National Aeronautics and Space Administration

HOUSTON, TEXAS

N 69-19892	
(ACCESSION NUMBER)	(THRU)
16	1
(PAGES)	(CODE)
NASA-CR-99577	19
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

FACILITY FORM 602

Manned Spacecraft Center

NASA CR 99577

INTERPOLATIONS WITH
A SPLINE FUNCTION

By

S. W. Kahng

Prepared by:

Lockheed Electronics Company
Houston Aerospace Systems Division
Houston, Texas

Under Contract NAS 9-5384

For

National Aeronautics and Space Administration
Manned Spacecraft Center
Houston, Texas
May, 1968

Revised June 1968

80026

INTERPOLATIONS WITH A SPLINE FUNCTION

Prepared by: S. W. Kahng
S. W. Kahng
Scientific Programmer Specialist
Lockheed Electronics Company

Approved by: J. A. Barnes
J. Barnes, Supervisor
Theory and Analysis
Lockheed Electronics Company

Approved by: H. P. Decell, Jr.
H. P. Decell, Jr.
Chief
Theory and Analysis Office

National Aeronautics and Space Administration
Manned Spacecraft Center
Houston, Texas

TABLE OF CONTENTS

Summary

Necessary and sufficient conditions for a spline term to interpolate a given function at two points, and to interpolate the function and its derivative at the given point are found. In both cases, the knot is chosen out of an interval. Conditions for the above cases are equivalent when the given function is differentiable.

1. Introduction

Interpolation with a spline function has been made in many cases by first preassigning knots and then using one spline term to interpolate the given function at one point. This preassignment of knots makes the interpolation problem relatively easy. On the other hand, it excludes the possibility of fitting two points with a spline term which contains two parameters.

In this paper, we consider the spline interpolation where each knot is selected from an interval and one spline term is used to interpolate either a function at two given points or a function and its derivative at a point. Necessary and sufficient conditions for the above interpolations are given.

2. Interpolations

Consider the problem of interpolating a given continuous function $f(x)$ with a spline function

$$(1)\qquad S(x) = \sum_{i=0}^{n} a_i x^i + \sum_{j=1}^{k} b_j (x - u_j)_+^n \quad ,$$

where $x_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}.$

Since $S(x)$ has $n + 2k + 1$ parameters, one could expect, sometimes, to interplate $f(x)$ at that many points. However this involves the solution of a system of nonanalytic and nonlinear equations, a difficult problem. To simplify the problem, we consider the interpolation of two successive points with a spline term, while the knot u is chosen from an interval.

It is known that a given function $f(x)$ can always be interpolated at $n+1$ or more given points with a polynomial $P(x)$ of degree n. After this is accomplished, we try to interpolate $F(x) = f(x) - P(x)$ with a spline term $b(x - u)_+^n$ at the next two points x_1 and x_2, while u lies between the last interpolated point x_0 and x_1, or $x_0 < u < x_1 < x_2$. We then proceed to interpolate $F(x) - b(x - u)_+^n$ at the next two points with a new spline term. If such a two-point fit is not possible, one can interpolate $F(x)$ at x_1 with $b(x - x_0)_+^n$ where $b = F(x_1)/(x_1 - x_0)^n$.

We also know that a function $f(x)$ and its first derivative can be interpolated at $[(n+1)/2]$ points with a polynomial P of odd degree n. After this fit is made, one will interpolate $f(x) - P(x)$ and its first derivative at the next point x_1 with $b(x - u)_+^n$, where u lies between the last interpolated point x_0 and x_1.

The following lemma shows the necessary and sufficient condition for the existence of a spline term that interpolates two given points, with the restriction mentioned above.

Lemma 1.

Given $n \geq 1$, a continuous function $F(x)$, and three points x_0, x_1 and x_2 with $F(x_0) = 0$, and $x_0 < x_1 < x_2$, then there exists a unique spline term $b(x - u)_+^n$ such that $F(x_1) = b(x_1 - u)_+^n$, $F(x_2) = b(x_2 - u)_+^n$, $b \neq 0$, and $x_0 < u < x_1$ if and only if

$$(2) \qquad 0 < F(x_1)/F(x_2) < (x_1 - x_0)^n/(x_2 - x_0)^n .$$

Proof:

Let

$$w(v) = \frac{(x_1 - v)^n}{(x_2 - v)^n}$$

Then $w(x_1) = 0$ and $w(v)$ is strictly decreasing in $[x_0, x_1]$. Therefore if the condition (2) holds, then there exists a unique u in (x_0, x_1) with

$$\frac{F(x_1)}{F(x_2)} = \left(\frac{x_1 - u}{x_2 - u}\right)^n$$

and also a unique

$$b = \frac{F(x_1)}{(x_1 - u)^n} \neq 0$$

with

$$F(x_1) = b(x_1 - u)^n$$

and

$$F(x_2) = b(x_2 - u)^n \quad .$$

Conversely let there exist $b \neq 0$, and u so that

$$F(x_1) = b(x_1 - u)^n$$

$$F(x_2) = b(x_2 - u)^n$$

and

$$x_0 < u < x_1 \quad .$$

Then we have

$$\frac{F(x_1)}{F(x_2)} = \left(\frac{x_1 - u}{x_2 - u}\right)^n \quad .$$

Since $w(v)$ defined above is strictly decreasing in $[x_0, x_1]$ and $w(x_1) = 0$, we conclude that

$$\frac{F(x_1)}{F(x_2)} < \left(\frac{x_1 - x_0}{x_2 - x_0}\right)^n .$$

The left side is greater than zero, since $(x_1 - u)/(x_2 - u) > 0$.

The above lemma suggests a step-by-step method of determining a spline function that interpolates the given points, in succession, provided the condition (2) is met at each step.

We now consider the interpolation of the $f(x)$ and its first derivative with $S(x)$. As in the previous case, each knot is to be chosen from a preassigned interval, and one spline term is to interpolate $f(x)$ and $f'(x)$ at a point.

Lemma 2.

Given $n \geq 1$, and a function $F(x)$ with a continuous first derivative and two points x_0 and x_1, with $F(x_0) = 0$, $F(x_1) \neq 0$, and $x_0 < x_1$. Then there exists a unique spline term $b(x - u)_+^n$ such that

$$F(x_1) = b(x_1 - u)_+^n$$

$$F'(x_1) = \frac{d}{dx}(b(x - u)_+^n)\Big|_{x=x_1}$$

$$b \neq 0 , \text{ and } x_0 < u < x_1$$

if and only if

$$(3) \qquad 0 < F(x_1)/F'(x_1) < (x_1 - x_0)/n \quad .$$

Proof

i. If (3) holds true, then there exists a unique $u \varepsilon (x_0, x_1)$ satisfying

$$\frac{F(x_1)}{F'(x_1)} = \frac{x_1 - u}{n} = \frac{(x_1 - u)^n}{n(x_1 - u)^{n-1}}$$

and there exists a unique $b \neq 0$ with

$$F(x_1) = b(x_1 - u)_+^n$$

$$F'(x_1) = \frac{d}{dx}(b(x - u)_+^n)\Big|_{x=x_1} \quad .$$

ii. If there exists $b \neq 0$ and u such that

$$F(x_1) = b(x_1 - u)_+^n$$

$$F'(x_1) = nb(x_1 - u)_+^{n-1}$$

and

$$x_0 < u < x_1 \quad ,$$

then

$$\frac{F(x_1)}{F'(x_1)} = \frac{x_1 - u}{n} < \frac{x_1 - x_0}{n}$$

The left term is positive and is less than $(x_1 - x_0)/n$ since $u \epsilon (x_0, x_1)$.

In the previous two lemmas we considered interpolations only at the preassigned points. If such interpolations are to be possible at all the points in some interval I , we will require that the conditions (2) or (3) hold true at all the points in I .

Lemma 3.

Given $n \geq 1$, and a function $F(x)$ with continuous first derivative in $I = [a,b]$ with $F(x_0) = 0$, $F(x_1) \neq 0$, and $x_0 < a$. Then

(2) $$0 < F(x_1)/F(x_2) < (x_1 - x_0)^n/(x_2 - x_0)^n$$

for all x_1 and x_2 in I with $x_1 < x_2$, if and only if

(3) $$0 < F(x)/F'(x) < (x - x_0)/n$$

for $x > x_0$ in I .

Proof:

Condition (2) is true for all $x_1 < x_2$ in I if and only if $F(x)$ does not change sign and $|F(x)|/(x - x_0)^n$ is a strictly increasing function in I. This is true if and only if the first derivative of $|F(x)|/(x - x_0)^n$ is positive, or $0 < F(x)/F'(x) < (x - x_0)/n$, in I.

3. Error Term

The error term of the spline function interpolations, described in the previous sections, can be derived in a manner similar to that of the polynomial interpolation.

Lemma 4.

a. Let

$$S(x) = \sum_{i=0}^{n} a_i x^i + \sum_{j=1}^{k} b(x - u_j)_+^n$$

interpolate $f(x) \varepsilon C^{(n+1)}$ at $x_1 < x_2 < \cdots < x_{n+2k+1}$, $\pi(x) = \prod_{i=1}^{n+2k+1} (x - x_i)$. Then for all z with $z \neq x_i$, $i = 1,2,\cdots,n + 2k + 1$

$$f(z) - S(z) = \lim_{t \to \xi} \frac{f^{(n+1)}(t)}{\pi^{(n+1)}(t)} \pi(z)$$

where ξ is in the open interval that connects $x_1, x_2, \cdots, x_{n+2k+1}$, and z .

b. Let

$$S(x) = \sum_{i=0}^{n} a_i x^i + \sum_{j=1}^{k} b_j (x - u_j)_+^n$$

and $S'(x)$ interpolate $f(x) \varepsilon C^{(n+1)}$ and $f'(x)$ respectively at $x_1 < x_2 < \cdots < x_{\frac{n+1}{2}+k}$, where n is an odd integer and $k > 1$. Let $\pi(x) = \prod_{i=1}^{\frac{n+1}{2}+k} (x - x_i)^2$. Then for all z with $z \neq x_i$, $i = 1,2,\cdots,\frac{n+1}{2} + k$

$$f(z) - S(z) = \left(\lim_{t \to \xi} \frac{f^{(n+1)}(t)}{\pi^{(n+1)}(t)}\right)\pi(z)$$

where ξ is in the open set that connects $x_1,x_2,\cdots,x_{\frac{n+1}{2}+k}$, and z.

Proofs of these two lemmas are almost identical and we omit the proof of the first one.

Proof of b.

Let $N = \frac{n+1}{2} + k$, Z be a point with $Z \neq x_i$, $i = 1,2,\cdots,N$, J be an open set that connects $x_1,x_2,\cdots,x_N$, and Z.

Let

$$G(x) = f(x) - S(x) - c\pi(x) ,$$

where

(4) $$c = f(z) - S(z)/\pi(z) .$$

Then $G(x)$ has double zeros at $x_1, x_2, \cdots, x_N$, and a single zero at Z, or $2N + 1$ zeros, counting multiplicities, in J. $G'(x)$ has N zeros at $x_1, x_2, \cdots, x_N$, and N zeros at other N distinct points in J. Similarly $G^{(n-1)}(x)$ vanishes at $2N - n + 2$ points in J. We notice that $G^{(n)}(x)$ does not exist at knots u_j, $j = 1, 2, \cdots, K$.

Therefore $G^{(n)}(x)$ vanishes at least at $2N - n - K + 1 = k + 2$ points in $J' = J - \{u_1, u_2, \cdots, u_n\}$. Since J' consists of $k + 1$ open intervals, $G^{(n)}(x)$ vanishes at two points in one of these $k + 1$ open intervals. Call this interval I. Then there is a point ξ in I where $G^{(n+1)}(\xi)$ vanishes, and $f^{(n+1)}(\xi) = c\pi^{(n+1)}(\xi)$. Notice that $S(x)$ is an *nth*-degree polynomial and $S^{(n+1)}(x) = 0$ if $x \neq u_i$, $i = 1, 2, \cdots, k$.

If $\pi^{(n+1)}(\xi) \neq 0$, then

$$c = \frac{f^{(n+1)}(\xi)}{\pi^{(n+1)}(\xi)} = \lim_{t \to \xi} \frac{f^{(n+1)}(t)}{\pi^{(n+1)}(t)}$$

since $f^{(n+1)}(t)$ and $\pi^{(n+1)}(t)$ are continuous in J.

If $\pi^{(n+1)}(\xi) = 0$, then $f^{(n+1)}(\xi)$ is also equal to zero since c is finite by (4), and

$$c = \lim_{t \to \xi} \frac{f^{(n+1)}(t)}{\pi^{(n+1)}(t)} .$$

Acknowledgement

The author is indebted to Dr. H. P. Decell, Jr., and Mr. J. L. Engvall, Manned Spacecraft Center, NASA, for valuable discussions concerning this paper.