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Abstract

The problem of a long, thin cylinder on a core, subjected to a finite band of axisymmetric radial load is considered. The core is analysed by elasticity theory. The cylinder is analysed by thin-shell theory. Appropriate displacements and stresses are matched at the cylinder-core interface. After careful numerical evaluation of improper integrals, which appear in the solution, non-dimensional displacement and stress ratios for the core and cylinder are presented for different widths of uniform pressure band load. A comparison to existing solutions for the limiting case of a line load is made and shows the maximum radial stress to be significantly larger than previously reported. It is also found that an increase in the band width from the limiting case of a line load decreases the stresses and displacements in the assembly significantly even though the external resultant force in kept constant.

Introduction

In recent years the development of solid propellant rocket technology has been very rapid. Considerable attention has been given to the structural safety and reliability of the motor case and its contents, which are a major component of such rockets. The present investigation is concerned with the effect of a uniform band of load intensity p and width 2δ applied to a motor case assembly, as shown in Figure 1. Such a loading may arise when a clamp is placed over the assembly. The motor case is taken as a long, thin cylinder and its contents, consisting of solid propellant grain, is taken as a long solid cylinder or core.

The core and cylinder are assumed to behave elastically. The core is examined by the use of equations from the theory of elasticity and the cylinder by the use of equations from classical thin shell theory. The elasticity equations for the core will be written in terms of displacements and solved with the aid of displacement functions. The shell theory equations used to analyze the cylinder, are written in terms of its middle surface displacements in the radial and in the axial directions. It is then required that continuity of appropriate stresses and displacements at the interface between the core and cylinder be satisfied. In this way the cylinder unknowns may be related to the elasticity functions.

The core displacement functions satisfy partial differential equations that will be solved by a separation of variables technique. The functions may be expressed in terms of coefficients of a Fourier integral representation. When the load is also represented by a Fourier integral, the coefficients may be evaluated from the shell theory equations which constitute the boundary conditions for the evaluation of the functions.

To obtain desired stresses and displacements, improper integrals have to be evaluated. These are evaluated numerically after appropriate non-dimensionalization. A convergence test that established accuracies of one percent in the final results is incorporated in the computer program written for the numerical integration.

In the limiting case as the width of band load becomes infinitely small, the assembly is loaded by an axisymmetric line load. The limiting case is of special interest and the results obtained are compared with those obtained by previous investigators. Comparison of the band load results obtained herein is also made to a plane strain solution for a finite cylinder.

Field Equations of the Core

The governing elasticity equations for the axisymmetric solid cylinder, or core, problem shown in Figure 1 are given below. For the cylindrical coordinate system (r,θ,x) and W and U representing the radial and axial displacement components respectively, the strain-displacements equations may be written[1]*

$$\varepsilon_r = \frac{\partial W}{\partial r} \;;\quad \varepsilon_\theta = \frac{W}{r} \;;\quad \varepsilon_x = \frac{\partial U}{\partial x} \;;\quad \varepsilon_{rx} = \frac{1}{2}\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial r}\right) \qquad \text{(1a-1d)}$$

where, ε_r, ε_θ, ε_x and ε_{rx} are the radial, circumferential, axial and shear strain components respectively. The stress-strain relations are

$$\sigma_r = \lambda e + 2G\varepsilon_r \;;\quad \sigma_\theta = \lambda e + 2G\varepsilon_\theta \qquad \text{(2a,2b)}$$

$$\sigma_x = \lambda e + 2G\varepsilon_x \;;\quad \tau_{rx} = 2G\varepsilon_{rx} \qquad \text{(2c,2d)}$$

where, σ_r, σ_θ, σ_x, and τ_{rx} are the radial, circumferential, axial and shear stress components respectively. The term e is three times the mean normal strain so that

$$e = \varepsilon_r + \varepsilon_\theta + \varepsilon_x \qquad (3)$$

The terms G and λ are the Lamé material property constants. They are related to the modulus of elasticity, E_c, and Poissons's ratio, ν, by the relations

$$\lambda = \frac{\nu E_c}{(1+\nu)(1-2\nu)} \qquad (4)$$

$$G = \frac{E_c}{2(1+\nu)} \qquad (5)$$

The stress components satisfy the equilibrium relations[1]

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rx}}{\partial x} - \frac{\sigma_\theta - \sigma_r}{r} = 0 \qquad \text{(6a)}$$

$$\frac{\partial \tau_{rx}}{\partial r} + \frac{\partial \sigma_x}{\partial x} + \frac{\tau_{rx}}{r} = 0 \qquad \text{(6b)}$$

When the strain-displacement relations of Equations (1) and (3) are substituted into the stress-strain relations of Equations (21) one obtains the stress-displacement equations.

$$\sigma_r = \frac{G}{\alpha-1}\left[\alpha\frac{\partial W}{\partial r} + (2-\alpha)\left(\frac{W}{r} + \frac{\partial U}{\partial x}\right)\right] \qquad \text{(7a)}$$

*Numbers in superscripted brackets [ ] designate references in the Bibliography.

$$\sigma_\theta = \frac{G}{\alpha-1}\left[(2-\alpha)\left(\frac{\partial W}{\partial r} + \frac{\partial U}{\partial x}\right) + \alpha\frac{W}{r}\right] \tag{7b}$$

$$\sigma_x = \frac{G}{\alpha-1}\left[(2-\alpha)\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right) + \alpha\frac{\partial U}{\partial x}\right] \tag{7c}$$

$$\tau_{rx} = G\left[\frac{\partial W}{\partial x} + \frac{\partial U}{\partial r}\right] \tag{7d}$$

where the quantity α is defined as

$$\alpha = 2(1-\nu) \tag{8}$$

Substitution of the stress-displacement relations of Equations (7) into the equilibrium relations of Equations (6) expresses the equilibrium equations in terms of the displacement components

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} - \frac{W}{r^2} + \left(1-\frac{1}{\alpha}\right)\frac{\partial^2 W}{\partial x^2} + \frac{1}{\alpha}\frac{\partial^2 U}{\partial r \partial x} = 0 \tag{9a}$$

$$\frac{\partial^2 U}{\partial x^2} + \left(1-\frac{1}{\alpha}\right)\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) + \frac{1}{\alpha}\left(\frac{\partial^2 W}{\partial r \partial x} + \frac{1}{r}\frac{\partial W}{\partial x}\right) = 0 \tag{9b}$$

These equations were previously given by Marguerre[2], with a different notation, as his Equations (23').

Two functions (f_1 and f_2) of the coordinates r and x are now introduced in the following manner

$$W = \frac{1}{2G}\left[-\frac{\partial f_1}{\partial r} - r\frac{\partial f_2}{\partial r} + (2\alpha-1) f_2\right] \tag{10a}$$

$$U = -\frac{1}{2G}\left[\frac{\partial f_1}{\partial x} + r\frac{\partial f_2}{\partial x}\right] \tag{10b}$$

The introduction of displacement functions in this manner has been referred to as a Two Function approach[3,4]. In these references it is shown that substitution of Equations (10) into (9) leads to the following relations that the functions f_1 and f_2 must satisfy

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} + \frac{\partial^2 f_1}{\partial x^2} = 0 \tag{11a}$$

$$\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{1}{r^2} f_2 + \frac{\partial^2 f_2}{\partial x^2} = 0 \qquad (11b)$$

The stress components may be written in terms of f_1 and f_2 by use of Equations (7). In view of Equations (10), the stress relations are

$$\sigma_r = -\frac{\partial^2 f_1}{\partial r^2} - r\frac{\partial^2 f_2}{\partial r^2} + \alpha\frac{\partial f_2}{\partial r} \qquad (12a)$$

$$\sigma_\theta = -\frac{1}{r}\frac{\partial f_1}{\partial r} + (1-\alpha)\frac{\partial f_2}{\partial r} + (1+\alpha)\frac{f_2}{r} \qquad (12b)$$

$$\sigma_x = -\frac{\partial^2 f_1}{\partial x^2} - r\frac{\partial^2 f_2}{\partial x^2} + (2-\alpha)\left(\frac{\partial f_2}{\partial r} + \frac{f_2}{r}\right) \qquad (12c)$$

$$\tau_{rx} = -\frac{\partial^2 f_1}{\partial r \partial x} - r\frac{\partial^2 f_2}{\partial r \partial x} + (\alpha-1)\frac{\partial f_2}{\partial x} \qquad (12d)$$

Equations (9) and (12) constitute the relationship of the displacement and stress components of the axisymmetric problem to the displacement functions f_1 and f_2 of the Two Function approach.

Thin Wall Cylinder

Classical thin-shell theory for a cylinder is often associated with the names of Timoshenko[5] and Donnell[6]. The derivation of the governing equations of this theory will be presented herein based on a simplification of the related problem as formulated by the theory of elasticity. This simplification is accomplished by the use of the following assumptions. In terms of geometry, stress, and displacement, the assumptions are made that

(i) the wall thickness is considered negligible in comparison to the mean radius.

(ii) the normal stress in the direction of small dimension is small as compared to the normal stresses in the other dimensions and may be neglected.

(iii) plane sections initially normal to the undeformed middle surface remain unextended and plane and normal to the deformed middle surface.

The elasticity Equations (1), (2), and (6) can be rewritten in terms of the shell theory notation for the z, y, x coordinate system shown in Figure 1, so that the strain-displacement relations are

$$\varepsilon_{zz} = \frac{\partial w_z}{\partial z} \quad ; \quad \varepsilon_{yy} = \frac{w_z}{R+z} \quad ; \quad \varepsilon_{xx} = \frac{\partial u_z}{\partial x} \tag{13a-13c}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w_z}{\partial x} + \frac{\partial u_z}{\partial z}\right) \tag{13d}$$

The stress-strain relations are

$$\sigma_{zz} = \lambda_s \bar{e} + 2G_s \varepsilon_{zz} \quad ; \quad \sigma_{yy} = \lambda_s \bar{e} + 2G_s \varepsilon_{yy} \tag{14a,14b}$$

$$\sigma_{xx} = \lambda_s \bar{e} + 2G_s \varepsilon_{xx} \quad ; \quad \tau_{zx} = 2G_s \varepsilon_{zx} \tag{14c,14d}$$

while the equilibrium relations are

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} - \frac{\sigma_{yy} - \sigma_{zz}}{R+z} = 0 \tag{15a}$$

$$\frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\tau_{zx}}{R+z} = 0 \tag{15b}$$

where, three times the mean normal strain $\bar{e}$ is related to the radial, circumferential, axial and shear strains respectively, by the relation

$$\bar{e} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{16}$$

The quantities w_z and u_z are the radial and axial displacements respectively and R is the radius to the middle surface of the cylinder. The terms G_s and λ_s are the material properties with the same definitions as given by Equations (4) and (5) where E is the modulus of elasticity of the cylinder and Poisson's ratio is given by μ.

Shell theory assumption (ii) indicates that $\sigma_{zz} \ll \sigma_{xx}$ and $\sigma_{zz} \ll \sigma_{yy}$ so that σ_{zz} may be neglected. Rather than deal directly with the surviving stress components σ_{xx}, σ_{yy}, and τ_{zx}, it is more convenient to deal with resultants. They are defined as forces or moments per unit length of middle surface. N_x, N_y are the normal force resultants, Q_x is the shear force resultant, while M_x, M_y are the bending moment resultants. They are related to the stress components by the definitions

$$N_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx}(1 + \frac{z}{R})dz \quad ; \quad N_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy}dz \qquad (17a,17b)$$

$$Q_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{zx}(1 + \frac{z}{R})dz \qquad (18)$$

$$M_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx}(1 + \frac{z}{R})zdz \quad ; \quad M_y = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{yy}zdz \qquad (19a,19b)$$

where, z is the radial coordinate measured radially outward from the mean surface of the cylinder and t is its wall thickness. The directions of these forces and bending moments are indicated in Figure 2. Shell theory assumption (i) states that $t/R << 1$ and since $|z| \leq \frac{t}{2}$ the quantities z/R may also be neglected in comparison to unity. Thus, Equations (17a), (18) and (19a) can be written simply as

$$N_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx}dz \qquad (20)$$

$$Q_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{zx}dz \qquad (21)$$

$$M_x = -\int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{xx}zdz \qquad (22)$$

Shell theory assumption (iii) is generally called the Love-Kirchoff hypothesis for shells. From the assumption of inextensibility of the cross-section, the radial displacement at a distance z from the middle surface is equal to the radial displacement at the middle surface. From the assumptions that plane sections remain plane and that sections initially normal remain normal, the axial displacement at a distance z from the middle surface can be related to the displacements at the middle surface. Denoting the radial and axial displacements at the middle surface by w and u respectively, one concludes that

$$w_z = w \quad ; \quad u_z = u - z\frac{\partial w}{\partial x} \tag{23a,23b}$$

Sutstitution of these equations into Equations (13) and neglecting z compared to R in Equation (13b) by shell assumption (i) the strain components become

$$\varepsilon_{zz} = 0 \quad ; \quad \varepsilon_{yy} = \frac{w}{R} \tag{24a,24b}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - \frac{z\partial^2 w}{\partial x^2} \quad ; \quad \gamma_{zx} = 0 \tag{24c,24d}$$

Since it has been assumed that $\sigma_{zz} << \sigma_{xx}$, and $\sigma_{zz} << \sigma_{yy}$ so that σ_{zz} vanishes, the constitutive elasticity Equations (14b) and (14c) together with Equation (14a) lead to

$$\sigma_{xx} = \frac{E}{1-\mu^2}\,[\varepsilon_{xx} + \mu\varepsilon_{yy}] \tag{25a}$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}\,[\varepsilon_{yy} + \mu\varepsilon_{xx}] \tag{25b}$$

The stress-strain equations which involve z quantities (Equations (14a), (14d)) can be seen to be violated. This is an expected consequence of the additional assumptions that have been made for shell theory for quantities in the z direction. Substitution of the strain-middle surface displacement relations of Equations (24b), (24c) into the stress-strain relations of Equations (25) gives the stress-middle surface displacement equations

$$\sigma_{xx} = \frac{E}{1-\mu^2}\,[\frac{\partial u}{\partial x} - \frac{z\partial^2 w}{\partial x^2} + \frac{\mu w}{R}] \tag{26a}$$

$$\sigma_{yy} = \frac{E}{1-\mu^2}\left[\frac{w}{R} + \frac{\mu\partial u}{\partial x} - \frac{\mu z\partial^2 w}{\partial x^2}\right] \tag{26b}$$

Substitution of the foregoing into Equations (17b), (19b), (20) and (22) which define the stress resultants, and performance of the indicated integrations, shows that the stress resultants are related to the middle surface displacements as

$$N_x = \frac{Et}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \frac{\mu w}{R}\right] \; ; \; N_y = \frac{Et}{1-\mu^2}\left[\frac{\mu\partial u}{\partial x} + \frac{w}{R}\right] \qquad (27a,27b)$$

$$M_x = \frac{D\partial^2 w}{\partial x^2} \; ; \; M_y = \mu\frac{D\partial^2 w}{\partial x^2} \qquad (27c,27d)$$

where, the flexural rigidity of the shell D is defined

$$D = \frac{Et^3}{12(1-\mu^2)} \tag{28}$$

Equilibrium requirements in the x and z directions as well as moment equilibrium about an axis in the y direction, as shown in Figure 2, demands

$$\frac{\partial N_x}{\partial x} + p_x = 0 \tag{29a}$$

$$\frac{N_y}{R} + \frac{\partial Q_x}{\partial x} - p_z = 0 \tag{29b}$$

$$\frac{\partial M_x}{\partial x} - Q_x = 0 \tag{29c}$$

where, p_z and p_z are the radial and axial loads per unit middle surface area. These are shown in Figure 2. Substitution of Equation (29c) into Equation (29b) gives

$$\frac{N_y}{R} + \frac{\partial^2 M_x}{\partial x^2} - p_z = 0 \tag{30}$$

Substitution of Equations (27a-27c) into Equations (29a) and (30) gives

$$\frac{R^2\partial^2 u}{\partial x^2} + \frac{\mu R\partial w}{\partial x} + (1-\mu^2)\,\frac{R^2}{Et}\,p_x = 0 \tag{31a}$$

$$\frac{\mu R\partial u}{\partial x} + w + \frac{R^2 t^2}{12}\,\frac{\partial^4 w}{\partial x^4} - (1-\mu^2)\,\frac{R^2}{Et}\,p_z = 0 \tag{31b}$$

When no axial surface load is present (i.e. $p_x = 0$) these equations reduce to the single familiar equations for axisymmetric cylindrical shell problems and beam on elastic foundation problems. This is shown and used in Appendix 1.

Boundary Conditions

Equations (10) and (12) for the core and Equations (26) and (31) for the cylinder are the relations that must be satisfied to obtain solutions for the stresses and displacements. The core unknowns have been expressed in terms of the displacement functions f_1 and f_2. The boundary conditions provide the means of relating the cylinder unknowns to these functions and necessary relations for the evaluation of the functions themselves.

The following boundary conditions for the cylinder-core problem must be satisfied. The stresses and displacements must all remain finite as $x \to \pm \infty$. Continuity of radial and shear stress as well as continuity of radial and axial displacements must exist at the cylinder-core interface for all values of axial coordinate. Let the core radial and shear stresses at the outer radius of the core ($r = a$) be designated σ_a and τ_{ax} and let the radial and axial displacements at the same surface be designated U_a and W_a. The stresses at the cylinder-core interface are assumed to act on the cylinder middle surface. For an axisymmetric externally applied radial load on the cylinder $p^*(x)$, chosen outward positive in accord with the coordinate system, the cylinder unit loads are

$$p_z = p^*(x) - \sigma_a \quad ; \quad p_x = -\tau_{ax} \qquad (32a,32b)$$

From Equations (23) the cylinder displacements at the interface are given by

$$w_z \Big|_{z = -\frac{t}{2}} = w \quad ; \qquad u_z \Big|_{z = -\frac{t}{2}} = u + \frac{t}{2}\frac{\partial w}{\partial x} \qquad (33a,33b)$$

Since the left hand side of Equations (33) must equal W_a and U_a respectively, by the displacement continuity conditions above, the middle surface displacements of the cylinder may be written as

$$w = W_a \quad ; \qquad u = U_a - \frac{t}{2}\frac{\partial W_a}{\partial x} \tag{34a,34b}$$

When the condition imposed by Equation (34b) is used the results obtained are referred to as an interface matching solution.

It is in the boundary condition expressed by Equation (34a) that previous investigators of the line load solution[7,8] assumed

$$u = U_a \tag{35}$$

When this boundary condition is used the results obtained are referred to as a middle surface matching solution. The axisymmetric line loading of a cylinder with no core helps to shed some light on the difference in the matching. In Figure 3 the axial variations of the axial displacement parameter $2Gu_z/q \times 10^3$ are plotted for the cylinder inner and middle surfaces. The quantity q is the resultant force per unit circumferential length. The results in this figure were obtained from shell theory expressions derived in Appendix 1. The effect of the difference in the use of Equation (34b) or (35) in the final results of the cylinder-core combination will be evidenced once numerical results are obtained.

In addition to the boundary conditions for the stresses and displacements it is necessary to relate the radii of the outer radius of the core, a, and the radius to the middle surface, R. Since

$$a = R\left(1 - \frac{t}{2R}\right) \tag{36}$$

and it is assumed that $t/R \ll 1$ for thin-shell theory it is taken that

$$a = R \tag{37}$$

The cylinder displacements can be expressed in terms of the elasticity stress functions f_1 and f_2 through the substitution of Equations (10) and

(34) into Equations (33). No additional information for the radial displacement would result since this can be evaluated as the core radial displacement at the outer core radius. When the following definitions are made

$$f_{1a} = f_1\Big|_{r=a} \quad ; \qquad f_{2a} = f_2\Big|_{r=a} \tag{38a,38b}$$

the axial displacement is given as

$$u_z = -\frac{1}{2G}\left[f_{1a,x} + af_{2a,x} - (z + \frac{t}{2})(f_{1a,rx} + af_{2a,rx} - (2\alpha-1)\, f_{2a,x})\right] \tag{39}$$

The cylinder stresses can similarly be expressed in terms of f_{1a} and f_{2a} defined in Equations (38). By the substitution of Equations (10), (34), (37) and (39) into Equations (26), it is found that

$$\sigma_{xx} = G_m\{-[f_{1a,xx}+af_{2a,xx}] + (z + \frac{t}{2})[f_{1a,rxx}+af_{2a,rxx}-(2\alpha-1)f_{2a,xx}] - \frac{\mu}{a}[f_{1a,r} + af_{2a,r} - (2\alpha-1)f_{2a}]\} \tag{40a}$$

$$\sigma_{yy} = G_m\{ -\frac{1}{a}[f_{1a,r} + af_{2a,r} - (2\alpha-1)f_{2a}] - \mu[f_{1a,xx} + af_{2a,xx}] + \mu(z + \frac{t}{2})[f_{1a,rxx} + af_{2a,rxx} - (2\alpha-1)f_{2a,xx}]\} \tag{40b}$$

where the quantity G_m is defined by

$$G_m = \frac{1}{2(G/E)(1-\mu^2)} \tag{41}$$

Equations (39) and (40) thus express the cylinder axial displacement and stresses in terms of the elasticity functions f_1 and f_2.

<u>Unknown Coefficients and Their Evaluation for a Band Load</u>

The solution for the core unknowns can be achieved with the aid of the functions f_1, f_2 which must satisfy Equations (11). These functions can be solved for by a separation of variables technique [3,4]. If even functions of the axial coordinate are considered the expressions for f_1 and f_2 can be written

$$f_1 = a \int_0^\infty A\, I_o\,(n\,\frac{r}{a})\,\cos\,(n\,\frac{x}{a})\,dn \tag{42a}$$

$$f_2 = \int_0^\infty B\, I_1\,(n\,\frac{r}{a})\,\cos\,(n\,\frac{x}{a})\,dn \tag{42b}$$

where A and B are functions of n and I_o and I_1 are modified Bessel Functions of the first kind or order one. The quantities A, B depend upon the continuity relations between the core and cylinder. Before the evaluation of these quantities is considered, it is beneficial to express all the displacements and stresses of interest in terms of A and B.

For the core, substitution of Equations (42) into the displacement an stress relations of Equations (10) and (12) and noting the derivatives of the Bessel functions

$$\frac{\partial[I_o\,(n\,\frac{r}{a})]}{\partial r} = \frac{n}{a}\,I_1(n\,\frac{r}{a}) \tag{43a}$$

$$\frac{\partial[I_1\,(n\,\frac{r}{a})]}{\partial r} = \frac{n}{a}\,I_o\,(n\,\frac{r}{a}) - \frac{1}{r}\,I_1(n\,\frac{r}{a}) \tag{43b}$$

it is found that the radial and axial displacements of the core are given by the expressions

$$2GU = \int_0^\infty n\,[AI_o\,(n\frac{r}{a}) + B\frac{r}{a}\,I_1\,(n\frac{r}{a})]\,\sin\,(n\frac{x}{a})\,dn \tag{44a}$$

$$2GW = \int_0^\infty \{-AnI_1(n\frac{r}{a}) + B\,[2\alpha\,I_1\,(n\frac{r}{a}) - n\frac{r}{a}\,I_o(n\frac{r}{a})]\}\,\cos\,(n\frac{x}{a})\,dn \tag{44b}$$

The radial, circumferential, axial and shear stresses within the core are given by the relations

$$\sigma_r = \int_0^\infty \{A\,[\,-\,\frac{n^2}{a}\,I_o(n\frac{r}{a}) + \frac{n}{r}\,I_1\,(n\frac{r}{a})]$$

$$-B\,[(\frac{n^2 r}{a^2} + \frac{2\alpha}{r})I_1\,(n\frac{r}{a}) - (1+\alpha)\,\frac{n}{a}\,I_o(n\frac{r}{a})]\}\,\cos\,(n\frac{x}{a})\,dn \tag{45a}$$

$$\sigma_\theta = \int_0^\infty \{-A\frac{n}{r} I_1(n\frac{r}{a}) + B[(1-\alpha)\frac{n}{a} I_o(n\frac{r}{a}) + \frac{2\alpha}{r} I_1(n\frac{r}{a})]\} \cos(n\frac{x}{a})\, dn \quad (45b)$$

$$\sigma_x = \int_0^\infty \{ A\frac{n^2}{a} I_o(n\frac{r}{a}) + B[(2-\alpha)\frac{n}{a} I_o(n\frac{r}{a}) + \frac{n^2 r}{a^2} I_1(n\frac{r}{a})]\} \cos(n\frac{x}{a})\, dn \quad (45c)$$

$$\tau_{rx} = \int_0^\infty \{A\frac{n^2}{a} I_1(n\frac{r}{a}) - B[\alpha\frac{n}{a} I_1(n\frac{r}{a}) - n^2 \frac{r}{a^2} I_o(n\frac{r}{a})]\} \sin(n\frac{x}{a})\, dn \quad (45d)$$

It can be noted that similar expressions for the displacements and the shear stress were given by Yao[7] for the line load solution. For the line load solution particular values of the coefficients A and B apply. A discrepancy in the radial stress expression he used exists. This was carried through by Yao in his subsequent equations and calculations.

Since the quantities U_a, W_a, σ_a and τ_{ax} are necessary for the boundary conditions expressed by Equations (32) and (34) they may be written simply as

$$2GU_a = \int_0^\infty n[AI_o(n) + BI_1(n)] \sin(n\frac{x}{a})\, dn \quad (46a)$$

$$2GW_a = \int_0^\infty \{-AnI_1(n) + B[2\alpha I_1(n) - nI_o(n)]\} \cos(n\frac{x}{a})\, dn \quad (46b)$$

$$\sigma_a = \frac{1}{a}\int_0^\infty \{A[nI_1(n) - n^2 I_o(n)] - B[(n^2+2\alpha) I_1(n) - (1+\alpha)nI_o(n)]\} \cos(n\frac{x}{a})\, dn \quad (46c)$$

$$\tau_{ax} = \frac{1}{a}\int_0^\infty \{An^2 I_1(n) - B[\alpha n I_1(n) - n^2 I_o(n)]\} \sin(n\frac{x}{a})\, dn \quad (46d)$$

The cylinder axial displacement and stress expressions in terms of A and B can be arrived at by the substitution of Equations (42) into Equations (39) and (40). This gives

$$u_z = \frac{1}{2G}\int_0^\infty \{A\ [nI_o(n) - \eta I_1(n)]$$
$$+B\ [nI_1(n) + \eta(2\alpha \frac{I_1(n)}{n} - I_o(n))]\}\ \sin\ (n\frac{x}{a})\ dn \qquad (47a)$$

$$\sigma_{xx} = G_m \int_0^\infty \{A\ [\ \frac{n^2}{a} I_o(n) - \frac{n}{a}(\mu+\eta)\ I_1(n)]$$
$$-B\ [\ \frac{n}{a}(\mu+\eta)\ I_o(n) - \frac{1}{a}(n^2+2\alpha(\mu+\eta))\ I_1(n)]\}\ \cos\ (n\frac{x}{a})\ dn \qquad (47b)$$

$$\sigma_{yy} = G_m \int_0^\infty \{A\ [\ \frac{\mu n^2}{a} I_o(n) - \frac{n}{a}(1+\mu\eta) I_1(n)]$$
$$-B\ [\ \frac{n}{a}(1+\mu\eta) I_o(n) - \frac{1}{a}(\mu n^2+2\alpha(1+\mu\eta)) I_1(n)]\}\ \cos\ (n\frac{x}{a})\ dn \qquad (47c)$$

where the quantity η is given by

$$\eta = \frac{n^2(2\frac{z}{t}+1)}{2(\frac{a}{t})} \qquad (48)$$

From Equations (44), (45) and (47) and a mathematical theorem presented in Sneddon[9] the displacements and stresses can be shown to satisfy the boundary condition requirement that they remain finite as the axial coordinate goes to infinity.

The radial load, which up to this point could be any general axisymmetric distribution, can be represented by a Fourier integral. In correspondence with the forms of the expressions for f_1 and f_2 given by Equations (42) the band load representation is taken as

$$p^*(x) = \frac{1}{\pi a}\int_0^\infty C\cos\ (n\frac{x}{a})\ dn \qquad (49)$$

where, C is a function of n given by

$$C = \int_{-\infty}^\infty p^*(x)\ \cos(n\frac{x}{a})\ dx \qquad (50)$$

This expression for the Fourier integral representation of an even function, to which the load is restricted, may be found in any mathematics text in which Fourier integrals are discussed (e.g. Kreyszig[10]). For the particular case of a band load of uniform pressure p, C reduces to

$$C = -q \frac{\sin(n\frac{\delta}{a})}{(n\frac{\delta}{a})} \tag{51}$$

where q is the resultant force per unit circumferential length as shown in Figure 1 and 2δ is the total axial width of the load with

$$q = 2\delta p \tag{52}$$

Substitution of Equations (32),(34), (37), (46), (49) and (51) into Equations (31) leads to two integral relations containing A and B, namely

$$\int_0^\infty - \frac{n}{2G} [\lambda_{11}A + \lambda_{12}B] \sin(n\frac{x}{a})\, dn = 0 \tag{53a}$$

$$\int_0^\infty \frac{1}{2G} [\lambda_{21}A + \lambda_{22}B - \zeta] \cos(n\frac{x}{a})\, dn = 0 \tag{53b}$$

where the parameters

$$\lambda_{11} = -n^2 I_o(n) + (g + \frac{n^2 t}{2a})n\, I_1(n) \tag{54a}$$

$$\lambda_{12} = (g + \frac{n^2 t}{2a})\, n\, I_o(n) - [n^2(1+ \frac{\alpha t}{a}) + \alpha(\mu+g)]\, I_1(n) \tag{54b}$$

$$\lambda_{21} = n^2 g\, I_o(n) - [1+ \frac{\mu n^2 t}{2a} + kn^4 + g-\mu]n\, I_1(n) \tag{54c}$$

$$\lambda_{22} = -[1 + \frac{\mu n^2 t}{2a} + kn^4 -(1+\alpha)(\mu-g)]n\, I_o(n)$$
$$+ [(n^2+2\alpha)g + 2\alpha\, (1-\mu+kn^4) + \frac{\mu n^2 \alpha t}{a}]I_1(n) \tag{54d}$$

and the material and geometric parameters ζ, k and g are given by

$$\zeta = \frac{g-\mu}{\pi} \frac{\sin(n\frac{\delta}{a})}{n\frac{\delta}{a}}\, q \tag{55a}$$

$$k = \frac{t^2}{12a^2} \tag{55b}$$

$$g = \mu - 2(1-\mu^2)\, \frac{G}{E}\, \frac{a}{t} \tag{55c}$$

If it is taken that the axial displacement matching condition is the middle surface matching given by Equation (35) rather than Equation (34b) some of the terms of Equations (54) would not apply, but instead one would find

$$\lambda_{11} = -n^2 I_0(n) + gn\, I_1(n) \tag{56a}$$

$$\lambda_{12} = gnI_0(n) - [n^2 + \alpha(\mu+g)]\, I_1(n) \tag{56b}$$

$$\lambda_{21} = n^2 g\, I_0(n) - [1 + kn^4 + g-\mu]n\, I_1(n) \tag{56c}$$

$$\lambda_{22} = -[1 + kn^4 + (1+\alpha)(\mu-g)]n\, I_0(n) + [(n^2+2\alpha)g + 2\alpha(1-\mu+kn^4)]\, I_1(n) \tag{56d}$$

To satisfy Equations (53) it is necessary that

$$\lambda_{11}A + \lambda_{12}B = 0 \tag{57a}$$

$$\lambda_{21}A + \lambda_{22}B = \zeta \tag{57b}$$

The direct evaluation of the coefficients A and B which are functions of n is not necessary. Substitution of the algebraic solution of the simple linear relations of Equations (57) shows that

$$A = -\frac{\lambda_{21}\zeta}{D_\lambda} \quad ; \quad B = \frac{\lambda_{11}\zeta}{D_\lambda} \tag{58a,58b}$$

where the denominator, D_λ is given by

$$D_\lambda = \begin{vmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{vmatrix} = \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} \tag{59}$$

Appropriate values of A and B can be substituted into the desired displacement and stress expressions. When the values of A and B given by Equations (59) are substituted into Equations (44), (45), and (47) the final forms of the expression to be evaluated numerically result. These are easily obtained and are all included here in nondimensional form for completeness.

The expressions for core displacements are given in nondimensional form as

$$\frac{2GU}{q} = \int_0^\infty Un\, \frac{\bar{\zeta}}{D_\lambda} \sin(n\frac{x}{a})\, dn \tag{60a}$$

$$\frac{2GW}{q} = \int_0^\infty W_n \frac{\bar{\zeta}}{D_\lambda} \cos\,(n\frac{x}{a})\,dn \qquad (60b)$$

where the terms under the integral are

$$U_n = n[-\lambda_{12} I_o(n\frac{r}{a}) + \lambda_{11}\frac{r}{a} I_1(n\frac{r}{a})] \qquad (61a)$$

$$W_n = \lambda_{12} n\, I_1(n\frac{r}{a}) + \lambda_{11}[2\alpha\, I_1(n\frac{r}{a}) - n\frac{r}{a} I_o(n\frac{r}{a})] \qquad (61b)$$

and the parameter $\bar{\zeta}$ is defined

$$\bar{\zeta} = \frac{\zeta}{q} \qquad (62)$$

The corresponding nondimensional core stress relations are

$$\frac{a\sigma_r}{q} = \int_0^\infty (\sigma_r)_n \frac{\bar{\zeta}}{D_\lambda} \cos(n\frac{x}{a})\,dn \qquad (63a)$$

$$\frac{a\sigma_\theta}{q} = \int_0^\infty (\sigma_\theta)_n \frac{\bar{\zeta}}{D_\lambda} \cos(n\frac{x}{a})\,dn \qquad (63b)$$

$$\frac{a\sigma_x}{q} = \int_0^\infty (\sigma_x)_n \frac{\bar{\zeta}}{D_\lambda} \cos(n\frac{x}{a})\,dn \qquad (63c)$$

$$\frac{a\tau_{rx}}{q} = \int_0^\infty (\tau_{rx})_n \frac{\bar{\zeta}}{D_\lambda} \sin(n\frac{x}{a})\,dn \qquad (63d)$$

where the terms under the integral are

$$(\sigma_r)_n = \lambda_{12}[n^2 I_o(n\frac{r}{a}) - \frac{n}{r/a} I_1(n\frac{r}{a})] - \lambda_{11}[(n^2\frac{r}{a} + \frac{2\alpha}{\frac{r}{a}})\, I_1(n\frac{r}{a}) - (1+\alpha) n I_o(n\frac{r}{a})] \qquad (64a)$$

$$(\sigma_\theta)_n = \lambda_{12}\frac{n}{\frac{r}{a}} I_1(n\frac{r}{a}) + \lambda_{11}[(1-\alpha) n I_o(n\frac{r}{a}) + \frac{2\alpha}{\frac{r}{a}} I_1(n\frac{r}{a})] \qquad (64b)$$

$$(\sigma_x)_n = -\lambda_{12} n^2 I_o(n\frac{r}{a}) + \lambda_{11}[(2-\alpha) n I_o(n\frac{r}{a}) + n^2\frac{r}{a} I_1(n\frac{r}{a})] \qquad (64c)$$

$$(\tau_{rx})_n = -\lambda_{12} n^2 I_1(n\frac{r}{a}) - \lambda_{11}[\alpha n I_1(n\frac{r}{a}) - n^2\frac{r}{a} I_o(n\frac{r}{a})] \qquad (64d)$$

The computed core values to be presented herein are to be evaluated at the interface, r = a, only. For this case, Equations (61) for the displacements and Equations (64) for the stresses reduce to

$$U_n\Big|_{r=a} = n[-\lambda_{12}I_o(n) + \lambda_{11}I_1(n)] \tag{65a}$$

$$W_n\Big|_{r=a} = \lambda_{12}nI_1(n) + \lambda_{11}[2\alpha I_1(n) - nI_o(n)] \tag{65b}$$

$$(\sigma_r)_n\Big|_{r=a} = \lambda_{12}[n^2 I_o(n) - nI_1(n)] - \lambda_{11}[(n^2+2\alpha)I_1(n) - (1+\alpha)nI_o(n)] \tag{65c}$$

$$(\sigma_\theta)_n\Big|_{r=a} = \lambda_{12}nI_1(n) + \lambda_{11}[(1-\alpha)nI_o(n) + 2\alpha I_1(n)] \tag{65d}$$

$$(\sigma_x)_n\Big|_{r=a} = -\lambda_{12}n^2 I_o(n) + \lambda_{11}[(2-\alpha\ nI_o(n) + n^2 I_1(n)] \tag{65e}$$

$$(\tau_{rx})_n\Big|_{r=a} = -\lambda_{12}n^2 I_1(n) - \lambda_{11}[\alpha nI_1(n) - n^2 I_o(n)] \tag{65f}$$

The nondimensional cylinder axial displacement and stresses are given by

$$\frac{2Gu_z}{q} = \int_o^\infty (u_z)_n \frac{\bar{\zeta}}{D_\lambda} \sin(n\frac{x}{a})\, dn \tag{66a}$$

$$\frac{a\sigma_{xx}}{q} = G_m \int_o^\infty (\sigma_{xx})_n \frac{\bar{\zeta}}{D_\lambda} \cos(n\frac{x}{a})\, dn \tag{66b}$$

$$\frac{a\sigma_{yy}}{q} = G_m \int_o^\infty (\sigma_{yy})_n \frac{\bar{\zeta}}{D_\lambda} \cos(n\frac{x}{a})\, dn \tag{66c}$$

where the terms under the integral are

$$(u_z)_n = -\lambda_{12}[nI_o(n) - \eta I_1(n)] + \lambda_{11}[nI_1(n) + \eta(\frac{2\alpha I_1(n)}{n} - I_o(n))] \tag{67a}$$

$$(\sigma_{xx})_n = -\lambda_{12}[n^2 I_o(n) - n(\mu+\eta)I_1(n)] - \lambda_{11}[n(\mu+\eta)\, I_o(n) - (n^2+2\alpha(\mu+\eta))I_1(n)] \tag{67b}$$

$$(\sigma_{yy})_n = -\lambda_{12}[\mu n^2 I_o(n) - n(1+\mu\eta)I_1(n)]$$

$$-\lambda_{11}[n(1+\mu\eta)I_o(n) - (\mu n^2 + 2\alpha(1+\mu\eta))I_1(n)] \qquad (67c)$$

The computed cylinder values to be presented herein are the stresses which are evaluated at the cylinder outer surface, $\frac{2z}{t} = 1$, only. For this, the quantity η from Equation (48) reduces to

$$\eta\Big|_{\frac{2z}{t}=1} = \frac{n^2}{(\frac{a}{t})} \qquad (68)$$

Computational Difficulties

The formal solution to the band of load on a cylinder and core assembly as derived in the previous section is sufficiently complicated to cause computational difficulties. The methods used to overcome these difficulties will be discussed in the present section.

It can be seen from Equations (60), (63) and (66) that a typical integral requiring solution may be expressed as the nondimensional quantity $\bar{S}$ defined as

$$\bar{S} = \int_0^\infty s\{(\frac{g-\mu}{\pi}) \frac{\sin(n\frac{\delta}{a})}{(n\frac{\delta}{a})}\}\left\{\begin{matrix}\sin(n\frac{x}{a}) \\ \text{or} \\ \cos(n\frac{x}{a})\end{matrix}\right\} dn \qquad (69)$$

where, s is the quotient of two expressions that involve powers and modified Bessel functions of n. It may be written as

$$s = \frac{s_n}{D_\lambda} \qquad (70a)$$

where s_n takes on values given by Equations (61), (64), and (67). General computational difficulties encountered in elasticity solutions for cylinders were discussed in a previous report by the authors[3]. In that report, it was indicated how the modified Bessel functions can be evaluated and the general method to be used herein for the accurate evaluation of the improper integrals was given. At this point the details of the numerical evaluation of the particular typical integral of the cylinder-core problem in Equation (59) are given. The difficulties that arise are caused by the function s and by the presence of the trigonometric functions for both band width and for axial location. They will be discussed in this order.

For both 'small' and 'large' values of Bessel function arguments, values of the numerator, s_n and denominator, D_λ of s too small or too large to be handled by a digital computer will occur. To elminate this difficulty both numerator and denominator in s are divided by $n^3 \, I_o{}^2(n)$. Equation (70a) is rewritten to read

$$s = \frac{\tilde{s}_n}{\tilde{D}} \qquad (70b)$$

where, the numerator and denominator are given by

$$\tilde{s}_n = \frac{s_n}{n^3 I_o{}^2(n)} \qquad (71)$$

$$\tilde{D}_\lambda = \frac{D_\lambda}{n^3 I_o{}^2(n)} = \tilde{\lambda}_{11}\tilde{\lambda}_{22} - \tilde{\lambda}_{12}\tilde{\lambda}_{21} \qquad (72)$$

The $\tilde{\lambda}$'s are taken as

$$\tilde{\lambda}_{11} = \frac{\lambda_{11}}{n^2 I_o(n)} = -1 + (g + n^2 \frac{t}{2a})\tilde{g} \qquad (73a)$$

$$\tilde{\lambda}_{12} = \frac{\lambda_{12}}{n I_o(n)} = (g = n^2 \frac{t}{2a}) - [n^2(1+\alpha\frac{t}{a}) + \alpha(\mu+g)]\tilde{g} \qquad (73b)$$

$$\tilde{\lambda}_{21} = \frac{\lambda_{21}}{n^2 I_o(n)} = g - [1 + \frac{n^2 t}{2a}(2\mu-g) + kn^4 + g-\mu]\tilde{g} \qquad (73c)$$

$$\tilde{\lambda}_{22} = \frac{\lambda_{22}}{n I_o(n)} = -[1 + \frac{n^2 t}{2a}(2\mu-g) + kn^4 - (1+\alpha)(\mu-g)]$$
$$+ [(n^2+2\alpha) g + 2\alpha(1-\mu+kn^4) + \frac{n^2 \alpha t}{2a}(3\mu-g)]\tilde{g} \qquad (73d)$$

where, $\tilde{g}$ is the ratio of modified Bessel functions given by

$$\tilde{g} = \frac{I_1(n)}{n I_o(n)} \qquad (74)$$

Again some terms would have been lost had displacements been matched with the middle surface so that the $\tilde{\lambda}$'s would be given as

$$\tilde{\lambda}_{11} = \frac{\lambda_{11}}{n^2 I_o(n)} = -1 + g\,\tilde{g} \tag{75a}$$

$$\tilde{\lambda}_{12} = \frac{\lambda_{12}}{n I_o(n)} = g - [n^2 + \alpha(\mu + g)]\tilde{g} \tag{75b}$$

$$\tilde{\lambda}_{21} = \frac{\lambda_{21}}{n^2 I_o(n)} = g - [1 + kn^4 + g - \mu]\tilde{g} \tag{75c}$$

$$\tilde{\lambda}_{22} = \frac{\lambda_{22}}{n I_o(n)} = -[1 + kn^4 - (1 + \alpha)(\mu - g)] + [(n^4 + 2\alpha)\,g + 2\alpha\,(1-\mu+kn^4)]\tilde{g} \tag{75d}$$

The values of the displacements necessary for computational purposes for small and large arguments from Equations (61), (71) and (73) are

$$\tilde{U}_n = \frac{1}{n}\,[-\tilde{\lambda}_{12}\tilde{g}_o + \tilde{\lambda}_{11}(n\tfrac{r}{a})^2\,\tilde{g}\,] \tag{76a}$$

$$\tilde{W}_n = \frac{r}{a}\,\{\tilde{\lambda}_{12}\tilde{g}_1 + \tilde{\lambda}_{11}[2\alpha\tilde{g}_1 - \tilde{g}_o]\} \tag{76b}$$

where $\tilde{g}_o$ and $\tilde{g}_1$ are the ratios of modified Bessel functions given by

$$\tilde{g}_o = \frac{I_o(n\frac{r}{a})}{I_o(n)} \quad ; \quad \tilde{g}_1 = \frac{I_1(n\frac{r}{a})}{(n\frac{r}{a})\,I_o(n)} \tag{77a,77b}$$

Similarly for the core stresses, from Equations (64), (71) and (73), the values for small and large arguments are

$$(\tilde{\sigma}_r)_n = \tilde{\lambda}_{12}[\tilde{g}_o - \tilde{g}_1] - \tilde{\lambda}_{11}\,[(n\tfrac{r}{a})^2 + 2\alpha]\,\tilde{g}_1 - (1 + \alpha)\,\tilde{g}_o\} \tag{78a}$$

$$(\tilde{\sigma}_\theta)_n = \tilde{\lambda}_{12}\tilde{g}_1 + \tilde{\lambda}_{11}[(1 - \alpha)\,\tilde{g}_o + 2\alpha\tilde{g}_1] \tag{78b}$$

$$(\tilde{\sigma}_x)_n = -\tilde{\lambda}_{12}\tilde{g}_o + \tilde{\lambda}_{11}[(2-\alpha)\tilde{g}_o + (n\tfrac{r}{a})^2\,\tilde{g}_1] \tag{78c}$$

$$(\tilde{\tau}_{rx})_n = n\frac{r}{a}\,[-\tilde{\lambda}_{12}\tilde{g}_1 - \tilde{\lambda}_{11}(\alpha\tilde{g}_1 - \tilde{g}_o)] \tag{78d}$$

At the outer radius ($r = a$) $\tilde{g}_o = 1$ and $\tilde{g}_1 = \tilde{g}$. The quantity $\tilde{g}$, given by Equation (74), is evaluated for small values of n (less than .001) by the division of Equation (42b) of [3] by Equation (42a) of [3]. For large values of n (greater than 50) $\tilde{g}$ is evaluated by the division of Equation (45b) of [3] by n times Equation (45a) of [3]. For $\frac{r}{a} < 1$ and $n < .001$ the quantity $\tilde{g}_o$ may be evaluated by the division of Equation (42a) of [3] for $\bar{\eta} = n\frac{r}{a}$ by the same Equation for $\bar{\eta} = n$. For the same condition $\tilde{g}_1$ may be evaluated by the division of Equation (45b) of[3] for $\bar{\eta}=n\frac{r}{a}$ by Equation (45a) of[3] for $\bar{\eta}=n$. For $r/a < 1$ and $n > 50$ two methods of evaluation of each of $\tilde{g}_o$ and $\tilde{g}_1$ must be considered. These methods depend upon the magnitude of n r/a. For $n\frac{r}{a} \leq 3.75$, $\tilde{g}_o$ and $\tilde{g}_1$ may be evaluated by the division of Equations (42a) and (42b) of [3] respectively for $\bar{\eta} = n\frac{r}{a}$ by Equation (44a) of[3] for $\bar{\eta} = n$. For $n\frac{r}{a} > 3.75$, $\tilde{g}_o$ and $\tilde{g}_1$ may be evaluated by the division of Equations (44a) and (44b) of [3] respectively for $\bar{\eta} = n\frac{r}{a}$ by Equation (44b) of[3] for $\bar{\eta} = n$. In this case $\tilde{g}_o$ and $\tilde{g}_1$ can be expressed more simply as

$$\tilde{g}_o = \frac{S_o(n\frac{r}{a})\ e^{-n(1 - \frac{r}{a})}}{\sqrt{\frac{r}{a}}\ S_o(n)} \tag{79a}$$

$$\tilde{g}_1 = \frac{S_1(n\frac{r}{a})\ e^{-n(1 - \frac{r}{a})}}{n(\frac{r}{a})^{3/2}\ S_o(n)} \tag{79b}$$

Since the computed core values to be presented herein are to be evaluated at the interface r = a only, the final expressions needed to evaluate the core displacement and stress integrals at the outer radius for small and large values of n become, from Equations (76) and (78).

$$\tilde{U}_n\Big|_{r=a} = \frac{1}{n}[-\tilde{\lambda}_{12} + \tilde{\lambda}_{11}\tilde{g}] \tag{80a}$$

$$\tilde{W}_n\Big|_{r=a} = \tilde{\lambda}_{12}\tilde{g} + \tilde{\lambda}_{11}[2\alpha\tilde{g} - 1] \tag{80b}$$

$$(\tilde{\sigma}_r)_n\Big|_{r=a} = \tilde{\lambda}_{12}[1 - \tilde{g}] - \tilde{\lambda}_{11}[(n^2 + 2\alpha)\tilde{g} - (1 + \alpha)] \tag{80c}$$

$$(\tilde{\sigma}_\theta)_n\Big|_{r=a} = \tilde{\lambda}_{12}\tilde{g} + \tilde{\lambda}_{11}[1 - \alpha + 2\alpha\tilde{g}] \tag{80d}$$

$$(\tilde{\sigma}_x)_n\Big|_{r=a} = -\tilde{\lambda}_{12} + \tilde{\lambda}_{11}[2 - \alpha + n^2\tilde{g}] \tag{80e}$$

$$(\tilde{\tau}_{rx})_n\Big|_{r=a} = n[-\tilde{\lambda}_{12}\tilde{g} - \tilde{\lambda}_{11}(\alpha\tilde{g} - 1)] \tag{80f}$$

Similarly for small and large values of n, the cylinder displacements and stresses given by Equations (67) are written

$$(\tilde{u}_z)_n = \frac{1}{n}\{-\tilde{\lambda}_{12}(1 - \eta\tilde{g}) + \tilde{\lambda}_{11}[n^2\tilde{g} + \eta(2\alpha\tilde{g} - 1)]\} \tag{81a}$$

$$(\tilde{\sigma}_{xx})_n = -\tilde{\lambda}_{12}[1 - (\mu + n)\tilde{g}] - \tilde{\lambda}_{11}\{\mu+\eta-[n^2+2\alpha(\mu+\eta)]\tilde{g}\} \tag{81b}$$

$$(\tilde{\sigma}_{yy})_n = -\tilde{\lambda}_{12}[\mu-(1+\mu\eta)\tilde{g}] - \tilde{\lambda}_{11}\{1+\mu\eta-[\mu n^2 + 2\alpha(1+\mu\eta)]\tilde{g}\} \tag{81c}$$

For evaluation at the outer radius of the cylinder Equation (68) appropriately gives the quantity η.

Equations (59), (61), (64) and (67) apply in the intermediate ranges of values of n. Equations (72), (76), (78) and (81) apply in the small and large ranges of values of n. These equations thus provide the necessary quantities for the evaluation of s given by Equations (70) in the typical integration of Equation (69). For evaluation at the outer core radius Equations (65) and (80) apply as the special cases of Equations (61) and (64); (76) and (78).

It should be noted that for small values of n the final bracketed quantity in the integrand of Equation (69) reduces to $\left\{\begin{matrix} n\frac{x}{a} \\ \text{or} \\ 1 \end{matrix}\right\}$. The $n\frac{x}{a}$ term applies to the core and cylinder axial displacement and core shear stress quantities. The multiplication of the n of $n\frac{x}{a}$ with $\tilde{U}_n$ and $(\tilde{u}_z)_n$ must be done prior to evaluation to avoid division by zero in the computations.

For a given nondimensional displacement or stress parameter, the quantity s in Equation (69) was found to remain the same sign and converge to zero with increasingly larger values of n. The general nature of the integrand of Equation (69) is effected by the periodicity produced by the trigonometric functions. Greater difficulty is encountered in the numerical integration for the band load problem than for the limiting line load problem. This is due to the presence of two periodic functions and the greater frequencies of oscillation associated with the consideration of larger axial length values.

Typical plots of the integrand as functions of n for the radial deflection for a line and two widths of band load are shown in Figure 4. From these plots it can be seen that the contributions to the total integral occur in a continuous fashion for $\frac{x}{a} = 0$, $\frac{2\delta}{a} = 0$ but in discrete intervals between the zeroes of the trigonometric functions for $\frac{x}{a} \neq 0$ and/or $\frac{2\delta}{a} \neq 0$. The method of arriving at a satisfactory evaluation of the integral was different for these cases. Basically, however, four groups were formed in both cases. These gave partial contributions towards the accurate evaluation of the integrals. These were obtained by numerical integration using Simpson's rule[11]. In the case of $\frac{x}{a} = 0$, $\frac{2\delta}{a} = 0$ the groups in Figure 5 were chosen. These are shown by Roman numerals. For $\frac{x}{a} \neq 0$ and/or $\frac{2\delta}{a} \neq 0$, the groups were formed as the contributions of successive intervals between the zeroes of the trigonometric functions. These are shown in Figure 6. Four groups were arbitrarily chosen, in that a second group may sometimes be small compared to the first and yet subsequent groups may contribute to the overall integral. This is shown to some extent in Figure 4 for $\frac{x}{a} = 0.1$ and $\frac{2\delta}{a} = 0.5$.

After these groups were formed a check was made comparing the second group with the first, the third with the sum of the first two and the fourth with the sum of the first three. Tests of 3%, 2%, and 1% were made to establish adequate convergence. If any group was not less than these specified percentages, the test was rerun with the formation of additional groups as necessary.

The computer program that was set up to perform the numerical integration is included as Table 1. This particular program was used to calculate the core deflection and stress distributions in the axial direction for different band load widths. For convenience an equilvalence table is included for the algebraic and computer symbols for the different variables. As compiled and listed by an IBM 360 Model 50 computer in Table 1, the equations of the Two Function approach with interface matching used herein are shown. This is designated Run (1) and used to obtain results for band load solutions and, in the limit, for the line load solution. By a simple change in some of the cards of the Fortran source deck, middle surface matching solutions can be obtained. The Two Function approach with middle surface matching represents an approach that should lead to results in agreement with those obtained by Yao[7]. This solution has been designated Run (2). The Love Function approach with middle surface matching, which is considered in Appendix 2, is designated Run (3). For the line load solution, Run (3), results obtained should correspond to those obtained by Yogananda[8].

The program as set up for an 'accuracy' of 1% took relatively large amounts of computer time, especially for the larger values of $2\delta/a$. For the most part, computer results were obtained on a Control Data G-20 digital computer. On this machine for the run printed in Table 1 the total time for $2\delta/a = .5$ was close to one-half hour. As rerun on the IBM 360 Model 50 this was cut down to six minutes. It was surprising that such a large reduction in time occurred in that, although the 360 is acknowledged to be faster, almost identical results were obtained even as far as the necessary number of groups to satisfy the convergence criterion.

Numerical Results

In all the results graphed, the material parameters are $G/E = 4 \times 10^{-5}$ and $\nu = \mu = 0.3$. The geometric parameter which controls the relative thickness of the cylinder was taken as $a/t = 400$. Although only these particular parameters are used, and results at the core and cylinder outer radii were obtained, the results for a wide range of values can easily be obtained.

Table 2 presents the core displacement and stress parameter results of Run (1) for the band load solution with $2\delta/a = 0.5$. These are a set of typical results from the computer program of Table 1. Table 3 presents typical results for the outer surface cylinder stress parameters. Tables 4 and 5 present the core displacement and radial and shear stress parameter results for the line load solutions of the interface matching Runs (2) and (3).

Figures 7 through 12 show the distribution of core displacements and stresses at the outer core radius for $2\delta/a = 0, 0.1, 0.25$ and 0.5. For these same band width values the outer surface cylinder axial and circumferential stress distributions are shown in Figure 13 and 14. Results for the limiting case of line load ($\frac{2\delta}{a} = 0$) are shown in all these Figures 7 through 14.

No measurable difference in results for the radial displacement and radial stress for the two different axial displacement assumptions was observed in the line load solutions of the different Runs (1), (2), and (3). The distributions of nondimensional axial displacement and shear stress for both interface and middle surface matching are shown in Figures 15 and 16. The cylinder axial displacement results for a line load when no core is present previously given in Figure 3 are also shown in the axial displacement plot of Figure 15 by the dashed curves. The significance of these curves is considered in greater detail in the discussion of results.

In Figures 17 through 20 a more general picture is given of how the displacements and stresses at the core outer radius (at the center of the load) decrease with increasing band width. Figures 21 and 22 show the same decrease for the cylinder stresses. In both these figures a dashed curve has been used to indicate the "plane strain" solution for a finite cylinder and core combination where the 'length' of the cylinder is equal to the band load 'width'. By plane strain, it is meant that the axial displacement of both the cylinder and core is zero. In the latter calculations the distinction between the interface and cylinder mean surface radius was recognized. The derivation is given in Appendix 3.

Discussion of Results

Equations have been derived for the stresses and displacements within a long cylinder encasing a core, subject to a band of compressive stress uniformly distributed over a finite width 2δ. Results that show displacement and stress distributions versus the axial coordinate for various band widths and versus band width size are shown graphically in a number of cases for which it was assumed that the material constants G/E is 4×10^{-5}, Poisson's ratio $\nu = \mu = 0.3$ and the core radius to wall thickness ratio a/t is 400.

For an increasing width of band load 2δ as considered in Figures 7 through 14 the decrease in magnitudes of the displacement and stress is apparent. The maximum axial displacement and corresponding maximum shear stress necessary to satisfy the interface boundary condition are at similar locations. The location of this maximum was at an axial length slightly greater than the distance to the edge of the band load. In accordance with St. Venant's hypothesis, the axial displacement at a large distance from the load is independent of the particular width of load.

The radial, circumferential and axial stresses, as well as the radial deflection, at the outer radius of the core, all have their maximum magnitudes at the center of the band width for small band widths. For wider band widths, however, the peaks occur at axial locations closer to the edge of the band. The radial tensile stress and shear stress at the cylinder to core interface are particularly important because they specify the bond strength required between the core and cylinder. The maximum radial tensile stress values as well as the maximum shear stress values occur at an axial location slightly beyond the edge of the load. The maximum radial stress tensile values are somewhat smaller than the maximum shear stress values.

In Figure 13 the axial displacements at the middle and inner surfaces of a cylinder with no core subjected to a line load are shown in the dashed curves. These curves show the difference in axial displacements at these two surfaces due to the presence of an axial bending moment. The corresponding cylinder-core combination line load solutions given by the solid curves in this figure can be seen to be a natural consequence of the particular matched boundary conditions. For the material and geometric properties considered the radial displacement and stress were not values effected by the different matching (interface or middle surface) boundary conditions.

Although a similar method of solution to that used herein was used for the limiting case line load solution by both Yao[7] and Yogananda[8] substantially different results from those herein and from each other were obtained in each case. For example for the same parameters as those chosen herein the maximum radial stress obtained by Yao and Yogananda is about 20% and 37% lower respectively, than the current results. The reason for this difference has been noted for Yao in terms of the error in the radial stress expression. Yogananda's equations were computer checked with the convergence criteria presented herein. Rather than the results he presented, the computer results were found to lead to the values obtained in the present investigation (with middle surface matching). Indeed, it can be shown that the Two Function approach used herein by Yao and the Love Function approach which Yogananda used, are algebraically equivalent. This is shown in Part B of Appendix 2.

It may be of interest to examine the stresses and radial displacements within a section which passes through the center of the externally applied pressure band in order to observe the changes that would occur as the band width increases while the external force is constant. Stresses and radial displacement at the midsection have been calculated by the use of equations derived in the present paper and are shown by solid lines in Figures 17 through 22 for a band width $2\delta/a$ which ranges from 0 to 2. By comparison, results derived in Appendix 3 for a simplified solution of a cylinder and core assembly under the plane strain assumption that the axial displacement vanishes are shown in the same figures by the dashed lines. As expected the greatest difference between solutions, as measured by the vertical intercept between the dashed and solid lines, occurs for small band widths. The difference decreases considerably once the band width exceeds a length equal to the core radius. It is interesting to note that except for the axial stress component, the simplified solution provides a reasonable approximation to stress and axial displacement at the midsection of the pressure band provided the band width exceeds a core radius.

All curves of Figures 17 through 22 show the rapid decrease in stress and radial displacement as the band width increases from that of a ring load. The rapid change emphasizes the advantage of designing away from very narrow bands of pressure.

In conclusion it should be noted that tensile stresses can be created in the bond between a case and its core even when the assembly is subject to a compressive band of pressure. An increase in the width of a narrow band of pressure significantly decreases the stresses and displacements in the assembly even though the external resultant force is kept constant.

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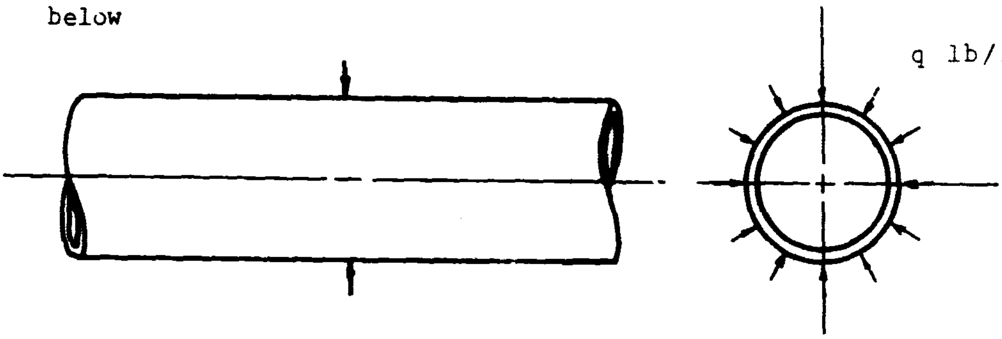
APPENDIX 1

Axial Displacement of a Cylinder Without a Core

The classical equations of thin shell theory are given by Equations (31). When a cylinder is subject to a line load as shown in the sketch below



the boundary conditions and symmetry conditions, to be used in conjunction with Equations (31), may be written

$$[w,_x]_{x=0} = 0 \quad ; \quad [Q_x]_{x=0} = -\frac{q}{2} \quad ; \quad \lim_{x \to \infty} w = 0 \qquad (1.1a\text{-}1.1c)$$

In addition the unit surface loadings p_x and p_z vanish. Equation (31a) can then be written

$$Ru,_x = -\mu w \qquad (1.2a)$$

or

$$u = -\frac{\mu}{R}\int w \, dx \qquad (1.2b)$$

Substitution of Equation (1.2a) into Equation (31b) yields the familiar equation for axisymmetric cylindrical shell problems, namely

$$w,_{xxxx} + 4\beta^4 w = 0 \qquad (1.3)$$

where the parameter

$$\beta = \left[\frac{3(1-\mu^2)}{R^2t^2}\right]^{1/4}$$

The solution for the radial displacement can easily be obtained from Equations (1.1), (1.3) (27c), and (29c). A solution is given, for example on pages 471-474 of Timoshenko and Woinowsky-Krieger[5] and can be written in the current notation as

$$w = -\frac{q}{8\beta^3 D} e^{-\beta x} [\cos \beta x + \sin \beta x] \tag{1.5}$$

With the substitution of Equation (1.4) into Equation (1.2b), performance of the indicated integration, and the use of the boundary condition that the axial displacement at the origin of the axial coordinate vanishes, the expression for axial displacement may be written as

$$u = \frac{\mu q R}{2Et} [1 - e^{-\beta x} \cos \beta x] \tag{1.6}$$

The axial displacement at any distance from the middle surface can be found from Equations (23b), (1.4) and (1.6). When the dimensionless quantities that have been used herein are introduced and use is made of Equation (37), which expresses the equivalence of the inner and middle surface radii of the cylinder, the axial displacement of the cylinder may be written as

$$\frac{2Gu_z}{q} \times 10^3 = \left(\frac{G}{E} \times 10^3\right)\left(\frac{a}{t}\right) \left\{\mu - e^{-\bar{\beta}\frac{x}{a}} \left[\mu \cos \bar{\beta}\frac{x}{a} + \left(\frac{2z}{t}\right)\frac{\bar{\beta}}{\left(\frac{a}{t}\right)} \sin \bar{\beta}\frac{x}{a}\right]\right\} \tag{1.7}$$

where,

$$\bar{\beta} = \beta a = \left[3(1-\mu^2)\left(\frac{a}{t}\right)^2\right]^{1/4} \tag{1.8}$$

The axial displacement ratio $2Gu_z/q \times 10^3$ vs. the dimensionless axial coordinate x/a is plotted in Figure 3 for values of 2z/t = -1 corresponding to the interface (no core) and 2z/t = 0 for the middle surface (no core). It is of interest to note that as x/a becomes infinitely large the axial displacement ratio is given as

$$\frac{2Gu_z}{q} \times 10^3 = (\frac{G}{E} \times 10^3)\ (\frac{a}{t})\ \mu \qquad (1.9)$$

and is independent of 2z/t. The asymptotic value of 4.8 is reached for the particular parameters considered herein as shown in Figure 3.

APPENDIX 2 - PART A

LOVE FUNCTION APPROACH - METHOD OF SOLUTION

Equations (9) express the elasticity equilibrium equations in terms of the displacement components, W and U. The Love function $\chi = \chi(r,x)$ may be introduced in the following manner[3,4]

$$W = \frac{-1}{2G} \frac{\partial^2 \chi}{\partial r \partial x} \tag{2A.1a}$$

$$U = \frac{1}{2G} [\alpha(\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial x^2}) - \frac{\partial^2 \chi}{\partial x^2}] \tag{2A.1b}$$

Substitution of Equations (2A.1) into the stress-displacement relations of Equations (7) then relate the radial and shear stress to the Love Function as

$$\sigma_r = \frac{\partial}{\partial x} [- \frac{\partial^2 \chi}{\partial r^2} + (1- \frac{\alpha}{2})(\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial x^2})] \tag{2A.2a}$$

$$\tau_{rx} = \frac{\partial}{\partial r} [- \frac{\partial^2 \chi}{\partial x^2} + \frac{\alpha}{2} (\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{\partial^2 \chi}{\partial x^2})] \tag{2A.2b}$$

Substitution of Equations (2A.1) into Equations (9) leads t the following differential equation which governs the Love function

$$\frac{\partial^4 \chi}{\partial r^4} + \frac{2}{r}\frac{\partial^3 \chi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r^3}\frac{\partial \chi}{\partial r} + \frac{2}{r}\frac{\partial^3 \chi}{\partial r \partial x^2} + 2\frac{\partial^4 \chi}{\partial r^2 \partial x^2} + \frac{\partial^4 \chi}{\partial x^4} = 0 \tag{2A.3}$$

The solution of this equation may be found by the use of a separation of the variables technique twice. This is done in Föppl and Föppl[12]. The solution may be written

$$\chi = a^2 \int_0^\infty \frac{1}{n^3} [A' I_o(n\frac{r}{a}) + B' n \frac{r}{a} I_1(n\frac{r}{a})] \sin (n\frac{x}{a})\, dn \tag{2A.4}$$

where A' and B' are functions of n analogous to the functions A and B introduced into Equations (42) of the text; I_o and I_1 are modified Bessel functions of the first kind of order zero and order one respectively.

Substitution of Equation (2A.4) into Equations (2A.1) and (2A.2) expresses the axial and radial displacements and the radial and shear stress components in terms of A' and B' as

$$2GU = \int_0^\infty \{A' \, I_o(n\frac{r}{a}) + B' \, [2\alpha \, I_o(n\frac{r}{a}) + n\frac{r}{a} \, I_1(n\frac{r}{a}) \,]\} \sin \, (n\frac{x}{a}) \, dn \qquad (2A.5a)$$

$$2GW = - \int_0^\infty \frac{1}{n} \, [A' \, I_1(n\frac{r}{a}) + B' \, n\frac{r}{a} \, I_o(n\frac{r}{a})] \cos \, (n\frac{x}{a}) \, dn \qquad (2A.5b)$$

$$\sigma_r = \frac{1}{a} \int_0^\infty \{A' \, [-I_o(n\frac{r}{a}) + \frac{I_1(n\frac{r}{a})}{n\frac{r}{a}} \,] + B' \, [(1-\alpha) \, I_o(n\frac{r}{a}) - n\frac{r}{a} \, I_1(n\frac{r}{a})]\} \cos \, (n\frac{x}{a}) \, dn \qquad (2A.5c)$$

$$\tau_{rx} = \frac{1}{a} \int_0^\infty \{A' \, I_1(n\frac{r}{a}) + B'[n\frac{r}{a} \, I_o(n\frac{r}{a}) + \alpha I_1(n\frac{r}{a})]\} \sin \, (n\frac{x}{a}) \, dn \qquad (2A.5d)$$

Substitution of Equations (32), (34a), (35), (37), (49), (51) and (2A.5) each evaluated at the outer core radius, into Equations (31) leads to two integral equations of the form given by Equations (53). In the present case the λ's have revised definitions now. Denoted by primes,

$$\lambda'_{11} = I_o(n) - g\frac{I_1(n)}{n} \qquad (2A.6a)$$

$$\lambda'_{12} = (2\alpha-g) \, I_o(n) + [n^2 + \alpha(\mu-g)] \, \frac{I_1(n)}{n} \qquad (2A.6b)$$

$$\lambda'_{21} = -g \, I_o(n) + (1 + kn^4 + g - \mu) \, \frac{I_1(n)}{n} \qquad (2A.6c)$$

$$\lambda'_{22} = [1 + kn^4 + g - \mu-\alpha(g+\mu)] \, I_o(n) - ng \, I_o(n) \qquad (2A.6d)$$

Expressions identical to Equations (57) through (60) and (63) hold for the primed quantities A', B', λ''s and for D'_λ equal to the determinant of the λ' quantities and where the terms U_n, W_n, $(\sigma_r)_n$ and $(\tau_{rx})_n$ now denoted by primes have the following definitions:

$$U'_n = \frac{1}{n} \{-\lambda'_{12} I_o(n\frac{r}{a}) + \lambda'_{11} [2\alpha I_o(n\frac{r}{a}) + n\frac{r}{a} I_1(n\frac{r}{a})]\} \quad (2A.7a)$$

$$W'_n = \lambda'_{12} \frac{I_1(n\frac{r}{a})}{n} - \lambda'_{11} \frac{r}{a} I_o(n\frac{r}{a}) \quad (2A.7b)$$

$$(\sigma_r)'_n = \lambda'_{12} [I_o(n\frac{r}{a}) - \frac{I_1(n\frac{r}{a})}{n\frac{r}{a}}] + \lambda'_{11} [(1-\alpha) I_o(n\frac{r}{a}) - n\frac{r}{a} I_1(n\frac{r}{a})] \quad (2A.7c)$$

$$(\tau_{rx})'_n = -\lambda'_{12} I_1(n\frac{r}{a}) + \lambda'_{11} [n\frac{r}{a} I_o(n\frac{r}{a}) + \alpha I_1(n\frac{r}{a})] \quad (2A.7d)$$

At this point, it is possible to compare the algebraic expressions obtained from the Two Function approach and the Love Function approach. This is done in Part B of this Appendix. That they are equivalent for the axial displacement parameter is shown for middle surface matching (i.e., it is shown that $U_n/D_\lambda = U'_n/D'_\lambda$). This can be shown in a similar fashion for the other nondimensional quantities. Rather than show this for the other nondimensional quantities, the numerical results, which are easily obtained for the different approaches have been used as a means of verification. Since numerical results are obtained at the core outer radius Equations (2A.7) reduce to

$$U'_n = \frac{1}{n} \{-\lambda'_{12} I_o(n) + \lambda'_{11}[2\alpha I_o(n) + n I_1(n)]\} \quad (2A.8a)$$

$$W'_n = \lambda'_{12} \frac{I_1(n)}{n} - \lambda'_{11} I_o(n) \quad (2A.8b)$$

$$(\sigma_r)'_n = \lambda'_{12}[I_o(n) - \frac{I_1(n)}{n}] + \lambda'_{11}[(1-\alpha) I_o(n) - nI_1(n)] \quad (2A.8c)$$

$$(\tau_{rx})'_n = -\lambda'_{12} I_1(n) + \lambda'_{11}[nI_o(n) + \alpha I_1(n)] \quad (2A.8d)$$

A similar numerical integration difficulty is encountered in dealing with small and large values of n in the present Love function approach as was considered in the Two Function approach. In this case the difficulty can be overcome by dividing numerator and denominator of the equivalent s quantity of Equation (70) by $I_o^2(n)$. The denominator becomes

$$\tilde{D}'_\lambda = \frac{D'_\lambda}{I_o^2(n)} = \tilde{\lambda}'_{11} \tilde{\lambda}'_{22} - \tilde{\lambda}'_{12}\tilde{\lambda}'_{21} \qquad (2A.9)$$

where the functions $\tilde{\lambda}'_{11}$, $\tilde{\lambda}'_{12}$, $\tilde{\lambda}'_{21}$, and $\tilde{\lambda}'_{22}$ are defined

$$\tilde{\lambda}'_{11} = \frac{\lambda'_{11}}{I_o(n)} = 1 - g\tilde{g} \qquad (2A.10a)$$

$$\tilde{\lambda}'_{12} = \frac{\lambda'_{12}}{I_o(n)} = 2\alpha - g + [n^2 + \alpha(\mu-g)]\,\tilde{g} \qquad (2A.10b)$$

$$\tilde{\lambda}'_{21} = \frac{\lambda'_{21}}{I_o(n)} = -g + (1 + kn^4 + g - \mu)\,\tilde{g} \qquad (2A.10c)$$

$$\tilde{\lambda}'_{22} = \frac{\lambda'_{22}}{I_o(n)} = [1 + kn^4 + g - \mu - \alpha(g + \mu)] - n^2\, g\tilde{g} \qquad (2A.10d)$$

The final expressions evaluated at r = a for the numerator in the small and large range of n are given by

$$\tilde{U}'_n\Big|_{r=a} = \frac{1}{n}\,\{-\tilde{\lambda}'_{12} + \tilde{\lambda}'_{11}\,[2\alpha + n^2 g]\,\} \qquad (2A.11a)$$

$$\tilde{W}'_n\Big|_{r=a} = \tilde{\lambda}'_{12}\,\tilde{g} - \tilde{\lambda}'_{11} \qquad (2A.11b)$$

$$(\tilde{\sigma}_r)'_n\Big|_{r=a} = \tilde{\lambda}'_{12}\,[1-\tilde{g}] + \tilde{\lambda}'_{11}\,[1-\alpha-n^2\tilde{g}] \qquad (2A.11c)$$

$$(\tau_{rx})'_n\Big|_{r=a} = n[-\tilde{\lambda}'_{12}\,\tilde{g} + \tilde{\lambda}'_{11}\,(1 + \alpha\tilde{g})] \qquad (2A.11d)$$

Equations (2A.6) and (2A.7) in the intermediate range of n and Equations (2A.10) and (2A.11) in the small and large range of n provide the necessary quantities for the evaluation of the typical equivalent nondimensional quantity $\overline{S}$ in Equations (69) for the Love Fucntion approach.

APPENDIX 2 - PART B

Love Function Approach - Check of Equivalence

It is the purpose of the present section of Appendix 2 to show, by examination of equations, the equivalence of the final expressions for the core axial displacement parameters of the Two Function and the Love Function approach. The middle surface matching solution is considered for both approaches.

The axial displacement parameter $\frac{2GU}{q}$ is given for the Two Function approach by Equation (60a). For the Love Function approach it may be written as

$$\frac{2GU}{q} = \int_0^\infty \frac{U'_n}{D'_\lambda} \, \overline{\zeta} \sin\left(n\frac{x}{a}\right) dn \qquad (2B.1)$$

where U'_n is given by Equation (2A.7a) and

$$D'_\lambda = \lambda'_{11}\,\lambda'_{22} - \lambda'_{12}\,\lambda'_{21} \qquad (2B.2)$$

where the λ''s are given by Equations (2A.6).

To show the equivalence of the two approaches it is sufficient to show that

$$\frac{U_n}{D_\lambda} = \frac{U'_n}{D'_\lambda} \qquad (2B.3)$$

where U_n and D_λ are given by Equations (61a) and (59) respectively. To do this, each side of this equation, numerator and denominator separately, will be put into slightly different form and it will be shown that

$$U'_n = \frac{U_n}{n^3} \quad ; \quad D'_\lambda = \frac{D_\lambda}{n^3} \qquad (2B.4b)$$

and therefore Equation (2B.3) is satisfied.

For convenience, let the following definitions for the quantitis h_1 and h_2 be made

$$h_1 = n^2 + \alpha(\mu + g) \; ; \; h_2 = 1 + kn^4 + g-\mu \qquad \text{(2B.5a,2B.5b)}$$

Then Equations (56) may be written

$$\lambda_{11} = -n^2 I_o(n) + ng I_1(n) \qquad \text{(2B.6a)}$$

$$\lambda_{12} = ng I_o(n) - h_1 I_1(n) \qquad \text{(2B.6b)}$$

$$\lambda_{21} = n^2 g I_o(n) - h_2 n I_1(n) \qquad \text{(2B.6c)}$$

$$\lambda_{22} = -[h_2-\alpha(\mu-g)]n I_o(n) + [n^2 g + 2\alpha h_2] I_1(n) \qquad \text{(2B.6d)}$$

Substitution of Equations (2B.6) into Equation (59), followed by rearranging and reducing terms lead to

$$D_\lambda = [h_2 -\alpha(\mu-g) -g^2] n^3 I_o^2(n) + [n^2 g^2 + 2\alpha h_2 g-h_1 h_2] nI_1^2(n)$$

$$+ 2\alpha n^2[\mu g - h_2] I_o(n) I_1(n) \qquad \text{(2B.7)}$$

Substitution of Equations (2B.5) into Equations (2A.6) yields

$$\lambda'_{11} = I_o(n) - \frac{g}{n} I_1(n) \qquad \text{(2B.8a)}$$

$$\lambda'_{12} = (2\alpha-g) I_o(n) + [n^2 + \alpha(\mu-g)] \frac{I_1(n)}{n} \qquad \text{(2B.8b)}$$

$$\lambda'_{21} = -gI_o(n) + \frac{h_2}{n} I_1(n) \qquad \text{(2B.8c)}$$

$$\lambda'_{22} = [h -\alpha(g + \mu) I_o(n) - ng I_1(n) \qquad \text{(2B.8d)}$$

Substitution of these equations into Equation (2B.2) verifies that Equation (2B.4b) is true. Equation (2A.7a) may be rewritten

$$U'_n = (-\frac{\lambda'_{12} + 2\alpha \lambda'_{11}}{n}) I_o(n\frac{r}{a}) + \lambda'_{11} \frac{r}{a} I_1(n\frac{r}{a}) \qquad \text{(2B.9)}$$

From Equations (2B.6a), (2B.6b) and (2B.8a), (2B.8b) is is easily shown that

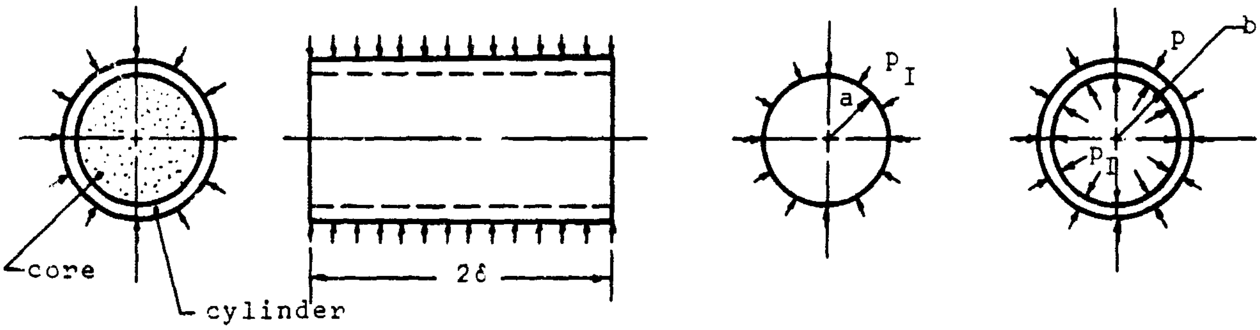
$$\frac{-\lambda'_{12} + 2\alpha\lambda'_{11}}{n} = -n\,\frac{\lambda_{12}}{n^3}\ ; \quad \lambda'_{11} = -\,\frac{n\lambda_{11}}{n^3} \qquad \text{(2B.10a,2B.10b)}$$

Therefore, from Equations (61a), (2B.9) and (2B.10) it follows that Equation (2B.4a) is true. Thus, the algebraic equivalence for the two different approaches for the axial displacement parameter is verified.

APPENDIX 3

Cylinder - Core Assembly for which Axial Displacement Vanishes

Let us analyze a core and cylinder assembly subject to a uniform pressure over its entire length 2δ as shown in sketch (a) below. The problem is



(a) (b) (c)

to be analyzed with the use of the plane strain assumption that the axial displacement vanishes. Hence for brevity the solution will be referred to as a plane strain solution, even though the axial dimension 2δ need not necessarily be large compared to the diametral dimension 2a. The assembly is separated into the two free bodies shown in Sketch (b) and Sketch (c). The interface pressure between both bodies designated p_I is initially unknown. It will be determined from the compatibility requirement between the radial displacement of the core and that of the cylinder.

The solutions to the problems depicted in Sketches (b) and (c) can be taken directly from any well known text on elasticity. In terms of the notations used in the present paper, the stresses and radial displacement of the core is given as

$$\sigma_r = -p_I \quad ; \quad \sigma_\theta = -p_I \quad ; \quad \sigma_x = -2\nu p_I \qquad (3.1a\text{-}3.1c)$$

$$W = -\frac{r p_I}{E_c}\left[(1-2\nu)(1+\nu)\right] \qquad (3.1d)$$

The solution for the cylinder was found by Lamé and may be found on page 144 of Love[22]. The stresses and radial displacements are found to be

$$\sigma_{zz} = \frac{a^2b^2(p-p_I)}{(b^2-a^2)} \frac{1}{r^2} + \frac{p_I a^2 - pb^2}{b^2-a^2} \qquad (3.2a)$$

$$\sigma_{yy} = - \frac{a^2b^2(p-p_I)}{(b^2-a^2)} \frac{1}{r^2} + \frac{p_I a^2 - pb^2}{b^2-a^2} \qquad (3.2b)$$

$$\sigma_{xx} = 2\mu \frac{p_I a^2 - pb^2}{b^2-a^2} \qquad (3.2c)$$

$$w = \frac{r}{E} \left[\frac{1+\mu}{r^2} \frac{(p_I-p)a^2b^2}{(b^2-a^2)} + \frac{p_I a^2 - pb^2}{b^2-a^2} (1-2\mu)(1+\mu) \right] \qquad (3.2d)$$

where b, is the outer radius of the cylinder.

Equating radial displacements at the interface

$$W\Big|_{r=a} = W_a = w\Big|_{r=a} = w_a \qquad (3.3)$$

the interface pressure p_I is found to be given by the expression

$$\frac{p_I}{p} = \frac{2K^2(1-\mu^2)}{(1+\mu)K^2+(1-2\mu)(1+\mu) + \dfrac{(1-2\nu)(K^2-1)}{2(\frac{G}{E})}} \qquad (3.4)$$

where, $K = \dfrac{b}{a} = \dfrac{\frac{a}{t}+1}{a/t}$

In order to compare the present solution with that presented in the main section of this report, let us introduce the resultant force per unti circumferential q. The term $q = 2\delta p$, as given by Equation (65). Hence the stress components of Equations (3.1) may be written

$$\frac{a\sigma_r}{q} = \frac{a\sigma_\theta}{q} = - \frac{(\frac{p_I}{p})}{(2\frac{\delta}{a})} \; ; \; \sigma_x = - \frac{\nu(\frac{p_I}{p})}{(\frac{\delta}{a})} \qquad (3.5a\text{-}3.5c)$$

The radial deflection at the cylinder middle surface, is then given from Equation (3.2d) as

$$\frac{2GW}{q}\bigg|_{r = R = \frac{a+b}{2}} = \frac{\left(\frac{G}{E}\right)}{\left(2\frac{\delta}{a}\right)}\left[\frac{4(1+\mu)\left(\frac{p_I}{p} - 1\right)K^2 + \left(\frac{p_I}{p} - K^2\right)(1-2\mu)(1+\mu)(K+1)^2}{(K+1)^2(K-1)}\right] \quad (3.6)$$

The difference between this radial deflection and that at the interface is very small. Equations (3.5) and (3.6) were used to obtain the dashed curves shown in Figures 17 through 20.

In a similar fashion the nondimensional axial and circumferential stress ratios can be expressed in terms of nondimensional quantities introduced herein. From Equations (3.2b) and (3.2c) one may write

$$\frac{a\sigma_{xx}}{q} = \frac{\mu}{\left(\frac{\delta}{a}\right)}\left[\frac{\frac{p_I}{p} - K^2}{K^2-1}\right] \quad (3.7a)$$

$$\frac{a\sigma_{yy}}{q} = \frac{\left\{\dfrac{-K^2\left[1 - \frac{p_I}{p}\right]}{\left[1 + \dfrac{\left(1 + \frac{2z}{t}\right)}{2\left(\frac{a}{t}\right)}\right]^2} + \frac{p_I}{p} - K^2\right\}}{\left(2\frac{\delta}{a}\right)(K^2-1)} \quad (3.7b)$$

These equations were used to find the dashed curves shown in Figures 22 and 23.

TABLE 1

COMPUTER PROGRAM TO CALCULATE AXIAL VARIATION OF CORE DEFLECTION AND STRESS PARAMETERS FOR DIFFERENT BAND LOAD WIDTHS

The program is given on the following pages. The algebraic and computer nomenclature for the significant variables in the problem considered are:

Algebraic	Computer
π	PI
G/E	GE
ν	VF
μ	V
a/t	AT
$2\delta/a$	DL2
δ/a	DL
α	AL
g	G
k	PK
x/a	XX
n	X
$I_o(n)$	XIO
$I_1(n)/n$	XI1N
$I_1(n)$	XI1
$S_o(n)$	XXO
$S_1(n)$	XX1
$\tilde{g}$	GS
λ_{11}	F1
λ_{12}	F2
λ_{21}	F3
λ_{22}	F4

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```
C REQUEST FOR STORAGE
      DIMENSION U(6),M(6),FF1(44),FF2(44),FF3(44),FF4(44),FF5(44),
     1          FF6(44),SP(6),XL(6),XD(6),S(4,6),A(4,6),B(4,6)
C CONSTANTS
      PI=3.141592654E0
C INPUT MATERIAL AND SIZE PARAMETERS
   10 READ (1,20) GE,V,VF,AT,DL2
   20 FORMAT (5F10.0)
C CALCULATIONS OF OTHER MATERIAL AND SIZE PARAMETERS
      AL=2.*(1-VF)
      G=V-2.*(1.-V*V)*GE*AT
      PK=1./(12.*AT**2.)
      DL = DL2/2.
C RUN NUMBER
      N = 1
C HEADING
      WRITE (3,30)N,DL2,GE,AT,VF,V
   30 FORMAT (1H1,57X,'TABLE 2'//37X,'ELASTIC ANALYSIS OF CYLINDER WITH
     1CORE - RUN ('I1,')'//41X,'UNIFORM BAND LOAD OF WIDTH 2D/A='F5.2//3
     10X,'G/E ='F12.8,3X,'A/T ='F7.2,3X,'POISSON RATIO - CORE ='F6.3/77X
     1,'CYLINDER ='F6.3//18X,'X/A'2X,'2GU/Q*10 3'2X,'M'5X,'2GW/Q'3X,'M'4
     1X,'TRXA/Q'3X,'M'5X,'SRA/Q'3X,'M'5X,'SXA/Q'3X,'M'5X,'STA/Q'3X,'M'/)
C OUTPUT LOCATIONS AS DEPENDENT UPON D/A
      IF (DL) 40,50,40
   40 IT = 19
      GO TO 60
   50 IT = 21
   60 DO 1280 I = 1,IT
      IF (I-1) 70,70,80
   70 KS = 0
   80 XI = I
      IF (DL) 100,90,100
   90 XX = (XI-1.)/100.
      GO TO 160
  100 IF (I-1) 130,130,110
  110 IF (I-5) 140,140,120
  120 IF (I-15) 150,150,140
  130 XX = 0
      GO TO 160
  140 XX = XX+DL/5.
      GO TO 160
  150 XX = XX+DL/25.
C SIX VALUES FOR DEFLECTIONS AND STRESSES DETERMINED
  160 DO 1260 LS = 1,6
      IF (LS-2) 200,170,180
  170 L = 3
      GO TO 210
  180 IF (LS-4) 190,200,200
  190 L = 2
      GO TO 210
  200 L = LS
  210 M(L)=0
```

```
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      IF (KS)230,220,230
  220 SP(L)=0
  230 KK=0
      A(1,L)=0
      KI=2
      JI=1
C  SELECTION OF NUMERICAL INTEGRATION LIMITS
  240 DO 250 J1 = JI,4
  250 S(J1,L)=0
      IF (XX) 450,260,450
  260 IF (L-1)270,1260,270
  270 IF (L-3)280,1260,280
  280 IF (DL)370,290,370
C  NO PERIODICITY -DL=0,XX=0
  290 IF (KK)310,300,310
  300 B(1,L)=50.
  310 DO 360 K1=KI,4
      IF (KK)320,340,320
  320 IF (K1-KI)340,330,340
  330 KT=4
      GO TO 350
  340 KT=K1-1
  350 A(K1,L)=B(KT,L)
  360 B(K1,L)=A(K1,L)+50.
      KO=1
      GO TO 810
C  ZEROES OF SIN (X*DL) - XX=0
  370 IF (KK)390,380,390
  380 B(1,L)=PI/DL
  390 DO 440 K3=KI,4
      IF (KK)400,420,400
  400 IF (K3-KI)420,410,420
  410 KT=4
      GO TO 430
  420 KT=K3-1
  430 A(K3,L)=B(KT,L)
  440 B(K3,L)=A(K3,L)+PI/DL
      KO=1
      GO TO 810
C  ZEROES OF COS  (X*XX) OR SIN (X*XX) - DL=0
  450 IF (DL)600,460,600
  460 IF (KK)510,470,510
  470 IF (L-1)480,490,480
  480 IF (L-3)500,490,500
  490 B(1,L)=PI/XX
      GO TO 510
  500 B(1,L)=PI/(2.*XX)
  510 DO 560 K2=KI,4
      IF (KK)520,540,520
  520 IF (K2-KI)540,530,540
  530 KT=4
      GO TO 550
  540 KT=K2-1
```

```
  550 A(K2,L)=B(KT,L)
  560 B(K2,L)=A(K2,L)+PI/XX
      IF (L-1)570,580,570
  570 IF (L-3)590,580,590
  580 KO=2
      GO TO 810
  590 KO=1
      GO TO 810
C  ZEROES OF BOTH COS (X*XX) OR SIN (X*XX) AND SIN  (X*DL)
  600 IF (KK)700,610,700
  610 XL(L)=PI/XX
      XD(L)=PI/DL
      IF (L-1)620,640,620
  620 IF (L-3)630,640,630
  630 XL(L)=XL(L)/2.
  640 IF (XL(L)-XD(L))650,670,680
  650 IF (ABS(XL(L)-XD(L))-.01) 670,670,660
  660 P(1,L)=XL(L)
      XL(L)=XL(L)+PI/XX
      GO TO 700
  670 XD(L)=2.*XD(L)
      GO TO 660
  680 IF (ABS(XL(L)-XD(L))-.01) 670,670,690
  690 B(1,L)=XD(L)
      XD(L)=2.*XD(L)
  700 DO 800 K4=KI,4
      IF (KK)710,730,710
  710 IF (K4-KI)730,720,730
  720 KT=4
      GO TO 740
  730 KT=K4-1
  740 A(K4,L)=B(KT,L)
      IF (XL(L)-XD(L))750,770,780
  750 IF (ABS(XL(L)-XD(L))-.01) 770,770,760
  760 B(K4,L)=          XL(L)
      XL(L)=XL(L)+PI/XX
      GO TO 800
  770 XD(L)=XD(L)+PI/DL
      GO TO 760
  780 IF (ABS (XL(L)-XD(L))-.01) 770,770,790
  790 B(K4,L)=          XD(L)
      XD(L)=XD(L)+PI/DL
  800 CONTINUE
      KO=2
  810 MI=0
C FORMATION OF FOUR GROUPS FOR CONVERGENCE TEST
      DO 1170 J4 = JI,4
      M(L)=M(L)+1
      IF (M(L)-98) 820,820,1260
  820 DO 1170 J = 1,11
      MI=MI+1
      IGN=(-1)**J
      IF (J-1) 830,830,840
```

```
10/09/67                FORTMAIN
  550 A(K2,L)=B(KT,L)
  560 B(K2,L)=A(K2,L)+PI/XX
      IF (L-1)570,580,570
  570 IF (L-3)590,580,590
  580 KO=2
      GO TO 810
  590 KO=1
      GO TO 810
C  ZEROES OF BOTH COS (X*XX) OR SIN (X*XX) AND SIN  (X*DL)
  600 IF (KK)700,610,700
  610 XL(L)=PI/XX
      XD(L)=PI/DL
      IF (L-1)620,640,620
  620 IF (L-3)630,640,630
  630 XL(L)=XL(L)/2.
  640 IF (XL(L)-XD(L))650,670,680
  650 IF (ABS(XL(L)-XD(L))-.01) 670,670,660
  660 B(1,L)=XL(L)
      XL(L)=XL(L)+PI/XX
      GO TO 700
  670 XD(L)=2.*XD(L)
      GO TO 660
  680 IF (ABS(XL(L)-XD(L))-.01) 670,670,690
  690 B(1,L)=XD(L)
      XD(L)=2.*XD(L)
  700 DO 800 K4=KI,4
      IF (KK)710,730,710
  710 IF (K4-KI)730,720,730
  720 KT=4
      GO TO 740
  730 KT=K4-1
  740 A(K4,L)=B(KT,L)
      IF (XL(L)-XD(L))750,770,780
  750 IF (ABS(XL(L)-XD(L))-.01) 770,770,760
  760 B(K4,L)=        XL(L)
      XL(L)=XL(L)+PI/XX
      GO TO 800
  770 XD(L)=XD(L)+PI/DL
      GO TO 760
  780 IF (ABS (XL(L)-XD(L))-.01) 770,770,790
  790 B(K4,L)=        XD(L)
      XD(L)=XD(L)+PI/DL
  800 CONTINUE
      KO=2
  810 M1=0
C FORMATION OF FOUR GROUPS FOR CONVERGENCE TEST
      DO 1170 J4 = JI,4
      M(L)=M(L)+1
      IF (M(L)-98) 820,820,1260
  820 DO 1170 J = 1,11
      M1=M1+1
      IGN=(-1)**J
      IF (J-1) 830,830,840
```

```
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  830 X=A(J4,L)
      GG=1.
      GO TO 890
  840 IF (J-11) 860,850,850
  850 X=B(J4,L)
      GG=1.
      GO TO 890
  860 X=X+(B(J4,L)-A(J4,L))/10.
      IF (IGN) 870,870,880
  870 GG = 2.
      GO TO 890
  880 GG = 4.
C SETUP FOR STORAGE
  890 IF (KK) 930,900,930
  900 IF (KU-1) 920,910,920
  910 IF (L-2) 930,930,1130
  920 IF (L-1) 930,930,1130
C LOAD
  930 IF (DL) 940,940,950
  940 F5=(G-V)/PI
      GO TO 970
  950 IF (X) 960,940,960
  960 F5=(G-V)/PI*SIN(X*DL)/(X*DL)
C MODIFIED BESSEL FUNCTIONS
  970 T = X/3.75
      IF (T) 990,980,990
  980 XI0 = 1.
      XI1N = .5
      XI1 = 0
      GO TO 1040
  990 IF (T-1.) 1000,1000,1010
 1000 XI0=1.+3.5156229E0*T*T+3.0899424E0*T**4+1.2067492E0*T**6+.2659732
     1T**8+.0360768*T**10+.0045813*T**12
      XI1N=(1./2.+.87890594E0*T*T+.51498869[illegible]T**4+.15084934E0*T**6+
     1.02658733E0*T**8+.00301532E0*T**10+.00[illegible]2411E0*T**12)
      XI1=XI1N*X
      GO TO 1030
 1010 XX0=(.39894228E0+.01328592E0*T**(-1)+.00225319E0*T**(-2)-
     1.00157565E0*T**(-3)+.00916281E0*T**(-4)-.02057706E0*T**(-5)+
     1.02635537E0*T**(-6)-.01647633E0*T**(-7)+.00392377E0*T**(-8))
      XX1=(.39894228E0-.03988024E0*T**(-1)-.00362018E0*T**(-2)+
     1.00163801E0*T**(-3)-.01031555E0*T**(-4)+.02282967E0*T**(-5)-
     1.02895312E0*T**(-6)+.01787654E0*T**(-7)-.00420059E0*T**(-8))
      IF (X-50.) 1020,1020,1050
 1020 XI0=XX0*EXP(X)/SQRT(X)
      XI1=XX1*EXP(X)/SQRT(X)
      XI1N=XI1/X
 1030 IF (X-.001) 1040,1040,1090
C SMALL AND LARGE VALUES OF X
 1040 GS=XI1N/XI0
      GO TO 1060
 1050 GS = XX1/(X*XX0)
 1060 F1 = -1.+(G+X*X/(2.*AT))*GS
```

```
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      F2=G+X*X/(2.*AT)-(X*X*(1.+AL/AT)+AL*(V+G))*GS
      F3 = G-(1.+V*X*X/(2.*AT)+PK*X**4+G-V)*GS
      F4 = -(1.+V*X*X/(2.*AT)+PK*X**4- (1.+AL)*(V-G))+((X*X+2.*AL)*G+2.
     1*AL*(1.-V+PK*X**4)+V*X*X*AL/AT)*GS
      D=F1*F4-F2*F3
      S1=(-F2*(GS-1.)-F1*((X*X+2.*AL)*GS-(1.+AL)))*F5/D
      W1=(F2*GS+F1*(2.*AL*GS-1.))*F5/D
      SX1=(-F2+F1*(2.-AL+X*X*GS     ))*F5/D
      ST1=(F2*GS+F1*(1.-AL+2.*AL*GS))*F5/D
      IF (X-25.) 1070,1080,1080
 1070 U1 = (-F2+F1*X*X*GS)*F5/D
      T1 = X*X*(-F2*GS-F1*(AL*GS-1.))*F5/D
      GO TO 1100
 1080 U1 = (-F2+F1*X*X*GS)*F5/(X*D)
      T1 =    X*(-F2*GS-F1*(AL*GS-1.))*F5/D
      GO TO 1100
C INTERMEDIATE VALUES OF X
 1090 F1=-X*X*XI0+(X*G+X**3/(2.*AT))*XI1
      F2=(X*G+X**3/(2.*AT))*XI0-(X*X*(1.+AL/AT)+AL*(V+G))*XI1
      F3 = X*X*G*XI0-X*(1.+V*X*X/(2.*AT)+PK*X**4+G-V)*XI1
      F4 = -X*(1.+V*X*X/(2.*AT)+PK*X**4-(1.+AL)*(V-G))*XI0+((X*X+2.*AL)
     1*G+2.*AL*(1.-V+PK*X**4)+V*X*X*AL/AT)*XI1
      D=F1*F4-F2*F3
      S1=(-F2*(X*XI1-X*X*XI0)-F1*((X*X+2.*AL)*XI1-(1.+AL)*X*XI0))*F5/D
      U1=X*(-F2*XI0+F1*XI1)*F5/D
      W1=(F2*X*XI1+F1*(2.*AL*XI1-X*XI0))*F5/D
      T1=(-F2*X*X*XI1-F1*(AL*X*XI1-X*X*XI0))*F5/D
      SX1=(-F2*X*X*XI0+F1*((2.-AL)*X*XI0+X*X*XI1))*F5/D
      ST1=(F2*XI1*X+F1*((1.-AL)*X*XI0+2.*AL*XI1))*F5/D
C STORAGE
 1100 IF (KK) 1110,1110,1120
 1110 FF1(M1)=U1
      FF2(M1)=W1
      FF3(M1)=T1
      FF4(M1)=S1
      FF5(M1)=SX1
      FF6(M1)=ST1
 1120 IF (X-.001) 1140,1140,1150
 1130 U1=FF1(M1)
      W1=FF2(M1)
      T1=FF3(M1)
      S1=FF4(M1)
      SX1=FF5(M1)
      ST1=FF6(M1)
      IF (X-.001) 1140,1140,1150
 1140 UN=U1*XX
      WN=W1
      TN=T1*XX
      SN=S1
      SXN=SX1
      STN=ST1
      GO TO 1160
 1150 UN=U1*SIN(X*XX)
```

```
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      WN=W1*COS(X*XX)
      TN=T1*SIN(X*XX)
      SN=S1*COS(X*XX)
      SXN=SX1*COS(X*XX)
      STN=ST1*COS(X*XX)
 1160 U(1)=UN*1000.
      U(2)=WN
      U(3)=TN
      U(4)=SN
      U(5)=SXN
      U(6)=STN
 1170 S(J4,L)=S(J4,L)+GG*U(L)*(B(J4,L)-A(J4,L))/30.
C  CONVERGENCE TEST
      DO 1180 II = 1,4
      IF (ABS(S(II,L))-.010*ABS(SP(L)))1180,1180,1190
 1180 CONTINUE
      S(1,L)=S(1,L)+S(2,L)+S(3,L)+S(4,L)
      GO TO 1260
 1190 DO 1210 L1 = 1,3
      XL1=L1
      IF (ABS(S(L1+1,L))-(.04-.01*XL1)*ABS(S(1,L)))1200,1200,1220
 1200 S(1,L)=S(1,L)+S(L1+1,L)
 1210 CONTINUE
      GO TO 1260
 1220 S(1,L)=S(1,L)+S(L1+1,L)
      KK=1
      IF (L1-2)1230,1240,1250
 1230 S(2,L)=S(3,L)
      S(3,L)=S(4,L)
      KI=4
      JI=4
      GO TO 240
 1240 S(2,L)=S(4,L)
      KI=3
      JI=3
      GO TO 240
 1250 KI=2
      JI=2
      GO TO 240
C  END CONVERGENCE TEST
 1260 CONTINUE
      DO 1270 IK=1,6
 1270 SP(IK)=AMAX1(ABS(S(1,IK)),SP(IK))
      KS=1
C PRINTOUT OF RESULTS
 1280 WRITE (3,1290)XX,((S(1,N1),M(N1)),N1=1,6)
 1290 FORMAT (15X,F7.3,6(E11.3,I3))
      GO TO 10
      END
```

TABLE 2

ELASTIC ANALYSIS OF CYLINDER WITH CORE - RUN (1)

UNIFORM BAND LOAD OF WIDTH 2D/A= 0.50

G/E = 0.00004000 A/T = 400.00 POISSON RATIO - CORE = 0.300
CYLINDER = 0.300

X/A	2GU/Q*10 3	M	2GW/Q	M	TRXA/Q	M	SRA/Q	M	SXA/Q	M	STA/Q	M
0.0	0.0	0	-0.592E-01	6	0.0	0	-0.146E 00	14	-0.623E-01	10	-0.140E 00	9
0.050	0.890E 00	10	-0.595E-01	8	-0.443E-03	40	-0.161E 00	14	-0.680E-01	10	-0.146E 00	10
0.100	0.178E 01	8	-0.545E-01	7	-0.156E-02	38	-0.799E-01	17	-0.393E-01	14	-0.106E 00	14
0.150	0.268E 01	10	-0.488E-01	9	0.702E-03	45	-0.165E 00	17	-0.706E-01	14	-0.134E 00	14
0.200	0.376E 01	9	-0.508E-01	10	0.451E-01	20	-0.389E 00	15	-0.151E 00	13	-0.229E 00	13
0.210	0.407E 01	6	-0.491E-01	10	0.644E-01	17	-0.408E 00	12	-0.154E 00	12	-0.232E 00	12
0.220	0.431E 01	8	-0.463E-01	10	0.914E-01	15	-0.395E 00	14	-0.147E 00	14	-0.223E 00	14
0.230	0.453E 01	8	-0.425E-01	10	0.116E 00	12	-0.351E 00	16	-0.129E 00	14	-0.199E 00	14
0.240	0.470E 01	8	-0.378E-01	10	0.138E 00	14	-0.267E 00	16	-0.998E-01	14	-0.158E 00	14
0.250	0.488E 01	6	-0.299E-01	11	0.147E 00	9	-0.986E-01	18	-0.428E-01	17	-0.811E-01	16
0.260	0.488E 01	8	-0.286E-01	11	0.137E 00	14	-0.523E-01	25	-0.299E-01	19	-0.633E-01	17
0.270	0.489E 01	8	-0.247E-01	11	0.118E 00	12	0.292E-01	25	-0.106E-02	19	-0.231E-01	23
0.280	0.486E 01	8	-0.210E-01	13	0.906E-01	17	0.775E-01	22	0.188E-01	19	0.122E-02	15
0.290	0.480E 01	8	-0.182E-01	13	0.634E-01	19	0.967E-01	22	0.292E-01	20	0.135E-01	26
0.300	0.473E 01	9	-0.160E-01	13	0.425E-01	19	0.982E-01	22	0.326E-01	20	0.182E-01	22
0.350	0.447E 01	9	-0.122E-01	17	-0.850E-03	15	0.336E-01	29	0.134E-01	24	-0.102E-02	19
0.400	0.453E 01	10	-0.982E-02	18	-0.551E-02	39	0.221E-01	36	0.888E-02	31	-0.343E-02	44
0.450	0.454E 01	13	-0.884E-02	19	-0.540E-02	42	0.180E-01	39	0.674E-02	33	-0.411E-02	47
0.500	0.458E 01	9	-0.658E-02	20	-0.598E-02	33	0.454E-01	38	0.170E-01	32	0.100E-01	41

TABLE 3

ELASTIC ANALYSIS OF CYLINDER WITH CORE - RUN (1)

UNIFORM BAND LOAD OF WIDTH 2D/A= 0.50

G/E = 0.00004000 A/T = 400.00 POISSON RATIO - CORE = 0.300
CYLINDER = 0.300

CYLINDER STRESSES AT Z BAR = 1.00

X/A	AXIAL SXX*A/Q	M	CIRCUMFERENTIAL SYY*A/Q	N
0.0	-0.338E 01	51	-0.745E 03	7
0.050	-0.303E 01	40	-0.740E 03	8
0.100	-0.299E 01	51	-0.733E 03	10
0.150	-0.241E 02	36	-0.729E 03	9
0.200	-0.129E 03	22	-0.708E 03	10
0.250	0.129E 01	22	-0.378E 03	13
0.300	0.129E 03	22	-0.560E 02	22
0.350	0.254E 02	46	-0.214E 02	34
0.400	0.458E 01	51	-0.169E 02	39
0.450	0.363E 01	51	-0.108E 02	50
0.500	0.359E 01	51	-0.817E 01	17
0.550	0.312E 01	51	-0.112E 01	18
0.600	0.264E 01	51	0.736E 00	16
0.650	0.314E 01	51	0.435E 01	16
0.700	0.298E 01	51	-0.367E 01	17
0.750	0.223E 01	51	0.482E 01	18
0.800	0.279E 01	51	0.421E 01	16
0.850	0.204E 01	51	-0.121E 01	19
0.900	0.259E 01	51	0.467E 01	20
0.950	0.277E 01	40	0.704E 01	16
1.000	0.206E 01	46	0.497E 00	21

TABLE 4

ELASTIC ANALYSIS OF CYLINDER WITH CORE - RUN (2)

UNIFORM BAND LCAD OF WIDTH 2D/A= 0.0

G/E = 0.00004000   A/T = 400.00   POISSON RATIO - CORE = 0.300
CYLINDER = 0.300

X/A	2GU/Q*10 3	M	2GW/Q	M	TRXA/Q	M	SRA/Q	M	SXA/Q	M	STA/Q	M
0.0	0.0	0	-0.316E 00	5	0.0	0	-0.623E 01	6	-0.267E 01	6	-0.308E 01	6
0.010	0.921E 00	4	-0.296E 00	4	0.800E 00	4	-0.517E 01	4	-0.222E 01	4	-0.260E 01	4
0.020	0.173E 01	4	-0.255E 00	4	0.107E 01	5	-0.330E 01	5	-0.141E 01	5	-0.174E 01	5
0.030	0.241E 01	4	-0.205E 00	5	0.112E 01	5	-0.164E 01	6	-0.703E 00	6	-0.969E 00	6
0.040	0.295E 01	4	-0.157E 00	5	0.102E 01	5	-0.395E 00	11	-0.170E 00	11	-0.375E 00	7
0.050	0.335E 01	4	-0.115E 00	5	0.842E 00	6	0.431E 00	12	0.184E 00	12	0.358E-01	8
0.060	0.363E 01	4	-0.812E-01	6	0.657E 00	7	0.902E 00	11	0.386E 00	11	0.280E 00	9
0.070	0.384E 01	4	-0.555E-01	6	0.483E 00	7	0.111E 01	9	0.476E 00	9	0.404E 00	9
0.080	0.397E 01	5	-0.372E-01	7	0.336E 00	11	0.114E 01	9	0.490E 00	9	0.445E 00	12
0.090	0.407E 01	5	-0.245E-01	8	0.223E 00	13	0.108E 01	10	0.463E 00	10	0.428E 00	13
0.100	0.413E 01	5	-0.164E-01	9	0.139E 00	15	0.947E 00	13	0.406E 00	13	0.385E 00	13
0.110	0.417E 01	5	-0.112E-01	10	0.810E-01	17	0.806E 00	14	0.346E 00	14	0.331E 00	14
0.120	0.420E 01	5	-0.835E-02	11	0.428E-01	22	0.660E 00	15	0.283E 00	15	0.274E 00	14
0.130	0.423E 01	5	-0.670E-02	14	0.197E-01	30	0.5[illegible]7E 00	18	0.230E 00	18	0.221E 00	18
0.140	0.425E 01	5	-0.590E-02	15	0.870E-02	11	0.431E 00	18	0.185E 00	18	0.177E 00	18
0.150	0.427E 01	5	-0.547E-02	16	0.259E-02	11	0.343E 00	21	0.147E 00	21	0.140E 00	21
0.160	0.428E 01	6	-0.524E-02	16	0.394E-03	11	0.279E 00	22	0.120E 00	22	0.112E 00	23
0.170	0.430E 01	6	-0.510E-02	17	-0.447E-02	12	0.229E 00	24	0.985E-01	24	0.909E-01	27
0.180	0.432E 01	6	-0.488E-02	16	-0.418E-02	12	0.191E 00	26	0.823E-01	26	0.750E-01	29
0.190	0.433E 01	6	-0.466E-02	18	-0.363E-02	12	0.163E 00	30	0.700E-01	30	0.632E-01	31
0.200	0.435E 01	6	-0.467E-02	9	0.197E-02	13	0.141E 00	32	0.609E-01	32	0.545E-01	33

TABLE 5

ELASTIC ANALYSIS OF CYLINDER WITH CORE - RUN (3)

UNIFORM BAND LOAD OF WIDTH 2D/A= 0.0

G/E = 0.00004000 A/T = 400.00 POISSON RATIO - CORE = 0.300
CYLINDER = 0.300

X/A	2GU/Q*10 3	M	2GW/Q	M	TRXA/Q	M	SRA/Q	M	SXA/Q	M	STA/Q	M
0.0.	0.0	0	-0.316E 00	5	0.0	0	-0.623E 01	6	0.0	4	0.0	4
0.010	0.920E 00	4	-0.296E 00	4	0.800E 00	4	-0.517E 01	4	0.0	4	0.0	4
0.020	0.173E 01	4	-0.255E 00	4	0.107E 01	5	-0.330E 01	5	0.0	4	0.0	4
0.030	0.241E 01	4	-0.205E 00	5	0.112E 01	5	-0.164E 01	6	0.0	4	0.0	4
0.040	0.294E 01	4	-0.157E 00	5	0.102E 01	5	-0.395E 00	11	0.0	4	0.0	4
0.050	0.334E 01	4	-0.115E 00	5	0.842E 00	6	0.431E 00	12	0.0	4	0.0	4
0.060	0.363E 01	4	-0.812E-01	6	0.657E 00	7	0.902E 00	11	0.0	4	0.0	4
0.070	0.383E 01	4	-0.555E-01	6	0.483E 00	7	0.111E 01	9	0.0	4	0.0	4
0.080	0.397E 01	5	-0.372E-01	7	0.336E 00	11	0.114E 01	9	0.0	4	0.0	4
0.090	0.406E 01	5	-0.245E-01	8	0.223E 00	13	0.108E 01	10	0.0	4	0.0	4
0.100	0.413E 01	5	-0.164E-01	9	0.139E 00	15	0.947E 00	13	0.0	4	0.0	4
0.110	0.417E 01	5	-0.112E-01	10	0.811E-01	17	0.806E 00	14	0.0	4	0.0	4
0.120	0.420E 01	5	-0.835E-02	11	0.428E-01	22	0.660E 00	15	0.0	4	0.0	4
0.130	0.423E 01	5	-0.670E-02	14	0.197E-01	30	0.537E 00	18	0.0	4	0.0	4
0.140	0.425E 01	5	-0.590E-02	15	0.875E-02	11	0.431E 00	18	0.0	4	0.0	4
0.150	0.427E 01	5	-0.547E-02	16	0.259E-02	11	0.343E 00	21	0.0	4	0.0	4
0.160	0.428E 01	6	-0.524E-02	16	0.412E-03	11	0.279E 00	22	0.0	4	0.0	4
0.170	0.430E 01	6	-0.510E-02	17	-0.447E-02	12	0.229E 00	24	0.0	4	0.0	4
0.180	0.432E 01	6	-0.488E-02	16	-0.418E-02	12	0.191E 00	26	0.0	4	0.0	4
0.190	0.433E 01	6	-0.466E-02	18	-0.363E-02	12	0.163E 00	30	0.0	4	0.0	4
0.200	0.435E 01	6	-0.467E-02	9	0.199E-02	13	0.141E 00	32	0.0	4	0.0	4

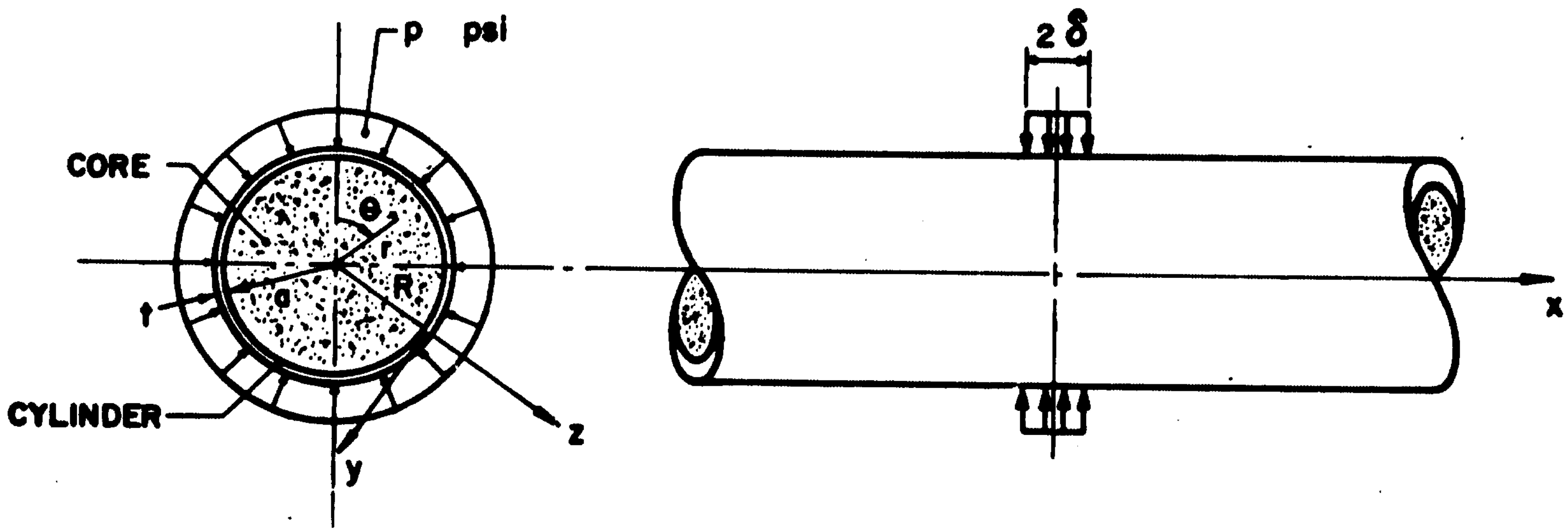

Figure 1. CYLINDER AND CORE GEOMETRY, COORDINATE SYSTEMS AND APPLIED LOAD

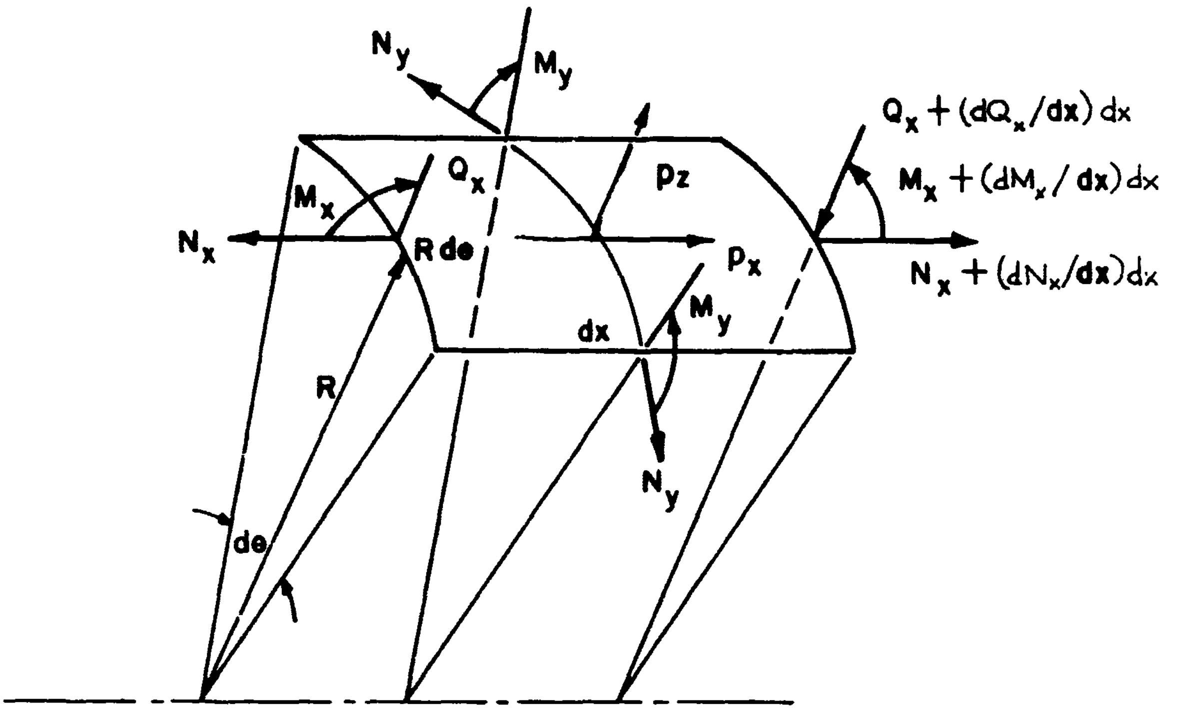

Figure 2. STRESS, LOAD AND BODY FORCE RESULTANTS ACTING ON MIDDLE SURFACE ELEMENT AND THEIR VARIATION

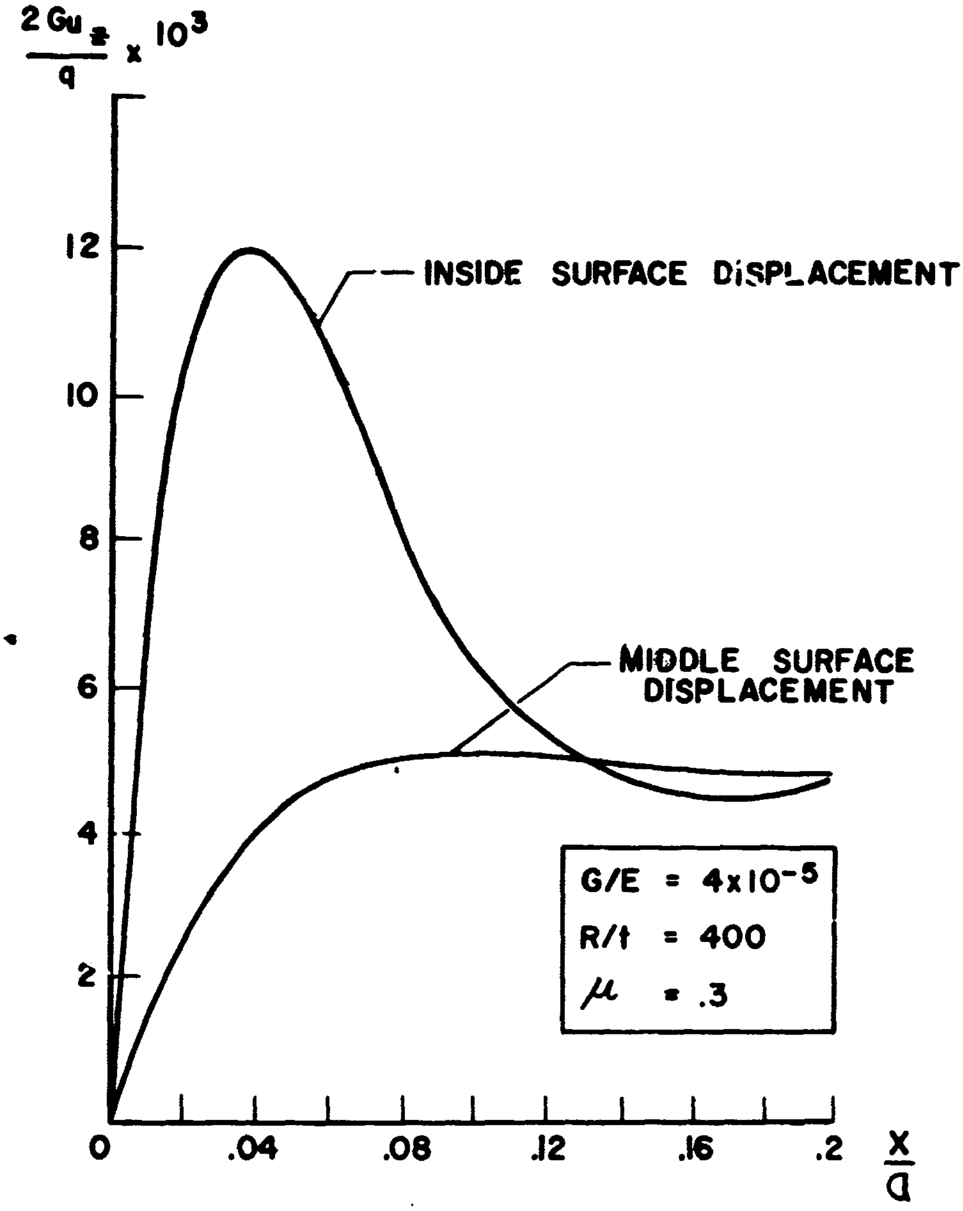

Figure 3. AXIAL VARIATION OF THE AXIAL DISPLACEMENTS OF CYLINDER SUBJECTED TO LINE LOAD AT THE INSIDE AND MIDDLE SURFACES

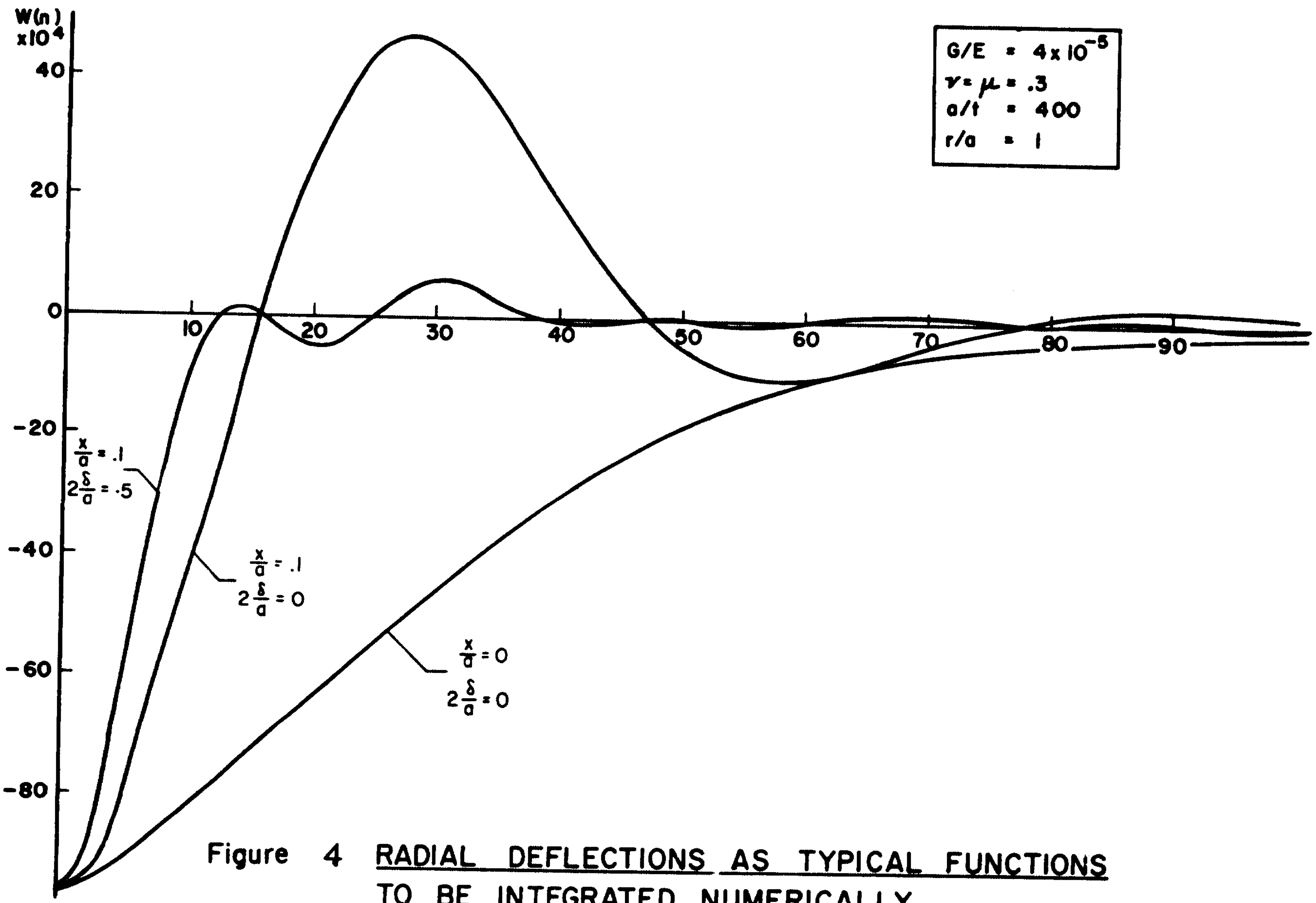

Figure 4 RADIAL DEFLECTIONS AS TYPICAL FUNCTIONS TO BE INTEGRATED NUMERICALLY

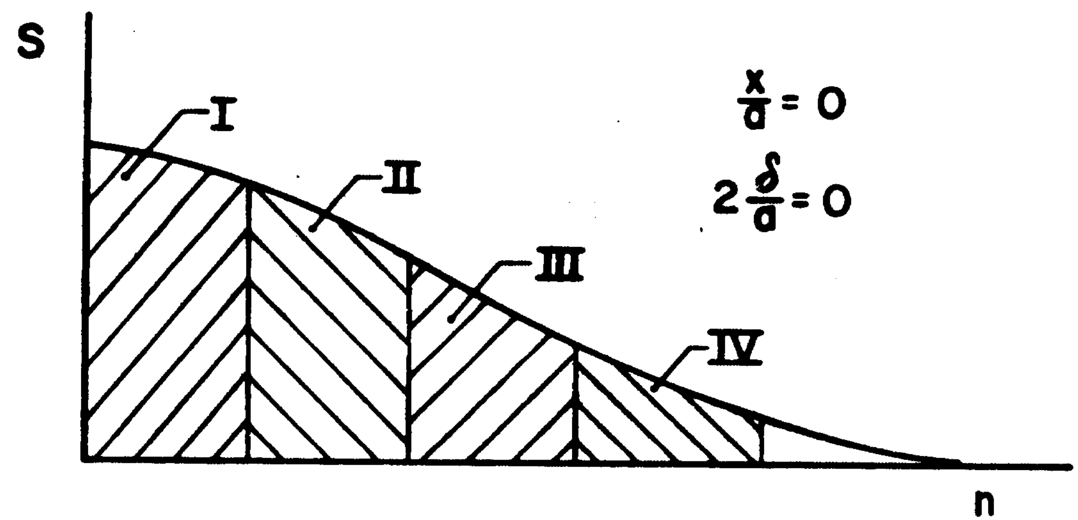

Figure 5. <u>NUMERICAL INTEGRATION SCHEME FOR NON TRIGONOMETRIC FUNCTIONS</u>

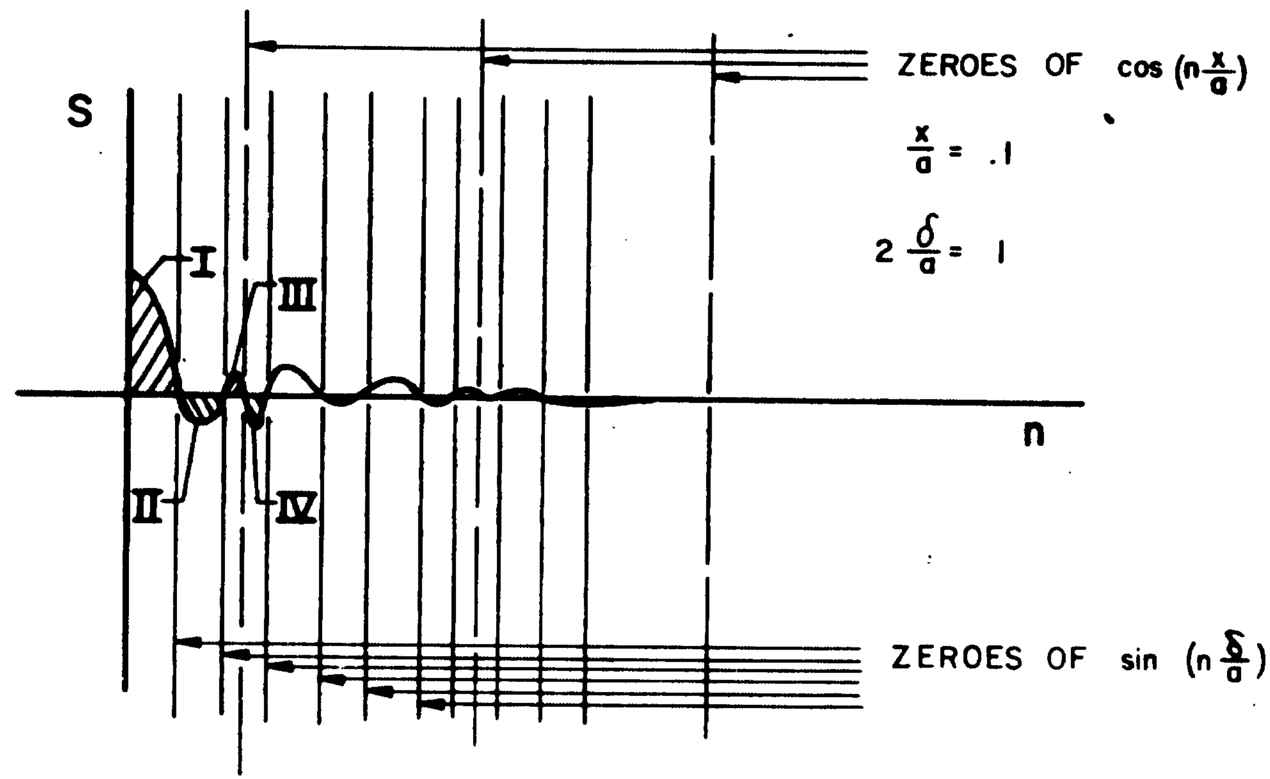

Figure 6. **NUMERICAL INTEGRATION SCHEME WHEN TRIGONOMETRIC FUNCTIONS ARE PRESENT**

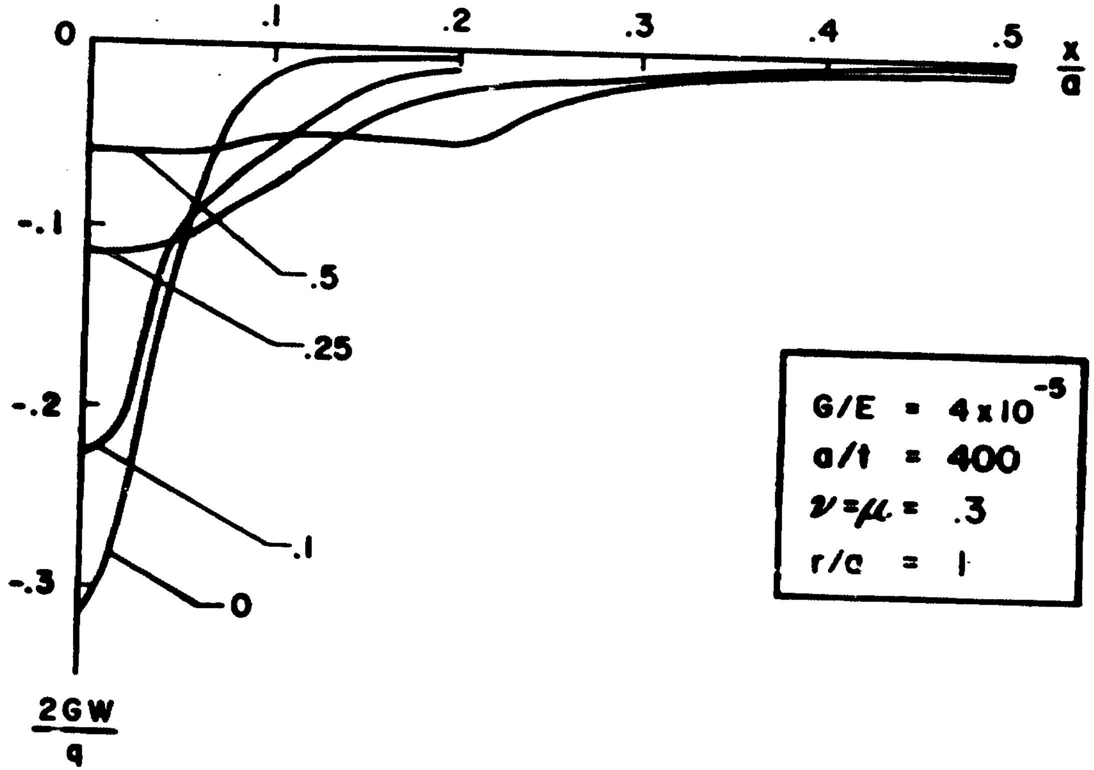

Figure 7. DISTRIBUTIONS OF RADIAL DISPLACEMENTS AT OUTER CORE RADIUS FOR $2\frac{\delta}{a}$ = 0, .1, .25, .5

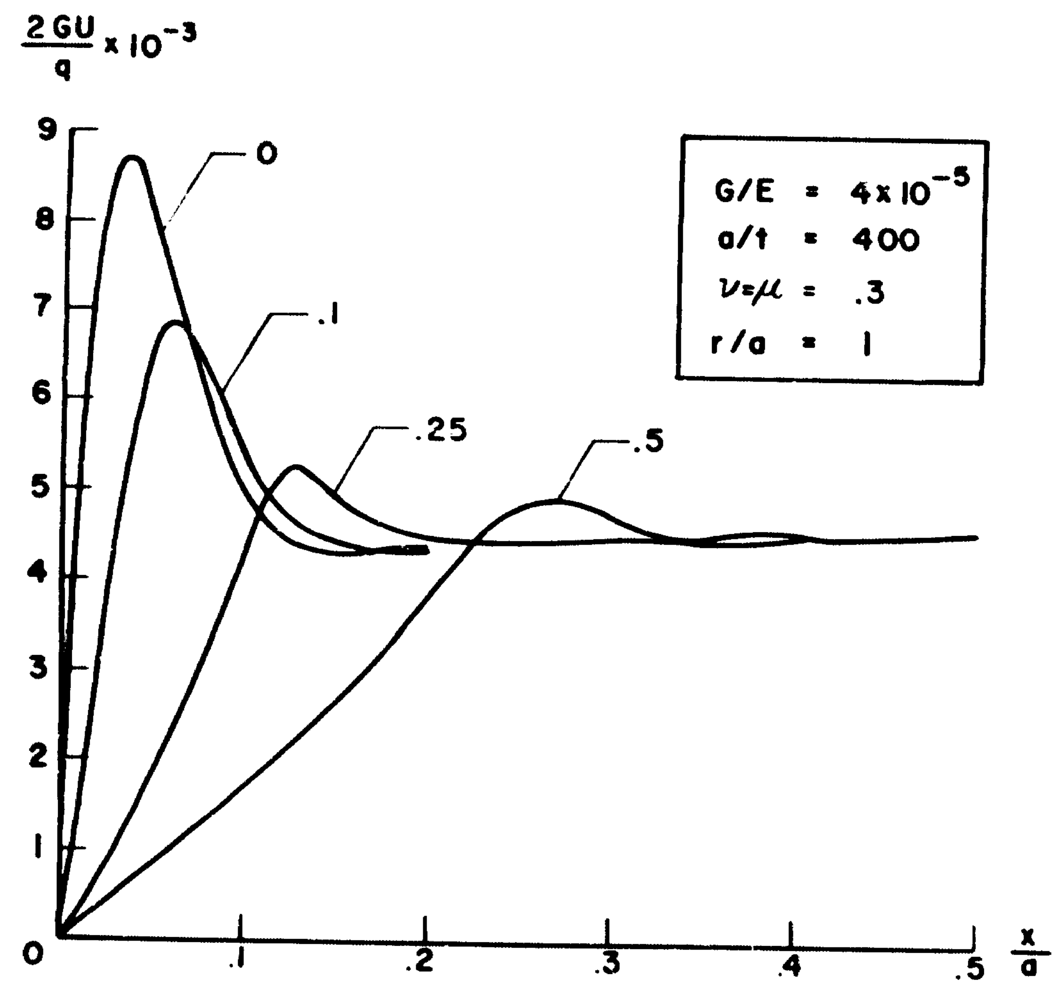

Figure 8. DISTRIBUTIONS OF AXIAL DISPLACEMENTS AT OUTER CORE RADIUS FOR $2\frac{\delta}{a} = 0, .1, .25, .5$

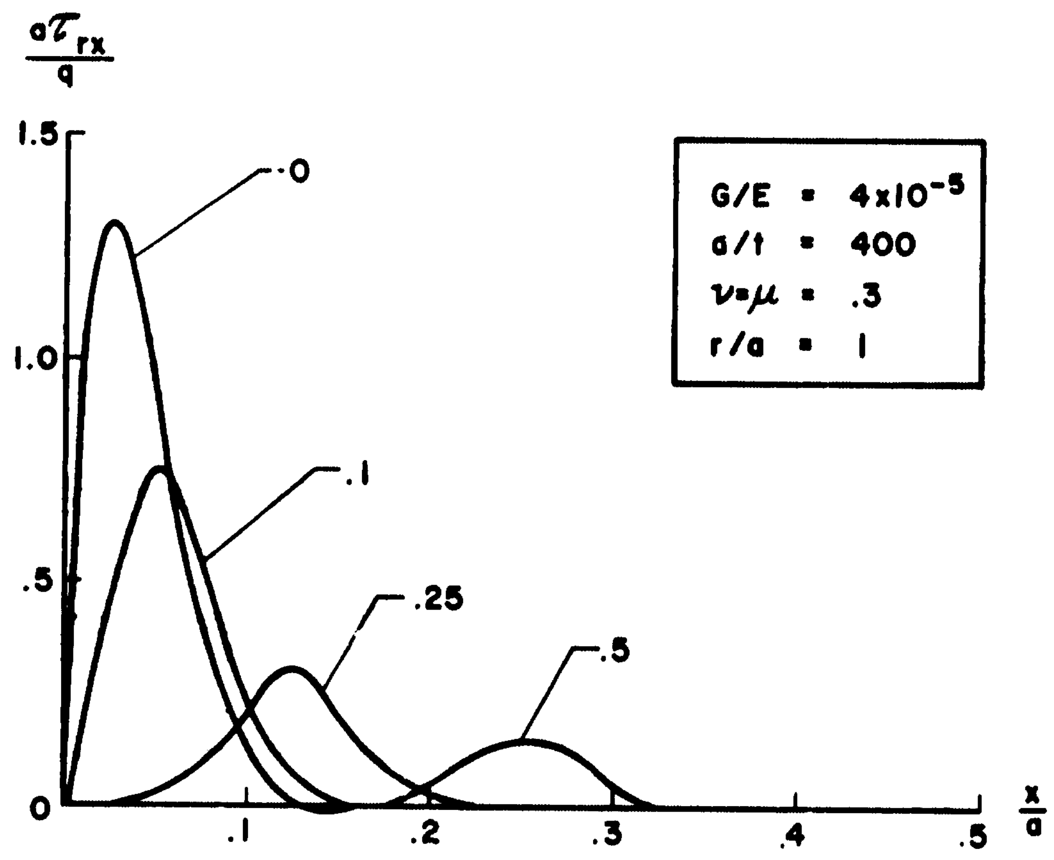

Figure 9. DISTRIBUTIONS OF SHEAR STRESSES AT OUTER CORE RADIUS FOR $2\frac{\varepsilon}{a}$ = 0, .1, .25, .5

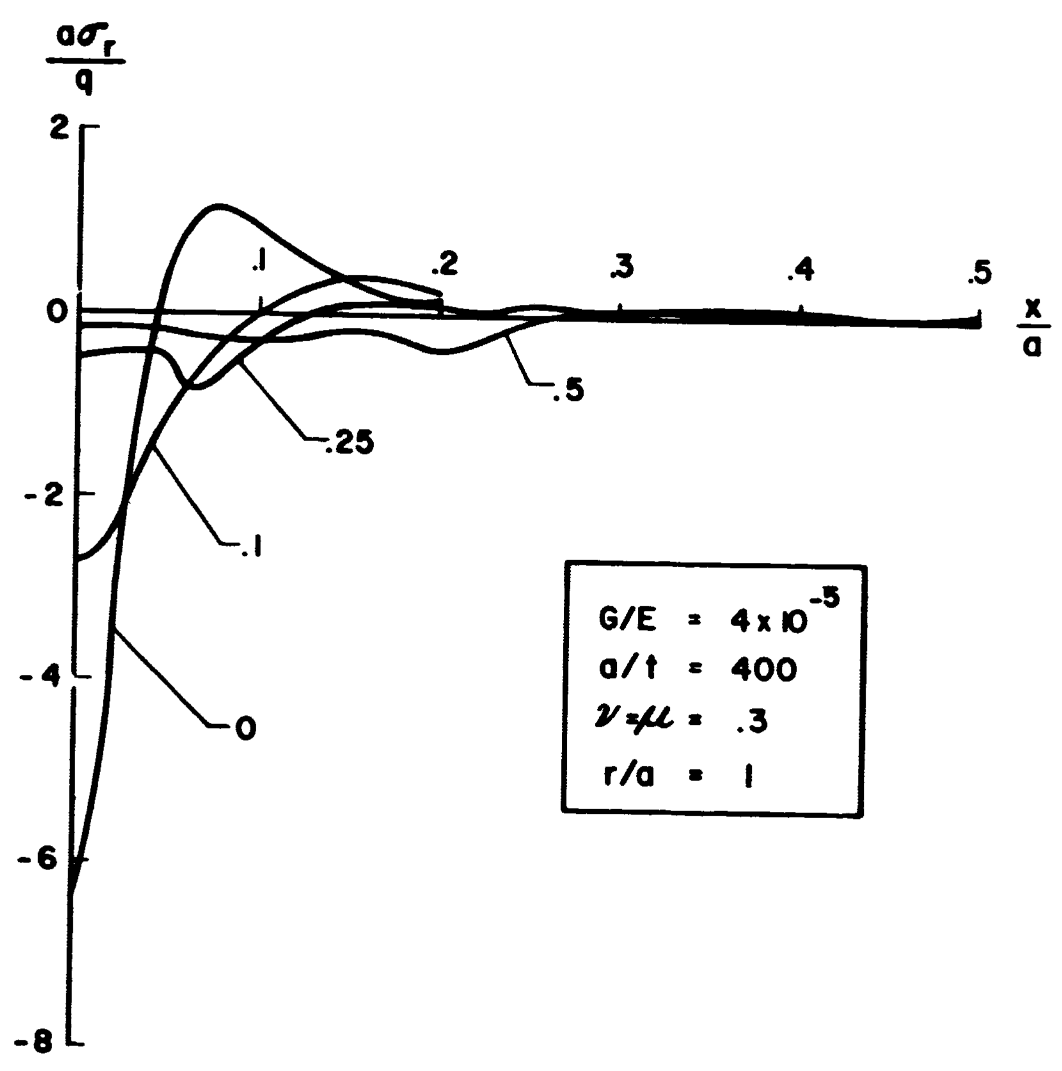

Figure 10. DISTRIBUTIONS OF RADIAL STRESSES AT OUTER CORE RADIUS FOR $2\frac{\delta}{a} = 0, .1, .25, .5$

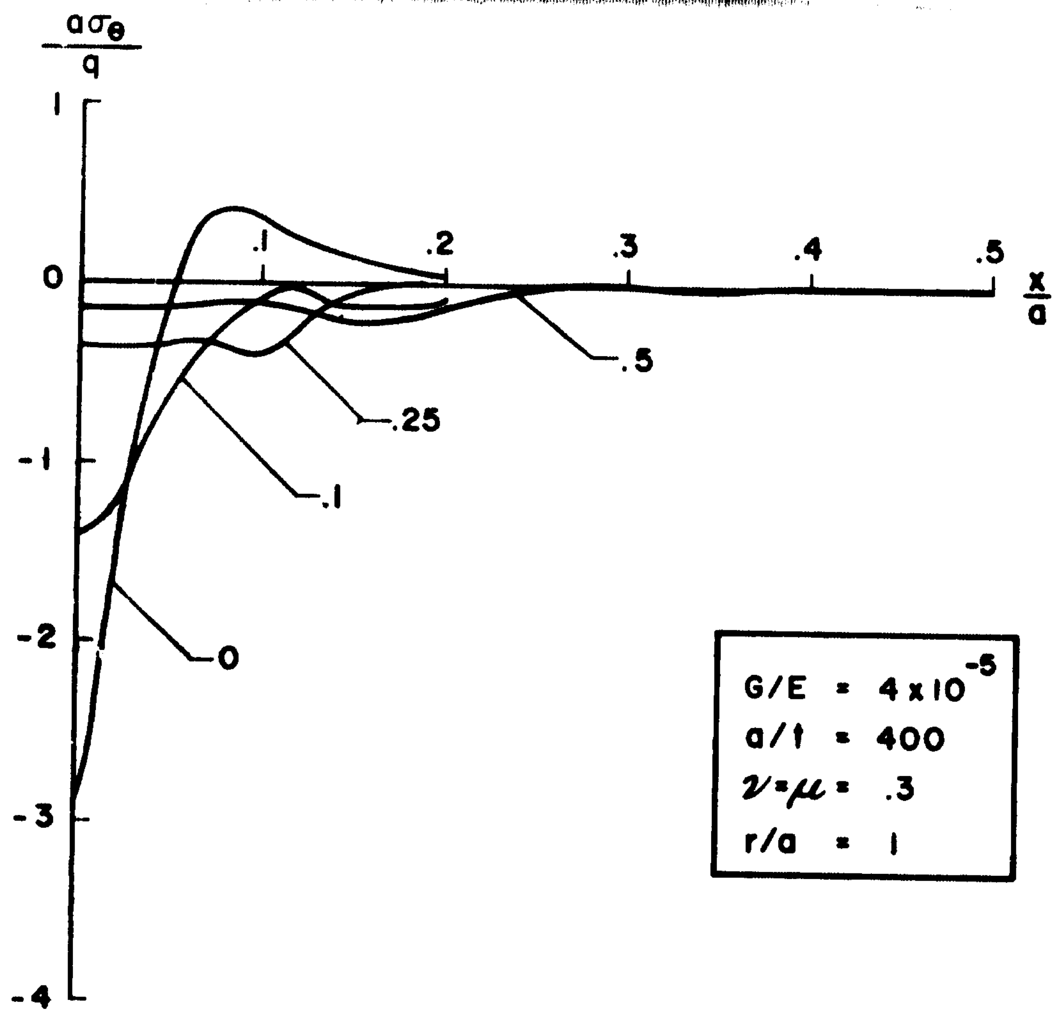

Figure 11. DISTRIBUTIONS OF CIRCUMFERENTIAL STRESSES AT OUTER CORE RADIUS FOR $2\frac{\xi}{a}$ = 0, .1, .25, .5

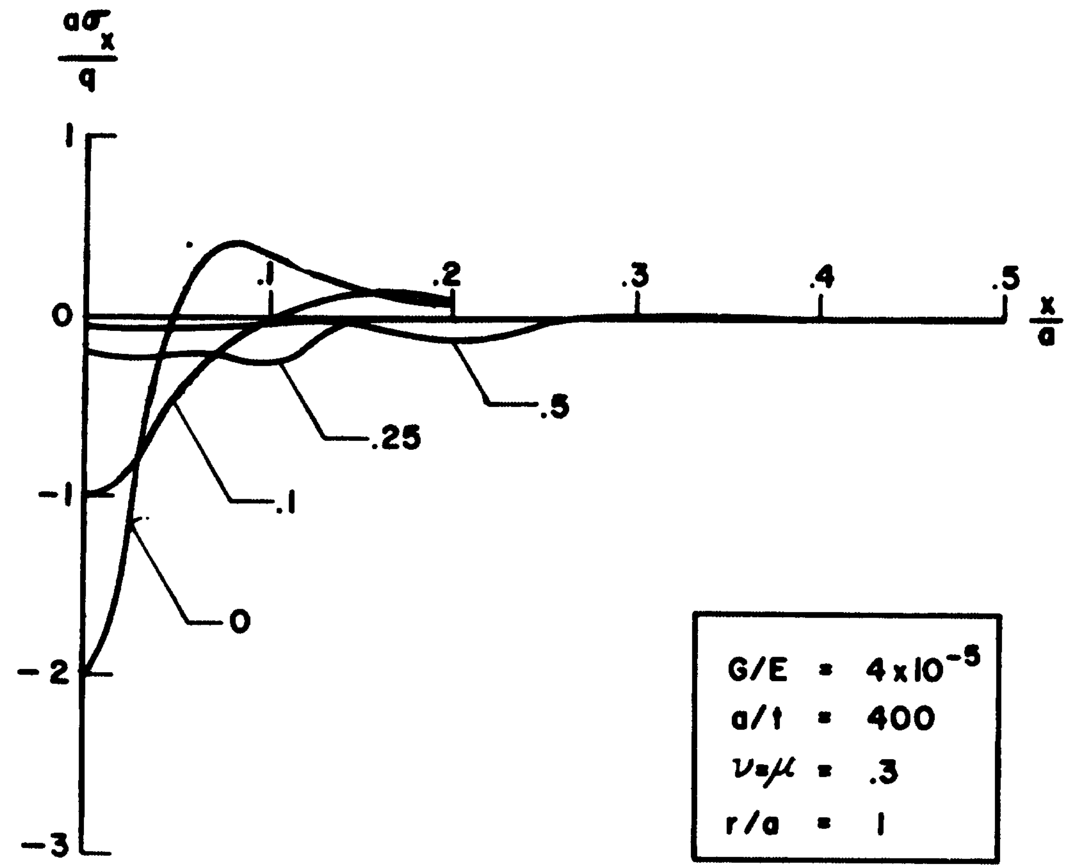

Figure 12. DISTRIBUTIONS OF AXIAL STRESSES AT OUTER CORE RADIUS FOR $2\frac{\delta}{a} = 0, .1, .25, .5$

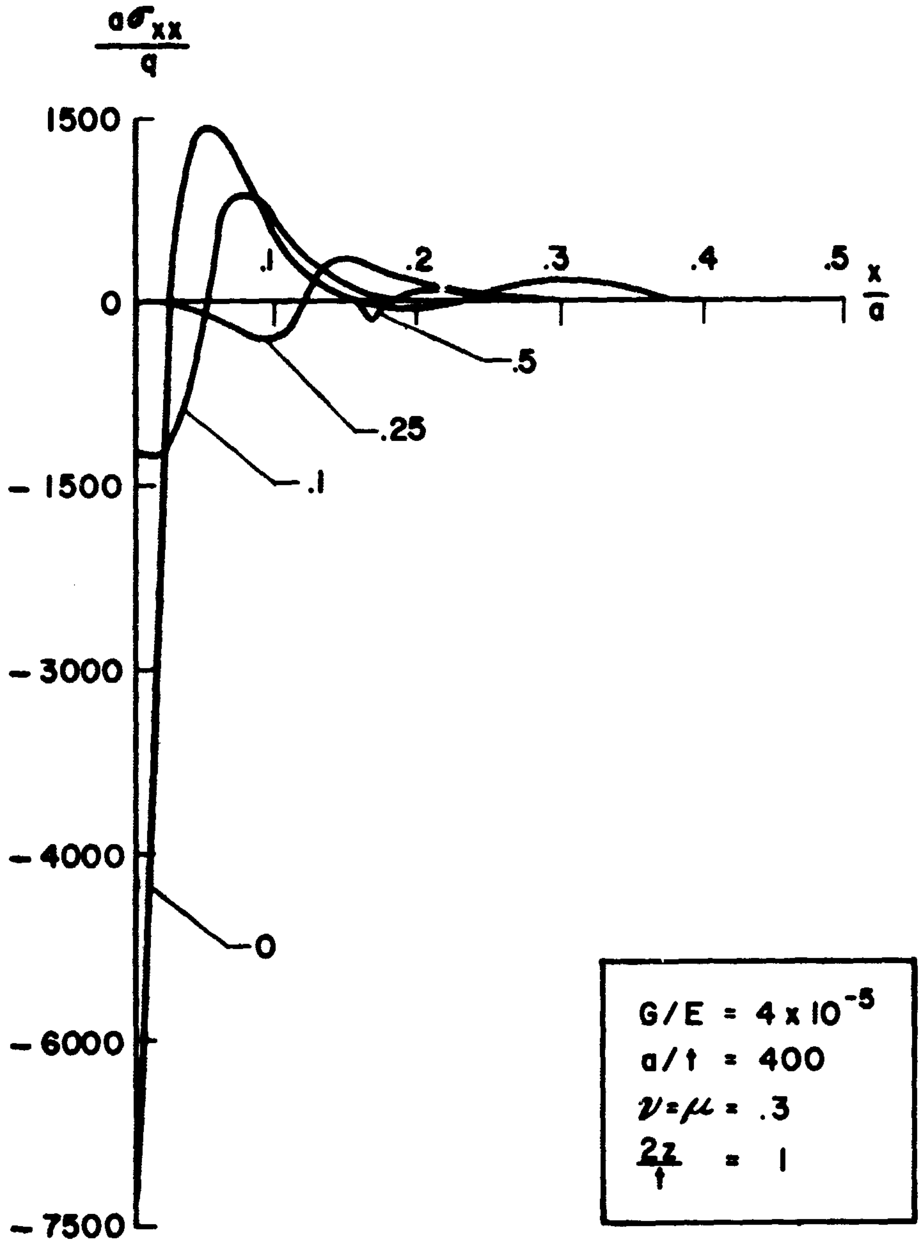

Figure 13. DISTRIBUTION OF AXIAL CYLINDER STRESSES FOR $2\frac{\delta}{a} = 0, .1, .25, .5$

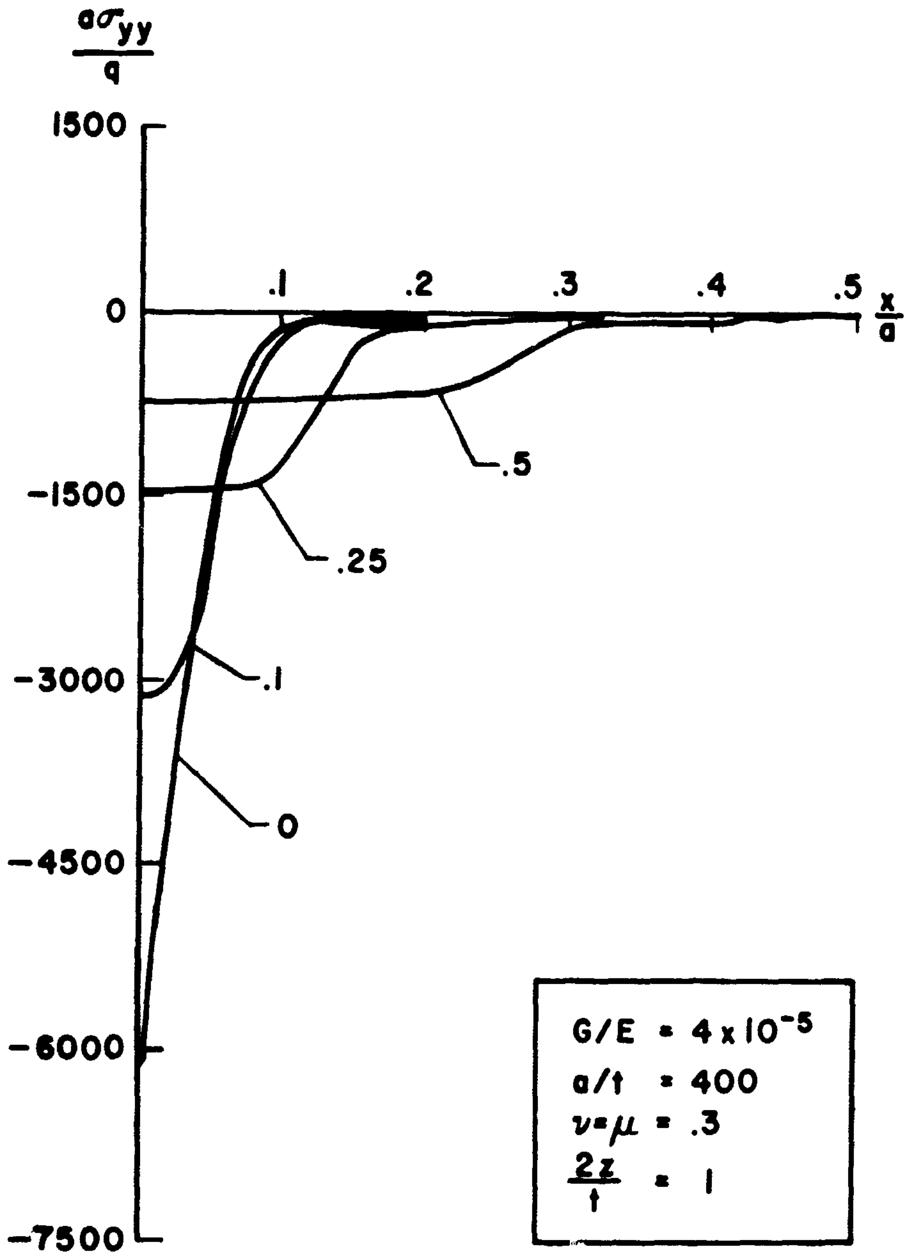

Figure 14. DISTRIBUTION OF CIRCUMFERENTIAL CYLINDER STRESSES FOR $2\frac{\delta}{a} = 0, .1, .25, .5$

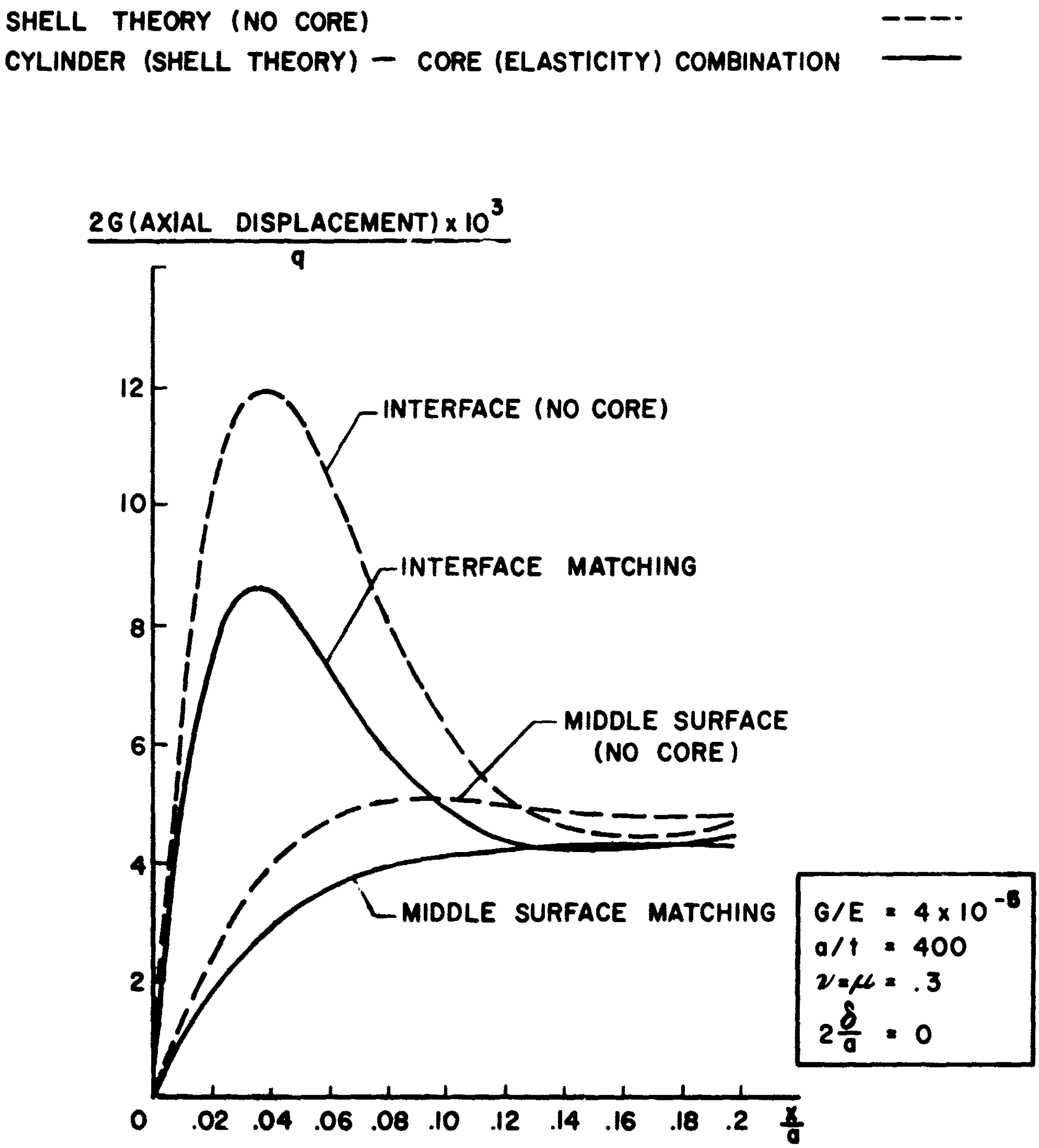

Figure 15. AXIAL DISPLACEMENTS FOR LINE LOAD— DIFFERENT MATCHING ASSUMPTIONS

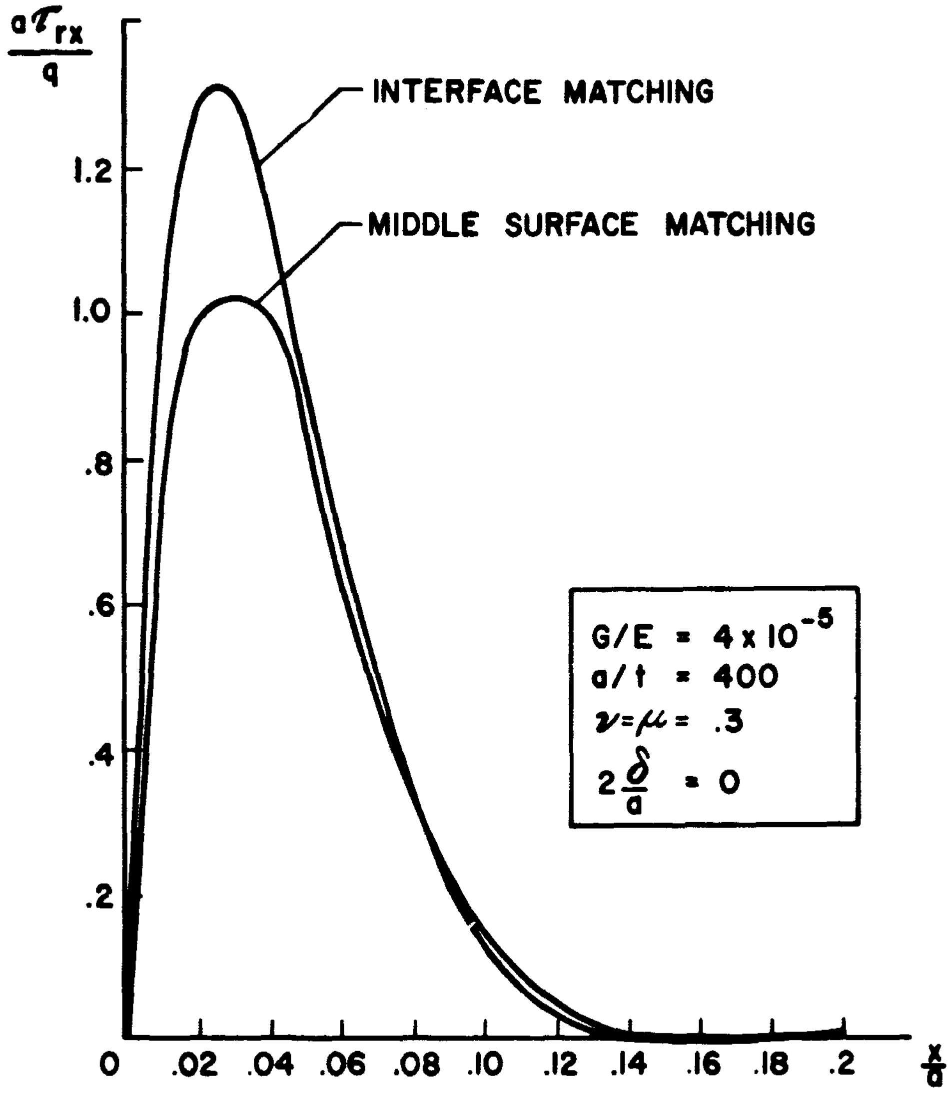

Figure 16. SHEAR STRESS DISTRIBUTIONS FOR LINE LOAD — DIFFERENT MATCHING ASSUMPTIONS

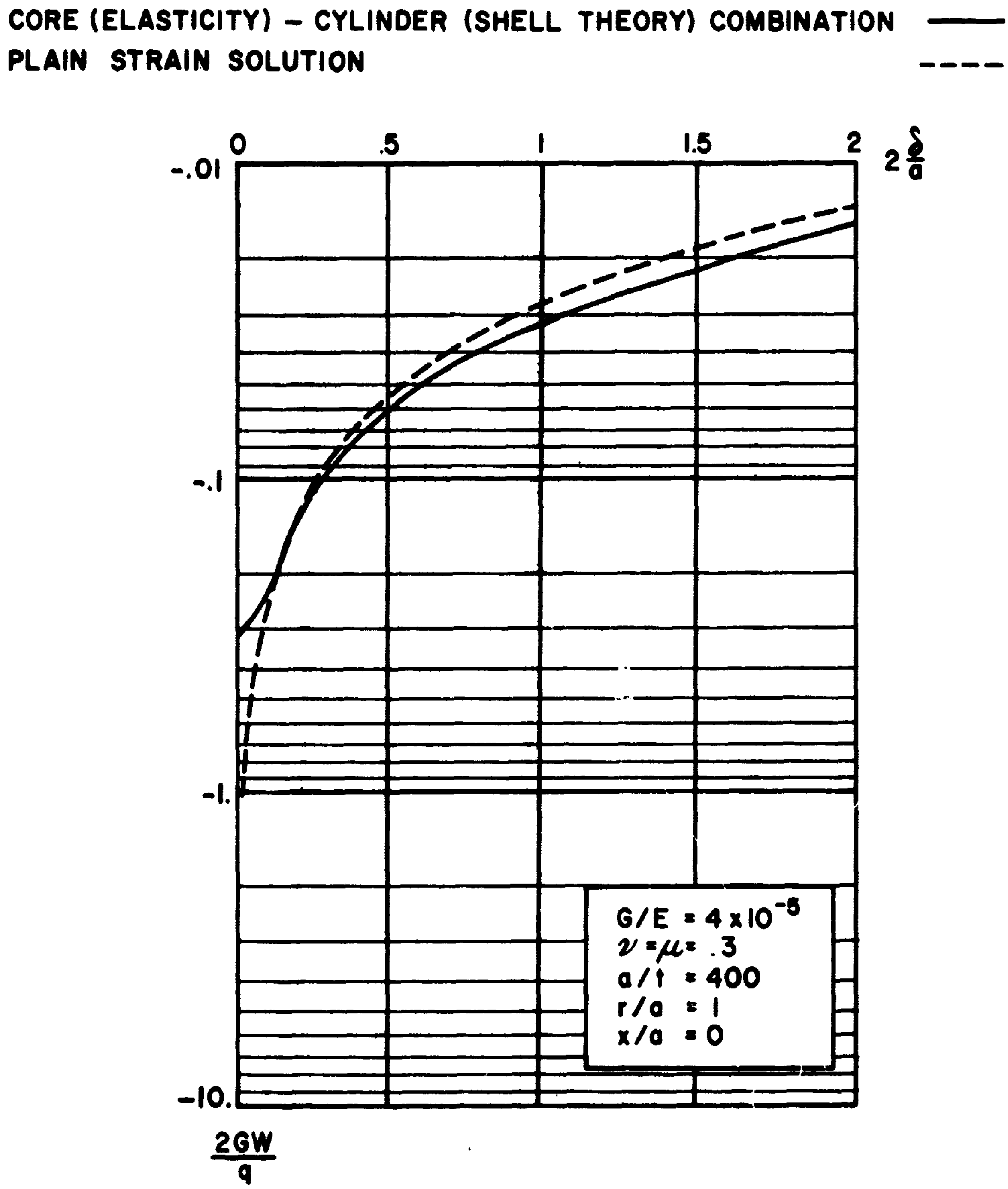

Figure 17. CHANGES IN RADIAL DISPLACEMENT OF CORE AT BAND LOAD CENTER vs BAND LOAD WIDTH

CORE (ELASTICITY) — CYLINDER (SHELL THEORY) COMBINATION ——
PLANE STRAIN SOLUTION ----

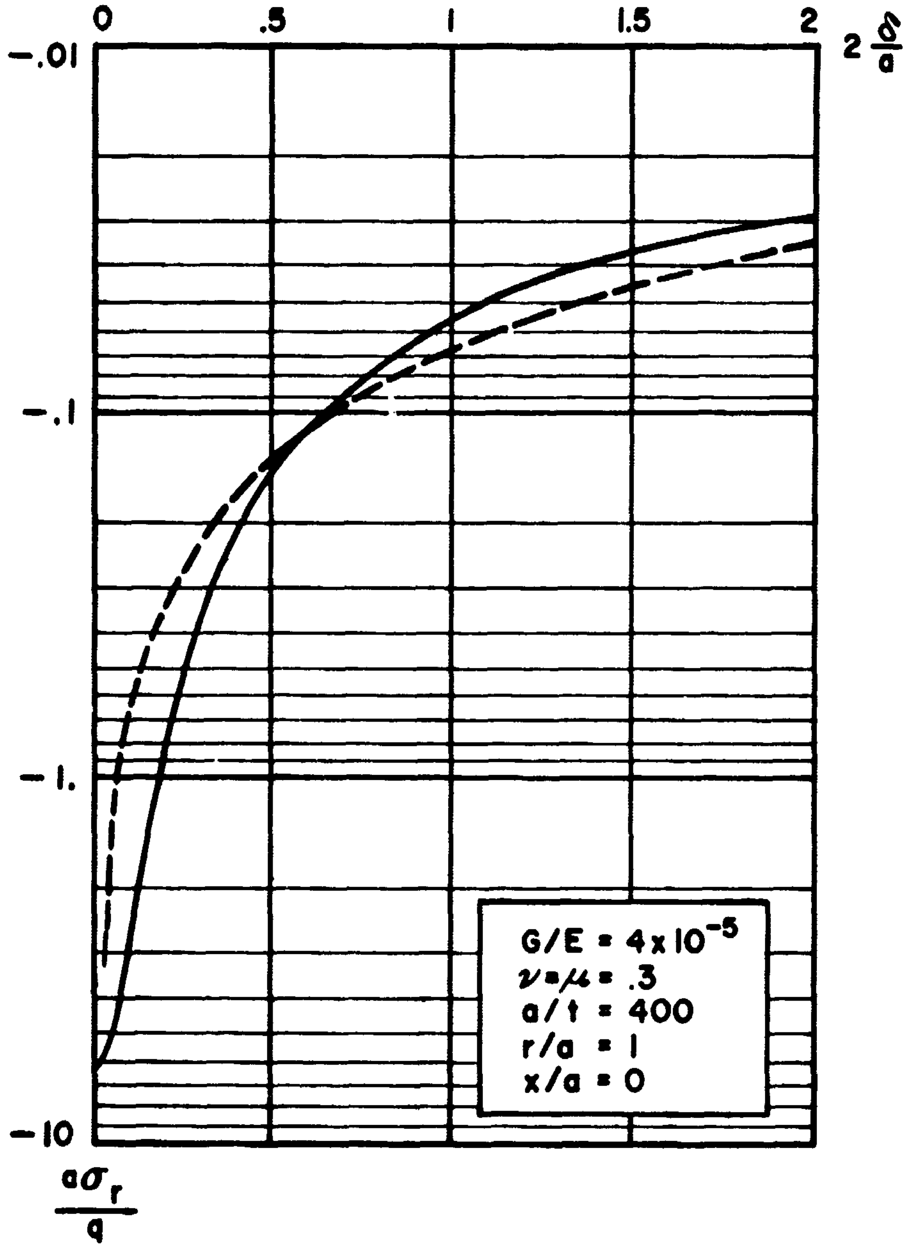

Figure 18. CHANGES IN RADIAL NORMAL STRESSES OF CORE AT BAND LOAD CENTER vs BAND LOAD WIDTH

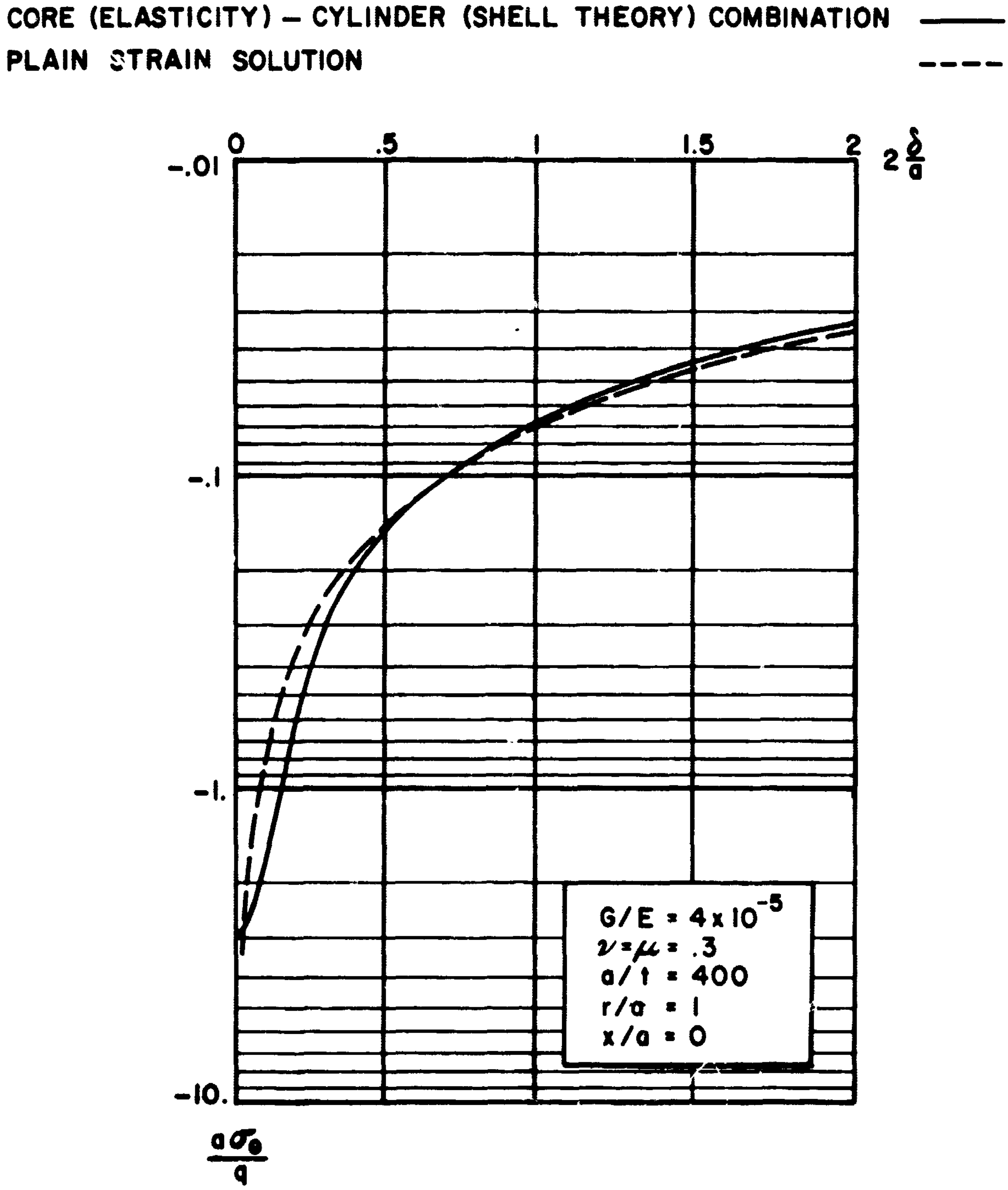

Figure 19. CHANGES IN CIRCUMFERENTIAL NORMAL STRESSES OF CORE AT BAND LOAD CENTER vs BAND LOAD WIDTH

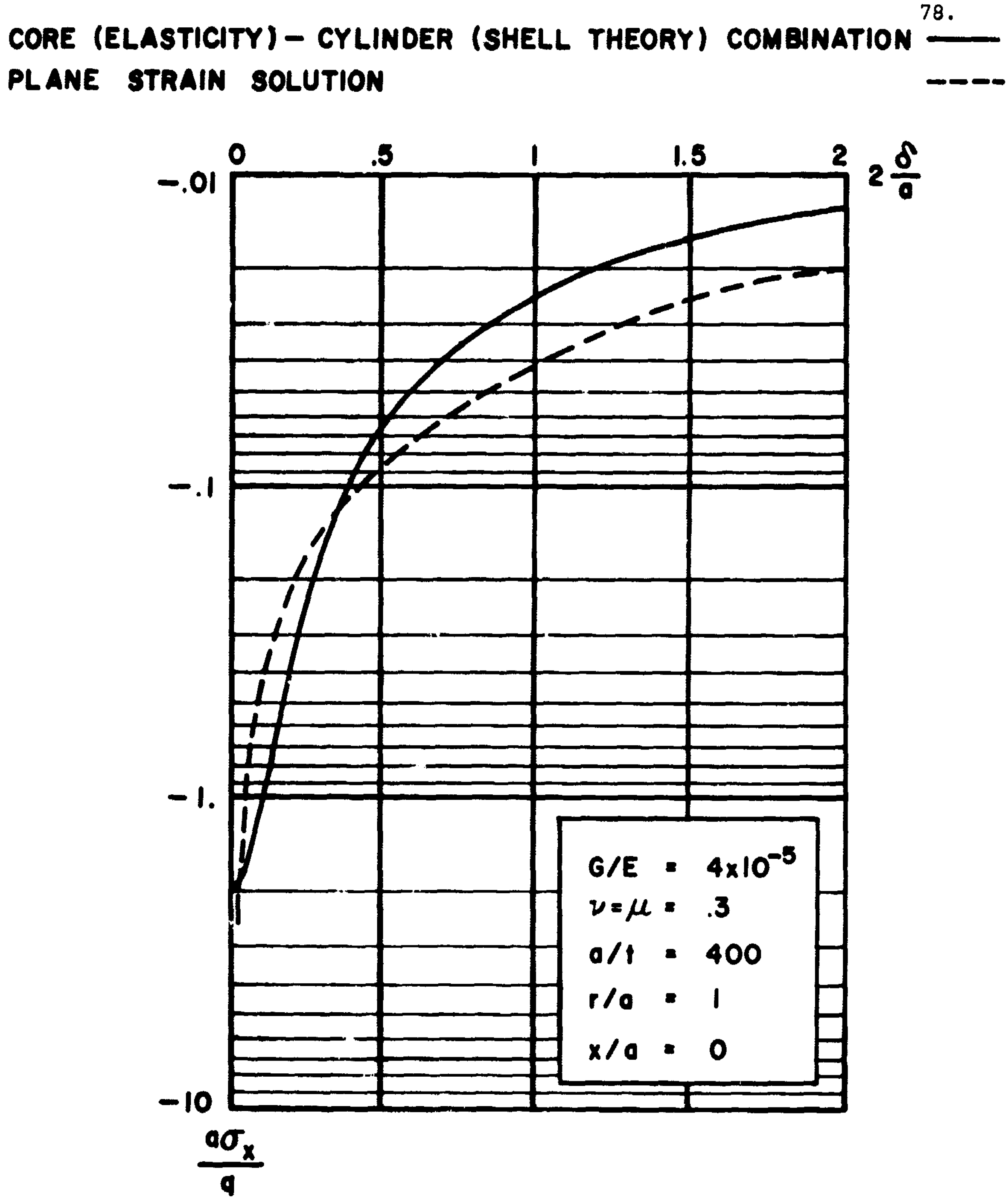

Figure 20. CHANGES IN AXIAL NORMAL STRESSES OF CORE AT BAND LOAD CENTER vs BAND LOAD WIDTH

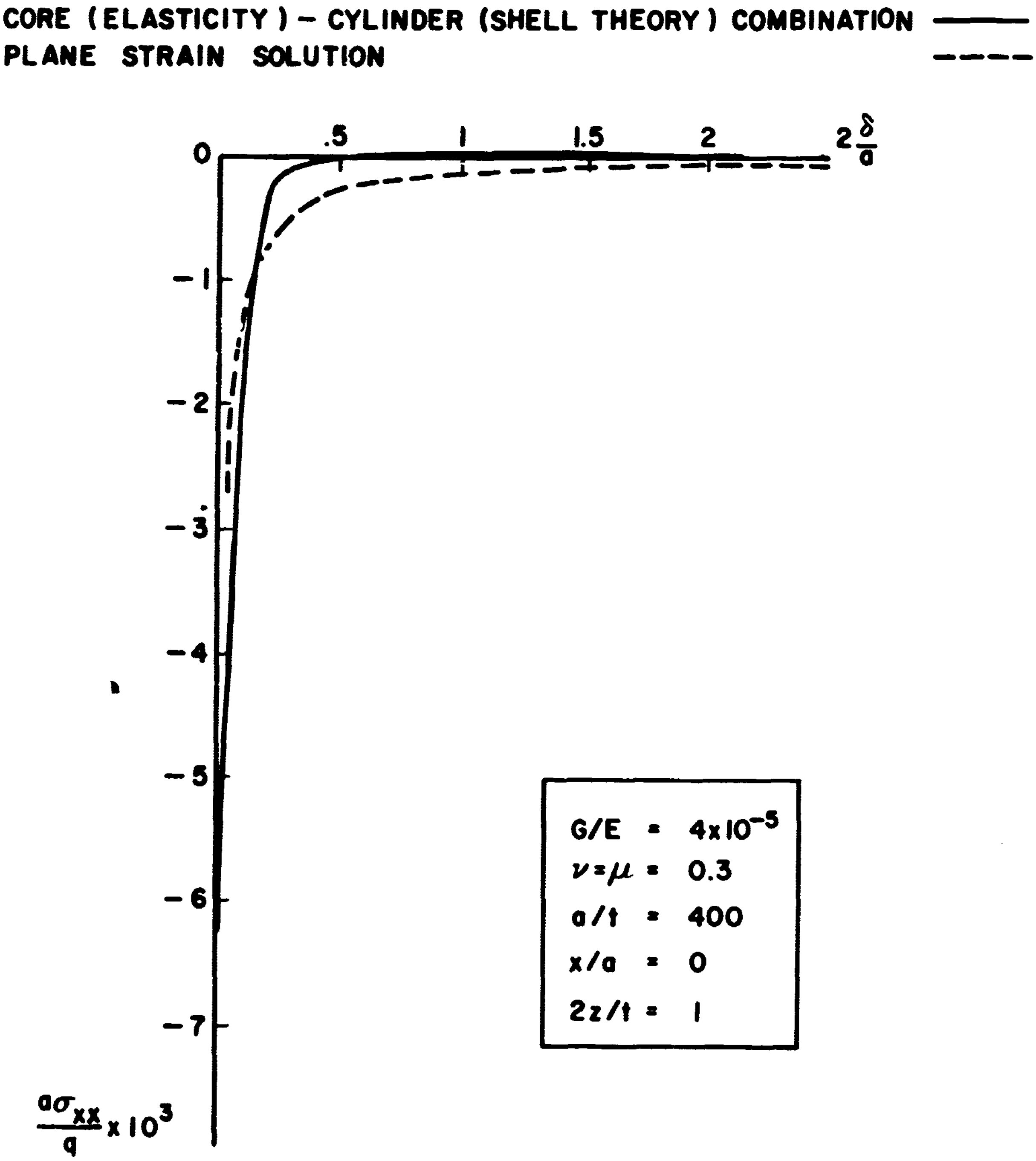

Figure 21. CHANGES IN AXIAL CYLINDER STRESSES AT BAND LOAD CENTER vs BAND LOAD WIDTH

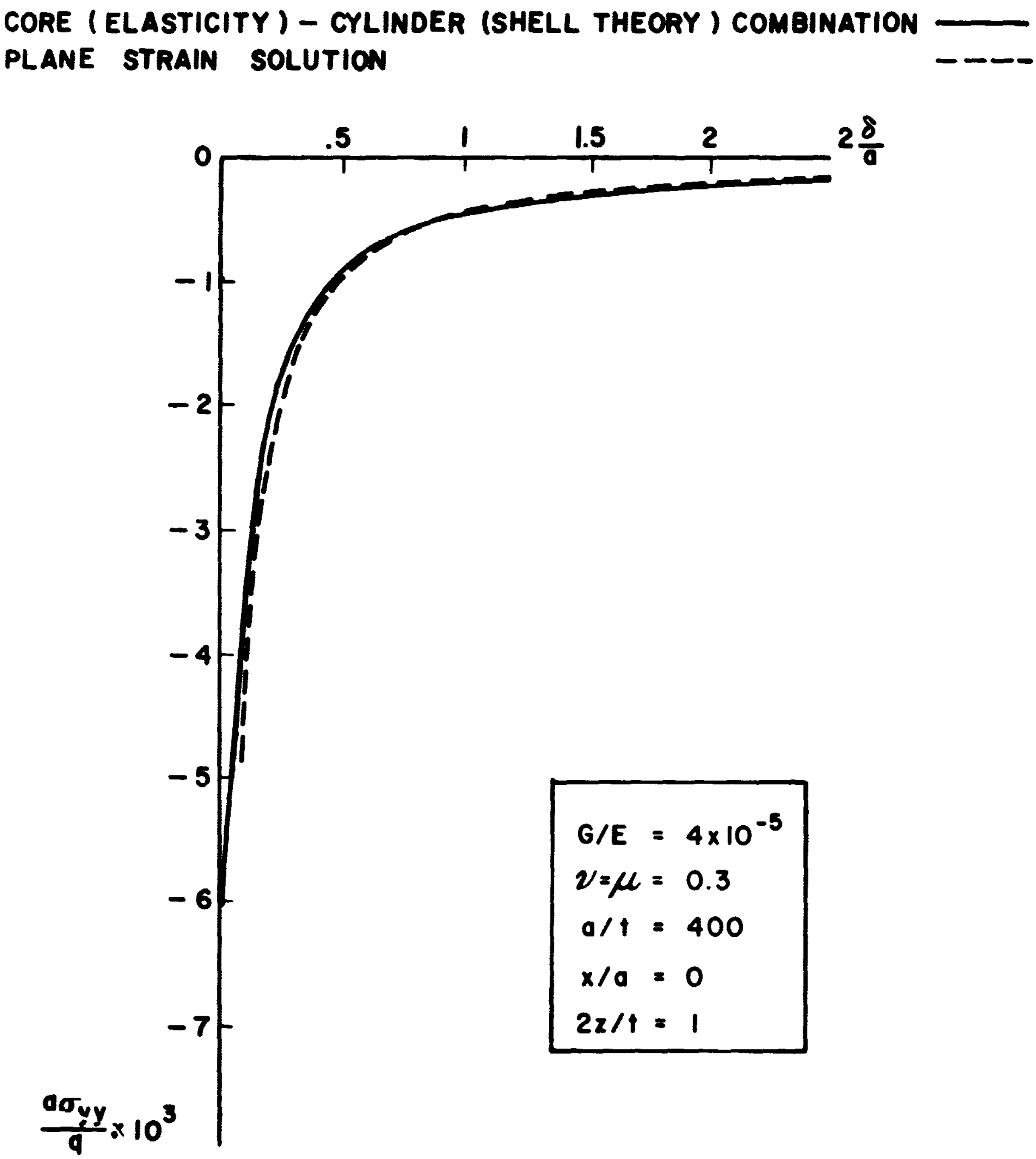

Figure 22. CHANGES IN CIRCUMFERENTIAL CYLINDER STRESSES AT BAND LOAD CENTER vs. BAND LOAD WIDTH