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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

An analysis of the operation of a mass measuring system, which is based on the oscillating spring-mass principle, aboard a spacecraft is made. This analysis considers the dynamic effects of the oscillating device on an uncontrolled spacecraft by developing the equations of motion for the system. An approximate solution of these equations is then deduced, and a simple analytic expression for the maximum attitude error is found.

Numerical integration of the equations of motion was performed on a digital computer for some typical measurement operations. Comparison between the approximate and exact solutions for these cases was very close. Results showed small (less than 0.02°) attitude errors arising from the operation of this device.

INTRODUCTION

With the advent of long-term spaceflight (from 2 weeks to a year or more) it becomes necessary to monitor various physiological indicators of an astronaut's well-being. One important factor is his weight, which can serve as a monitor for other functions (such as cardiovascular effort or muscle condition). It then is important to find some means of determining astronaut weight in the zero-gravity environment of manned spacecraft. Studies have shown that an oscillating spring-mass system is perhaps an optimum method of determining the unknown mass of astronauts and other spacecraft apparatus (ref. 1). Life support expendables (e.g., food) and waste products (CO_2 canisters and urine samples) could be weighed by using this technique. The required spring-mass system could be lightweight and easily operable by a single crew member.

A prototype mass measuring system (MMS), employing the oscillating spring-mass principle, has been designed and fabricated under NASA contract. (See ref. 2 and fig. 1.) This report discusses the effects of operating a device similar to the prototype aboard an uncontrolled spacecraft with results presented for some typical operations.

SYMBOLS

C_1, C_2 constants of integration

F_X, F_Y, F_Z force along X-, Y-, and Z-axis, respectively

H_{cm} angular momentum of spacecraft about its center of mass

H_O angular momentum of spacecraft in geometric coordinates

H_S angular momentum of spacecraft mass center

$[I]$ inertia tensor

I_X, I_Y, I_Z spacecraft moments of inertia

I_{XY}, I_{XZ}, I_{YZ} spacecraft products of inertia

K spring constant

M_X, M_Y, M_Z moment about X-, Y-, and Z-axis, respectively

m single moving mass

m_j arbitrary mass

m_s spacecraft mass

Q mass factor

R_O radius of geometric origin with respect to inertial axes

r radius to mass measuring system

r_{mj}, R_{mj} radii of arbitrary mass

r_{ms}, R_{ms} radii of spacecraft mass center

T_X, T_Y, T_Z torque about X-, Y-, and Z-axis, respectively

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t	time
X_b, Y_b, Z_b	body-fixed spacecraft axes
X_I, Y_I, Z_I	inertial axes
X_i, Y_i, Z_i	moving axes parallel to inertial set
x, y, z	rectangular coordinates
$\alpha_x, \alpha_y, \alpha_z$	attitude angles of spacecraft obtained from approximate solution
$\overline{\Delta r}$	vector representing oscillation of mass measuring system
$\Delta r_x, \Delta r_y, \Delta r_z$	amplitude of mass measuring system oscillation in x-, y-, and z-direction, respectively
φ, θ, ψ	modified Euler angles
ω_n	natural frequency of spring-mass system
$\omega_x, \omega_y, \omega_z$	angular velocities of spacecraft about body axis

Subscripts:

c	quantity resulting from cocking of mass measuring system
o	referenced to geometric axes
s	referenced to spacecraft mass center

Dots over symbols indicate derivatives with respect to time.

A bar over a symbol indicates a vector quantity.

ANALYSIS

Measurement Technique

The principle of the oscillating spring-mass system is applied by placing the unknown subject mass in a carriage that is constrained (mounted on a rail, for example)

to move in one degree of freedom. The carriage is connected to the spacecraft structure by two springs in parallel (one at each end of the carriage). The carriage, with its subject mass, is displaced from its equilibrium position (cocked) and then released to oscillate at the natural frequency of the carriage-spacecraft system. By measuring the frequency (or the period) of oscillation and by knowing the spring constant of the springs the unknown mass can be determined. Consider the following derivation.

From figure 2, the following expressions are evident:

$$m_1 \ddot{x}_1 + Kx = 0 \tag{1}$$

$$m_2 \ddot{x}_2 - Kx = 0 \tag{2}$$

where spring displacement is

$$x = x_1 - x_2 \tag{3}$$

Multiplying equation (1) by m_2 and equation (2) by m_1 and subtracting equation (2) from equation (1) give

$$m_1 m_2 \ddot{x} + K(m_2 + m_1)x = 0 \tag{4}$$

$$\ddot{x} + K \frac{m_2 + m_1}{m_1 m_2} x = 0 \tag{5}$$

Equation (5) is the equation of motion for harmonic vibrations having a natural frequency ω_n of

$$\omega_n = \sqrt{\frac{K(m_2 + m_1)}{m_1 m_2}} \tag{6}$$

For the present system (see appendix)

$$m_1 = m \tag{7}$$

$$m_2 = m_s - m \tag{8}$$

and

$$\frac{m_1 m_2}{m_2 + m_1} = \frac{m(m_s - m)}{m_s} = Q \tag{9}$$

Equation (6) can thus be written

$$\omega_n = \sqrt{\frac{K}{Q}} \tag{10}$$

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The symbol ω_n is the frequency that would be measured if there were no non-linearities in the system. However, friction will always be present to a certain extent, and spacecraft body rates will probably exist (introducing additional degrees of freedom). In the following analyses, these rates are assumed small, so that a one-degree-of-freedom system is maintained. In addition friction is assumed negligible. In actual space-flight use, empirical calibration would be used to reduce measurement data.

Applied Torques

The disturbance moments $\overline{M}_0$ (see appendix) resulting from operation of the mass measuring system are caused by the forces exerted by the springs. These spring forces are

$$
\left.
\begin{aligned}
F_X &= K(x - x_0) \\
F_Y &= K(y - y_0) \\
F_Z &= K(z - z_0)
\end{aligned}
\right\} \tag{11}
$$

where x_0, y_0, z_0 are body coordinates for the equilibrium position of the device. The x, y, and z coordinates for the moving mass are given by (assuming that velocity of the MMS following release is in a positive direction)

$$
\left.
\begin{aligned}
x &= x_0 - \Delta r_x \cos \omega_n t \\
y &= y_0 - \Delta r_y \cos \omega_n t \\
z &= z_0 - \Delta r_z \cos \omega_n t
\end{aligned}
\right\} \tag{12}
$$

The components of $\overline{M}_0$ due to the spring forces are written

$$
\left.
\begin{aligned}
M_X &= yF_Z - zF_Y \\
M_Y &= zF_X - xF_Z \\
M_Z &= xF_Y - yF_X
\end{aligned}
\right\} \tag{13}
$$

By using equations (11) and (12), equations (13) may be rewritten as

$$
\left.
\begin{aligned}
M_X &= K \cos \omega_n t \left(z_0 \Delta r_y - 2\Delta r_y \Delta r_z \cos \omega_n t - y_0 \Delta r_z \right) \\
M_Y &= K \cos \omega_n t \left(x_0 \Delta r_z - 2\Delta r_z \Delta r_x \cos \omega_n t - z_0 \Delta r_x \right) \\
M_Z &= K \cos \omega_n t \left(y_0 \Delta r_x - 2\Delta r_x \Delta r_y \cos \omega_n t - x_0 \Delta r_y \right)
\end{aligned}
\right\} \tag{14}
$$

Substitution of equations (14) into the set (A39) yields the spacecraft equations of motion for the present problem. Other disturbances such as gravity gradients and aerodynamic

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torques have been neglected since they cause negligible rate and attitude changes in the short time (10 to 15 sec) needed for a measurement operation.

Approximate Solution

A linearized solution to the equations of motion can be obtained, which will prove useful in many applications not requiring the exact solution. If it is assumed that products of inertia are negligible (=0), then the equations of motion are Euler's equations for principal axes, that is

$$[I]\dot{\overline{\omega}} + \overline{\omega} \times [I]\overline{\omega} = \overline{T} \tag{15}$$

or, expanding and arranging terms

$$\left.\begin{aligned}
\dot{\omega}_x &= \frac{T_X}{I_X} + \frac{I_Y - I_Z}{I_X} \omega_y \omega_z \\[2mm]
\dot{\omega}_y &= \frac{T_Y}{I_Y} + \frac{I_Z - I_X}{I_Y} \omega_x \omega_z \\[2mm]
\dot{\omega}_z &= \frac{T_Z}{I_Z} + \frac{I_X - I_Y}{I_Z} \omega_x \omega_y
\end{aligned}\right\} \tag{16}$$

If small angles and angular rates are assumed ($\alpha < 0.1$ rad, $\omega < 0.1$ rad/sec) then the terms containing products of angular velocities will be negligible in comparison with the T/I terms and the equations of motion can be written as

$$[I]\dot{\overline{\omega}} = \overline{T} \tag{17}$$

Accordingly, rewriting equation (17) for the mass measurement application gives

$$[I]\dot{\overline{\omega}} + \overline{r} \times Q\ddot{\overline{r}} = \overline{M} \tag{18}$$

But

$$\overline{M} = \overline{r} \times \overline{F} \tag{19}$$

and

$$\overline{F} = K(\overline{r} - \overline{r}_0) \tag{20}$$

so that

$$\dot{\overline{\omega}} = [I]^{-1}\overline{r} \times \left[K(\overline{r} - \overline{r}_0) - Q\ddot{\overline{r}}\right] \tag{21}$$

Now

$$\overline{r} = \overline{r}_0 + (\cos \omega_n t)\overline{\Delta r} \tag{22}$$

$$\ddot{\overline{r}} = -(\omega_n^2 \cos \omega_n t)\overline{\Delta r} \tag{23}$$

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Substituting equations (22) and (23) into equation (21) and collecting terms gives

$$\dot{\bar{\omega}} = [I]^{-1} \bar{r}_o \times \left[\left(K + Q\omega_n{}^2 \right) \cos \omega_n t \right] \overline{\Delta r} \qquad (24)$$

Integrating twice to find rate $\bar{\omega}$ and attitude $\bar{\alpha}$ results in

$$\bar{\omega} = [I]^{-1} \bar{r}_o \times \left(\frac{K + Q\omega_n{}^2}{\omega_n} \sin \omega_n t \right) \overline{\Delta r} + \overline{C}_1 \qquad (25)$$

and (since angular rotations can be treated as vectors for small angles)

$$\bar{\alpha} = -[I]^{-1} \bar{r}_o \times \left(\frac{K + Q\omega_n{}^2}{\omega_n{}^2} \cos \omega_n t \right) \overline{\Delta r} + \overline{C}_1 t + \overline{C}_2 \qquad (26)$$

At $t = 0$ for any measurement operation $\omega = 0$ and $\alpha = \alpha_o$ (static angle caused by cocking the MMS), and the constants of integration $\overline{C}_1$ and $\overline{C}_2$ can be evaluated as follows:

$$\overline{C}_1 = 0$$

$$\overline{C}_2 = [I]^{-1} \bar{r}_o \times \frac{K + Q\omega_n{}^2}{\omega_n{}^2} \overline{\Delta r} + \overline{\alpha}_o$$

Equations (25) and (26) can now be written as

$$\bar{\omega} = [I]^{-1} \bar{r}_o \times \left(\frac{K + Q\omega_n{}^2}{\omega_n} \sin \omega_n t \right) \overline{\Delta r} \qquad (27)$$

$$\bar{\alpha} = -[I]^{-1} \bar{r}_o \times \left[\frac{K + Q\omega_n{}^2}{\omega_n{}^2} \left(\cos \omega_n t - 1 \right) \right] \overline{\Delta r} + \overline{\alpha}_o \qquad (28)$$

In cocking the MMS, the cocking force is equal to or greater than the spring force, so that

$$\dot{\bar{\omega}}_c = [I]^{-1} \bar{r} \times \left(-Q\ddot{\bar{r}}_c \right) \qquad (29)$$

where

$$\bar{r} = \bar{r}_o + \bar{r}_{\ddot{\bar{r}}}$$

$$\bar{\alpha}_o = -Q[I]^{-1} \int\int \left(\bar{r}_o \times \ddot{\bar{r}}_c + \bar{r}_{\ddot{\bar{r}}} \times \ddot{\bar{r}}_c \right) dt \qquad (30)$$

The cocking acceleration vector $\ddot{\bar{r}}_c$ is collinear with the incremental cocking radius $r_{\ddot{\bar{r}}}$ and hence

$$\overset{\cdots}{r} \times \overset{\cdots}{r}_c = 0$$
$$_{\overset{\cdots}{r}}$$

$$\alpha_o = -Q[I]^{-1} \iint \overset{\cdots}{r}_o \times \overset{\cdots}{r}_c \, dt = -Q[I]^{-1} \overset{\cdots}{r}_o \times \overline{\Delta r} \tag{31}$$

Equation (28) can now be written to include equation (31)

$$\alpha = -[I]^{-1} \overset{\cdots}{r}_o \times \left[\frac{K + Q\omega_n^2}{\omega_n^2} (\cos \omega_n t - 1) + Q \right] \overline{\Delta r} \tag{32}$$

Orientation of MMS to Yield Minimum Disturbances

Since small spacecraft attitude angles have been assumed, α can be treated as a vector, and solutions for ω and α can be expressed in terms of the components of $[I]$, $\overset{\cdots}{r}_o$, and $\overline{\Delta r}$, as

$$\omega = -\frac{K + Q\omega_n^2}{\omega_n} \left[\frac{1}{I_X^2} \left(r_{o,y} \Delta r_z - r_{o,z} \Delta r_y \right)^2 + \frac{1}{I_Y^2} \left(r_{o,z} \Delta r_x - r_{o,x} \Delta r_z \right)^2 \right.$$
$$\left. + \frac{1}{I_Z^2} \left(r_{o,x} \Delta r_y - r_{o,y} \Delta r_x \right)^2 \right]^{1/2} \sin \omega_n t \tag{33}$$

$$\alpha = \left[\frac{K + Q\omega_n^2}{\omega_n^2} (\cos \omega_n t - 1) + Q \right] \left[\frac{1}{I_X^2} \left(r_{o,y} \Delta r_z - r_{o,z} \Delta r_y \right)^2 \right.$$
$$\left. + \frac{1}{I_Y^2} \left(r_{o,z} \Delta r_x - r_{o,x} \Delta r_z \right)^2 + \frac{1}{I_Z^2} \left(r_{o,x} \Delta r_y - r_{o,y} \Delta r_x \right)^2 \right]^{1/2} \tag{34}$$

For any time t the magnitude of α and ω for a measurement operation is dependent upon the quantity

$$\frac{1}{I_X^2} \left(r_{o,y} \Delta r_z - r_{o,z} \Delta r_y \right)^2 + \frac{1}{I_Y^2} \left(r_{o,z} \Delta r_x - r_{o,x} \Delta r_z \right)^2 + \frac{1}{I_Z^2} \left(r_{o,x} \Delta r_y - r_{o,y} \Delta r_x \right)^2$$

in equations (33) and (34). This quantity is represented by N in subsequent equations. This quantity is a function of the spacecraft moments of inertia and the location and orientation of the MMS within the spacecraft. For a given spacecraft the moments of inertia are constant, and rate and attitude errors are functions only of position of the MMS and the orientation of its line of action.

If operation of the mass measurement device is assumed in a plane perpendicular to one of the spacecraft axes, then one component of $\overline{\Delta r}$ is zero. Accordingly, let the plane of operation be perpendicular to the spacecraft X-axis, so that $\overline{\Delta r}$ can be expressed as

$$\overline{\Delta r} = \left\{ \begin{array}{c} 0 \\ \Delta r_y \\ \sqrt{\Delta r^2 - \Delta r_y^2} \end{array} \right\} \tag{35}$$

It can be seen from equations (33) and (34) that ω and α are minimized whenever the quantity N is a minimum. With the use of equations (35),

$$N = \frac{1}{I_X^2}\left[r_{0,y}^2\left(\Delta r^2 - \Delta r_y^2\right) - 2r_{0,y}r_{0,z}\Delta r_y\left(\Delta r^2 - \Delta r_y^2\right)^{1/2} + r_{0,z}^2\Delta r_y^2 \right]$$

$$+ \frac{1}{I_Y^2}r_{0,x}^2\left(\Delta r^2 - \Delta r_y^2\right) + \frac{1}{I_Z^2}r_{0,x}^2\Delta r_y^2 \tag{36}$$

If a specific location within the spacecraft is considered, then $r_{0,x}$, $r_{0,y}$, and $r_{0,z}$ are constant, and the minimum values of ω and α are influenced by the magnitude of Δr_y (since $\overline{\Delta r}$ has a constant magnitude) or by the orientation of the mass measurement device with respect to the y and z axes (since $\Delta r_x = 0$). This orientation can be specified by the angle η whose cosine is $\Delta r_y / \Delta r$ or

$$\eta = \cos^{-1} \frac{\Delta r_y}{\Delta r} \tag{37}$$

To find the value of Δr_y (and, hence, the value of η) which will minimize N, take

$$\frac{\partial N}{\partial \Delta r_y} = 0 \tag{38}$$

and solve for $\Delta r_y / \Delta r$. To obtain a closed-form solution, the binomial expansion of $\left(\Delta r^2 - \Delta r_y^2\right)^{1/2}$ must be used; this results in a (2n)th degree algebraic equation (where n is the number of terms used in the expansion). A digital computer was used to solve this equation for $\cos \eta$ as a parametric function of $r_{0,x}$, $r_{0,y}$, and $r_{0,z}$. Figure 3 presents $\cos \eta$ as a function of MMS location within the spacecraft for a typical inertia distribution $g - h = 0.1$ where

$$g - h = \left(\frac{I_X}{I_Y}\right)^2 - \left(\frac{I_X}{I_Z}\right)^2 \tag{39}$$

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Digital Computer Simulation and Results for Typical Operations

A digital computer program has been written that includes equations (A2) and (A39) along with equations (11), (12), and (13). This program employs a fourth-order Rungë-Kutta integration scheme to integrate equations (A39) and (A2) to yield spacecraft angular velocities and Euler angles, respectively. All initial rates and Euler angles, as well as inertia properties and MMS properties can be read into the program as initial conditions and changed at specified times throughout the program. Program output consists of the Euler angles φ, θ, and ψ, as functions of time, and ω_X, ω_Y, and ω_Z, also as functions of time. In simulating the MMS operations to be described, all equations were solved exactly. Initial spacecraft rates and Euler angles were set equal to zero (prior to cocking of the device). First examine the response (rigid body) of a spacecraft during a typical mass measuring system operation. An Apollo Applications Program spacecraft, consisting of an Apollo command and service module docked with an S-IVB Orbital Workship/Multiple Docking Adapter combination, is considered. This configuration, with the orientation of its body axes, is shown in figure 4. The MMS is assumed operating in the Orbital Workshop. Moments of inertia of the spacecraft about its center of mass are

$$I_{X,o} = 115\ 000 \text{ slug-ft}^2 = 155\ 917 \text{ kg-m}^2$$

$$I_{Y,o} = 1\ 970\ 000 \text{ slug-ft}^2 = 2\ 670\ 926 \text{ kg-m}^2$$

$$I_{Z,o} = 1\ 930\ 000 \text{ slug-ft}^2 = 2\ 616\ 694 \text{ kg-m}^2$$

All products of inertia are taken to be zero. The coordinates of the MMS equilibrium position are

$$r_{o,x} = 15.0 \text{ ft} = 4.5720 \text{ m}$$

$$r_{o,y} = 2.0 \text{ ft} = 0.6096 \text{ m}$$

$$r_{o,z} = 3.0 \text{ ft} = 0.9144 \text{ m}$$

and orientation of the device is such that (with $\Delta r = 0.5 \text{ ft} = 0.1524 \text{ m}$)

$$\Delta r_X = 0$$

$$\Delta r_Y = -0.3535 \text{ ft} = -0.1077 \text{ m}$$

$$\Delta r_Z = 0.3535 \text{ ft} = 0.1077 \text{ m}$$

Characteristics of the device are those of the final prototype design (ref. 2). The weighing of a 160-lb (72.544 kg) man was simulated (including cocking and carriage return) with results shown in figure 5. This figure gives the time histories of the Euler angles ϕ, θ, and ψ. Also shown in this figure are the approximate solutions to the

equations of motion for this case — $\alpha_x(\approx \varphi)$, $\alpha_y(\approx \theta)$, and $\alpha_z(\approx \psi)$ — given by the dashed curves on the plots. It can be seen, from comparison of the exact and approximate solutions, that the assumptions made in the derivation of the approximate solution (small angular velocities, etc.) are valid; this solution provides an acceptable means of estimating spacecraft error.

The operation just considered did not take into account the optimum orientation of the mass measuring device within the spacecraft. For the given location (with g - h essentially zero) figure 3 can be used to determine the proper value of the orientation parameter $\cos \eta$ which in this case is 0.554. Thus, since $\Delta r_x = 0$, Δr_y and Δr_z can be found from the following equations:

$$\Delta r_y = \Delta r \cos \eta = 0.227 \text{ ft} = 0.0692 \text{ m}$$

$$\Delta r_z = \Delta r \sqrt{1 - \cos^2 \eta} = 0.416 \text{ ft} = 0.1268 \text{ m}$$

With this orientation, the spacecraft response is given in figure 6 for a similar operation as the previous case. Again, the approximate solution is shown to be in good agreement with the exact solution. It can be seen from comparison of figures 5 and 6 that reorienting the device significantly reduces spacecraft attitude excursions.

CONCLUDING REMARKS

This report summarizes the effects of operating a spring-mass device onboard a spacecraft for the purpose of determining mass. It should be recognized that this analysis considers only a rigid subject mass. From the results of typical man-weighing operations the frequency of oscillation was approximately 25 cycles per minute, which is sufficiently low so that a properly restrained subject will behave very nearly as a rigid body.

Since the subject mass and spacecraft properties were arbitrary throughout the theoretical development, it is evident that the analysis presented herein can be applied to masses of any magnitude in any spacecraft configuration. Because mass measuring requirements aboard a manned spacecraft may range from a fraction of a gram (as in chemical analysis) to several hundred pounds (recoverable satellites, for example) the technique and theory presented in this report may be applied to a wide variety of mass measurement problems.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., October 4, 1968,
127-53-16-05-23.

APPENDIX

DERIVATION OF SPACECRAFT EQUATIONS OF MOTION

The motion of the spacecraft is defined with respect to a set of moving body-fixed axes X_b, Y_b, Z_b and these body axes will then be related to space-fixed axes X_i, Y_i, Z_i by means of a set of modified Euler angles (ref. 3). These modified Euler angles, shown in figure 7, result from three consecutive rotations. The first, about the Z_i-axis, carries the X_i- and Y_i-axes through an angle ψ measured in a horizontal plane. The second rotation, about the new Y_i-axis, carries the X_i- and Z_i-axes through an angle θ, measured in a vertical plane. The final rotation, about the new X_i-axis, takes the Y_i- and Z_i-axes through an angle φ, measured in an inclined plane, to give the X_b-, Y_b-, and Z_b-axes.

The modified Euler angles may be expressed mathematically in terms of the angular rates about the body axes. These rates, as a function of Euler angle derivatives, are

$$\left.\begin{aligned}
\omega_X &= \dot\varphi - \dot\psi \sin\theta \\
\omega_y &= \dot\theta \cos\varphi + \dot\psi \sin\varphi \cos\theta \\
\omega_z &= \dot\psi \cos\theta \cos\varphi - \dot\theta \sin\varphi
\end{aligned}\right\} \tag{A1}$$

Solving for the Euler rotations gives

$$\left.\begin{aligned}
\dot\varphi &= \omega_X + \omega_y \sin\varphi \tan\theta + \omega_z \cos\varphi \tan\theta \\
\dot\theta &= \omega_y \cos\varphi - \omega_z \sin\varphi \\
\dot\psi &= \omega_z \sec\theta \cos\varphi + \omega_y \sec\theta \sin\varphi
\end{aligned}\right\} \tag{A2}$$

Integration of the set (A2) yields the Euler angles for the spacecraft.

Before developing the equations of motion, the coordinate systems must be defined. Figure 8 illustrates the notations to be used. The X_I-, Y_I-, and Z_I-axes are fixed in inertial space. The X_i-, Y_i-, and Z_i-axes have their origin at the spacecraft while remaining parallel to the X_I-, Y_I-, and Z_I-axes. The X_b, Y_b, Z_b system is a set of body-fixed spacecraft axes to which all coordinates within the spacecraft are referenced.

In order to obtain the Euler rates, then, it is first necessary to determine the spacecraft body rates. In the following analysis, these body rates will be determined from momentum considerations (ref. 4).

APPENDIX

The angular momentum of the spacecraft, in its geometric coordinates is defined as

$$\bar{H}_O \equiv \sum \bar{r}_{mj} \times m_j \dot{\bar{R}}_{mj} \qquad (A3)$$

and the momentum of its mass center is

$$\bar{H}_S \equiv \bar{r}_{ms} \times m_s \dot{\bar{R}}_{ms} \qquad (A4)$$

The total angular momentum about the mass center is then

$$\bar{H}_{cm} = \bar{H}_O - \bar{H}_S \qquad (A5)$$

or

$$\bar{H}_{cm} = \sum \bar{r}_{mj} \times m_j \left(\dot{\bar{R}}_O + \dot{\bar{r}}_{mj} + \bar{\omega} \times \bar{r}_{mj} \right) - \bar{r}_{ms} \times m_s \left(\dot{\bar{R}}_O + \dot{\bar{r}}_{ms} + \bar{\omega} \times \bar{r}_{ms} \right) \qquad (A6)$$

Now,

$$\bar{r}_{ms} = \frac{\sum m_j \bar{r}_{mj}}{\sum m_j} = \frac{\sum m_j \bar{r}_{mj}}{m_s} \qquad (A7)$$

Rewriting equation (A6) by using equation (A7) results in

$$\bar{H}_{cm} = \sum \bar{r}_{mj} m_j \times \dot{\bar{R}}_O - \frac{\sum m_j \bar{r}_{mj}}{m_s} \times m_s \dot{\bar{R}}_O + \sum \bar{r}_{mj} \times m_j \dot{\bar{r}}_{mj} - \bar{r}_{ms} \times m_s \dot{\bar{r}}_{ms}$$

$$+ \sum \bar{r}_{mj} \times m_j \left(\bar{\omega} \times \bar{r}_{mj} \right) - \bar{r}_{ms} \times m_s \left(\bar{\omega} \times \bar{r}_{ms} \right) \qquad (A8)$$

or

$$\bar{H}_{cm} = \sum \bar{r}_{mj} \times m_j \dot{\bar{r}}_{mj} - \bar{r}_{ms} \times m_s \dot{\bar{r}}_{ms} + \sum \bar{r}_{mj} \times m_j \left(\bar{\omega} \times \bar{r}_{mj} \right) - \bar{r}_{ms} \times m_s \left(\bar{\omega} \times \bar{r}_{ms} \right) \qquad (A9)$$

For a rigid body, $\dot{\bar{r}}_{mj} = 0$ and $\dot{\bar{r}}_{ms} = 0$, and the angular momentum for the rigid body $\bar{H}_{rb}$ becomes

$$\bar{H}_{rb} = \sum \bar{r}_{mj} \times m_j \left(\bar{\omega} \times \bar{r}_{mj} \right) - \bar{r}_{ms} \times m_s \left(\bar{\omega} \times \bar{r}_{ms} \right) \qquad (A10)$$

or, from elementary vector analysis

$$\overline{H}_{rb} = \sum m_j \left[\left(\overline{r}_{mj} \cdot \overline{r}_{mj} \right) \overline{\omega} - \left(\overline{r}_{mj} \cdot \overline{\omega} \right) \overline{r}_{mj} \right] - m_s \left[\left(\overline{r}_{ms} \cdot \overline{r}_{ms} \right) \overline{\omega} - \left(\overline{r}_{ms} \cdot \overline{\omega} \right) \overline{r}_{ms} \right] \quad \text{(A11)}$$

Expanding equation (A11) and writing in matrix form gives

$$\overline{H}_{rb} = \begin{bmatrix} H_{rb,x} \\ H_{rb,y} \\ H_{rb,z} \end{bmatrix} = \begin{bmatrix} I_X & -I_{XY} & -I_{XZ} \\ -I_{XY} & I_Y & -I_{YZ} \\ -I_{XZ} & -I_{YZ} & I_Z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \text{(A12)}$$

Here, the moments of inertia are about axes passing through the center of mass of the spacecraft (by the parallel axis theorem).

Again consider the origin of the geometric coordinates at the center of mass of the spacecraft without any moving masses, and the moment of inertia about the X centroidal axis becomes

$$I_X = \underbrace{\sum_{j=1}^{n-K} m_j \left(y_j^2 + z_j^2 \right)}_{\substack{\text{Spacecraft} \\ \text{inertia without} \\ \text{moving masses} \\ = \text{Constant } I_{X,o}}} + \underbrace{\sum_{j=K}^{n} m_j \left(y_j^2 + z_j^2 \right)}_{\substack{\text{Inertia of} \\ (n - K) \text{ moving} \\ \text{masses}}} - \underbrace{m_s \left(y_s^2 + z_s^2 \right)}_{\substack{\text{Inertia shift to} \\ \text{mass center}}} \quad \text{(A13)}$$

Now, if it is assumed that there is only one moving mass m located at a distance r from the geometric origin, the moment of inertia about the X-axis (centroidal) can be written

$$I_X = I_{X,o} + m \left(y^2 + z^2 \right) - m_s \left(y_s^2 + z_s^2 \right) \quad \text{(A14)}$$

Since r_s is now

$$r_s = \frac{mr}{m_s} \quad \text{(A15)}$$

$$I_X = I_{X,o} + \left(m - \frac{m^2}{m_s} \right) \left(y^2 + z^2 \right) \quad \text{(A16)}$$

and

$$I_X = I_{X,o} + Q \left(y^2 + z^2 \right) \quad \text{(A17)}$$

where

$$Q = \frac{m(m_s - m)}{m_s} \tag{A18}$$

All inertias can be written in the same manner as follows:

$$\left.\begin{aligned}
I_X &= I_{X,o} + Q\left(y^2 + z^2\right) \\
I_Y &= I_{Y,o} + Q\left(x^2 + z^2\right) \\
I_Z &= I_{Z,o} + Q\left(x^2 + y^2\right) \\
I_{XY} &= I_{XY,o} + Q(xy) \\
I_{XZ} &= I_{XZ,o} + Q(xz) \\
I_{YZ} &= I_{YZ,o} + Q(yz)
\end{aligned}\right\} \tag{A19}$$

The total moment about the geometric origin is

$$\overline{M}_o = \sum \bar{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} \tag{A20}$$

The force equation, through the mass center, is

$$\overline{F}_s = m_s \ddot{\overline{R}}_{ms} \tag{A21}$$

The symbol $\overline{F}_s$ represents the total external force vector acting through the mass center. From equation (A3)

$$\frac{d\overline{H}_o}{dt} = \frac{d}{dt}\left(\sum \bar{r}_{mj} \times m_j \dot{\overline{R}}_{mj}\right) \tag{A22}$$

$$\frac{d\overline{H}_o}{dt} = \sum \frac{d\bar{r}_{mj}}{dt} \times m_j \dot{\overline{R}}_{mj} + \sum \bar{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} \tag{A23}$$

$$\frac{d\overline{H}_o}{dt} = \sum \frac{d\bar{r}_{mj}}{dt} \times m_j \left(\dot{\overline{R}}_O + \frac{d\bar{r}_{mj}}{dt}\right) + \sum \bar{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} \tag{A24}$$

$$\frac{d\overline{H}_o}{dt} = \sum m_j \frac{d\bar{r}_{mj}}{dt} \times \dot{\overline{R}}_O + \sum \bar{r}_{mj} \times m_j \overline{R}_{mj} \tag{A25}$$

$$\frac{d\overline{H}_o}{dt} = m_s \frac{d\bar{r}_{ms}}{dt} \times \dot{\overline{R}}_O + \sum \bar{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} \tag{A26}$$

Similarly, from equation (A4)

$$\frac{d\overline{H}_s}{dt} = m_s \frac{d\overline{r}_{ms}}{dt} \times \dot{\overline{R}}_O + \overline{r}_{ms} \times m_s \ddot{\overline{R}}_{ms} \qquad (A27)$$

Also, from equation (A5)

$$\frac{d\overline{H}_{cm}}{dt} = \frac{d\overline{H}_O}{dt} - \frac{d\overline{H}_s}{dt} \qquad (A28)$$

$$\frac{d\overline{H}_{cm}}{dt} = \sum \overline{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} - r_{ms} \times m_s \ddot{\overline{R}}_{ms} \qquad (A29)$$

Now, by substituting for $\ddot{\overline{R}}_{ms}$ from equation (A21), equation (A29) will become

$$\frac{d\overline{H}_{cm}}{dt} = \sum \overline{r}_{mj} \times m_j \ddot{\overline{R}}_{mj} - \overline{r}_{ms} \times \overline{F}_s \qquad (A30)$$

Referring back to equation (A20) and considering equation (A30) then

$$\overline{M}_O = \frac{d\overline{H}_{cm}}{dt} + \overline{r}_{ms} \times \overline{F}_s \qquad (A31)$$

The applied moment about the geometric axes will equal the derivative of the angular momentum about the mass center if no external forces act through the center of mass. For most rotational problems the external force contributions due to mass center shifts can be neglected. In this case

$$\overline{M}_O = \frac{d\overline{H}_{cm}}{dt} \qquad (A32)$$

Referring back to equation (A9) and remembering that by letting $\dot{\overline{r}}_{mj}$ and $\dot{\overline{r}}_{ms}$ both equal 0, an expression for $\overline{H}_{rb}$ was obtained, and the angular momentum about the center of mass can be written as

$$\overline{H}_{cm} = \sum \overline{r}_{mj} \times m_j \dot{\overline{r}}_{mj} - \overline{r}_{ms} \times m_s \dot{\overline{r}}_{ms} + H_{rb} \qquad (A33)$$

Again, only one moving mass m is considered and equation (A32) can be written

$$\overline{M}_O = \frac{d\overline{H}_{cm}}{dt} = \frac{d}{dt}\left(\overline{r} \times m\dot{\overline{r}} - \overline{r}_{ms} \times m_s \dot{\overline{r}}_{ms} + H_{rb}\right) \qquad (A34)$$

Substituting for $\overline{r}_{ms}$ and $\dot{\overline{r}}_{ms}$ by using equation (A15) gives

$$\overline{M}_O = \frac{d}{dt}\left[\overline{r} \times m\dot{\overline{r}} - \frac{m\overline{r}}{m_s} \times m_s\left(\frac{m}{m_s}\dot{\overline{r}}\right) + H_{rb}\right] \qquad (A35)$$

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$$\overline{M}_O = \frac{d}{dt}\left[\frac{m(m_s - m)}{m}(\bar{r} \times \dot{\bar{r}}) + H_{rb}\right] \quad (A36)$$

$$\overline{M}_O = \frac{d}{dt}\left[Q(\bar{r} \times \dot{\bar{r}}) + H_{rb}\right] \quad (A37)$$

Expanding equation (A37) results in the following equation:

$$\overline{M}_O = Q\left[\bar{r} \times \ddot{\bar{r}} + \overline{\omega} \times (\bar{r} \times \dot{\bar{r}})\right] + \dot{\overline{H}}_{rb} + \overline{\omega} \times \overline{H}_{rb} \quad (A38)$$

Now, by substituting for $\overline{H}_{rb}$ $\left(\text{and } \dot{\overline{H}}_{rb}\right)$ by making use of equation (A22) the following components of $\overline{M}_O$ about the X_b-, Y_b-, and Z_b-axes through the geometric origin are obtained:

$$
\left.
\begin{aligned}
M_X &= Q\left\{(y\ddot{z} - z\ddot{y}) + \left[\omega_y(x\dot{y} - y\dot{x}) - \omega_z(z\dot{x} - x\dot{z})\right]\right\} \\
&\quad + \left(I_X\dot{\omega}_x + \omega_x\dot{I}_X - I_{XY}\dot{\omega}_y - \omega_y\dot{I}_{XY} - I_{XZ}\dot{\omega}_z - \omega_z\dot{I}_{XZ}\right) \\
&\quad + \left[\omega_y\left(I_Z\omega_z - I_{XZ}\omega_x - I_{YZ}\omega_y\right) - \omega_z\left(I_Y\omega_y - I_{YZ}\omega_z - I_{XY}\omega_x\right)\right] \\
\\
M_Y &= Q\left\{(z\ddot{x} - x\ddot{z}) + \left[\omega_z(y\dot{z} - z\dot{y}) - \omega_x(x\dot{y} - y\dot{x})\right]\right\} \\
&\quad + \left(I_Y\dot{\omega}_y + \omega_y\dot{I}_Y - I_{YZ}\dot{\omega}_z - \omega_z\dot{I}_{YZ} - I_{XY}\dot{\omega}_x - \omega_x\dot{I}_{XY}\right) \\
&\quad + \left[\omega_z\left(I_X\omega_x - I_{XY}\omega_y - I_{XZ}\omega_z\right) - \omega_x\left(I_Z\omega_z - I_{XZ}\omega_x - I_{YZ}\omega_y\right)\right] \\
\\
M_Z &= Q\left\{(x\ddot{y} - y\ddot{x}) + \left[\omega_x(z\dot{x} - x\dot{z}) - \omega_y(y\dot{z} - z\dot{y})\right]\right\} \\
&\quad + \left(I_Z\dot{\omega}_z + \omega_z\dot{I}_Z - I_{XZ}\dot{\omega}_x - \omega_x\dot{I}_{XZ} - I_{YZ}\dot{\omega}_y - \omega_y\dot{I}_{YZ}\right) \\
&\quad + \left[\omega_x\left(I_Y\omega_y - I_{YZ}\omega_z - I_{XY}\omega_x\right) - \omega_y\left(I_X\omega_x - I_{XY}\omega_y - I_{XZ}\omega_z\right)\right]
\end{aligned}
\right\} \quad (A39)
$$

If the moments M_X, M_Y, M_Z and coordinates for the moving mass x, y, z are expressed as functions of time, then equations (A39) can be integrated to find ω_x, ω_y, and ω_z with numerical methods. The spacecraft rates, then, are used in equations (A2) to find the Euler angle rates.

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3. Kurzhals, Peter R.; and Grantham, Carolyn: A System for Inertial Experiment Pointing and Attitude Control. NASA TR R-247, 1966.

4. Kurzhals, Peter R.: An Approximate Solution of the Equations of Motion for Arbitrary Rotating Spacecraft. NASA TR R-269, 1967.
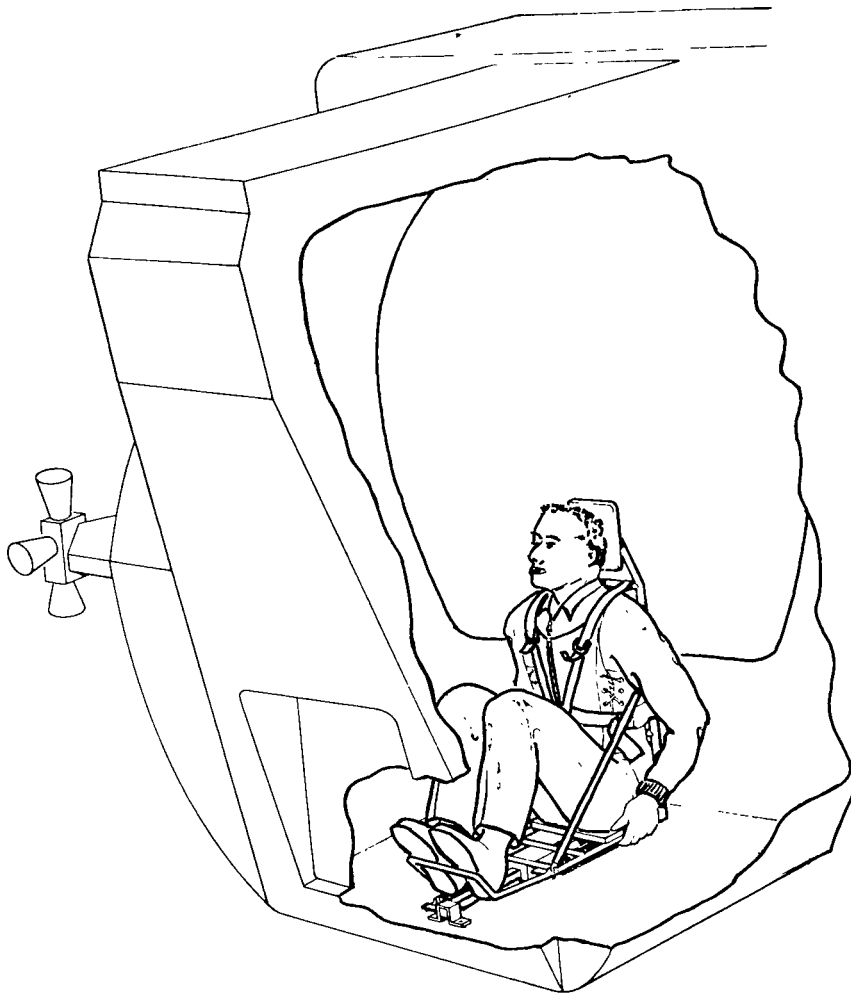
Figure 1.- Schematic of mass measuring system in spacecraft.
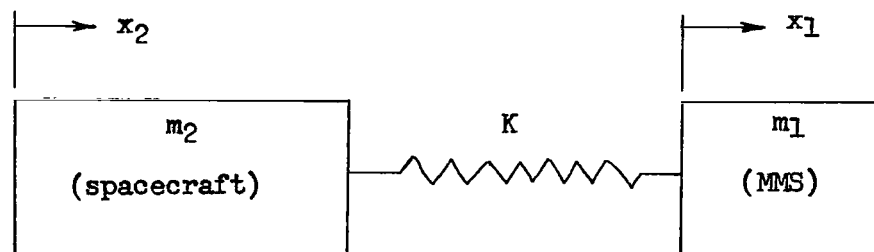
Figure 2.- Schematic of MMS-spacecraft dynamic system.

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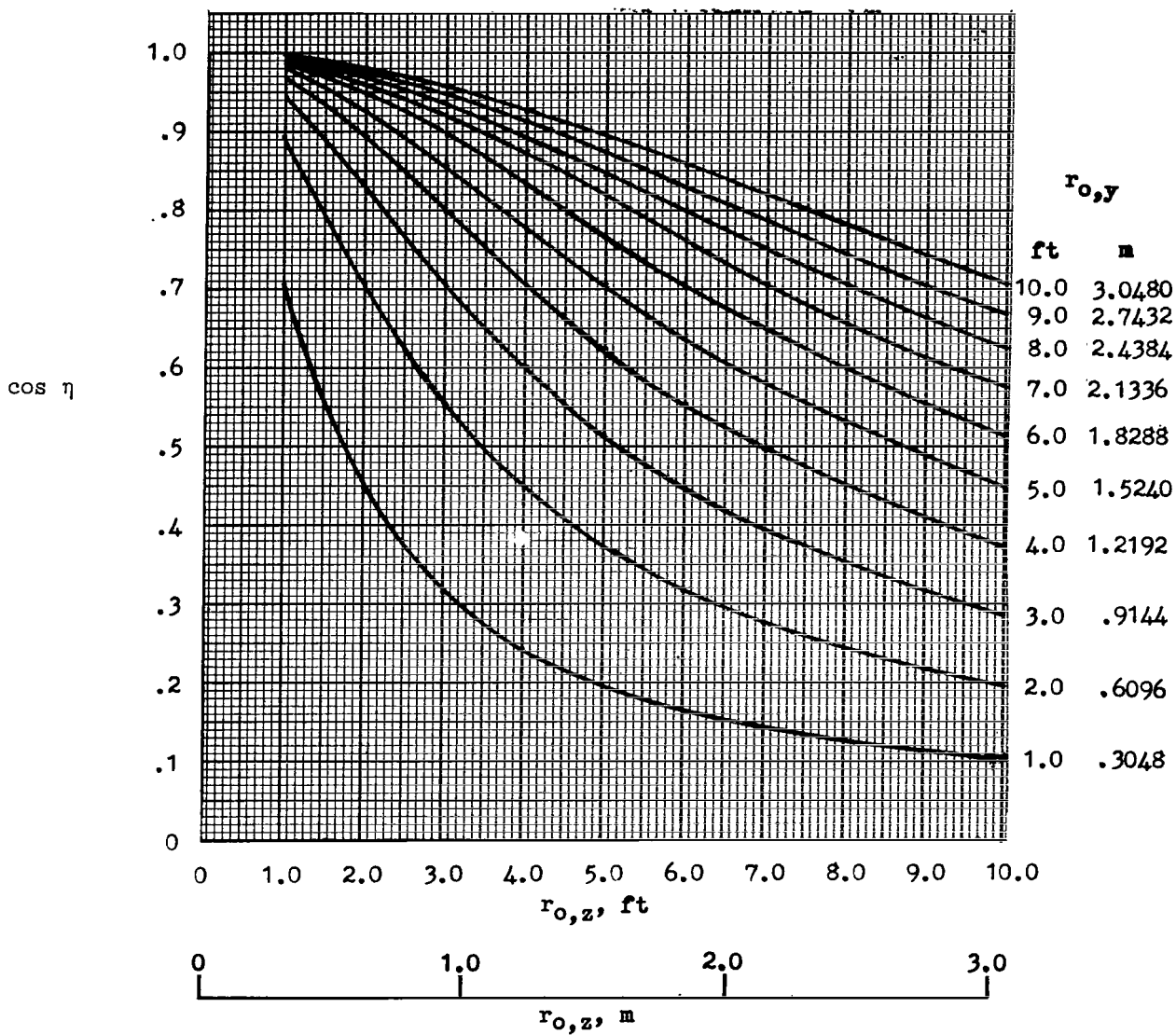
(a) g - h ≥ 0, $r_{0,x}$ = 0; g - h = 0, $r_{0,x}$ ≥ 0.

Figure 3.- MMS orientation parameter cos η as a function of location within the spacecraft.
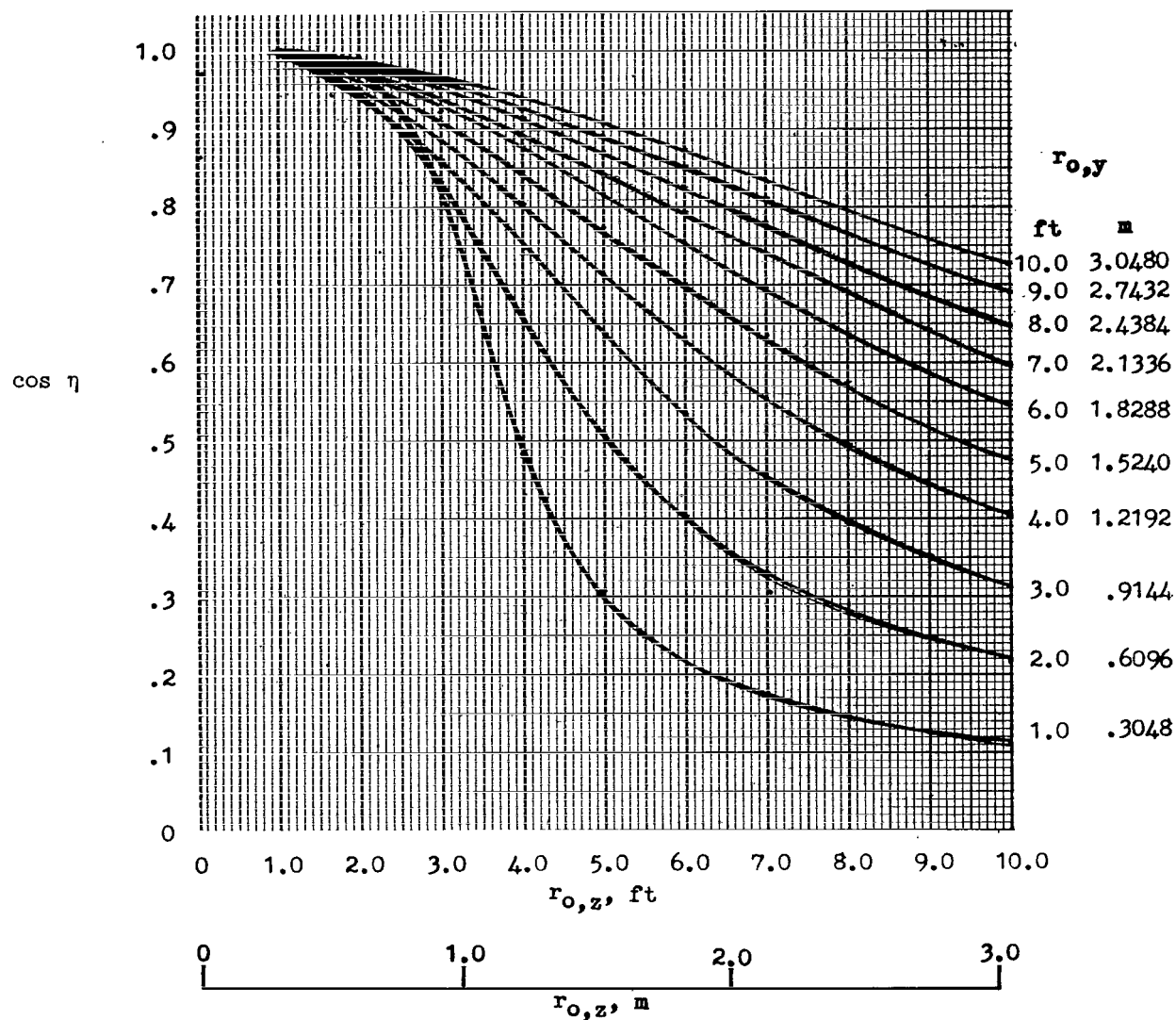
(b) g – h = 0.1, $r_{o,x}$ = 10.0 ft (3.0480 m).

Figure 3.- Continued.
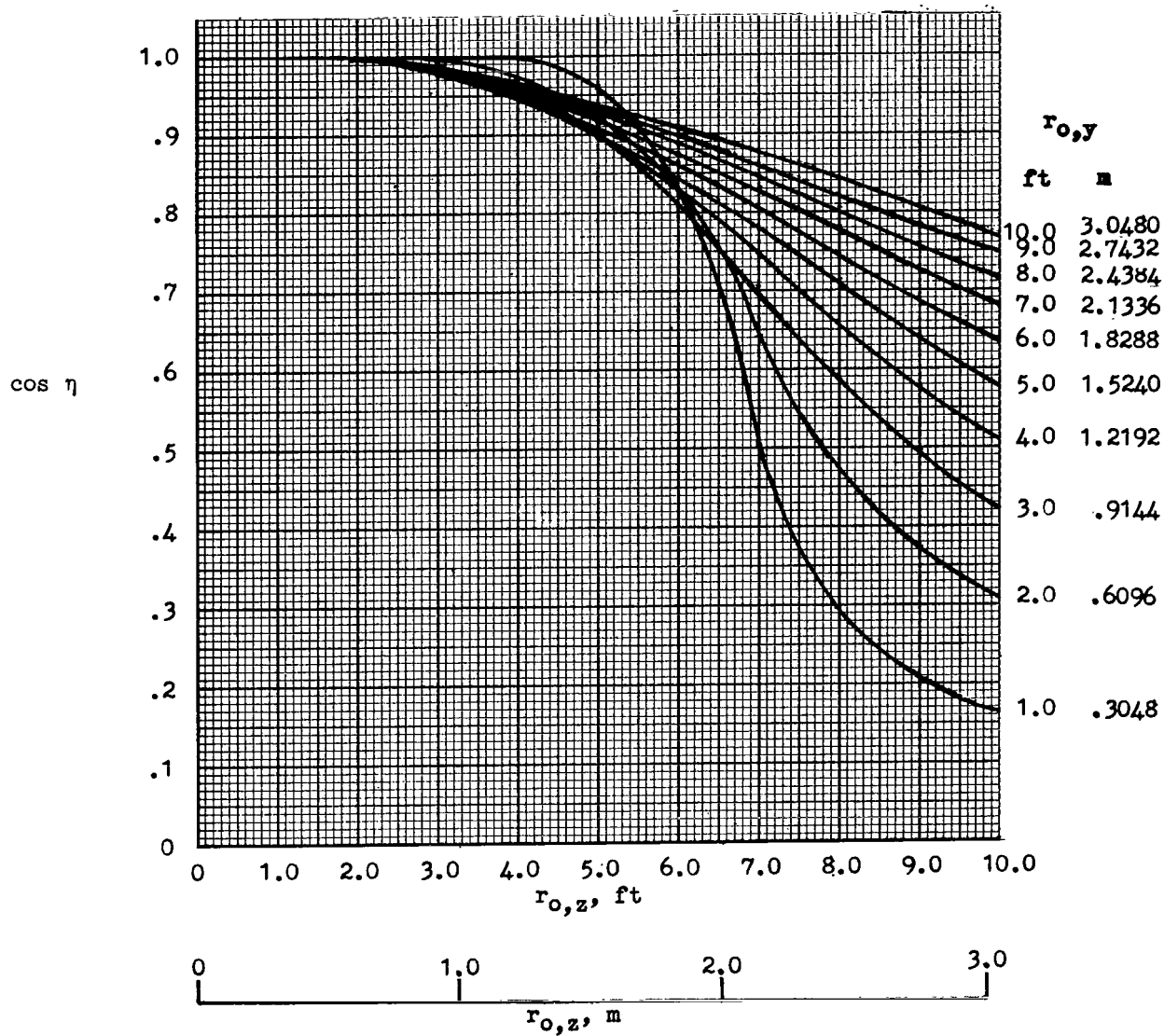
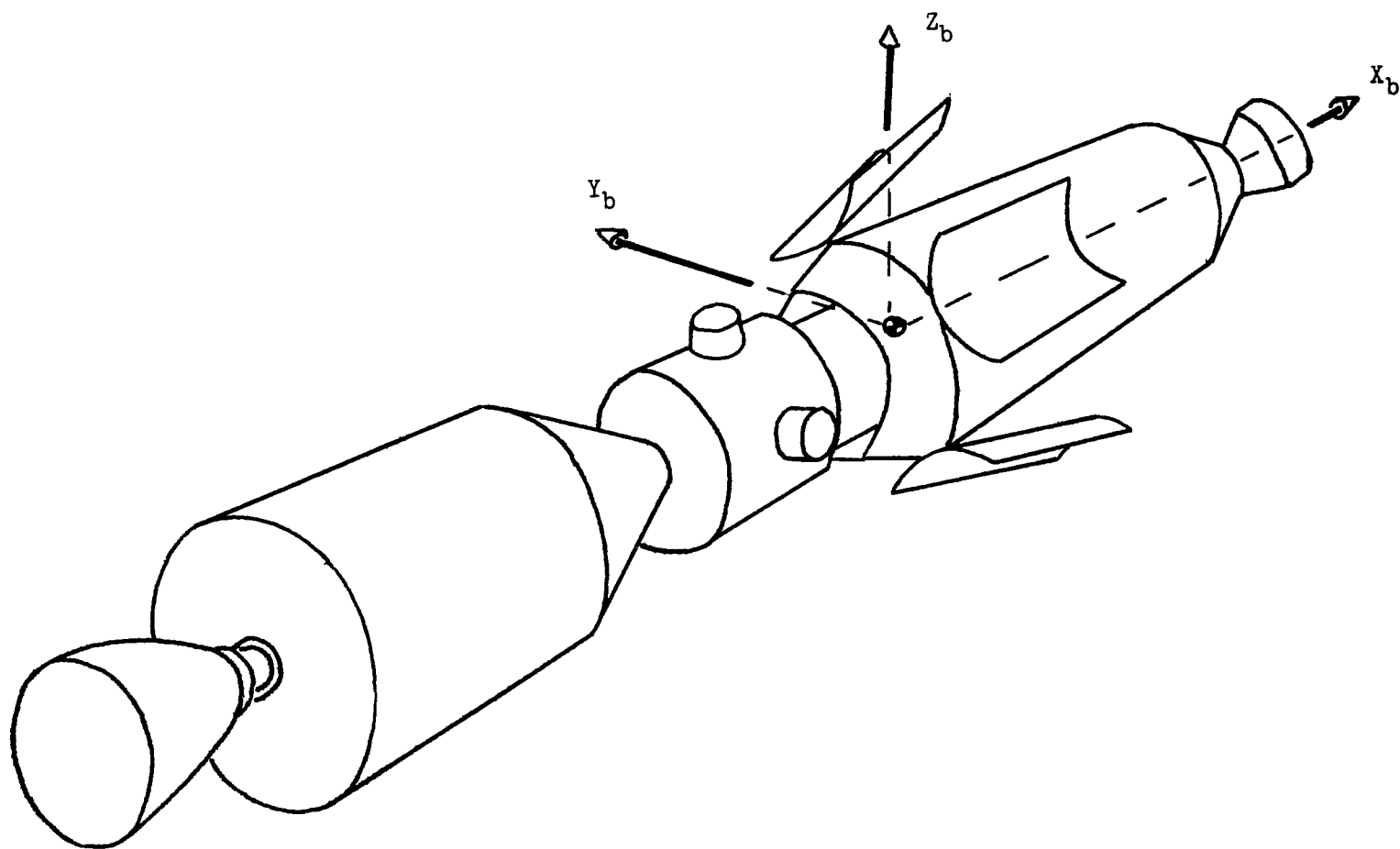(c) g - h = 0.1, $r_{0,x}$ = 20.0 ft (6.096 m).

Figure 3.- Concluded.

Figure 4.- Schematic of Apollo Applications Program Orbital Workshop cluster, showing axis orientation.
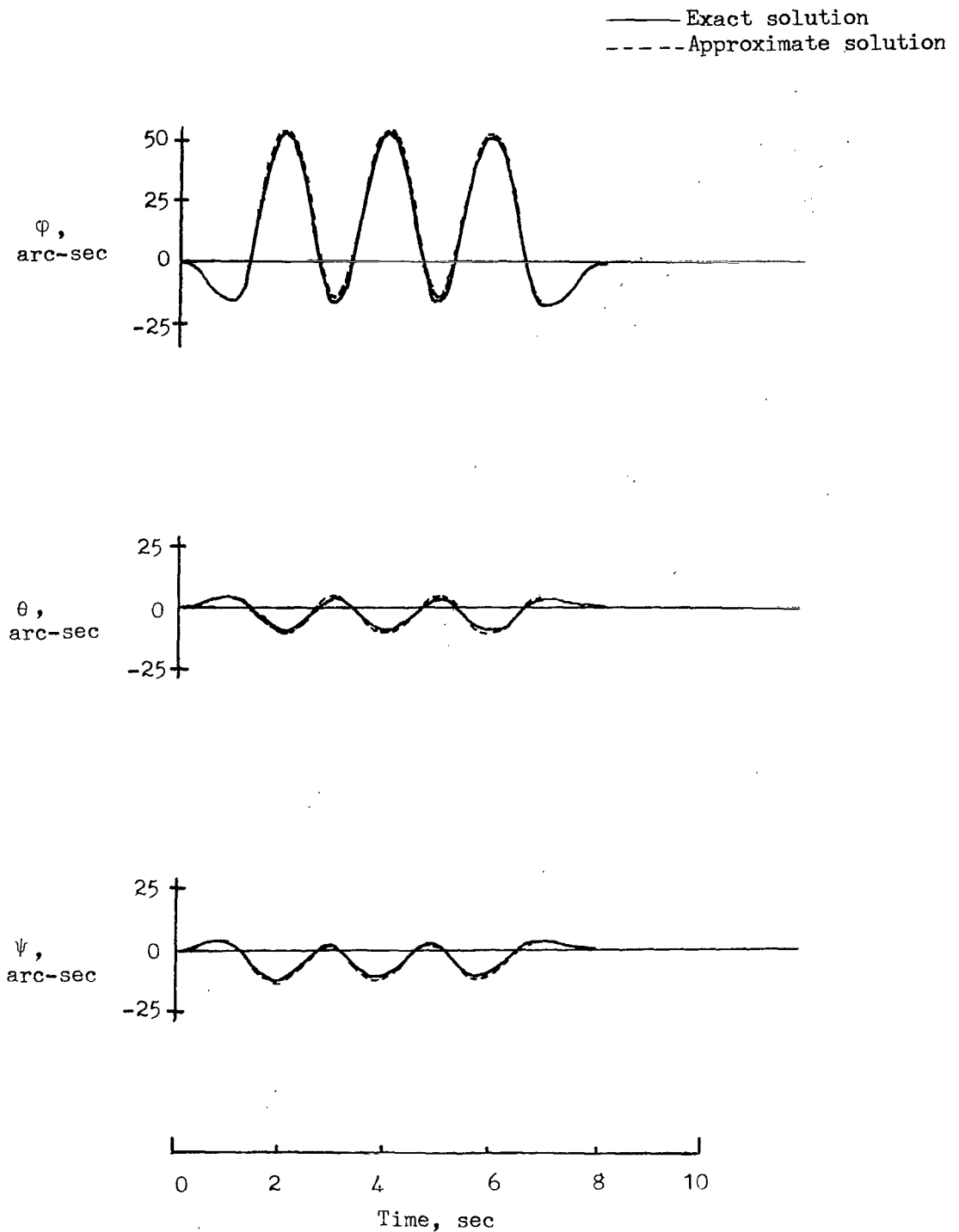
φ ,
arc-sec

50

25

0

-25

θ ,
arc-sec

25

0

-25

ψ ,
arc-sec

25

0

-25

0   2   4   6   8   10

Time, sec

Figure 5.- Response of spacecraft to typical MMS operation.
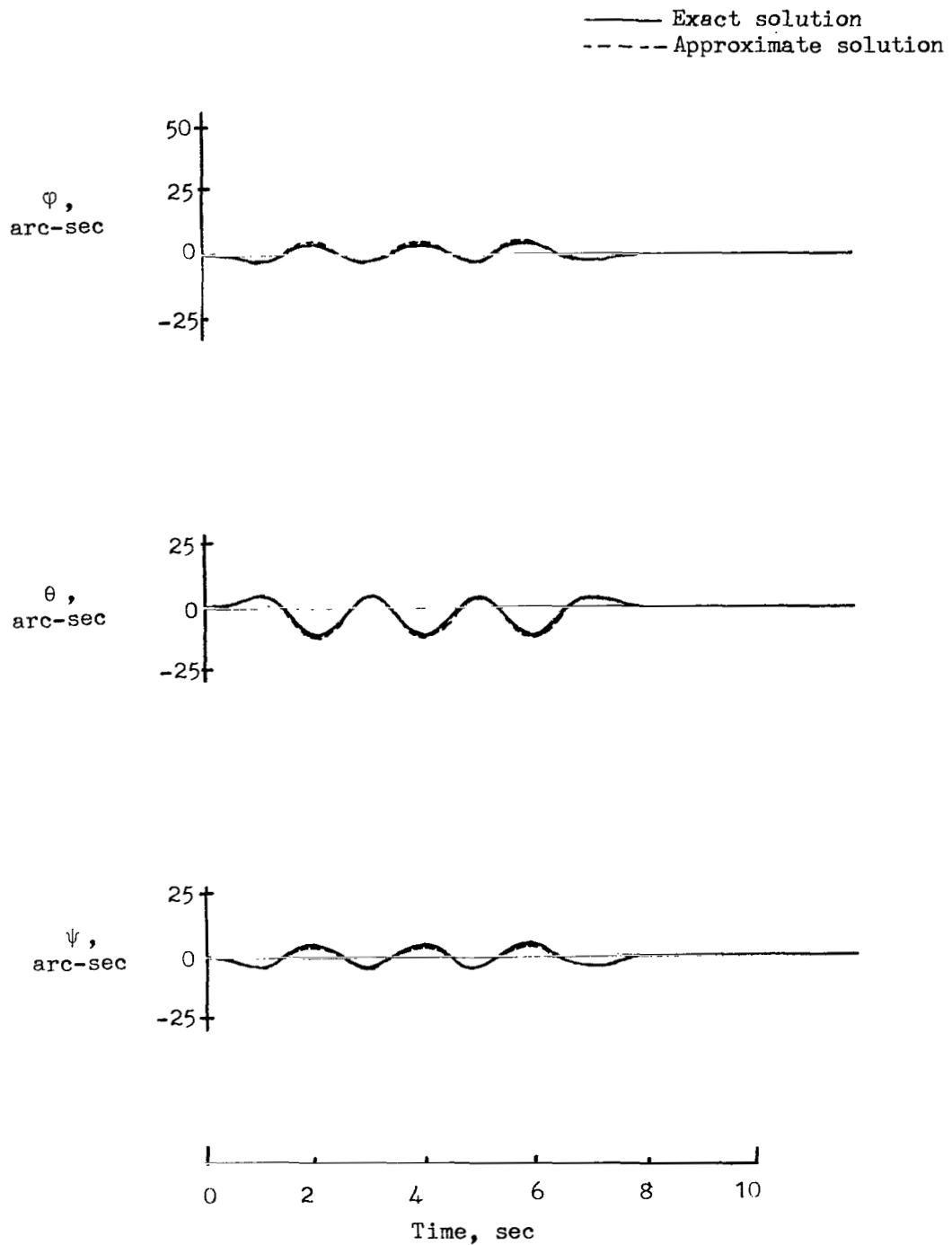
Figure 6.- Response of spacecraft to typical MMS operation (with optimum orientation).
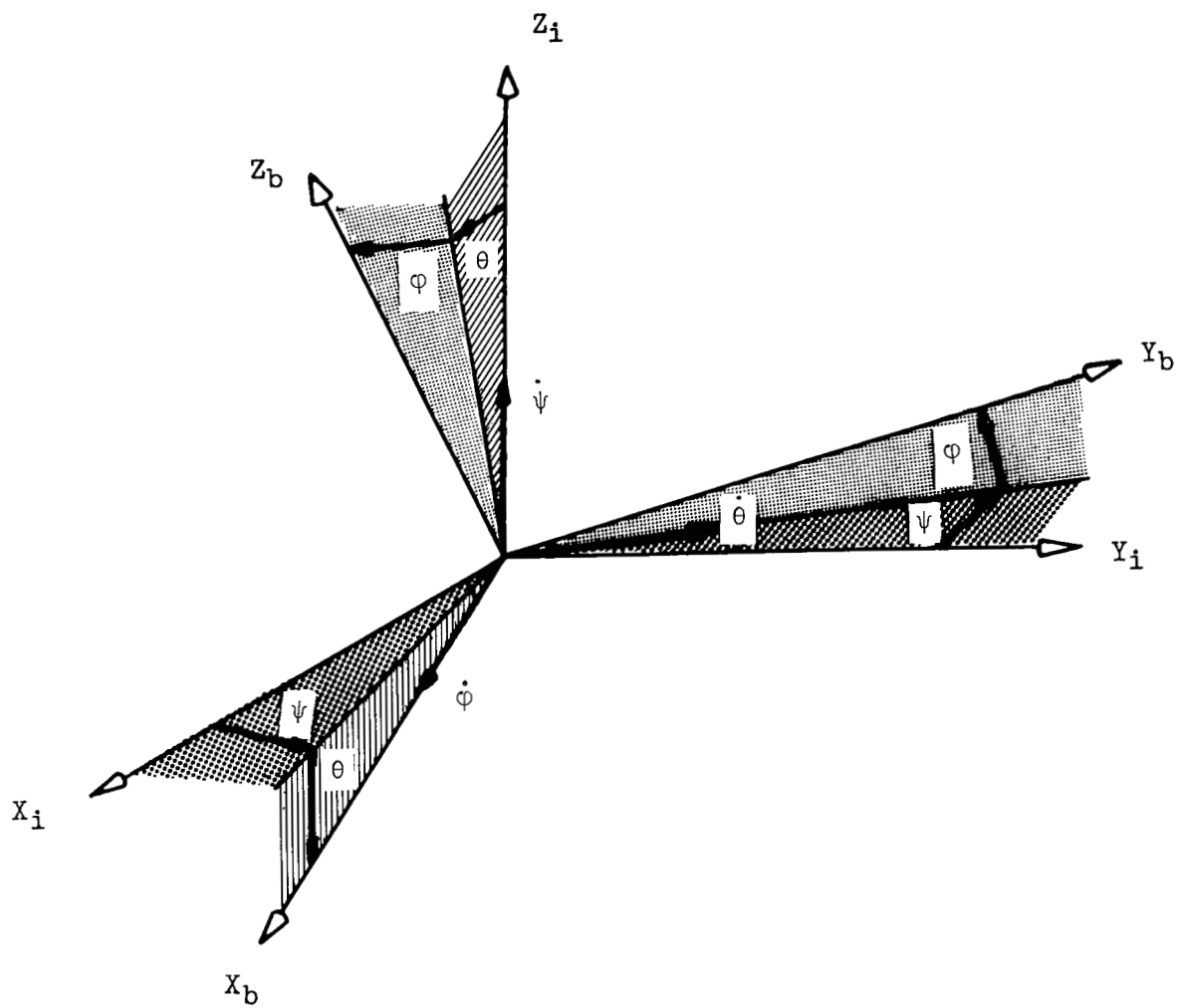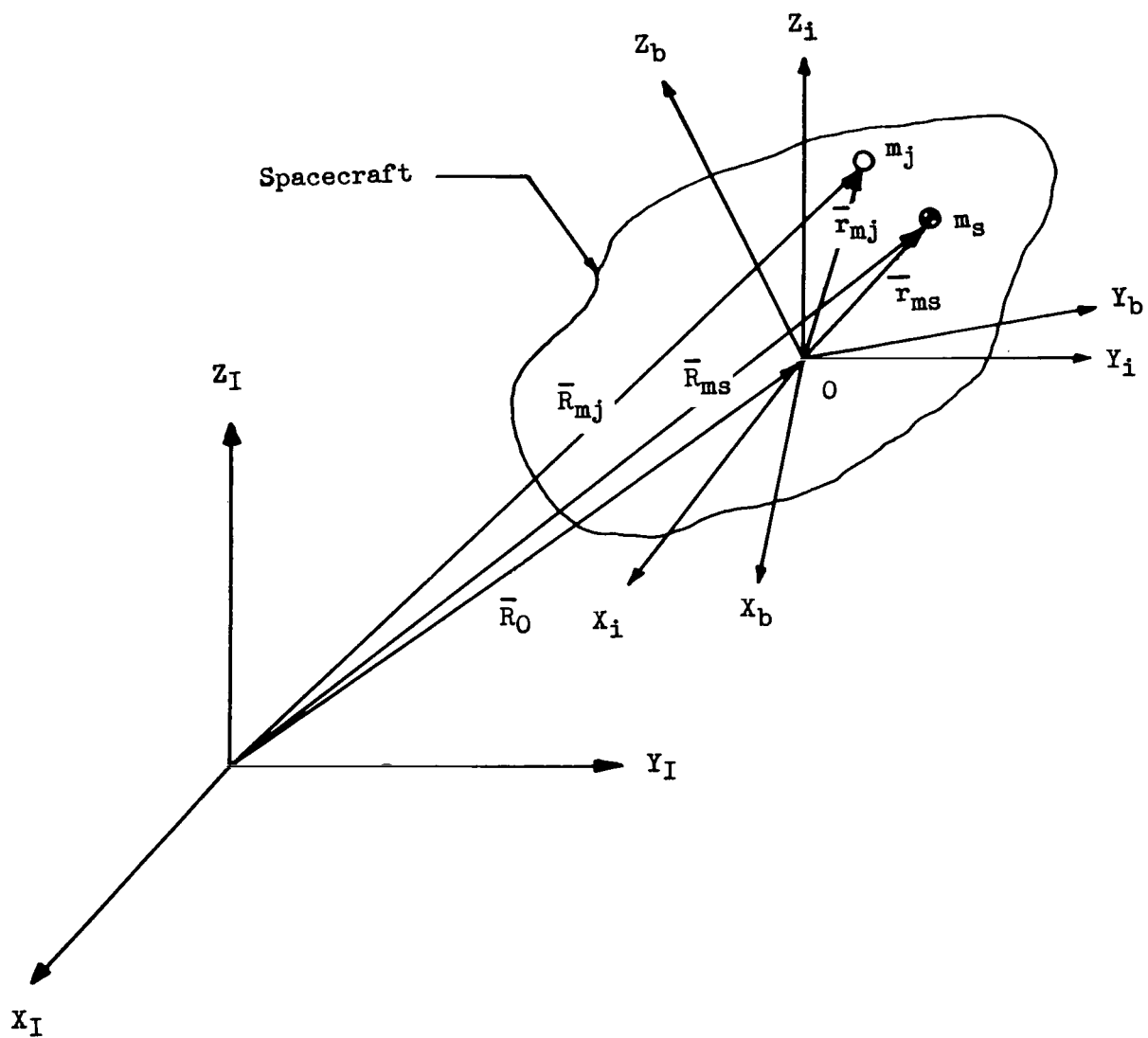
Figure 7.- Modified Euler angles.

Figure 8.- Spacecraft axis and coordinate notation.