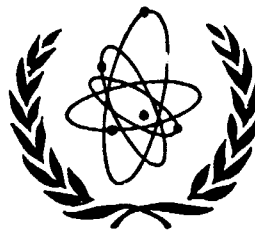


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INTERNATIONAL ATOMIC ENERGY AGENCY

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PHYSICS

**INTERNAL GRAVITY WAVES
WITH VELOCITY GRADIENTS**

A. McLELLAN IV

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PIAZZA OBERDAN

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INTERNAL GRAVITY WAVES WITH VELOCITY GRADIENTS *

Alden McLellan IV **

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** On leave of absence from the Desert Research Institute and Dept. of Physics, University of Nevada, Reno, Nevada, USA.

ABSTRACT

Low-frequency waves with small amplitudes are treated in a two-dimensional isothermal compressible fluid medium with a velocity gradient and within a constant gravitational field. In general, there are two modes of wave motion which we called the + mode and the - mode. A local dispersion relation governing the wave propagation for constant velocity gradients is derived from the linearized equations of fluid dynamics and thermodynamics, and arranged in a non-dimensional form independent of the constants describing a particular atmosphere. The wave modes are discussed and the amplitude growth rates which depend on the angle of wave propagation and velocity gradient strength are determined for each wave mode. Linear and parabolic velocity gradients are treated by a generalized W.K.B. method. Equations are found which give the height at which wave reflection occurs. This determines the maximum heights up to which the wave modes can propagate. Dissipative effects arising from viscosity and heat conduction will be neglected.

INTERNAL GRAVITY WAVES WITH VELOCITY GRADIENTS

1. INTRODUCTION

Internal gravity waves have been discussed previously in some detail in regard to investigating certain phenomena such as the irregular motion in the earth's higher atmosphere¹⁾²⁾³⁾⁴⁾ and the heating of the solar corona⁵⁾⁶⁾⁷⁾. These waves are also of considerable significance in geophysics for the internal atmospheric wave motion above mountains, internal waves through the ocean, and the earth's core⁸⁾. Landau has considered the problem of internal gravity waves in an incompressible fluid⁹⁾. We also would like to refer to some earlier work in this field¹⁰⁾. It can be shown that, in general, two modes of wave propagation exist in a compressible medium under the influence of gravity. The purpose of this paper is to give an analysis of these wave modes within a two-dimensional fluid which has an initial velocity gradient. The functional forms of the velocity gradient will be of the following three forms: constant, linear and parabolic.

2. FUNDAMENTAL EQUATIONS

The fundamental equations necessary to describe the wave motions are

a) Euler equation

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla) \underline{v} \right] = -\nabla p + \rho \underline{g} \quad (2.1)$$

In this equation $\underline{v}$ is the velocity vector of a particle of the oscillating medium, p is the pressure, ρ the density, and $\underline{g}$ the gravity vector. For convenience we choose $\underline{g}$ to be along the negative z -direction (downward). Thus, we have

$$\underline{g} = -g \underline{e}_z \quad (2.2)$$

where $\underline{e}_z$ is the unit vector in the z-direction.

b) Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad (2.3)$$

c) Equation of state (adiabatic approximation)

$$\frac{d}{dt} (\rho p^{-\gamma}) = 0, \quad (2.4)$$

where γ is the ratio of specific heats. Expressing the time derivative in eq. (2.4) in terms of quantities referring to points fixed in space, we have for eq. (2.4)

$$\frac{\partial p}{\partial t} - \gamma \frac{p}{\rho} \frac{\partial \rho}{\partial t} + \rho^{\gamma} (\underline{v} \cdot \nabla) (\rho p^{-\gamma}) = 0. \quad (2.5)$$

We will assume that for the equilibrium conditions the variation of density and pressure with height is exponential, as is the case for an isothermal ideal gas under the influence of a constant gravitational field. Thus, we have

$$\rho_{eq} = \rho_0 e^{-z/h}$$

and

$$p_{eq} = p_0 e^{-z/h}, \quad (2.6)$$

where ρ_0 and p_0 are constants, and h is the so-called "scale height".

3. THE LINEARIZED EQUATION OF MOTION

We assume that all derivations of the perturbed quantities from their equilibrium values are small. Hence, we may put to first order

$$\rho(\underline{r}, t) = [\rho_0 + \rho'(\underline{r}, t)] e^{-z/h} \quad (3.1)$$

$$p(\underline{r}, t) = [p_0 + p'(\underline{r}, t)] e^{-z/h} \quad (3.2)$$

and

$$\underline{v}(\underline{r}, t) = \underline{v}_0(z) + \underline{v}'(\underline{r}, t), \quad (3.3)$$

where $\underline{v}_0(z)$ is the velocity profile. The quantities ρ' , p' , and $\underline{v}'$ are considered to be perturbing quantities of first order.

The fundamental eqs. (2.1) and (2.3) when considered at zeroth order, yields the equilibrium condition. At equilibrium, we have from eqs. (2.6) and (3.3)

$$\rho = \rho_0 e^{-z/h} \quad , \quad (3.4)$$

$$p = p_0 e^{-z/h} \quad ,$$

and

$$\underline{v} = \underline{v}_0(z) \quad .$$

By substituting eqs. (3.4) into eq. (2.1), the Euler equation becomes upon cancelling the exponential

$$\rho_0 (\underline{v}_0 \cdot \nabla) \underline{v}_0 = \frac{\rho_0}{h} \underline{e}_z + \rho_0 \underline{g} \quad (3.5)$$

Eq. (3.5) is the steady state Euler equation for the system. By substituting eqs. (3.4) into the continuity equation (2.3), we have, upon cancelling the exponential,

$$\rho_0 (\nabla \cdot \underline{v}_0) - \frac{\rho_0}{h} (\underline{v}_0 \cdot \underline{e}_z) = 0 \quad . \quad (3.6)$$

Eq. (3.6) is the continuity equation for the steady state condition, or in the language of perturbation theory, it is the zeroth order continuity equation.

We substitute the quantities (3.1), (3.2), and (3.3) into the fundamental eqs. (2.1), (2.3), and (2.4). By neglecting the products of the perturbing quantities and with the help of eqs. (3.5) and (3.6), there results the linearized fundamental equations.

The linearized Euler equation is

$$\rho_0 \frac{\partial \underline{v}'}{\partial t} + \rho_0 (\underline{v}_0 \cdot \nabla) \underline{v}' + \rho_0 (\underline{v}' \cdot \nabla) \underline{v}_0 + \rho' (\underline{v}_0 \cdot \nabla) \underline{v}_0$$

$$- \frac{p'}{h} \underline{e}_z + \nabla p' - \rho' \underline{g} = 0 \quad .$$

(3.7)

The linearized continuity equation is

$$\frac{\partial \rho'}{\partial t} + \rho_0 (\nabla \cdot \underline{v}') + \rho' (\nabla \cdot \underline{v}_0) + \underline{v}_0 \cdot (\nabla \rho')$$

$$- \frac{\rho_0}{h} (\underline{v}' \cdot \underline{e}_z) - \frac{\rho'}{h} (\underline{v}_0 \cdot \underline{e}_z) = 0$$

(3.8)

The linearized equation of state is

$$\frac{\partial p'}{\partial t} - \gamma \frac{p_0}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\gamma - 1}{h} p_0 (\underline{v}_0 \cdot \underline{e}_z) + \frac{\gamma - 1}{h} p_0 (\underline{v}' \cdot \underline{e}_z)$$

$$+ \frac{\gamma - 1}{h} p' (\underline{v}_0 \cdot \underline{e}_z) + (\underline{v}_0 \cdot \nabla) p' - \gamma \frac{p_0}{\rho_0} (\underline{v}_0 \cdot \nabla) \rho' = 0 \quad .$$

(3.9)

Eqs. (3.7), (3.8), and (3.9) form a set of linear equations for the unknowns $\underline{v}'$, ρ' and p' . Eq. (3.7) is a vector equation which consists of three scalar equations. (We have yet to make the restriction to the

two-dimensional case). Thus we have five equations and five unknowns. From these equations and the plane wave assumption, it is possible to derive a local dispersion relation.

4. PERFECT GAS ASSUMPTION AND PLANE WAVE ASSUMPTION

It is quite obvious that the case of a perfect gas deserves special consideration for the reason of being a very good approximation in most situations and for its mathematical simplicity. The assumption of a perfect gas under adiabatic compression is expressed in the equation of state (2.4). For a monatomic gas, gamma, the ratio of specific heats at constant pressure and density has the value 5/3. For a diatomic gas, gamma has the value 7/5. The equation of state may be written in another form:

$$p \rho^{-\gamma} = p_0 \rho_0^{-\gamma} \quad (4.1)$$

From eq. (4.1) and the definition of sound, we have

$$a^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0} \quad (4.2)$$

At this point we assume that the velocity profile has only a x-component. Thus, it is represented by

$$\underline{v}_0(z) = v_0(z) \underline{e}_x \quad (4.3)$$

Hence, the steady state Euler equation (3.5) becomes

$$0 = \frac{p_0}{h} e_z + \rho_0 g$$

or, upon re-arranging,

$$h = \frac{p_0}{\rho_0 g} \quad (4.4)$$

With the help of eq. (4.2), the scale height may be expressed as follows:

$$h = \frac{a^2}{g\gamma} \quad (4.5)$$

The set of linear eqs. (3.7), (3.8) and (3.9) are differential equations with non-constant coefficients. If we assume that the coefficients are exactly constant, which means that we assume a velocity profile independent of z , that is, the velocity gradient is zero, then our system under a linear transformation reduces to that of internal gravity waves in a stationary medium, which has already been solved¹⁾. However, for the next lowest non-trivial approximation, we assume that the wavelength is small compared to the change in velocity profile. This assumption allows us to treat k as large compared to the change in velocity profile when taking the fourier transform of the set of linear eqs. (3.7), (3.8) and (3.9). In a better approximation, such as the W.K.B. approximation, we shall assume in a later section the velocity gradient to be slowly varying.

For plane waves, the perturbations describing the wave motion varies sinusoidally as follows:

$$\rho' = \rho_c e^{i(k \cdot r - \omega t)} \quad (4.6)$$

$$\underline{p}' = \underline{p}_c e^{i(\underline{k} \cdot \underline{r} - \omega t)} \quad (4.7)$$

and

$$\underline{v}' = \underline{v}_c e^{i(\underline{k} \cdot \underline{r} - \omega t)} \quad (4.8)$$

where ρ_c and p_c are constants with units of density and pressure respectively. $\underline{v}_c$ is a constant vector in the direction of $\underline{v}'$, $\underline{k}$ is the propagating wave number vector, and ω is the frequency of the wave. Assuming a two-dimensional space for simplicity, substituting eqs. (4.6), (4.7) and (4.8) into eqs. (3.7), (3.8) and (3.9), and carrying out the mathematical operations, we have upon re-arranging

$$-i\omega \rho_0 \underline{v}_c + i k_x \rho_0 \underline{v}_c \underline{v}_0 + \rho_0 \frac{dv_0}{dz} (\underline{v}_c \cdot \underline{e}_z) \underline{e}_x$$

$$- \frac{p_c}{h} \underline{e}_z + i \underline{k} p_c - \rho_c \underline{g} = 0, \quad (4.9)$$

$$\omega \rho_c - \rho_0 (\underline{k} \cdot \underline{v}_c) - v_0 (\underline{k} \cdot \underline{e}_x) \rho_0 - i \frac{\rho_0}{h} (\underline{v}_c \cdot \underline{e}_z) = 0 \quad (4.10)$$

and

$$\begin{aligned} \omega p_c - \omega a^2 \rho_c + i \frac{\gamma-1}{h} p_0 (\underline{v}_c \cdot \underline{e}_z) - v_0 p_c (\underline{k} \cdot \underline{e}_x) \\ + a^2 (\underline{k} \cdot \underline{e}_x) v_0 \rho_c = 0, \end{aligned} \quad (4.11)$$

where $\underline{e}_x$ is the unit vector in the x-direction.

5. THE LOCAL DISPERSION RELATION

By keeping in mind that eq. (4.9) is a vector equation consisting of two scalar equations, we see that the system is determined locally by the four eqs. (4.9), (4.10) and (4.11) containing the four unknowns v_{cx} , v_{cz} , ρ_c and p_c . The condition for a non-trivial solution is obtained by setting the determinant of the coefficients of the unknowns equal to zero. This gives us the local dispersion relation. However, we may eliminate ρ_c and p_c by substituting as follows: Solve eq. (4.10) for ρ_c and eq. (4.11) for p_c . Then, substitute ρ_c of eq. (4.10) into eq. (4.11) and substitute ρ_c of eq. (4.10) and p_c of eq. (4.11) into the vector eq. (4.9). This gives upon re-arranging

$$\begin{aligned} [-i\omega \underline{v}_c + i(\underline{k} \cdot \underline{e}_x) v_0 \underline{v}_c + \frac{dv_0}{dz} (\underline{v}_c \cdot \underline{e}_z) \underline{e}_x] [\omega - v_0 (\underline{k} \cdot \underline{e}_x)] - \\ - \underline{k} [g(\underline{v}_c \cdot \underline{e}_z) - i a^2 (\underline{k} \cdot \underline{v}_c)] + \\ + g(\gamma-1) (\underline{k} \cdot \underline{v}_c) = 0 \end{aligned} \quad (5.1)$$

Separating eq. (5.1) into components, we have, for the x-component

$$\begin{aligned}
& (\underline{v}_c \cdot \underline{e}_x) \left[\omega^2 - 2\omega v_0 (\underline{k} \cdot \underline{e}_x) + (\underline{k} \cdot \underline{e}_x)^2 v_0^2 - a^2 (\underline{k} \cdot \underline{e}_x)^2 \right] \\
& + (\underline{v}_c \cdot \underline{e}_z) \left[i\omega \frac{dv_0}{dz} - i v_0 \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) - ig (\underline{k} \cdot \underline{e}_x) \right. \\
& \quad \left. - a^2 (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) \right] = 0
\end{aligned}
\tag{5.2}$$

and for the z-component

$$\begin{aligned}
& (\underline{v}_c \cdot \underline{e}_x) \left[-a^2 (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) - ig (\gamma - 1) (\underline{k} \cdot \underline{e}_x) \right] \\
& + (\underline{v}_c \cdot \underline{e}_z) \left[\omega^2 - 2\omega v_0 (\underline{k} \cdot \underline{e}_x) + v_0^2 (\underline{k} \cdot \underline{e}_x)^2 \right. \\
& \quad \left. - a^2 (\underline{k} \cdot \underline{e}_z)^2 - ig \gamma (\underline{k} \cdot \underline{e}_z) \right] = 0
\end{aligned}
\tag{5.3}$$

Thus, we see that from eqs. (5.2) and (5.3) the local dispersion is given by the determinant

$$\begin{vmatrix}
\omega^2 - 2\omega v_0 (\underline{k} \cdot \underline{e}_x) + v_0^2 (\underline{k} \cdot \underline{e}_x)^2 - a^2 (\underline{k} \cdot \underline{e}_x)^2 & i\omega \frac{dv_0}{dz} - i v_0 \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) - ig (\underline{k} \cdot \underline{e}_x) - a^2 (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) \\
-i a^2 (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) - ig (\gamma - 1) (\underline{k} \cdot \underline{e}_x) & \omega^2 - 2\omega v_0 (\underline{k} \cdot \underline{e}_x) + v_0^2 (\underline{k} \cdot \underline{e}_x)^2 - a^2 (\underline{k} \cdot \underline{e}_z)^2 - ig \gamma (\underline{k} \cdot \underline{e}_z)
\end{vmatrix} = 0
\tag{5.4}$$

Upon solving the determinant (5.4), we have

$$\begin{aligned}
& \omega^4 + \omega^3 [-4 v_0 (\underline{k} \cdot \underline{e}_x)] + \omega^2 [6 v_0^2 (\underline{k} \cdot \underline{e}_x)^2 - a^2 k^2 \\
& - i g \gamma (\underline{k} \cdot \underline{e}_z)] + \omega [-4 v_0^3 (\underline{k} \cdot \underline{e}_x)^3 + 2 a^2 v_0 k^2 (\underline{k} \cdot \underline{e}_x) \\
& + 2 i g \gamma v_0 (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) + i a^2 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) \\
& - g (\gamma - 1) \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)] + [v_0^4 (\underline{k} \cdot \underline{e}_x)^4 - a^2 v_0^2 k^2 (\underline{k} \cdot \underline{e}_x)^2 \\
& - i g \gamma v_0^2 (\underline{k} \cdot \underline{e}_x)^2 (\underline{k} \cdot \underline{e}_z) - i a^2 v_0 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)^2 (\underline{k} \cdot \underline{e}_z) \\
& + g (\gamma - 1) v_0 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)^2 + g^2 (\gamma - 1) (\underline{k} \cdot \underline{e}_x)^2] = 0
\end{aligned}$$

(5.5)

This is the local dispersion relation. If we put

$$v_0 = 0 = \frac{dv_0}{dz} ,$$

eq. (5.5) reduces to the limiting case

$$\omega^4 + \omega^2 [-a^2 k^2 - ig\gamma(\underline{k} \cdot \underline{e}_z)] + g^2(\gamma-1)(\underline{k} \cdot \underline{e}_x)^2 = 0 , \quad (5.6)$$

which is the dispersion relation for internal gravity waves used by HINES¹).

To eliminate the amplitude growth due to the exponential decay of density with height, we make the linear transformation in k-space

$$k_x = K_x \quad (5.7)$$

$$\begin{aligned} k_z &= K_z - i \frac{1}{2h} \\ &= K_z - i \frac{g\gamma}{2a^2} \end{aligned} \quad (5.8)$$

and furthermore $K^2 = K_x^2 + K_z^2$.

By substituting eqs. (5.7) and (5.8) into the dispersion relation (5.5), we have

$$\begin{aligned}
& \omega^4 + \omega^3 \left[-4 v_0 (\underline{k} \cdot \underline{e}_x) \right] + \omega^2 \left[6 v_0^2 (\underline{k} \cdot \underline{e}_x)^2 - \right. \\
& \left. - a^2 K^2 - \frac{g^2 \gamma^2}{4 a^2} \right] + \omega \left[-4 v_0^3 (\underline{k} \cdot \underline{e}_x)^3 + 2 a^2 v_0 K^2 (\underline{k} \cdot \underline{e}_x) + \right. \\
& \left. + \frac{g^2 \gamma^2}{2 a^2} v_0 (\underline{k} \cdot \underline{e}_x) + i a^2 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) - \right. \\
& \left. - \frac{g \gamma}{2} \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x) + g \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x) \right] + \\
& + \left[v_0^4 (\underline{k} \cdot \underline{e}_x)^4 - a^2 v_0^2 K^2 (\underline{k} \cdot \underline{e}_x)^2 - \frac{g^2 \gamma^2}{4 a^2} v_0^2 (\underline{k} \cdot \underline{e}_x)^2 - \right. \\
& - i a^2 v_0 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)^2 (\underline{k} \cdot \underline{e}_z) + \frac{g \gamma}{2} v_0 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)^2 - \\
& \left. - g v_0 \frac{d v_0}{d z} (\underline{k} \cdot \underline{e}_x)^2 + g^2 (\gamma-1) (\underline{k} \cdot \underline{e}_x)^2 \right] = 0 \quad .
\end{aligned}
\tag{5.9}$$

Eq. (5.9) is the local dispersion relation without the wave-amplitude-increase effect due to the equilibrium density decrease in the z-direction. If we again put

$$v_0 = 0 = \frac{d v_0}{d z}$$

eq. (5.9) reduces to the limiting case

$$\omega^4 + \omega^2 \left[-a^2 K^2 - \frac{g^2 \gamma^2}{4a^2} \right] + g^2 (\gamma - 1) (\underline{K} \cdot \underline{e}_x)^2 = 0 \quad , \quad (5.10)$$

which is the non-growth dispersion relation for internal gravity waves in a stationary medium used by HINES¹⁾

Upon re-arranging the terms in eq. (5.9), we obtain

$$\begin{aligned} & [\omega - v_0 (\underline{K} \cdot \underline{e}_x)]^4 + [\omega - v_0 (\underline{K} \cdot \underline{e}_x)] \left[-a^2 K^2 - \frac{g^2 \gamma^2}{4a^2} \right] + \\ & + [\omega - v_0 (\underline{K} \cdot \underline{e}_x)] \left[i a^2 \frac{dv_0}{dz} (\underline{K} \cdot \underline{e}_x) (\underline{K} \cdot \underline{e}_z) - \right. \\ & \left. - \frac{g\gamma}{2} \frac{dv_0}{dz} (\underline{K} \cdot \underline{e}_x) + g \frac{dv_0}{dz} (\underline{K} \cdot \underline{e}_x) \right] + g^2 (\gamma - 1) (\underline{K} \cdot \underline{e}_x)^2 = 0 \end{aligned} \quad (5.11)$$

Consider the frequency transformation

$$\bar{\omega} = \omega - v_0 (\underline{K} \cdot \underline{e}_x) \quad . \quad (5.12)$$

By substituting the frequency (5.12) into the dispersion relation (5.11), we have

$$\begin{aligned}
& \bar{\omega}^4 + \bar{\omega}^2 \left[-a^2 k^2 - \frac{g^2 \gamma^2}{4a^2} \right] + \\
& + \bar{\omega} \left[i a^2 \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) - \frac{g\gamma}{2} \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) + \right. \\
& \left. + g \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) \right] + g^2 (\gamma-1) (\underline{k} \cdot \underline{e}_x)^2 = 0
\end{aligned} \tag{5.13}$$

Eq. (5.13) is the transformed dispersion relation without height-growth. "Transformed", in the sense of the frequency transformation (5.12), and "without height-growth" refers to the fact that the wave-amplitude-increase effect due to the decrease in equilibrium density in the z-direction has been extracted by means of the wave number transformation given by eqs. (5.7) and (5.8). In order to obtain the transformed dispersion relation with height-growth, we reverse the transformation (5.7) and (5.8) and we keep in mind that

$$\bar{\omega} = \omega - v_0 (\underline{k} \cdot \underline{e}_x) , \tag{5.14}$$

where $\bar{\omega}$, the transformed frequency, is the same as $\bar{\omega}$ appearing in eq. (5.12). Applying this to eq. (5.13), we have

$$\begin{aligned}
& \bar{\omega}^4 + \bar{\omega}^2 \left[-a^2 k^2 - i g \gamma (\underline{k} \cdot \underline{e}_z) \right] + \bar{\omega} \left[i a^2 \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) (\underline{k} \cdot \underline{e}_z) - \right. \\
& \left. - g \gamma \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) + g \frac{dv_0}{dz} (\underline{k} \cdot \underline{e}_x) \right] + \\
& + g^2 (\gamma-1) (\underline{k} \cdot \underline{e}_x)^2 = 0 .
\end{aligned} \tag{5.15}$$

6. THE DISPERSION RELATIONS IN NON-DIMENSIONAL FORM

In order to obtain the angular dependence of the local dispersion relations (5.13) and (5.14), we define the two angles θ and ϕ as follows

$$(\underline{k} \cdot \underline{e}_z) = k \cos \theta \quad (6.1)$$

and

$$(\underline{K} \cdot \underline{e}_z) = K \cos \phi \quad (6.2)$$

θ is the angle between the wave propagation vector and the vertical, and therefore it is a real angle. ϕ is an imaginary angle, as can be seen by substituting the wave number transformation eq. (5.8) into the definition of ϕ , given by eq. (6.2). We have

$$\cos \phi = \frac{k \cos \theta + i \frac{g\gamma}{2a^2}}{\left[k^2 - \frac{g^2 \gamma^2}{4a^4} + i \frac{g\gamma}{a^2} k \cos \theta \right]^{\frac{1}{2}}} \quad (6.3)$$

Upon substituting eqs. (6.1) and (6.2) into the local dispersion relations (5.13) and (5.14), respectively, we obtain

$$\begin{aligned}
& \bar{\omega}^4 + \bar{\omega}^2 \left[-a^2 k^2 - \frac{g^2 \gamma^2}{4 a^2} \right] + \\
& + \bar{\omega} \left[i a^2 k \sin \phi - \frac{g}{2} (\gamma - 2) \right] \frac{dv_0}{dz} k \cos \phi + \\
& + g^2 (\gamma - 1) k^2 \cos^2 \phi = 0 \quad , \\
& \hspace{25em} (6.4)
\end{aligned}$$

and

$$\begin{aligned}
& \bar{\omega}^4 + \bar{\omega}^2 \left[-a^2 k^2 - i g \gamma k \cos \theta \right] + \\
& + \bar{\omega} \left[i a^2 k \sin \theta - g (\gamma - 1) \right] \frac{dv_0}{dz} k \cos \theta + \\
& + g^2 (\gamma - 1) k^2 \cos^2 \theta = 0 \quad . \\
& \hspace{25em} (6.5)
\end{aligned}$$

Eqs. (6.4) and (6.5) give the angular dependence of the local dispersion relation without and with "height-growth", respectively.

In order to put these dispersion relations into a non-dimensional form, we must first define a characteristic frequency, characteristic wave number and a non-dimensional velocity gradient. Toward this end, we define

$$\bar{\omega}_c = g/a, \quad (6.6)$$

$$K_c = g/a^2, \quad (6.7)$$

$$S = \frac{d}{dz} \left(\frac{a}{g} v_0 \right), \quad (6.8)$$

and, furthermore,

$$W = \frac{\bar{\omega}}{\bar{\omega}_c} \quad (6.9)$$

$$K = \frac{K}{K_c} \quad (6.10)$$

and
$$\bar{K} = \frac{k}{K_c} \quad (6.11)$$

By substitution of the definitions (6.6)-(6.11) into eqs. (6.4) and (6.5), we have, respectively

$$W^4 + W^2 \left[-K^2 - \left(\frac{\gamma}{2} \right)^2 \right] +$$

$$+W \left[(1 \kappa \sin \phi - \frac{\gamma}{2} + 1) S \kappa \cos \phi \right] +$$

$$+ (\gamma - 1) \kappa^2 \cos^2 \phi = 0 \quad ,$$

(6.12)

and

$$W^4 + W^2 \left[-\bar{\kappa}^2 - 1 \gamma \bar{\kappa} \cos \theta \right] +$$

$$+ W \left[(1 \bar{\kappa} \sin \theta - \gamma + 1) S \bar{\kappa} \cos \theta \right] +$$

$$+ (\gamma - 1) \bar{\kappa}^2 \cos^2 \theta = 0 \quad .$$

(6.13)

Eqs. (6.12) and (6.13) are algebraic equations of the fourth degree (quartic), and the solutions can be expressed in an analytic form (see Appendix I).

Also, they can be expressed in a form by which the wave numbers are a function of the frequencies. By re-arranging eqs. (6.12) and (6.13) we have, respectively

$$\kappa^2 \left[i W S \sin \phi \cos \phi + (\gamma - 1) \cos^2 \phi - W^2 \right] +$$

$$+ K \left[WS \cos \phi \left(1 - \frac{\gamma}{2} \right) \right] + \left[W^4 - W^2 \left(\frac{\gamma}{2} \right)^2 \right] = 0 ,$$

(6.14)

and

$$\bar{K}^2 \left[i WS \sin \theta \cos \theta + (\gamma - 1) \cos^2 \theta - W^2 \right] +$$

$$+ \bar{K} \left[-i \gamma W^2 \cos \theta + (1 - \gamma) WS \cos \theta \right] +$$

$$+ [W^4] = 0 .$$

(6.15)

Eqs. (6.14) and (6.15) are quadratic equations each having two roots indicating that there are two distinct modes of wave propagation.

We will call them the + mode and the - mode. Upon solving eqs. (6.14) and (6.15) for the wave number, we have, respectively

$$K_{\pm} = \frac{\frac{1}{2} \left(\frac{\gamma}{2} - 1 \right) WS \cos \phi}{i WS \sin \phi \cos \phi + (\gamma - 1) \cos^2 \phi - W^2} \quad \pm$$

$$+ \left\{ \frac{\left[WS \cos \phi \left(\frac{\gamma}{2} - 1 \right) \right]^2 - 4 \left[i WS \sin \phi \cos \phi + (\gamma - 1) \cos^2 \phi - W^2 \right] \left[W^4 - W^2 \left(\frac{\gamma}{2} \right)^2 \right]}{4 \left[i WS \sin \phi \cos \phi + (\gamma - 1) \cos^2 \phi - W^2 \right]^2} \right\}^{\frac{1}{2}}$$

(6.16)

and

$$\overline{\kappa}_{\pm} = \frac{\frac{1}{2} \left[(\gamma - 1) WS \cos \theta + i \gamma W^2 \cos \theta \right]}{i WS \sin \theta \cos \theta + (\gamma - 1) \cos^2 \theta - W^2} \pm \left\{ \frac{\left[(\gamma - 1) WS \cos \theta + i \gamma W^2 \cos \theta \right]^2 - 4 \left[i WS \sin \theta \cos \theta + (\gamma - 1) \cos^2 \theta - W^2 \right] \left[W^4 \right]}{4 \left[i WS \sin \theta \cos \theta + (\gamma - 1) \cos^2 \theta - W^2 \right]^2} \right\}^{\frac{1}{2}}$$

(6.17).

Eqs. (6.16) and (6.17) are the non-dimensional transformed local dispersion relations, respectively, without and with the wave-amplitude-growth effect due to the equilibrium density decrease in the z-direction.

7. W.K.B. APPROXIMATION

We assumed that v_0 and dv_0/dz could be considered a constant in the lowest approximation if the wavelength was small compared to a significant change in the velocity profile. A better approximation is obtained by taking into account the slowly varying change of the velocity profile and velocity gradient by a generalized W.K.B. method, which will be presented here.

It is known, for instance, from the application of the W.K.B. approximation to the solution of Schrödinger's equation, that the W.K.B. method gives reasonably good results even if the condition of a small wavelength is no longer satisfied. Therefore, we can also expect that our results, which are derived by using the W.K.B. method, are applicable with reasonable accuracy even if the wavelength cannot be considered small against the scale height.

In the plane wave analysis, eqs. (4.6), (4.7) and (4.8), the z -dependence of the solution was automatically taken into account. In the W.K.B. method, however, we assume for the perturbed density, the perturbed pressure and the velocity components a dependence of the form

$$f(z) e^{i(k_x x - \omega t)} \quad (7.1)$$

$f(z)$ describe the unknown z -dependence to be treated by the W.K.B. method. In eqs. (4.6), (4.7) and (4.8) it is seen that we had chosen

$$f(z) = e^{ik_z z} \quad (7.2)$$

Now, however, we have, in keeping with the W.K.B. analysis,

$$f(z) = A(z) e^{\Phi(z)} \quad (7.3)$$

where $-i\Phi(z)$ is the phase, and $A(z)$ is a slowly varying function of z . We can consider d^2A/dz^2 small against A , and therefore neglect it.

We begin the W.K.B. analysis by introducing the following space and time dependence for the perturbing quantities:

$$\rho'(\underline{r}, t) = \rho_c f(z) e^{i(k_x x - \omega t)} \quad , \quad (7.4)$$

$$p'(\underline{r}, t) = p_c f(z) e^{i(k_x x - \omega t)} \quad , \quad (7.5)$$

and

$$\underline{v}'(\underline{r}, t) = \underline{v}_c f(z) e^{i(k_x x - \omega t)} \quad , \quad (7.6)$$

where again $\underline{v}_c$ is a constant vector in the $\underline{v}'$ direction and ρ_c and p_c are constants with the units of density and pressure, respectively.

By substituting the expressions (7.4), (7.5) and (7.6) into eqs. (3.7), (3.8) and (3.9) we perform the various mathematical operations indicated in these equations as follows:

$$\frac{dv'}{dt} = -i\omega \underline{v}_c f e^{i(k_x x - \omega t)} \quad , \quad (7.7)$$

$$\nabla p' = i p_c f e^{i(k_x x - \omega t)} \underline{k}_x + \frac{df}{dz} p_c e^{i(k_x x - \omega t)} \underline{e}_z, \quad (7.8)$$

$$\nabla p' = i p_c f e^{i(k_x x - \omega t)} \underline{k}_x + \frac{df}{dz} p_c e^{i(k_x x - \omega t)} \underline{e}_z, \quad (7.9)$$

$$\nabla \cdot \underline{v}' = i(\underline{v}_c \cdot \underline{e}_x) k_x f e^{i(k_x x - \omega t)} + (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} e^{i(k_x x - \omega t)}, \quad (7.10)$$

and

$$(\underline{v}_0 \cdot \nabla) p' = i k_x f v_0 p_c e^{i(k_x x - \omega t)} \quad (7.11)$$

By substitution of eqs. (7.7)-(7.11) into eqs. (3.7), (3.8) and

(3.9), we have, respectively, cancelling the exponential terms

$$\begin{aligned}
 & -i\omega \rho_c \underline{v}_c f + \rho_0 v_0 i k_x \underline{v}_c f + \rho_0 f (\underline{v}_c \cdot \underline{e}_z) \frac{d v_0}{d z} \underline{e}_x \\
 & - \frac{\rho_c f}{h} \underline{e}_z + i \rho_c f k_x \underline{e}_x + \frac{d f}{d z} \rho_c \underline{e}_z - \rho_c f \underline{g} = 0 \quad , \\
 & \hspace{15em} (7.12)
 \end{aligned}$$

$$\begin{aligned}
 & -i\omega \rho_c f + \rho_0 (\underline{v}_c \cdot \underline{e}_x) i k_x f + \rho_0 (\underline{v}_c \cdot \underline{e}_z) \frac{d f}{d z} + \\
 & + v_0 \rho_c i k_x f - \frac{\rho_0}{h} (\underline{v}_c \cdot \underline{e}_z) f = 0 \quad , \\
 & \hspace{15em} (7.13)
 \end{aligned}$$

and

$$\begin{aligned}
 & -i\omega \rho_c f + i\omega \gamma \frac{\rho_0}{\rho_0} \rho_c f + \frac{\gamma-1}{h} \rho_0 (\underline{v}_c \cdot \underline{e}_z) f + \\
 & + v_0 \rho_c i k_x f - \gamma \frac{\rho_0}{\rho_0} i k_x f v_0 \rho_c = 0 \quad , \\
 & \hspace{15em} (7.14)
 \end{aligned}$$

Since eq. (7.12) is a two-dimensional vector equation, we have four equations and four unknowns v_{cx} , v_{cz} , ρ_c and p_c . The condition for a non-trivial solution of eqs. (7.12), (7.13) and (7.14) will give an equation for $f(z)$. It is obtained by setting the determinant of the coefficients of the unknowns equal to zero. However, we may eliminate ρ_c and p_c by substituting as follows: Solve eq. (7.13) for ρ_c and eq. (7.14) for p_c . Then, substitute ρ_c of eq. (7.13) into eq. (7.14), and substitute ρ_c of eq. (7.13) and p_c of eq. (7.14) into the vector eq. (7.12). This gives, upon re-arranging and collecting terms,

$$\begin{aligned} & \underline{v}_c [\bar{\omega}^2 f] + \underline{e}_x \left[i \bar{\omega} \frac{dv_0}{dz} (\underline{v}_c \cdot \underline{e}_z) f - i g k_x (\underline{v}_c \cdot \underline{e}_z) \dot{f} + \right. \\ & + i a^2 k_x (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} - a^2 k_x^2 (\underline{v}_c \cdot \underline{e}_x) f \left. \right] + \underline{e}_z \left[-\gamma g (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} - \right. \\ & - i \gamma g k_x (\underline{v}_c \cdot \underline{e}_x) f + a^2 (\underline{v}_c \cdot \underline{e}_z) \frac{d^2 f}{dz^2} + i a^2 k_x (\underline{v}_c \cdot \underline{e}_x) \frac{d\dot{f}}{dz} + \\ & \left. + i g k_x (\underline{v}_c \cdot \underline{e}_x) \dot{f} \right] = 0 \quad , \end{aligned} \tag{7.15}$$

where $\bar{\omega}$ is given by the transformation, similar to (5.14),

$$\bar{\omega} = \omega - v_0 k_x \quad , \tag{7.16}$$

and use has been made of eqs. (4.2) and (4.4).

From eq. (7.3), we have, on taking the derivative of f ,

$$\frac{df}{dz} = \left(\frac{dA}{dz} + A \frac{d\Phi}{dz} \right) e^{\Phi} \quad (7.17)$$

and

$$\frac{d^2f}{dz^2} = \left[\frac{d^2A}{dz^2} + 2 \frac{dA}{dz} \frac{d\Phi}{dz} + A \left(\frac{d\Phi}{dz} \right)^2 + A \frac{d^2\Phi}{dz^2} \right] e^{\Phi} \quad (7.18)$$

By inserting eqs. (7.17) and (7.18) into eq. (7.15), we obtain terms multiplied by A , $A \frac{d\Phi}{dz}$, $A \frac{d^2\Phi}{dz^2}$, $A \left(\frac{d\Phi}{dz} \right)^2$, $\frac{dA}{dz}$, $\left(\frac{dA}{dz} \right) \left(\frac{d\Phi}{dz} \right)$ and $\frac{d^2A}{dz^2}$. In the spirit of the W.K.B. method, we consider $A(z)$ to be a function that is slowly varying over a wavelength, $1/k$, and for this reason we neglect terms multiplied by $\frac{d^2A}{dz^2}$ and $\frac{dA}{dz}$. However, since $\frac{d\Phi}{dz} \propto k$, terms multiplied by $\left(\frac{dA}{dz} \right) \left(\frac{d\Phi}{dz} \right)$ cannot be neglected in the W.K.B. approximation, which becomes exact in the mathematical limit $k \rightarrow \infty$.

We separate eq. (7.15) into two equations; one equation containing terms multiplied by $\left(\frac{dA}{dz} \right) \left(\frac{d\Phi}{dz} \right)$ and $A \frac{d^2\Phi}{dz^2}$, and the other equation containing terms multiplied by A , $A \left(\frac{d\Phi}{dz} \right)^2$ and $A \left(\frac{d\Phi}{dz} \right)^2$. Hence, we have, respectively,

$$\underline{e}_z \left[2 a^2 (\underline{v}_0 \cdot \underline{e}_z) \left(\frac{dA}{dz} \right) \left(\frac{d\bar{\Phi}}{dz} \right) + a^2 (\underline{v}_0 \cdot \underline{e}_z) A \frac{d^2 \bar{\Phi}}{dz^2} \right] = 0$$

(7.19)

and

$$\begin{aligned} \underline{v}_c [\bar{\omega}^2] + \underline{e}_x \left[i \bar{\omega} \frac{dv_0}{dz} (\underline{v}_c \cdot \underline{e}_z) - i g k_x (\underline{v}_c \cdot \underline{e}_z) + \right. \\ \left. + i a^2 k_x (\underline{v}_c \cdot \underline{e}_z) \frac{d\bar{\Phi}}{dz} - a^2 k_x^2 (\underline{v}_c \cdot \underline{e}_x) \right] + \\ + \underline{e}_z \left[-\gamma g (\underline{v}_c \cdot \underline{e}_z) \frac{d\bar{\Phi}}{dz} - i \gamma g k_x (\underline{v}_c \cdot \underline{e}_x) + \right. \\ \left. + a^2 (\underline{v}_c \cdot \underline{e}_z) \left(\frac{d\bar{\Phi}}{dz} \right)^2 + i a^2 k_x (\underline{v}_c \cdot \underline{e}_x) \frac{d\bar{\Phi}}{dz} + \right. \\ \left. + i g k_x (\underline{v}_c \cdot \underline{e}_x) \right] = 0 . \end{aligned}$$

(7.20)

From eq. (7.19), we obtain

$$2 \left(\frac{dA}{dz} \right) \left(\frac{d\bar{\Phi}}{dz} \right) = - A \frac{d^2 \bar{\Phi}}{dz^2} ,$$

which gives

$$A = \left(\frac{d\bar{\Phi}}{dz} \right)^{-\frac{1}{2}} \quad (7.21)$$

This is the expected relation between the amplitude and phase from employing the W.K.B. method.

Eq. (7.20) is a two-dimensional vector equation, which is equivalent to two scalar equations for the two velocity components of $\underline{V}_c$, which are V_{cx} and V_{cz} . The condition for the solution of these two equations is the vanishing of the determinant of the coefficients of these unknowns. However, here it is advantageous to take the dot product of eq. (7.20) with $\underline{e}_x$ and $\underline{e}_z$. The two scalar equations thus formed are

$$\begin{aligned} & (\underline{V}_c \cdot \underline{e}_x) [\bar{\omega}^2 - a^2 k_x^2] \\ & + (\underline{V}_c \cdot \underline{e}_z) \left[i\bar{\omega} \frac{dV_0}{dz} - igk_x + ia^2 k_x \frac{d\bar{\Phi}}{dz} \right] = 0 \end{aligned} \quad (7.22)$$

and

$$\begin{aligned} & (\underline{V}_c \cdot \underline{e}_x) \left[ia^2 k_x \frac{d\bar{\Phi}}{dz} - (\gamma-1) igk_x \right] \\ & + (\underline{V}_c \cdot \underline{e}_z) \left[\bar{\omega}^2 - \gamma g \frac{d\bar{\Phi}}{dz} + a^2 \left(\frac{d\bar{\Phi}}{dz} \right)^2 \right] = 0 \end{aligned} \quad (7.23)$$

Setting the determinant of the coefficients of $(\underline{v}_c \cdot \underline{e}_x)$ and $(\underline{v}_c \cdot \underline{e}_z)$ of eqs. (7.22) and (7.23) equal to zero, gives

$$\begin{aligned} & \left(\frac{d\bar{\Phi}}{dz} \right)^2 [\bar{\omega}^2 a^2] + \left(\frac{d\bar{\Phi}}{dz} \right) \left[-\bar{\omega}^2 \gamma g + \bar{\omega} a^2 k_x \frac{dv_0}{dz} \right] + \\ & + \left[\bar{\omega}^4 - \bar{\omega}^2 a^2 k_x^2 - (\gamma-1) \bar{\omega} g k_x \frac{dv_0}{dz} + \right. \\ & \left. + (\gamma-1) g^2 k_x^2 \right] = 0 \end{aligned} \quad (7.24)$$

Eq. (7.24) is a first-order differential equation of the second degree. The theory of differential equations allows us to solve it by using the binomial theorem. This gives

$$\begin{aligned} \left(\frac{d\bar{\Phi}}{dz} \right)_{\pm} = & \frac{\gamma g}{2a^2} - \frac{k_x \frac{dv_0}{dz}}{2\bar{\omega}} \pm \left[\frac{k_x^2 \left(\frac{dv_0}{dz} \right)^2}{4\bar{\omega}^2} + \frac{\gamma^2 g^2}{4a^4} \right. \\ & \left. - \frac{\bar{\omega}^2}{a^2} + k_x^2 + \frac{\left(\frac{\gamma}{2} - 1 \right) g k_x \frac{dv_0}{dz}}{\bar{\omega} a^2} - \frac{(\gamma-1) g^2 k_x^2}{\bar{\omega}^2 a^2} \right]^{\frac{1}{2}} \end{aligned} \quad (7.25)$$

The solution of the differential eq. (7.25) gives the phase for the W.K.B. solution of the + and - modes. As a limiting case, if we let

$$\frac{d\bar{\phi}}{dz} \Rightarrow ; k_z , \quad (7.26)$$

and use the definitions (6.1), (6.2) and (6.6)-(6.11), eq. (7.25) goes over into the local dispersion relation (6.17), as it should.

8. VERTICAL PENETRATION DEPTHS

The condition $\frac{d\bar{\phi}}{dz} = 0$ implies that the z-component of the wave number, k , is zero. This gives the condition for the termination of the wave propagation in the z-direction, that is, the vertical penetration depth.

Setting $\frac{d\bar{\phi}}{dz} = 0$ in eq. (7.24), we have

$$\bar{\omega}^4 - \bar{\omega}^2 a^2 k_x^2 - (\gamma-1) \bar{\omega} g k_x \frac{dv_0}{dz} + (\gamma-1) g^2 k_x^2 = 0 . \quad (8.1)$$

To put eq. (8.1) in non-dimensional form, we again employ the definitions (6.6)-(6.11), and we keep in mind that $\underline{k} = k_x \underline{e}_x$ when $\frac{d\bar{\phi}}{dz} = 0$. We obtain

$$W^4 - W^2 K^2 - (\gamma-1) W K S + (\gamma-1) K^2 = 0 . \quad (8.2)$$

We shall consider the following three cases: constant velocity gradients, linear velocity gradients, and parabolic velocity gradients.

Case I: Constant velocity gradients

We consider S to be independent of z , and we solve eq. (8.2) for the wave number K . This gives

$$K_{\pm} = \frac{(\gamma - 1) W S}{2[(\gamma - 1) - W^2]}$$

$$\pm \left\{ \left[\frac{(\gamma - 1) W S}{2[(\gamma - 1) - W^2]} \right]^2 - \frac{W^4}{\gamma - 1 - W^2} \right\}^{\frac{1}{2}}$$

(8.3)

Eq. (8.3) gives the non-dimensional wave number for the two modes at the point at which the z -component of the wave number vanishes for the constant velocity gradient case. One must keep in mind that W is the transformed frequency, and it is a function of z . In order to find the penetration depths, one must make the inverse frequency transformation and solve the resulting cubic algebraic equation for z_{pd} by means of Appendix I. z_{pd} is the penetration depth.

Case II: Linear velocity gradients

We consider

$$S = \alpha + \beta z, \quad (8.4)$$

where α is an arbitrary unitless constant, and β is an arbitrary constant with units $[\text{cm}^{-1}]$. Substituting expression (8.4) into eq. (8.2), gives, upon solving for z , the height to which the wave can propagate. This is the vertical penetration depth, z_{pd} .

$$z_{pd\pm} = \frac{W_{\pm}^3}{(\gamma-1)K\beta} - \frac{W_{\pm}K_{\pm}}{(\gamma-1)\beta} + \frac{K_{\pm}}{W_{\pm}\beta} - \frac{\alpha}{\beta} \quad (8.5)$$

Again we keep in mind that W is the transformed frequency and can be made independent of z by a coordinate transformation, which we will not do here.

Case III: Parabolic velocity gradients

We consider

$$S = \alpha + \beta z + \delta z^2, \quad (8.6)$$

where α and β are defined in eq. (8.4), and δ is an arbitrary constant with units $[\text{cm}^{-2}]$. Substituting the expression (8.6) into eq. (8.2), gives, upon solving for the vertical penetration depth,

$$z_{pd\pm} = -\frac{\beta}{2\delta} + \left[\left(\frac{\beta}{2\delta} \right)^2 + \frac{W_{\pm}^3}{(\gamma-1)K_{\pm}\delta} - \frac{W_{\pm}K_{\pm}}{(\gamma-1)\delta} + \frac{K_{\pm}}{W_{\pm}\delta} \right]^{\frac{1}{2}} \quad (8.7)$$

Again, we keep in mind that W is the transformed frequency and can be made independent of z by a coordinate transformation, which we will not do here.

9. CONCLUSION

We have considered small amplitude low frequency waves in an isothermal compressible medium with a velocity gradient and within a constant gravitational field. It was found that there are two distinct modes of wave motion which we called the + mode and the - mode. Under the assumption that the wavelength be small compared to a change in the velocity profile, a local dispersion relation governing the wave propagation for constant velocity gradients was derived from the linearized equations of fluid dynamics and thermodynamics, and arranged in a non-dimensional form independent of the constants describing a particular atmosphere. By employing a generalized W.K.B. method, a better approximation was obtained by taking into account the slowly varying change of the velocity profile and the velocity gradient. From the generalized W.K.B. method, equations were found which give the height at which wave reflection occurs. This determines the maximum heights up to which the wave modes can propagate. However, the normalized transformed frequency W is a function of z , but can be made independent of z by an appropriate transformation of coordinates, which was not done here. Dissipative effects arising from viscosity and heat conduction was not considered.

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APPENDIX I

The local dispersion relations (6.12) and (6.13) may be solved analytically in the following manner. In general, we have

$$w^4 + A w^2 + B w + C = 0 \quad (I.1)$$

where for eq. (6.12)

$$A = -\kappa^2 - \left(\frac{\gamma}{2}\right)^2, \quad (I.2)$$

$$B = (i \kappa \sin \phi - \frac{\gamma}{2} + 1) S \kappa \cos \phi, \quad (I.3)$$

and

$$C = (\gamma - 1) \kappa^2 \cos^2 \phi; \quad (I.4)$$

and where for eq. (6.13)

$$A = -\bar{\kappa}^2 - i \gamma \bar{\kappa} \cos \theta, \quad (I.5)$$

$$B = (i \bar{\kappa} \sin \theta - \gamma + 1) S \bar{\kappa} \cos \theta \quad (I.6)$$

and

$$c = (\gamma - 1) \bar{\kappa}^2 \cos^2 \theta \quad (I.7)$$

From the quartic eq. (I.1), we consider the resolvent cubic equation

$$\xi^3 + \frac{A}{2} \xi^2 + \frac{1}{4} \left(\frac{1}{4} A^2 - c \right) \xi - \frac{B^2}{64} = 0 . \quad (I.8)$$

It is convenient to transform eq. (I.8) into Euler's resolvent cubic, thereby eliminating the quadratic term. Using the transformation

$$\xi = \eta - \frac{A}{6} \quad (I.9)$$

we have

$$\eta^3 + q\eta + r = 0 \quad (I.10)$$

where

$$q = \frac{1}{4} \left(\frac{1}{4} A^2 - c \right) - \frac{A^2}{12} \quad (I.11)$$

and

$$r = -\frac{B^2}{64} - \frac{A}{24} \left(\frac{1}{4} A^2 - C \right) + \frac{A^3}{108} \quad (\text{I.12})$$

The three roots of eq. (I.10) may be expressed in the form

$$\eta_1 = -\frac{r}{q} f_1(y) \quad , \quad (\text{I.13})$$

$$\eta_2 = \frac{r}{q} f_2(y) \quad , \quad (\text{I.14})$$

and

$$\eta_3 = -\frac{r}{q} f_3(y) \quad , \quad (\text{I.15})$$

where

$$y = \frac{r^2}{q^3} \quad (\text{I.16})$$

and

$$f_1(y) = \frac{\left[1 + \left(1 + \frac{4}{27y}\right)^{\frac{1}{2}}\right]^{1/3} + \left[1 - \left(1 + \frac{4}{27y}\right)^{\frac{1}{2}}\right]^{1/3}}{(2y)^{1/3}} \quad (I.17.)$$

$$f_2(y) = -\frac{f_1(y)}{2} + \left[\frac{f_1^2(y)}{4} - \frac{1}{yf_1(y)} \right]^{\frac{1}{2}} \quad (I.18)$$

and

$$f_3(y) = -\frac{f_1(y)}{2} - \left[\frac{f_1^2(y)}{4} - \frac{1}{yf_1(y)} \right]^{\frac{1}{2}} \quad (I.19)$$

For the condition $-\frac{4}{27} < y < 0$, all roots are real, and for $y > 0$ or $y = \frac{-4}{27}$, there is one real root and a pair of complex roots. By different proper choices of the phase angle for the $1/3$ powers in eq. (I.17), $f_1(y)$ may be always made real, even though it may not be continuous at $y = \frac{-4}{27}$.

Thus, from eq. (I.9) the roots of eq. (I.8) are

$$\xi_i = \eta_i - \frac{A}{6} \quad (i = 1, 2, 3) , \quad (I.20)$$

the roots of eq. (I.1) are

$$w_i = \pm \sqrt{\xi_1} \pm \sqrt{\xi_2} \pm \sqrt{\xi_3} , \quad (I.21)$$

such that $m = \sqrt{\xi_1} \sqrt{\xi_2} \sqrt{\xi_3}$ is an element of $R(\xi_1, \xi_2, \xi_3)$. This means that m must be always positive or always negative. We will choose the former. Thus, the four roots of the quartic eq. (I.1) are

$$w_1 = \sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{\xi_3} \quad (I.22)$$

$$w_2 = \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} \quad (I.23)$$

$$w_3 = -\sqrt{\xi_1} - \sqrt{\xi_2} + \sqrt{\xi_3} \quad (I.24)$$

and

$$w_4 = -\sqrt{\xi_1} + \sqrt{\xi_2} - \sqrt{\xi_3} \quad (I.25)$$

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