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CONTROLLABILITY AND THE SINGULAR PROBLEM

By George W. Haynes

December 1968

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Semi-Annual Report

Prepared under Contract No. NAS2-4895 by
Martin Marietta Corporation
Denver, Colorado
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for

AMES RESEARCH CENTER
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Semi-Annual Report

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Program Manager

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FOREWORD

This interim report outlines the researches performed during the past six months on Controllability and the Singular Problem under NASA Contract NAS2-4898, Ames Research Center. The research was performed by G. W. Haynes of the Martin Marietta Corporation, Denver, in collaboration with Professor H. Hermes of the Mathematics Department, Colorado University.

I. INTRODUCTION

The concept of complete controllability of linear systems was introduced by Kalman in 1960. This concept immediately provided the rationale for many of the assumptions invoked in the study of linear systems. For autonomous linear systems, a simple algebraic test determines both necessary and sufficient conditions for complete controllability. However, for the complete controllability of nonautonomous linear systems, Kalman derived an integral criterion involving the fundamental solution of a homogeneous linear system, which diminished the usefulness of the test. In 1961, Hermann in a difficult and obscure work derived an algebraic test for the complete controllability of nonautonomous linear systems. Hermann's approach was based on the geometrical interpretation of nonintegrability of pfaffians and neighborhoods of attainable points as demonstrated by Caratheodory and subsequently extended by Chow.

Unfortunately the full generality of Chow's Theorem cannot be applied to nonlinear systems since the coordinate system is endowed with one special coordinate, namely time, which has to evolve. This restriction invalidates Chow's Theorem and limits its applicability. For certain control systems where the control actuator vectors do not generate an "involutive distribution", Chow's Theorem can be used to generate criteria for global controllability.

In general, for nonlinear systems, Chow's Theorem can only be used to determine the existence or nonexistence of integral manifolds and establish whether or not some points are obviously inaccessible.

II. MATHEMATICAL PRELIMINARIES AND NOTATION

We shall briefly review some of the notions and symbolism associated with the geometry of manifolds [1] that is pertinent to the geometric differential approach to controllability. For convenience all manifolds, vector fields, curves, maps, etc., will be assumed to be smooth, that is, differentiable as often as we please. Any exceptions to this rule will be stated in the text.

Composition of mappings will be denoted either by $\phi \circ \psi$ when brevity of notation is required or by the obvious notation $\phi(\psi(\sigma))$. All manifolds M are assumed to be connected and paracompact (finite covering) with coordinate systems covering M denoted by $(x_1, x_2, \ldots, x_n)$. If M is a manifold, then at a point $m \in M$ we shall denote by F(M,m) the set of smooth functions with domain a neighborhood N(m) of m. A curve $\gamma(\sigma)$ (with parameter σ) in M is a map of a closed interval [a,b] into M which can be extended to a smooth map of an open interval.

Tangent Vector - The directional derivative of a function $f \in F(M,m)$ at m in the direction of a given curve γ gives rise to the notion of a tangent vector. The curve $\gamma(\sigma)$ generates a tangent vector $\gamma_*(\sigma)$ which maps F(M,m) into the reals R' by

$$\gamma_*(\sigma)(f) = \frac{d}{d\sigma}(f \circ \gamma)(\sigma)$$

for a parameter σ given by $m = \gamma(\sigma)$. The tangents at a point $m \in M$ form a linear space denoted by M_m. The dimension of M_m is n, the dimension of M; in fact, if $(x_1, x_2, \ldots, x_n)$ is a coordinate system in a neighborhood N(m) of M, then $(D_{x_1}, D_{x_2}, \ldots, D_{x_n})$ is a basis for M_m, where D_{x_i} means partial differentiation with respect to the x_i coordinate. Tangents are completely characterized by the following.

<u>Theorem 2.1</u> - If $(x_1, x_2, \ldots, x_n)$ is a coordinate system at $m \in M$ and $t \in M_m$ a tangent vector at m then t can be given the representation

$$t = \sum t(x_i)\, D_{x_i}(m)$$

<u>Differentials</u> - If $\phi : M \longrightarrow N$ is a smooth mapping of manifolds, then we define the differential of ϕ, $d\phi$: $M_m \longrightarrow N_{\phi(m)}$ as follows. If $\gamma \in M_m$ and $f \in F(N, \phi(m))$ then $d\phi(\gamma)(f) = \gamma(f \circ \phi)$.

<u>Differentials of Functions</u> - Each element f of $F(M,m)$ generates by its differential an element of the dual space M_m^* of M_m by df: $M_m \longrightarrow R$. By this association if $t \in M_m$, then $df(t) = t(f)$. If $(x_1, x_2, \ldots, x_d)$ is a coordinate system at m, then $(dx_1, dx_2, \ldots, dx_n)$ forms a basis of M_m^* dual to $D_{x_i}(m)$. The differential of a function $f \in F(M,m)$ has a more obvious representation as

$$df = \sum D_{x_i} f(m) dx_i.$$

<u>Vector Fields</u> - A vector field X, is a function defined on a subset E of a manifold M which assigns at each point $m \in E$ an element $X(m)$ of M_m. If $(x_1, x_2, \ldots x_n)$ is a coordinate system then D_{x_i} is a smooth vector field. If X is a vector field with its domain contained in the coordinate system then X may be given the representation

$$X = \sum f_i D_{x_i},$$

where the f_i are real valued functions. If X and Y are smooth vector fields, $(x_1, x_2, \ldots, x_n)$ a coordinate system, and if X and Y have the representations

$$X = \sum f_i D_{x_i}, \quad Y = \sum g_i D_{x_i}$$

then the Lie Bracket is a vector field defined by

$$[X,Y] = \sum (f_i D_{x_i} g_i - g_i D_{x_i} f_i) D_{x_j}$$

The bracket operation is bilinear with respect to real coefficients and is also skew-symmetric

$$[X,Y] = - [Y,X]$$

Integral Curves - If X is a smooth vector field then γ is the integral curve of X starting at m if $\gamma(o) = m$ and for every σ in the domain of γ, we have $\dot{\gamma}_*(\sigma) = X(\gamma(\sigma))$. For the definition of X given above, this is equivalent to the system of differential equations

$$\frac{d\gamma(\sigma)}{d\sigma} = f(\gamma(\sigma)).$$

Differential One Forms - If $(x_1, x_2, \ldots, x_n)$ is a coordinate system at $m \in M$, then $(dx_1, dx_2, \ldots, dx_n)$ at m forms a basis for the cotangent space M_m^* so that every element f of $G'(M_m^*)$, the space of linear functions on M_m into the reals, can be represented by $f = \sum g_i(m) dx_i$. A differential one form of

M is a function θ defined on some subset E of M, whose value at each $m \in E$ is an element of $G'(M_m^*)$. It should be noted that every differential one form is not necessarily the differential of a smooth function.

III DIFFERENTIAL GEOMETRIC APPROACH TO CONTROLLABILITY

The basis of Hermann's [2] differential geometric approach to controllability is the use of Chow's theorem [3] which relates the accessibility of points to integral curves of a pfaffian system. Chow's theorem in turn, is a generalization of the important theorem due to Caratheo'dory [4] for a single pfaffian. We cite the following contrapositive form of Carathe'odory's theorem since it appeals directly to the notion of controllability.

3.1 Theorem - If the differential one form $\omega(x) = \sum a_i(x)dx_i$, defined on a manifold M with coordinate structure $(x_1, x_2, \ldots, x_n)$, is not integrable then there exists some neighborhood $N(x_o)$ of a given point $x_o \in M$ in which all points are accessible by integral curves $\gamma(\sigma)$ satisfying $\omega(\gamma) = 0$. As previously mentioned, this result was extended by Chow to systems of pfaffians or differential one forms. In the application of Chow's theorem to the controllability problem, Hermann's approach is based on the proposition that every point is accessible that is not obviously inaccessible. To prevent some points in $N(x_o)$ from being obviously inaccessible it is evident that we must negate the existence of any integral manifolds to the system of differential one forms. The existence of integral manifolds for the pfaffian system can be determined by using Frobenius' integration theorem [5]. Equivalently, the integrability conditions has a dual formulation in terms of distributions of vector fields being involutive.

If θ is a p-dimensional distribution on a manifold M ($p \leq \dim(M)$) which assigns to each $m \in M$ a p-dimensional linear subspace $\theta(m)$ of M_m the tangent space, then a vector field X belongs to the distribution θ denoted by $X \in \theta$ if for every point m in the domain of X, $X(m) = \theta(m)$. A distribution θ is involutive if for all vector fields X, Y which belong to θ the lie bracket $[X, Y]$ also belongs to the distribution θ. An integral manifold V of θ is a submanifold of M such that $di(V_m) = \theta(i(m))$ where i defines the map of $N \to M$ and di denotes the differential of i. A distribution θ is integrable if for every $m \in M$ there is an integral manifold of θ containing m. The following theorem is pertinent to the existence of integral manifolds for involutive distributions.

3.2 Theorem - An involutive distribution θ on M is integrable. Furthermore, through every $m \in M$ there passes a unique maximal connected integral manifold of θ and every other connected integral manifold containing m is an open submanifold of this maximal one.

Hermann applied these results to the controllability problem as follows. Consider the control system

$$\dot{x}_i = f_i(t, x, u)$$

where the state x is an n-vector, the control u is an s vector ($s \leq n$) and the functions f are assumed to be smooth. In the (t, x, u) space we can associate with the control system a codistribution of one forms defined by

$$dx_i - f_i(t, x, u)dt = 0$$

The dual space of vector fields is spanned by

$$X = D_t + f_i(t, x, u)Dx_i$$

$$Y = Du_\alpha$$

It is now a routine matter to demonstrate whether or not this distribution of vector fields is involutive under the lie bracket operation and thus determine the existence or nonexistence of an integral manifold.

The relation of Hermann's proposition regarding avoidance of obviously inaccessible points as well as Carathe'odory's theorem and Chow's generalization on integrability and inaccessible points to the converse problem of accessible points follows from the geometrical interpretation of the lie bracket of vector fields. The tangent vectors associated with the integral curves of the vector fields do not span the tangent space M_m. However, if the distribution is not involutive then the tangent vectors associated with the derived system of vector fields under the lie bracket operation do span the tangent space M_m. If the tangent vectors span M_m, then all points in some neighborhood $N(x_o)$ can be attained by integral curves of the vector fields, provided we can identify integral curves with those vector fields that are generated by the lie bracket. The following theorem resolves this problem [1].

<u>3.3 Theorem</u> - Let X and Y be smooth vector fields both defined at $m \in M$. If $\gamma(\sigma)$ denotes the final point obtained by traversing in sequence the integral curves to the vector fields X, Y, -X, -Y for a fixed parameter σ and initial point m, then γ has $[X, Y](m)$ as the limit of its tangents.

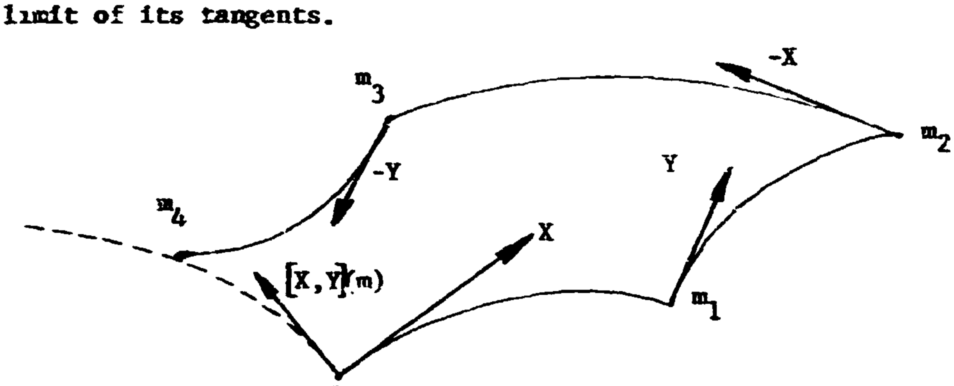

Figure 3.1

Proof - Let X and Y have the representations $X = \sum f_i(x) Dx_i$, $Y = \sum g_i(x) Dx_i$. If $\phi(\sigma)$ and $\psi(\sigma)$ are the integral curves to the vector fields X and Y respectively so that $\phi_*(\sigma) = X(\phi(\sigma))$, $\psi_*(\sigma) = Y(\psi(\sigma))$, then ϕ and ψ satisfy the differential equations

$$\frac{d\phi}{d\sigma} = f(\phi); \qquad \frac{d\psi}{d\sigma} = g(\psi).$$

As we traverse a rectangle of integral curves (Figure 3.1) we obtain the following relations

$$m_1 = \phi(\sigma, m)$$

$$m_2 = \psi(\sigma, m_1)$$

$$m_3 = \phi(-\sigma, m_2)$$

$$m_4 = \psi(-\sigma, m_3)$$

Since we shall compare the point m_4 to the point m_1 for small σ, we have on expanding the integral curves in a Taylor's series in σ

$$m_1 = m + f(m)\sigma + X(m)\, f(m)\frac{\sigma^2}{2} + O(\sigma^3)$$

$$m_2 = m_1 + g(m_1)\sigma + Y(m_1)g(m_1)\frac{\sigma^2}{2} + O(\sigma^3)$$

$$m_3 = m_2 - f(m_2)\sigma + X(m_2)f(m_2)\frac{\sigma^2}{2} + O(\sigma^3)$$

$$m_4 = m_3 - g(m_3)\sigma + Y(m_3)g(m_3)\frac{\sigma^2}{2} + O(\sigma^3)$$

Expanding these terms about the point m and only retaining terms in σ^2 or lower, yields

$$m_1 = m + f(m)\sigma + X(m)f(m)\frac{\sigma^2}{2}$$

$$m_2 = m + (f(m) + g(m))\sigma + (X(m)f(m) + Y(m)g(m) + 2X(m)g(m))\frac{\sigma^2}{2}$$

$$m_3 = m + g(m)\sigma + (2X(m)g(m) + Y(m)g(m) - 2Y(m)f(m))\frac{\sigma^2}{2}$$

$$m_4 = m + (2X(m)g(m) - 2Y(m)f(m))\frac{\sigma^2}{2}$$

Therefore the curve generated by the rectangle of integral curves of the vector fields X and Y is, for small σ, given by

$$\gamma(\sigma) = m_4 - m = (X(m)g(m) - Y(m)f(m))\sigma^2$$

The relation between the integral curve γ and the lie bracket is obvious since

$$[X, Y] = \{X(x)g_i(x) - Y(x)f_i(x)\} Dx_i$$

The lie bracket creates a second order tangent rather than a first order tangent since

$$\frac{d\gamma(o)}{d\sigma} = 0$$

Therefore for any function h we have

$$2[X, Y]h(m) = \frac{d^2}{d\sigma^2} (ho\gamma)(o)$$

This geometrical interpretation of the lie bracket gives insight into the local attainability of points. Traversing a one parameter family of rectangles whose sides are tangent to a distribution might yield locally a curve whose tangent is not in the distribution. When the distribution is not involutive, we can generate an independent set of tangent vectors which span the manifold and implies local attrinability of points by integral curves to the distribution.

IV. LINEAR SYSTEMS

As an example of the differential geometric approach, Hermann derived the following algebraic test for the controllability of the linear system.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{4.1}$$

4.1 Theorem - If the rank of $\left[B(t), \Gamma B(t), \ldots\ldots, \Gamma^{n-1} B(t)\right]$, where $\Gamma = A(t)-D_t$, is n for each t then every point of the x-space is accessible from the origin on paths that are solutions of the linear system (4.1) for some choice of the control u(t).

The proof of this theorem is fairly trivial and proceeds as follows. With the vector fields defined by

$$X = D_t + (A(t)x + B(t)u)^T Dx$$

$$Y = Du$$

successive application of the Lie Bracket yields

$$[Y,X] = B(t)^T Dx$$

$$[X, [Y,X]] = (\Gamma B(t))^T Dx$$

$$[X, [X, [Y,X]]] = (\Gamma^2 B(t))^T Dx \qquad \text{etc.}$$

The distribution is not involutive if rank $\left[B(t), \Gamma B(t), \ldots, \Gamma^{n-1}B(t)\right]$ is n, which completes the proof.

Subsequent to this result, Kalman et.al. [6], derived the following integral test for controllability.

<u>4.2 Theorem</u> - The linear system (4.1) is completely controllable at t_o if and only if there exists a $t_1 > t_o$ such that $W(t_o,t_1)$ is nonsingular, where $W(t_o,t_1)$ is the nxn matrix defined by

$$W(t_o,t_1) = \int_{t_o}^{t_1} \Phi(t_o,t)B(t)B^T(t)\,\Phi^T(t_o,t)dt$$

and $\Phi(t,t_o)$ is the fundamental solution to the homogeneous differential equation.

Central to the proof of this theorem is the demonstration that there exists one interval $[t_o,t_1]$ on which the functions $\Phi(t_o,t)B(t)$ are linearly independent. The reason for this is obvious. If the functions $\Phi(t_o,t)B(t)$ are not independent on any interval, then this implies the existence of a constant vector C such that

$$C^T\Phi(t_o,t)B(t) = 0$$

for all t. This, in turn, implies that the control system

$$\dot{y} = \Phi(t_o,t)B(t)u$$

derived from (4.1) by the nonsingular transformation Φ is not controllable since the integral manifolds would be given by $C^T y$.

The equivalence between Hermann's algebraic test and Kalman's integral test follows from the demonstration of the linear independence of the functions $\Phi(t_o,t)B(t)$. If we assume that $A(t) \in C^{n-2}$ and $B(t) \in C^{n-1}$, then by formally differentiating the expression $C^T \Phi(t_o,t)B(t)(n-1)$ times, we obtain Hermann's algebraic test on negating the existence of the constant vector C. In fact, there is an equivalence between this method and the differential geometric method of showing that the distribution is not involutive. However, the algebraic test implies that the functions $\Phi(t_o,t)B(t)$ are linearly independent for all intervals $[t_o t_1]$. Kalman's integral test on the other hand requires that we find <u>one</u> interval where this is true for the linear system to be completely controllable. Having found one such interval $[t_o,t_1]$ on which the functions are linearly independent, there can exist subintervals of this interval on which the functions are not independent and integral manifolds exist. Since the integral manifolds are different on each subinterval, otherwise the functions would not be independent, it follows that the integral manifolds must span the manifold M for points to be accessible over the interval $[t_o,t_1]$. The algebraic test, therefore, constitutes only a sufficient condition for complete controllability of linear systems. However, if the matrices A(t) and B(t) are analytic, then Charg [7] has proven that the algebraic test is also necessary. Obviously, in this case, we cannot piece together integral manifolds to span the manifold since if any row or combination of rows of the matrix functions $\Phi(t_o,t)B(t)$ are zero on some interval, then they are zero everywhere by the analyticity condition.

The following algebraic test is fully equivalent to Kalman's integral test for controllability.

4.3 Theorem - Consider the linear system $\dot{y} = H(t)u$ where $H(t)$ is an nxr matrix composed of C^{n-1} elements. This system is completely controllable at t_o if and only if there exists n times $t_1, \ldots, t_n \geq t_o$ such that

$$\text{rank} \quad [H(t_1), \dot{H}(t_2), \ddot{H}(t_3), \ldots, H^{(n-1)}(t_n)] \text{ is n.}$$

Proof. To show sufficiency we prove the contrapositive. If the linear system is not completely controllable at t_o, this implies

$$\text{rank} \quad [H(t_1), \ldots, H^{(n-1)}(t_n)] < n$$

for any set $t_1, \ldots, t_n \geq t_o$. In fact, if the linear system is not completely controllable at t_o, this implies there exists a nonzero vector C such that $C^T H(t) = 0$ for all $t \geq t_o$. This in turn implies that

$$C^T \dot{H}(t), \ldots, C^T H^{(n-1)}(t),$$

are also zero for all $t \geq t_o$ Hence, for any set $t_1, \ldots, t_n \geq 0$

$$\text{rank} \quad [H(t_1), \dot{H}(t_2), \ldots, H^{(n-1)}(t_n)] < n.$$

For necessity we shall assume that the linear system is completely controllable at t_o and demonstrate the existence of a set $t_1, \ldots, t_n \geq t_o$ such that

$$\text{rank} \quad [H(t_1), \dot{H}(t_2), \ldots \quad H^{(n-1)}(t_n)] \text{ is n.}$$

This is equivalent to showing that for any nonzero vector C the nr dimensional vector

$C^T[H(t_1), \ldots, H^{(n-1)}(t_n)] \neq 0$. Let e_1 be any nonzero vector. Since the system is assumed to be controllable and H(t) is continuous then there exists a $t_1 \geq t_o$ such that $e_1^T H(t_1) \neq 0$. If rank $H(t_1)$ is n then the proof is finished. If not, there exists a nonzero vector e_2 such that $e_2^T H(t_1) = 0$, so that e_2 and e_1 are linearly independent. Now there exists a $t_2 \geq t_o$ such that $e_2^T \dot{H}(t_2) \neq 0$, if not, then $e_2^T \dot{H}(t) = 0$ for all $t \geq t_o$. This implies

$$\int_{t_1}^{t} e_2^T \dot{H}(t)dt = e_2^T H(t) = 0$$

for all $t \geq t_o$ and contradicts the assumption that the system is completely controllable. Next consider $[H(t_1), \dot{H}(t_2)]$, if the rank of this matrix is n the proof is finished. If not, then there exists a nonzero vector e_3 such that

$$e_3^T[H(t_1), H(t_2)] = 0; \text{ and a } t_3 > t_o \text{ such that } e_3^T \ddot{H}(t_3) \neq 0.$$

Clearly e_1, e_2 and e_3 are linearly independent. Continuing inductively, either for some

$$j < n, \text{ rank } [H(t_1), \dot{H}(t_2), \ldots, H^{(j-1)}(t_j)] \text{ is } n$$

or else we generate n linearly independent vectors $e_1, \ldots, e_n$ such that

$$e_{j+1}^T\left[H(t_1),\dot{H}(t_2), \quad H^{(j-1)}(t_j)\right] = 0$$

In the first instance, we are finished. In the second, any nonzero vector C can be expressed as $C = \sum \gamma_i e_i$ with not all the γ_i zero. From the property that the e_i satisfies, it follows that

$$C^T\left[H(t_1).\dot{H}(t_2), \; . \; . \; . \; . \; , H^{(n-1)}(t_n)\right] \neq 0$$

This completes the proof.

<u>4.4 Corollary</u> - Consider the linear system

$$\dot{x} = A(t)x + B(t)u,$$

where A(t) is an nxn matrix of C^{n-2} elements, and B(t) is an nxr matrix of C^{n-1} elements. This system is completely controllable at t_o if and only if there exists n times $t_1, \; . \; . \; . \; . \; , t_n \geq t_o$ such that

$$\text{rank} \qquad \left[B(t_1), \Phi(t_1,t_2)\,\Gamma B(t_2), \; . \; . \; . \; . \; , \Phi(t_1,t_n)\,\Gamma^{n-1}B(t_n)\right] = n$$

where $\Gamma = A(t) - D_t$ and Φ is the fundamental solution to the homogeneous equation.

By means of the following lemma we can derive an even simpler algebraic test for complete controllability.

4.5 Lemma - Let $H(t)$ be an $n \times r$ matrix valued function. Suppose rank $[H(t_1), \ldots, H(t_n)] < n$ for all sets $\{t_1, t_2, \ldots, t_n\}$ with $t_i \geq t_o$. Then there exists a nontrivial constant vector C such that $C^T H(t) = 0$ for all $t \geq t_o$.

Proof. Let $t_1 \geq t_o$ be chosen so that rank $H(t_1)$ is maximal, and call this rank r_1. Select t_2 so that rank $[H(t_1), H(t_2)]$ is maximal and call this rank r_2. We continue this process to the choice of t_{n-1} such that rank $[H(t_1), \ldots, H(t_{n-1})] = r_{n-1} < n$ is maximal. Now either $r_j = r_{j+1}$ for some $j = 1, 2, \ldots, (n-1)$ or $r_{n-1} = n-1$. In the first case, let j be the smallest integer such that $r_j = r_{j+1}$. The columns of $H(t)$, therefore, must lie in the r_j dimensional subspace spanned by the columns of $H(t_1), \ldots, H(t_j)$ for all t. Hence, if C is a nontrivial vector orthogonal to this subspace then $C^T H(t) = 0$ for all $t \geq t_o$. In the second case, since by hypothesis we cannot increase the rank of $[H(t_1), \ldots, H(t_{n-1})]$ by adjoining $H(t)$, the columns of $H(t)$ lie in the $(n-1)$ dimensional subspace spanned by the columns of $H(t_1), \ldots, H(t_{n-1})$ and a nontrivial vector C orthogonal to this subspace satisfied $C^T H(t) = 0$ for all $t \geq t_o$. We are now in a position to prove the following theorem.

4.6 Theorem - The system $\dot{y} = H(t)u$ is completely controllable at t_o if and only if there exists n times $t_1, \ldots, t_n \geq t_o$ such that rank $[H(t_1), \ldots, H(t_n)]$ is n.

Proof. The system $\dot{y} = H(t)u$ is completely controllable at t_o if and only if $C^T H(t) = 0$ for $t \geq t_o$ implies $C = 0$. Suppose the system is not completely controllable at t_o. Then there exists a nontrivial vector C such that $C^T H(t) = 0$, $t \geq t_o$, which implies $C^T [H(t_1), \ldots, H(t_n)] = 0$ for all sets $\{t_1, \ldots, t_n\}$. The contrapositive of this shows, if the rank $[H(t_1), \ldots, H(t_n)]$ is n for some sets $\{t_1, \ldots, t_n\}$ then the system is completely controllable at t_o.

Next suppose that rank $[H(t_1), \ldots, H(t_n)] < n$ for all sets $\{t_1, \ldots, t_n\}$. Then by lemma (4.5) there exists a nontrivial vector C such that $C^T H(t) = 0$ for all $t \geq t_o$ which shows that the system is not completely controllable.

V. NONLINEAR CONTROL SYSTEMS

We shall be concerned with nonlinear systems in which the control appears linearly as defined by

$$\dot{x} = A(x) + B(x)u \qquad 5.1$$

where $A(x)$ is an $n \times 1$ matrix of smooth functions; $B(x)$ is an $n \times r$ $(r \leq n)$ matrix of smooth functions, and is assumed to be of maximal rank for all x in some neighborhood $N(x)$ of a given point $x_o \epsilon M$. The state x is an n-vector, and the control u is an r-vector. Associated with the control system (5.1) is a codistribution ω of (n-r) differential one forms,

$$\omega = \Psi^T(x)dx - \Psi^T(x)A(x)dt \qquad 5.2$$

where Ψ is orthogonal to B for all $x \epsilon N(x_o)$. The dual space of vector fields is spanned by the distribution θ

$$D_t + A^T(x)D_x$$

$$B^T(x)D_x$$

We can now apply Hermann's method to the distribution θ to derive algebraic criteria for the controllability of the nonlinear system (5.1). However, we must point out the fallacy in Hermann's approach, which we have delayed until now because it does not present any problems for linear systems. In the case of nonlinear systems, demonstrating that a given distribution is not involutive does not imply, since there are no obviously inaccessible points, that all points are accessible. This is because we deal with coordinate systems that are endowed with one special coordinate, namely time, which has to evolve, or if parametrized it must be monotone increasing. This restriction partially invalidates Chow's theorem; however, demonstrating that there are no obviously inaccessible points is important to controllability.

For a certain class of nonlinear control systems Chow's theorem does have application. Let us associate with ω a reduced pfaffian system

$$\omega_R = \Psi^T(x)dx \qquad 5.4$$

Since time is no longer present, the full generality of Chow's theorem can be applied to determine the attainability of points. The dual space of vector fields are now spanned by

$$\theta_R = \mathcal{J}B^T(x)D_x$$

If θ_R is not an involutive distribution then all points in some neighborhood of a given point can be attained instantaneously. This is where the geometric interpretation of the Lie bracket gives insight into the validity of this result.

Being able to achieve desired states instantaneously is now defined as being totally controllable [8] and had its genesis in LaSalle's [9] concept of a "proper" control system. This concept requires the control system to be completely controllable on every interval of time.

We can now summarize this result in the following theorem.

5.1 Theorem - The nonlinear system (5.1) is totally controllable in some region $N(x) \subset M$ if the control actuator vectors $B(x)$ do not generate an involutive system of vector fields over $N(x)$.

This condition is sufficient but not necessary, since there are obvious cases where the control actuator vectors define an involutive system of vector fields but the system is totally controllable.

Henceforth, we shall confine our attention to the case where the control actuator vectors do generate a involutive distribution. Furthermore, we shall assume that the dimension of the distribution is equal to the number of control actuator vectors. If this is not the case ($\dim \mathcal{B} > r$) then we shall augment the control actuator vectors with the independent vector fields derived by the Lie bracket so that the assumption is satisfied. This assumption is introduced purely as a matter of convenience, since by virtue of the geometric interpretation of the Lie bracket, the original control system will be fully equivalent to the augmented control system. For the "involutive" control systems we have the following decomposition theorem.

5.2 Theorem - If the control actuator vectors define an involutive distribution of dimension r, then there exists a coordinate transformation which decomposes the control system (5.1) into

$$\dot{y} = F(y,z)$$

$$\dot{z} = G(y,z) + H(z)u. \qquad 5.5$$

where y is an n-r vector, z an r vector, F an n-r vector of smooth functions, G an r vector of smooth functions and H is a nonsingular matrix of smooth functions.

Proof. Let θ represent the distribution of vector fields B_j where

$$B_j = \sum B_{ij}(x)D_{x_i} = B_j^T(x)D_x$$

Since θ is an involutive distribution then $[B_i, B_j] \epsilon \theta$ for all vector fields B_i, B_j. Also by assumption, for each $m \epsilon M$, $B_i(m)$ spans an r dimensional linear subspace of the tangent space M_m. The proof, in part, follows from the representation of involutive distributions as given by a theorem of Frobenius [10] which says that for each point $m \epsilon M$ we can find a coordinate system $(z_1, z_2, \ldots, z_n)$ such that the vector fields $(D_{z_1}, \ldots, D_{z_r})$ generates θ on M. We have to modify this result somewhat since it assumes that we can always choose a basis X_i for θ such that $[X_i, X_j] = 0$. In general, this is not true for the basis B_i.

Let $(y_1, \ldots, y_{n-r})$ and $(z_1, \ldots, z_r)$, denoted by (y,z), represent a partitioned coordinate system of M'. If ϕ is a mapping of M' into M as given by $x = \phi(y,z)$, then the tangent vectors transform by

$$D_z = (D_z^T \phi) D_x \qquad 5.6$$

By virtue of the transformation ϕ we have

$$H^T(z)D_z = B^T(x)D_x \qquad 5.7$$

where H(z) is nonsingular and can be chosen to be I if $[B_i, B_j] = 0$ for all B_i, B_j. This result can be verified by deriving the integrability conditions for the nonhomogeneous partial differential equations obtained from (5.7) by using (5.6).

$$D_z^T \phi = B^T(\phi)\left\{H^T(z)\right\}^{-1} \qquad 5.8$$

The integrability conditions are, in fact

$$[B_i, B_j] \epsilon \Theta$$

Differentiating the transformation ϕ with respect to time yields,

$$D_y^T \phi \dot{y} + D_z^T \phi \dot{z} = A(\phi) + B(\phi)u$$

so that from (5.8) this reduces to

$$D_y^T \phi \dot{y} + D_z^T \phi(\dot{z} - H(z)u) = A(\phi)$$

Since the Jacobian of the transformation ϕ is assumed to be different from zero in some neighborhood of M then the decomposition (5.5) follows.

The significance of the decomposition theorem is that it isolates out those transformed states that are obviously totally controllable. In fact, there is no loss of generality if we choose

$$u = H^{-1}(z)\left[v - G(y,z)\right]$$

so that the control system now assumes the form

$$\dot{y} = F(y,z)$$

$$\dot{z} = v$$

and the controllability problem viewed as determining the attainability of the states y(t) for given inputs z(t).

It should be noted that the pfaffian system (5.2) transforms to

$$dy - F(y,z)dt = 0$$

so that the reduced pfaffian system $\Psi^T(x)dx$ is completely integrable and defines an integral manifold by y(x) = constant.

We shall now develop some equivalences between "involutive" control systems and "totally singular" control system. In many optimal control problems a singular problem can arise that is characterized by the fact that the maximum principle fails to yield any information regarding the choice of the optimal controls. To distinguish this condition, the controls and trajectories are termed singular. The controllability problem is intimately connected with the time optimal problem. This connection will be shown for the nonlinear system (5.1). The time optimal problem consists of finding the optimal controls u(t) that transfer the state from some specified initial condition x_o to some specified final condition x_f in minimum time.

Following the Kalman-Caratheodory approach [11], we define the system Hamiltonian as,

$$H(t, x, p, u) = 1 + p^T A(x) + p^T B(x)u,$$

where p is an n vector describing the costate. For each x and p, the Hamiltonian is minimized with respect to the controls over the set of admissible controls Ω. If, for example, the set of admisible controls are constrained to an r-dimensional hypercube described by

$$\Omega = \left\{ u : \left| u_i \right| \leq 1,\ i\ 1,2, \ldots ,r, \right\}$$

the maximum principle yields

$$u = -\mathrm{sgn}(B^T(x)p) \qquad 5.9$$

for the choice of the optimal controls.

Any solution of the nonlinear system (5.1) with controls (5.9) that pass through the desired terminal points x_o and x_f, with the costate satisfying

$$\dot{p} = -D_x A^T(x)p - u^T D_x B^T(x)p \qquad 5.10$$

would be regarded as a minimizing trajectory. However, singular controls $u_s(t)$ that are not necessarily bang-bang can exist so that the corresponding solutions to (5.1) and (5.10) $x = \phi(t)$ and $p = \psi(t)$ make some or all of the r components of the vector $B^T(\phi)\psi$ vanish over some measurable time interval. It is immediately obvious that this situation would invalidate the maximum principle for the singular components of the control vector. The singular components of the control vector, if they exist, are obtained by repeatedly differentiating the appropriate components of the singular condition $B^T(\phi)\psi = 0$ with respect to time. The problem is said to be totally singular if all components of $B^T(\phi)\psi$ vanish. The equivalence between the "involutive" control system and the "totally singular" control system is summarized in the following Lemma which is a generalization of the results first proven for (n-1) components of controls [12].

5.3 Lemma - A necessary condition that an optimal totally singular vector control exists is that the vector fields generated by the control actuator vector B(x) be involutive.

Proof: The proof is trivial and simply follows from the fact that if B(x) does not define an involutive distribution then the reduced pfaffian system is not integrable and Theorem 5.1 applies.

The geometric equivalence between "involutive" and "totally controllable" control systems can be established as follows. For ease of treatment we shall consider the decomposed control system

$$\dot{y} = G(y,z)$$
$$\dot{z} = v \qquad 5.11$$

It is evident from the pfaffian system dy - F(y,z)dt, that the integral manifolds y associated with the reduced pfaffian system stratify M, and motion on these integrals y(x) = constant, can be achieved in zero time by appropriate choice of z. The minimum time to traverse from one state to some other state will depend strictly on the vector y. Let us denote this cost by V(y). The rate of change V with respect to t time is given by

$$\frac{dV}{dt} = D_y^T V(y) G(y,z) \tag{5.12}$$

Obviously, we have to determine those points z(y) on the stratification which extremizes the cost derivative. These points are determined by

$$D_z G^T(y,z) D_y V = 0 \tag{5.13}$$

so that the system (5.11) can be integrated to yield $y = \phi(t)$. Along each integral curve ϕ, we require the cost $V(\phi)$ to be time like so that from (5.12) we obtain

$$D_y^T V(\phi) G(\phi, z(\phi)) = 1$$

Differentiating, this identity with respect to time yields

$$G^T \left\{ (D_y G^T) D_y V + D_y^T (D_y V) G \right\} = 0 \tag{5.14}$$

If we now define the costate p by

$$p = D_y V$$

then (5.13) becomes $D_z G^T p = 0$ which is the condition for the control vector to be totally singular; furthermore, (5.14) is satisfied by the costate equations since it reduces to

$$G^T \left\{ (D_y G^T) p + \dot{p} \right\} = 0$$

To obtain necessary and sufficient conditions for the controllability of nonlinear systems, we need to combine the differential geometric techniques of Section III to rule out obviously inaccessible points, with a concept of local neighborhoods of attainability. One result pertinent to local neighborhoods of controllability is the following theorem of Markus [13] .

5.4 Theorem - Consider the control process in E^n

$$\dot{x} = f(x, u),$$

with $f \in C^1$ in E^{n+m} and restraint $u(t) \subset \Omega \subset E^m$.

Assume:

(1) $f(0, 0) = 0$,

(ii) Ω contains m+1 vectors $u_1, u_2, \ldots, u_{m+1}$, which span an m-simplex with u = 0 in its interior, and also contains $\epsilon u_1, \epsilon u_2, \ldots, \epsilon u_{m+1}$ for certain arbitrary small $\epsilon > 0$,

(iii) rank $\left[B, AB, \ldots, A^{n-1}B\right] = n$,

where $A = D_x f(o) \quad B = D_u f(o)$

Then there exists some open neighborhood of the origin in E^n, in which all states can be steered to the origin in finite time by admissible controllers. Therefore, Theorem (5.4) can be regarded as defining local controllability in some neighborhood of the trajectory x = 0. Since the proof of this theorem is in part based on the linear approximating system

$$\dot{x} = Ax + Bu,$$

then this suggests applying the linear system controllability results to linearized versions of the control system about some specified trajectory.

It would appear therefore that we can generate an infinity of algebraic criteria for controllability, each one depending on the particular choice of the trajectory defining the linear approximating system. This raises the question of what trajectory should be chosen. Does there exist, for example, a trajectory defining a linear approximating system whose algebraic controllability criteria immediately determines the complete controllability of the original control system. This question leads to a paradox that was first observed by Hermes [14] and is contained in the following theorem which we cite without proof.

5.5 Theorem - The linear approximating system describing motions in some neighborhood of the totally singular trajectories associated with the time optimal problem is not completely controllable.

An obvious conclusion of this theorem is the following.

5.6 Corollary - The distribution of linearized vector fields about the totally singular trajectories associated with the time optimal problem is involutive.

A remarkable facet of Theorem (5.5) is that the result is independent of the optimality of the totally singular vector control. If the totally singular arc is truly a minimizing arc, one would expect it to persist as a natural boundary to the set of reachable points, since by definition it would be better than any bang-bang control irrespective of the magnitude of the control bounds. On the other hand, if the control system is completely controllable, one would expect all motions to completely fill the n-dimensional manifold by virtue of the system being linear in the control vector.

Therefore, one could conjecture that if the totally singular arc for the time optimal problem is not a minimizing arc, then the control system is completely controllable.

Unfortunately, the conjecture is insufficient as it stands to resolve the controllability of nonlinear systems. That the conjecture is false is reflected in the following example which possesses a non-optimal singular arc for the time optimal problem. Consider the system [14]

$$\dot{x}_1 = u$$

$$\dot{x}_2 = 1 + x_2 x_1^2 u$$

then it is trivial to verify by the Green's Theorem approach that that non-optimal singular arc for the time optimal problem is described by $x_1(t) = 0$. Contrary to the proposed conjecture, the lack of controllability for the above system can be demonstrated by the following transformation which is nonsingular for all finite regions of Euclidean two space. With the transformation

$$y_1 = x_1$$

$$y_2 = x_2 e^{\frac{-x_1^3}{3}}$$

the control system (1) becomes

$$\dot{y}_1 = u$$

$$\dot{y}_2 = e^{\frac{-y_1^3}{3}}$$

The lack of global controllability is evident since $\dot{y}_2 = e^{-y_1^3/3} > 0$ irrespective of the choice of the control.

Let us return to Theorem 4.6 and give a geometric proof thereof, since it has obvious applications to nonlinear system. We recall that the theorem stated, "The system $\dot{y} = H(t)u$ is completely controllable at t_0 if and only if there exist n times $t_1, \ldots, t_n \geq t_0$ such that rank $[H(t_1), H(_2), \ldots, H(t_n)]$ is n".

The conclusion of the theorem follows from the integrability of the reduced pfaffian system and provides a technique for generating necessary conditions for the controllability of nonlinear systems. For the linear system $\dot{y} = H(t)u$ we can associate the hyperplanes $\Psi^t(t)y$ where $\Psi(t)$ is an n x n-r matrix of vectors orthogonal to H(t). Since the impulsive motions are confined to these hyperplanes the controllability of the linear system ca. be ¯tned in terms of the hyperplanes admitting a coordinate structure in M. That is to say, the hyperplanes span M by suitable choice of the essential constants (time) of the hyperplanes by $z = \Phi^t(t\ )y$, then the hyperplanes will span M if the normal vectors $\Phi^t(t\ )$ form a basis, or equivalently if the tangent vectors $H(t\ )$ form a basis. This requires that rank $H(t_1), H(t_2), \ldots, H(t_n)$ is n. Alternatively this condition can be derived by considering a sequence of n delta functions having measures ξ at the points $t_1, t_2, \ldots, t_n$. The rank condition defines a one to one mapping between the state y and the measures ξ.

This theorem can be applied to the nonlinear system

$$\dot{x} = A(x) + b(x)u$$

by the following transformation. Let $\phi(t,y)$ denote the solution to $\dot{x} = A(x)$, then by a variation of parameters we obtain

$$\dot{y} = \left[D_y^T\phi(t,y)\right]^{-1}B(\phi(t,y))u = H(t,y)u.$$

5.7 Theorem - A necessary condition for the system $\dot{y} = H(t,y)u$ to be completely controllable at t_o in some neighborhood N of y_o is that there exist n times $t_1, \ldots, t_n \geq t_o$ such that rank $[H(t_1,y), H(t_2,y), \ldots, H(t_n,y)]$ is n for almost all $y \in N$, and the orientation of the transformation $[H(t_1,y), \ldots, H(t_n,y)]$ is preserved.

Proof. Let $C(t,y)$ define (n-r) vectors orthogonal to $H(t,y)$. $H(t,y)$ is an involutive system of order r, then it follows that the pfaffian system $C^T(t,y)dy$ is integrable for fixed t. If we now assume that the rank $[H(t_1,y), \ldots, H(t_n,y)]$ is less than n for all sets $\{t_1, \ldots, t_n\}$ and all $y \in N$ then this implies that there exists a nontrivial vector $C(y)$ such that $C^T(y)H(t,y) = 0$. Since the pfaffian system $C^T(t,y)dy$ is integrable for fixed t it follows that $C^T(y)dy$ is integrable so that an integral manifold exists. Hence, the control system is not completely controllable. The contrapositive of this yields the result of the theorem.

Since we admit the vanishing of rank $[H(t_1,y), \ldots, H(t_n,y)]$ on sets of measure zero, then we have to make sure that the transformation does not fold up on itself, i.e., the orientation of the transformation must be preserved.

There is a unique relation between the singular arc and the points of measure zero where $[H(t_1,y), \ldots, H(t_n,y)]$ vanishes and is summarized in the following theorem.

5.7 Theorem - Suppose there exists a smooth curve $y(t)$ such that rank $[H(t_1,y(t)), H(t_2,y(t)), \ldots, H(t_n,y(t))]$ n for all sets $\{t_1, t_2, \ldots, t_n\}$ then $y(t)$ is a singular arc for the control system $\dot{y} = H(t,y)u$.

Proof.

Since rank $[H(t_1,y(t)), \ldots, H(t_n,y(t))]$ is less than n for all sets

$$\{t_1, t_2, \ldots t_n\}$$

then there exists a nontrivial vector $\Psi(t)$ such that

$$\Psi^t(t)H(\tau,y(t)) \overset{t,\tau}{=} 0 \qquad 5.15$$

Since this is an identity in t and τ then differentiating with respect to t yields

$$\dot{\Psi}^t(t)H(\tau,y(t))+\Psi^t(t)H_y(\tau,y(t))H(t,y(t))u(t) = 0$$

The r columns of H(t,y), for fixed t, define a complete set of tangent vectors of order r. Therefore, since the Lie bracket does not generate new tangent vectors, the order of differentiation with respect to y in equation 2. can be changed, on substituting $\tau = t$, to give

$$[\dot{\Psi}^t(t) + \Psi^t(t)H_y(t,y(t))u(t)]\, H(t,y(t)) = 0$$

The result is now obvious since $\Psi(t)$ can be identified with the costate, and equation (5.15) is the singular condition.

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