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Clemson University
Department of Mathematics

STATISTICAL FILTER THEORY APPLIED TO A MODEL
WITH APPROXIMATE TRANSITION MATRICES
PART II

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STATISTICAL FILTER THEORY APPLIED TO A MODEL
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PART II

December 1967

A Report on Contract NAS 8-11259
Advanced Studies Office
Astrionics Laboratory
Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

CLEMSON UNIVERSITY INVESTIGATORS

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Abstract

Members of the Mathematics Department of Clemson University, under an extension of NASA contract NAS 8-11259 investigated an application of statistical filter theory. R. E. Kalman's statistical filter was applied to a nonlinear mathematical model. A simulation model was constructed so that two linearization techniques; first, linearization by neglecting the nonlinear terms and second, linearization at each time step could be investigated.

Summary

A mathematical model is constructed to illustrate the concepts of The Kalman Filter. The model is a nonlinear model of a well known example of a system that has a limit cycle. The limit cycle is a circle of arbitrary radius which is obtained from the general solution. Other solutions yield paths which approach the limit cycle from within or outside the circle.

Two simulation models are constructed. Both are similar with the exception of the computation of the transition matrix. One simulation is obtained when the nonlinear terms in the system are neglected. This yields an analytic solution to the corresponding linear system and a transition matrix which is a function of the time step. The other simulation requires solving the nonlinear system at each time step by linearizing at each time step. This simulation involves numerical solutions of the corresponding linear system at each time step.

Illustrations

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List of Symbols by Chapters

Introduction

t	time in general; present time
$X(t_n)$	state vector
$I(t_{n+1};t_n)$	transition matrix
$Y(t_n)$	observation vector
$M(t_n)$	mapping matrix
$N(t_n)$	input vector (random)
$Q(t_n)$	covariance matrix of $N(t_n)$
E	expectation
$XB(t_n)$	updated estimate of $X(t_n)$
$XU(t_n)$	estimate of $X(t_n)$
$P(t_n)$	covariance matrix of the error in the estimate $XU(t_n)$
$XA(t_n)$	actual state vector
$YO(t_n)$	observation vector
$YU(t_n)$	updated estimate of $Y(t_n)$
$P^*(t_n)$	covariance matrix of the error in estimation
$XB(t_n)$	initial best estimate

Mathematical Model

$F(x, dx/dt)$	function
x	position
dx/dt	velocity
$P(x,y)$	function
$Q(x,y)$	function
k, c, t_0	arbitrary constants

Simulation Details

$X(t)$	state vector
$F(t)$	matrix
$A(t)$	matrix
K	vector
$\Phi(t_n; t_n)$	transition matrix
Δt	time interval
$XB(t_0)$	initial best estimate of the state
t_0	initial time
t_f	final time
$XN(t_0)$	random noise vector
$XA(t_0)$	actual state
$YU(t_n)$	updated observation
$YO(t_n)$	contaminated observation
$XN(t_n)$	random noise vector
$XB(t_n)$	best estimate of state
$P^*(t_{n+1})$	covariance matrix of error in estimation
$F(t_{n+1})$	matrix
$x_1 B(t_n), x_2 B(t_n)$	components of $IB(t_n)$
$\phi_{11}, \phi_{12}, \phi_{21}, \phi_{22}$	components of $\Phi(t_{n+1}; t_n)$
$P(t_{n+1})$	covariance matrix of error in estimate of
	$XU(t_n)$

Chapter I

Introduction

In general, data filtering entails the extraction of a signal from observations that include random noise. A solution to this problem, assuming the equations of motion are linear, was formulated by R. F. Kalman [1] in 1960. In Kalman's formulation the mathematical model consisted of a differential equation with additive noise and an equation relating the observations to the state of the system.

A mathematical model that is more applicable to trajectory analysis has additive random noise in the observations and no additive random excitations in the linear system, i.e., the relations are

$$\begin{aligned} X(t_{n+1}) &= \Phi(t_{n+1}; t_n) X(t_n) \\ Y(t_n) &= M(t_n) X(t_n) + N(t_n), \end{aligned} \quad (1.1)$$

where $X(t_{n+1})$ is a state vector, $M(t_n)$ is a mapping matrix, $Y(t_n)$ is an observation vector, and $N(t_n)$ is a vector valued independent random process with zero mean and known covariance matrix $Q(t_n)$, i.e.,

$$\begin{aligned} Q(t_n) &= E[N(t_n) N'(t_n)] \\ E[N(t_n) N'(t_m)] &= 0 \quad n \neq m \\ E[N(t_n)] &= 0 \quad \text{for all } n. \end{aligned}$$

The problem is then: given system (1.1) to find the "best" estimate $XB(t_n)$ of the updated state $XU(t_n)$ by using the observation $Y(t_n)$ and an updated estimate $XU(t_n)$ based on the observations

$Y(t_0), Y(t_1), \dots, Y(t_{n-1})$. The covariance matrix $P(t_n)$ of the error in the updated estimate $XU(t_n)$ is assumed to be known. The criteria used in establishing the "best" estimate $XB(t_n)$ is to minimize the trace of the covariance matrix

$$E[\{XA(t_n) - XB(t_n)\} \{XA(t_n) - XB(t_n)\}']$$

where $XA(t_n)$ is the actual state. Note that $XA(t_n)$ is the same as $X(t_n)$ given in equation (1.1).

The solution was formulated by C. B. Solloway [2]. The solution, using R. E. Kalman's result, can be represented by

$$XB(t_n) = XU(t_n) + P(t_n)M'(t_n) (M(t_n) P(t_n)M'(t_n) + Q(t_n))^{-1} (Y(t_n) - YU(t_n))$$

$$(1.2)$$

where

$$YU(t_n) = M(t_n)XU(t_n) \quad (1.3)$$

$$XU(t_n) = \phi(t_n; t_{n-1}) XB(t_{n-1}) \quad (1.4)$$

$$P(t_n) = \phi(t_n; t_{n-1}) P^*(t_{n-1}) \phi'(t_n; t_{n-1}) \quad (1.5)$$

and where $Y(t_n)$ is the observation vector $Y(t_n)$. The covariance matrix $P^*(t_n)$ of the error in the estimate is

$$P^*(t_n) = P(t_n) - P(t_n) M'(t_n) (M(t_n) P(t_n) M'(t_n) + Q(t_n))^{-1} M(t_n) P(t_n). \quad (1.6)$$

An initial best estimate $XB(t_0)$ and the covariance matrix $P(t_0)$ of the error in the estimate is required to initiate this iterative scheme.

However, the mathematical formulation of many physical problems result in mathematical models which are nonlinear. In many cases it is possible to replace such a nonlinear system of equations by a related linear system which approximates the nonlinear equations close enough to give useful results. It is true that such a "linearization" is not always feasible; and when it is not, the original nonlinear equation itself must be

considered. For example, the equations of motion for trajectory calculations which are nonlinear [3] can be treated by linearization techniques. Trajectory problems since the time of Gauss have been treated with linear techniques by restricting the calculations to deviates. These deviates are taken to be the difference between the actual and reference trajectory. In this approach, the trajectory problem is reduced to solving linear differential equations that represent perturbations of the actual trajectory from the reference trajectory. [4]

The derivation of Kalman's filter is dependent upon a linear dynamical system. The specific conclusion that Kalman gives is a minimum mean square error estimate for the linear system as a function of the sequence of observables.

Thus, to apply the Kalman filter to a nonlinear system, one approach would be to linearize the nonlinear system. However, when one linearizes a nonlinear system, one must be cognizant of the linearization process.

An alternate approach would be to use numerical methods on automatic digital computers to solve the actual nonlinear estimation problem. Even though there has been recent efforts in this direction, the theory is still incomplete [5]. Consequently most nonlinear problems are approached through linearization techniques.

The authors of this report have concentrated on the linearization technique in order to apply Kalman's filter to a nonlinear problem. The material is essentially tutorial. A nonlinear mathematical model is structured, and emphasis is placed on the linearization techniques. The system is flexible in that it allows variations in the inputs, different noise contaminations,

and alternate transition matrix calculations.

Two computerized programs have been developed. The first one linearizes by neglecting the nonlinear terms. The second program linearizes with respect to the updated nominal trajectory.

Chapter II

Mathematical Model

The general theory and methods of linear equations are highly developed. However, very little of a general character is known about nonlinear equations. Some basic definitions and concepts are reviewed at this time. [b].

A second order nonlinear differential equation of the form

$$\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt}) \quad (2.1)$$

is considered. A specific example would be van der Pol's equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0 \quad (2.2)$$

since equation (2.2) can be put in the form of equation (2.1) where

$$F(x, \frac{dx}{dt}) = -\mu(x^2 - 1)\frac{dx}{dt} - x.$$

Suppose equation (2.1) describes a certain dynamical system having one degree of freedom. Then the state of this system at time t can be determined by the values of x (position) and dx/dt (velocity). The plane of the variables x and dx/dt is called a phase plane.

Let $y = dx/dt$. Then equation (2.1) can be replaced by the following equivalent system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= F(x, y). \end{aligned} \quad (2.3)$$

Information pertaining to equation (2.1) can be determined from a study of system (2.3). The solutions of equation (2.3) will yield configurations (curves) which will be of interest. If t is assumed to be a parameter then the configurations will appear in the xy plane, that is, the $(x, \frac{dx}{dt})$ phase plane since $y = dx/dt$.

A more general system considered would have the form

$$\begin{aligned}\frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y),\end{aligned}\tag{2.4}$$

where $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives for all (x, y) . Such a system, in which the independent variable t appears only in the differentials dt of the left members and not explicitly in the functions P and Q on the right, is called an autonomous system.

An existence theorem states that given any number t_0 and any pair (x_0, y_0) of real numbers, there exists a unique solution

$$\begin{aligned}x &= f(t) \\ y &= g(t),\end{aligned}\tag{2.5}$$

of the system (2.4) such that

$$\begin{aligned}f(t_0) &= x_0 \\ g(t_0) &= y_0.\end{aligned}$$

If both $f(t)$ and $g(t)$ are not both constant functions, then equation (2.5) defines a curve in the xy plane which is called a path of the system (2.4). With each path there is associated a definite direction, the direction of increase of the parameter t .

Eliminating t from the system (2.4) yields

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)} . \quad (2.6)$$

This equation gives the slope of the tangent to the path described by system (2.4) passing through the point (x, y) provided the functions P and Q are not both zero at this point. The general solution of equation (2.6) thus provides a one parameter family of paths of system (2.4).

At a point (x_0, y_0) at which both P and Q are zero, the slope of the tangent to the path as defined by equation (2.6) is indeterminate. Given the autonomous system (2.4), a point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$ is called a critical point of the system (2.4). A critical point (x_0, y_0) of the system (2.4) is called isolated if there exists a circle, $(x - x_0)^2 + (y - y_0)^2 = r^2$, about the point (x_0, y_0) such that (x_0, y_0) is the only critical point of system (2.4) within this circle.

The nonlinear model that is considered in this report is a well-known example of a system having a limit cycle. A closed path C of the system (2.4) which is approached spirally from either the inside or the outside by a nonclosed path C_1 of system (2.4) either as $t \rightarrow +\infty$ or as $t \rightarrow -\infty$ is called a limit cycle.

Consider the mathematical model characterized by the following system

$$\begin{aligned}\frac{dx}{dt} &= y + x[1 - k^2(x^2 + y^2)] \\ \frac{dy}{dt} &= -x + y[1 - k^2(x^2 + y^2)],\end{aligned}\tag{2.7}$$

where k is an arbitrary constant. This is an autonomous system whose only critical point $(0, 0)$ is isolated.

Using polar coordinates, that is, let

$$\begin{aligned}x &= r \cos \theta \\ y &= r \sin \theta,\end{aligned}$$

in system (2.7). Then, after some manipulation, system (2.7) reduces to

$$\begin{aligned}\frac{dr}{dt} &= r(1 - k^2 r^2) \\ \frac{d\theta}{dt} &= -1.\end{aligned}\tag{2.8}$$

The general solution of system (2.8) is determined to be

$$\begin{aligned}r(t) &= \frac{1}{(k^2 + c_2 e^{-2t})^{\frac{1}{2}}} \\ \theta(t) &= -t + c_1,\end{aligned}\tag{2.9}$$

where c_1 and c_2 are arbitrary constants.

The solution of system (2.7) becomes

$$\begin{aligned}x(t) &= \frac{\cos(-t + c_1)}{(k^2 + c_2 e^{-2t})^{\frac{1}{2}}} \\ y(t) &= \frac{\sin(-t + c_1)}{(k^2 + c_2 e^{-2t})^{\frac{1}{2}}}.\end{aligned}\tag{2.10}$$

The solutions given by equation (2.10) defines paths of system (2.7) in the xy plane. Examining these paths for various values of c yield the following conclusions:

(i) If $c = 0$ the path defined by equation (2.10) is the circle $x^2 + y^2 = \frac{1}{k^2}$ described in a clockwise direction.

(ii) If $c \neq 0$, the paths defined by equation (2.10) are not closed paths but rather paths having a spiral behavior. If $c > 0$ the paths are spirals lying inside the circle $x^2 + y^2 = \frac{1}{k^2}$. As $t \rightarrow +\infty$, they approach this circle; while as $t \rightarrow -\infty$, they approach the critical point $(0, 0)$ of system (2.7). If $c < 0$, the paths lie outside the circle $x^2 + y^2 = \frac{1}{k^2}$. These outer paths also approach this circle as $t \rightarrow +\infty$; while as $t \rightarrow \ln\sqrt{|c|}$, both $|x|$ and $|y|$ become infinite. Thus, the circle $x^2 + y^2 = \frac{1}{k^2}$ is the limit cycle of the system (2.7). See Figure 1.

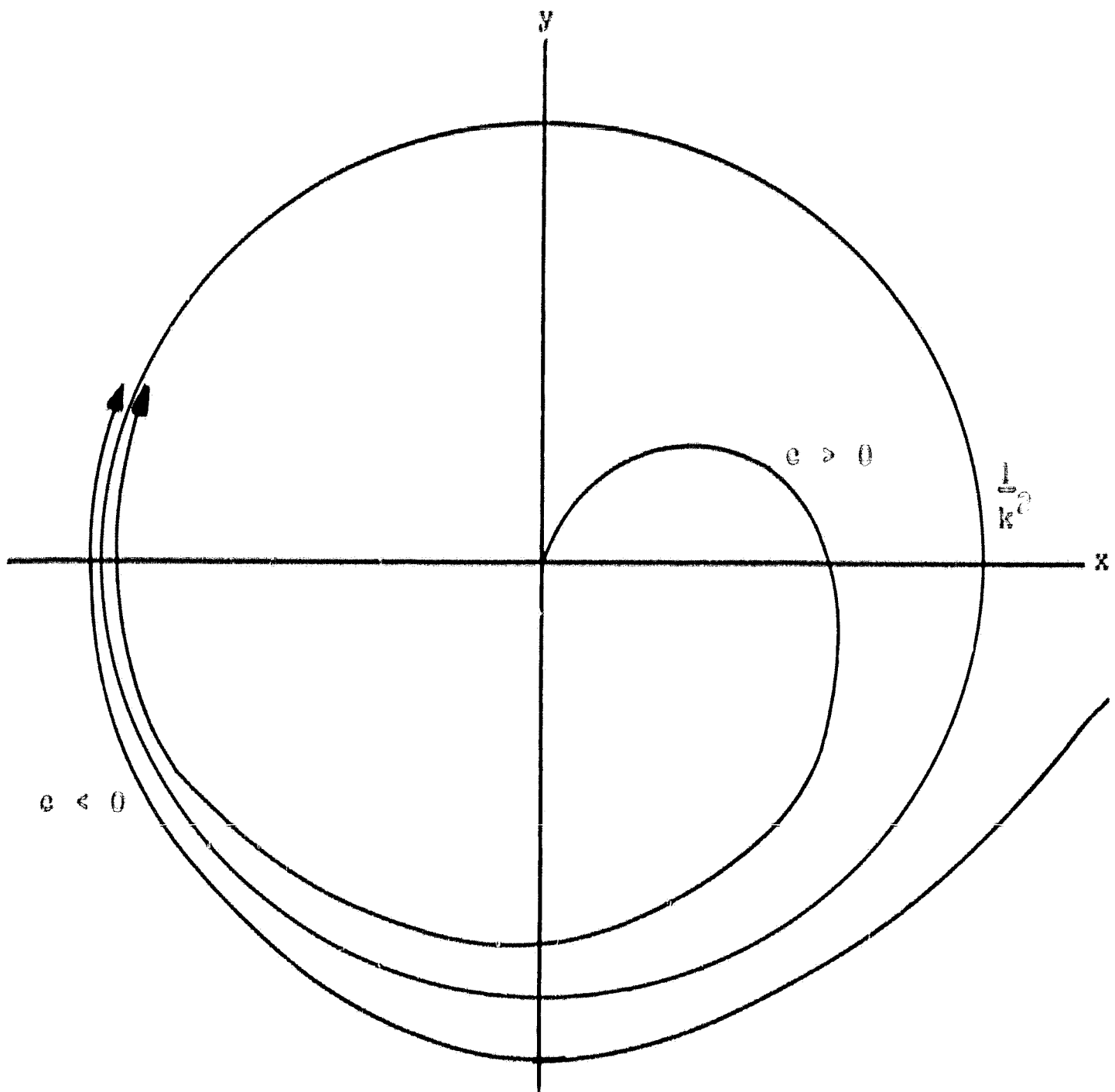


Figure 1. Limit Cycle of System 2.7.

Chapter III

Simulation Details

The following simulation is an attempt to generate an estimation model that allows only one primary hypothesis weakening. The previous mathematical model, that is, system (2.7) is adapted to give a simulation with known solution trajectories. The noise contaminated observations are used to estimate the actual positions. In one case nonlinear terms are neglected at first in order to apply linear filter techniques and in the other, linearization takes place at each time step.

Neglecting the nonlinear terms reduces system (2.7) to

$$\begin{aligned} \dot{x}(t) &= x(t) + y(t) \\ \dot{y}(t) &= -x(t) + y(t). \end{aligned} \quad (3.1)$$

Let $x(t) = x_1(t)$ and $y(t) = x_2(t)$. Then equation (3.1) can be written in vector notation, i.e.,

$$\dot{X}(t) = F(t) X(t) \quad (3.1')$$

where

$$F(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This linearization is equivalent to applying the Taylor series expansion in two variables to the right member of system (2.7) and linearizing about the origin (0,0).

The general solution of equation (3.1') is

$$X(t) = A(t) K \quad (3.2)$$

where

$$A(t) = e^t \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} k_0 \\ k_1 \end{bmatrix}$$

At an initial time t_0 equation (3.2) becomes

$$X(t_0) = A(t_0) K$$

and one then obtains

$$K = A^{-1}(t_0) X(t_0)$$

where

$$A^{-1}(t_0) = e^{-t_0} \begin{bmatrix} \cos t_0 & -\sin t_0 \\ \sin t_0 & \cos t_0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} X(t) &= A(t) A^{-1}(t_0) X(t_0) \\ &= \phi(t; t_0) X(t_0) \end{aligned}$$

where

$$\phi(t; t_0) = e^{(t-t_0)} \begin{bmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{bmatrix} \quad (3.3)$$

After some manipulation one further can show that

$$\phi(t_n; t_{n-1}) = e^{\Delta t} \begin{bmatrix} \cos \Delta t & \sin \Delta t \\ -\sin \Delta t & \cos \Delta t \end{bmatrix} \quad (3.4)$$

where

$$\Delta t = t_n - t_{n-1}. \quad (3.5)$$

The state transition matrix $\phi(t_n; t_{n-1})$ given by equation (3.4) is the one utilized in computing the Kalman filter for the case when the nonlinear terms are neglected.

To linearize at each time step is equivalent to computing the transition matrix $\phi(t_n; t_{n-1})$ at each time step, that is, the system of equations need to be solved at each time step. This linearization is equivalent to applying the Taylor series expansion in two variables to the right member of system (2.7) and linearizing with respect to the derivation between the actual and reference trajectories. This scheme requires a linear perturbation technique in order to compute the transition matrix. Details of this scheme is given in conjunction with the simulation flow chart

illustrated in Figure 2. It is noted that in the simulation model that the observables are the state variables. Thus, in Kalman's linear filter the mapping matrix is reduced to the identity matrix.

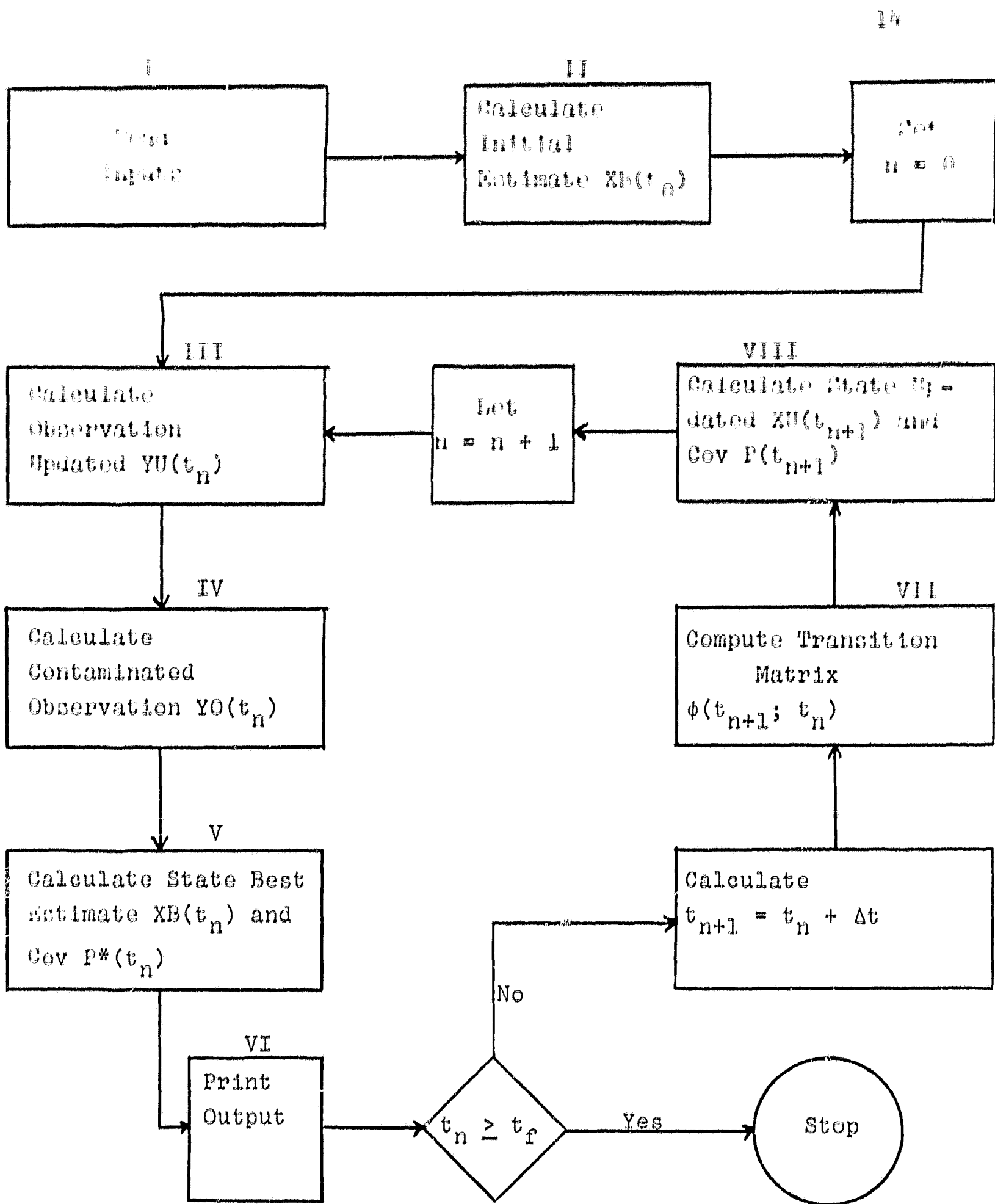


Figure 2. Block Diagram of the Simulation Models.

I. Read Inputs

Parameters: k, t_0, c_1, c_2 Time interval: $\Delta t = t_n - t_{n-1}$ Final time: t_f Covariance matrix of noise: $Q(t)$ Covariance matrix of error in initial estimate: $P(t_0)$ Mapping matrix: $M(t)$ II. Calculate Initial Estimate $XB(t_0)$ $XB(t_0) = XA(t_0) + XN(t_0)$ where

$$XA(t_0) = \begin{bmatrix} \frac{\cos(-t_0 + c_1)}{(k^2 + c_2 e^{-2t_0})^{\frac{1}{2}}} \\ \frac{\sin(-t_0 + c_1)}{(k^2 + c_2 e^{-2t_0})^{\frac{1}{2}}} \end{bmatrix}$$

and where $XN(t_0)$ is a random white noise vector with mean zero and standard deviation the square root of the diagonal elements of $P(t_0)$.

III. Calculate Observation Updated $YU(t_n)$ $YU(t_n) \triangleq M(t) XU(t_n)$ where $XU(t_0) \triangleq XB(t_0)$ IV. Calculate Contaminate Observation $YO(t_n)$ $YO(t_n) = XA(t_n) + XN(t_n)$

where

$$XA(t_n) = \begin{bmatrix} x_1(t_n) \\ x_2(t_n) \end{bmatrix} = \begin{bmatrix} \frac{\cos(-t_n + e_1)}{(k^2 + e_2 e^{-2t_n})^{1/2}} \\ \frac{\sin(-t_n + e_1)}{(k^2 + e_2 e^{-2t_n})^{1/2}} \end{bmatrix}$$

and where $XN(t_n)$ is a random white noise vector with mean zero and standard deviation the square roots of the diagonal elements of $Q(t_n)$ [which is for this simulation $Q(t)$ at each time step].

V. Calculate State Best Estimate $XB(t_n)$ and Cov $P^*(t_n)$

$$XB(t_n) = XU(t_n) + P(t_n)M'(t_n)[M(t_n)P(t_n)M'(t_n) + Q(t_n)]^{-1}(YO(t_n) - YU(t_n))$$

$$P^*(t_n) = P(t_n) - P(t_n)M'(t_n)[M(t_n)P(t_n)M'(t_n) + Q(t_n)]^{-1}M(t_n)P(t_n).$$

VI. Print Output

$$YO(t_n), XB(t_n), P^*(t_n), XA(t_n)$$

VII. Compute Transition Matrix $\Phi(t_{n+1}; t_n)$

(a) Neglecting nonlinear terms

$$\Phi(t_{n+1}; t_n) = e^{\Delta t} \begin{bmatrix} \cos \Delta t & \sin \Delta t \\ -\sin \Delta t & \cos \Delta t \end{bmatrix}$$

(b) At each time step

Compute

$$F(t_{n+1}) = k \begin{bmatrix} 1-3k^2(x_1B(t_n))^2-k^2(x_2B(t_n))^2 & 1-2k^2x_1B(t_n)x_2B(t_n) \\ -1-2k^2x_1B(t_n)x_2B(t_n) & 1-k^2(x_1B(t_n))^2-3k^2(x_2B(t_n))^2 \end{bmatrix}$$

↓

Integrate Numerically

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F(t_{n+1}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

↓

$$\begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}$$

↓

Integrate Numerically

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F(t_{n+1}) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

↓

$$\begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix}$$

↓

$$\Phi(t_{n+1}; t_n) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

VIII. Calculate State Updated $XU(t_{n+1})$ and Cov $P(t_{n+1})$

$$XU(t_{n+1}) = \Phi(t_{n+1}; t_n) XB(t_n)$$

$$P(t_{n+1}) = \Phi(t_{n+1}; t_n) P^*(t_n) \Phi'(t_{n+1}; t_n) .$$

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