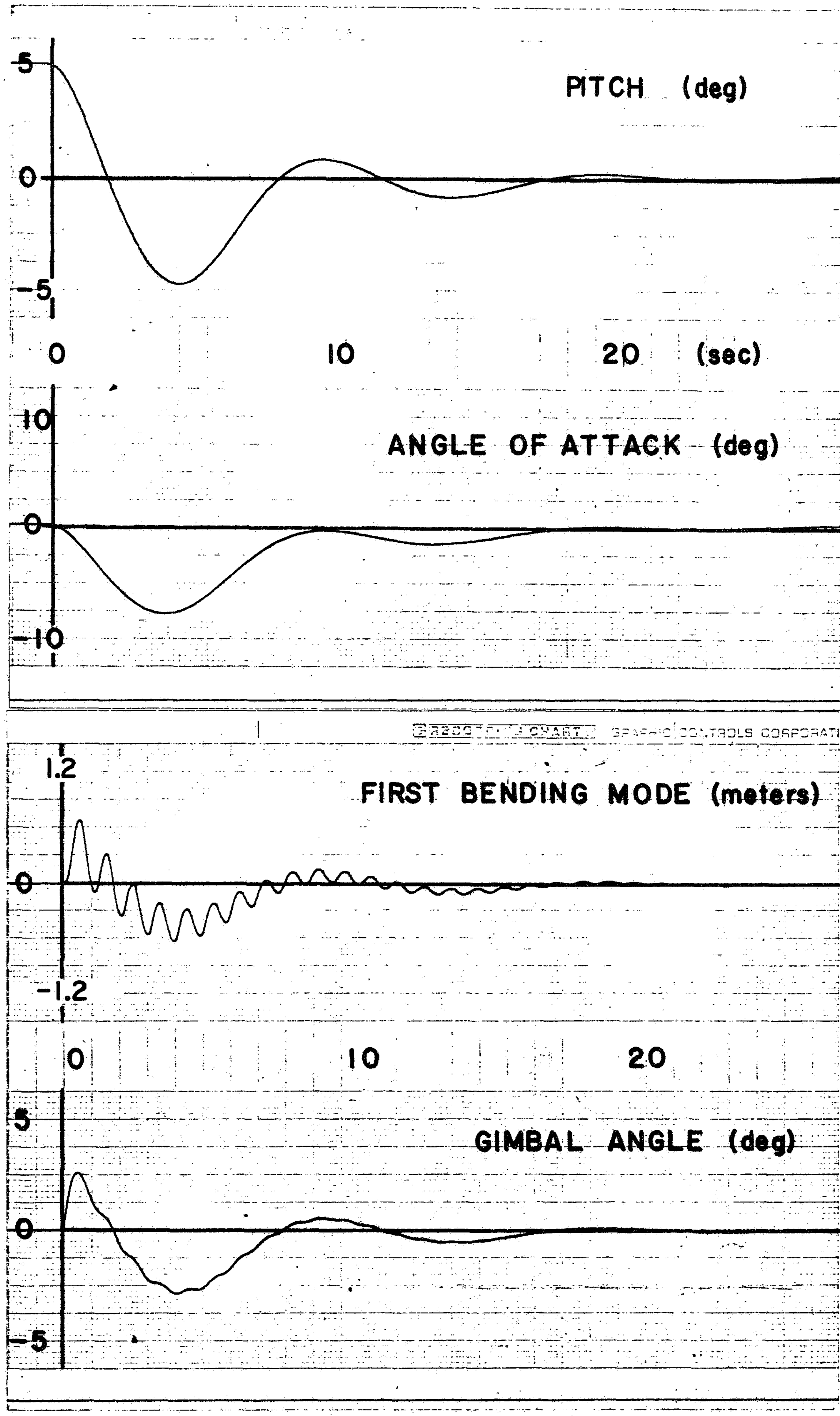


FIG. 5-14 DESENSITIZED BOOSTER $\omega = \omega_0$

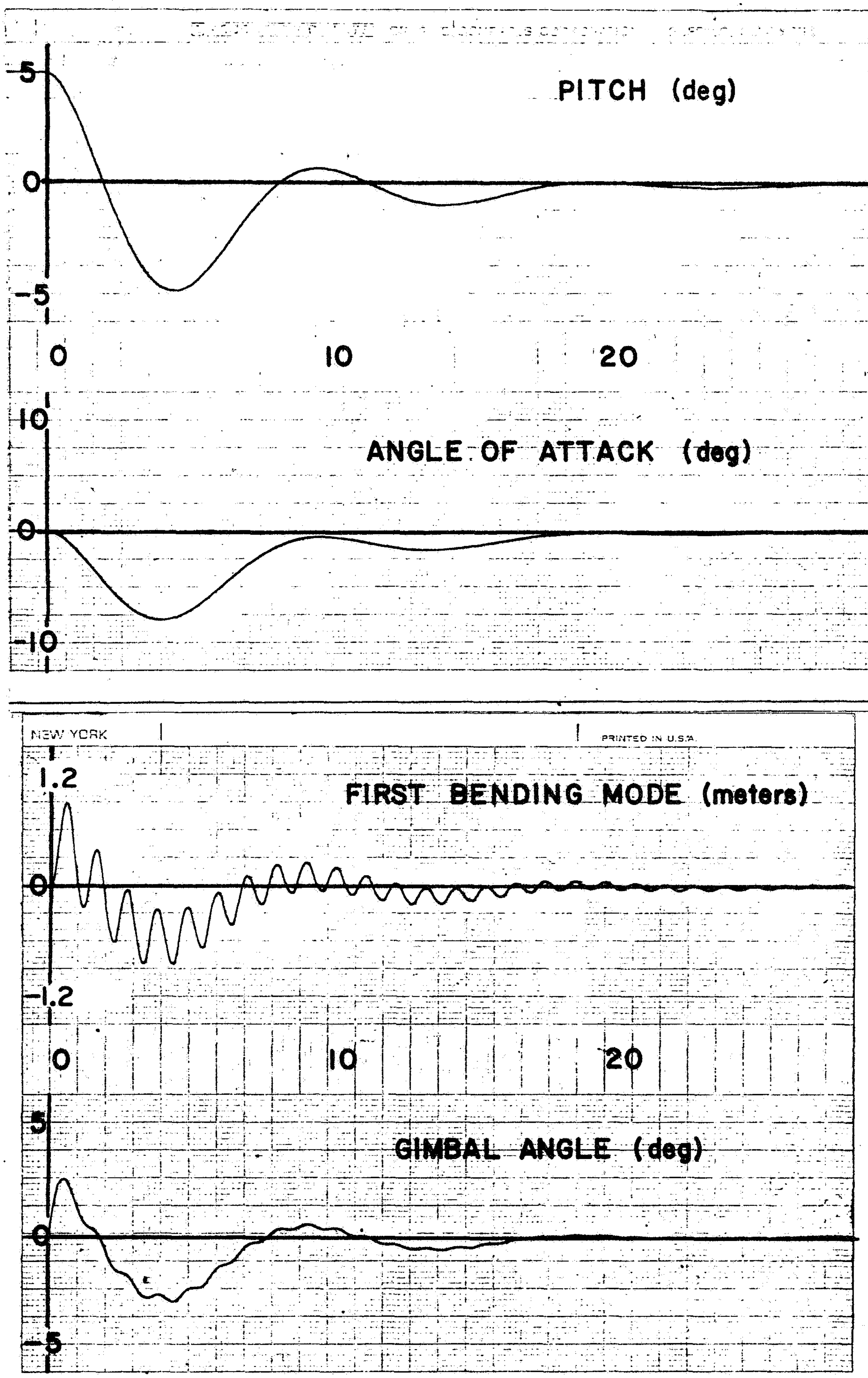


FIG. 5-15 DESENSITIZED BOOSTER $\omega = 0.9\omega_0$

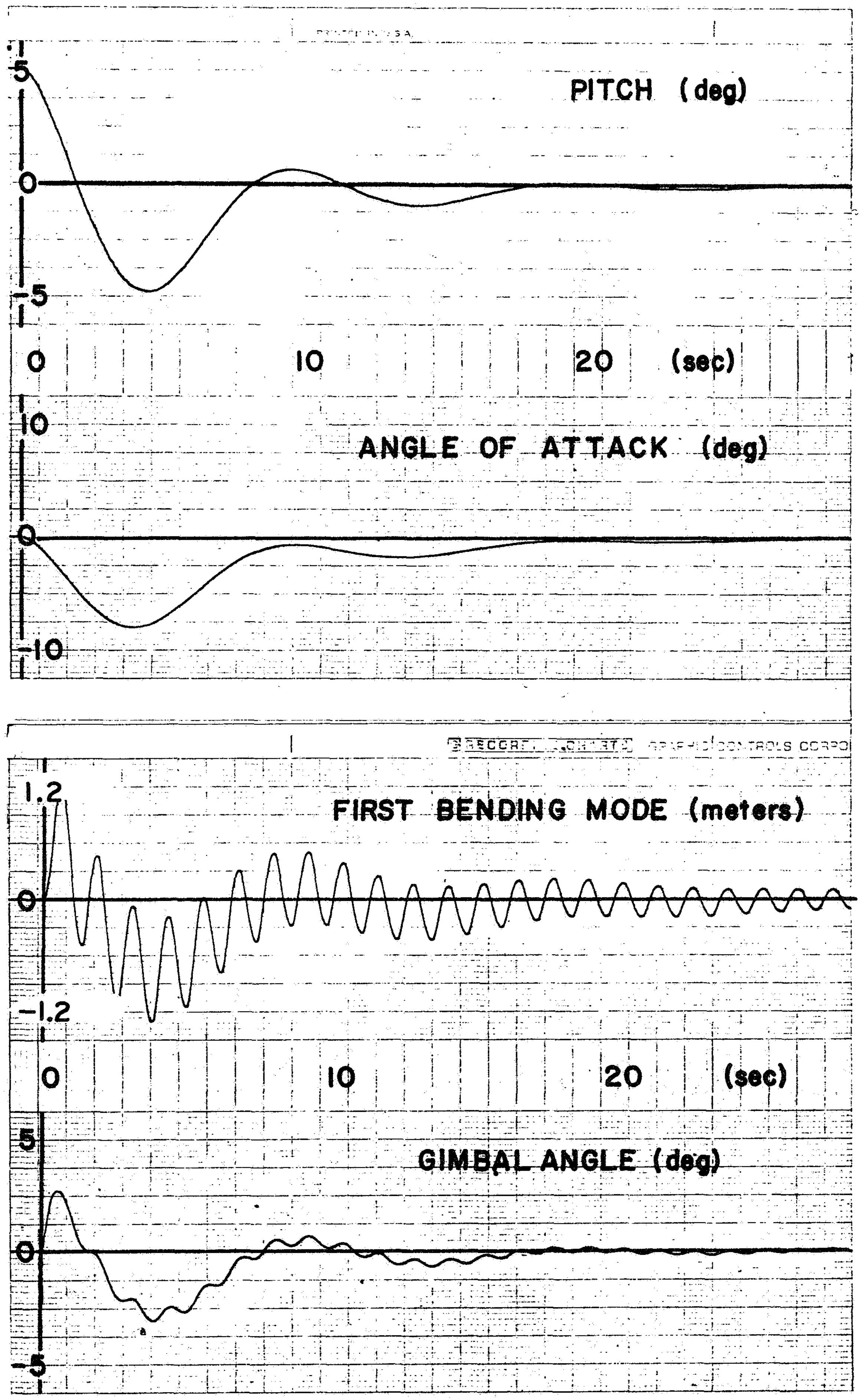


FIG. 5-16 DESENSITIZED BOOSTER $\omega = 0.8\,\omega_0$

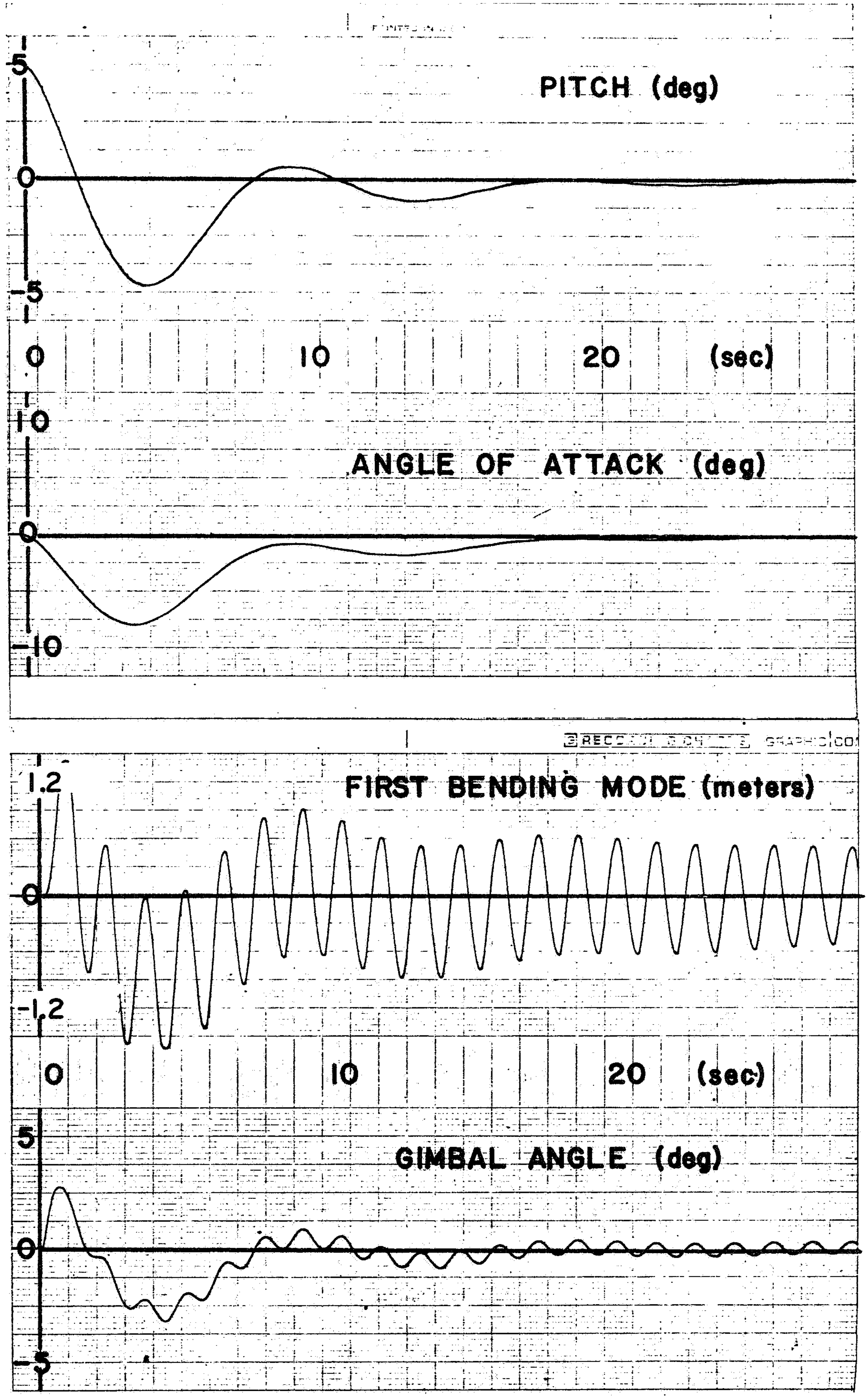


FIG. 5-17 DESENSITIZED BOOSTER $\omega = 0.7\omega_0$

at the bending frequency indicates that stability is becoming marginal, but the system is still stable.

The recordings of Figure 5-10 through Figure 5-17 demonstrate the usefulness and ease of application of Analog Sensitivity Design to realistic systems of high order, in this case seventh order. It is clear from the recordings that the frozen time point model of the Saturn V has indeed been desensitized to changes in the frequency of the first bending mode. Designing the control system to limit the values of the sensitivity coefficients resulted in a system that gave satisfactory response with parameter variations large enough to drive the optimal system unstable.

5.8 Time Varying Booster Simulation

Section 5.7 showed the results obtained when Analog Sensitivity Design was applied to a seventh order frozen time point model of the Saturn V - Apollo. It remains to evaluate this control system on a more realistic time varying model with three bending modes included.

Appendix E gives the complete description of the time varying eleventh order model used to represent the Saturn V during the boost phase of flight. This model was implemented by a time varying digital simulation routine on an IBM 360/50 digital computer. The output of the simulator was used as input to an analog plotting board to make the curves shown in Figure 5-18 through Figure 5-23.

The model was started off with zero initial conditions on all states. The disturbance vector, $u(t)$, was constructed using the design wind profile discussed in Section 5.5 and shown in Figure 5-6. For evaluation of the off nominal response of the vehicle, all three bending mode frequencies were perturbed by the same percentage. This represents as closely as possible the actual situation when the bending parameters are inadequately known.

Figure 5-18a and Figure 5-18b show the rigid body behavior and bending mode response respectively of the Saturn V with nominal values of the bending frequencies, $\omega_i = \omega_{i_0}$, for the optimal value of feedback gains (5.7-1). The vehicle is obviously stable and the response of all states remain within satisfactory limits. It is interesting to note how closely the rigid body states follow the wind induced angle of attack of Figure 5-7 for the design wind with peak values of the response occuring at max q. This indicates that exact knowledge of the wind would make possible an almost unperturbed response by simply adding a signal into the control system to cancel the wind. If the wind was not exactly what was expected, however, this additive signal could easily be worse than none at all. For this reason all the design work was done assuming zero wind and the design wind used only for evaluation of the control systems.

As soon as the bending mode frequencies are decreased by ten percent, $\omega_i = 0.9\ \omega_{i_0}$, the optimal system goes unstable. This is shown in Figure 5-19a and Figure 5-19b. By comparing these two figures it appears that the bending becomes unstable first. This is then fed to the gimbal through the gyro coupling terms in the booster equations and the rigid body driven unstable. For clarity in this and succeeding figures, when an unstable response goes off scale on the graph, plotting of it is

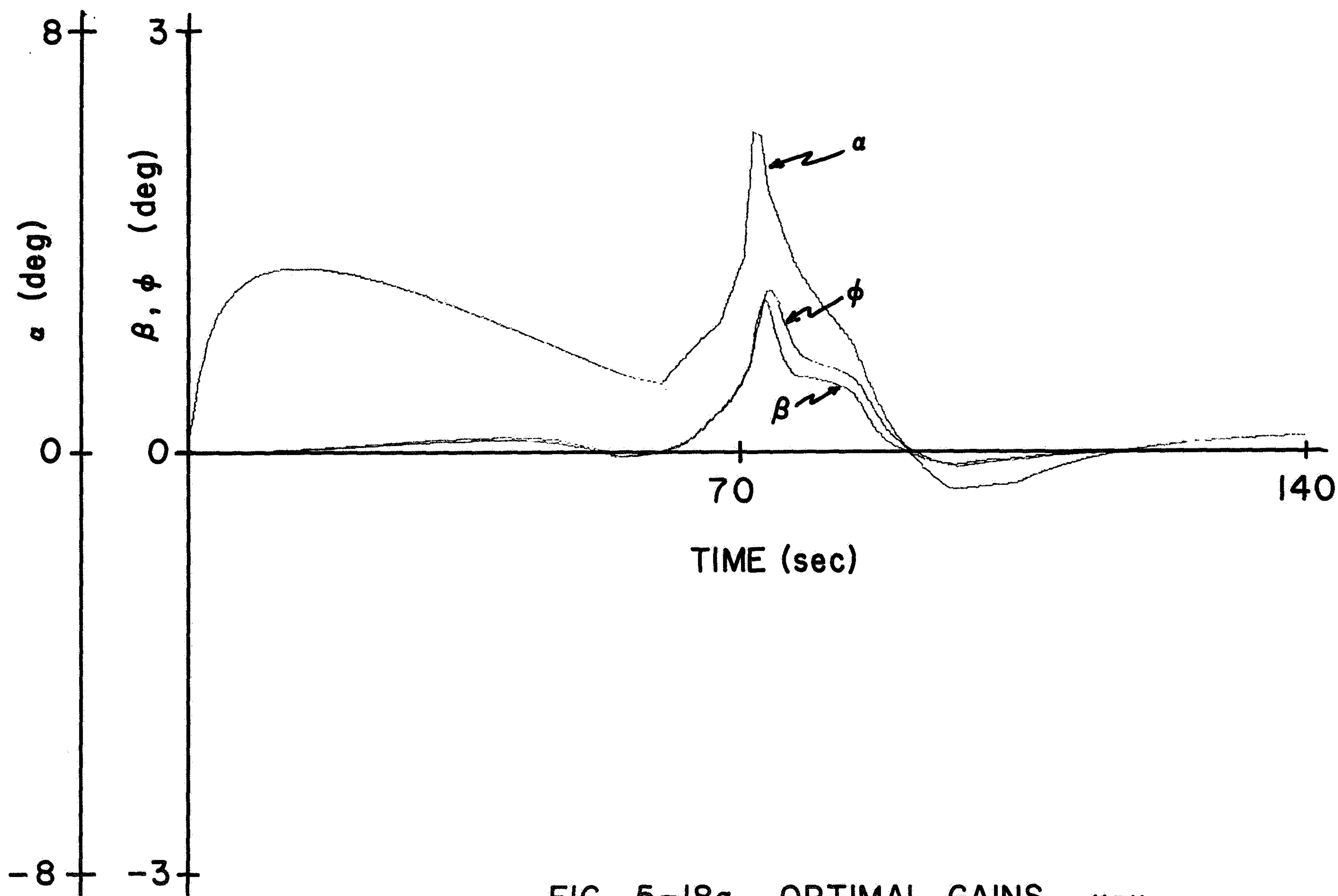


FIG. 5-18a OPTIMAL GAINS $\omega = \omega_o$

RIGID BODY

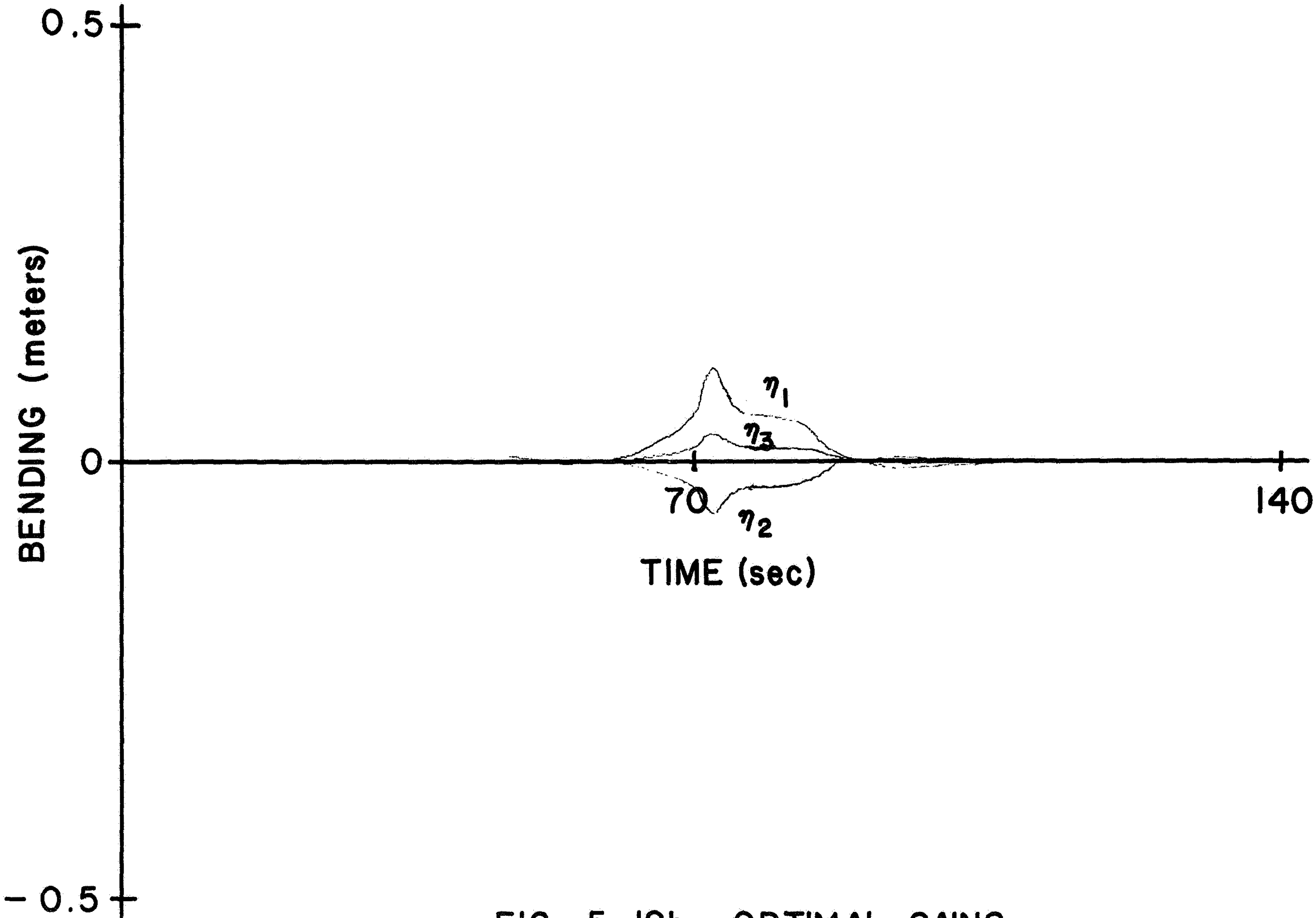


FIG. 5-18b OPTIMAL GAINS $\omega=\omega_o$

BENDING

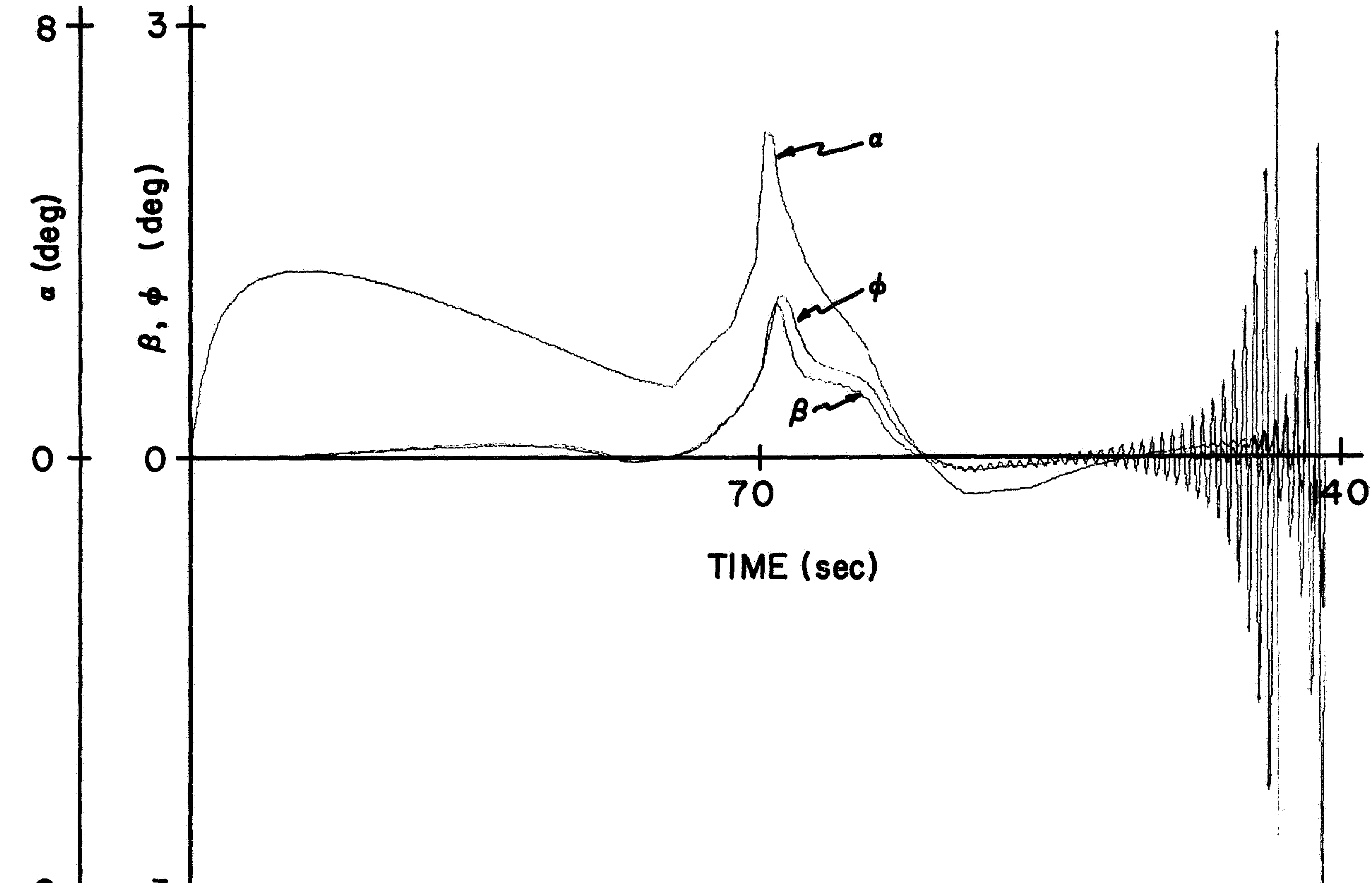


FIG. 5-19a OPTIMAL GAINS $\omega = 0.9\omega_o$

RIGID BODY

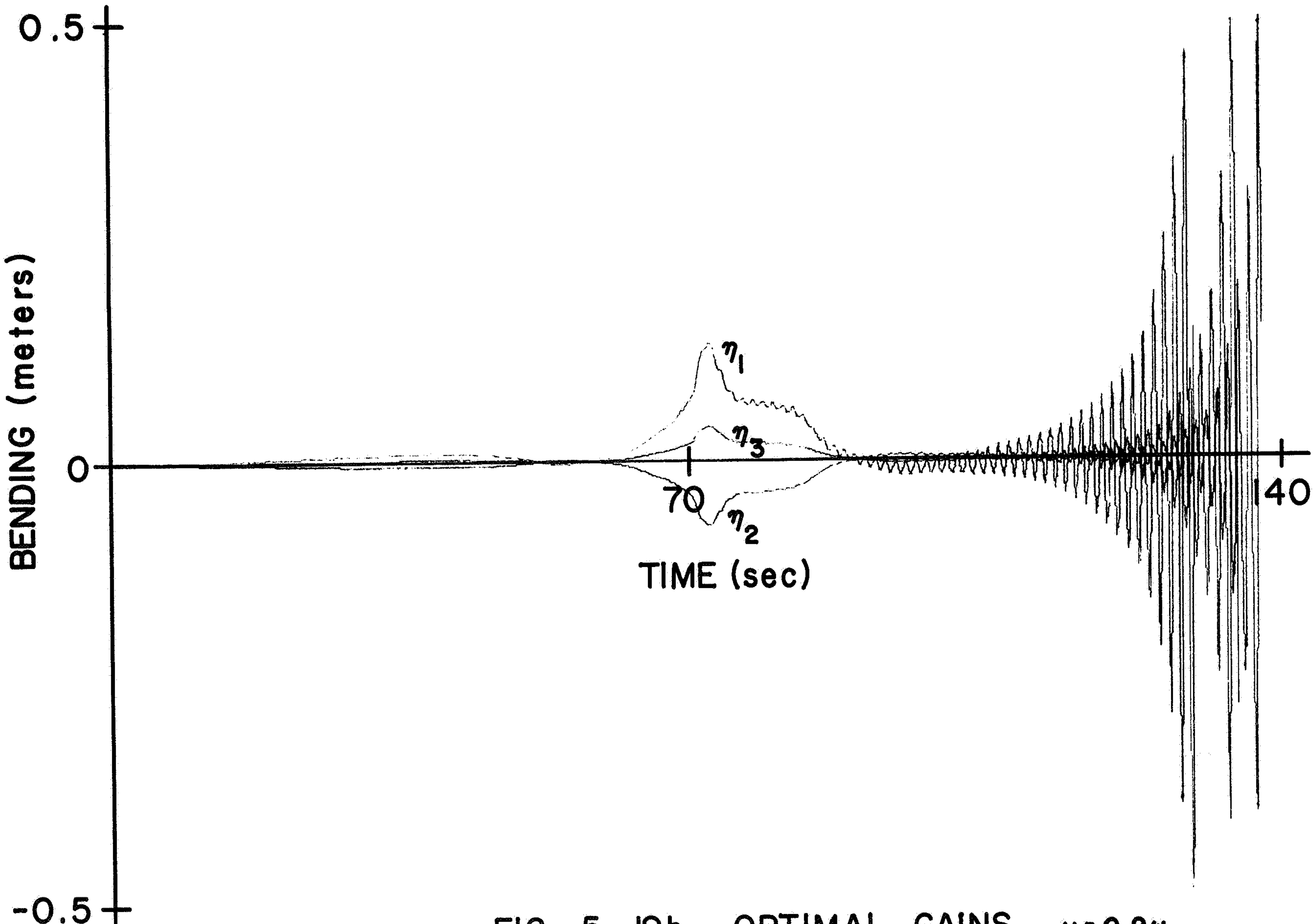


FIG. 5-19b OPTIMAL GAINS $\omega = 0.9\omega_0$

BENDING

terminated so that the remaining quantities may be seen more readily.

Figure 5-20a and Figure 5-20b show the response of the optimal system for $\omega_i = 0.85\ \omega_{i_0}$. For this larger parameter perturbation instability has come more quickly as could be expected.

From these curves it is apparent that while the optimal system using the gains (5.7-1) gives satisfactory control for nominal values of the bending mode frequencies, just a small decrease in these frequencies from the nominal value is sufficient to cause the system to go destructively unstable. Desensitization of the vehicle control system is definitely needed.

Next the time varying simulation of the Saturn V - Apollo was run using the desensitized gains of (5.7-3). The response for nominal bending frequencies is shown in Figure 5-21a and Figure 5-21b. If these are compared with the nominal response for the optimal system of Figure 5-18 it can be seen that the pitch and gimbal angle response are somewhat larger for the desensitized system. The pitch angle, in particular, is almost twice as large at its maximu, although still within acceptable limits. This decrease in the desirability of the nominal response must be expected and, as discussed in Section 4.4, is the price that must be paid for decreased system sensitivity.

That the system has indeed been desensitized is apparent from Figure 5-22a and Figure 5-22b which show the system response for $\omega_i = 0.9\ \omega_{i_0}$. It is very difficult to distinguish this from the nominal response except for a very slight increase in the magnitude of the bending at max q.

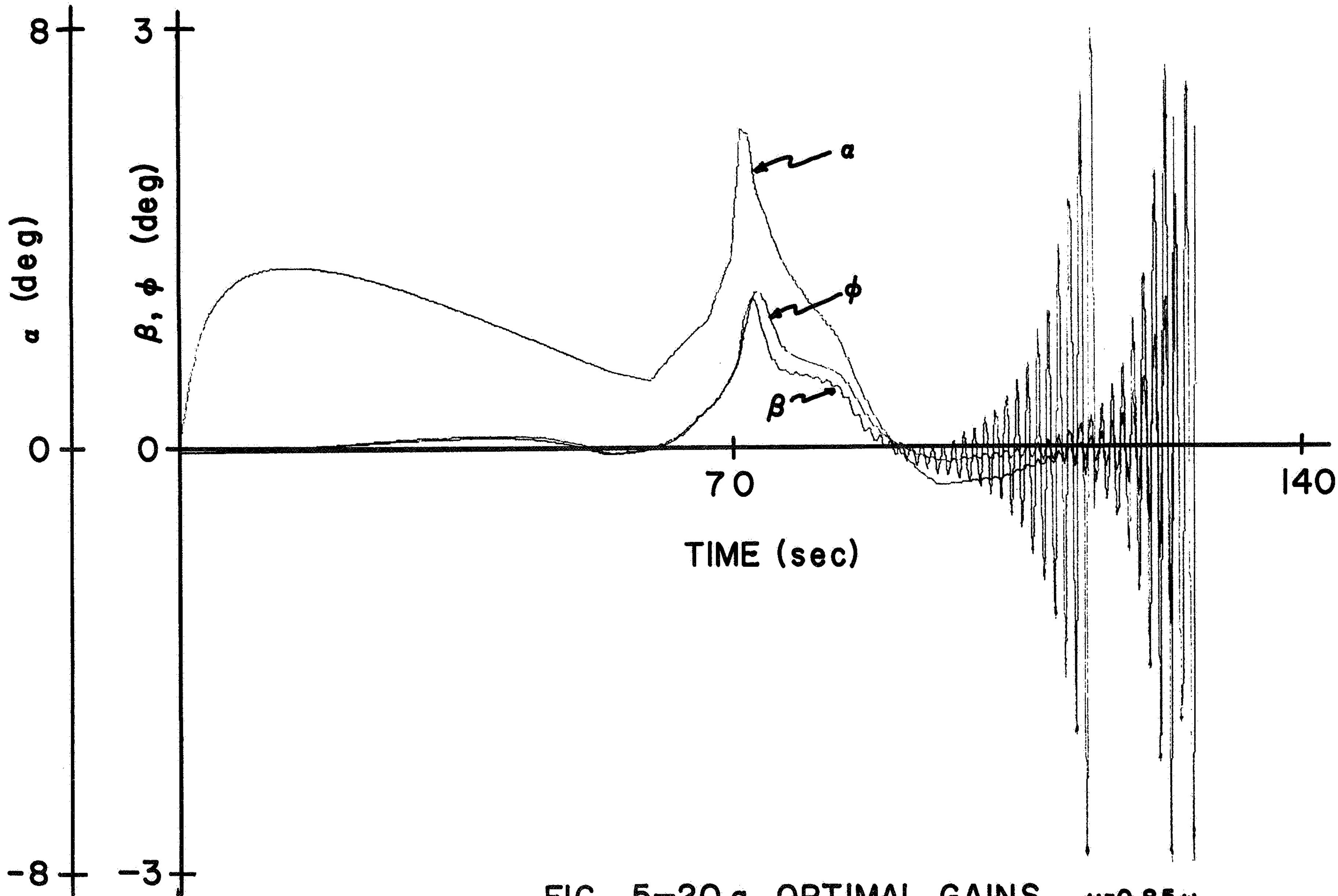


FIG. 5-20a OPTIMAL GAINS $\omega = 0.85\omega_0$

RIGID BODY

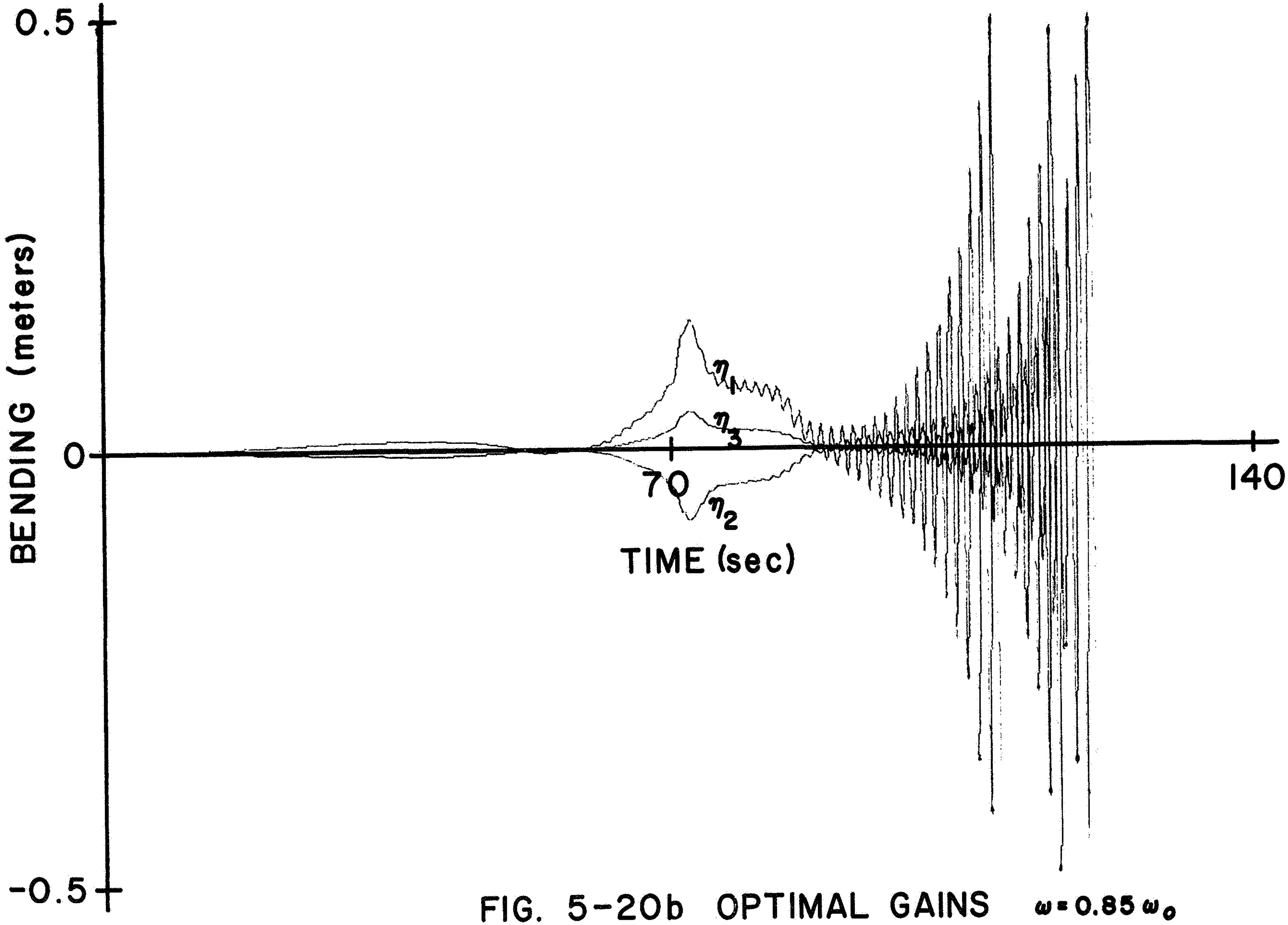


FIG. 5-20b OPTIMAL GAINS $\omega = 0.85\,\omega_o$

BENDING

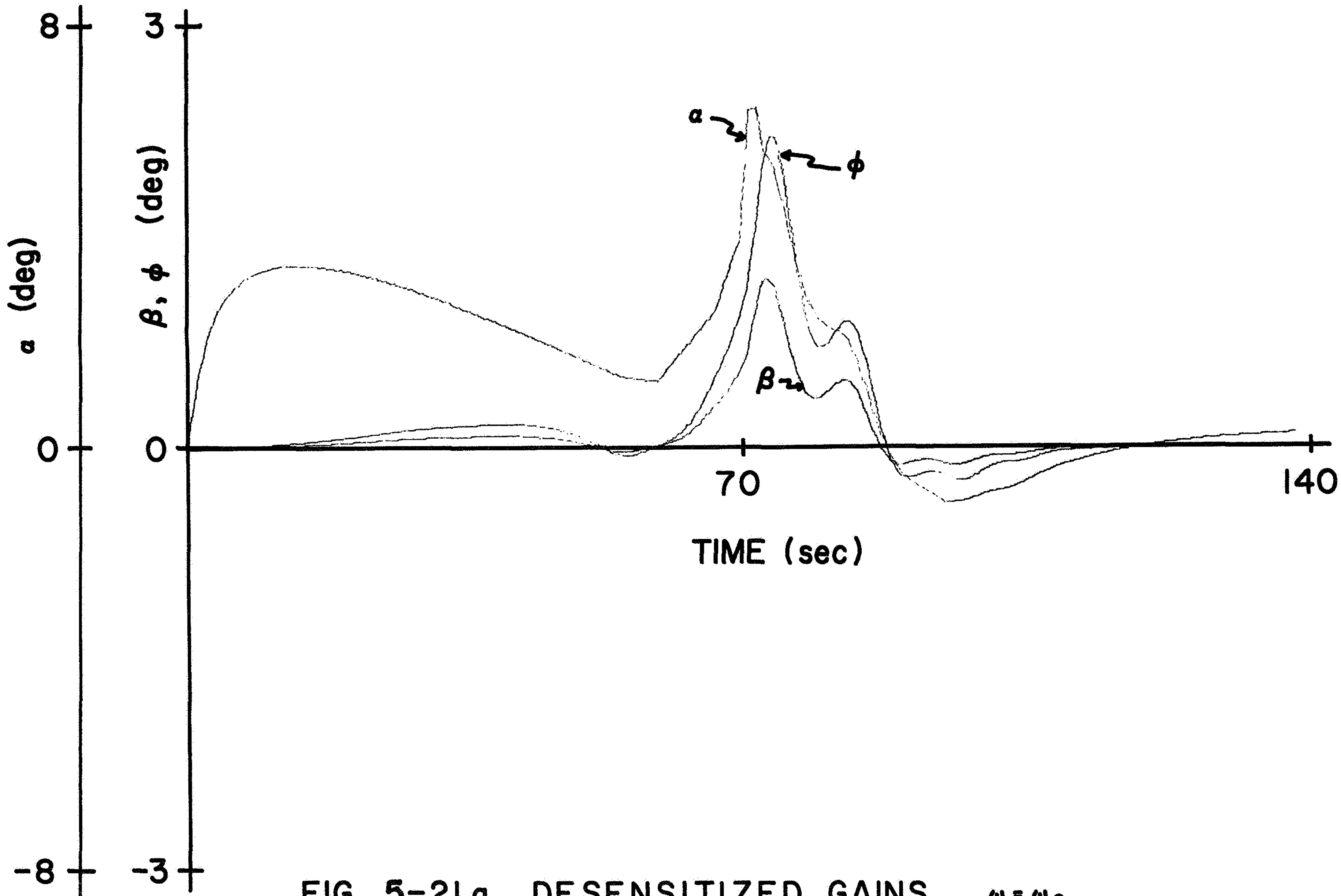


FIG. 5-21a DESENSITIZED GAINS $\omega = \omega_o$

RIGID BODY

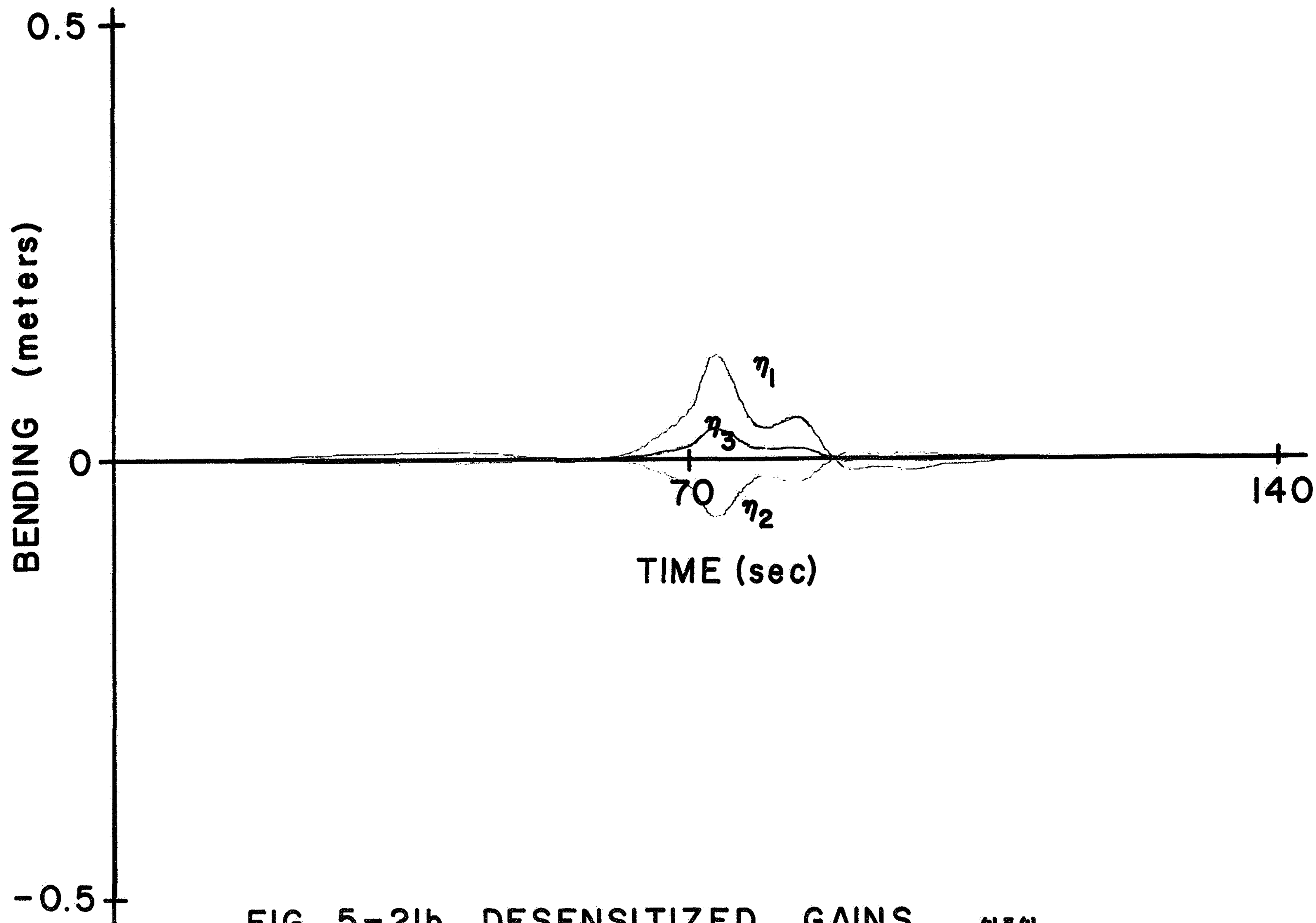


FIG. 5-21b DESENSITIZED GAINS $\omega = \omega_o$

BENDING

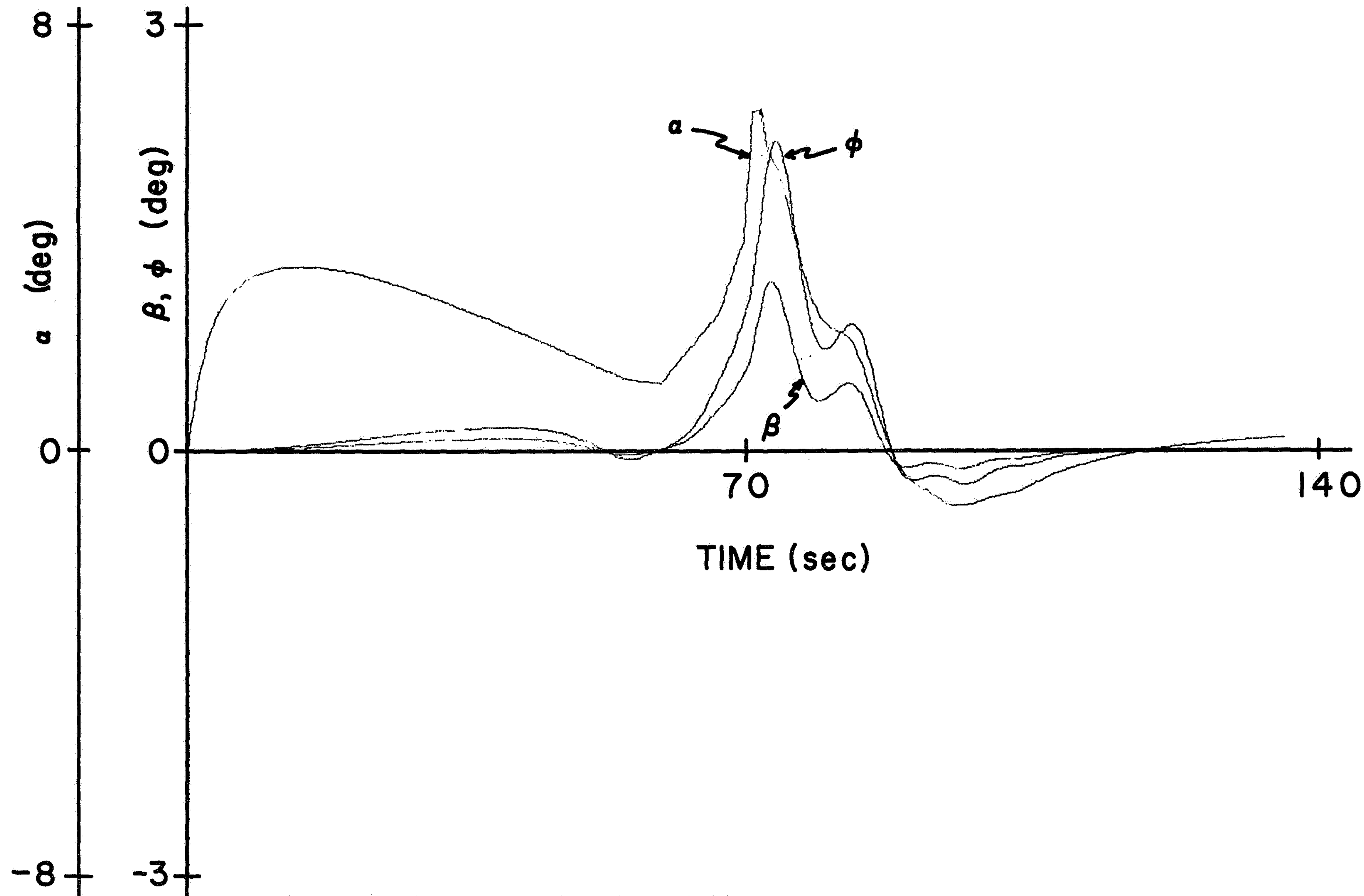


FIG. 5-22a DESENSITIZED GAINS $\omega = 0.9\omega_0$

RIGID BODY

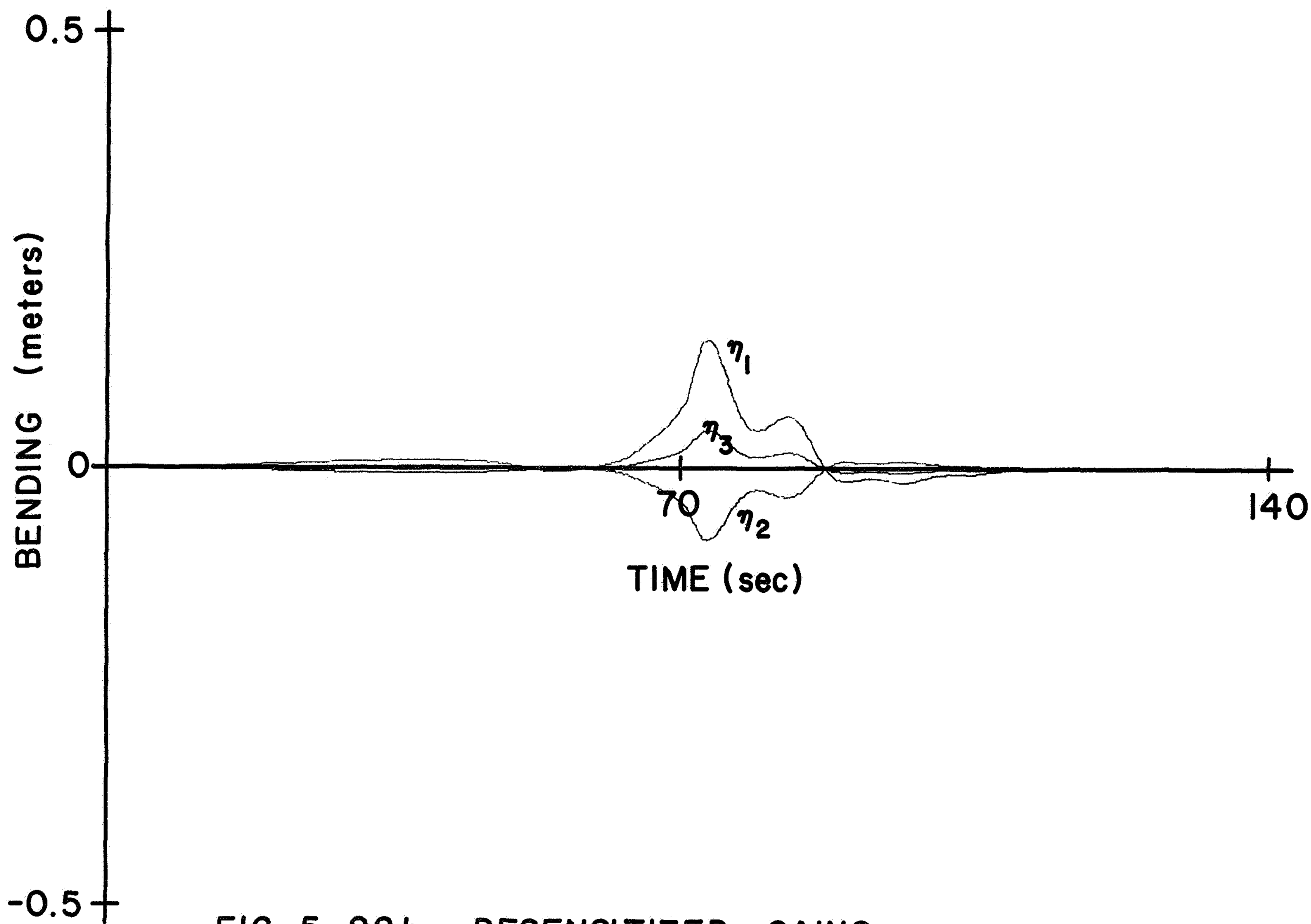


FIG. 5-22b DESENSITIZED GAINS $\omega = 0.9\omega_0$

BENDING

Figure 5-23a and Figure 5-23b, with $\omega_i = 0.85\ \omega_{i_0}$ offer further proof of the system insensitivity. Again the only difference between this response and the nominal response for the desensitized gains is an increase in the bending around the max q region. Even for this fifteen percent perturbation in bending frequency of all three modes, the response of the vehicle is excellent. This should be compared with the violently unstable behavior of the optimal system with this large a parameter perturbation as shown in Figure 5-20.

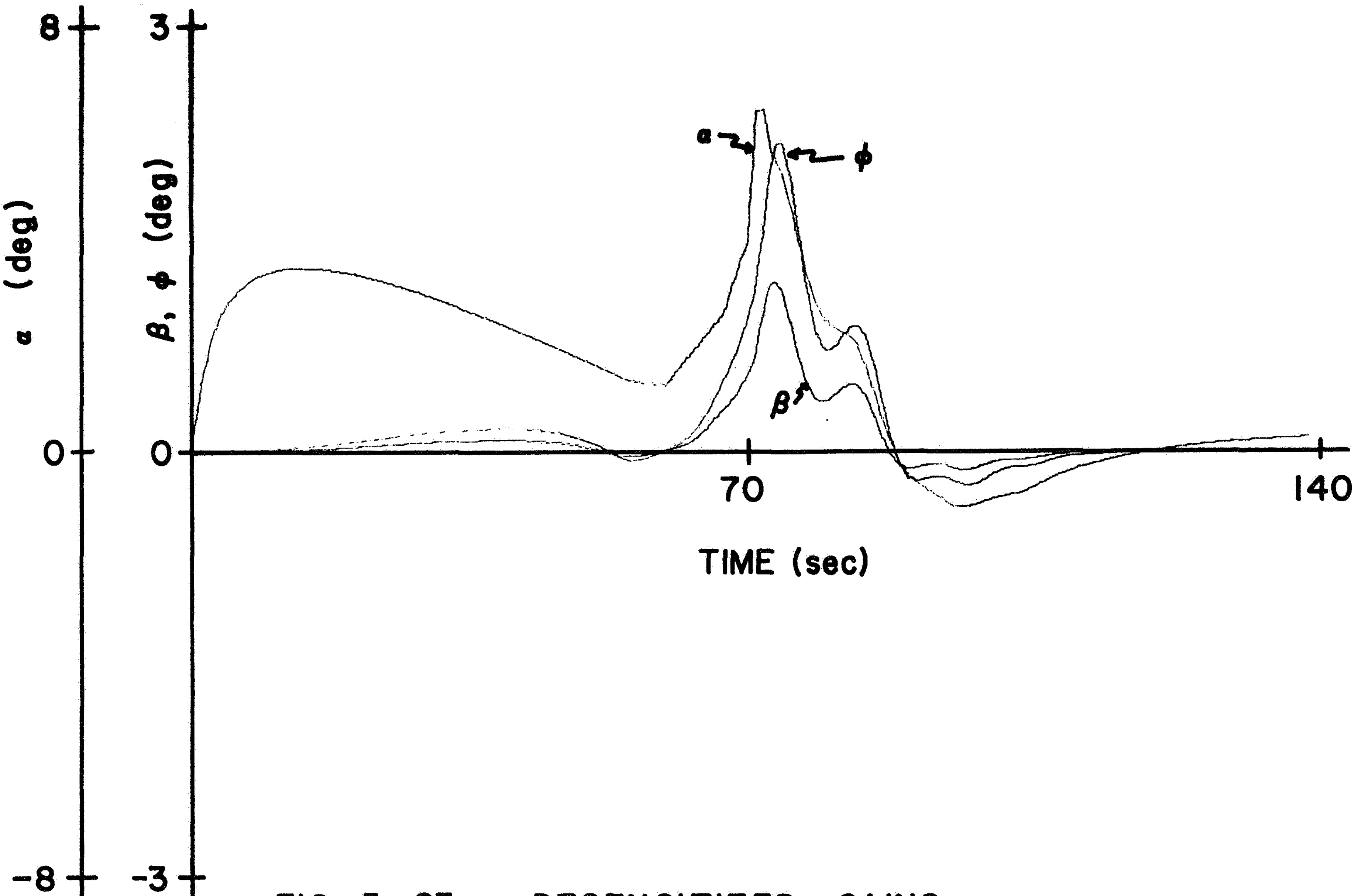


FIG. 5-23a DESENSITIZED GAINS $\omega = 0.85\omega_0$

RIGID BODY

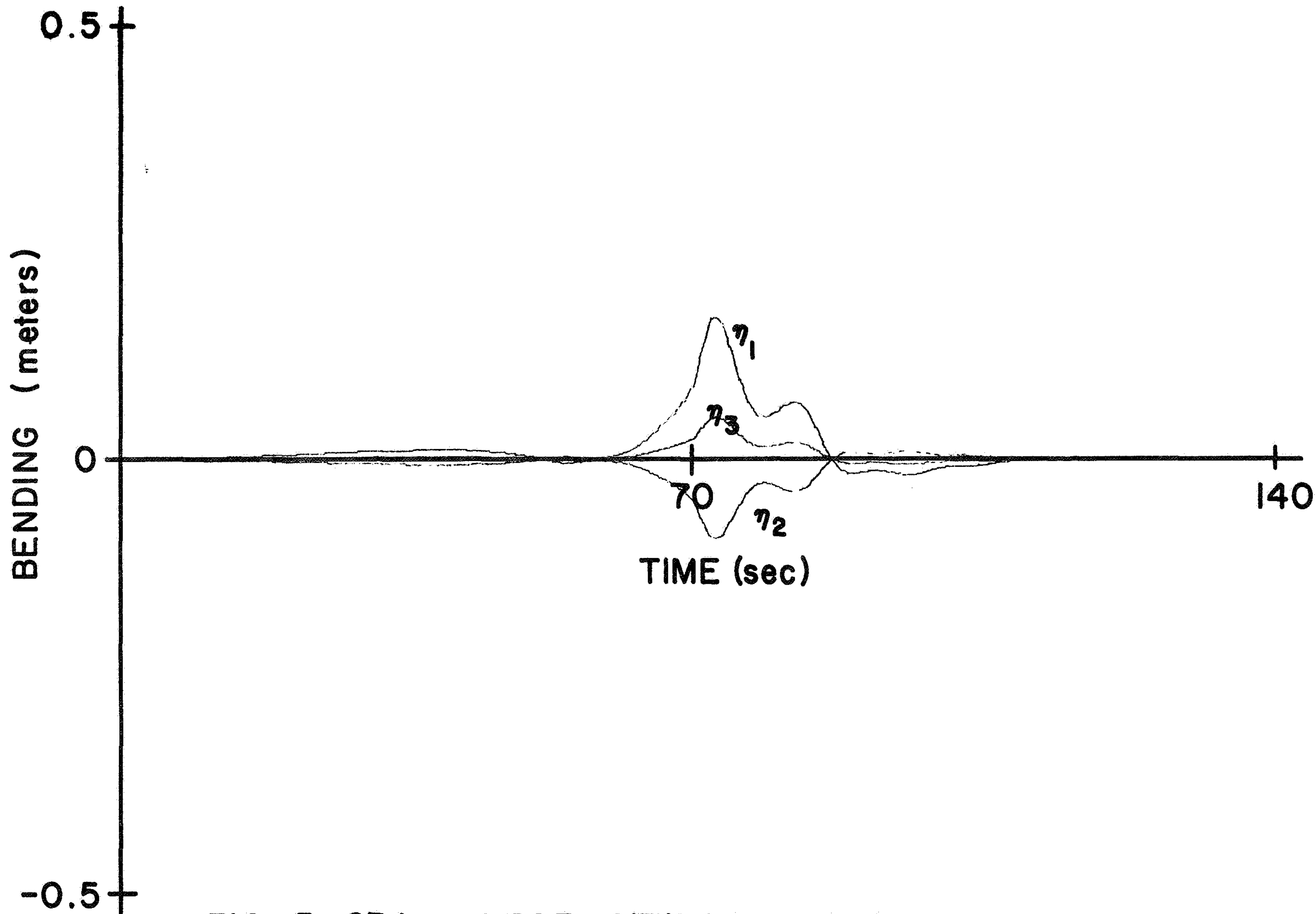


FIG. 5-23 b DESENSITIZED GAINS $\omega = 0.85\omega_0$

BENDING

CHAPTER 6

SUMMARY AND RECOMENDATIONS

6.1 Summary

This paper has demonstrated that until now there was a definite lack of a truly useful sensitivity design technique which enabled the designer to factor into the design of a dynamic system the desired insensitivity to certain parameters. Several methods of sensitivity analysis were examined but none proved to be extendable to a tractable design procedure. Dougherty's technique of combining the sensitivity equations with the standard parameter optimization problem was shown to require solution of a two point boundary value problem. Thus, the method is slow to converge and likely to suffer from numerical problems for high order systems.

Analog Sensitivity Design, developed in Chapter 4, was shown to satisfy this lack. It provides a versitile, easy to apply method of accomplishing the desensitization of virtually any system even if the describing equations of motion are unknown. As one of its distinct advantages, the designer can directly observe the tradeoff between system nominal response and insensitivity to parameter variations.

Using Analog Sensitivity Design it was possible to design a control system for a seventh order model of the Saturn V - Apollo that was insensitive to changes in bending mode frequency. This was compared with an optimal design which became destructively unstable for identical perturbations in bending mode frequency. The control system was then successfully used to control an eleventh order, time varying model of the booster disturbed by a design wind for various perturbations of the bending frequencies of three modes.

6.2 Recomendations for Future Work

There are several extensions of this work that would be of interest in determining the range of applicability of Analog Sensitivity Design in practice.

First, because of the limited amount and sophistication of the analog computation equipment available it was impossible to work with a time varying dynamic system. For the booster problem a frozen time point model had to be used. Theoretically, Analog Sensitivity Design should work just as well using a time varying model and the results would certainly be more closely related to the response of an actual time varying dynamic system.

Also because of the analog computation equipment available it was necessary to perform the hill climbing iterations manually. Implementation of the automated technique, described in Chapter 4, using logic elements would make the mechanics of problem solution easier.

The structural method of Section 4.3 discuss the use of Analog Sensitivity Design on physical devices with unknown mathematical descriptions. It would be interesting to see how well this actually worked in practice. Two identical physical systems consisting of the interconnections of linear blocks of elements would be needed, but actual implementation should not prove to difficult.

Kokotovic[11] has developed a technique called the method of "Sensitivity Points" whereby all the sensitivity coefficients of a system can be obtained simultaneously from a single sensitivity model. Using this method and defining the additional sensitivity coefficients

(6.2-1)

$$y_i = \frac{\partial I}{\partial K_i}$$

where I is the performance index of Section 4.2 and the K_i are the adjustable gains, it should be possible to develop a gradient technique for the solution of the Analog Sensitivity Design problem. This would speed the convergence to a solution and would require very little additional analog computation equipment.

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APPENDIX A

SENSITIVITY EQUATION INITIAL CONDITIONS

When a parameter variation of the dynamic system (A-1) causes the initial conditions $x(0)$ to change,

$$\begin{aligned} \dot{x} &= f(x,\ t,\ q_o) \\ x(0) &= c(q) \end{aligned} \qquad \text{(A-1)}$$

the analysis of Section 3.3 is no longer valid. To examine the effect of such parameter variation, consider the following case

$$\begin{aligned} \dot{x} &= f(x,\ t,\ q_o + \Delta q) \\ x(0) &= c + \Delta c(q) \end{aligned} \qquad \text{(A-2)}$$

where the change in initial conditions is due to the perturbation in the parameter q.

Using the alternative definition of the sensitivity coefficients developed in Section 3.3, the initial condition of the sensitivity equation of (A-2) is

$$\begin{aligned} z(0) = \lim_{\Delta q \to 0} \frac{x(q_o + \Delta q,\ 0) - x(q_o, 0)}{\Delta q} &= \lim_{\Delta q \to 0} \frac{c + \Delta c - c}{\Delta q} \qquad \text{(A-3)} \\ &= \frac{\partial c}{\partial q}(q_o) \end{aligned}$$

Thus in general, when the initial condition of a dynamic system can be changed by a parameter variation, the initial condition of the sensitivity equation is

$$z(0) = \frac{\partial x(0)}{\partial q}(q_o) \qquad \text{(A-4)}$$

APPENDIX B

ANALYTICITY OF SYSTEM EQUATIONS

The development of the sensitivity coefficients of Chapter 3 depends on the fact that the differential equation of the dynamic system is analytic in the parameter q. This appendix shows that this does not hold when a variation of the parameter can change the order of the system.

Consider the linear system of (B-1).

$$\sum_{i=0}^{n} a_i \frac{\partial x^i}{\partial t^i} = 0 \tag{B-1}$$

The stability of this system depends on the sign of the real parts of the roots of the characteristic equation (B-2)

$$\sum_{i=0}^{n} a_i \, s^i = 0 \tag{B-2}$$

Equation (B-1) is stable if the roots of (B-2) all have negative real parts and unstable otherwise.

Now assume a particular variation q of the parameter q increases the order of the system by one. The characteristic equation then becomes

$$\Delta q \, s^{n+1} + \sum_{i=0}^{n} a_i' \, s^i = 0 \tag{B-3}$$

where a_i' $i = 1, 2, \ldots, n$ are the new values of the a_i caused by the parameter variation.

Dividing (B-3) through by $q \neq 0$ gives

$$s^{n+1} + \sum_{i=0}^{n} \frac{a_i'}{\Delta q} s^i = 0 \tag{B-4}$$

Thus as $\Delta q \rightarrow 0$, $s^{n+1} \rightarrow -\infty$. Therefore for sufficiently small Δq lower order terms can be neglected leaving

$$s = -\frac{a_i'}{\Delta q} \tag{B-5}$$

which is another way of saying that the $(n+1)^{st}$ eigenvalue λ_{n+1} is given by

$$\lambda_{n+1} = \frac{-a_n'}{\Delta q} \tag{B-6}$$

Thus the solution of (B-1) does not depend analytically on the small parameter Δq.

APPENDIX C

EIGENVALUE SENSITIVITY

C.1 General

In Section 4.2 the eigenvalue sensitivity of a linear dynamic system was shown to be

$$\frac{\partial \lambda_i}{\partial q}(q_o) = \frac{v_i^T(q_o)\,\frac{\partial A}{\partial q}(q_o)\,u_i(q_o)}{v_i^T(q_o)\,u_i(q_o)} \tag{C.1-1}$$

where (λ_i, v_i) are eigenvalue-eigenvector pairs of the system matrix A and the u_i are the corresponding eigenvectors of A^T.

This appendix evaluates the eigenvalue sensitivity of the two desensitized systems of Section 4.6.

The dynamic equations of the system are given by

$$x = \begin{pmatrix} (q-1) - K_1 & -(K_2+1) \\ 1 & 0 \end{pmatrix} x \tag{C.1-2}$$

$$x(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

where $K^T = (K_1 \quad K_2)$ are the feedback gains and q is the varying parameter with nominal value $q_o = 1$.

The first set of gains, denoted by K_d was obtained using Dougherty's digital desensitization scheme

$$K_d = \begin{pmatrix} 4.16 \\ 2.76 \end{pmatrix} \tag{C.1-3}$$

The second set of gains, K_a, was obtained using Analog Sensitivity Design.

$$K_a = \begin{pmatrix} 6.0 \\ 3.7 \end{pmatrix} \tag{C.1-4}$$

C.2 Digital Gains

Using the digital gains (C.1-3) gives a characteristic equation of

$$|\lambda I - A| = \begin{vmatrix} \lambda + 4.16 & 3.76 \\ -1 & \lambda \end{vmatrix} = 0 \tag{C.2-1}$$

which yields eigenvalues of

$$\lambda_1 = -2.82 \qquad \lambda_2 = -1.34 \tag{C.2-2}$$

The corresponding eigenvectors are then given by

$$\begin{pmatrix} 1.34 & 3.76 \\ -1 & -2.82 \end{pmatrix} v_1 = 0$$

$$\begin{pmatrix} 2.82 & 3.76 \\ -1 & -1.34 \end{pmatrix} v_2 = 0 \tag{C.2-3}$$

which solve to yield

$$v_1 = \begin{pmatrix} -2.82 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1.34 \\ 1 \end{pmatrix} \tag{C.2-4}$$

The eigenvectors of the transposed system matrix are given by

$$\begin{pmatrix} 1.34 & -1 \\ 3.76 & -2.82 \end{pmatrix} u_1 = 0$$

$$\begin{pmatrix} 2.82 & -1 \\ 3.76 & -1.34 \end{pmatrix} u_2 = 0 \qquad (C.2\text{-}5)$$

Which, when solved give

$$u_1 = \begin{pmatrix} 1 \\ 1.34 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 2.82 \end{pmatrix} \qquad (C.2\text{-}6)$$

Using the values from (C.2-2), (C.2-4) and (C.2-6) in the formula for eigenvalue sensitivity (C.1-1) gives

$$\frac{\partial \lambda_1}{\partial q}(q_o) = 1.9$$

$$\frac{\partial \lambda_2}{\partial q}(q_o) = -\ 0.9 \qquad (C.2\text{-}7)$$

C.3 Analog Sensitivity Design Gains

An analysis similar to that of Section C.2 can be made for the analog gains (C.1-4). The resulting eigenvalues are

$$\lambda_1 = -\ 5.07 \qquad \lambda_2 = -\ 0.93 \qquad (C.3\text{-}1)$$

The eigenvectors of A are

$$v_1 = \begin{pmatrix} -5.07 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -0.93 \\ 1 \end{pmatrix} \qquad (C.3\text{-}2)$$

and the eigenvectors of A^T are

$$u_1 = \begin{pmatrix} 1 \\ 0.93 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 5.07 \end{pmatrix} \tag{C.3-3}$$

Using these values in the eigenvalue sensitivity formula gives

$$\frac{\partial \lambda_1}{\partial q}(q_o) = 1.23 \qquad \frac{\partial \lambda_2}{\partial q}(q_o) = -\ 0.23 \tag{C.3-4}$$

APPENDIX D

TIME DERIVATIVE OF ROTATING UNIT VECTORS

Figure D-1 shows two coordinate systems. Assume the xyz system is rotating with angular velocity with respect to the fixed XYZ coordinate system. Thus unit vectors in the x, y, z directions are $\underline{i}$, $\underline{j}$, $\underline{k}$ respectively.

Consider one of the unit vectors, call it $\underline{r}$, and assume the axis of rotation is that of Figure D-2. The magnitude of the tangential velocity $v = |\underline{v}|$ at point P is given by

$$v = \lim_{\Delta t \to 0} \frac{a \, \Delta \theta}{\Delta t} = \frac{a \, d\theta}{dt} = a\omega \tag{D-1}$$

But the radius is given by

$$a = r \sin \alpha \tag{D-2}$$

Thus combining equations (D-1) and (D-2) gives an expression for the magnitude of the velocity

$$v = \omega \, r \sin \alpha \tag{D-3}$$

Using (D-3) together with the fact that $\underline{v}$ is perpendicular to the plane formed by $\underline{\omega}$ and $\underline{r}$ gives

$$\underline{v} = \underline{\omega} \times \underline{r} \tag{D-4}$$

But since $\underline{v}$ is the time derivative of $\underline{r}$ this can be written as

$$\underline{r} = \underline{\omega} \times \underline{r} \tag{D-5}$$

Then using (D-5) with the unit vectors i, j, k in place of the general unit vector r gives

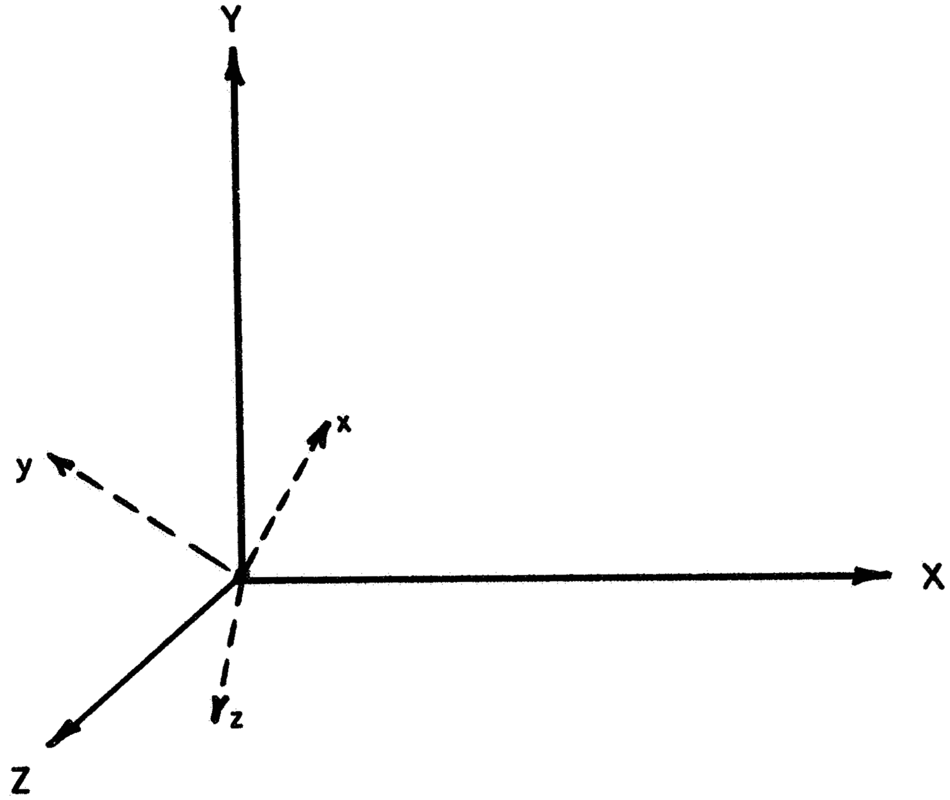

FIG. D-1 ROTATING SYSTEM

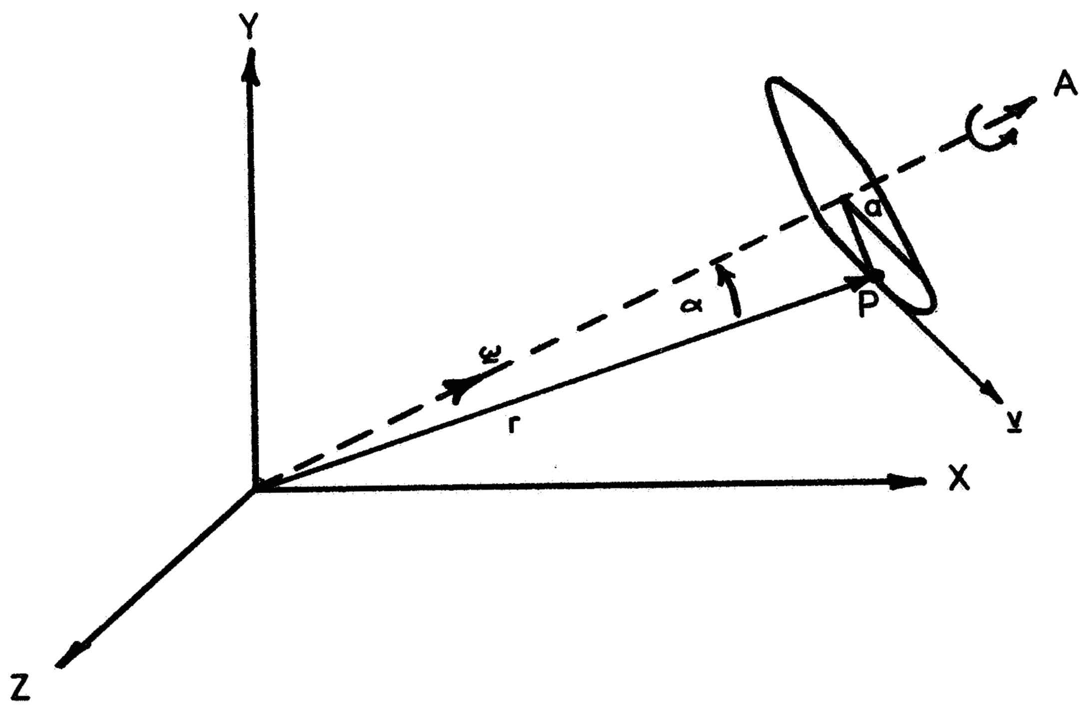

FIG. D-2 ROTATING VECTOR

$$\frac{d\underline{i}}{dt} = \underline{\omega} \times \underline{i}$$

$$\frac{d\underline{j}}{dt} = \underline{\omega} \times \underline{j} \tag{D-6}$$

$$\frac{d\underline{k}}{dt} = \underline{\omega} \times \underline{k}$$

APPENDIX E

ANALOG SIMULATION OF BOOSTER

E.1 Booster Equations

The differential equations of the seventh order model of the Saturn V - Apollo used in the Analog Sensitivity Design of Section 5.7 are

$$\ddot{\phi} = -\left(\frac{R' lcg}{I}\right)\beta - \left(\frac{N' lcp}{I}\right)\alpha \tag{E.1-1a}$$

$$\dot{\alpha} = -\left(\frac{T-D}{mv} - \frac{v}{v}\right)\phi + \dot{\phi} - \left(\frac{N'}{mv} + \frac{v}{v}\right)\alpha - \frac{R'}{mv}\beta \tag{E.1-1b}$$

$$\ddot{\eta}_1 = -\omega_1^2 \eta_1 - 2\zeta_1 \omega_1 \dot{\eta}_1 + \frac{R' Y_1(x_\beta)}{m_1}\beta \tag{E.1-1c}$$

$$\beta_c = -K_1\left[\phi + Y_1'(x_D)\eta_1\right] - K_2\left[\phi + Y_1'(x_R)\dot{\eta}_1\right] \tag{E.1-1d}$$

$$\ddot{\beta} = -50\,\beta - 10\,\dot{\beta} + 50\,\beta_c \tag{E.1-1e}$$

For ease in notation these equations will be rewritten as

$$\ddot{\phi} = a_{23}\,\alpha + b_2\,\beta \tag{E.1-2a}$$

$$\dot{\alpha} = a_{31}\,\phi + \dot{\phi} + a_{33}\,\alpha + b_3\,\beta \tag{E.1-2b}$$

$$\ddot{\eta}_1 = a_{54}\eta_1 + a_{55}\dot{\eta}_1 + b_5\,\beta \tag{E.1-2c}$$

$$\beta_c = -K_1(\phi + c_1\eta_1) - K_2(\phi + c_2\dot{\eta}_1) \tag{E.1-2d}$$

$$\ddot{\beta} = d_1\,\beta + d_2\,\dot{\beta} + d_3\,\beta_c \tag{E.1-2e}$$

These equations completely describe the seventh order booster model.

E.2 Scaling the Model

Before the booster model (E.1-1) can be put on the analog computer it must be magnitude scaled. This involves choosing maximum values for all the variables. The assumed maxima are

$$
\begin{array}{ll}
\phi \text{ max} = 0.5 \text{ radians} & \beta \text{ max} = 0.1 \text{ radians} \\
\dot{\phi} \text{ max} = 0.5 \text{ radians} & \phi \text{ max} = 0.5 \text{ radians/sec} \\
\alpha \text{ max} = 0.5 \text{ radians} & \alpha \text{ max} = 0.5 \text{ radians/sec} \\
\eta \text{ max} = 2.0 \text{ meters} & \ddot{\eta} \text{ max} = 20.0 \text{ meters/sec} \\
\dot{\eta} \text{ max} = 20.0 \text{ meters/sec} &
\end{array}
\qquad (E.2\text{-}1)
$$

These figures are quite conservative for a satisfactory control system hence saturation problems are avoided even for rather large deviations in response.

Since an EAI TR 20 analog computer has a maximum dynamic range of $\pm$ 10 volts, these maxima of (E.2-1) should correspond to 10 volts. Therefore define the scaled or computer variables as

$$
\begin{array}{lll}
\phi^s = 20\,\phi & \alpha^s = 20\,\alpha & \beta^s = 100\,\beta \\
\dot{\phi}^s = 20\,\dot{\phi} & \eta^s = 5\,\eta & \\
\ddot{\phi}^s = 20\,\ddot{\phi} & \dot{\eta}^s = 0.5\,\dot{\eta} & \\
 & \ddot{\eta}^s = 0.5\,\ddot{\eta} &
\end{array}
\qquad (E.2\text{-}2)
$$

It is most important at this point to note that since η and $\dot{\eta}$ are scaled differently,

$$
\int \dot{\eta}^s \, dt = \int \frac{1}{2} \dot{\eta} \, dt = \frac{1}{2} \eta = \frac{1}{10} \eta^s \qquad (E.2\text{-}3)
$$

or leaving out the middle steps

$$\eta^{s} = 10 \int \dot{\eta}^{s}\, dt \qquad \text{(E.2-4)}$$

Inserting the values of the variables in terms of the scaled variables from (E.2-2) into the original model equations (E.1-2) gives the scaled equations

$$\ddot{\phi}^{s} = a_{23}\, \alpha^{s} + 0.2\, b_{2}\, \beta^{s} \qquad \text{(E.2-5a)}$$

$$\dot{\alpha}^{s} = a_{31}\, \phi^{s} + \dot{\phi}^{s} + a_{33}\, \alpha^{s} + 0.2\, b_{3}\, \beta^{s} \qquad \text{(E.2-5b)}$$

$$\ddot{\eta}_{1}^{s} = 0.1\, a_{54} \eta_{1}^{s} + a_{55} \dot{\eta}_{1}^{s} + 0.005\, b_{5}\, \beta^{s} \qquad \text{(E.2-5c)}$$

$$\beta_{c}^{s} = - K_{1}\left[5\phi^{s} + 20\, c_{1} \eta_{1}^{s} \right] - K_{2}\left[5\dot{\phi}^{s} + 200\, c_{2} \dot{\eta}_{1}^{s} \right] \qquad \text{(E.2-5d)}$$

$$\ddot{\beta}^{s} = d_{1}\, \beta^{s} + d_{2}\, \dot{\beta}^{s} + d_{3}\, \beta_{c}^{s} \qquad \text{(E.2-5e)}$$

These are the scaled equations that are implemented on the analog computer. The simulation diagram is shown in Figure 5-9a.

APPENDIX F

TIME VARYING BOOSTER MODEL

F.1 Mathematical Model

The time varying model of the Saturn V - Apollo used in this study consists of eleven linearized, first order state equations with time varying coefficients. The linearized state variables are

x_1	rigid body pitch angle ϕ
x_2	rigid body pitch rate $\dot{\phi}$
x_3	rigid body angle of attack α
x_4	normalized first bending mode displacement η_1
x_5	normalized first bending mode rate $\dot{\eta}_1$
x_6	normalized second bending mode displacement η_2
x_7	normalized second bending mode rate $\dot{\eta}_2$
x_8	normalized third bending mode displacement η_3
x_9	normalized third bending mode rate $\dot{\eta}_3$
x_{10}	engine gimbal angle β
x_{11}	engine gimbal rate $\dot{\beta}$

In terms of these state variables the vehicle equations analogous to those of Chapter 5 can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a_{23}\, x_3 + a_{2,10}\, x_{10} \\
\dot{x}_3 &= a_{31}\, x_1 + x_2 + a_{33}\, x_3 + a_{3,10}\, x_{10} + u(t) \\
\dot{x}_4 &= x_5 \\
\dot{x}_5 &= a_{54}\, x_4 + a_{55}\, x_5 + a_{5,10} x_{10} \\
\dot{x}_6 &= x_7 \\
\dot{x}_7 &= a_{76}\, x_6 + a_{77}\, x_7 + a_{7,10}\, x_{10} \\
\dot{x}_8 &= x_9 \\
\dot{x}_9 &= a_{98}\, x_8 + a_{99}\, x_9 + a_{9,10}\, x_{10} \\
\dot{x}_{10} &= x_{11} \\
\dot{x}_{11} &= a_{11,10}\, x_{10} + a_{11,11}\, x_{11} + b_{11}\, \beta_c
\end{aligned}
\tag{F.1-1}
$$

where the time varying coefficients are given by

$$
\begin{aligned}
a_{23} &= -\, N' lcp/I \\
a_{2,10} &= -\, R' lcg/I \\
a_{31} &= -\left(\frac{T-D}{mv} - \frac{\dot{v}}{v}\right) \\
a_{3,10} &= -\, R'/mv \\
a_{33} &= -\left(\frac{N'}{mv} - \frac{\dot{v}}{v}\right) \\
a_{54} &= -\, \omega_1^{\,2} \\
a_{55} &= -\, 2\, \zeta_1\, \omega_1 \\
a_{5,10} &= R' Y_1(x_\beta)/m_1 \\
a_{76} &= -\, \omega_2^{\,2} \\
a_{77} &= -\, 2\, \zeta_2\, \omega_2 \\
a_{7,10} &= -\, R' Y_2(x_\beta)/m_2 \\
a_{98} &= -\, \omega_3^{\,2} \\
a_{99} &= -\, 2\, \zeta_3\, \omega_3
\end{aligned}
$$

$$
\begin{aligned}
a_{9,10} &= -\, R' Y_3(x_\beta)/m_3 \\
a_{11,10} &= -\, \omega_f^{\,2} \\
a_{11,11} &= -\, 2\, \zeta_f\, \omega_f \\
b_{11} &= \omega_f^{\,2} \\
u(t) &= \frac{\dot{v}}{v}\, \alpha_w + \dot{\alpha}_w
\end{aligned}
\tag{F.1-2}
$$

As discussed in Chapter 5, only the outputs of the pitch and pitch rate gyros are available for control information. Thus these measurable variables, y_1 and y_2 respectively, are given by

$$\begin{aligned} y_1 &= x_1 + \Gamma_{14}\, x_4 + \Gamma_{16}\, x_6 + \Gamma_{18}\, x_8 \\ y_2 &= x_2 + \Gamma_{25}\, x_5 + \Gamma_{27}\, x_7 + \Gamma_{29}\, x_9 \end{aligned} \qquad \text{(F.1-3)}$$

where the time varying coefficients are given by

$$\begin{aligned} \Gamma_{14} &= Y_1'(x_D) \qquad & \Gamma_{25} &= Y_1'(x_R) \\ \Gamma_{16} &= Y_2'(x_D) \qquad & \Gamma_{27} &= Y_2'(x_R) \\ \Gamma_{18} &= Y_3'(x_D) \qquad & \Gamma_{29} &= Y_3'(x_R) \end{aligned} \qquad \text{(F.1-4)}$$

The feedback control law is of the form

$$\beta_c = - K_1\, y_1 - K_2\, y_2 \qquad \text{(F.1-5)}$$

where K_1 and K_2 are the pitch and pitch rate gyro feedback gains, respectively.

The gimbal angle β is related to the control signal β_c by the roll off filter characteristic

$$\ddot{\beta} + 2\zeta_f\, \omega_f\, \dot{\beta} + \omega_f^2\, \beta = \omega_f^2\, \beta_c \qquad \text{(F.1-6)}$$

where ω_f is the natural frequency of the filter and ζ_f is the damping ratio.

This eleventh order model was implemented on an IBM 360/50 digital computer using BPS FØRTRAN. The output of the program was used to drive a CALCØMP plotter to give the curves shown in Chapter 5.

F.2 Numerical Data

The following pages give the numerical data used in the booster model. The data are given at 4 second intervals and linear interpolation was used between these points. Nomenclature follows that used in Section F.1 and Chapter 5 with the exception that the measurement matrix Γ is here denoted by the symbol C.

TIME	A(2,3)	A(3,1)	A(3,3)	A(2,10)	A(3,10)
0.00	-0.00000E-01	-3.89717E 01	-1.00000E 01	-8.72556E-01	-3.92183E 01
4.00	2.34665E-04	-9.51120E-01	-2.63210E-01	-8.74762E-01	-9.71986E-01
8.00	1.01622E-03	-4.53424E-01	-1.38907E-01	-8.78044E-01	-4.73305E-01
12.00	2.43095E-03	-2.88416E-01	-9.79386E-02	-8.82565E-01	-3.07714E-01
16.00	4.57627E-03	-2.06503E-01	-7.81349E-02	-8.87882E-01	-2.25471E-01
20.00	7.60375E-03	-1.57548E-01	-6.69383E-02	-8.94789E-01	-1.76404E-01
24.00	1.14311E-02	-1.24987E-01	-6.01712E-02	-9.02719E-01	-1.43915E-01
28.00	1.62591E-02	-1.01944E-01	-5.58516E-02	-9.11922E-01	-1.20837E-01
32.00	2.15078E-02	-8.46425E-02	-5.30970E-02	-9.22882E-01	-1.03582E-01
36.00	2.64925E-02	-7.11557E-02	-5.14360E-02	-9.35202E-01	-9.01960E-02
40.00	3.22049E-02	-6.02938E-02	-5.05306E-02	-9.49016E-01	-7.95108E-02
44.00	3.62459E-02	-5.14219E-02	-4.99302E-02	-9.64189E-01	-7.08068E-02
48.00	3.51612E-02	-4.40860E-02	-4.96738E-02	-9.81029E-01	-6.36027E-02
52.00	2.11236E-02	-3.78825E-02	-4.96559E-02	-9.99619E-01	-5.75869E-02
56.00	-1.38871E-02	-3.26468E-02	-5.00528E-02	-1.01913E 00	-5.25287E-02
60.00	-8.21618E-03	-2.79962E-02	-5.04673E-02	-1.03976E 00	-4.82414E-02
64.00	3.69351E-02	-2.43069E-02	-4.94450E-02	-1.06234E 00	-4.45763E-02
68.00	1.12482E-01	-2.10898E-02	-4.79278E-02	-1.08613E 00	-4.13469E-02
72.00	1.62098E-01	-1.82865E-02	-4.60360E-02	-1.11129E 00	-3.84289E-02
76.00	1.86803E-01	-1.57953E-02	-4.27376E-02	-1.13864E 00	-3.57815E-02
80.00	2.03047E-01	-1.36615E-02	-4.06825E-02	-1.16693E 00	-3.33820E-02
84.00	2.39399E-01	-1.18564E-02	-3.96747E-02	-1.19690E 00	-3.12185E-02
88.00	2.08929E-01	-1.02619E-02	-3.57209E-02	-1.22966E 00	-2.92647E-02
92.00	1.85940E-01	-8.86826E-03	-3.31805E-02	-1.26529E 00	-2.75001E-02
96.00	1.57643E-01	-7.70149E-03	-3.09094E-02	-1.30489E 00	-2.59155E-02
100.00	1.31395E-01	-6.22695E-03	-2.93635E-02	-1.34873E 00	-2.44987E-02
104.00	1.08228E-01	-5.52359E-03	-2.73510E-02	-1.40055E 00	-2.31645E-02
108.00	8.61555E-02	-5.01682E-03	-2.56583E-02	-1.45446E 00	-2.21193E-02
112.00	6.73236E-02	-4.78110E-03	-2.38922E-02	-1.52292E 00	-2.10694E-02
116.00	5.00602E-02	-4.68305E-03	-2.23819E-02	-1.59491E 00	-2.02576E-02
120.00	3.89616E-02	-4.04329E-03	-2.16080E-02	-1.68970E 00	-1.94422E-02
124.00	2.85525E-02	-3.60106E-03	-2.09329E-02	-1.79097E 00	-1.88293E-02
128.00	2.32690E-02	-2.85572E-03	-2.06856E-02	-1.92912E 00	-1.82079E-02
132.00	1.78008E-02	-2.33237E-03	-2.04568E-02	-2.08038E 00	-1.77667E-02
136.00	1.64635E-02	-2.27325E-03	-1.98447E-02	-2.29853E 00	-1.73135E-02
140.00	1.43081E-02	-2.35353E-03	-1.93036E-02	-2.54732E 00	-1.70297E-02

TIME	A(5,4)	A(5,5)	A(5,10)	C(1,4)	C(2,5)
0.00	-3.89433E 01	-1.24809E-01	1.99244E 02	1.50000E-02	7.00000E-03
4.00	-3.92419E 01	-1.25287E-01	2.00990E 02	1.50000E-02	7.00000E-03
8.00	-3.95337E 01	-1.25752E-01	2.02669E 02	1.50000E-02	7.00000E-03
12.00	-3.98190E 01	-1.26205E-01	2.04929E 02	1.50000E-02	7.00000E-03
16.00	-4.01126E 01	-1.26669E-01	2.07314E 02	1.50000E-02	7.00000E-03
20.00	-4.04555E 01	-1.27209E-01	2.08774E 02	1.50000E-02	7.00000E-03
24.00	-4.07920E 01	-1.27737E-01	2.11242E 02	1.50000E-02	7.00000E-03
28.00	-4.10170E 01	-1.28089E-01	2.13627E 02	1.50000E-02	7.00000E-03
32.00	-4.13153E 01	-1.28554E-01	2.16306E 02	1.50000E-02	7.00000E-03
36.00	-4.16228E 01	-1.29031E-01	2.18875E 02	1.50000E-02	7.00000E-03
40.00	-4.19477E 01	-1.29534E-01	2.21923E 02	1.50000E-02	7.00000E-03
44.00	-4.22249E 01	-1.29961E-01	2.24808E 02	1.50000E-02	7.00000E-03
48.00	-4.25358E 01	-1.30439E-01	2.27871E 02	1.50000E-02	7.00000E-03
52.00	-4.27984E 01	-1.30841E-01	2.30702E 02	1.50000E-02	7.00000E-03
56.00	-4.30866E 01	-1.31281E-01	2.34065E 02	1.50000E-02	7.00000E-03
60.00	-4.33510E 01	-1.31683E-01	2.37979E 02	1.50000E-02	7.00000E-03
64.00	-4.36078E 01	-1.32073E-01	2.41442E 02	1.50000E-02	7.00000E-03
68.00	-4.38572E 01	-1.32450E-01	2.44845E 02	1.50000E-02	7.00000E-03
72.00	-4.41072E 01	-1.32827E-01	2.47996E 02	1.50000E-02	7.00000E-03
76.00	-4.43580E 01	-1.33204E-01	2.51663E 02	1.50000E-02	7.00000E-03
80.00	-4.46681E 01	-1.33668E-01	2.54610E 02	1.50000E-02	7.00000E-03
84.00	-4.49289E 01	-1.34058E-01	2.58204E 02	1.50000E-02	7.00000E-03
88.00	-4.51988E 01	-1.34460E-01	2.61315E 02	1.50000E-02	7.00000E-03
92.00	-4.54526E 01	-1.34837E-01	2.65393E 02	1.50000E-02	7.00000E-03
96.00	-4.56902E 01	-1.35189E-01	2.69548E 02	1.50000E-02	7.00000E-03
100.00	-4.59368E 01	-1.35553E-01	2.73145E 02	1.50000E-02	7.00000E-03
104.00	-4.62183E 01	-1.35968E-01	2.78995E 02	1.50000E-02	7.00000E-03
108.00	-4.64921E 01	-1.36370E-01	2.84057E 02	1.50000E-02	7.00000E-03
112.00	-4.68182E 01	-1.36848E-01	2.91524E 02	1.50000E-02	7.00000E-03
116.00	-4.71542E 01	-1.37338E-01	2.99606E 02	1.50000E-02	7.00000E-03
120.00	-4.74827E 01	-1.37815E-01	3.10272E 02	1.50000E-02	7.00000E-03
124.00	-4.79427E 01	-1.38481E-01	3.19662E 02	1.50000E-02	7.00000E-03
128.00	-4.83701E 01	-1.39097E-01	3.42572E 02	1.50000E-02	7.00000E-03
132.00	-4.89926E 01	-1.39989E-01	3.74430E 02	1.50000E-02	7.00000E-03
136.00	-4.97874E 01	-1.41120E-01	4.02145E 02	1.50000E-02	7.00000E-03
140.00	-5.07675E 01	-1.42503E-01	4.34076E 02	1.50000E-02	7.00000E-03

TIME	A(7,6)	A(7,7)	A(7,10)	C(1,6)	C(2,7)
0.00	-1.22483E 02	-2.21344E-01	-3.23933E 02	4.69000E-03	-3.36000E-03
4.00	-1.23207E 02	-2.21997E-01	-3.27875E 02	4.61000E-03	-3.42000E-03
8.00	-1.23948E 02	-2.22664E-01	-3.32040E 02	4.53000E-03	-3.48000E-03
12.00	-1.24760E 02	-2.23392E-01	-3.36842E 02	4.43000E-03	-3.54800E-03
16.00	-1.25674E 02	-2.24209E-01	-3.42340E 02	4.31000E-03	-3.62400E-03
20.00	-1.26578E 02	-2.25013E-01	-3.48188E 02	4.19000E-03	-3.70000E-03
24.00	-1.27683E 02	-2.25994E-01	-3.54858E 02	4.04600E-03	-3.78400E-03
28.00	-1.28793E 02	-2.26974E-01	-3.61956E 02	3.90200E-03	-3.86800E-03
32.00	-1.29993E 02	-2.28029E-01	-3.69626E 02	3.74600E-03	-3.95600E-03
36.00	-1.31301E 02	-2.29173E-01	-3.77912E 02	3.57800E-03	-4.04800E-03
40.00	-1.32614E 02	-2.30316E-01	-3.86685E 02	3.41000E-03	-4.14000E-03
44.00	-1.34109E 02	-2.31611E-01	-3.95892E 02	3.21400E-03	-4.24000E-03
48.00	-1.35612E 02	-2.32905E-01	-4.05584E 02	3.01800E-03	-4.34000E-03
52.00	-1.37212E 02	-2.34275E-01	-4.15422E 02	2.81600E-03	-4.44000E-03
56.00	-1.38910E 02	-2.35720E-01	-4.25364E 02	2.60800E-03	-4.54000E-03
60.00	-1.40618E 02	-2.37165E-01	-4.35646E 02	2.40000E-03	-4.64000E-03
64.00	-1.42442E 02	-2.38698E-01	-4.44946E 02	2.17600E-03	-4.74400E-03
68.00	-1.44293E 02	-2.40244E-01	-4.54430E 02	1.95200E-03	-4.84800E-03
72.00	-1.46171E 02	-2.41802E-01	-4.63115E 02	1.72600E-03	-4.94600E-03
76.00	-1.48107E 02	-2.43398E-01	-4.70877E 02	1.49800E-03	-5.03800E-03
80.00	-1.50055E 02	-2.44994E-01	-4.78571E 02	1.27000E-03	-5.13000E-03
84.00	-1.51985E 02	-2.46565E-01	-4.83702E 02	1.04600E-03	-5.21800E-03
88.00	-1.53928E 02	-2.48136E-01	-4.88564E 02	8.22000E-04	-5.30600E-03
92.00	-1.55820E 02	-2.49656E-01	-4.91951E 02	6.04000E-04	-5.38400E-03
96.00	-1.57693E 02	-2.51152E-01	-4.93844E 02	3.92000E-04	-5.45200E-03
100.00	-1.59560E 02	-2.52634E-01	-4.95491E 02	1.80000E-04	-5.52000E-03
104.00	-1.61263E 02	-2.53979E-01	-4.95429E 02	-2.00000E-05	-5.57200E-03
108.00	-1.62975E 02	-2.55324E-01	-4.95336E 02	-2.20000E-04	-5.62400E-03
112.00	-1.64583E 02	-2.56580E-01	-4.95516E 02	-4.10000E-04	-5.67600E-03
116.00	-1.66119E 02	-2.57774E-01	-4.96251E 02	-5.90000E-04	-5.72800E-03
120.00	-1.67661E 02	-2.58968E-01	-4.96867E 02	-7.70000E-04	-5.78000E-03
124.00	-1.69047E 02	-2.60036E-01	-5.02431E 02	-9.18000E-04	-5.81200E-03
128.00	-1.70438E 02	-2.61104E-01	-5.08067E 02	-1.06600E-03	-5.84400E-03
132.00	-1.71869E 02	-2.62197E-01	-5.21646E 02	-1.20800E-03	-5.86800E-03
136.00	-1.73321E 02	-2.63303E-01	-5.43285E 02	-1.34400E-03	-5.88400E-03
140.00	-1.74780E 02	-2.64409E-01	-5.65285E 02	-1.48000E-03	-5.90000E-03

TIME	A(9,8)	A(9,9)	A(9,10)	C(1,8)	C(2,9)
0.00	-2.57615E 02	-3.21008E-01	3.16400E 02	-8.75000E-03	-4.52000E-03
4.00	-2.57736E 02	-3.21083E-01	3.17442E 02	-8.74200E-03	-4.51600E-03
8.00	-2.57837E 02	-3.21146E-01	3.18714E 02	-8.73400E-03	-4.51200E-03
12.00	-2.58019E 02	-3.21259E-01	3.21105E 02	-8.73800E-03	-4.50400E-03
16.00	-2.58281E 02	-3.21422E-01	3.24621E 02	-8.75400E-03	-4.49200E-03
20.00	-2.58523E 02	-3.21573E-01	3.28407E 02	-8.77000E-03	-4.48000E-03
24.00	-2.59009E 02	-3.21875E-01	3.34164E 02	-8.80200E-03	-4.45600E-03
28.00	-2.59494E 02	-3.22177E-01	3.40201E 02	-8.83400E-03	-4.43200E-03
32.00	-2.60143E 02	-3.22579E-01	3.47436E 02	-8.88000E-03	-4.39800E-03
36.00	-2.60974E 02	-3.23094E-01	3.55841E 02	-8.94000E-03	-4.35400E-03
40.00	-2.61807E 02	-3.23609E-01	3.64488E 02	-9.00000E-03	-4.31000E-03
44.00	-2.63089E 02	-3.24401E-01	3.75254E 02	-9.08800E-03	-4.24200E-03
48.00	-2.64375F 02	-3.25192E-01	3.86157E 02	-9.17600E-03	-4.17400E-03
52.00	-2.65991E 02	-3.26185E-01	3.98055E 02	-9.28400E-03	-4.08800E-03
56.00	-2.67922E 02	-3.27366E-01	4.10859E 02	-9.41200E-03	-3.98400E-03
60.00	-2.69859E 02	-3.28548E-01	4.23577E 02	-9.54000E-03	-3.88000E-03
64.00	-2.72611E 02	-3.30219E-01	4.37679E 02	-9.70800E-03	-3.72400E-03
68.00	-2.75378E 02	-3.31890E-01	4.51507E 02	-9.87600E-03	-3.56800E-03
72.00	-2.78682E 02	-3.33876E-01	4.65440E 02	-1.00700E-02	-3.38200E-03
76.00	-2.82492E 02	-3.36150E-01	4.79386E 02	-1.02900E-02	-3.16600E-03
80.00	-2.86328E 02	-3.38425E-01	4.92799E 02	-1.05100E-02	-2.95000E-03
84.00	-2.91347E 02	-3.41378E-01	5.04290E 02	-1.07780E-02	-2.62600E-03
88.00	-2.96410E 02	-3.44331E-01	5.15275E 02	-1.10460E-02	-2.30200E-03
92.00	-3.02061E 02	-3.47598E-01	5.23000E 02	-1.13300E-02	-1.92000E-03
96.00	-3.08273E 02	-3.51155E-01	5.27618E 02	-1.16300E-02	-1.48000E-03
100.00	-3.14549E 02	-3.54711F-01	5.31960E 02	-1.19300E-02	-1.04000E-03
104.00	-3.21248E 02	-3.58468E-01	5.24521E 02	-1.22300E-02	-5.24000E-04
108.00	-3.28018E 02	-3.62225E-01	5.16982E 02	-1.25300E-02	-8.00110E-06
112.00	-3.34467E 02	-3.65769E-01	5.02933E 02	-1.27920E-02	4.90000E-04
116.00	-3.40632E 02	-3.69124E-01	4.82598E 02	-1.30160E-02	9.70000E-04
120.00	-3.46853E 02	-3.72480E-01	4.61991E 02	-1.32400E-02	1.45000E-03
124.00	-3.51549E 02	-3.74993E-01	4.37139E 02	-1.33960E-02	1.83800E-03
128.00	-3.56301E 02	-3.77519E-01	4.12472E 02	-1.35520E-02	2.22600E-03
132.00	-3.60321E 02	-3.79642E-01	3.91770E 02	-1.36780E-02	3.01000E-03
136.00	-3.63596E 02	-3.81364E-01	3.75070E 02	-1.37740E-02	4.19000E-03
140.00	-3.66886E 02	-3.83086E-01	3.58701E 02	-1.38700E-02	5.37000E-03

TIME (SEC)	MASS (KG)	XCG (M)	XCP (M)	LCP (M)	IXX (KG-M)
0.00	2.76205E 06	2.73900E 01	3.62103E 01	-8.82031E 00	8.50078E 08
4.00	2.70854E 06	2.73900E 01	3.61097E 01	-8.71973E 00	8.48227E 08
8.00	2.65511E 06	2.74000E 01	3.60091E 01	-8.60915E 00	8.46271E 08
12.00	2.60150E 06	2.74200E 01	3.58080E 01	-8.38797E 00	8.44119E 08
16.00	2.54799E 06	2.74400E 01	3.56068E 01	-8.16679E 00	8.41937E 08
20.00	2.49447E 06	2.74800E 01	3.55062E 01	-8.02621E 00	8.39623E 08
24.00	2.44095E 06	2.75200E 01	3.53051E 01	-7.78505E 00	8.37146E 08
28.00	2.38743E 06	2.75700E 01	3.51039E 01	-7.53387E 00	8.34599E 08
32.00	2.33392E 06	2.76400E 01	3.48021E 01	-7.16212E 00	8.31848E 08
36.00	2.28040E 06	2.77200E 01	3.42992E 01	-6.57921E 00	8.28952E 08
40.00	2.22688E 06	2.78100E 01	3.38969E 01	-6.08687E 00	8.25766E 08
44.00	2.17337E 06	2.79100E 01	3.32934E 01	-5.38336E 00	8.22359E 08
48.00	2.11985E 06	2.80300E 01	3.21869E 01	-4.15694E 00	8.18694E 08
52.00	2.06633E 06	2.81700E 01	3.01753E 01	-2.00526E 00	8.14624E 08
56.00	2.01282E 06	2.83200E 01	2.72583E 01	1.06169E 00	8.10440E 08
60.00	1.95930E 06	2.84800E 01	2.79624E 01	5.17593E-01	8.05866E 08
64.00	1.90578E 06	2.86700E 01	3.07788E 01	-2.10876E 00	8.00737E 08
68.00	1.85227E 06	2.88800E 01	3.50033E 01	-6.12329E 00	7.95254E 08
72.00	1.79875E 06	2.91100E 01	3.78197E 01	-8.70966E 00	7.89244E 08
76.00	1.74523E 06	2.93800E 01	4.02337E 01	-1.08537E 01	7.82626E 08
80.00	1.69172E 06	2.96600E 01	4.16419E 01	-1.19819E 01	7.75421E 08
84.00	1.63820E 06	2.99700E 01	4.33518E 01	-1.33818E 01	7.67645E 08
88.00	1.58468E 06	3.03200E 01	4.45588E 01	-1.42388E 01	7.58893E 08
92.00	1.53117E 06	3.07100E 01	4.51629E 01	-1.44523E 01	7.49251E 08
96.00	1.47765E 06	3.11500E 01	4.52629E 01	-1.41129E 01	7.38585E 08
100.00	1.42413E 06	3.16300E 01	4.53635E 01	-1.37335E 01	7.26793E 08
104.00	1.37061E 06	3.21900E 01	4.55143E 01	-1.33243E 01	7.13040E 08
108.00	1.31710E 06	3.27500E 01	4.56652E 01	-1.29152E 01	6.99287E 08
112.00	1.26358E 06	3.34400E 01	4.56149E 01	-1.21749E 01	6.82292E 08
116.00	1.21006E 06	3.41300E 01	4.55646E 01	-1.14346E 01	6.65296E 08
120.00	1.15655E 05	3.49950E 01	4.57155E 01	-1.07205E 01	6.44063E 08
124.00	1.10303E 06	3.58600E 01	4.58664E 01	-1.00064E 01	6.22830E 08
128.00	1.04951E 06	3.69500E 01	4.66208E 01	-9.67078E 00	5.95878E 08
132.00	9.95996E 05	3.80400E 01	4.73752E 01	-9.33516E 00	5.68926E 08
136.00	9.42480E 05	3.94400E 01	4.91354E 01	-9.69537E 00	5.33912E 08
140.00	8.88963E 05	4.08400E 01	5.08956E 01	-1.00556E 01	4.98897E 08

TIME (SEC)	THRUST (N)	DRAG (N)	VELOCITY (M/SEC)	Q (N/M)	CNALPHA
0.00	3.38509E 07	3.53040E 04	0.00000E-01	0.00000E-01	4.60000E 00
4.00	3.38626E 07	4.09890E 04	1.02900E 01	6.25000E 01	4.60000E 00
8.00	3.38989E 07	5.97750E 04	2.15800E 01	2.73500E 02	4.60000E 00
12.00	3.39620E 07	9.35160E 04	3.39400E 01	6.69800E 02	4.60000E 00
16.00	3.40534E 07	1.33627E 05	4.74200E 01	1.28890E 03	4.61000E 00
20.00	3.41742E 07	1.78357E 05	6.21300E 01	2.16840E 03	4.62000E 00
24.00	3.43254E 07	2.24660E 05	7.81700E 01	3.34370E 03	4.63000E 00
28.00	3.45071E 07	2.59467E 05	9.56900E 01	4.84720E 03	4.68000E 00
32.00	3.47186E 07	2.82588E 05	1.14890E 02	6.70820E 03	4.69000E 00
36.00	3.49584E 07	2.84786E 05	1.35970E 02	8.94460E 03	4.70000E 00
40.00	3.52241E 07	3.16057E 05	1.59150E 02	1.15600E 04	4.76000E 00
44.00	3.55119E 07	3.83726E 05	1.84610E 02	1.45280E 04	4.80000E 00
48.00	3.58171E 07	5.02170E 05	2.12520E 02	1.77990E 04	4.90000E 00
52.00	3.61339E 07	7.73218E 05	2.42930E 02	2.12750E 04	5.08000E 00
56.00	3.64560E 07	1.14050E 06	2.75840E 02	2.48160E 04	5.38000E 00
60.00	3.67763E 07	1.71583E 06	3.11270E 02	2.82650E 04	5.70000E 00
64.00	3.70882E 07	2.02296E 06	3.49260E 02	3.14300E 04	5.62000E 00
68.00	3.73853F 07	2.06992E 06	3.90520E 02	3.41980E 04	5.38000E 00
72.00	3.76622F 07	1.91976E 06	4.35880E 02	3.64170E 04	5.08000E 00
76.00	3.79139E 07	1.76683E 06	4.85710E 02	3.76990E 04	4.50000E 00
80.00	3.81348E 07	1.61020E 06	5.40220E 02	3.76130E 04	4.40000E 00
84.00	3.83214E 07	1.44991E 06	5.99450E 02	3.58840E 04	4.82000E 00
88.00	3.84723E 07	1.24739E 06	6.63670E 02	3.26150E 04	4.30000E 00
92.00	3.85877E 07	1.02566E 06	7.33130E 02	2.83000E 04	4.29000E 00
96.00	3.86746E 07	8.28414E 05	8.07950E 02	2.42770E 04	4.28000E 00
100.00	3.87389E 07	6.70182E 05	8.88270E 02	2.05580E 04	4.26000E 00
104.00	3.87795E 07	5.43185E 05	9.77135E 02	1.74090E 04	4.19000E 00
108.00	3.88201E 07	4.16187E 05	1.06600E 03	1.42600E 04	4.12000E 00
112.00	3.88411E 07	3.26879E 05	1.16715E 03	1.17617E 04	4.04000E 00
116.00	3.88621F 07	2.37570E 05	1.26830E 03	9.26339E 03	3.96000E 00
120.00	3.88724F 07	1.83636E 05	1.38300E 03	7.50129E 03	3.93000E 00
124.00	3.88828E 07	1.29702E 05	1.49770E 03	5.73920E 03	3.90000E 00
128.00	3.88877E 07	9.87330E 04	1.62800E 03	4.59475E 03	3.93000E 00
132.00	3.88927E 07	6.77640E 04	1.75830E 03	3.45030E 03	3.96000E 00
136.00	3.88950E 07	5.08205E 04	1.90690E 03	2.76475E 03	4.13000E 00
140.00	3.88973E 07	3.38770E 04	2.05550E 03	2.07920E 03	4.30000E 00